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$$F(x) = \int_0^\infty e^{-xt} \sum_{k=0}^\infty \frac{(-1)^k}{2k!} t^{2k}\, dt = \sum_{k=0}^\infty \frac{(-1)^k}{2k!} \int_0^\infty t^{2k} e^{-xt}\, dt \qquad x \gg 1$$

Series converges uniformly $(t \in (0,M))$ $M < \infty$ bounded interval.

Now $\int_0^\infty t^{2k} e^{-xt}\, dt$ let $v = xt$

$dt = \frac{dv}{x}$

$$= \int_0^{\infty\ (x>0)} \frac{v^{2k}}{x^{2k}} e^{-v} \frac{dv}{x} = \frac{1}{x^{2k+1}} \int_0^{\infty\ (x>0)} v^{2k} e^{-v}\, dv$$

Do Integration by Parts on last integral, ... or remember def of Gamma fn

$$\int_0^\infty e^{-t} t^{z-1}\, dt = (z-1)! \quad \text{when } z \in \mathbb{N}.$$

$\therefore$ Above integral is $2k!$

$$\therefore\ F(x) = \sum_{k=0}^\infty \frac{(-1)^k}{x^{2k+1}} = \frac{1}{x} \sum_{k=0}^\infty \left(\frac{(-1)}{x^2}\right)^k$$

$$= \frac{1}{x}\frac{1}{\left(1 - \frac{(-1)}{x^2}\right)} \qquad \text{if } |x| > 1$$

$$= \frac{x}{x^2+1}$$ But consider the orig integral

$$F(x) = \int_0^\infty e^{-xt} \cos t \, dt = \cos t \frac{e^{-xt}}{(-x)} \Bigg|_0^\infty \mp \frac{1}{x} \int_0^\infty \sin t \, e^{-xt} dt$$

$(x>0)$

$$= 0 + \frac{1}{x} - \frac{1}{x} \int_0^\infty \sin t \, e^{-xt} dt$$

$(x>0)$

$$= \frac{1}{x} - \frac{1}{x} \left[\frac{\sin t \, e^{-xt}}{(-x)} \Bigg|_0^\infty - \frac{1}{(-x)} \int_0^\infty \cos t \, e^{-xt} dt \right]$$

$$= \frac{1}{x} - \frac{1}{x} \left[0 - 0 + \frac{1}{x} F(x) \right]$$

$$= \frac{1}{x} - \frac{1}{x^2} F(x) \Rightarrow \left(1 + \frac{1}{x^2}\right) F(x) = \frac{1}{x}$$

$$\Rightarrow F(x) = \frac{1/x}{1 + 1/x^2} = \frac{x}{x^2+1}$$

Do other order of integration by parts to see if you get the same result.

$$G(x) = \int_0^\infty e^{-xt} \sum_{k=0}^\infty (-1)^k t^k \, dt$$

series only converges uniformly $t \in (0,1)$

$$= \sum_{k=0}^\infty (-1)^k \underbrace{\int_0^\infty t^k e^{-xt} \, dt}_{\frac{1}{x^{k+1}} k!}$$ From before

$$= \sum_{k=0}^\infty \frac{(-1)^k k!}{x^{k+1}}$$

$$G(x) - g_n(x) = G(x) - \sum_{k=0}^{(n-1)} \frac{(-1)^k k!}{x^{k+1}}$$

$$\frac{1}{1-t} = 1 + t + t^2 + \cdots + t^n$$

$$\sum_{k=0}^{N} t^k = \frac{1}{t-1} \sum_{k=0}^{N} \Delta t^k = \frac{1}{t-1} \, t^{k} \Big|_0^{N+1}$$

$$\Delta t^k = t^k (t-1)$$

$$= \frac{1}{t-1}\left(t^{N+1} - 1\right) = \frac{1}{1-t} + \frac{t^{N+1}}{t-1}$$

$$\rightarrow \quad \frac{1}{1-t} = \sum_{k=0}^{N} t^k + \frac{t^{N+1}}{1-t}$$

Replace $t \to -t$

$$\frac{1}{1+t} = \sum_{k=0}^{n-1} (-1)^k t^k + \frac{(-1)^n t^n}{1+t}$$

$$\mathcal{E}_n(x) = \int_0^\infty \frac{e^{-xt}}{1+t}\,dt - \sum_{k=0}^{(n-1)} \frac{(-1)^k k!}{x^{k+1}}$$

$$= \int_0^\infty e^{-xt}\left[\cancel{\sum_{k=0}^{n-1} (-1)^k t^k} + \frac{(-1)^n t^n}{1+t}\right] dt - \cancel{\sum_{k=0}^{n-1} \frac{(-1)^k k!}{x^{k+1}}}$$

$$= (-1)^n \int_0^\infty \frac{t^n}{1+t} e^{-xt}\,dt$$

$$|\mathcal{E}_n(x)| \le \frac{1}{1} \int_0^\infty t^n e^{-xt}\,dt = \frac{n!}{x^{n+1}}$$

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$x \in \mathbb{R}$

1

$$e^{ix}\{1+o(1)\} + e^{-ix}\{1+o(k)\} =$$

$$= e^{ix} + o(1) + e^{-ix} + o(1) = 2\cos x + o(1)$$

Pg 6 Olver

" $f = o(\phi)$
"f is of order less than ϕ"

① Pr: $x^{\nu} = o(e^{x})$ $x \to \infty$

$$\lim_{x\to\infty} \frac{x^{\nu}}{e^{x}} = \lim_{x\to\infty} \frac{e^{\nu \log x}}{e^{x}} = \lim_{x\to\infty} e^{\nu \log x - x} = 0$$

Any sign $Re(\nu)$

$e^{-x} = o(x^{\nu})$ $x \to \infty$

$$\lim_{x\to\infty} \frac{e^{-x}}{x^{\nu}} = \lim_{x\to\infty} \frac{e^{-x}}{e^{\nu \log x}} = \lim_{x\to\infty} e^{-x-\nu\log x} = 0$$

Any sign $Re(\nu)$

$\ln x = o(x^{\nu})$ ∞/∞

$$\lim_{x\to\infty} \frac{\ln x}{x^{\nu}} = \lim_{x\to\infty} \frac{1/x}{\nu x^{\nu-1}} = \lim_{x\to\infty} \frac{1}{\nu x^{\nu}} = 0 \qquad Re(\nu) > 0$$

$f = O(\phi)$
"f is of order not exceeding"

Ex 2.2 $x + o(x) = O(x)$

$$\lim_{x\to\infty} \frac{x+o(x)}{x} = \lim_{x\to\infty} 1 + \frac{o(x)}{x} = 1 \quad \therefore$$

$$\left|\frac{x+o(x)}{x}\right| < 1 \quad x \to \infty$$

1st eq 1st

$$\{O(x)\}^2 = O(x^2) = o(x^3)$$

$$\cancel{\lim_{x\to\infty} \frac{\{O(x)\}^2}{x^2}} = \lim_{x\to\infty} \frac{O(x)}{x}\frac{O(x)}{x} = \cancel{OO} = 1\cdot 1$$

$$\left|\frac{[O(x)]^2}{x^2}\right| = \frac{O(x)}{x}\frac{O(x)}{x} \le K_1^2 \quad \text{\& is} \therefore \text{ Bounded.}$$

So $O(x^2) = o(x^3)$

$$\lim_{x\to\infty} \frac{O(x^2)}{x^3} \le \lim_{x\to\infty} \frac{Kx^2}{x^3} = 0.$$

(2.3) Pr:

$$\cos(O(x^{-1})) = O(1) \qquad x \to \infty$$

~~let $\phi(x) = O(x^{-1})$~~

~~Then~~

$$\left|\frac{\cos(O(x^{-1}))}{1}\right| \le 1 \quad \checkmark$$

Pr:

$$\sin(O(x^{-1})) = O(x^{-1})$$

As $|\sin x| \le |x| \; \forall x$

$$\left|\frac{\sin(O(x^{-1}))}{x^{-1}}\right| \le \frac{O(x^{-1})}{x^{-1}} \le K.$$

Pr: $\cos\{x+\alpha+o(1)\} = \cos(x+\alpha) + o(1) \qquad \alpha \in \mathbb{R}.$

$\Rightarrow$ let $f_1, f_2 = o(1)$ order less than 1.

Then $\cos(x+\alpha + f_1(x)) = \cos(x+\alpha) + -\sin(x+\alpha$

$\Leftrightarrow$ Pr:

$$\cos(x+\alpha+o(1)) - \cos(x+\alpha) = o(1)$$

Thus consider

$$\left| \underbrace{\cos(x+\alpha+o(1))} - \cos(x+\alpha) \right|$$

$$\cos\cancel{(x+\alpha) + -\sin(x+\alpha)o}(1)$$

$$\left| \cos(x+\alpha) - \sin(x+\alpha+\xi)\, o(1) - \cos(x+\alpha) \right| \leq o(1) = o(1)$$

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Ex 2.4

$$\{1+o(1)\}\cosh x - \{1+o(1)\}\sinh x = \{1+o(1)\}e^{-x} \qquad x\to\infty.$$

Claim no

$$\{1+o(1)\}\tfrac{1}{2}(e^x+e^{-x}) - \{1+o(1)\}\tfrac{1}{2}(e^x-e^{-x})$$

Now both occurrences of $o(1)$ are not the same fn

$\therefore \quad 1+o(1) = 1+f_1(x) \quad$ w/ $\quad \frac{f(x)}{1}\to 0 \quad x\to\infty$

$\therefore$ LHS

$$= (1+f_1(x))\tfrac{1}{2}(e^x+e^{-x}) - (1+f_2(x))\tfrac{1}{2}(e^x-e^{-x})$$

$$\tfrac{1}{2}(\cancel{e^x}+f_1e^x+e^{-x}+e^{-x}f_1) - \tfrac{1}{2}(\cancel{e^x}-e^{-x}+f_2e^x-f_2e^{-x})$$

$$e^{-x}+\frac{f_1(x)}{2}e^x+\frac{f_1(x)}{2}e^{-x}-\frac{f_2(x)}{2}e^x+\frac{f_2(x)}{2}e^{-x}$$

$$= e^{-x}+\tfrac{1}{2}(f_1(x)-f_2(x))e^x+\tfrac{1}{2}(f_1(x)+f_2(x))e^{-x}$$

$$= e^{-x}\{1+\underbrace{\tfrac{1}{2}(f_1(x)+f_2(x))}\} + \tfrac{1}{2}(f_1-f_2)e^{2x}e^{-x}$$

Now $o(1)$ $x\to\infty$ But There is no gaurantee That $\tfrac{1}{2}(f_1(x)-f_2(x))e^{2x} = o(1)$ $x\to\infty$.

Ex 2.5

$O(\phi)O(\psi) = O(\phi\psi)$

$\left|\frac{f}{\phi}\right| < M \quad x \to \infty$

$\Rightarrow f = O(\phi)$

f is of order not exceeding ϕ.

let $f_1(x) = O(\phi)$

$f_2(x) = O(\psi)$

The $\left|\frac{f_1 \cdot f_2}{\phi\psi}\right| = \left|\frac{f_1}{\phi}\right|\left|\frac{f_2}{\psi}\right| \leq M_1 M_2 = M$

$\therefore \quad f_1 f_2 = O(\phi\psi) \Rightarrow O(\phi)O(\psi) \subset O(\phi\psi)$

If $g \in O(\phi\psi)$ ....

Classes of fns yes but equality in this definition is sided from left to right.

$\Rightarrow \left|\frac{g}{\phi\psi}\right| \leq M$

$\sqrt{g}$

- - -

$\underbrace{O(\phi)}_{} \underbrace{o(\psi)}_{} = o(\phi\psi)$

if $f_1 \quad f_2$

Then $\left|\frac{f_1}{\phi}\right| < M \quad x \to \infty.$

$+ \quad \left|\frac{f_2}{\psi}\right| \to 0 \quad x \to \infty$

Then $\left|\frac{f_1 f_2}{\phi\psi}\right| = \left|\frac{f_1}{\phi}\right| \cdot \left|\frac{f_2}{\psi}\right| \leq \left|\frac{f_2}{\psi}\right| \cdot M \longrightarrow 0 \quad \therefore \quad f_1 f_2 = o(\phi\psi)$

$$\underbrace{O(\phi)}_{f_1} + \underbrace{O(f)}_{f_2} = O(|\phi| + |f|)$$

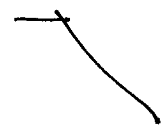
$$\cdot \left|\frac{f_1 + f_2}{|\phi| + |f|}\right| = \left|\frac{f_1}{|\phi| + |f|}\right| + \frac{|f_2|}{|\phi| + |f|} \le \frac{|f_1|}{|\phi|} + \frac{|f_2|}{|f|} \le M_1 + M_2 = M$$

$$\Rightarrow f_1 + f_2 = O(|\phi| + |f|).$$

<u>Ex</u> 2.6 $\quad (x+1)^2 = O(x^2) \qquad x \to \infty \qquad x \geq 1$

$$\left|\frac{(x+1)^2}{x^2}\right| = \frac{x^2 + 2x + 1}{x^2} = 1 + \frac{2}{x} + \frac{1}{x^2} =: f(x)$$

$$f' = -\frac{2}{x^2} - \frac{2}{x^3} = -2\left(\frac{1}{x^2} + \frac{1}{x^3}\right) < 0 \quad \forall x$$



$$\therefore f(x) \leq f(1) = 1 + 2 + 1 = 4.$$

$$\therefore \left|\left(\frac{x+1}{x}\right)^2\right| \le 4 \qquad \forall x \geq 1$$

- - -

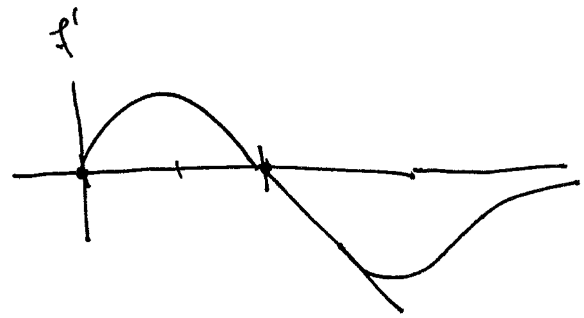
$$\left|\frac{(x^2 - 1/2)^{1/2}}{x}\right| = \left(1 - \frac{1}{2x^2}\right)^{1/2}$$

$$f' = \frac{1}{2}\left(1 - \frac{1}{2x^2}\right)^{-1/2} \cdot \frac{\cancel{-}1\,\cancel{(-2)}}{\cancel{2}4\,x^3} = \frac{1}{2x^3\sqrt{1 - 1/2x^2}} > 0 \qquad x > 1$$

$\therefore \quad f(x) \leq f(\infty) = 1.$

$x^2 = O(e^x)$

$x^2 e^{-x} =: f$



$2xe^{-x} - x^2 e^{-x} = f'$

$f' = x(2-x)e^{-x} = 0 \Rightarrow x=0 \quad \text{or} \quad x=2$

$f'' = (2-x)e^{-x} + x(-1)e^{-x} - x(2-x)e^{-x}$

$= e^{-x}\left[(2-x) - x + x^2 - 2x\right]$

$f(0) = 0 \qquad f(2) = 4e^{-2}$

$f(\infty) = 0$

$\therefore \quad x^2 \leq 4e^{-2}e^{+x}$

Pg 6 Olver   o = order less than.

Ex 2.7 $f \sim \phi \Rightarrow \frac{f}{\phi} \longrightarrow 1 \quad x \to \infty$

let $\frac{f(x)}{\phi(x)} = 1 + t(x)$ I claim $t(x) \longrightarrow 0$ i.e. $t(x) = o(1)$

$\underbrace{\left|\frac{f}{\phi} - 1\right| < \epsilon}_{x > \bar{X}} \quad \therefore \quad |t(x)| < \epsilon \quad \forall x > \bar{X}$ As $\epsilon > 0$ and

$t(x) \to 0 \quad x \to \infty \Rightarrow t = o(1) \quad x \to \infty$

If $\quad f = \{1 + o(1)\}\phi$ Th $\frac{f(x)}{\phi(x)} = 1 + t(x)$

w/ $t = o(1) \quad x \to \infty$

Then $\frac{f(x)}{\phi(x)} \to 1 \quad x \to \infty$ iff $\phi(x) \not\to 0$ As $x \to \infty$.

Ex 2.8 $f \sim \phi$ As $x \to \infty$

$\Rightarrow \quad f = \phi(x)(1 + o(1))$

$\Rightarrow \quad \sup_{t \in (x,\infty)} f(t) = \sup_{t \in (x,\infty)} \left(\phi(x) + \phi(x)o(1)\right) =$

$\leq \sup_{t \in (x,\infty)} \phi(x) + \sup_{t \in (x,\infty)} \phi(x) t(x) \quad$ w/ $\phi(x) = o(1) \quad x \to \infty$

Then $$\frac{\sup_{t\in(x,\infty)} f(t)}{f(x)} = \frac{\sup_{t\in(x,\infty)} \phi(t)}{\phi(x)} + \frac{\sup_{t\in(x,\infty)} \phi(t)f(t)}{\phi(x)} \quad ?$$

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$$\sin\left(n\pi + \frac{1}{n}\right) = O\left(\frac{1}{n}\right)$$

$$\lim_{n\to\infty} \frac{\sin\left(n\pi + \frac{1}{n}\right)}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\sin(n\pi)\cos\frac{1}{n} + \sin\frac{1}{n}\cos(n\pi)}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} (-1)^n \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} \qquad \frac{0}{0}$$

$$= \lim_{n\to\infty} (-1)^n \frac{\cos\left(\frac{1}{n}\right) \cancel{\frac{d}{dn}\left(\frac{1}{n}\right)}}{\cancel{\frac{d}{dn}\left(\frac{1}{n}\right)}} = \pm 1 \Rightarrow \left|\frac{\sin\left(n\pi + \frac{1}{n}\right)}{\frac{1}{n}}\right| \le 2 \text{ say.}$$

$$\therefore \sin\left(n\pi + \frac{1}{n}\right) = O\left(\frac{1}{n}\right) \quad n\to\infty$$

$$\frac{x^2 - c^2}{x^2} = \frac{(x-c)(x+c)}{x^2} \sim \frac{(x-c)\,2c}{c^2} = \frac{2(x-c)}{c} = O(x-c)$$

$$= o(1)$$

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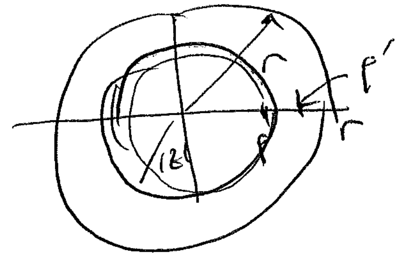


$$a_s \rho'^s \to 0 \qquad s \to \infty$$

$$\Rightarrow \quad |a_s \rho'^s| \le A \qquad \forall\ s \in \mathbb{N}$$

$$\Rightarrow \quad |a_s| \rho'^s \le A$$

$$\left|\sum_{s=n}^{\infty} a_s z^s\right| \le \sum_{s=n}^{\infty} |a_s||z|^s \le \sum_{s=n}^{\infty} A\left|\frac{z}{\rho'}\right|^s = A\sum_{s=n}^{\infty}\left|\frac{z}{\rho'}\right|^s$$

$$= A\,\cancel{\frac{1-(|z/\rho'|)^{n+1}}{1-(|z/\rho'|)}} = A\,\frac{\left|\frac{z}{\rho'}\right|^s}{|z/\rho'|-1}\Bigg|_{s=n}^{\infty}$$

$$= A\left(0 - \frac{\left|\frac{z}{\rho'}\right|^n}{|z/\rho'|-1}\right) = \frac{A|z|^n \rho'^{-n}}{1-|z|/\rho'}\cdot\frac{\rho'}{\rho'}$$

$$= \frac{A|z|^n \rho'^{-n+1}}{\rho' - |z|} \le \frac{A\rho'^{(1-n)}|z|^n}{\rho' - \rho}$$

$$\ln(1+z) \overset{?}{=} \underbrace{\sum_{k=0}^{\infty}\frac{(-1)^k z^{k+1}}{k+1}}_{O(z)} = z + \underbrace{\sum_{k=1}^{\infty}\frac{(-1)^k z^{k+1}}{k+1}}_{=\,O(z^2)}$$

$u \in [0, a]$

$$f(z) = e^{(z-u)^2}$$

$$= e^{z^2 - 2zu + u^2}$$

$$\frac{|f(z)|}{|e^{z^2}|} = |e^{-2zu+u^2}| = e^{u^2} e^{-2uz} \xrightarrow[\text{part}]{z \to \infty\ z\ \text{w/ pos real}} 0$$

But if $u \in [-a, 0]$ then $e^{(z-u)^2} = O(e^{z^2})$ in

left half plane.

(

pg 8 Arev

Ex 3,1

$\delta > 0$

$\cosh z \sim \frac{1}{2}e^{z} \qquad |\mathrm{ph}\, z| \le \frac{1}{2}\pi - \delta.$

$\qquad \|$

$\frac{e^{z}+e^{-z}}{2}$

The $\frac{\cosh z}{\frac{1}{2}e^{z}} = 1 + e^{-2z} \longrightarrow 1$ if $|\mathrm{ph}\, z| \le \frac{\pi}{2} - \delta$

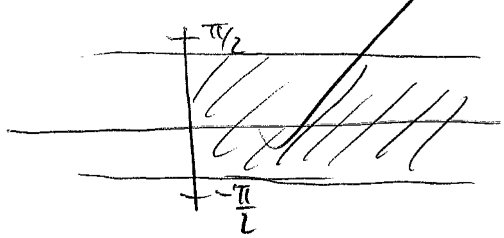
If $|\mathrm{ph}\, z| < \frac{\pi}{2}$ The convergence to 1 is not uniform as required by the def. $\sim$. Because As we take ~~θ~~ θ's closer & closer to ~~$\frac{\pi}{2}$~~

Ex 3.2 ~~$e^{-\sinh z} = o(1) \quad z \to \infty \quad \mathrm{Re}(z) \ge 0$~~

~~$|\mathrm{Im}\, z| \le \frac{\pi}{2} - \delta < \frac{\pi}{2}$~~

~~$\lim_{z \to \infty} e^{-\left(\frac{e^{z}}{2} - \frac{e^{-z}}{2}\right)} =$~~ $\quad \sinh z$

let $z = a + ib$



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Ex 3,2

Pr: $e^{-\sinh z} = o(1) \quad z \to \infty$

$\Longleftrightarrow$ $Re(\sinh z) \to \infty$ As $z \to \infty$

for $Re(z) \geq 0$ (Don't think $Re(z) = 0$).
$|Im z| \leq \frac{\pi}{2} - \delta < \frac{\pi}{2}$.

Now $\sinh z = \frac{e^{z} - e^{-z}}{2}$ w/ $z = a + ib$

$= \frac{1}{2}\left(e^{a}(\cos b + i \sin b) - e^{-a}(\cos b - i \sin b)\right)$

$= \frac{1}{2}\left(e^{a}\cos b - e^{-a}\cos b + i\left(e^{a}\sin b + e^{-a}\sin b\right)\right)$

$= \frac{1}{2}\left(2\sinh a \cos b + 2i \cosh a \sin b\right)$

$= \sinh a \cos b + i \cosh a \sin b$

$\Rightarrow Re(\sinh z) = \sinh a \cos b$

If $a \geq 0$ $|b| \leq \frac{\pi}{2} - \delta$ Then $\cos b \underset{\neq}{>} 0$

$\therefore$ $a \to \infty$ w/ $a > 0$ $\sinh z \to +\infty$. ✓.

$$\lim_{z\to\infty} \exp\left\{-\frac{1}{2}\left(e^{a}e^{ib} - e^{-a}e^{-ib}\right)\right\}$$

$$= \lim_{z\to\infty} \exp\left\{-\frac{1}{2}\left(e^{a}(\cos b + i\sin b) - e^{-a}(\cos b - i\sin b)\right)\right\}$$

$$= \lim_{a\to\infty,\, b\to\infty} \exp\left\{-\frac{1}{2}\left((e^{a}\cos b - e^{-a}\cos b) + i(e^{a}\sin b + e^{-a}\sin b)\right)\right\}$$

$$= \lim_{a\to\infty,\, b\to\infty} \exp\left\{-\frac{1}{2}\right.$$

~~$a \neq 0$~~ ~~$a = 0$~~

⋮

Ex 33 $\quad e^{-z} = O(z^{-p})$

$$\Rightarrow \left|\frac{e^{-z}}{z^{-p}}\right| \le M \quad \text{uniformly in sector } |\mathrm{ph}\, z| \le \frac{\pi}{2} - \delta < \frac{\pi}{2}$$

$$= \frac{z^{p}}{e^{z}} \qquad \text{let } z = \rho e^{i\theta}$$

$$\Rightarrow \left|\frac{\rho^{p} e^{pi\theta}}{e^{\rho e^{i\theta}}}\right| = \left|\frac{\rho^{p}}{e^{\rho(\cos\theta + i\sin\theta)}}\right| = \left|\frac{\rho^{p}}{e^{\rho\cos\theta}}\right| \Big\} \text{ find max as fn of } \theta$$

$$= \frac{\rho^{p}}{\left|e^{\rho\cos\theta}\right|} \qquad \rho > 0 \quad \text{find Max of } e^{-\rho\cos\theta} \text{ w.r.t. } \theta$$

$$\frac{d}{d\theta} e^{-\rho\cos\theta} = e^{-\rho\cos\theta}(+\rho\sin\theta) = 0$$

$$\Rightarrow \theta = 0, \pi$$

$$e^{-\rho\cos 0} = e^{-\rho}$$

$$e^{-\rho\cos\pi} = e^{\rho} \Longleftarrow \text{Max}$$

$$\therefore \frac{|z^{p}|}{|e^{z}|} \le$$

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Ex 3.4 $\quad \phi(x) > 0 \quad p \in \mathbb{R} \quad f(x) \sim \phi(x) \quad x \to \infty$

$$\Rightarrow f(x) = \phi(x)(1 + \eta(x)) \quad \text{w/} \quad \eta(x) = o(1) \quad x \to \infty$$

$$= \phi(x)(1 + o(x))$$

Then $\quad f(x)^p = (\phi(x)(1+\eta(x)))^p = \phi(x)^p (1+\eta(x))^p$

But By Thm 3.1 $\quad (1+o(1))^p = \sum_{k=0}^{\infty} \binom{p}{k} 1^{p-k} o(1)^k$

$$= \cancel{1 + o(1)}$$

$$= 1 + \underbrace{\sum_{k=1}^{\infty} \binom{p}{k} 1^{p-k} o(1)^k}_{o(1) \text{ By Thm 3.1}}$$

$$\Rightarrow f^p = \phi^p (1 + o(1))$$

$\Rightarrow$ By ex. 2.7 $\Rightarrow f^p \sim \phi^p$.

$$\ln f(x) = \ln(\phi(x)(1+\eta(x)))$$

$$= \ln \phi(x) + \ln(1 + \eta(x))$$

$$= \ln \phi + o(1) \quad x \to \infty$$

$$= \ln\phi\Big(1 + \underbrace{\frac{o(1)}{\ln\phi}}_{?\ o(1)}\Big)$$

iff $\phi(x)$ is Bounded away from 1

$$e^{f(x)} \sim e^{\phi(x)}$$

let $\phi(x) = -x$ $\qquad f \sim \phi \quad x \to \infty$

$f(x) = -x + \sqrt{x}$

But $e^{\phi(x)} = e^{-x} \to 0 \quad x \to \infty$

$e^{f(x)} = e^{-x} \cdot e^{\sqrt{x}} \to 1 \cdot e^{\infty} \to \infty \quad x \to \infty$

$e^{-\infty + \infty}$ $\qquad \dfrac{e^{\sqrt{x}}}{e^{x}} \qquad \dfrac{\infty}{\infty}$

$$= \frac{\frac{d}{dx}(\sqrt{x})\, e^{\sqrt{x}}}{e^{x}} =$$

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$f(u,x) = O(u) \quad u \to \infty$

Ex 3,5

Pr:

$$(x + f(u,x))^{1/2} = x^{1/2} + O(u^{1/2}) \quad \cancel{u \to \infty}$$

$$x^{1/2}\left(1 + \frac{f(u,x)}{x}\right)^{1/2} \qquad \text{But} \quad \frac{f(u,x)}{x} \Rightarrow O(u)$$

$$\therefore \quad x^{1/2}\left(1 + O\left(\frac{f(u,x)}{x}\right)\right)$$

$$= x^{1/2}$$

Pg 8 over

$$\int_x^\infty f(t)\,dt \approx \int_x^\infty t^\nu\,dt = \left.\frac{t^{\nu+1}}{\nu+1}\right|_x^\infty = -\frac{x^{\nu+1}}{\nu+1}$$

$Re(\nu+1) < 0$

$Re(\nu) < -1$

$$\int_a^x f(t)\,dt \overset{?}{\approx} \int_a^x t^\nu dt = \left.\frac{t^{\nu+1}}{\nu+1}\right|_a^x = \frac{x^{\nu+1}}{\nu+1} - \frac{a^{\nu+1}}{\nu+1}$$

$Re(\nu+1) < 0$

$Re(\nu) < -1$

$$\Rightarrow \int_a^x f(t)\,dt \sim -\frac{a^{\nu+1}}{\nu+1}$$

$$\int_a^x f(t)\,dt \sim \int_a^x t^{-1}\,dt = \ln x - \ln a \sim \ln x$$

$$\int_a^x f(t)\,dt \sim \int_0^x t^\nu\,dt = \frac{x^{\nu+1}}{\nu+1} \qquad x \to \infty \qquad Re(\nu+1) > 0$$

$Re(\nu) > -1$

$$\int_a^x f(t)\,dt = \int_a^{\bar{X}} t^\nu \{1+\eta(t)\}\,dt + \int_{\bar{X}}^x t^\nu \{1+\eta(t)\}\,dt$$

$$= \int_a^{\bar{X}} f(t)\,dt + \underbrace{\int_{\bar{X}}^x t^\nu\,dt}_{\left.\frac{t^{\nu+1}}{\nu+1}\right|_{\bar{X}}^{x}} + \int_{\bar{X}}^x t^\nu \eta(t)\,dt$$

$$\int_a^{\bar{X}} f(t)\,dt + \frac{1}{\nu+1}\left(x^{\nu+1} - \bar{X}^{\nu+1}\right) + \int_{\bar{X}}^x t^\nu \eta(t)\,dt$$

$$\Rightarrow \frac{\nu+1}{x^{\nu+1}}\int_a^x f(t)\,dt = \frac{\nu+1}{x^{\nu+1}}\int_a^{\bar{X}} f(t)\,dt + 1 - \frac{\bar{X}^{\nu+1}}{x^{\nu+1}}$$

$$+ \frac{\nu+1}{x^{\nu+1}}\int_{\bar{X}}^x t^\nu \eta(t)\,dt$$

$$\Rightarrow \frac{\nu+1}{x^{\nu+1}}\int_a^x f(t)\,dt - 1 = \overbrace{\cancel{\frac{\nu+1}{x^{\nu+1}}\int_a^{\bar{X}} f(t)\,dt}}^{\xrightarrow{x\to\infty} 0} - \overbrace{\cancel{\frac{\bar{X}^{\nu+1}}{x^{\nu+1}}}}^{\xrightarrow{x\to\infty} 0} + \frac{\nu+1}{x^{\nu+1}}\int_{\bar{X}}^x t^\nu \eta(t)\,dt$$

Last term is ...

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$$\left| \frac{\nu+1}{x^{\nu+1}} \int_X^x t^\nu \eta(t)\,dt \right| \le |\nu+1| \left| \frac{1}{x^{\nu+1}} \int_X^x t^\nu \eta(t)\,dt \right|$$

$$\frac{|\nu+1|}{|x|^{\nu+1}} \int_X^x |t|^\nu |\eta(t)|\,dt \le \frac{\epsilon|\nu+1|}{|x|^{\nu+1}} \left. \frac{t^{\nu+1}}{\nu+1} \right|_X^x$$

$$= \frac{\epsilon|\nu+1|}{|x|^{\nu+1}(\nu+1)} \left(x^{\nu+1} - X^{\nu+1} \right) = \frac{\epsilon|\nu+1|}{\nu+1} \left(\frac{x^{\nu+1}}{|x|^{\nu+1}} - \cancel{\frac{X^{\nu+1}}{|x|^{\nu+1}}} \right) \xrightarrow{} 0$$

$\nu \in$ Complex $t > 0$

$$= \underbrace{\frac{\epsilon|\nu+1|}{\nu+1}}_{\text{a constant.} \propto \epsilon} e^{i\theta(\nu+1)} \quad (x \to +\infty)$$

$$|t|^\nu \le t^{\operatorname{Re}\nu}$$

$$= \int_X^x |t|^\nu\,dt \le \int_X^x t^{\operatorname{Re}\nu}\,dt$$

$$= \left. \frac{t^{\operatorname{Re}\nu+1}}{\operatorname{Re}\nu+1} \right|_X^x \cdots$$

$$\therefore \quad \frac{\nu+1}{x^{\nu+1}} \int_a^x f(t)\,dt - 1 \le \frac{\epsilon|\nu+1|}{\nu+1} e^{i\theta(\nu+1)}$$

$$\lim_{x\to\infty} \left(\frac{\nu+1}{x^{\nu+1}} \int_a^x f(t)\,dt - 1 \right) = 0.$$

$$\Rightarrow \lim_{x\to\infty} \int_a^x f(t)\,dt = \lim_{x\to\infty} \frac{x^{\nu+1}}{\nu+1}$$

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$$\left| \frac{\nu+1}{x^{\nu+1}} \int_{\bar{X}}^{X} t^{\nu} \eta(t)\, dt \right| \le \frac{|\nu+1|}{|x|^{\nu+1}} \epsilon \int_{\bar{X}}^{X} |t|^{\nu} dt$$

$$= \frac{|\nu+1|\epsilon}{|x|^{\nu+1}} \left. \frac{t^{\nu+1}}{\nu+1} \right|_{\bar{X}}^{X} = \frac{|\nu+1|\epsilon}{|x|^{\nu+1}} \frac{X^{\nu+1} - \bar{X}^{\nu+1}}{\nu+1}.$$

$f'(x)$ is nondecreasing $\Rightarrow f'(x+h) > f'(x) \quad \forall h>0.$

(f' is increasing)

Then $f'(x) < f'(x+\xi) \quad \forall\ \xi>0$

$$\Rightarrow \int_{x}^{x+h} f'(\xi)\, d\xi \geq f'(x) \int_{x}^{x+h} d\xi = f'(x)h.$$

$$\Rightarrow hf'(x) \le \int_{x}^{x+h} f'(\xi)\, d\xi = f(x+h) - f(x)$$

$f = x^p + x^p \eta(x)$

$f' = px^{p-1} + \frac{d}{dx}(x^p \eta)$

$f(x) = x^p \{1+\eta(x)\} \Rightarrow f' = px^{p-1}\{1+\eta(x)\} + x^p \eta'(x)$ or $= px^{p-1} + \frac{d}{dx}(x^p \eta(x))$

$$= \int_{x}^{x+h} pt^{p-1} dt + \int_{x}^{x+h} \frac{d}{dt}(t^p \eta(t))\, dt$$

$$= \int_{x}^{x+h} pt^{p-1} dt + (x+h)^p \eta(x+h) - x^p \eta(x)$$

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As $p \geq 1$

$p-1 \geq 0$

$\therefore t^{p-1}$ is a increasing fn $\therefore$ $\displaystyle\int_x^{x+h} t^{p-1}\,dt \leq (x+h)^{p-1}\int_x^{x+h} dt$

$$\therefore \int_x^{x+h} t^{p-1}\,dt \leq (x+h)^{p-1} h$$

+ ~~$(x+h)^p$~~ x^p is increasing fn also.

$\therefore$ ~~$x+h$~~ $|x^p \eta(x)| \leq \epsilon(x+h)^p$ As x^p is increasing.

$$\therefore \left|(x+h)^p\eta(x+h) - x^p\eta(x)\right| \leq \left|(x+h)^p\eta(x+h)\right| + \left|x^p\eta(x)\right|$$

$$\leq 2\epsilon(x+h)^p.$$

3. ~~if $h = \epsilon^{1/2} x$~~

$$\cancel{f'(x) \leq p(x+\epsilon^{1/2}x)^{p-1} + \frac{2\epsilon(x+\epsilon^{1/2}x)^p}{x\epsilon^{1/2}} = px^{p-1}(1+\epsilon^{1/2})^{p-1} + 2\epsilon^{1/2}}$$

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$\cancel{h=\epsilon^{1/2}x}$ $f'(x) \le p(x+h)^{p-1} + \frac{2\epsilon}{h}(x+h)^p$

$\Rightarrow$ $\cancel{f'(x) \le \epsilon^{1/2}}x$ let $h = \epsilon^{1/2}x$

$\Rightarrow f'(x) \le p(x+\epsilon^{1/2}x)^{p-1} + 2\epsilon(x+\epsilon^{1/2}x)^p / \epsilon^{1/2}x$

$px^{p-1}(1+\epsilon^{1/2})^{p-1} + \frac{2\epsilon x^p}{\epsilon^{1/2}x}(1+\epsilon^{1/2})^p$

$\Rightarrow f'(x) \le px^{p-1}(1+\epsilon^{1/2})^{p-1} + 2\epsilon^{1/2}x^{p-1}(1+\epsilon^{1/2})^p$

$px^{p-1}\{(1+\epsilon^{1/2})^{p-1} + 2\epsilon^{1/2}p^{-1}(1+\epsilon^{1/2})^p\}$ $x > X$

Sim $\cancel{f'(x)h} \ge$ $\cancel{\int_{x}^{x+h}}$ $\int_{x-h}^{x} f'(t)\,dt = f(x) - f(x-h)$

If $f'(t)$ nondecreasing x large enough $\{f'(t)\nearrow$

$\Rightarrow$ $f(x) > f(x-h)$

$hf'(x) \ge \int_{x-h}^{x} f'(t)\,dt = f(x) - f(x-h)$ But $f(x) = x^p\{1+\eta(x)\}$

$\Rightarrow f' = px^{p-1} + \frac{d}{dx}(x^p\eta(x))$ for $x > X$

$\cancel{x-h > X}$

$= x^p\cancel{\{1+\eta(x)\} - (x-h)^p\{1+\eta(x-h)\}}$ if $x-h > X$.

$= x^p \cancel{- (x-h)^p + x^p\eta(x) - (x-h)^p\eta(x-h)}$

$$h f'(x) \geq \int_{x-h}^{x} f'(t)\,dt$$

$$= \int_{x-h}^{x} p t^{p-1}\,dt + t^{p} \eta(t)\Big|_{x-h}^{x}$$

$$= \int_{x-h}^{x} p t^{p-1}\,dt + x^{p}\eta(x) - (x-h)^{p}\eta(x-h)$$

Now t^{p-1} is an increasing fn $\therefore$ $\int_{x-h}^{x} t^{p-1}\,dt \geq (x-h)^{p-1}(h)$

$$\Rightarrow h f'(x) \geq p(x-h)^{p-1} h + x^{p}\eta(x) - (x-h)^{p}\eta(x-h)$$

~~let h~~

$$\Rightarrow f'(x) \geq p(x-h)^{p-1} + \frac{x^{p}\eta(x)}{h} - \frac{(x-h)^{p}}{h}\eta(x-h)$$

Let $h = \epsilon^{1/2} x$ Then

$$f'(x) \geq p(x - x\epsilon^{1/2})^{p-1} + \frac{x^{p}\eta(x)}{\epsilon^{1/2}x} - \frac{(x-\epsilon^{1/2}x)^{p}}{\epsilon^{1/2}x}\eta(x - x\epsilon^{1/2})$$

$$p x^{p-1} \left\{ (1-\epsilon^{1/2})^{p-1} + p^{-1}\frac{\eta(x)}{\epsilon^{1/2}} - p^{-1} \cancel{x}^{\cancel{p}} \frac{(1-\epsilon^{1/2})^{p}}{\epsilon^{1/2}} \eta(x - x\epsilon^{1/2}) \right\}$$

Now:

$$|\eta(x)| < \epsilon \quad \text{if} \quad x > \overline{X}$$

$$\therefore \quad |\eta(x(1-\epsilon^{1/2}))| < \epsilon \quad \text{if} \quad x(1-\epsilon^{1/2}) > \overline{X}$$

$$\Rightarrow \quad x > \frac{\overline{X}}{(1-\epsilon^{1/2})}$$

& Thus

$$\eta(x) > -\epsilon.$$

$$\Rightarrow f'(x) \geq p x^{p-1} \left\{ (1-\epsilon^{1/2})^{p-1} - p^{-1}\epsilon^{1/2} - \frac{p^{-1}}{\epsilon^{1/2}}\epsilon(1-\epsilon^{1/2})^{p} \right\}$$

$0<\epsilon<1$

for $p \geq 1$

& $(1-\epsilon^{1/2})^p$ is an decreasing fn of ϵ. $\therefore$ $(1-\epsilon^{1/2})^p < 1^p = 1$

$$\therefore -(1-\epsilon^{1/2})^p > -1$$

$$\Rightarrow f'(x) \geq p x^{p-1} \left\{ (1-\epsilon^{1/2})^{p-1} - p^{-1}\epsilon^{1/2} - p^{-1}\epsilon^{1/2}\, 1 \right\}$$

$$\Rightarrow f'(x) \geq p x^{p-1} \left\{ (1-\epsilon^{1/2})^{p-1} - 2p^{-1}\epsilon^{1/2} \right\}, \quad \text{when } x > \frac{\overline{X}}{1-\epsilon^{1/2}}.$$

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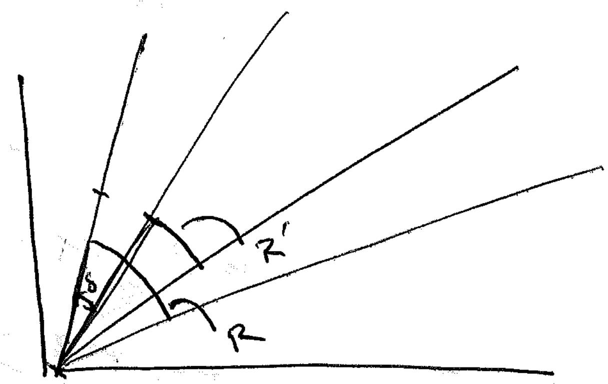
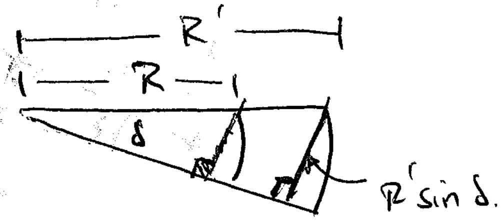
$|z-c|^p \sim |z|^p$?

$$\frac{|z-c|^p}{|z|^p} = \left|1-\frac{c}{z}\right|^p \longrightarrow 1$$



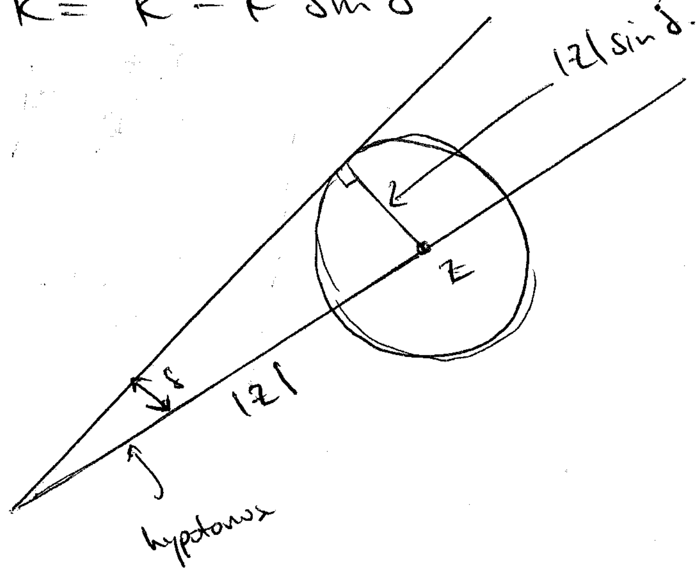


$R = R' - R'\sin\delta$

Let R' be choosen ∋ $R' = \frac{R}{1-\sin\delta}$

$\Rightarrow \quad R = R' - R'\sin\delta$

pick radius ~~about~~ around z to be of size $|z|\sin\delta$



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$|t-z| = |z|\sin\delta$

Then $\iff$ $\big||t|-|z|\big| \le |z|\sin\delta$ $\quad |x|\le 4$

$\Rightarrow -4 \le x \le 4$

$\iff -|z|\sin\delta \le |t|-|z| \le |z|\sin\delta$

$|z|(1-\sin\delta) \le |t| \le |z|(1+\sin\delta)$

$4.03 \Rightarrow \quad |f(z)| \le k|z|^p \quad z\to+\infty \quad z\in S(R)$

Then $|f^{(m)}(z)| \le \dfrac{m!}{2\pi}\displaystyle\int_C \frac{k|t|^p}{|t-z|^{m+1}}\,dt$

only if $p\geq 0$.

$= \dfrac{m!\,k}{2\pi}\displaystyle\int_C \frac{|t|^p}{|z|^{m+1}|\sin\delta|^{m+1}}\,dt \le \frac{km!}{\cancel{2\pi}}\frac{|z|^p(1+\sin\delta)^p\cancel{2\pi}\,|z|\sin\delta}{|z|^{m+1}|\sin\delta|^{m+1}}$

$\therefore |f^{(m)}(z)| \le \dfrac{km!(1+\sin\delta)^p}{|\sin\delta|^m}\sin\delta\,|z|^{p-m}$ $\qquad p\geq 0$

~~$x = u - \tanh x$~~ $\quad x \gg 1$

~~$|\tanh x| < 1 \Rightarrow x \sim u \quad (u\to\infty)$.~~

$$|f^{(m)}(z)| \le \frac{m!}{2\pi} \int_C \frac{k\,|t|^p\,dt}{|t-z|^{m+1}}$$

$$= \frac{m!\,k}{2\pi} \int_C \frac{|t|^p\,dt}{|z|^{m+1}\,|\sin\delta|^{m+1}}$$

But $|t|^p \le |z|^p(1+\sin\delta)^p$ if $p \ge 0$

$|t|^p \le |z|^p(1-\sin\delta)^p$ if $p < 0$

$$|f^{(m)}(z)| \le \frac{k\,m!\,|z|^p(1\pm\sin\delta)^p \cdot \cancel{2\pi}|z||\sin\delta|}{\cancel{2\pi}\,|z|^{m+\cancel{1}}\,|\sin\delta|^{m+\cancel{1}}}$$

$$= \frac{m!}{(|z|\sin\delta)^m}\,k\,|z|^p\,(1\pm\sin\delta)^p$$

1

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Ex 4.1 $\quad f(x) = o(\phi(x)) \quad x \to \infty \qquad |f(x)| < k\,|\phi(x)|$

$$\int_a^x f(t)\,dt = o(x\,\phi(x)) \quad x \to \infty$$

$$\left| \frac{\int_a^x f(t)\,dt}{x\,\phi(x)} \right| \le \frac{\max\limits_{t\in[a,x]} |f(t)|\,(x-a)}{|x\,\phi(x)|} = \cancel{\left(1-\frac{a}{x}\right)}$$

$x > 0$

$$= \left(1 - \frac{a}{x}\right) \frac{\max\limits_{t\in[a,x]} |f(t)|}{\phi(x)}$$

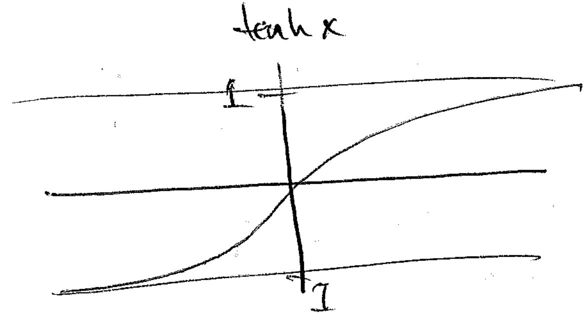
$$= \cancel{\left(1-\frac{a}{x}\right)\max_{t\in}}$$

How show $\dfrac{\max\limits_{t\in[a,x]} |f(t)|}{\phi(x)} \to 0.$

$f = o(\phi(x))$ w/ ϕ non decreasing fn ... Ex 2.8 pg 6?

$x = U - \tanh x$ pg 11 Oliver

$x \sim U \quad (U \to \infty)$



$\leftarrow \tanh(x) = 1 + o(1)$

$\Rightarrow \tanh(x) - 1 = o(1)$

$$\lim_{x \to \infty} \frac{\tanh(x) - 1}{1} = 0 \quad \checkmark.$$

$x(U) = U - \tanh x(U)$

$\therefore \quad x = U - 1 + o(1) \quad (U \to \infty)$

~~if $x \sim U$~~ x ~~$\sim U - \tanh U \sim U - 1 + o(1)$~~ $U \to \infty$.

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \cdot \frac{e^x}{e^x} = \frac{e^{2x} - 1}{e^{2x} + 1}$$

let $\xi = e^{2x}$

$$\tanh \xi = \frac{\xi - 1}{\xi + 1} = (\xi - 1)\sum_{k=0}^{\infty} (-1)^k \xi^k = \sum_{k=0}^{\infty} (-1)^k \xi^{k+1} - \sum_{k=0}^{\infty} (-1)^k \xi^k$$

$$= \sum_{k=1}^{\infty} (-1)^{k-1} \xi^k - \sum_{k=0}^{\infty} (-1)^k \xi^k = -1 + \sum_{k=1}^{\infty} (-1)^k (-1-1) \xi^k$$

$$= -1 - 2\sum_{k=1}^{\infty} (-1)^k \xi^k.$$

$\therefore$

~~$\tanh x$~~ $=$ $x = U - \tanh x = U - 1 + 2e^{-2x} - 2e^{-4x} \ldots$ exact ...

But $x \sim U - 1 + o(1) \quad (U \to \infty)$

1st $e^{-2x} = e^{-2(u-1+o(1))} = e^{-2u+2+o(1)}$ $u\to\infty$

$= O(e^{-2u})$ $u \to \infty$

Now All terms are $O(e^{-2x})$ By Thm 3.1 & ∴

$x = u-1+O(e^{-2x}) = u-1+O(e^{-2u})$ By calc. Above

Now consider $x = u-1+O(e^{-2u})$ As the starting point & put this into the exact e^{-2x} term.

$$\therefore \quad x = u-1+2\exp\{-2(u-1+O(e^{-2u}))\} + O(e^{-4x})$$

Now $e^{-4x} = e^{-4(u-1+O(e^{-2u}))} = e^{-4u+4+O(e^{-2u})} = O(e^{-4u})$ $(u\to\infty)$

$$\therefore \quad x = u-1+2\exp\{-2u+2\}e^{O(e^{-2u})} + O(e^{-4u}) \qquad (u\to\infty)$$

$$\Rightarrow \quad x = u-1+2\exp\{-2u+2\} + O(e^{-4u}) \qquad *$$

Do one more step ...

consider * The starting point & put this expression into next 2 e terms i.e. $e^{-2x} + e^{-4x}$ ...

$\Rightarrow \; x = u - 1 + 2e^{-2x} - 2e^{-4x} + O(e^{-6x})$

$$\Rightarrow \; x = u - 1 + 2\exp\left\{-2(u-1+2e^{-2u+2} + O(e^{-4u}))\right\}$$
$$-2\exp\left\{-4(u-1+2e^{-2u+2} + O(e^{-4u}))\right\} + O(e^{-6x})$$

As Before $O(e^{-6x}) = O(e^{-6u}) \quad (u \to \infty)$

$\therefore$

$$x = u - 1 + 2\exp\left\{-2u+2-4e^{-2u+2}\right\}$$
$$-2\exp\left\{-4u+4-8e^{-2u+2}\right\} + O(e^{-6u}) \quad (u \to \infty)$$

$$\Rightarrow \; x = u - 1 + 2e^{-2u+2} - 2e^{-4u+4} + O(e^{-6u})$$

How Argue this last step. ...

$$\frac{1}{1+x^2} = \sum_{k=0}^{\infty} (-1)^k x^{2k} \qquad |x| < 1$$

$$\int_0^x \frac{1}{1+t^2}\,dt = \tan^{-1} x - \cancel{\tan^{-1} 0}^{\,0} = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}.$$

$$\therefore \tan^{-1}\frac{1}{x} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)} \frac{1}{x^{2k+1}} = \frac{1}{x} - \frac{1}{3x^3} + \frac{1}{5x^5} - \cdots \qquad \text{for } |x| > 1$$

Thus: $x_n = n\pi + \tan^{-1}(1/x_n) = n\pi + \frac{1}{x_n} + O\left(\frac{1}{x_n^3}\right)$ $(n \to \infty)$

$x = n\pi + O(1/x)$ By Thm 3.1

$\Rightarrow x = n\pi + \frac{1}{n\pi} + O\left(\frac{1}{n^3}\right)$ ✓ 2nd Approx.

$+ \quad x = n\pi + \frac{1}{x} - \frac{1}{3x^3} + O\left(\frac{1}{x^5}\right)$ rewrite.

$\Rightarrow \cancel{x = n\pi + \frac{1}{n\pi} - \frac{1}{3(n\pi)^3} + O\left(\frac{1}{n^5}\right)}$

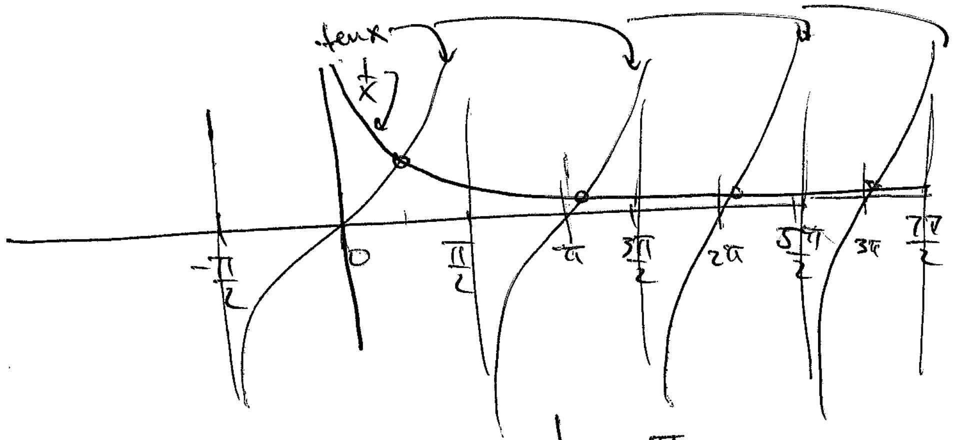
3rd Approx

$$x = n\pi + \frac{1}{\left(n\pi + \frac{1}{n\pi}\right)} - \frac{1}{3\left(n\pi + \frac{1}{n\pi}\right)^3} + O\left(\frac{1}{n^5}\right)$$

$$= n\pi + \frac{n\pi}{(n\pi)^2 + 1} - \frac{1}{3(n\pi)^3} + O\left(\frac{1}{n^5}\right)$$

$x \tan x = 1$

$\Rightarrow \quad \tan x = \frac{1}{x}$



Note: intersections ~~get~~ get closer & closer to $n\pi$.

$x_n = n\pi + \tan^{-1}\left(\frac{1}{x_n}\right)$ want x_n $n \to \infty$

$\left|\tan^{-1}\left(\frac{1}{x_n}\right)\right| \leq \frac{\pi}{2} \Rightarrow x_n \sim n\pi$

By Thm 3.1 $x = n\pi + O(1/x) = n\pi + O(1/n)$ $n \to \infty$ *

Then consider * As exact & put this into 1st true term

After 1st

$x = n\pi + \frac{1}{x} + O\left(\frac{1}{x^3}\right)$

$\Rightarrow x = n\pi + \frac{1}{n\pi + O(1/n)} + O\left(\frac{1}{(n\pi + O(1/n))^3}\right)$ $n \to \infty$

$\Rightarrow x = n\pi + \frac{1}{n\pi} + O\left(\frac{1}{n^3}\right)$ $n \to \infty$.

Now take $x = n\pi + \frac{1}{n\pi} + O\left(\frac{1}{n^3}\right)$ as a starting point

+ put it into 2 term expansion of $\tan^{-1}\frac{1}{x}$ i.e.

$$x = n\pi + \frac{1}{x} - \frac{1}{3x^3} + O\left(\frac{1}{x^5}\right)$$

$$x = n\pi + \frac{1}{n\pi + \frac{1}{n\pi} + O(n^{-3})} - \frac{1}{3\left(n\pi + \frac{1}{n\pi} + O(n^{-3})\right)^3} + O\left(\frac{1}{n^5}\right) \quad n\to\infty$$

$$= n\pi + \frac{1}{n\pi} - \frac{1}{3(n\pi)^3} + O\left(\frac{1}{n^5}\right) \quad n\to\infty$$

WRONG...

$$\frac{1}{n\pi + \frac{1}{n\pi} + O(n^{-3})}$$

has a term of order $\frac{1}{n^3}$ in it. That is dropped by assuming $\frac{1}{n\pi}$ is subdominant to $n\pi$

$$x = n\pi + \frac{1}{n\pi\left(1 + \frac{1}{(n\pi)^2} + O(n^{-4})\right)} - \frac{1}{3(n\pi)^3\left(1 + \frac{1}{(n\pi)^2} + O(n^{-4})\right)^3}$$

$$+ O\left(\frac{1}{n^5}\right)$$

$$= n\pi + \frac{1}{n\pi}\left(1 + \frac{1}{(n\pi)^2} + O(n^{-4})\right)^{-1} - \frac{1}{3(n\pi)^3}\left(1 + \frac{1}{(n\pi)^2} + O(n^{-4})\right)^{-3}$$

$$+ O(n^{-5})$$

$$= n\pi + \frac{1}{n\pi}\left(1 - \frac{1}{(n\pi)^2} + O(n^{-4})\right) - \frac{1}{3(n\pi)^3}\left(1 + O\left(\frac{1}{n^2}\right)\right)$$

$$+ O(n^{-5})$$

$$= n\pi + \frac{1}{n\pi} - \frac{1}{(n\pi)^3} + O(n^{-5}) - \frac{1}{3(n\pi)^3} + O(n^{-5}) + O(n^{-5})$$

$$= n\pi + \frac{1}{n\pi} - \frac{4}{3(n\pi)^3} + O(n^{-5})$$ *

Doing one more iteration would ~~[illegible]~~ amount to putting in *

for 3 terms $\frac{1}{x} - \frac{1}{3x^3} + \frac{1}{5x^5}$ & taylor expanding &

collecting powers ~~of~~ see pg 8 & 9.

8

$$-\frac{1}{3n^3\pi^3}\left(1-3\left(\frac{1}{n^2\pi^2}\right)+O(n^{-4})\right)\Bigg)$$

$$= n\pi + \frac{1}{n\pi} - \frac{1}{n^3\pi^3} - \frac{1}{3n^3\pi^3} + \cancel{\frac{1}{3n^3\pi^3}} + O(n^{-5})$$

$$= n\pi + \frac{1}{n\pi} - \left(1+\frac{1}{3}\right)\frac{1}{n^3\pi^3} + O(n^{-5})$$

$$= n\pi + \frac{1}{n\pi} - \frac{4}{3}\frac{1}{n^3\pi^3} + O(n^{-5})$$

Nxt order:

$$x = n\pi + \frac{1}{x} - \frac{1}{3x^3} + \frac{1}{5x^5} + O(x^{-7})$$

Then

$$x = n\pi + \frac{1}{\left(n\pi + \frac{1}{n\pi} - \frac{4}{3}\frac{1}{n^3\pi^3} + O(n^{-5})\right)} - \frac{1}{3}\frac{1}{\left(n\pi + \frac{1}{n\pi} - \frac{4}{3}\frac{1}{n^3\pi^3} + O(n^{-5})\right)^3}$$

$$+ \frac{1}{5\left(n\pi + \frac{1}{n\pi} - \frac{4}{3}\frac{1}{n^3\pi^3} + O(n^{-5})\right)^5} + O(n^{-7}).$$

$$x = n\pi + \frac{1}{n\pi\left(1+\frac{1}{n^2\pi^2} - \frac{4}{3}\frac{1}{(n\pi)^4} + O(n^{-6})\right)} - \cancel{\frac{1}{3n^3\pi^3}}$$

$$-\frac{1}{3(n\pi)^3}\,\frac{1}{\left(1+\frac{1}{(n\pi)^2}-\frac{4}{3}\frac{1}{(n\pi)^4}+O(n^{-6})\right)^3}$$

$$+\frac{1}{5(n\pi)^5}\,\frac{1}{\left(1+\frac{1}{(n\pi)^2}-\frac{4}{3}\frac{1}{(n\pi)^4}+O(n^{-6})\right)^5}+O(n^{-7})$$

$$x = n\pi + \frac{1}{n\pi}\left(1-\left(\frac{1}{n^2\pi^2}-\frac{4}{3}\frac{1}{(n\pi)^4}+\right.\right.$$

Pg 11 Olver

Ex 4.2 $\mathrm{Re}\,\nu = -1 \quad \mathrm{Im}\,\nu \neq 0$

$$\int_a^x f(t)\,dt = O(1)$$ As one could just "perform" the integration & get $\left. \frac{t^{i\mathrm{Im}\nu}}{} \right|_a^x = O(1)$

let $f(t)$

$$= \left. e^{i\,\mathrm{Im}\,\nu \ln t} \right|_a^x \Rightarrow \text{an oscillatory fn}$$

let $f(x) = x^{i\mu - 1} + (x\ln x)^{-1}$

Then $f(x) \sim x^{i\mu-1} \quad x \to \infty$

$$\frac{f(x)}{x^{i\mu-1}} = 1 + \frac{1}{x\ln x\, x^{i\mu-1}} = 1 + \frac{1}{x^{i\mu}\ln x} \xrightarrow{x\to\infty} 1 \quad \checkmark$$

Now $$\int_a^x f(t)\,dt = \left.\frac{t^{i\mu}}{i\mu}\right|_a^x + \int_a^x \frac{dt}{t\ln t}$$

$v = \ln x$

$dv = \frac{dx}{x}$

$$= \frac{(x^{i\mu} - a^{i\mu})}{i\mu} + \int_{\ln a}^{\ln x} \frac{dv}{v}$$

$$= \frac{x^{i\mu} - a^{i\mu}}{i\mu} + \ln(\ln x) - \ln(\ln a) \neq O(1) \quad x \to \infty \;!$$

Pg 11 Olver

Ex 4.3

$$\int_x^\infty \frac{dt}{t(t^2+t+v^2)^{1/2}}$$

let $f(t) = \dfrac{1}{t(t^2+t+v^2)^{1/2}} = \dfrac{1}{t}(t^2+t+v^2)^{-1/2}$

$$= \frac{1}{t}(t^{-1})(1+t^{-3}+v^2t^{-2})^{-1/2} = \frac{1}{t^2}(1+v^2t^{-2}+t^{-3})^{-1/2}$$

$$= \frac{1}{t^2}\sum_{k=0}^{\infty}\binom{-1/2}{k}(v^2t^{-2}+t^{-3})^k$$

$$= \frac{1}{t^2}\left[1-\frac{1}{2}(v^2t^{-2}+t^{-3})+\frac{3}{8}(v^2t^{-2}+t^{-3})^2+O(t^{-6})\right]$$

$O(t^{-4}), O(t^{-5}), O(t^{-6})$

$$\binom{-1/2}{2} = \frac{(-1/2)(-1/2-1)}{2!} = \frac{1/2(3/2)}{2} = \frac{3}{8}$$

$$= \frac{1}{t^2}\left[1-\frac{v^2}{2t^2}-\frac{1}{2t^3}+\frac{3}{8}O(t^{-4})\right]$$

$$= \frac{1}{t^2}-\frac{v^2}{2t^4}-\frac{1}{2t^5}+O(t^{-6})$$

Then $$\int_x^\infty f(t)\,dt = -\frac{1}{t} - \frac{u^2}{2t^3}\frac{1}{(-3)} - \frac{1}{2t^4}\frac{1}{(-4)} + O(t^{-5}) \Big|_x^\infty$$

$$= \frac{1}{x} - \frac{3u^2}{2x^3} + \frac{1}{8}\frac{1}{x^4} + O(x^{-5})$$

$$= \frac{1}{x} + O\left(\frac{u^2}{x^3}\right) + O\left(\frac{1}{x^4}\right) + \qquad x \to \infty$$

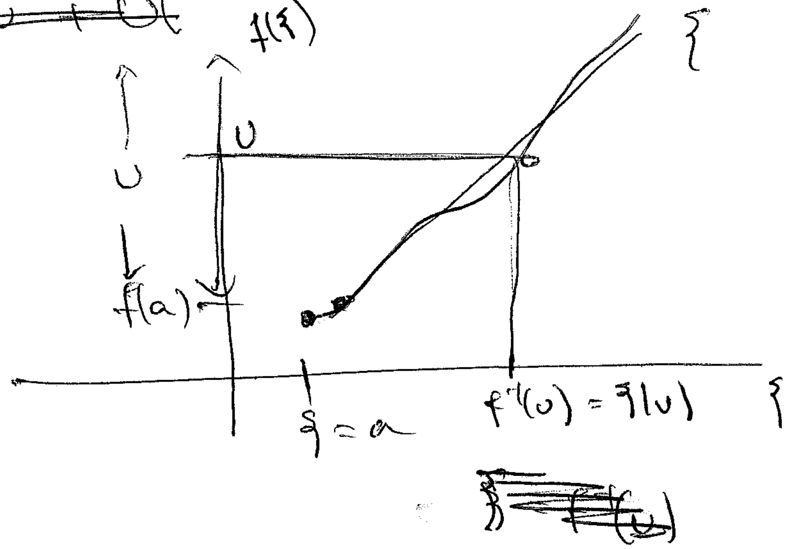
pg 12 Olver

$x^2 - \ln x = U$ $\quad \boxed{x = x(U)}$ invert

$\cancel{x^2 \approx U}$ $\quad$ ~~large x~~ U.

$\to$ ~~[illegible]~~ $\quad x \cancel{\approx \pm\sqrt{U} + O(}$

$\cancel{\xi(U) \sim U}.$ $\quad \cancel{(U \to \infty)}$



$f(\xi) \sim \xi \qquad \xi \to \infty$

$U = \xi(1 + o(1)) \qquad \xi \to \infty$

$$\xi = \frac{U}{1 + o(1)} = U(1 + o(1))$$

$\xi = x^2 \qquad f(\xi) = \xi - \frac{1}{2}\ln\xi \qquad f(\xi) \sim \xi \quad \xi \to +\infty$

$f'(\xi) = 1 - \frac{1}{2\xi} > 0 \qquad \xi > 1/2$

$U \sim \xi$ by Thm 5.1. $\quad U \to +\infty$.

$\Rightarrow U \sim x^2 \qquad U \to +\infty$

$\Rightarrow x^2 = U(1 + o(1)) \qquad U \to +\infty$ "Def" of $\sim$

$x = U^{1/2}(1 + o(1)) \qquad U \to \infty.$

Now As $x^2 - \frac{1}{2}\ln x^2 = x^2 - \ln x \sim x^2 \quad x \to \infty$

$\ln x$ can be thought of as a correction even though it is obviously not as $x \to \infty$

$\Rightarrow \quad x^2 = U - \ln x$

w/ $x = U^{1/2}(1 + o(1)) \qquad U \to \infty$ from Above

we get

$$x^2 = U + \ln(U^{1/2}(1 + o(1))) \qquad U \to \infty$$

$$= U + \ln U^{1/2} + \ln(1 + o(1))$$

$$= U + \frac{1}{2}\ln U + o(1) \qquad U \to \infty$$

$$\therefore \quad x = U^{1/2}\left\{1 + \frac{1}{2}\frac{\ln U}{U} + \frac{o(1)}{U}\right\}^{1/2} \qquad U \to \infty$$

$$= U^{1/2}\left(1 + \frac{1}{4}\frac{\ln U}{U} + o\left(\frac{1}{U}\right)\right) \qquad U \to \infty \qquad *$$

As $\frac{o(1)}{U} = o\left(\frac{1}{U}\right) \qquad U \to \infty$

~~As~~ Now consider $*$ as the starting point

Then $x^2 = U + \ln U^{1/2}\left(1 + \frac{1}{4}\frac{\ln U}{U} + o\left(\frac{1}{U}\right)\right) \qquad U \to \infty$

$$\rightarrow \quad x^2 = U + \ln U^{1/2} + \ln\left(1 + \frac{\ln U}{4U} + o\left(\frac{1}{U}\right)\right) \qquad U \to \infty$$

$$= U + \frac{\ln U}{2} + \cancel{\frac{1}{1+\frac{\ln U}{4U}}} + \ln\left(1 + \frac{\ln U}{4U} + o\left(\frac{1}{U}\right)\right)$$

$$\ln(1+x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{k+1}}{k+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + O(x^5)$$

$$\frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k$$

$$\therefore \quad x^2 = U + \frac{\ln U}{2} + \frac{\ln U}{4U} + o\left(\frac{1}{U}\right) - O\left(\frac{\ln^2 U}{U^2}\right)$$

$$x = U^{1/2}\left\{1 + \frac{\ln U}{2U} + \frac{\ln U}{4U^2} + o\left(\frac{1}{U^2}\right) + O\left(\frac{\ln^2 U}{U^3}\right)\right]^{1/2}$$

can be absorbed in here

$$= U^{1/2}\left\{1 + \frac{\ln U}{4U} + \frac{\ln U}{16U^2} + o\left(\frac{1}{U^2}\right)\right.$$ As $\frac{\ln^2 U}{U^3} = \frac{\ln^2 U}{U} \cdot \frac{1}{U^2}$

goes to zero also.

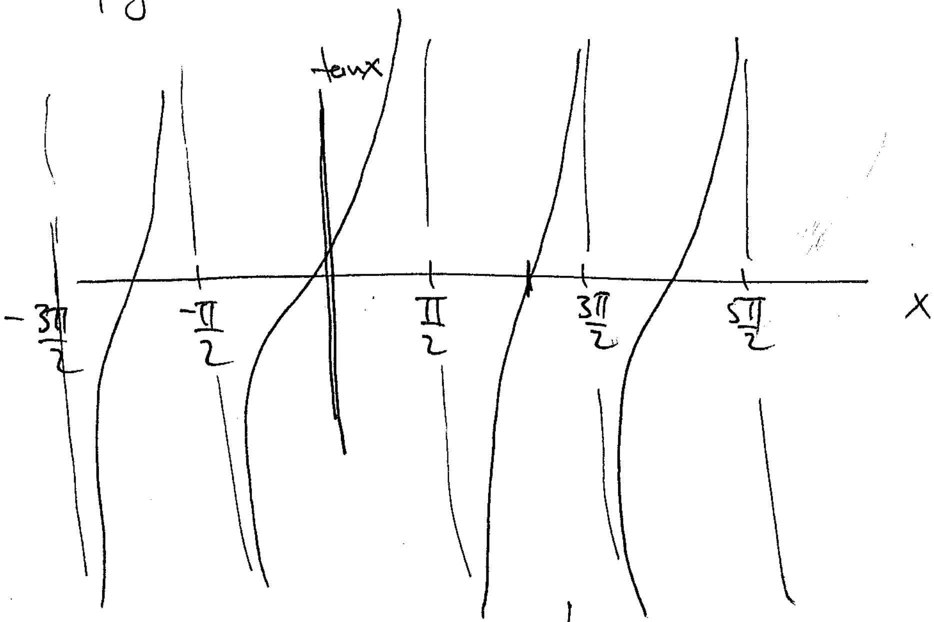
$$\left. + \alpha \frac{\ln^2 U}{4U^2} + \cdots \right]$$ is this correct?

Pg 13 Olver

Ex 5.1

$$x \tan x = U$$

$U \to \infty$ Behavior in $(0, \pi/2)$ dominated by $\tan x$.

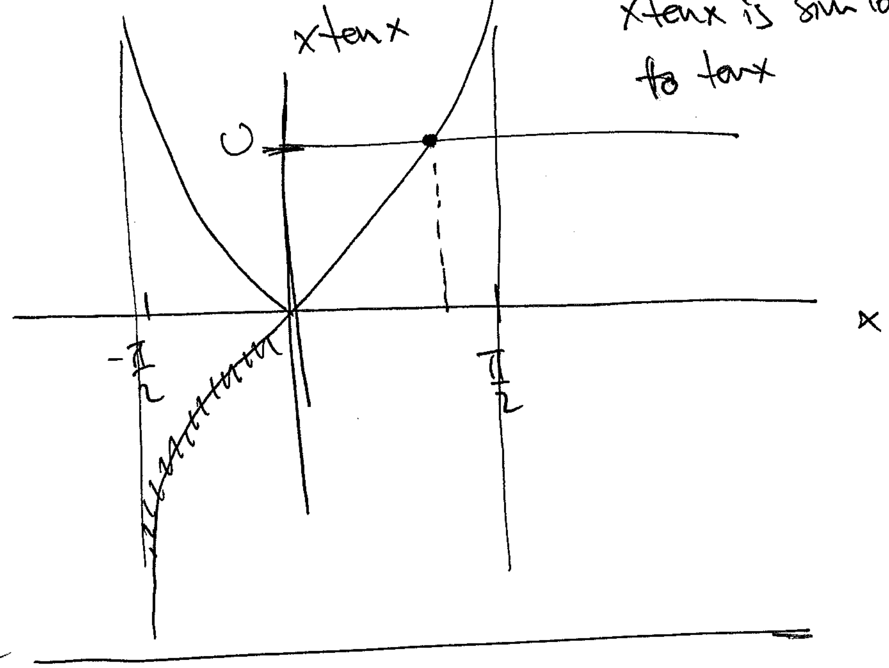


$\therefore \quad x \sim \frac{\pi}{2} +$

$$x(U) \sim \frac{\pi}{2}$$

$$\Rightarrow \quad x(U) = \frac{\pi}{2} + o(1) \qquad U \to \infty$$

x tan x is sim looking to tan x



Then

~~x tan(π/2 +~~

$$X = \tan^{-1}\left(\frac{U}{x}\right)$$

$$x = n\pi + \tan^{-1}\left(\frac{U}{x}\right)$$

$n = 0$ for x to lie in $(0, \pi/2)$

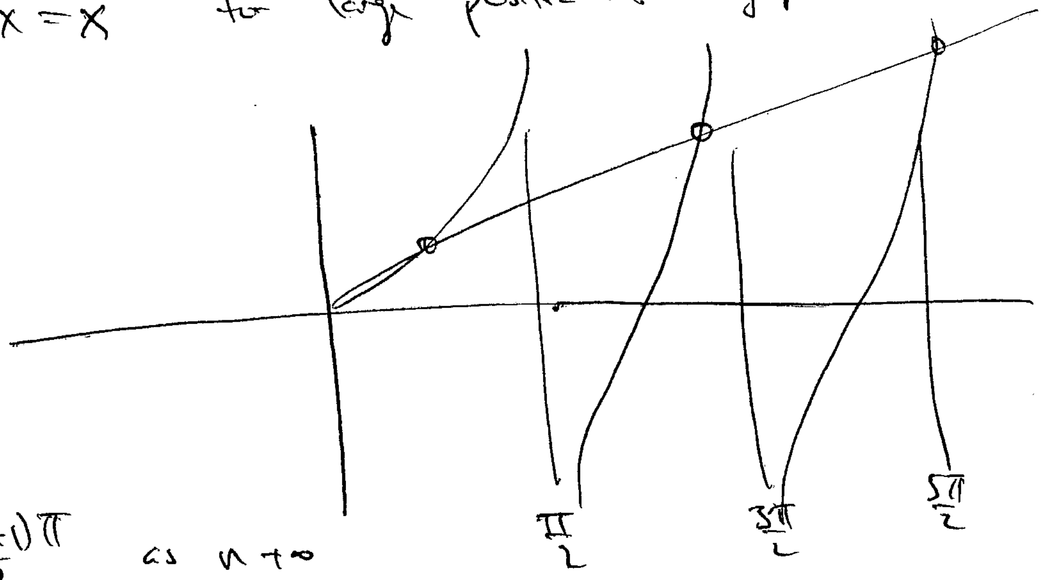
$$\tan x = \frac{U}{x}$$

$$\tan x = \frac{U}{\frac{\pi}{2} + o(1)} = \frac{2}{\pi} U \left(1 + o(1)\right)^{-1} = \frac{2}{\pi} U \left(1 - o(1)\right)$$

$$= \frac{2}{\pi} U - o(U)$$

Pg 13 Olver
& should go in diff folder

Ex 5.2 $\tan x = x$ for large positive x. A graphic



analysis shows

That $x_n \sim \frac{(2n+1)\pi}{2}$ as $n \to \infty$

$\Rightarrow \quad x = \frac{(2n+1)\pi}{2} + O\left(\frac{1}{n}\right)$

$\Rightarrow \quad x = \frac{(2n+1)\pi}{2} + \tan^{-1}(x) \qquad x \to +\infty$

check $\tan(x) = \tan\left(\frac{(2n+1)\pi}{2} + \tan^{-1}(x)\right)$

$$= \frac{\tan\left(\frac{(2n+1)\pi}{2}\right) + x}{1 - \tan\left(\frac{(2n+1)\pi}{2}\right)\tan(\tan^{-1}(x))}$$

Assume $\tan(\tan^{-1}(x)) = x$

$= -\frac{1}{x}$ Not correct?

Then it must be

$$x = \frac{(2n+1)\pi}{2} - \tan^{-1}(x^{-1}) \qquad x \to \infty$$

Now $\tan^{-1}(x^{-1})$ is expanded as pg 12

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)x^{(2k+1)}} = \frac{1}{x} - \frac{1}{3x^3} + \frac{1}{5x^5} - \frac{1}{7x^7} + \cdots$$

$$= \frac{1}{x} + O(x^{-3})$$

Thus $\quad x = \mu - \frac{1}{x} + O(x^{-3}) \qquad \mu = (n+\tfrac{1}{2})\pi$

$$= \mu - \frac{1}{\mu - \frac{1}{x} + O(x^{-3})} + O(\mu^{-3})$$

$$= \mu - \frac{1}{\mu + O(\mu^{-1})} + O(\mu^{-3}) \qquad \mu \to \infty$$

$$= \mu - \frac{1}{\mu}\underbrace{\left(\frac{1}{1 + O(\mu^{-2})}\right)}$$

$$= \mu - \mu^{-1}\left(1 + O(\mu^{-2})\right) + O(\mu^{-3})$$

$$= \mu - \mu^{-1} + O(\mu^{-3}) . \qquad \mu \to \infty$$

Now keep one more term

$$+ O(x^{-5}) \qquad x \to \infty$$

Now $\tan^{-1}(x^{-1})$ is expanded in $1/x$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)x^{(2k+1)}} = \frac{1}{x} - \frac{1}{3x^3} + \frac{1}{5x^5} - \frac{1}{7x^7} + \cdots$$

$$= \frac{1}{x} + O(x^{-3})$$

Thus $\quad x = \mu - \frac{1}{x} + O(x^{-3}) \qquad \mu = (n+\tfrac{1}{2})\pi$

$$= \mu - \frac{1}{\mu - \frac{1}{x} + O(x^{-3})} + O(\mu^{-3})$$

$$= \mu - \frac{1}{\mu + O(\mu^{-1})} + O(\mu^{-3}) \qquad \mu \to \infty$$

$$= \mu - \frac{1}{\mu}\left(\frac{1}{1 + O(\mu^{-2})}\right)$$

$$= \mu - \mu^{-1}(1 + O(\mu^{-2})) + O(\mu^{-3})$$

$$= \mu - \mu^{-1} + O(\mu^{-3}). \qquad \mu \to \infty$$

Now keep one more term

$$x = \mu - \frac{1}{x} + \frac{1}{3x^3} + O(x^{-5}) \qquad x \to \infty$$

$$x = \mu - \frac{1}{\mu-\mu^{-1}+O(\mu^{-3})} + \frac{1}{3(\mu-\mu^{-1}+O(\mu^{-3}))^{3}} + O(\mu^{-5}) \qquad \mu\to\infty$$

$$= \mu - \frac{1}{\mu(1-\mu^{-2}+O(\mu^{-4}))} + \frac{1}{3\mu^{3}(1-\mu^{-2}+O(\mu^{-4}))^{3}} + O(\mu^{-5})$$

$$= \mu - \mu^{-1}\left[1+(\mu^{-2}+O(\mu^{-4}))+\dots\right]$$

$$+ \frac{1}{3}\mu^{-3}(1-\mu^{-2}+O(\mu^{-4}))^{-3} + O(\mu^{-5})$$

$$\boxed{\frac{1}{1-x} = 1+x+x^{2}+\dots}$$

$$= \mu-\mu^{-1}-\mu^{-3}+O(\mu^{-5}) + \frac{1}{3}\mu^{-3}(1+3\mu^{-2}+O(\mu^{-4})) +$$

$$= \mu-\mu^{-1}-\mu^{-3}+\frac{1}{3}\mu^{-3} + O(\mu^{-5}) \qquad \mu\to\infty$$

$$= \mu-\mu^{-1}-\frac{2}{3}\mu^{-3}+O(\mu^{-5}) \qquad \mu\to\infty$$

To do the next order keep 3 terms & ~~then~~ in this expansion & expand 3 terms in orig expansion

Pg 15 Over

$f(z) = z + g(z)$ Now As $f(z) \sim z$ $z \to \infty$

$\Rightarrow g(z) = \cancel{O(1)}$ $\cancel{z\to\infty}$

$\cancel{o(1/z)}$

$\Rightarrow \dfrac{g(z)}{z} = o(1)$ $z \to \infty$

$\Rightarrow g(z) = o(z)$ $z \to \infty$

By Ritts Thm

$g'(z) = o(1)$ $z \to \infty$

Now $f(z_2) - f(z_1) = z_2 + g(z_2) - z_1 - g(z_1)$

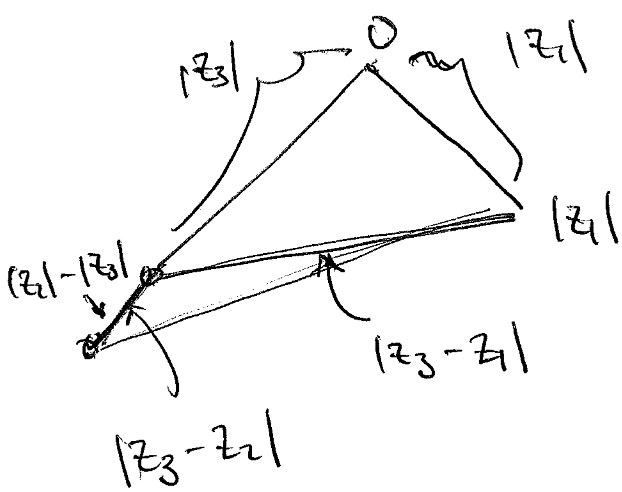
$$= (z_2 - z_1)\Big(1 + \underbrace{\frac{g(z_2) - g(z_1)}{z_2 - z_1}}_{\Theta}\Big)$$

Note Θ is diff quotient of g's & ∴ As $g' = o(1)$

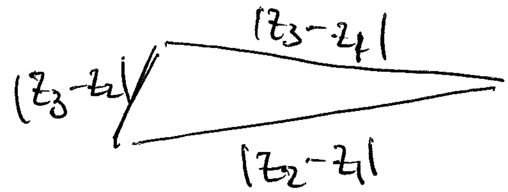
It should be made as small as we like.

$|z-$

$|\Theta| \le$



∴ Δ is



where does condition on $\measuredangle z_1 z_3 0$ less than $\frac{\pi}{2}$ come?

But By Δ ineq.

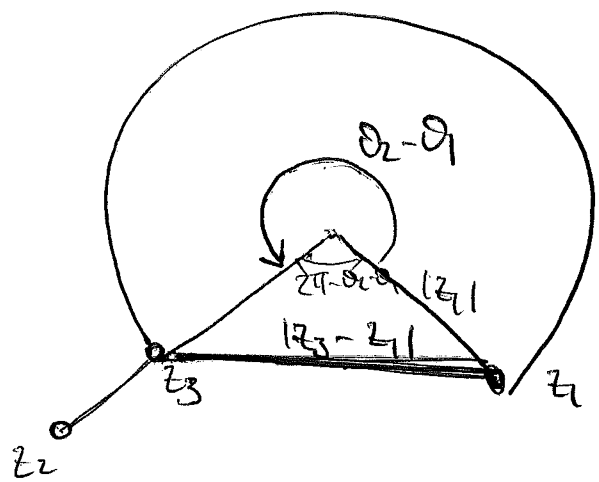
$$|z_3 - z_2| < |z_2 - z_1|$$
$$|z_3 - z_1| < |z_2 - z_1|$$

Then

$$\frac{l(z_1, z_2)}{|z_2 - z_1|} = \frac{\text{length } J}{|z_2 - z_1|} + \frac{\text{length } C}{|z_2 - z_1|} \leq \frac{\text{length } J}{|z_3 - z_2|} + \frac{\text{length } C}{|z_3 - z_1|}$$

But length $J = |z_3 - z_2|$

$$\frac{l(\ )}{|\ \ |} \leq 1 + \frac{\text{length } C}{|z_3 - z_1|}$$



$$\underbrace{\frac{|\theta_2 - \theta_1|}{2\pi}}_{\text{fraction part of circle}} \cdot 2\pi |z_1|$$

$\therefore$

$$\frac{l(\ )}{|z_2 - z_1|} \leq \frac{|\theta_2 - \theta_1|\,|z_1|}{|z_3 - z_1|}$$

Now $\dfrac{|z_1|}{|z_3 - z_1|} = \dfrac{1}{\sin\left|\frac{\theta_2}{2} - \frac{\theta_1}{2}\right|}$

How get

$\Rightarrow \quad \sin\left|\frac{\theta_2}{2} - \frac{\theta_1}{2}\right| = \frac{|z_3 - z_1|}{|z_1|}$ How?

$f = z(1 + \xi(z) z^{-1})$

$\mathrm{ph} f = \mathrm{ph} z + \mathrm{ph}(1 + z^{-1}\xi) = \mathrm{ph} z + o(1) \qquad z \to \infty$

Becaus $\quad z^{-1}\xi = o(1)$

$\therefore \mathrm{ph}(1 + z^{-1}\xi) = \mathrm{ph}(1 + o(1)) = o(1)$

$|f(z)| = |z||1 + z^{-1}\xi(z)| = a_1|1 + z^{-1}\xi(z)| \le a_1(|1| + |z^{-1}\xi(z)|)$

$\le 2a_1$

$\frac{\epsilon}{1+\epsilon} \longrightarrow 0 \quad \epsilon \to 0$

$\therefore \quad |z^{-1}\xi(z)| < \epsilon(1+\epsilon)^{-1}$

$\therefore \left|\frac{z(u)}{u} - 1\right| = \left|\frac{z(u) - u}{u}\right| = \left|\frac{z(u) - (z + \xi(z))}{z + \xi(z)}\right|$

But $u = f(z)$

$= \left|\frac{-\xi(z)}{z(1 + z^{-1}\xi(z))}\right| = \left|\frac{z^{-1}\xi(z)}{1 + z^{-1}\xi(z)}\right| \le \frac{\epsilon(1+\epsilon)^{-1}}{1 - \epsilon(1+\epsilon)^{-1}}$

As ~~$z^{-1}\xi(z)$~~ If $|z^{-1}\xi(z)| < \epsilon(1+\epsilon)^{-1}$

$\Rightarrow \quad z^{-1}\xi(z) > -\epsilon(1+\epsilon)^{-1}$

$1 + z^{-1}\xi(z) > 1 - \epsilon(1+\epsilon)^{-1}$

$$= \frac{\epsilon}{1+\epsilon-\epsilon} = \epsilon$$

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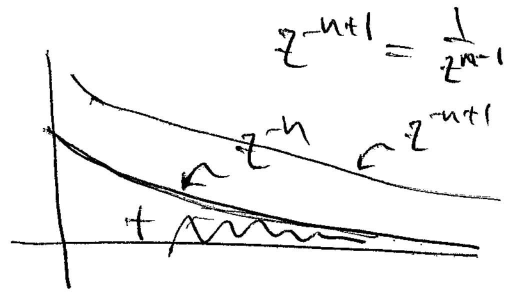
pg 17 Olver

$n \le N \to R_n(z) = \underbrace{O(z^{-n})}_{f} \quad n \le N$

$\left|\frac{f}{z^{-n}}\right| = |f z^n| < M \quad z \to \infty$

Then $\left|\frac{f}{z^{-n+1}}\right| = \left|\frac{f z^n}{z}\right| \to 0$



Thus if $R_n(z) = O(z^{-n})$

Then $R_n(z) = o(z^{-n+1})$

If $\sum a_s z^{-s}$ converge

$\sum_{s=0}^{N-1} a_s z^{-s} + R_N(z) \qquad + R_N = O(z^{-N})$ By Thm 3.1

Thm 7.1:

It is ~~rest~~ subtract That $z^n \left\{ f(z) - \sum_{s=0}^{n-1} \frac{a_s}{z^s} \right\} \to a_n$

For given an asymptotic exp of $f(t)$ By the def

We have $f(z) \simeq a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots \frac{a_{n-1}}{z^{n-1}} + R_n(z)$

w/ $R_n(z) = O(z^{-n})$

$$\therefore \left\{ f(z) - \sum_{s=0}^{n-1} \frac{a_s}{z^s} \right\} = O(z^{-n})$$

$$\Rightarrow \quad z^n \left\{ f(z) - \sum_{s=0}^{n-1} \frac{a_s}{z^s} \right\} = O(1)$$

But $z^n \left\{ f(z) - \sum_{s=0}^{n-1} \frac{a_s}{z^s} \right\} \to a_n = O(1)$ so the Above is sat.

That $z^n \left\{ f(z) - \sum_{s=0}^{n-1} \frac{a_s}{z^s} \right\} \to a_n$ is nss

7.01 $\Rightarrow$

$$z^n f(z) = z^n \sum_{s=0}^{n-1} \frac{a_s}{z^s} + z^n R_n(z)$$

~~$z^n R_n(z)$~~

$$\| $$

$$\sum_{s=0}^{n} \frac{a_s}{z^s} + R_{n+1}(z)$$

$$\Rightarrow \quad z^n \sum_{s=0}^{n} \frac{a_s}{z^s} + z^n R_{n+1} = z^n \sum_{s=0}^{n-1} \frac{a_s}{z^s} + z^n R_n$$

$$\Rightarrow \quad z^n \frac{a_n}{z^n} + z^n R_{n+1} = z^n R_n$$

$$\Rightarrow \quad z^n R_n = z^n \left\{ \frac{a_n}{z^n} + R_{n+1} \right\} \cancel{\Rightarrow a_n + z^n R_{n+1}}$$

$$= a_n + z^n O(z^{-n-1}) = a_n + O(z^{-1}) \to a_n$$

(i) $f(z) \to$ an series

(ii) $z^n R_n(z) \to a_n$

$\Rightarrow \quad R_n(z) \to a_n z^{-n} \Rightarrow |a_n|$ must be greater than or equal to the nth implied constant

---

$z^0 e^{-z} \longrightarrow 0 = a_0$

$z^1 e^{-z} \longrightarrow 0 = a_1$

$\vdots$

If $f(z) \sim \phi(z) \sum_{s=0}^{\infty} \frac{a_s}{z^s} = \phi(z)\left(a_0 + \sum_{s=1}^{\infty} \frac{a_s}{z^s}\right)$

Then $\lim_{z \to \infty} \frac{f(z)}{a_0 \phi(z)} =$

look at $\dfrac{f(z)}{a_0 \phi(z)} \sim \displaystyle\sum_{s=0}^{\infty} \frac{a_s}{a_0 z^s}$

$a_0 \neq 0$

def of asymptotic expansion of f/ϕ

evaluate $\displaystyle\lim_{z \to \infty} \frac{f(z)}{a_0 \phi(z)} = ?$

By def of asymptotic expansion of f/ϕ

$$\frac{a_s}{a_0} \neq z^n \left\{ \frac{f(z)}{a_0 \phi(z)} - \sum_{s=0}^{n-1} \frac{a_s}{a_0 z^s} \right\} \longrightarrow \frac{a_n}{a_0} = \begin{cases} 1 & n = 0 \\ \frac{a_n}{a_0} & n \neq \end{cases}$$

$$\Rightarrow \quad \frac{f(z)}{a_0 \phi(z)} \longrightarrow \frac{a_n}{a_0 z^n} + \sum_{s=0}^{n-1} \frac{a_s}{a_0 z^s} = \underline{1}$$

Try $f(z) = z$. pole order 1 at ∞.

Try

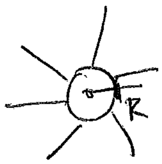
$f = z + \frac{1}{z}$

The $f - z = \frac{1}{z}$ Is a Taylor expansion $+ \therefore$ ~~$\frac{1}{z}$ has an asymptotic~~

$\frac{1}{z}$ has an asymptotic expansion

$\therefore \; f(z) \sim z + \frac{1}{z}$ ✓. $f\ \phi =$ poly deg 1.

$$f \sim \sum_{s=n}^{\infty} \frac{a_s}{z^s} \qquad z \to \infty \qquad \forall \ \mathrm{ph}\, z.$$



$$f(z) = \sum_{s=-\infty}^{\infty} \frac{b_s}{z^s} \qquad |z| > R$$

$$b_s = \frac{1}{2\pi i}\int_{|z|=\rho} f(z) z^{s-1}\, dz = \frac{1}{2\pi i}\int_{|z|=\rho} \frac{f(z)}{z^{1-s}}$$

$$b_{-s} = \frac{1}{2\pi i}\int_{|z|=\rho} \frac{f(z)}{z^{s+1}}\, dZ$$ = These are the coeff of z^s.

$$b_s = \frac{1}{2\pi i}\int_{|z|=\rho} f(z) z^{s-1}\, dz$$ = These are the coeff of z^{-s}.

$$f(z) = \sum_{s=-\infty}^{n-1} \frac{b_s}{z^s} + \sum_{s=n}^{\infty} \frac{b_s}{z^s} -$$

How get $f(z) = O(z^{-n})$ if $f \neq O(z^{-n})$

Then >

If $f(z) \sim \sum_{s=0}^{n} \frac{a_s}{z^s} \Rightarrow f(z) - \sum_{s=0}^{n} \frac{a_s}{z^s} = O(z^{-n})$ By def.

$$(7.06) \Rightarrow \quad f(z) \sim \sum_{s=n}^{\infty} \frac{a_s}{z^s} \Rightarrow \quad f(z) - \sum_{s=n}^{\infty} \frac{a_s}{z^s} = O(z^{-n})$$

But $$f(z) = \sum_{s=-\infty}^{n-1} \frac{b_s}{z^s} + \sum_{s=n}^{\infty} \frac{b_s}{z^s}$$

$$\therefore \quad f(z) - \sum_{s=n}^{\infty} \frac{a_s}{z^s} = \sum_{s=-\infty}^{n-1} \frac{b_s}{z^s} + \sum_{s=n}^{\infty} \frac{b_s - a_s}{z^s} = O(z^{-n})$$

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$$\lambda f(z) + \mu g(z) - \sum_{s=0}^{\infty} (\lambda f_s + \mu g_s) z^{-s}$$

$$= \lambda\left(f(z) - \sum_{s=0}^{\infty} f_s \quad\right) + \mu\left(g(z) - \sum_{s=0}^{n} g_s \quad\right) = O(z^{-n}) \checkmark$$

$$F_n(z) = f(z) - \sum_{s=0}^{n-1} \frac{f_s}{z^s} \qquad G_n(z) = g(z) - \sum_{s=0}^{n-1} g_s z^{-s}$$

$$H_n(z) = h(z) - \sum_{s=0}^{n-1} h_s z^{-s}$$

$$= f(z) g(z) - \sum_{s=0}^{n-1} \left(\sum_{k=0}^{s} f_k g_{s-k} \right) z^{-s}$$

How get result in the book?

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Consider:

$$\sum_{s=0}^{n-1} \frac{f_s}{z^s} G_{n-s} + g(z) F_n$$

$$= \sum_{s=0}^{n-1} \frac{f_s}{z^s}\left(g(z) - \sum_{k=0}^{n-s-1} g_k z^{-k}\right) + g(z) F_n(z)$$

$$= \cancel{g(z)\sum_{s=0}^{n-1} \frac{f_s}{z^s}} - \sum_{s=0}^{n-1} \frac{f_s}{z^s} \sum_{k=0}^{} g_k z^{-k} + g(z)\left(f(z) - \cancel{\sum_{s=0}^{n-1} \frac{f_s}{z^s}}\right)$$

$$= gf - \sum_{s=0}^{n-1}\left(\sum_{k=0}^{n-s-1} f_s g_k\right) z^{-(s+k)}$$

want to be c_s of z^{-s}.

let $v = s+k$ leave outer sum the same.

Change inner sum $k=0 \Rightarrow v=s$

$k = n-s-1 \Rightarrow v = n-1$

$$\Rightarrow gf - \sum_{s=0}^{n-1} \underbrace{\sum_{v=s}^{n-1} f_s g_{v-s} z^{-v}}$$

$f_s g_0 + f_{s+1} g_1 + f_{s+2} g_2 + \cdots + f_{n-1} g_{n-1-s}$

Consider

$$gf - \sum_{s=0}^{n-1}\left(\sum_{k=0}^{n-s-1} f_s g_k z^{-(k+s)}\right)$$

$\cancel{f_s g_0 +}$

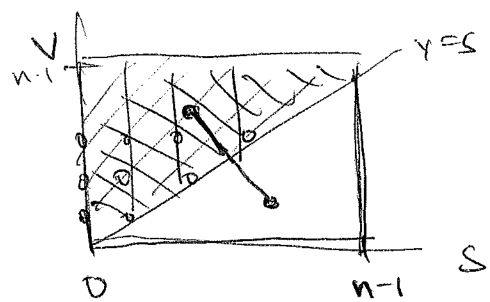
$\cancel{s=0}$

$s=0 \quad f_0\left(g_0 z^0 + g_1 z^{-1} + g_2 z^{-2} + \cdots + g_{n-1} z^{-(n-1)}\right)$

$s=1 \quad + f_1\left(g_0 z^{-1} +\right.$

$$gf - \sum_{s=0}^{n-1} \sum_{k=0}^{n-s-1} f_s g_k z^{-(s+k)}$$

$$\sum_{k=0}^{n-1} f_k z^{-k} \sum_{p=0}^{n-1} g_p z^{-p} =$$



$$= gf - \sum_{s=0}^{n-1} \sum_{v=s}^{n-1} f_s g_{v-s} z^{-v}$$

$$\cancel{= gf - \sum_{s=0}^{n-1} \left\{ f_s g_0 + f_s g \;\; f_s g_0 z^{-s} + f_s g_1 z^{-(s+1)} \right.}$$

$$\cancel{+ f_s g_2 z^{-(s-2)} + \cdots +}$$

Map pts through line
$V = S$

$\rightarrow$

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$$\frac{1}{f(z)} = \frac{1}{f_0 + F_1(z)} = \frac{1}{f_0} \frac{1}{\left(1 + \frac{F_1(z)}{f_0}\right)}$$

$$= \frac{1}{f_0}\left(\sum_{s=0}^{n-1} \left(\frac{-F_1(z)}{f_0}\right)^{k} + \frac{\left(\frac{-F_1(z)}{f_0}\right)^n}{1 + \frac{F_1(z)}{f_0}} \right)$$

$$\frac{1}{1+x} = \sum_{k=0}^{n-1} (-1)^k x^k$$

$$\Delta x^t = x^t(x-1)$$

$$\sum x^k = \frac{1}{x-1} \sum \Delta x^k = \frac{1}{x-1} x^k \Big|_{\square}^{E(\ )}$$

$$\therefore \sum_{k=0}^{n-1} x^k = \frac{1}{x-1} x^k \Big|_0^n = \frac{x^n - 1}{x-1} = \frac{x^n}{x-1} + \frac{1}{1-x}$$

$$\rightarrow \frac{1}{1-x} = \sum_{k=0}^{n-1} x^k + \frac{x^n}{1-x}$$

$$= \frac{1}{f_0} \sum_{s=0}^{n-1} \frac{(-1)^s}{f_0^s} (F_1(z))^s + \frac{(-1)^n F_1(z)^n}{f_0^n (f_0 + F_1(z))}$$

But $f = f_0 + \underbrace{f_1 z^{-1} + f_2 z^{-2} + f_3 z^{-3} + \cdots + f_{n-1} z^{-n+1} + F_n(z)}_{F_1(z)}$

$$\therefore \frac{1}{f(z)} = \sum_{s=0}^{n-1} \frac{(-1)^s}{f_0^{s+1}} \left\{ \frac{f_1}{z} + \cdots + \frac{f_{n-1}}{z^{n-1}} + F_n(z) \right\}^s + \frac{(-1)^n F_1^n}{f_0^n (f_0 + F_1(z))}$$

$F_1 = O(z^{-1})$ $\qquad F_n = O(z^{-n})$

$F_1^n = O(z^{-n})$

$\therefore$ last term is $O(z^{-n})$

Now $\left\{ \frac{f_1}{z} + \cdots + \frac{f_{n-1}}{z^{n-1}} + F_n \right\}^s = \sum_{p=0}^{s} \binom{s}{p} \left(\frac{f_1}{z} + \cdots + \frac{f_{n-1}}{z^{n-1}} \right)^{s-p} F_n^p$

$F_n^p = \cancel{O(z^{-n+p})}\ O(z^{-np}) = O(z^{-n}) \quad p \geq 0$

$\therefore \left\{ \frac{f_1}{z} + \cdots + \frac{f_{n-1}}{z^{n-1}} + F_n \right\}^s = \cancel{O(z^{-n})}$

$= \binom{s}{0} \left\{ \frac{f_1}{z} + \cdots + \frac{f_{n-1}}{z^{n-1}} \right\}^s + O(z^{-n})$

$$\therefore \frac{1}{f(z)} = \sum_{s=0}^{n-1} \frac{(-1)^s}{f_0^{s+1}} \binom{s}{0} \left\{ \frac{f_1}{z} + \cdots + \frac{f_{n-1}}{z^{n-1}} \right\}^s + O(z^{-n})$$

$\therefore$

$$\frac{1}{f(z)} \sim \sum_{s=0}^{\infty} \frac{k_s}{z^s}$$

$$1 \sim f(z) \sum_{s=0}^{\infty} \frac{k_s}{z^s} \qquad \text{w/} \quad f(z) \sim \sum f_s/z^s$$

$$1 \sim \sum_{s_1=0}^{\infty} \frac{f_{s_1}}{z^{s_1}} \sum_{s_2=0}^{\infty} \frac{k_{s_2}}{z^{s_2}} = \sum_{s_1=0}^{\infty} \sum_{s_2=0}^{\infty} \frac{f_{s_1} k_{s_2}}{z^{s_1+s_2}}$$

let $V = s_1 + s_2$

$s_2 = V - s_1$

$$\cancel{= \sum_{s_1=0}^{\infty} \sum_{V=0}^{\infty} \frac{f_{s_1} k_{V-s_1}}{z^V} = \sum_{V=0}^{\infty} \sum_{s_1=0}^{\infty} \frac{f_{s_1} k_{V-s_1}}{z^V}}$$

$$= \cancel{f_0 k_0 +} \quad \sum_{s_1=0}^{\infty} \sum_{V=s_1}^{\infty} \frac{f_{s_1} k_{V-s_1}}{z^V}$$

$$= f_0 k_0 + \frac{f_0 k_1}{z} + \frac{f_0 k_2}{z^2} + \frac{f_0 k_3}{z^3} + \cdots$$

$$\frac{f_1 k_0}{z} + \frac{f_1 k_1}{z^2} + \frac{f_1 k_2}{z^3} + \cdots \qquad + \frac{f_2 k_1}{z^1}$$

If

$$f_0 k_s = -(f_1 k_{s-1} + f_2 k_{s-2} + \cdots + f_s k_0) \qquad s = 0, 1, 2, 3, \ldots$$

$$f_0 k_1 = -f_1 k_0 \quad \forall \quad k_0 = \frac{1}{f_0} \qquad \text{(By observation)}$$

$$f_0 k_1 = -f_1 \frac{1}{f_0} \Rightarrow \quad k_1 = -\frac{f_1}{f_0^2}$$

$$f_0 k_2 = -(f_1 k_1 + f_2 k_0)$$

$$k_2 = \frac{-1}{f_0}\left(f_1\left(-\frac{f_1}{f_0^2}\right) + \frac{f_2}{f_0}\right) = \frac{(f_1^2 - f_0 f_2)}{f_0^3}$$

$$k_3 = \frac{-1}{f_0}(f_1 k_2 + f_2 k_1 + f_3 k_0)$$

$$= \frac{-1}{f_0}\left(\frac{f_1}{f_0^3}(f_1^2 - f_0 f_2) + f_2\left(-\frac{f_1}{f_0^2}\right) + f_3 \frac{1}{f_0}\right)$$

$$= \frac{(-f_1^3 + f_1 f_0 f_2 + f_1 f_2 f_0 - f_0^2 f_3)}{f_0^4}$$

$$= \frac{(-f_1^3 + 2 f_0 f_1 f_2 - f_0^2 f_3)}{f_0^4}$$

Get $f_0 k_s = -(f_1 k_{s-1} + f_2 k_{s-2} + \cdots + f_s k_0)$

$$f(z)\left\{\frac{1}{f(z)}\right\} = 1$$

$$\sum_{s=0}^{\infty} f_s z^{-s} \sum_{p=0}^{\infty} k_p z^{-p} = 1$$

$$\sum_{p,s=0}^{\infty} f_s k_p z^{-(s+p)} = 1$$

let $s+p = v$

$p = v-s$

$$\sum_{k=0}^{\infty}\left(\sum_{p=0}^{k} f_k k_{k-p}\right) z^{k}$$

$$\Rightarrow \sum_{s=0}^{\infty} \sum^{v} f_s k_{v-s} z^{-v}$$

$f_0 k_0$

$$f(z)g(z) \sim \sum_{s=0}^{\infty} h_s z^{-s}$$

w/ $h_s = \sum_{k=0}^{s} f_s g_{s-k}$ if $s=0 \Rightarrow 1$

$\Rightarrow f_0 k_0 = 1$

if $s \neq 0$

$$0 = \cancel{f_0 g} \quad f_0 k_s + f_1 g_{s-1} + \cdots + f_2 g_{s-2} + \cdots + f_s k_0$$

$$\Rightarrow f_0 k_s = -\left(f_1 k_{s-1} + f_2 k_{s-2} + \cdots + f_s k_0 \right)$$

$$\frac{f_2 x^{-1}}{-1} + \frac{f_3 x^{-2}}{-2} + \cdots \Big|_x^{\infty}$$

$$= \frac{f_2}{x} + \frac{f_3}{2x^2} + \cdots$$

$$\int_a^x f(t)\,dt = \int_a^x \left(f(t) - f_0 - \frac{f_1}{t} \right) dt + f_0(x-a) + f_1 \ln(x/a)$$

$$= \left(\int_a^{\infty} - \int_x^{\infty} \right) \left\{ f(t) - f_0 - \frac{f_1}{t} \right\} dt + f_0(x-a) + f_1 \ln(x/a)$$

$$\sim \left(\frac{f_2}{a} + \frac{f_3}{2a^2} + \frac{f_4}{3a^3} + \cdots \right) - \left(\frac{f_2}{x} + \frac{f_3}{2x^2} + \frac{f_4}{3x^3} + \cdots \right)$$

$$+ f_0(x-a) + f_1 \ln(x) \qquad \sim A + f_0 x + f_1 \ln x$$

$$- \frac{f_2}{x} - \frac{f_3}{2x^2} - \frac{f_4}{3x^3} + \cdots$$

$$f = e^{-x}\sin(e^x)$$

$$a_0 = \lim_{z\to\infty}\{e^{-x}\sin(e^x)\} = 0.$$

$$a_1 = \lim_{x\to\infty} x\{e^{-x}\sin(e^x)\} = 0$$

$\vdots$

$$a_n = 0$$

$$f' = -e^{-x}\sin(e^x) + \cos(e^x)$$

Attempt an asymptotic expansion for this fn

$$a_0 = \lim_{x\to\infty} f' = \text{undef.}$$ Thus 7.04 doesn't wk for n=0

$$a_1 =$$

If f' is cont f cont.

$$\int_a^x f' = f(x) - f(a)$$

$$z \sim \xi \quad \Rightarrow \quad z = \{1 + o(1)\}\xi$$

$$z = \zeta - a_0 - \frac{a_1}{z} - \frac{a_2}{z^2} - \cdots - \frac{a_{n-1}}{z^{n-1}} + O\left(\frac{1}{z^n}\right)$$

$$z = \zeta - a_0 - \frac{a_1}{(1+o(1))\zeta} - \frac{a_1}{(1+o(1))^2\zeta^2} - \cdots$$

$$b_0 = a_0$$

$$z = \zeta - a_0 - \frac{a_1}{\zeta} + O\left(\frac{1}{\zeta^2}\right)$$

$$b_1 = a_1$$

$$z = \zeta - a_0 - \frac{a_1}{\left(\zeta - a_0 - \frac{a_1}{\zeta} + O\left(\frac{1}{\zeta^2}\right)\right)^1} - \frac{a_2}{\left(\zeta - a_0 - \frac{a_1}{\zeta} + O(1/\zeta^2)\right)^2} + O(1/\zeta^3)$$

$$= \zeta - a_0 - \frac{a_1}{\zeta\left(1 - \frac{a_0}{\zeta} - \frac{a_1}{\zeta^2} + O(1/\zeta^3)\right)} - \frac{a_2}{\zeta^2\left(1 - \frac{a_0}{\zeta} - \frac{a_1}{\zeta^2} + O(1/\zeta^3)\right)^2}$$

$$= \zeta - a_0 - \frac{a_1}{\zeta}\left(1 + \frac{a_0}{\zeta} + \frac{a_1}{\zeta^2} + O(1/\zeta^3)\right)$$

$$- \frac{a_2}{\zeta^2}\left(1 + \frac{2a_0}{\zeta} + \frac{2a_1}{\zeta^2} + O(1/\zeta^3)\right)$$

$$= \zeta - a_0 - \frac{a_1}{\zeta} - \frac{a_1 a_0}{\zeta^2} - \frac{a_1 a_1}{\zeta^3} + O\left(\frac{1}{\zeta^3}\right) - \frac{a_2}{\zeta^2} + O(1/\zeta^3)$$

$$= \xi - a_0 - \frac{a_1}{\xi} - \frac{(a_1 a_0 + a_2)}{\xi^2} + O(1/\xi^3)$$

$$b_2 = a_0 a_1 + a_2$$

$$\nu(|z|) \le |z| - \Big(\sum_{k=} |a_k|$$

If $|z| > z_n$ $\quad \nu(|z|) \geq n+1$

Becaus. $\quad |a_0| + |a_1| + \cdots + |a_{n+1}| + \nu(|z|)$

$$\le |a_0| + \cdots + |a_{n+1}| + n+1 = z_n < |z| \checkmark$$

$$\left| f(z) - \sum_{s=0}^{n-1} \frac{a_s}{z^s} \right| = \left| \sum_{s=0}^{\nu(|z|)} \frac{a_s}{z^s} - \sum_{s=0}^{n-1} \frac{a_s}{z^s} \right| = \left| \sum_{s=n}^{\nu(|z|)} \frac{a_s}{z^s} \right|$$

??

$$\le \frac{|a_n|}{|z|^n} + \frac{1}{|z|^{n+1}} \sum_{s=n+1}^{\nu(|z|)} \frac{a_s}{z}$$

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$$\left|\sum_{s=n}^{\nu(|z|)} \frac{a_s}{z^s}\right| \le \frac{|a_n|}{|z|^n} + \sum_{s=n+1}^{\nu(|z|)} \frac{|a_s|}{|z|^s}$$

$|z| > 1$

$\therefore \dfrac{1}{|z|^p} < 1$

$$\le \frac{|a_n|}{|z|^n} + \frac{1}{|z|^{n+1}} \sum_{s=n+1}^{\nu(|z|)} \frac{|a_s|}{|z|^{s-n-1}}$$

each $\dfrac{1}{|z|^{s-n-1}} < 1$

$$\le \frac{|a_n|}{|z|^n} + \frac{1}{|z|^{n+1}} \sum_{s=n+1}^{\nu(|z|)} |a_s|$$

$$(9.03) \rightarrow \quad \sum_{s=0}^{n-1} |a_s| + |a_n| + \sum_{s=n+1}^{\nu(|z|)} |a_s| \le |z|$$

$$\Rightarrow \quad \frac{|a_n|}{|z|} + \frac{1}{|z|} \sum_{s=0}^{n-1} |a_s| + \frac{1}{|z|} \sum_{s=n+1}^{\nu(|z|)} |a_s| \le 1$$

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$\phi_{s+1}(z) = o(\phi_s(z))$ $\phi_{s+1}(z)$ is of order less than $\phi_s(z)$.

$$\frac{\phi_{s+1}(z)}{\phi_s(z)} \longrightarrow 0 \quad z \to \infty$$

10.3

$$\left| \frac{\cos(tx)}{x^t} \right| \leq \frac{1}{x^t} \quad \text{w. M test}$$

$$\frac{\frac{\cos((t+1)x)}{x^{t+1}}}{\frac{\cos(tx)}{x^t}} = \frac{\cos(t+1)x}{\cos tx} \frac{1}{x}$$

As $x \to \infty$ tx will hit the zeros of $\cos tx$ & we will have divergence.

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$$\sum_{s=0}^{n-1} |f(x_{s+1}) - f(x_s)| = \sum \left| \int_{x_s}^{x_{s+1}} f'(t)\,dt \right| \leq \sum \int_{x_s}^{x_{s+1}} |f'(t)|\,dt$$

$$= \leq \int_{x_0}^{x_n} |f'(t)|\,dt$$

$$V_{a,b}(f) \leq \int_{x_0}^{x_n} |f'(t)|\,dt \leftarrow \int_a^b |f'(t)|\,dt$$

As $|f'(t)| > 0$ Thus increasing limits of integral makes larger value

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$$\Gamma(z) = \int_0^1 e^{-t} t^{z-1}\,dt + \int_1^\infty e^{-t} t^{z-1}\,dt$$

The on 1st int

$$|e^{-t} t^{z-1}| \le e^{-t} t^{\delta-1} < t^{\delta-1} \qquad t \in (0,1)$$

On 2nd

$$|e^{-t} t^{z-1}| \le e^{-t} t^{\Delta-1}$$

$+ \int_0^1 t^{\delta-1}\,dt = \left.\dfrac{t^\delta}{\delta}\right|_0^1$ conv iff $\delta > 0$

$+ \int_1^\infty e^{-t} t^{\Delta-1}\,dt < \infty$ By Ratio test, or int by parts.

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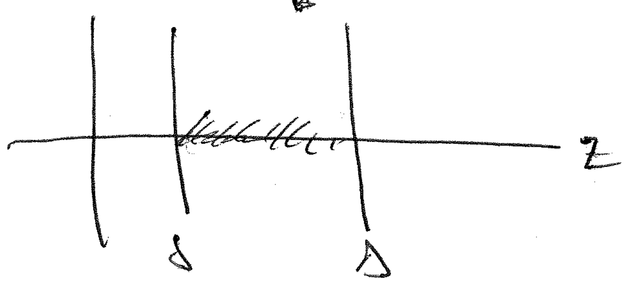
$0 < \delta \leq \operatorname{Re} z \leq \Delta$

$0 = \operatorname{Re} z > 0$



$t \in \mathbb{R}^+$

$$|t^{z-1}| = |e^{(z-1)\log t}| = |e^{(\operatorname{Re}(z) + i\operatorname{Img}(z) - 1)\log t}|$$

$$= |e^{(\operatorname{Re}(z)-1)\log t}| = |\underset{\substack{\shortparallel \\ t^{\operatorname{Re}(z)-1}}}{t^{\operatorname{Re}(z)-1}}| \leq \begin{cases} t^{\delta-1} & 0 < t < 1 \\ t^{\Delta-1} & t > 1 \end{cases}$$

As $t \geq 0$ (see previous pages)

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$\Gamma(z+1) = 1 + z f(z)$

$= \cancel{1+tz\Gamma}\quad 1 + z\Gamma(\tilde{z}) = 1 + z f(z)$ Properties of $f(z)$? holomorphic.

$f(z)$

is this true for complex fns? i.e. can the remainder be written as $\frac{f^{(n+1)}(\xi(z))(z-z_0)^{n+1}}{(n+1)!}$?

As in real variables? Don't think exactly so?

$\Rightarrow \quad z\Gamma(z) = 1 + z f(z)$

$\Gamma(z) = \frac{1}{z} + f(z)$

defn of simple pole of $f(z)$ is

$f(z) = \frac{1}{z} + \phi(z)$ w/ ϕ analytic.

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~~$\Gamma(z-n) =$~~

$\Gamma(z) = \cancel{(z+1)\Gamma(z-1) = (z-}$

$$\Gamma(z+1) = z\Gamma(z) = z(z-1)\Gamma(z-1) = \cdots = z(z-1)(z-2)\cdots(z-n)\cdot\Gamma(z-n)$$

$$\therefore \Gamma(z-n) = \frac{\Gamma(z+1)}{z(z-1)(z-2)\cdots(z-n)} = \frac{1+zf(z)}{z(z-1)(z-2)\cdots(z-n)}$$

$$= \frac{(1+zf(z))}{z}\left[\underbrace{\frac{1}{(z-1)(z-2)\cdots(z-n)}}_{\text{By taylor call this } F(z)}\right]$$

$$= F(0) + zg(z)\frac{(-1)^n}{n!} \quad \text{w/ } g(z) \text{ analytic at } z=0$$

$$F(0) = \frac{1}{(-1)(-2)\cdots(-n)} = \frac{(-1)^n}{n!}$$

$\therefore$

$$\frac{(1+zf(z))}{z}\frac{(-1)^n}{n!}(1+zg(z))$$

$$\Rightarrow \Gamma(z) = \frac{(-1)^n}{n!}\frac{(1+(z+n)f(z+n))(1+(z+n)g(z+n))}{(z+n)}$$

$\therefore$ poles at $z=-n$ are all simple

$\Rightarrow \Gamma(z) = \frac{(-1)^n}{n!} \frac{1}{(z+n)} \left\{ 1 + (z+n)(f+g)(z+n) + (z+n)^2 f \cdot g \right\} \}$ z

$\Rightarrow$ residue at $z = -n$ is $\frac{(-1)^n}{n!}$

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(1.3) Remember

$$t^z = e^{z \log t}$$

$$\lim_{n\to\infty}\left(1-\frac{t}{n}\right)^n = e^{-t}$$

& $$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt$$

let $$\Gamma_n(z) = \int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1}\, dt \qquad \operatorname{Re} z > 0$$

Do integration by parts (integrate t^{z-1} & take derivative of $\left(1-\frac{t}{n}\right)^n$)

$$\Gamma_n(z) = \left(1-\frac{t}{n}\right)^n \frac{t^z}{z}\Bigg|_0^n - \int_0^n \frac{t^z}{z}\cancel{n}\left(1-\frac{t}{n}\right)^{n-1}\left(\frac{-1}{\cancel{n}}\right) dt$$

$\operatorname{Re} z > 0$

$$= 0 - 0 + \frac{1}{z}\int_0^n \left(1-\frac{t}{n}\right)^{n-1} t^z\, dt$$

$$= \frac{1}{z}\left\{ \left(1-\frac{t}{n}\right)^{n-1}\frac{t^{z+1}}{z+1}\Bigg|_0^n - \frac{1}{z+1}\int_0^n (n-1)\left(\frac{-1}{n}\right)\left(1-\frac{t}{n}\right)^{n-2} t^{z+1}\, dt \right\}$$

$$= \frac{1}{z}\left\{ 0 - 0 + \frac{(n-1)}{(z+1)\,n} \int_0^n \left(1-\frac{t}{n}\right)^{n-2} t^{z+1}\,dt \right\}$$

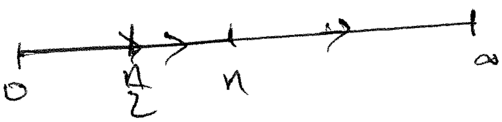
$$= \frac{n-1}{z(z+1)\,n} \int_0^n \left(1-\frac{t}{n}\right)^{n-2} t^{z+1}\,dt$$

$$= \cdots = \frac{(n-1)(n-2)(n-3)\cdots(1)}{z(z+1)n\,(z+2)n \cdots (z+n-1)n} \int_0^n t^{z+n-1}\,dt$$

$$= \frac{(n-1)!}{z(z+1)\cdots(z+n-1)\,n^{n-1}} \left(\frac{t^{z+n}}{z+n} \right|_0^n$$

$$= \frac{(n-1)!}{z(z+1)\cdots(z+n-1)\,n^{n-1}(z+n)}\, n^{z+n} \qquad \left\{ \frac{n^n}{n^{n-1}} = \frac{1}{n^{-1}} \right.$$

$$= \frac{n!\,n^z}{z(z+1)\cdots(z+n)}$$



$$\Gamma(z) - \Gamma_n(z) = I_1 + I_2 + I_3$$

$$= \underbrace{\int_n^\infty e^{-t} t^{z-1}\,dt}_{\text{From } \Gamma(z)} + \int_0^{n/2} \left\{ e^{-t} - \left(1-\frac{t}{n}\right)^n \right\} t^{z-1}\,dt$$

$$+ \int_{n/2}^{n} \left\{ e^{-t} - \left(1-\frac{t}{n}\right)^n \right\} t^{z-1}\, dt$$

$$t \in [0, n) \implies 1 - \frac{t}{n} > 0 \Rightarrow n > t$$

& From integrals limit $t \geq 0$

$$\ln\left\{\left(1-\frac{t}{n}\right)^n\right\} = n \ln\left(1 - \frac{t}{n}\right)$$

$$= \cancel{n} \sum_{k=0}^{\infty} \frac{(-1)^k (-1)^{k+1}}{(k+1)} \frac{t^{k+1}}{n^{k+\cancel{1}}}$$

$$\left\{ \ln(1+x) = \sum_{k=} \right.$$

$$\frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k$$

$$= - \sum_{k=0}^{\infty} \frac{t^{k+1}}{n^k (k+1)}$$

$$\left\{ \Rightarrow \ln(1+x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{k+1}}{k+1} \right.$$

$$= - \left[\frac{t}{1} + \frac{t^2}{n(2)} + \frac{t^3}{n^2 3} + \frac{t^4}{n^3 4} + \cdots \right]$$

$$= -t - \underbrace{\left(\frac{t^2}{2n} + \frac{t^3}{3n^2} + \frac{t^4}{4n^3} + \cdots \right)}_{\text{postive}}$$

$$\therefore \ln\left\{\left(1-\frac{t}{n}\right)^n\right\} = -t - T$$

$$\Rightarrow \left(1-\frac{t}{n}\right)^n = e^{-t-T} = e^{-t}\cdot e^{-T} \le e^{-t}$$

$$\therefore |I_3| \le \int_{n/2}^{n} e^{-t}|t^{z-1}|\,dt = \int_{n/2}^{n} e^{-t}t^{\operatorname{Re} z-1}\,dt$$

But from befor $|t^{z-1}| = t^{\operatorname{Re} z-1}$

Thus for integral w/o n limits now $\int_{n/2}^{n} e^{-t}t^{\operatorname{Re} z-1}\,dt \longrightarrow 0$

why? See pgs 1 & 2 pg 33 diver

For I_2 $\quad \frac{t}{n} \le \frac{1}{2}$ $\quad$ As $\quad t\in(0,n/2)$

Then $\quad T = t\left[\frac{t}{2n} + \frac{1}{3}\left(\frac{t}{n}\right)^2 + \frac{1}{4}\left(\frac{t}{n}\right)^3 + \cdots\right]$

$$= t\sum_{k=1}^{\infty}\frac{1}{(k+1)}\left(\frac{t}{n}\right)^k$$

could do this but might not be what is wanted

$$\therefore\ T \le t\underbrace{\sum_{k=1}^{\infty}\frac{1}{(k+1)}\left(\frac{1}{2}\right)^k} =$$

$$\frac{t}{n} \le \frac{1}{2}$$

$$T = \frac{t^2}{n}\left[\frac{1}{2} + \frac{t}{3n} + \frac{t^2}{4n^2} + \cdots\right]$$

$$= \frac{t^2}{n}\sum_{k=1}^{\infty} \frac{1}{(k+1)}\left(\frac{t}{n}\right)^{k-1}$$

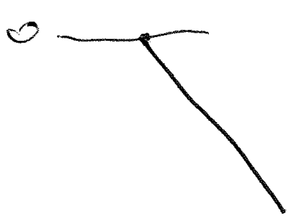
$$\therefore\ T \le \frac{t^2}{n}\underbrace{\sum_{k=1}^{\infty} \frac{1}{(k+1)}\left(\frac{1}{2}\right)^{k-1}}_{\text{converges}} = \frac{t^2}{n}\left[\frac{1}{2\cdot 1} + \frac{1}{3\cdot 2} + \frac{1}{4\cdot 2^2} + \cdots\right]$$

$$\therefore\ 0 \le e^{-t} - \left(1 - \frac{t}{n}\right)^n = e^{-t}\left(1 - e^{-T}\right)$$

But $1 - e^{-T} \le T$

$$f(T) = 1 - e^{-T} - T \qquad f(0) = 0$$

$$f'(T) = e^{-T} - 1 < 0 \quad T > 0$$



$$\therefore\ f(T) < 0 = f(0) \quad \forall\ T > 0$$

$$\Rightarrow\ 1 - e^{-T} < T \quad \forall\ T > 0$$

$$e^{-t}(1-e^{-T}) \le e^{-t}T \le e^{-t}\frac{ct^2}{n}$$

$$\therefore \quad |F_2| \le \frac{c}{n}\int_0^{n/2} |t^{z+1}|\, dt = \frac{c}{n}\int_0^{n/2} t^{Re(z)+1} e^{-t} dt$$

$Re(z) > 0$ ~~Re(z)+1 > 1~~ $\therefore$ ~~we have~~

convergence at ∞ is guaranteed w/ e^{-t}

check

At 0 $\quad Re\, z > 0 \Rightarrow Re(z)+1 > 1$

$\Rightarrow t^{Re(z)+1} \to 0 \quad t \to 0.$

$$\therefore \quad \lim_{n\to\infty} \frac{n!\, n^z}{z(z+1)(z+2)\cdots(z+n)} = \Gamma(z)$$

$Re\, z \in (-m, -m+1]$ Then $Re(z+m) \in (0, 1]$

$$\Gamma(z) = \frac{\Gamma(z+1)}{z} = \frac{\Gamma(z+2)}{z(z+1)} = \cdots = \frac{\Gamma(z+m)}{z(z+1)\cdots(z+m-1)}$$

$$= \frac{1}{z(z+1)\cdots(z+m-1)} \lim_{n\to\infty} n! \qquad \text{How get}$$

$$\lim_{n\to\infty} \frac{(n-m)!\,(n-m)^{z+m}}{(z+m)(z+m+1)\cdots(z+n)} \quad ?$$

go to 7c

$$|I_2| \leq \int_0^{n/2} \left| e^{-t} - \left(1 - \frac{t}{n}\right)^n \right| |t^{z-1}| \, dt$$

$$\leq \frac{c}{n} \int_0^{n/2} e^{-t} |t^2| |t^{z-1}| \, dt$$

$$= \frac{c}{n} \int_0^{n/2} e^{-t} |t^{z+1}| \, dt \leq \frac{c}{n} \int_0^{n/2} e^{-t} t^{\mathrm{Re}\, z + 1} \, dt$$

$$\leq \frac{c}{n} \int_0^{\infty} e^{-t} t^{\mathrm{Re}\, z + 1} \, dt \xrightarrow{n \to \infty} 0$$

$$\Gamma(z) = \frac{1}{z(z+1)(z+2)\cdots(z+m-1)} \Gamma(z+m)$$

$$\Gamma(z+m) = \lim_{n\to\infty} \frac{n!\, n^{z+m}}{(z+m)(z+m+1)(z+m+2)\cdots(z+m+n)}$$

$$= \lim_{n\to\infty} \frac{(n-m)!\,(n-m)^{z+m}}{(z+m)(z+m+1)\cdots(z+m+n-m)}$$

Shift every n index down by m. This makes no diff in the limit

$$\therefore \Gamma(z) = \lim_{n\to\infty} \frac{(n-m)!\,(n-m)^{z+m}}{z(z+1)\cdots(z+n)}$$

pg 33 Olver

For $I_3 = \int_{n/2}^{n} \{e^{-t} - (1-\frac{t}{n})^n\} t^{z-1}\, dt$

$$|I_3| \le \int_{n/2}^{n} \left|e^{-t} - (1-\tfrac{t}{n})^n\right| t^{\mathrm{Re}(z)-1}\, dt$$

see pg 31 eq 1.01

$$t^{z-1} := \cancel{e^{t\log(z-1)}} \qquad e^{(z-1)\log t}$$

$$= \cancel{e^{t(\log|z-1| + i\,\mathrm{ph}(z-1))}}$$

$$\therefore \cancel{|t^{z-1}| = |e^{t\log|z-1|}|} = e^{(\mathrm{Re}(z)-1)\log t + i\,\mathrm{Im}(z)\log t}$$

$$\therefore |t^{z-1}| = e^{(\mathrm{Re}(z)-1)\log t} = t^{\mathrm{Re}(z)-1}$$

$\therefore (1-\frac{t}{n})^n \nearrow e^{-t}$ from below

Thus $\left|e^{-t} - (1-\frac{t}{n})^n\right| \le |e^{-t}| + \left|(1-\frac{t}{n})^n\right|$

But $|(1-\frac{t}{n})^n| < e^{-t}$

$$\therefore \quad |I_3| \leq 2\int_{n/2}^{n} e^{-t}\, t^{Re(z)-1}\, dt$$

Both e^{-t} & $t^{Re(z)-1} > 0$

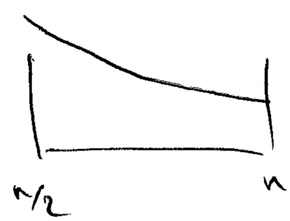
But for $Re(z) > 0$ $f(t) = t^{Re(z)-1}$ ~~is an~~

has $f' = (Re(z)-1)\, t^{Re(z)-2}$

Thus for $Re(z) < 1$ $t^{Re(z)-1}$ is an decreasing fn of t

so is e^{-t}

$$\Rightarrow \quad |I_3| \leq 2e^{-n/2}\left(\frac{n}{2}\right)^{Re(z)-1} \cdot \left(\frac{n}{2}\right)$$



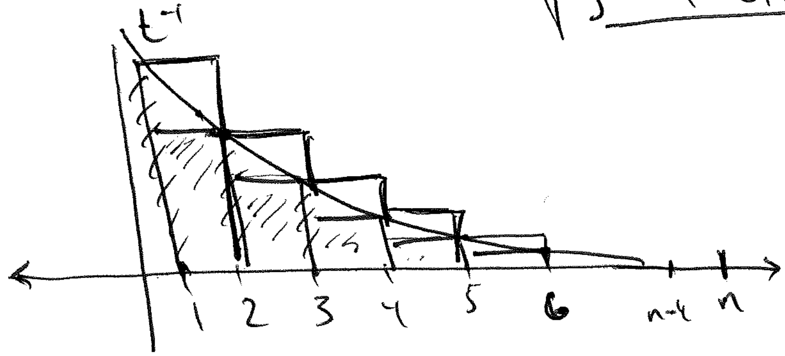
$$\longrightarrow 0 \quad n \to \infty$$

But if $Re(z) > 1$ $t^{Re(z)-1}$ is an increasing fn of t

$$\therefore \quad |I_3| \leq 2e^{-n/2}(n)^{Re(z)-1}\, \frac{n}{2} \longrightarrow 0 \quad n \to \infty$$

$$\therefore \quad |I_3| \xrightarrow{n \to \infty} 0$$

PS 34 clear



Then

$$\underbrace{\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}}_{\text{use } \cancel{\text{left}} \text{ right side of}} \le \underbrace{\int_1^n \frac{dt}{n}}_{\text{full integral}} \le \underbrace{1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n-1}}_{\text{Use left side of each box}}$$

use ~~left~~ right side of the graph, we are obviously under approximating;

full integral $= \ln n$;

Use left side of each box we are obviously over compensating

$$\therefore \underbrace{\frac{1}{2}+\dots+\frac{1}{n}-\ln n}_{u_n-1} \le 0 \le \underbrace{1+\frac{1}{2}+\dots+\frac{1}{n-1}-\ln n}_{u_n-\frac{1}{n}}$$

$$u_n - 1 \le 0 \le u_n - \frac{1}{n}$$

$$\Rightarrow -1 \le -u_n \le -\frac{1}{n}$$

$\frac{1}{n} \le u_n \le 1$ so u_n is bounded by 0 & 1.

$$u_{n+1} - u_n = \frac{1}{n+1} - \ln(n+1) + \ln n$$

$$= \frac{1}{n+1} + \ln\left(\frac{n}{n+1}\right) = \frac{1}{n+1} + \ln\left(1-\frac{1}{n+1}\right)$$

But $\ln\left(1-\frac{1}{n+1}\right) = -\frac{1}{n+1} + \frac{1}{(n+1)^2} C_1$

$$= \sum_{k=0}^{\infty} \frac{1}{(k+1)} \frac{1}{(n+1)^{k+1}}$$

Finish

$\therefore \cancel{\frac{1}{n+1}}$

$$\ln(1-x) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(k+1)} x^{k+1}$$

$$\ln(1-x) = \ln(1) + \frac{d}{dx}\ln(1-x)\Big|_{x=0}(x-0) + \frac{d^2}{dx^2}\ln(1-x)\Big|_{x=\xi}\frac{(x-0)^2}{2}$$

$$= 0 + \frac{(-1)}{1-x}\Big|_{x=0} x + \frac{(-1)(-1)(-1)}{(1-x)^2}\Big|_{x=\xi}\frac{x^2}{2}$$

$$= -x - \frac{1}{2(1-\xi)^2}x^2 = -x - C_1(\xi)x^2 \quad \text{w/ } C(\xi) > 0$$

Then

$$\frac{1}{n+1} + \ln\left(1-\frac{1}{n+1}\right) = \cancel{\frac{1}{n+1}} + \left[-\cancel{\frac{1}{n+1}} - \frac{C_1(\xi)}{(n+1)^2}\right] = -\frac{C_1(\xi)}{(n+1)^2}$$

$$+ \frac{-C'(\xi)}{(n+1)^2} < 0$$

$$\Gamma_n(z) = \frac{n!\, n^z}{z(z+1)\cdots(z+n)}$$

$$\therefore \quad \frac{1}{\Gamma_n(z)} = \frac{z(z+1)\cdots(z+n)}{n!\, n^z}$$

$$= z\,\frac{(z+1)}{1}\,\frac{(z+2)}{2}\,\frac{(z+3)}{3}\cdots\frac{(z+n)}{n}\; n^{-z}$$

$$= z\left(1+\frac{z}{1}\right)\left(1+\frac{z}{2}\right)\left(1+\frac{z}{3}\right)\cdots\left(1+\frac{z}{n}\right) n^{-z}$$

$$= z\, e^{-z/1}\left(1+\frac{z}{1}\right) e^{-z/2}\left(1+\frac{z}{2}\right) e^{-z/3}\left(1+\frac{z}{3}\right)\cdots e^{-z/n}\left(1+z/n\right) n^{-z}$$

$$\cdot\; e^{z/1} e^{z/2} e^{z/3} \cdots e^{z/n}$$

$$= z \prod_{s=1}^{n}\left\{\left(1+\frac{z}{s}\right)e^{-z/s}\right\} \underbrace{n^{-z}}_{e^{-z\log n}} \cdot e^{z\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)}$$

$$= \cancel{z\exp}$$

$$= z\exp\left\{z\left[\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)-\log n\right]\right\} \prod_{s=1}^{n}\left\{\left(1+\frac{z}{s}\right)e^{-z/s}\right\}$$

$\lim_{n\to\infty}$

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{s=1}^{\infty} \left\{ \left(1+\frac{z}{s}\right) e^{-z/s} \right\}$$

$$\frac{1}{z\Gamma(z)} e^{-\gamma z} = \prod_{s=1}^{\infty} \left\{ (1+z/s) e^{-z/s} \right\}$$

$$\log \longrightarrow \qquad \sum_{s=1}^{\infty} \log\left((1+z/s) e^{-z/s}\right)$$

$$\log\left((1+z/s)e^{-z/s}\right) = -\frac{z}{s} + \underbrace{\log\left(1+\frac{z}{s}\right)}_{\frac{z}{s} - \frac{1}{2}\left(\frac{z}{s}\right)^2 + O\left(z^3/s^3\right)}$$

$\therefore$

$$\log\left((1+z/s)e^{-z/s}\right) = \underbrace{-\frac{1}{2}\left(\frac{z}{s}\right)^2}_{\text{conv. By p-test.}} + O\left((z/s)^3\right)$$

$$\left|\left(\frac{z}{s}\right)^2\right| \leq \frac{M}{s^2} \qquad \text{conv Abs}$$

if $|z|^2 \leq M$ A compact domain excluding $0, -1, -2, \ldots$

5

$$\frac{1}{\Gamma(z)\Gamma(1-z)} = \lim_{n\to\infty}\left(\frac{z(z+1)(z+2)\cdots(z+n)}{n!\,\cancel{n^z}}\cdot\frac{(1-z)(2-z)\cdots(n+1-z)}{n!\,n^{\cancel{1-z}}}\right)$$

$$= z\lim_{n\to\infty}\left(\frac{(z+1)(z+2)\cdots(z+n)}{1\cdot 2\cdot\ \cdots\ n}\ \frac{(1-z)}{1}\frac{(2-z)}{2}\cdots\left(\frac{n-z}{n}\right)\frac{(n+1-z)}{\underset{\uparrow\ \text{extra } n \text{ from Above.}}{n}}\right)$$

$$= z\lim_{n\to\infty}\left((1+z)\left(1+\frac{z}{2}\right)\left(1+\frac{z}{3}\right)\cdots\left(1+\frac{z}{n}\right)(1-z)\left(1-\frac{z}{2}\right)\cdots\left(1-\frac{z}{n}\right)\left(1+\frac{1}{n}-\frac{z}{n}\right)\right)$$

$$= z\lim_{n\to\infty}(1-z^2)\left(1-\frac{z^2}{2^2}\right)\left(1-\frac{z^2}{3^2}\right)\cdots\left(1-\frac{z^2}{n^2}\right)\underbrace{\left(1+\frac{1}{n}-\frac{z}{n}\right)}_{\text{goes to } 1}$$

$$\lim_{n\to\infty} z\prod_{s=1}^{n}\left(1-\frac{z^2}{s^2}\right)$$

$$= z\prod_{s=1}^{\infty}\left(1-\frac{z^2}{s^2}\right)$$

$z = 1/2$

$$\frac{1}{\Gamma(1/2)\Gamma(1/2)} = \frac{1}{\pi} \Rightarrow \Gamma(1/2) = \pm\sqrt{\pi} \quad \text{take + by 1.05}$$

~~couldn't get...~~

$$\frac{2^{2z}\Gamma(z)\Gamma(z+1/2)}{\Gamma(2z)} = \lim_{n\to\infty}\left\{\cancel{2}^{2z}\frac{n!\,n^z}{z(z+1)(z+2)\cdots(z+n)}\ \frac{n!\,n^{z+1/2}}{(z+1/2)(z+3/2)\cdots(z+n+1/2)}\cdot\frac{(2z)(2z+1)(2z+2)\cdots(2z+2n)}{(2n)!\,(2n)^{2z}}\right.$$

Note: for $\Gamma(2z)$ we perform Euler limit as

$$\lim_{n\to\infty} \frac{(2z)(2z+1)\cdots(2z+2n)}{(2n)!\,(2n)^{2z}}$$

$$= \lim_{n\to\infty} \frac{(n!)^2\, n^{\cancel{2z}+1/2}}{z(z+1)\cdots(z+n)(z+1/2)(z+3/2)\cdots(z+n+1/2)} \frac{(2z)(2z+1)(2z+2)\cdots(2z+2n-1)(2z+2n)}{(2n)!\,\cancel{n^{2z}}}$$

$$= \lim_{n\to\infty} \frac{(n!)^2}{(2n)!} \frac{n^{1/2}\, 2^{2n+1}\cancel{(z)}(\cancel{z+1/2})(\cancel{z+1})\cdots(\cancel{z+n-1/2})(\cancel{z+n})}{\cancel{z}(\cancel{z+1})\cdots(\cancel{z+n})(\cancel{z+1/2})(\cancel{z+3/2})\cdots\underset{\cancel{(z+n-1/2)}}{,}\underbrace{(z+n+1/2)}_{\text{still left?}}}$$

Now:

$$(2z)(2z+1)(2z+2)\cdots(2z+2n)$$

Becomes

$$(2z)(2z+1)(2z+2)\cdots\quad(2z+n)(2z+n+1)\cdots(2z+2n-1)(2z+2n)$$

$$= 2^{2n+1}\; z\left(z+1/2\right)(z+1)\cdots(z+n/2)\left(z+\frac{n+1}{2}\right)$$

Think it should be $\Gamma(z-\frac{1}{2})$ instead.

Try:

$$\frac{2^{2z}\Gamma(z)\Gamma(z-\frac{1}{2})}{\Gamma(2z)} = \lim_{n\to\infty}\left\{ \cancel{2^{2z}} \frac{n!\,n^z}{z(z+1)(z+2)\cdots(z+n)} \cdot \frac{n!\,n^{z-1/2}}{\cancel{z(z+1)(z+2)\cdots+z}\;(z-\frac{1}{2})(z+\frac{1}{2})(z+\frac{3}{2})\cdots(z-\frac{1}{2}+n)} \right.$$

$$\left. \cdot \underbrace{\frac{(2z)(2z+1)\cdots(2z+2n)}{(2n)!\,(\cancel{2}n)^{2z}}} \right\}$$

Note: same trick.

$$= \lim_{n\to\infty}\left\{ \frac{(n!)^2\, n^{2z-1/2}\; 2^{2n+1}\cancel{z}(\cancel{z+\frac{1}{2}})(\cancel{z+1})\cdots(\cancel{z+n})}{\cancel{z(z+1)(z+2)}\cdots(\cancel{z+n})\underbrace{(z-\frac{1}{2})}_{\text{still left.}}(\cancel{z+\frac{1}{2}})(\cancel{z+\frac{3}{2}})\cdots(\cancel{z-\frac{1}{2}+n})\;(2n)!} \right\}$$

$z=\frac{1}{2}$

$$\Rightarrow \frac{2^1\sqrt{\pi}\,\cancel{\Gamma(1)}}{\cancel{\Gamma(1)}} = 2\sqrt{\pi} = \frac{2^{2z}\Gamma(z)\Gamma(z+\frac{1}{2})}{\Gamma(2z)}$$

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma(z+\tfrac{1}{2})$$

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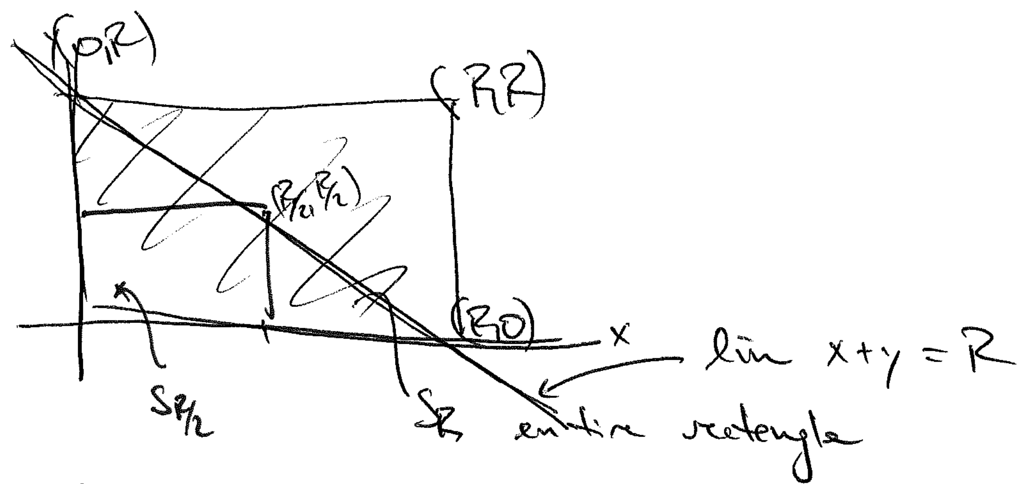


$$\iint_{S_{R/2}} \le \iint_{T_R} \le \iint_{S_R}$$ as integrand is positive over the entire range.

$$\begin{array}{ccc} \iint_{S_{R/2}} & & \iint_{S_R} \\ \big\downarrow & & \big\downarrow \; R \to \infty \\ \Gamma(p)\Gamma(q) & & \Gamma(p)\Gamma(q) \end{array}$$

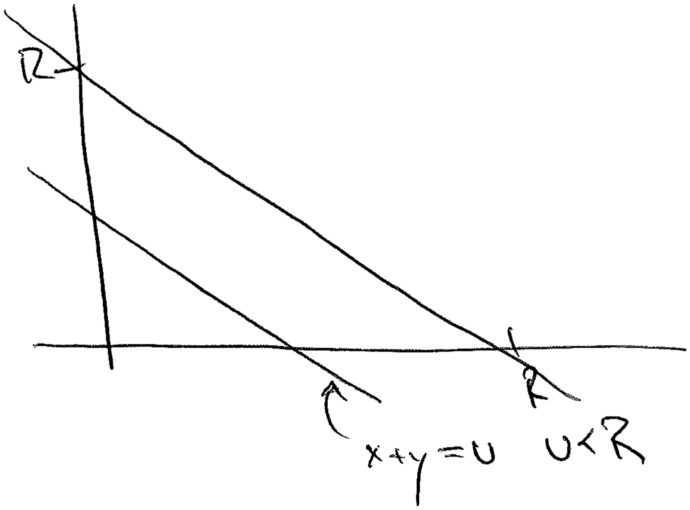
$$\Rightarrow \iint_{T_R} \to \Gamma(p)\Gamma(q)$$

let $x+y = u \qquad y = uv$

$\Rightarrow x = u - y$

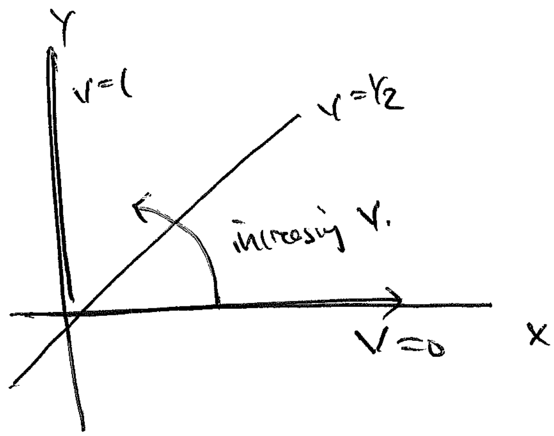
$x = u - uv = u(1-v)$

Then $$\iint_{T_R} dx\,dy = \iint_{``T_R"(u,v)} \frac{\partial(x,y)}{\partial(u,v)}\, du\,dv$$



$x+y=u$ w/ $u<R$

$\cancel{\frac{y}{x}} =$ $\cancel{1+\frac{y}{x} = \frac{u}{x}}$ from Above



$$\frac{y}{x} = \frac{uv}{u(1-v)} = \frac{v}{1-v}$$

Rays through the origin

$v=0 \Rightarrow y=0$

$v=1 \Rightarrow y=\infty\cdot x$

$v=\frac{1}{2} \Rightarrow y=x$

$$\frac{v}{1-v} = 1$$

$$v = 1-v$$

$$2v = 1 \Rightarrow v = 1/2$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 1-v & v \\ -u & u \end{vmatrix} = u - vu + vu = u.$$

$$\iint_{T_R} e^{-x-y} x^{q-1} y^{p-1}\,dx\,dy =$$

$$= \int_{u=0}^{R}\int_{v=0}^{1} e^{-u} u^{q-1}(1-v)^{q-1} u^{p-1} v^{p-1} u\,du\,dv$$

$\therefore$

$$\Gamma(p)\Gamma(q) = \underbrace{\lim_{R\to\infty}\left[\int_{u=0}^{R} e^{-u} u^{q+p-1}\,du\right]}_{} \int_{v=0}^{1} v^{p-1}(1-v)^{q-1}\,dv$$

$$\Gamma(q+p)\int_{0}^{1} v^{p-1}(1-v)^{q-1}\,dv$$

Normally: $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\,dt \qquad Re(z) > 0$

$$I(z) = \int_{-\infty}^{(0+)} e^{t} t^{-z}\,dt$$

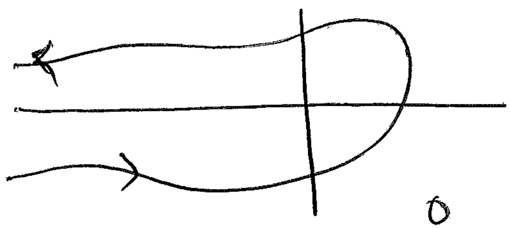
If $Re(z) < 1$

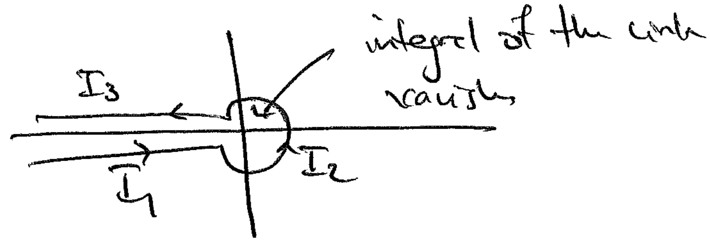
$-Re(z) > -1$

$1 - Re(z) > 0$

Th $\quad I(z) \approx \dfrac{t^{-z+1}}{-z+1} \longrightarrow 0$

for small circle $\quad t \to 0$





$$I(z) = \int_{I_1} + \cancel{\int_{I_2}}^{\,0} + \int_{I_3}$$

on I_1 $\quad z = \tau e^{-i\pi} \quad dz = d\tau e^{-i\pi}$

$$\int_{I_1} = \int_\infty^0 e^{\tau e^{-i\pi}} \tau^{-z} e^{i\pi z} d\tau e^{-i\pi} = -\int_\infty^0 e^{-\tau} \tau^{-z} e^{i\pi z} d\tau$$

on I_3 $\quad z = \tau e^{i\pi} \quad dz = d\tau e^{i\pi}$

$$\int_{I_3} = \int_0^\infty e^{\tau e^{i\pi}} \tau^{-z} e^{-i\pi z} d\tau e^{-i\pi} = -\int_0^\infty e^{-\tau} \tau^{-z} d\tau$$

$$\therefore I(z) = \underbrace{\left(e^{+i\pi z} - e^{-i\pi z}\right)}_{2i\sin(\pi z)} \underbrace{\int_0^\infty e^{-\tau} \tau^{-z} d\tau}_{\Gamma(1-z)}$$

$-z = x - 1$

$x = 1 - z$

But $\quad \sin(\pi z)\Gamma(1-z) = \dfrac{\pi}{\Gamma(z)}$

$$\therefore \quad I(z) = \frac{2i\pi}{\Gamma(z)} = \frac{2\pi i}{\Gamma(z)}$$

$$\therefore \quad \int_{-\infty}^{(0+)} e^{t} t^{-z}\, dt = \frac{1}{2\pi i}\frac{1}{\Gamma(z)}.$$

$$f(z) = \frac{\Gamma'(z)}{\Gamma(z)} = \underbrace{\frac{1}{\Gamma(z)}}_{\text{entire}} \Gamma'(z)$$

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$$E_1(z) = \int_z^\infty \frac{e^{-t}}{t}\,dt$$

let ~~$t = \eta - z$~~ $\quad$ ~~$t - z = \eta \Rightarrow t = \eta + z\eta$~~

~~$dt = d\eta$~~ $\quad$ ~~$dt = d\eta$~~

~~$E_1(z) = \int_0^\infty e^{-}$~~ $\quad$ let ~~$t - \eta z = \eta$~~

~~$t = \eta(z+1)$~~

let $\eta = t - z \Rightarrow t = \eta + z$

$d\eta = dt$

$$E_1(z) = \int_0^\infty \frac{e^{-\eta - z}}{\eta + z}\,d\eta = e^{-z}\int_0^\infty \frac{e^{-\eta}}{\eta + z}\,d\eta$$

let $\eta = tz$ w/ $z > 0$ (other values of z will be obtained by analytic cont)

$d\eta = dt \cdot z$

$$E_1(z) = e^{-z}\int_0^\infty \frac{e^{-tz}}{\cancel{z}(1+t)}\cancel{z}\,dt = e^{-z}\int_0^\infty \frac{e^{-tz}}{1+t}\,dt$$

$$\mathrm{Ein}(z) = \int_0^z \frac{1-e^{-t}}{t}\,dt$$

Now

$$1 - e^{-t} = 1 - \{1 - t + O(t^2)\}$$
$$= t + O(t^2)$$

How get Ein(z) is entire?

$$\frac{1-e^{-t}}{t} = \frac{1}{t}\left(1-\sum_{s=0}^{\infty} \frac{(-1)^s t^s}{s!}\right) = \frac{1}{t}\left(-\sum_{s=1}^{\infty} \frac{(-1)^s t^s}{s!}\right)$$

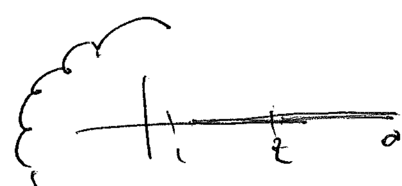
$$= -\sum_{s=1}^{\infty} \frac{(-1)^s t^{s-1}}{s!}$$ then 'integrate

$$\int_0^z \;''\; dt = -\sum_{s=1}^{\infty} \frac{(-1)^s}{s!}\frac{t^s}{s}\Bigg|_0^z$$

$$= \sum_{s=1}^{\infty} \frac{(-1)^{s-1}}{s}\frac{z^s}{s!}$$ eq 3.04 ✓.

$$\mathrm{Ein}(z) = \int_0^z \frac{1-e^{-t}}{t}\,dt = \int_0^1 \frac{1-e^{-t}}{t}\,dt + \int_1^z \frac{1-e^{-t}}{t}\,dt$$

$$= \int_0^1 \frac{1-e^{-t}}{t}\,dt + \underbrace{\ln t\Big|_1^z}_{\ln z} - \underbrace{\int_1^z \frac{e^{-t}}{t}\,dt}_{\left[\int_1^{\infty} \frac{e^{-t}}{t}\,dt - \int_z^{\infty} \frac{e^{-t}}{t}\,dt\right]}$$



$$Ein(z) = \int_0^1 \frac{1-e^{-t}}{t}\,dt + \ln z - \int_1^\infty \frac{e^{-t}}{t}\,dt + \int_z^\infty \frac{e^{-t}}{t}\,dt$$

By Ex 2b

$$\int_0^1 \frac{1-e^{-t}}{t}\,dt - \int_1^\infty \frac{e^{-t}}{t}\,dt = \gamma$$

$\therefore$

$$Ein(z) = E_1(z) + \ln z + \gamma$$

$$E_1(z) = -\ln z - \gamma + \sum_{s=1}^{\infty} \frac{(-1)^{s-1}}{s}\frac{z^s}{s!}$$

$$\cancel{x \in \mathbb{R} \qquad \int_x^\infty \frac{e^{-t}}{t}\,dt = \int_{-x}}$$

$$Ei(-x) = \int_{-\infty}^{-x} \frac{e^t}{t}\,dt \qquad v = -t, \quad dv = -dt$$

$$= \int_\infty^x \frac{e^{-v}}{-v}(-dv) = -\int_x^\infty \frac{e^{-v}}{v}\,dv = -E_1(x)$$

$$li(x) = \int_0^x \frac{dt}{\ln t}$$

let $v = \ln t \to t = e^v$

$dv = \frac{1}{t} dt \Rightarrow dt = t\,dv$

$$\therefore li(x) = \int_{-\infty}^{\ln x} \frac{e^v\,dv}{v} = Ei(\ln x) \qquad \text{eq } 3.10$$

$$\int \frac{e^{it}}{t} dt = \int_\pi^0 \exp\{ire^{i\theta}\} \frac{\cancel{r}i\cancel{e^{i\theta}}\,d\theta}{\cancel{r}\cancel{e^{i\theta}}} = i\int_\pi^0 \exp\{ire^{i\theta}\}\,d\theta$$

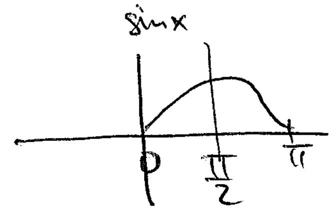
on $t = re^{i\theta}$

$dt = ire^{i\theta}\,d\theta$

$$\text{Now} = -i\int_0^\pi \exp\{ire^{i\theta}\}\,d\theta \xrightarrow{r \to 0} -i\int_0^\pi d\theta = -i\pi$$

$$\left|\int_{|t|=R} \frac{e^{it}}{t} dt\right| = \left|\int_0^\pi \exp\{iRe^{i\theta}\} \frac{\cancel{R}i\cancel{e^{i\theta}}\,d\theta}{\cancel{R}\cancel{e^{i\theta}}}\right| = \left|i\int_0^\pi \exp\{iRe^{i\theta}\}\,d\theta\right|$$

$$\cancel{\leq} \int_0^{\cancel{\pi}} = \left|i\int_0^\pi \exp\{R(i\cos\theta - \sin\theta)\}\,d\theta\right|$$



$$\le 1 \int_0^{\pi} 1\,|e^{-R\sin\theta}|\,|d\theta|$$

$$= 2\int_0^{\pi/2} e^{-R\sin\theta}\,d\theta \le 2\int_0^{\pi/2} e^{-R2\theta/\pi}\,d\theta$$

But Jordan's ineq: $\sin\theta \ge \frac{2\theta}{\pi}$ $\quad 0\le\theta\le\pi/2$

$$-\sin\theta \le -\frac{2\theta}{\pi}$$

$$= \cancel{2}\,\frac{e^{-R2\theta/\pi}}{-\frac{R\cancel{2}}{\pi}}\Bigg|_0^{\pi/2} = -\frac{\pi}{R}\left(e^{-R}-1\right) = \frac{\pi}{R}\left(1-e^{-R}\right)$$

Thus $$\int_C \frac{e^{it}}{t}\,dt = 0$$

$$\|$$

$$\int_{-R}^{-r} \frac{e^{it}}{t}\,dt + \underbrace{\int_{\curvearrowright}}_{= -i\pi} + \int_r^R + \cancelto{0}{\int_{\frown}} = 0$$

Then $\quad v = -t$

$\quad\quad\quad dv = -dt$

$$= \int_R^r \frac{e^{-iv}}{-v}(-dv) + \int_r^R \frac{e^{it}}{t}\,dt = i\pi$$

$$= + - \int_r^R \frac{e^{-iv}}{v}\,dv + \int_r^R \frac{e^{it}}{t}\,dt = i\pi$$

$$= \int_r^R \frac{2i\sin(t)\,dt}{t} = i\pi \qquad \text{As } r \to 0,\ R \to \infty$$

$$\Rightarrow \int_0^\infty \frac{\sin(t)}{t}\,dt = \frac{\pi}{2}.$$

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$$Si(z) = \int_0^z \frac{\sin t}{t}\,dt \qquad si(z) = -\int_z^\infty \frac{\sin t}{t}\,dt$$

Thus $\quad Si(z) - si(z) = \int_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2}$

$\rightarrow \quad Si(z) = si(z) + \frac{\pi}{2}$

---

eq 3.11 $\quad Si(z) = \int_0^z \frac{\sin t}{t}\,dt \qquad$ let $t = iv$

$dt = i\,dv$

$$= \int_0^{z/i} \frac{\sin(iv)}{\cancel{i}v}\,\cancel{i}\,dv = \int_0^{-iz} \frac{\sin(iv)}{v}\,dv = \frac{1}{2i}\int_0^{-iz} \frac{\left(e^{i(iv)} - e^{-i(iv)}\right)}{v}\,dv$$

$$= \frac{1}{2i}\int_0^{-iz} \frac{e^{-v}}{v} - \frac{1}{2i}\int_0^{-iz} \frac{e^{v}}{v}\,dv$$

$$\Rightarrow 2i\,Si(z) = \int_0^{-iz} \frac{1-e^{v}}{v}\,dv - \underbrace{\int_0^{-iz} \frac{1-e^{-v}}{v}\,dv}_{-Ein(-iz)}$$

let $t = -v$

& $dt = -dv$

$$2i\,Si(z) = \underbrace{\int_0^{iz} \frac{1-e^{-t}}{-t}\,(-dt)} - Ein(-iz)$$

$$= \quad Ein(iz)$$

$$\therefore \quad 2i\,Si(z) = Ein(iz) - Ein(-iz)$$

$\|$ By 3.14 $\qquad$ By ~~3.06~~ 3.05 $\qquad$ $\cancel{E_1(x) = -E}$ $\quad Ein(z) = E_1(z) + \ln z + \gamma$

$$2i\left(\frac{\pi}{2} + si(z)\right) = E_1(iz) + \ln(iz) + \gamma$$

$$- \left(E_1(-iz) + \ln(-iz) + \gamma\right)$$

$$\Rightarrow \quad \cancel{\pi i} + 2i\,si(z) = E_1(iz) - E_1(-iz) + \ln(iz) - \ln(-iz)$$

$$?$$

$$= E_1(iz) - E_1(-iz) + \underbrace{\ln\left(\frac{iz}{-iz}\right)}$$

$$\underbrace{\ln(-1)}$$

$$= \ln|-1| + i\,\mathrm{Arg}(-1)$$

$$= 0 + \cancel{i\pi}$$

$$\Rightarrow \quad 2i\,si(z) = E_1(iz) - E_1(-iz)$$

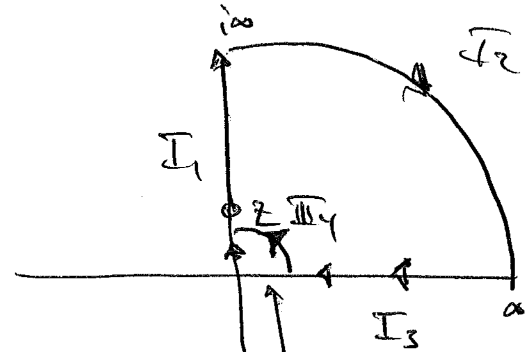
From 3.01

$$E_1(-iz) = \int_{-iz}^{\infty} \frac{e^{-t}}{t}\,dt$$

$$-t = iv \Rightarrow t = -iv \qquad \infty = -iv$$

$$-dt = i\,dv \qquad v = \infty i$$

$$= \int_{z}^{\infty i} \frac{e^{iv}}{\cancel{-}iv}(\cancel{-}i\,dv) = \int_{z}^{i\infty} \frac{e^{it}}{t}\,dt$$



~~z∞~~ ~~goes to~~ int goes to $-i\frac{\pi}{2}$

$z \in$ imag #

$$\int_{C} = 0$$

0 As in Lemma 3.1 w/ Jordan's ineq.

$$= \underbrace{\int_{z}^{i\infty}}_{I_1} + \underbrace{\cancel{\int}}_{I_2} + \underbrace{\int_{\infty}}_{I_3} + \int_{I_4} = 0$$

$$\Rightarrow \int_{z}^{i\infty} = \int_{z}^{\infty} + \int_{\infty}^{z} = -\int_{z}^{\infty}$$

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$$Ci(z) = -\int_z^\infty \frac{\cos t}{t}\,dt = -\int_z^\infty \frac{e^{it}+e^{-it}}{2t}\,dt$$

$$\Rightarrow 2Ci(z) = -\int_z^\infty \frac{e^{it}}{t}\,dt - \int_z^\infty \frac{e^{-it}}{t}\,dt$$

$$-E_1(-iz) \quad - \int_z^\infty \frac{e^{-it}}{t}\,dt$$

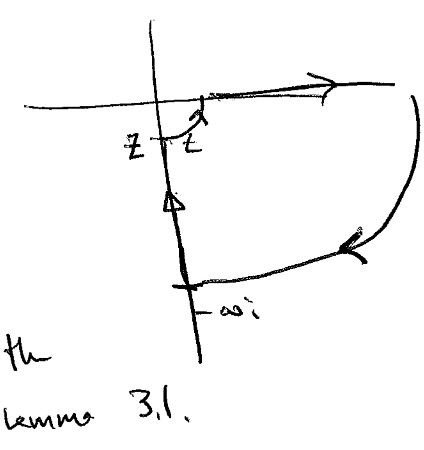
from line Above

Also from: 3.01

$$E_1(iz) = \int_{iz}^\infty \frac{e^{-t}}{t}\,dt \qquad +t = iv \Rightarrow t = +iv$$

$$dt = +i\,dv$$

$$= \int_{+z}^{-\infty i} \frac{e^{-iv}}{\cancel{i}v}\,\cancel{i}\,dv$$



$$= \int_z^{+\infty} \frac{e^{-iv}}{v}\,dv$$ By deforming the contour as in lemma 3.1.

$\therefore \quad 2\mathrm{Ci}(z) = -E_1(-iz) - E_1(iz)$

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$$\mathrm{Ein}(z) = \int_0^z \frac{1-e^{-t}}{t}\,dt$$

let $t = \pm iv$
$dt = \pm i\,dv$

$$= \int_0^{\frac{z}{\pm i} = \mp zi} \frac{1-e^{\mp iv}}{\cancel{\pm i}v}\,\cancel{\pm i}\,dv = \int_0^{\mp zi} \frac{1-e^{\mp iv}}{v}\,dv$$

$\therefore$

$$2\,\mathrm{Cin}(z) = 2\int_0^z \frac{1-(\cos t)}{t}\,dt = 2\int_0^z \frac{1-\frac{e^{it}}{2}-\frac{e^{-it}}{2}}{t}\,dt$$

$$= \underbrace{\int_0^z \frac{1-e^{it}}{t}\,dt}_{\mathrm{Ein}(iz)} + \int_0^z \frac{1-e^{-it}}{t}\,dt$$

$$2(\mathrm{Ci}(z) + \mathrm{Cin}(z)) = -E_1(iz) + \mathrm{Ein}(iz) - E_1(-iz) + \mathrm{Ein}(-iz)$$

But from 305

$$\mathrm{Ein}(z) - E_1(z) = \ln z + \gamma$$

$\therefore$

$$2(\mathrm{Ci}(z) + \mathrm{Cin}(z)) = \ln(iz) + \gamma + \ln(-iz) + \gamma$$

$$\overset{?}{=} 2\gamma + \cancel{\ln(i) + \ln(z)}$$

$$\ln(iz \cdot (-iz))$$

$$\ln(z^2)$$

$$2\ln z$$

$$\Rightarrow \mathrm{Ci}(z) + \mathrm{Cin}(z) = \gamma + \ln z$$

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$$\operatorname{erf} z + \operatorname{erfc} z = \frac{2}{\sqrt{\pi}} \underbrace{\int_0^\infty e^{-t^2} dt}_{\frac{\sqrt{\pi}}{2}} = 1$$

$$\operatorname{erf} z = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt$$

$$= \frac{2}{\sqrt{\pi}} \int_0^z \sum_{k=0}^\infty \frac{(-1)^k t^{2k}}{k!} dt = \frac{2}{\sqrt{\pi}} \sum_{k=0}^\infty \frac{(-1)^k}{k!} \frac{t^{2k+1}}{(2k+1)}$$

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$$F(z) = e^{-z^2} \int_0^z e^{t^2} dt = e^{-z^2} \int_0^{zi} e^{-v^2} (-i\, dv)$$

let $t^2 = -v^2$ or $\boxed{t = -iv}$

$\Rightarrow$ $dt = -i\, dv$

$$= \frac{e^{-z^2}}{i} \underbrace{\int_0^{zi} e^{-v^2} dv}_{\frac{\sqrt{\pi}}{2} \operatorname{erf}(zi)}$$

$$\therefore \quad F(z) = \frac{e^{-z^2}}{2i} \sqrt{\pi} \operatorname{erf}(iz)$$

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$C(z) + iS(z)$

$$= \int_0^z \cos\left(\frac{\pi}{2}t^2\right) dt + \int_0^z i\sin\left(\frac{\pi}{2}t^2\right) dt$$

$$= \int_0^z e^{i\frac{\pi}{2}t^2}\, dt$$

let $i\frac{\pi}{2}t^2 = -v^2$

$\Rightarrow t^2 = -\frac{2}{\pi} i v^2$

$\sqrt{i} = ?$

$\cancel{i = e^{\frac{\pi}{2}i + 2\pi n}}$

$\cancel{i^{1/2} = e^{\frac{\pi}{4}i + \pi n}} = \pm e^{\frac{\pi}{4}i} = \cancel{\pm\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) = \pm\frac{1}{\sqrt{2}}(1+i)}$

~~Take ± i~~

$\cancel{t = \pm\frac{1}{\sqrt{2}}(1+i)\sqrt{2}\sqrt{\pi}}$

$\sqrt{-i} = ?$

$-i = e^{\left(-\frac{\pi}{2} + 2\pi n\right)i}$

$$(-i)^{1/2} = e^{-\frac{\pi}{2}i + n\pi i} = \begin{cases} e^{-\frac{\pi}{4}i} & n=0 \\ -e^{-\frac{\pi}{4}i} & n=1 \end{cases}$$

$e^{-\frac{\pi}{4}i} = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$

$e^{\frac{3\pi}{4}i} = -\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$

Thus $$t^2 = -i\,\frac{2}{\pi}\,v^2$$

let $$t = +\frac{1}{\sqrt{\cancel{2}}}(1-i)\frac{\sqrt{\cancel{2}}}{\sqrt{\pi}}\,v = \frac{(1-i)v}{\sqrt{\pi}}$$

$$dt = \frac{(1-i)\,dv}{\sqrt{\pi}}$$

$$\therefore \int_0^{\frac{\sqrt{\pi}}{(1-i)}z} e^{-v^2}\left(\frac{1-i}{\sqrt{\pi}}\right)dv = \frac{1-i}{\sqrt{\pi}}\int_0^{\frac{\sqrt{\pi}}{1^2+1^2}(1+i)z} e^{-v^2}\,dv = \frac{1-i}{\sqrt{\pi}}\int_0^{\frac{\sqrt{\pi}}{2}(1+i)z} e^{-v^2}\,dv$$

$$= \frac{1-i}{\sqrt{\cancel{\pi}}}\cdot\frac{\sqrt{\cancel{\pi}}}{2}\,\mathrm{erf}\left(\frac{\sqrt{\pi}}{2}(1+i)z\right) = \frac{1-i}{2}\,\mathrm{erf}\left(\frac{\sqrt{\pi}}{2}(1+i)z\right)$$

If I choose minus ...

$$t = -\frac{1}{\sqrt{\cancel{2}}}(1-i)\frac{\sqrt{\cancel{2}}}{\sqrt{\pi}}\,v = -\frac{(1-i)}{\sqrt{\pi}}\,v$$

$$dt = -\frac{(1-i)}{\sqrt{\pi}}\,dv$$

$$\int_0^{-\frac{\sqrt{\pi}}{(1-i)}}$$

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$$\gamma(\alpha, z) = \int_0^z e^{-t} t^{\alpha-1}\, dt$$

$$= \int_0^z \sum_{k=0}^{\infty} \frac{(-1)^k t^k}{k!} t^{\alpha-1}\, dt = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_0^z t^{k+\alpha-1}\, dt$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{t^{k+\alpha}}{k+\alpha} \Big|_0^z$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k z^{k+\alpha}}{k!\,(k+\alpha)} = z^\alpha \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{k!\,(k+\alpha)}$$

$$\Gamma(\alpha, z e^{2\pi m i}) + e^{2\pi m i \alpha} \gamma(\alpha, z) = \Gamma(\alpha)$$

$$\cancel{\Gamma(\alpha, z e^{2\pi m i})} =$$

$$\therefore \Gamma(\alpha, z e^{2\pi m i}) = \Gamma(\alpha) - e^{2\pi m i \alpha} \underbrace{\gamma(\alpha, z)}_{\Gamma(\alpha) - \Gamma(\alpha, z)}$$

$$= \Gamma(\alpha) - e^{2\pi m i \alpha}\Gamma(\alpha) + e^{2\pi i m \alpha}\Gamma(\alpha, z)$$

$$= e^{2\pi i m \alpha}\Gamma(\alpha, z) + (1 - e^{2\pi m i \alpha})\Gamma(\alpha)$$

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$$c_{n,s}\int_a^b w(x)\{\phi_s(x)\}^2\,dx = \cancel{\int_a^b w(x)\phi_{n+1}\phi_s\,dx}^{\,0} - A_n\underbrace{\int_a^b x\phi_n(\phi_s(x))^2\,dx}_{\text{don't get 1 here As } x\phi_n \text{ is not}}$$

But $x\phi_s$ is poly degree $s+1$

$c_{n,s}$ vanish exc pos $c_{n,n-1}$ & $c_{n,n}$

$\Rightarrow$ $c_n \Rightarrow \phi_{n+1} - A_n x\phi_n = c_{n,n-1}\phi_{n-1} + c_{n,n}\phi_n$

$\Rightarrow$ $\phi_{n+1} - (A_n x + c_{n,n})\phi_n - c_{n,n-1}\phi_{n-1} = 0$

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$$\frac{1}{2^n}\binom{2n}{n} = \frac{(2n)!}{2^n(n!)^2}$$

Leibniz thm

$$D^n(uv) = \sum_{k=0}^{n}\binom{n}{k}D^k u\, D^{n-k} v$$

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$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}\phi_n(x)\,\omega(x)\,dx$$

$$= \frac{(-1)^n}{2^n n!}\int_{-1}^{1}\frac{d^n}{dx^n}\left\{(1-x)^{n+\alpha}(1+x)^{n+\beta}\right\}\omega(x)\,dx$$

$$= \frac{(-1)^n}{2^n n!}\left[\cancel{\frac{d^{n-1}}{dx^{n-1}}\left\{(1-x)^{n+\alpha}(1+x)^{n+\beta}\right\}\omega(x)\Big|_{-1}^{1}}^{\,0} - \int_{-1}^{1}\frac{d^{n-1}}{dx^{n-1}}\left\{(1-x)^{n+\alpha}(1+x)^{n+\beta}\right\}\omega'(x)\,dx\right]$$

$$= \frac{(-1)^n}{2^n n!}(-1)^n\int_{-1}^{1}\left\{(1-x)^{n+\alpha}(1+x)^{n+\beta}\right\}\omega^{n}(x)\,dx$$

$$= \frac{1}{2^n n!}\int_{-1}^{1}(1-x)^{n+\alpha}(1+x)^{n+\beta}\,\omega^{(n)}(x)\,dx = 0 \text{ if}$$

$\deg\ \omega < n$

$$\therefore \int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}\phi_n(x)\,\omega(x)\,dx = 0 \quad \text{if } \deg\omega < n.$$

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$$h_n = \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta}$$

From Above

$$\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta}\phi_n(x)\tilde{\omega}(x)\,dx = \frac{1}{2^n n!}\int_{-1}^{1}(1-x)^{\alpha+n}(1+x)^{n+\beta}\tilde{\omega}^{(n)}(x)\,dx$$

w/ $\tilde{\omega} = P_n^{(\alpha,\beta)}(x)$

The $\tilde{\omega}^{(n)} = a_{n,n}\, n!$ As $P_n^{(\alpha,\beta)}(x)$ is a polynomial of degree n

$\Rightarrow$

$$h_n = \int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}\phi_n^2\,dx = \frac{\cancel{n!}}{2^n \cancel{n!}}\, a_{n,n}\int_{-1}^{1}(1-x)^{\alpha+n}(1+x)^{n+\beta}\,dx$$

~~let $v = 1-x$ $\rightarrow$ $x = 1-v$~~
~~$dv = -dx$~~

$$= \frac{a_{n,n}}{2^n}\int_{-1}^{1}(1-x)^{\alpha+n}(1+x)^{n+\beta}\,dx$$

2

~~look at~~ $(1-x)^{\alpha+n}(1+x)^{n+\beta}$ "even fn on $-1,1$"?

let $v = 1+x \Rightarrow x = v-1$
$dv = dx$

$$\Rightarrow \frac{a_{n,n}}{2^n}\int_0^2 (1-(v-1))^{\alpha+n}(v)^{n+\beta}\,dv$$

$$= \frac{a_{n,n}}{2^n}\int_0^2 (2-v)^{\alpha+n}\,v^{n+\beta}\,dv$$

let ~~$v = w/2$~~ ~~$dv = \frac{dw}{2}$~~ $w = \frac{v}{2}$ $dv = 2\,dw$

$$= \frac{a_{n,n}}{2^n}\int_0^1 (2-2w)^{\alpha+n}(2w)^{n+\beta}\,2\,dw$$

$$= \frac{a_{n,n}\,2^{\alpha+n+\cancel{n}+\beta+1}}{\cancel{2^n}}\int_0^1 (1-w)^{\alpha+n}\,w^{n+\beta}\,dw$$

$$= a_{n,n}\,2^{n+\alpha+\beta+1}\int_0^1 w^{n+\beta}(1-w)^{\alpha+n}\,dw =$$

$$= \cancel{a_{n,n} 2^{n+\alpha+\beta+1}} \; \frac{\cancel{\Gamma(n+\beta)\Gamma(\alpha+n)}}{\cancel{\Gamma(}}$$

let $v = 1-\omega$ $\Rightarrow$ $a_{n,n} 2^{n+\alpha+\beta+1} \displaystyle\int_1^0 (1-v)^{n+\beta} v^{\alpha+n} (-dv)$

$dv = -d\omega$

$\{\omega = 1-v\}$

$$= a_{n,n} 2^{n+\alpha+\beta+1} \int_0^1 v^{\alpha+n} (1-v)^{n+\beta} dv$$

$$= a_{n,n} 2^{n+\alpha+\beta+1} \int_0^1 v^{(\alpha+n+1)-1} (1-v)^{(n+\beta+1)-1}$$

$$= a_{n,n} 2^{n+\alpha+\beta+1} \frac{\Gamma(\alpha+n+1)\Gamma(n+\beta+1)}{\Gamma(\alpha+\beta+2n+2)}$$

put in $a_{n,n}$

$$= \frac{1}{\cancel{2^{n}}} \binom{2n+\alpha+\beta}{n} 2^{\cancel{n}+\alpha+\beta+1} \frac{\Gamma(\alpha+n+1)\Gamma(n+\beta+1)}{\Gamma(\alpha+\beta+2n+2)}$$

$$= \frac{(2n+\alpha+\beta)!}{(n+\alpha+\beta)!\, n!} \frac{\Gamma(\alpha+n+1)\Gamma(n+\beta+1)}{\Gamma(\alpha+\beta+2n+2)} 2^{\alpha+\beta+1}$$

$$= \frac{2^{\alpha+\beta+1}\,\Gamma(\alpha+n+1)\,\Gamma(n+\beta+1)}{n!}\;\underbrace{\frac{(2n+\alpha+\beta)!}{(n+\alpha+\beta)!\,\Gamma(\alpha+\beta+2n+2)}}$$

$$= \quad '' \quad \frac{(2n+\alpha+\beta)(2n+\alpha+\beta-1)(2n+\alpha+\beta-2)\cdots\overset{(n+1+\alpha+\beta)}{\wedge}(n\cancel{+\alpha+\beta})!}{(n\cancel{+\alpha+\beta})!\,(\alpha+\beta+2n+1)\,\Gamma(\alpha+\beta+2n+1)}$$

$$= \frac{''}{(\alpha+\beta+2n+1)}\;\frac{(2n\cancel{+\alpha}+\beta)(2n\cancel{+\alpha}+\beta-1)\cdots(n+1\cancel{+\alpha}+\beta)}{(2n\cancel{+\alpha}+\beta)(2n\cancel{+\alpha}+\beta-1)\cdots\underset{(\alpha\cancel{+\beta}+n+1)}{\wedge}\;\Gamma(\alpha+\beta+n+1)}$$

$$= \frac{2^{\alpha+\beta+1}\,\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\,\Gamma(\alpha+\beta+n+1)}$$

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$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n (1-x)^{-\alpha}(1+x)^{-\beta}}{2^n n!} \frac{d^n}{dx^n}\left\{(1-x)^{n+\alpha}(1+x)^{n+\beta}\right\}$$

$$= \frac{(-1)^n (1-x)^{-\alpha}(1+x)^{-\beta}}{2^n n!} \sum_{k=0}^{n} \binom{n}{k} \frac{d^k}{dx^k}(1-x)^{n+\alpha} \frac{d^{n-k}}{dx^{n-k}}(1+x)^{n+\beta}$$

$$= \frac{(-1)^n (1-x)^{-\alpha}(1+x)^{-\beta}}{2^n n!} \sum_{k=0}^{n} \binom{n}{k} (n+\alpha)(n+\alpha-1)(n+\alpha-2)\cdots(n+\alpha-k+1)(-1)^k (1-x)^{n+\alpha-k}$$

$$\cdot (n+\beta)(n+\beta-1)(n+\beta-2)\cdots(n+\beta-(n-k)+1)(+1)^{n-k}(1+x)^{n+\beta-n+k}$$

$$= \frac{(-1)^n}{2^n n!} \sum_{k=0}^{n} \binom{n}{k} (n+\alpha)(n+\alpha-1)(n+\alpha-2)\cdots(n+\alpha-k+1)(-1)^k (1-x)^{n-k} \cdot$$

$$(n+\beta)(n+\beta-1)(n+\beta-2)\cdots(\beta+k+1)(1+x)^k$$

Thus The coefficient of x^n comes from ~~2 parts~~ the coefficient of $(1-x)^n$ i.e. $k=0$.

+ the coe

For P_n $\alpha=0,\ \beta=0$

$$\Rightarrow \quad h_n = \int_{-1}^{1} P_n^2\,dx = \frac{2}{2n+1} \frac{\cancel{\Gamma(\alpha+1)}\ \cancel{\Gamma(n+1)}\ \cancel{\Gamma(n+1)}}{\cancel{n!}\ \cancel{\Gamma(n+1)}}$$

$$= \frac{2}{2n+1}$$

For Laguerre:

$$\int_0^\infty e^{-x} x^\alpha \underbrace{\frac{e^x x^{-\alpha}}{n!} \frac{d^n}{dx^n}\left(e^{-x} x^{n+\alpha}\right)}_{L_n^{(\alpha)}(x)} \omega(x)\,dx$$

$$= \frac{1}{n!} \int_0^\infty \frac{d^n}{dx^n}\left(e^{-x} x^{n+\alpha}\right) \omega(x)\,dx = \frac{1}{n!} \frac{d^{n-1}}{dx^{n-1}}\left(e^{-x} x^{n+\alpha}\right)\omega(x)\Big|_0^\infty - \frac{1}{n!}\int_0^\infty \frac{d^{n-1}}{dx^{n-1}}\left(e^{-x} x^{n+\alpha}\right)\omega'\,dx$$

$$= \frac{1}{n!}(-1)^n \int_0^\infty e^{-x} x^{n+\alpha}\, \omega^{(n)}(x)\,dx \qquad \text{By I.B.P.}$$

Then

$$h_n = \cancel{\frac{1}{n!}(-1)^n}\int_0^\infty$$

$$h_n = \int_0^\infty e^{-x} x^\alpha \phi_n^2 \, dx = \int_0^\infty \frac{(-1)^n}{n!} e^{-x} x^{n+\alpha} \underbrace{\frac{d^n}{dx^n} L_n^{(\alpha)}(x)}_{\substack{n!\, a_{n,n} \\ = \cancel{n!} \frac{(-1)^n}{\cancel{n!}}}} \, dx$$

$$= \frac{1}{n!} \int_0^\infty e^{-x} x^{n+\alpha} \, dx = \frac{1}{n!} \int_0^\infty e^{-x} x^{(n+\alpha+1)-1} \, dx = \frac{\Gamma(n+\alpha+1)}{n!}$$

For Hermite poly's. consider

$$\int_{-\infty}^\infty e^{-x^2} H_n(x) w(x) \, dx = \int_{-\infty}^\infty \cancel{e^{-x^2}} (-1)^n \cancel{e^{x^2}} \frac{d^n}{dx^n}(e^{-x^2}) w(x) \, dx$$

$$= (-1)^n \int_{-\infty}^\infty \frac{d^n}{dx^n}(e^{-x^2}) w(x) \, dx = (-1)^n (-1)^n \int_{-\infty}^\infty e^{-x^2} w^{(n)}(x) \, dx$$

if $w(x) = H_n(x)$ $\quad w^{(n)}(x) = n!\, a_{n,n}$

$\therefore$

$$h_n = \int_{-\infty}^\infty e^{-x^2} H_n^2(x) \, dx = n!\, a_{n,n} \int_{-\infty}^\infty e^{-x^2} dx = n!\, a_{n,n} \sqrt{\pi}$$

$= \sqrt{\pi}\ 2^n n!$

$L_n^{(0)}(x) =$ laguerre poly, Then $h_n = \frac{\Gamma(n+1)}{n!} = \frac{\Gamma(n+\alpha+1)}{n!}$

$= 1$

$$P_n(x) = \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n}\left[(1-x^2)^n\right]$$

$$= \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n} \sum_{k=0}^{n} \binom{n}{k} 1^{n-k}(-x^2)^k = \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n} \sum_{k=0}^{n} \binom{n}{k}(-1)^k x^{2k}$$

$$= \frac{(-1)^n}{2^n n!} \sum_{k=0}^{n} \binom{n}{k}(-1)^k (2k)(2k-1)(2k-2)\cdots(2k-n+1)\, x^{2k-n}$$

coeff of $x^n \rightarrow k=n$

$$\Rightarrow \frac{(-1)^n}{2^n n!} \binom{n}{n}(-1)^n \underbrace{(2n)(2n-1)(2n-2)\cdots(2n-n+1)}_{(n+1)} x^n$$

$$= \frac{1}{2^n n!}(2n)(2n-1)(2n-2)\cdots(n+2)(n+1)$$

$$= \frac{1}{2^n n!} \frac{(2n)!}{n!} = \frac{(2n)!}{2^n (n!)^2}$$

coeff of x^{n-1} $\Rightarrow$ $k = n-1$

$$\Rightarrow \cancel{\frac{(-1)^n}{2^n n!}\binom{n}{n-1}(-1)^{n-1}}$$

$$2k - n = n-1$$
$$2k = 2n-1 \quad \text{impossible}$$

only even/odd terms in $P_n(x)$ coeff $x^{n-1} = 0$.

coeff of x^{n-2} $\rightarrow$ $2k - n = n-2$

$$2k = 2n-2 = 2(n-1)$$

$$\Rightarrow k = n-1$$

$$= \frac{(-1)^n}{2^n n!}\underbrace{\binom{n}{n-1}}_{n}(-1)^{n-1}\overbrace{(2(n-1))}^{2n-2}(2n-3)(2n-4)\cdots\underbrace{(2n-2-n+1)}_{(n+1)(n)(n-1)}\left\{x^{n-2}\right.$$

$$= \frac{(-1)n\,(2n-2)!}{2^n\, n!\,(n-2)!}$$

$$= \frac{(-1)(2n-2)!}{2^n (n-1)!\,(n-2)!}$$

From 6.03

$$P_{n+1} - (A_n x + B_n) P_n + C_n P_{n-1} = 0$$

Taking coeff of x^n $\Rightarrow$

$$\cancel{0 - B_n \frac{(2n)!}{2^n (n!)^2} + 0} = 0$$

$$0 - 0 - B_n\left(\frac{(2n)!}{2^n(n!)^2}\right) + 0 = 0$$

P_{n-1} starts at x^{n-1}

P_{n+1} has no x^n term only x^{n+1} + x^{n-1}.

P_n has x^n term + x^{n-2}.

$\therefore$ xP_n has x^{n+1} term + x^{n-1}

$\Rightarrow$ $B_n = 0$

Take coeff of x^{n-1}

$\Rightarrow$

$$\underbrace{\frac{-(2n)!}{2^{n+1}(n-1)!\,n!}} - A_n(-)\underbrace{\frac{(2n-2)!}{2^n(n-2)!(n-1)!}}_{\text{coeff of } x^{n-2} \text{ in } P_n} + C_n\underbrace{\frac{(2n-2)!}{2^{n-1}((n-1)!)^2}}_{\text{coeff of } x^{n-1} \text{ in } P_{n-1}} = 0$$

$$\Rightarrow \; -\frac{(\cancel{2n})(2n-1)}{\cancel{2}\,\cancel{x}} + \frac{A_n}{(n-2)} + \frac{2C_n}{} = 0$$

$$\left\{ \frac{(2n+1)}{(n+1)} + \frac{2(n-2)n}{n+1} \overset{?}{=} (2n-1)(n-2) \right.$$

$$= \quad \frac{A_n}{n-2} + 2C_n = (2n-1)$$

$$\frac{2n+1+2n^2-4n}{n+1} =$$

$$\left\{ \frac{2n^2-2n+1}{n+1} = 2n \right.$$

Take coeff of x^{n-2} from

$$\Rightarrow \; 0 - \underbrace{A_n \cdot 0}_{\substack{\text{coeff of } x^{n-3} \text{ in } P_n \\ =0}} + C_n$$

P_n has $x^n \neq 0$

P_n " $x^{n-2} \neq 0$

↑ Find mistake.

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If $A_n = \cdots$

$$P_{n+1} - (A_n x + B_n) P_n + C_n P_{n-1} = 0$$

$$\Rightarrow \quad P_{n+1} - \frac{(2n+1)x}{n+1} P_n + \frac{n}{n+1} P_{n-1} = 0$$

$$(n+1) P_{n+1} - (2n+1) x P_n + n P_{n-1} = 0$$

$$\frac{d}{dx}\left\{(1-x^2) P_n'(x)\right\} = (1-x^2) P_n'' - 2x P_n'$$

$$= \sum_{s=0}^{n} C_{n,s} P_s(x)$$

Mult P_s

$$\int_{-1}^{1} \frac{d}{dx}\left\{(1-x^2) P_n'\right\} P_s \, dx = C_{n,s} \frac{2}{2s+1}$$

$$\Rightarrow \quad C_{n,s} = \cancel{\frac{2s+1}{2} \int_{-1}^{1} dx} \quad \left.\cancel{(1-x^2) P_n'}^{\,0} P_s\right|_{-1}^{1} - \int_{-1}^{1} (1-x^2) P_n' P_s' \, dx$$

$$= C_{n,s} \frac{2}{2s+1}$$

IBP once again. integrate P_n'

$$= -\cancel{(1-x^2)}P_s' P_n \Big|_{-1}^{1} + \underbrace{\int_{-1}^{1} \underbrace{\frac{d}{dx}((1-x^2)P_s')}_{\text{poly of degree } s} P_n \, dx}_{= 0 \quad s<n}$$

If $s=n$

$\therefore \quad (1-x^2)P_n'' - 2xP_n' = C_{n,n} P_n \qquad \left\{ P_n = \sum_{k=0}^{n} a_k x^k \right.$

look at x^nth terms

$$\Rightarrow \quad C_{n,n} \frac{\cancel{(2n)!}}{\cancel{2^n (n!)^2}} = (-1)n(n-1)\frac{\cancel{(2n)!}}{\cancel{2^n (n!)^2}} - 2n\frac{\cancel{(2n)!}}{\cancel{2^n (n!)^2}}$$

$$\Rightarrow \quad C_{n,n} = (-1)n(n-1) - 2n = -n^2 + n - 2n = -n^2 - n$$

$$= -n(n+1)$$

$$\therefore \quad (1-x^2)P_n'' - 2xP_n' + n(n+1)P_n = 0$$

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$$P_n = \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n} \{(1-x^2)^n\}$$

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(t)}{(t-z)^{n+1}} dt$$ Cauchy integ formula

$$\frac{d^n}{dx^n} f(x) = \frac{n!}{2\pi i} \int_C \frac{f(t)}{(t-x)^{n+1}} dt$$

$$\therefore \frac{d^n}{dx^n} \{(1-x^2)^n\} = \frac{n!}{2\pi i} \int_C \frac{(1-t^2)^n}{(t-x)^{n+1}} dt$$

$$\therefore P_n(x) = \frac{\cancel{(-1)^n}}{2^{n+1} \cancel{n!}} \frac{\cancel{n!}}{\pi i} \int_C \frac{\cancel{(-1)^n}(t^2-1)^n}{(t-x)^{n+1}} dt$$

$$= \frac{1}{2^{n+1} \pi i} \int_C \frac{(t^2-1)^n}{(t-x)^{n+1}} dt$$ C must enclose the pt x.

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consider $\displaystyle\sum_{n=0}^{\infty} \frac{(t^2-1)^n h^n}{2^{n+1}\pi i (t-x)^{n+1}}$

Then $\left| \dfrac{(t^2-1)^n h^n}{2^{n+1}\pi i (t-x)^{n+1}} \right| \le \dfrac{|h|^n}{2^n}$ ~~for small enough~~

Fix a contour C, then

$\exists$ Max of $\dfrac{(t^2-1)^n}{(t-x)^{n+1}} \le M_n(x)$ for $t \in C$.

Then $\left| \dfrac{(t^2-1)^n h^n}{} \right.$ How do I show convergence is uniform?

But sum is

$$\frac{1}{2\pi i} \frac{\displaystyle\sum_{n=0}^{\infty} \frac{(t^2-1)^n h^n}{2^n (t-x)^n}}{(t-x)} = \frac{1}{2\pi i (t-x)} \underbrace{\sum_{n=0}^{\infty} \frac{(t^2-1)^n h^n}{2^n (t-x)^n}}_{\frac{1}{1-\frac{(t^2-1)h}{2(t-x)}}}$$

$$= \frac{1}{2\pi i} \frac{1}{(t-x) - \frac{(t^2-1)h}{2}} =$$

$\therefore \int_C \rightarrow$ onto sum gives

$$\underbrace{\frac{1}{2\pi i}\int_C \left\{1-\frac{(t^2-1)h}{2(t-x)}\right\}\frac{dt}{t-x}}_{\text{summing 1st}} = \sum_{n=0}^{\infty}\int_C \frac{(t^2-1)^n h^n}{2^{n+1}\pi i(t-x)^{n+1}}\,dt$$

$$= \sum_{n=0}^{\infty}\underbrace{\frac{1}{2^{n+1}\pi i}\int_C \frac{(t^2-1)^n}{(t-x)^{n+1}}\,dt}_{\substack{\text{Schläfli's integ}\\ P_n(x)}}\, h^n = \sum_{n=0}^{\infty} P_n(x)h^n = G(x,h)$$

But left hand side is

$$\frac{1}{\pi i}\int_C \frac{dt}{2t-2x-t^2h+th} = \frac{-1}{\pi i}\int_C \frac{dt}{ht^2-2t+(2x-h)}$$

$$= \frac{-1}{\pi i h}\int_C \frac{dt}{\left(t^2-\frac{2}{h}t+\left(\frac{2x}{h}-1\right)\right)}$$

$$t = \frac{\frac{2}{h}\pm\sqrt{\frac{4}{h^2}-4\left(\frac{2x}{h}-1\right)}}{2} = \frac{1}{h}\pm\frac{1}{h}\sqrt{1-2xh+h^2}$$

$$= \left(1\pm\sqrt{1-2xh+h^2}\right)/h$$

~~let $h \to 0$~~ $$t_1 = \frac{1-\sqrt{1-2xh+h^2}}{h} = \frac{1-\sqrt{1-(2xh-h^2)}}{h}$$

$$t_2 = \frac{1+\sqrt{1-2xh+h^2}}{h} = \frac{1+\sqrt{1-(2xh-h^2)}}{h}$$

$h \to 0$ $$t_1 = \frac{1-(1-(2xh-h^2)/2 + O(h^2))}{h}$$

$$t_2 = \frac{1+(1-(2xh-h^2)/2 + O(h^2))}{h}$$

$$\Rightarrow t_1 = \frac{\frac{2xh-h^2}{2} + O(h^2)}{h} = x - \frac{h}{2} + O(h)$$

$$t_2 = \frac{2 - xh + h^2/2 + O(h^2)}{h} = \frac{2}{h} - x + \frac{h}{2} + O(h)$$

$$\therefore G(x,t) = \frac{-1}{h\pi i}\int_C \frac{dt}{(t-t_1)(t-t_2)} = -\frac{2\cancel{\pi i}}{h\cancel{\pi i}} \sum \text{Res in } C$$

is t_1 for small h

C ← contour containing x, for small h $t_1 \to x$, $t_2 \to \infty$

Thus $t_2(x)$ moves outside the contour containing x

$$= \frac{-2}{h} \lim_{t\to t_1} \cancel{(t-t_1)} \frac{1}{\cancel{(t-t_1)}(t-t_2)} = \frac{-2}{h(t_1-t_2)}$$

$$= \frac{\cancel{2}}{\cancel{h}} \frac{1}{\frac{\cancel{2}\sqrt{1-2xh+h^2}}{\cancel{h}}} = \frac{1}{\sqrt{1-2xh+h^2}}$$

Sing at $1-2xh+h^2=0$

$$h = \frac{2x \pm \sqrt{4x^2-4}}{2} = x \pm \sqrt{x^2-1} \qquad x\in[-1,1]$$

Sing are imag.

$$h = x \pm i\sqrt{1-x^2}$$

Then

$$|h|^2 = x^2 + (1-x^2) = \underline{1} \quad \text{lie on unit cr}$$

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$$\int^t \cos\left(\tfrac{1}{3}t^3+xt\right)dt = \int^t \cos\left(\cancel{t\left(\tfrac{t^2}{3}+x\right)}\right)dt$$

Assum inner $\cancel{\frac{t^2}{3}+x}$ is ~~constant~~

$$= \cancel{\frac{\sin\left(t\left(\frac{t^2}{3}+x\right)\right)}{\frac{t^2}{3}+x}} +$$

$$\frac{d}{dt}\left(\frac{t^3}{3}+xt\right) = t^2+x \Rightarrow \frac{dv}{dt} = t^2+x \quad w/\ v = \frac{t^3}{3}+xt$$

$$\Rightarrow dt = \frac{dv}{t^2+x}$$

$$\Rightarrow \int^t \cos\left(\frac{t^3}{3}+xt\right) \cancel{\frac{d}{dt}\left(\frac{t^3}{3}\right.} + dt$$

$$= \frac{\int \cos(v)}{}$$

$$\frac{d}{dt} \Rightarrow$$

$$\cos\left(\frac{t^3}{3}+xt\right) = \cos\left(\frac{t^3}{3}+xt\right) + \cancel{\frac{\sin\left(\frac{t^3}{3}+xt\right)(-1)2t}{(t^2+x)^2}}$$

$$+ \cancel{\frac{2\sin\left(\frac{t^3}{3}+xt\right)t}{(t^2+x^2)^2}} \quad \text{equality } \checkmark$$

$$\int \cos(v(t))\,dt \qquad \text{w/} \quad \frac{dv}{dt} = t^2 + x$$

$$\Rightarrow \quad dt = \frac{dv}{t^2+x} = \frac{dv}{t^2(v)+x}$$

$$= \int \frac{\cos(v(t))}{t^2(v)+x}\,dv = \frac{1}{t^2+x}\ \sin(v(t)) - \int \sin(v(t))\frac{-2t}{(t^2+x)^2}\frac{dt}{dv}\cdot dv$$

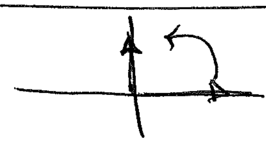
hold this constant ↑ integrate the next on ↑ derivative of $\frac{1}{t^2+x}$ w.r.t. v

$$\frac{d}{dv}\frac{1}{(t^2+x)} = \frac{(-1)}{(t^2+x)^2}\,2t\,\frac{dt}{dv}$$

$$= \frac{\sin\left(\frac{t^3}{3}+xt\right)}{t^2+x} + 2\int \frac{\sin\left(\frac{t^3}{3}+xt\right)}{(t^2+x)^2}\,t\,dt$$

---

let $t = v/i$



$$Ai(x) = \frac{1}{\pi}\int_0^{\infty i} \cos\left(\frac{1}{3}\frac{v^3}{(-i)} + \frac{xv}{i}\right)\frac{dv}{i} = \frac{1}{\pi i}\int_0^{i\infty}\cos\left(\frac{v^3}{3}i - xvi\right)dv$$

$\cos(i\theta) = \cosh\theta$ ✓

$$= \frac{1}{\pi i}\int_0^{i\infty}\cosh\left(\frac{v^3}{3} - xv\right)dv$$

$e^{i(i\theta)} + e^{-i(i\theta)}/2 =$

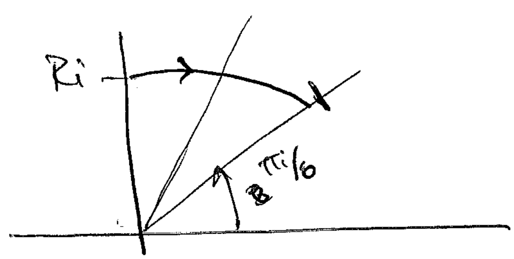
$$= \frac{1}{2\pi i}\int_0^{i\infty}\left(e^{v^3/3 - xv} + e^{-v^3/3 + xv}\right)dv$$

$$= \frac{1}{2\pi i}\int_0^{i\infty}\exp\left(\frac{v^3}{3} - xv\right)dv + \frac{1}{2\pi i}\int_0^{i\infty}\exp\left(-\left(\frac{v^3}{3} - xv\right)\right)dv$$

$$p = -v$$
$$dp = -dv$$
$$-\int_0^{-i\infty}\exp\left(\frac{p^3}{3} - xp\right)dv$$

$$= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\exp\left(\frac{v^3}{3} - xv\right)dv \qquad \text{let } x > 0$$

$$I(R) = \int_{iR}^{Re^{\pi i/6}}\left|\exp\left(\frac{v^3}{3} - xv\right)dv\right|$$



$$\text{let } v = iRe^{-i\theta/3} \qquad \cancel{R}e^{i\pi/6} = i\cancel{R}e^{-i\theta/3}$$

$$I(R) = \int_{\theta=0}^{\pi} \qquad \Rightarrow e^{-i\theta/3} = -ie^{i\pi/6} = e^{-\pi i/2}e^{i\pi/6} = e^{i\left(-\frac{\pi}{2}+\frac{\pi}{6}\right)}$$

$$= e^{-i\pi/3} \Rightarrow \theta = \pi$$

How do I know that $\theta \in [0, \pi]$ gives the original contour in the same direction? Assuming so

$\cancel{dv = (-1)}$

$\cancel{I(R)} =$

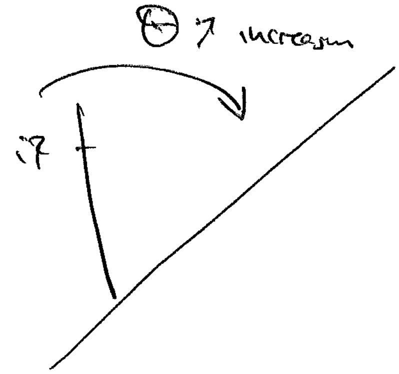
$1e^{i\theta}$ ⇒ start at 1 & go θ in red coord

clockwise sin $e^{-i\theta}$ " " " " " " "

counter clockwise

$iRe^{-i\theta/3}$ ⇒ start at iR & go $\theta/3$ in red coord origin

counter-clockwise



Thus it gives the same direction.

$$dv = iR\left(-i\frac{d\theta}{3}\right)e^{-i\theta/3} = \frac{R}{3}e^{-i\theta/3}\,d\theta$$

$$\Rightarrow I(R) = \frac{R}{3}\int_0^{\pi} \left|\exp\left(\underbrace{-\frac{iR^3e^{i\theta}}{3}} - \underbrace{xiRe^{-i\theta/3}}\right)\right| \underbrace{\left|e^{-i\theta/3}\right|}_{1}\, d\theta\Big|$$

$$\underbrace{-\frac{iR^3}{3}(\cos\theta - i\sin\theta)} \quad \underbrace{-xiR\left(\cos\frac{\theta}{3} - i\sin\frac{\theta}{3}\right)}$$

$$-\frac{iR^3}{3}\cos\theta - \frac{R^3}{3}\sin\theta \quad -xiR\cos\frac{\theta}{3} - xR\sin\frac{\theta}{3}$$

Taking the modulus

$$\Rightarrow \frac{R}{3}\int_0^{\pi} \exp\left\{-\frac{R^3}{3}\sin\theta - xR\sin\theta/3\right\} d\theta$$

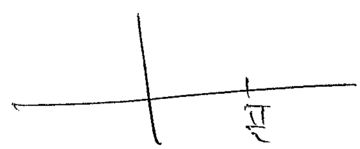
But Jordans ineq $\quad \sin\theta \geq \frac{2}{\pi}\theta \qquad \theta\in[0,\pi/2]$

$$\Rightarrow -\sin\theta \leq -\frac{2}{\pi}\theta$$

Don't need Jordan for this suff
$\sin\theta/3 > 0$ on $(0,\pi)$

$$e^{-xR\sin\theta/3} \leq e^{xR\left(-\frac{2}{\pi}\frac{\theta}{3}\right)} = e^{-\frac{2}{3}\frac{R}{\pi}x\theta} < 1$$

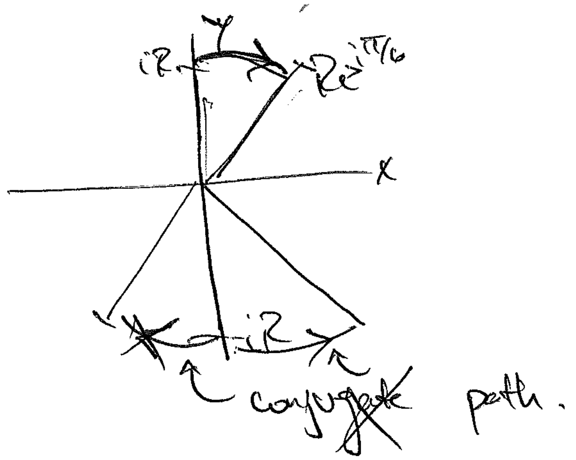
$$\leq \frac{R}{3}\int_0^{\pi} \exp\left\{-\frac{R^3}{3}\sin\theta\right\} d\theta$$



$$= \frac{2R}{3}\int_0^{\pi/2} \exp\left\{-\frac{R^3}{3}\sin\theta\right\} d\theta \leq \frac{2R}{3}\int_0^{\pi/2} e^{-\frac{R^3}{3}\cdot\frac{2\theta}{\pi}}\, d\theta$$

$$= \frac{2R}{\cancel{3}}\left(\frac{e^{-\frac{R^3}{3}\frac{2\theta}{\pi}}}{-\frac{R^3}{\cancel{3}}\frac{\cancel{2}}{\pi}}\right|_0^{\pi/2} = -\frac{\pi R}{R^3}\left(e^{-R^3/3} - 1\right)$$

$$= \frac{\pi}{R^2}\left(1 - e^{-R^3/3}\right) \leq \frac{\pi}{R^2}$$



Conjugation is reflection through $y = 0$

As All i terms were disregarded in the norm I would Believe that the integral would vanish along this path also.