Notes and Solutions for the Book: Signals And Systems by Alan V. Oppenheim and Alan S. Willsky with S. Hamid Nawab.

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Chapter 1: Signals and Systems

Problem Solutions

Problem 1.3 (computing P_∞ and E_∞ for some sample signals)

Recall that P_∞ and E_∞ (the total power and total energy) in the case of continuous and discrete signals are defined as

$$E_\infty = \int_{-\infty}^{\infty} |x(t)|^2 dx \quad \text{and} \quad E_\infty = \sum_{n=-\infty}^{\infty} |x[n]|^2 \,,$$

and

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dx \quad \text{and} \quad P_\infty = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 \,.$$

Using these we can compute each part of the given problem.
Part (a): For this signal we find

$$E_\infty = \int_0^\infty e^{-4t} dt = \left.\frac{e^{-4t}}{-4}\right|_0^\infty = \frac{1}{4} \,,$$

and

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T} \int_0^T e^{-4t} dt = \lim_{T\to\infty} \frac{1}{2T} \left(\left.\frac{e^{-4t}}{-4}\right|_0^T \right. = \lim_{T\to\infty} \frac{1}{8T}(1 - e^{-4T}) = 0 \,,$$

which we could have known without any calculation since $x(t)$ has finite energy it must have 0 time-averaged power.
Part (b): For this signal we find $E_\infty = \int_{-\infty}^{+\infty} 1 dt = +\infty$, and $P_\infty = \lim_{T\to\infty} \int_{-T}^{+T} 1 dt = 1$.
Part (c): For this signal we find

$$E_\infty = \int_{-\infty}^{\infty} |\cos(t)|^2 dt = \int_{-\infty}^{\infty} \cos(t)^2 dt = +\infty \,.$$

and

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} \cos(t)^2 dt = \lim_{T\to\infty} \frac{1}{2T} \left(T + \cos(T)\sin(T)\right) = \frac{1}{2} \,.$$

Part (d): For this signal we find

$$\begin{aligned} E_\infty &= \sum_{n=-\infty}^{\infty} |x[n]|^2 = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n = \frac{1}{1-1/4} = \frac{4}{3} \\ P_\infty &= 0\,, \end{aligned}$$

since E_∞ is finite.
Part (e): For this signal we find

$$\begin{aligned} E_\infty &= \sum_{n=-\infty}^{\infty} |x[n]|^2 = \sum_{n=0}^{\infty} 1 = \infty \\ P_\infty &= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 \\ &= \lim_{N\to\infty} \frac{1}{2N+1}(N-(-N)+1) = 1\,. \end{aligned}$$

Part (f): For this signal we find

$$\begin{aligned} E_\infty &= \sum_{n=-\infty}^{\infty} |\cos(\frac{\pi}{4}n)|^2 \\ &= \sum_{k=-\infty}^{\infty} |\cos(\frac{\pi}{4}(4k+0))|^2 + \sum_{k=-\infty}^{\infty} |\cos(\frac{\pi}{4}(4k+1))|^2 \\ &+ \sum_{k=-\infty}^{\infty} |\cos(\frac{\pi}{4}(4k+2))|^2 + \sum_{k=-\infty}^{\infty} |\cos(\frac{\pi}{4}(4k+3))|^2 \\ &\geq \sum_{k=-\infty}^{\infty} |\cos(\pi k)|^2 = \infty\,. \end{aligned}$$

To compute P_∞ we have

$$P_\infty = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |\cos(\frac{\pi}{4}n)|^2\,.$$

Now recall that $\cos(x) = \frac{1}{2}(e^{jx}+e^{-jx})$, so that

$$|\cos(x)|^2 = \frac{1}{4}(e^{jx}+e^{-jx})(e^{-jx}+e^{jx})$$

$$
\begin{aligned}
&= \frac{1}{4}(1 + e^{j(2x)} + e^{-j(2x)} + 1) \\
&= \frac{1}{2} + \frac{1}{2}\cos(2x)\,.
\end{aligned}
$$

Thus $|\cos(\frac{\pi}{4}n)|^2 = \frac{1}{2} + \frac{1}{2}\cos(\frac{\pi}{2}n)$. Now the $\cos(\cdot)$ in this later expression evaluates to

$$
\cos(\frac{\pi}{2}n) = \left\{ \begin{array}{cc} 0 & n \text{ odd} \\ (-1)^m & n \text{ even say } n = 2m \end{array} \right. .
$$

With this then the sum above needed in computing P_∞ is given by

$$
\lim_{T\to\infty} \frac{1}{2T} \sum_{n=-T}^{T} |x[n]|^2 = \lim_{M\to\infty} \frac{1}{2(2M)} \left((2M) + \frac{1}{2} \sum_{m=-M}^{M} (-1)^m \right) \to \frac{1}{2}\,,
$$

as $M \to \infty$

Problem 1.4 (when is $x[n]$ zero)

Part (a): $x[n-3]$ shifts $x[n]$ to the right by 3 so $x[n-3]$ will be zero when $n < 1$ and $n > 7$.
Part (b): $x[n+4]$ shifts $x[n]$ to the left by 4 so $x[n+4]$ will be zero when $n < -6$ and $n > 0$.
Part (c): $x[-n]$ reflects $x[n]$ about zero so $x[-n]$ will be zero for $n < -4$ and $n > 2$.
Part (d): Note that $x[-n+2]$ equals $x[-(n-2)]$ which is a right shift of $x[-n]$ by 2, so this function will be zero when $n < -2$ and $n > 4$.
Part (e): Note that $x[-n-2]$ equals $x[-(n+2)]$ which is a left shift of $x[-n]$ by 2, so this function will be zero when $n < -6$ and $n > 0$.

Problem 1.5 (when is $x(t)$ zero)

Part (a): Now $x(1-t) = x(-(t-1))$ which is a right shift by one of the function $x(-t)$. The function $x(-t)$ is zero when $t > -3$. So shifting this function one unit to the right means that $x(1-t)$ will be zero when $t > -2$.
Part (b): Now that $x(1-t) + x(2-t) = x(-(t-1)) + x(-(t-2))$. The first function, $x(-(t-1))$, was considered in Part (a) above and is zero when $t > -2$ by the logic above. The second function $x(-(t-2))$ is zero when $t > -1$ so the sum will be guaranteed to be zero for $t > -1$.

Part (c): In the product $x(1-t)x(2-t)$ the first function is zero when $t > -2$, while the second function is zero when $t > -1$, thus the product will be zero when $t > -2$.
Part (d): If $x(t)$ is zero when $t < 3$ then $x(3t)$ will be zero when $3t < 3$ or $t < 1$.
Part (e): If $x(t)$ is zero when $t < 3$ then $x(t/3)$ will be zero when $t/3 < 3$ or $t < 9$.

Problem 1.6 (are these functions periodic)

Part (a): No. The function $u(t)$ in the definition of $x(t)$ is not periodic.
Part (b): Note that $x_2[n] = 2$ when $n = 0$ and $x_2[n] = 1$ when $n \neq 0$, which shows that $x_2[n]$ is not periodic because of the value at $n = 0$.
Part (c): For the function $x_3[n] = \sum_{k=-\infty}^{\infty}\{\delta[n-4k] - \delta[n-1-4k]\}$, note that

$$\begin{aligned}
x_3[0] &= \sum_{k=-\infty}^{\infty}\{\delta[-4k] - \delta[-1-4k]\} = 1 \\
x_3[1] &= \sum_{k=-\infty}^{\infty}\{\delta[1-4k] - \delta[-4k]\} = -1 \\
x_3[2] &= \sum_{k=-\infty}^{\infty}\{\delta[2-4k] - \delta[1-4k]\} = 0 \\
x_3[3] &= \sum_{k=-\infty}^{\infty}\{\delta[3-4k] - \delta[2-4k]\} = 0\,,
\end{aligned}$$

and this cycle of numbers $1, -1, 0, 0$ repeats, showing that the sequence $x_3[n]$ is periodic.

Problem 1.7 (when is the even part zero)

For this problem recall that the *even* part of a discrete signal $x[n]$ is defined as

$$\mathcal{E}v\{x[n]\} = \frac{1}{2}(x[n] + x[-n])\,. \quad (1)$$

Part (a): For this signal we find

$$\mathcal{E}v\{x_1[n]\} = \frac{1}{2}(u[n] - u[n-4]) + \frac{1}{2}(u[-n] - u[4-n])\,.$$

Now the first expression $u[n] - u[n-4]$ represents the difference between $u[n]$ and a right shift of $u[n]$ by four units. This expression then is only non-zero when $0 \le n < 3$ where for those four points it has the value 1. The second expression $u[-n] - u[-(n-4)]$ represents the difference between $u[-n]$ and a right shift of $u[-n]$ by four. This expression will be non-zero only for $1 \le n \le 3$ where it will have the value of -1. When we add these two parts together (and multiply by $1/2$) we see that $\mathcal{E}v\{x_1[n]\}$ is zero for all n but $n = 0$ where it has the value of $1/2$ and for $n = 4$ where it has the value $-1/2$. Plotting the functions $u[n] - u[n-4]$ and $u[-n] - u[4-n]$ can help visualize this.

Part (b): For this signal we find

$$\mathcal{E}v\{x(t)\} = \frac{1}{2}(\sin(\frac{t}{2}) + \sin(-\frac{t}{2})) = 0\,,$$

for all t.

Part (c): For this signal we find

$$\mathcal{E}v\{x[n]\} = \frac{1}{2}\left(\left(\frac{1}{2}\right)^n u[n-3] + \left(\frac{1}{2}\right)^{-n} u[-n-3]\right)\,.$$

The first term is zero when $n < 3$. The second term is zero when $n > -3$. Thus the total combined expression is zero when $|n| < 3$. The given even signal from $x[n]$ also vanishes as $|n| \to \infty$.

Part (d): For this signal we find

$$\mathcal{E}v\{x(t)\} = \frac{1}{2}\left(e^{-5t}u(t+2) + e^{5t}u(-t+2)\right)\,.$$

The first term is zero when $t < -2$. The second term is zero when $t > 2$ so there is no part guaranteed to be zero but the entire expression goes to zero as $|t| \to \infty$.

Problem 1.9 (finding periods)

Part (a): The function je^{j10t} is periodic with a period T that is required to satisfy

$$10(t+T) = 10t + 2\pi \quad \text{or} \quad T = \frac{2\pi}{10} = \frac{\pi}{5}\,.$$

Part (b): The function $e^{-t}e^{-jt}$ is not periodic, since e^{-t} causes the periodic part e^{-jt} to decay.

Part (c): For the discrete function $x[n] = e^{j7\pi n}$ to be periodic requires there exist integers m and N such that

$$7\pi N = 2\pi m \quad \text{so} \quad N = \frac{2}{7}m\,.$$

If we take $m = 7$ then we get a fundamental period of $N = 2$.
Part (d): The given function $x[n]$ can be written as

$$x[n] = 3e^{j3\pi(n+\frac{1}{2})/5} = 3e^{j\frac{3\pi}{5}}e^{j\frac{3\pi n}{5}}\,,$$

and will be periodic if there exists integers m and N such that

$$N = m\frac{2\pi}{\left(\frac{3\pi}{5}\right)} = m\left(\frac{10}{3}\right)\,.$$

Thus we take $m = 3$ to get a fundamental period of $N = 10$.
Part (e): For the signal $x_5[n] = 3e^{j(3/5)(n+1/2)} = 3e^{j(3/10)}e^{j(3/5)n}$, to be periodic we require there exist integers m and N such that

$$\frac{3}{5}(n+N) = \frac{3}{5}n + 2\pi m\,,$$

or

$$\frac{3}{5}N = 2\pi m \quad \text{so} \quad N = \frac{10}{3}\pi m\,.$$

From this last equation we see that there is no integer value of m that will result in an integer value of N. Thus the signal $x_5[n]$ is not periodic.

Problem 1.10 (the fundamental period of a trigonometric sum)

To find the fundamental period of $x(t) = 2\cos(10t+1) - \sin(4t-1)$ we look for the least common multiple of the fundamental periods for each of the component terms in $x(t)$. The fundamental period of the first term $\cos(10t+1)$ is $T = \frac{2\pi}{10} = \frac{\pi}{5}$, while the fundamental period of the second term $\sin(4t-1)$ is $T = \frac{2\pi}{4} = \frac{\pi}{2}$. If we take five multiples of the first period and two multiples of the second period the elapsed time of both is π, which is the fundamental period of the combined expression $x(t)$.

Problem 1.11 (the fundamental period of a trigonometric sum)

For the discrete signal $x[n] = 1 + e^{j\frac{4\pi}{7}n} - e^{j\frac{2\pi n}{5}}$ the fundamental period for the function $e^{j\frac{4\pi n}{7}}$ is given by the smallest integer value of N such that

$$N = \frac{2\pi}{\left(\frac{4\pi}{7}\right)} m = \frac{7}{2} m \,,$$

so if we take $m = 2$ we get $N = 7$. In the same way, the fundamental period for $e^{j\frac{2\pi n}{5}}$ is given by selecting a m so that N in

$$N = \frac{2\pi}{\left(\frac{2\pi}{5}\right)} m = 5m \,,$$

is as small as possible and an integer. Taking $m = 1$ we get a fundamental period $N = 5$. To get the fundamental period of the combined expression we see the least common multiple of 7 and 5 or 35.

Problem 1.12 (summing shifted delta functions)

Note that $x[n]$ can be written as

$$x[n] = 1 - \sum_{k=3}^{\infty} \delta[n-1-k] = 1 - \sum_{k=3}^{\infty} \delta[n-(k+1)] \,,$$

or the sum of shifted $\delta[n]$ functions, each shifted by $k+1$ to the right of the origin. Since the range of $k+1$ for the summation values given is $4, 5, 6, \cdots$ when we view $-\sum_{k=3}^{\infty} \delta[n-(k+1)]$ as a function n we see that it is equal to $-u[n-4]$. When we add 1 to $-u[n-4]$ to get the sum above we get the function $u[-(n-3)] = u[-n+3]$. To match the given expression of $u[Mn - n_0]$ we should take $M = -1$ and $n_0 = -3$.

Problem 1.13 (the integral of some delta functions)

Note that the given function $x(t) = \delta(t+2) - \delta(t-2)$ results in a function $y(t)$ that can be expressed as

$$y(t) = \int_{-\infty}^{t} x(\tau) d\tau = \begin{cases} 0 & t < -2 \\ 1 & -2 < t < 2 \\ 0 & 2 < t \end{cases} \,.$$

Using this expression we can compute

$$E_\infty = \int_{-\infty}^{\infty} |y(\tau)|^2 d\tau = \int_{-2}^{2} d\tau = 4\,.$$

Problem 1.14 (the derivative of a discontinuous periodic function)

For the given periodic extension of the function $x(t)$ defined over a fundamental period $T = 2$ as

$$x(t) = \begin{cases} 1 & 0 \le t \le 1 \\ -2 & 1 < t < 2 \end{cases}\,,$$

by drawing this function over $(-\infty, +\infty)$ we see that the derivative is given by

$$\begin{aligned}
\frac{dx}{dt} &= 3\delta(t) \\
&- 3\delta(t-1) + 3\delta(t-2) - 3\delta(t-3) + \cdots \\
&- 3\delta(t+1) + 3\delta(t+2) - 3\delta(t+3) + \cdots \\
&= 3\delta(t) \\
&+ 3\delta(t-2) + 3\delta(t+2) + 3\delta(t-4) + 3\delta(t+4) + \cdots \\
&- 3\delta(t-1) - 3\delta(t+1) - 3\delta(t-3) - 3\delta(t+3) - \cdots \\
&= 3\sum_{k=-\infty}^{\infty} \delta(t-2k) - 3\sum_{k=-\infty}^{\infty} \delta(t-2k-1) \\
&= 3g(t) - 3g(t-1)\,.
\end{aligned}$$

So to match the expression suggested in the book we need to take $A_1 = 3$, $t_1 = 0$, $A_2 = -3$, and $t_2 = 1$.

Problem 1.15 (serial systems)

Part (a): After processing the input signal $x[n]$ by the system S_1 the output $y_1[n]$ is

$$y_1[n] = 2x[n] + 4x[n-1]\,,$$

so that after system S_2 processes $y_1[n]$ as an input it produces the output $y[n]$ given by

$$y[n] = y_1[n-2] + \frac{1}{2} y_1[n-3]$$

$$
\begin{aligned}
&= 2x[n-2] + 4x[n-3] + \frac{1}{2}(2x[n-3] + 4x[n-4]) \\
&= 2x[n-2] + 5x[n-3] + 2x[n-4]\,.
\end{aligned}
$$

Part (b): If we reverse the order of the two systems then after the second system S_2 processes the input $x[n]$ we obtain an output $y_1[n]$ given by

$$
y_1[n] = x[n-2] + \frac{1}{2}x[n-3]\,.
$$

Passing $y_1[n]$ as an input through the system S_1 we get a final output $y[n]$ of

$$
\begin{aligned}
y[n] &= 2(x[n-2] + \frac{1}{2}x[n-3]) + 4(x[n-3] + \frac{1}{2}x[n-4]) \\
&= 2x[n-2] + 5x[n-3] + 2x[n-4]\,,
\end{aligned}
$$

which is the same result as obtained earlier.

Problem 1.16 (memoryless systems)

Part (a): The system $x[n]x[n-2]$ is not memoryless since it depends on a past value in $x[n-2]$.
Part (b): The output when the input to this system is $A\delta[n]$ is $y[n] = 0$ for all n.
Part (c): This system cannot be invertible since a non-zero input signal gives the zero signal.

Problem 1.17 (some properties of the system $x(\sin(t))$)

Part (a): This system would *not* be causal if $\sin(t) > t$ for any t because then we would be observing the input $x(t)$ at a point in time in the future. Since when $t = -\pi$ we have $\sin(-\pi) = 0 > -\pi$ this system is not causal.
Part (b): This system is linear.

Problem 1.18 (the centered averaging system)

Part (a): This is a linear system.
Part (b): To see if this system is time invariant consider the output produced by the input $x[n-l]$ or a time shift by l units of $x[n]$ to the right. The output

to this system to this input $x[n-l]$ is given by

$$\sum_{k=n-n_0}^{n+n_0} x[k-l] = \sum_{k=n-l-n_0}^{n-l+n_0} x[k] = y[n-l]\,,$$

so this system *is* time invariant.
Part (c): We have the following

$$|y[n]| \leq \sum_{k=n-n_0}^{n+n_0} |x[k]| \leq B(n+n_0-(n-n_0)+1) = B(2n_0+1)\,.$$

Thus we see that $C = (2n_0+1)B$.

Problem 1.19 (some more properties of systems)

Part (a): This system $t^2x(t-1)$ is linear but not time invariant.
Part (b): This system $x[n-2]^2$ is not linear but is time invariant since the output from the input $x[n-n_0]$ is $x^2[n-n_0-2] = y[n-n_0]$.
Part (c): This system is linear and time invariant.
Part (d): Since

$$y[n] = \mathcal{O}d\{x[n]\} = \frac{1}{2}(x[n]-x[-n])\,,$$

we see that this is a linear system. To see if it is time invariant, consider the output to the time-shifted input $x[n-n_0]$. We would get (applying $\mathcal{O}d\{\cdot\}$ to the input $x[n-n_0]$)

$$\frac{1}{2}(x[n-n_0]-x[-n-n_0])\,.$$

While the shifted output would be

$$y[n-n_0] = \frac{1}{2}(x[n-n_0]-x[-(n-n_0)])\,,$$

which is *not* the same as the system operating on $x[n-n_0]$ so this system is not time invariant.

Problem 1.20 (using linearity)

Part (a): We recognized that $x_1(t) = \frac{1}{2}(e^{j2t} + e^{-j2t})$ and then use the linearity property of our system to get that $x_1(t)$ will be mapped to

$$x_1(t) \to \frac{1}{2}(e^{j3t} + e^{-j3t}) = \cos(3t) \, .$$

Part (b): We begin by writing $x_2(t)$ as

$$\begin{aligned} x_2(t) &= \cos(2t-1) = \frac{1}{2}(e^{j(2t-1)} + e^{-j(2t-1)}) \\ &= \frac{1}{2}(e^{-j}e^{j2t} + e^{j}e^{-j2t}) \, . \end{aligned}$$

Again using linearity, we have that the mapping of $x_2(t)$ under our system (denoted as $y_2(t)$) is given by

$$\begin{aligned} y_2(t) &= \frac{1}{2}(e^{-j}e^{j3t} + e^{j}e^{-j3t}) = \frac{1}{2}(e^{j(3t-1)} + e^{-j(3t-1)}) \\ &= \cos(3t-1) \, . \end{aligned}$$

Problem 1.21 (shifting and scaling continuously)

Part (a): This is a shift of the signal $x(t)$ one unit to the right.
Part (b): Since $x(2-t) = x(-(t-2))$ which is $x(t)$ flipped about the $t = 0$ axis (to produce the function $x(-t)$) and then shifted by two units to the right.
Part (c): Since $x(2t+1) = x(2(t+1/2))$ and this later function is $x(2t)$ shifted by $1/2$ to the left. The function $x(2t)$ is a contraction of the t axis in the original function $x(t)$ by two.
Part (d): Now $x(4-\frac{t}{2}) = x(-\frac{1}{2}(t-8))$ and this later function is a shift by 8 units to the right of the function $x(-\frac{t}{2})$. The function $x(-\frac{t}{2})$ is an expanded reflection about the $t = 0$ axis.
Part (e): Note that $(x(t) + x(-t))u(t)$ is the scaled even part of $x(t)$ but only for $t > 0$.
Part (f): Note that the given function can be written

$$x(t)(\delta(t+\frac{3}{2}) - \delta(t-\frac{3}{2})) = x(-\frac{3}{2})\delta(t+\frac{3}{2}) - x(\frac{3}{2})\delta(t-\frac{3}{2}) \, ,$$

is the superposition of two delta functions.

Problem 1.22 (shifting and scaling discreetly)

Since the support of $x[n]$ is so small most of these example functions can be plotted by just evaluating the proposed function at several values of n. One thing to note about these examples is that in the discrete case when we perform scaling of the time axis via multiplication of the independent variable n by a constant we may end up getting a signal that is defined for fewer points than the original function. See Part (c) below for an example of this

Part (a): This is a shift of $x[n]$ by four units to the right.

Part (b): This is a shift of the discrete function $x[-n]$ by three units to the right when we write $x[3-n] = x[-(n-3)]$.

Part (c): This is a contraction of the time axis by three and will represent fewer samples than the original $x[n]$ since points like $x[2]$ in the original discrete function will never be observed by new function $x[3n]$.

Part (d): The same comments as for Part (c) hold here.

Part (e): Note that $x[n]u[3-n] = x[n]u[-(n-3)]$. Now since $u[-(n-3)]$ is a shift by three units to the right of the function $u[-n]$, which is itself a reflection across the point $n = 0$ we see that $x[n]u[3-n]$ represents $x[n]$ for $n \leq 3$ and is zero for $n > 4$.

Part (f): This is a scaled single delta function since

$$x[n-2]\delta[n-2] = x[0]\delta[n-2]\,.$$

Part (g): Note that the suggested function

$$\frac{1}{2}x[n] + \frac{1}{2}(-1)^n x[n]\,,$$

will be $x[n]$ when n even and will be zero when n is odd.

Part (h): Note that $x[(n-1)^2]$ will be a nonlinear mapping of the time domain of $x[n]$ due to the expression $(n-1)^2$.

Problem 1.25 (some fundamental periods)

Part (a): This is a periodic function with a fundamental period given by $T = \frac{2\pi}{4} = \frac{\pi}{2}$.

Part (b): This is a periodic function with a fundamental period given by $T = \frac{2\pi}{\pi} = 2$.

Part (c): One might think that because $\cos(x)$ is periodic with a period 2π that $\cos(x)^2$ would be periodic with period also with a period of 2π. The function $\cos(x)^2$ *is* indeed periodic but with a period that is *half* the period of $\cos(x)$. To see this recall the identity

$$\cos(x)^2 = \frac{1+\cos(2x)}{2}\,, \tag{2}$$

where the expression on the right-hand-side has a period of $\frac{2\pi}{2} = \pi$ as claimed. For this problem write the given expression as

$$\cos(2t - \frac{\pi}{3})^2 = \frac{1}{2}(1 + \cos(2(2t - \frac{\pi}{3}))) = \frac{1}{2}(1 + \cos(4t - \frac{2\pi}{3}))\,,$$

From which we see that this is a periodic function with a fundamental period given by $T = \frac{2\pi}{4} = \frac{\pi}{2}$.

Part (d): Since the function we are too consider is given by

$$\begin{aligned}\mathcal{E}v\{\cos(4\pi t)u(t)\} &= \frac{1}{2}(\cos(4\pi t)u(t) + \cos(4\pi t)u(-t)) \\ &= \frac{1}{2}\cos(4\pi t)\,,\end{aligned}$$

which is periodic with a fundamental period given by $\frac{2\pi}{4\pi} = \frac{1}{2}$.

Part (e): Since the given expression is equal to

$$\begin{aligned}\mathcal{E}v\{\sin(4\pi t)u(t)\} &= \frac{1}{2}(\sin(4\pi t)u(t) - \sin(4\pi t)u(-t)) \\ &= \begin{cases} \frac{1}{2}\sin(4\pi t) & t > 0 \\ -\frac{1}{2}\sin(4\pi t) & t < 0 \end{cases}\,.\end{aligned}$$

We see that this function is not periodic since across $t = 0$ it is not.

Part (f): This function cannot be periodic since it is the sum of an infinite number of functions of $v(t) = e^{-2t}u(2t) = e^{-2t}u(t)$ each one individually shifted by n units to the right. Note that due to the discontinuous nature of the unit step $u(t)$ we have that $u(2t) = u(t)$. When the total signal $x(t)$ is viewed in this way we see that because of the unit step $u(t)$ functions in each of the component functions $v(t)$, there a great number of jump discontinuities to the function $x(t)$. For example, at every integer $n \geq 1$ a new term $e^{-(2t-n)}$ gets added to the sum resulting in a discontinuity.

Problem 1.26 (discrete periodic functions)

For a discrete signal to be periodic requires that there exist integers N and m such that

$$\omega_0(n+N) = \omega_0 n + 2\pi m \,,$$

or

$$\omega_0 N = 2\pi m \,.$$

Part (a): For this function N and m must satisfy $\frac{6\pi}{7}N = 2\pi m$ or $N = \frac{7m}{3}$. If we take $m = 3$ then $N = 7$ will be the fundamental period of this function.
Part (b): For this function N and m must satisfy $\frac{1}{8}N = 2\pi m$ or $N = 16\pi m$. Since there are no two integers N and m which will make this an identity this function is *not* periodic.
Part (c): For this function $x[n] = \cos(\frac{\pi}{8}n^2)$ to be periodic with period N means that $x[n+N] = x[n]$ for all n. This requires

$$\cos(\frac{\pi}{8}(n^2 + 2nN + N^2)) = \cos(\frac{\pi}{8}n^2) \,,$$

which in tern requires

$$\frac{\pi}{4}nN + \frac{\pi}{8}N^2 \,,$$

proportional to 2π for all n. If we take $N = 8$ we see that this is indeed true and thus $x[n]$ is periodic with fundamental period $N = 8$.
Part (d): Use the fact that

$$\cos(x)\cos(y) = \frac{1}{2}(\cos(x+y) + \cos(x-y)) \,. \tag{3}$$

we can write the given function as

$$x[n] = \frac{1}{2}(\cos(\frac{3\pi}{4}n) + \cos(\frac{\pi}{4}n)) \,.$$

The first function $\cos(\frac{3\pi}{4}n)$ has a fundamental period given by $\frac{3\pi}{4}N = 2\pi m$ or $N = \frac{8}{3}m$. If we take $m = 3$ then we get a fundamental period of $N = 8$. The second function $\cos(\frac{\pi}{4}n)$ has a fundamental period given by $\frac{\pi}{4}N = 2\pi m$ or $N = 8m$. If we take $m = 1$ then we get a fundamental period of $N = 8$. Thus $x[n]$ is periodic with fundamental period of $N = 8$.
Part (e): For the function

$$x[n] = 2\cos(\frac{\pi}{4}n) + \sin(\frac{\pi}{8}n) - 2\cos(\frac{\pi}{2}n + \frac{\pi}{8}) \,.$$

We will compute the fundamental period of each term. The first term has a fundamental period of $N = 8$. The second term has a fundamental period of $N = 16$. The third term has a fundamental period of $N = 4$. The total function $x[n]$ then will be periodic with a period of 16.

Problem 1.27 (some properties of systems)

Part (a): This system is not memoryless since it has a term $x(2-t)$ which indicates that the output depends on the past. This system is not time invariant, since the output to a time shifted input $x(t-t_0)$ is

$$x(t-2-t_0) + x(2-t-t_0)\,,$$

while the time shifted output would be

$$x(t-t_0-2) + x(2-(t-t_0)) = x(t-t_0-2) + x(2-t+t_0)\,,$$

which is not the same so this system is *not* time invariant. This system is linear. This system is not causal since it depends on $x(2-t)$ which for sufficiently negative time requires knowledge of $x(t)$ for $t > 0$. This system is stable since if $x(t)$ is bounded then

$$|y(t)| \leq 2|x(t)| \leq 2M\,,$$

and $y(t)$ is bounded also.
Part (b): This is linear, not time invariant, memoryless, causal and stable.
Part (c): This system is linear, not memoryless, not time invariant since if we consider a time shifted input of $x(t-t_0)$ we get an output of

$$\int_{-\infty}^{2t} x(\tau - t_0)d\tau = \int_{-\infty}^{2t-t_0} x(v)dv\,,$$

when we take $v = \tau - t$. If we compare this to $y(t-t_0)$ we would get

$$\int_{-\infty}^{2(t-t_0)} x(\tau)d\tau = \int_{-\infty}^{2t-2t_0} x(\tau)d\tau\,,$$

which is not the same. This system is not causal since the output $y(1)$ depends on integrating $x(t)$ up to the time 2 which requires knowing $x(t)$ for $t > 1$. This system is not stable since the lower limit of the integral is $-\infty$.

Part (d): For the system

$$y(t) = \begin{cases} 0 & t < 0 \\ x(t) + x(t-2) & t \geq 0 \end{cases}$$

We see that this system is linear, not memoryless, is causal, and is stable. To see if it is time invariant consider what the input $x(t - t_0)$ is mapped to. We see that it is mapped to the function $y(t; t_0)$ given by

$$y(t; t_0) = \begin{cases} 0 & t < 0 \\ x(t - t_0) + x(t - 2 - t_0) & t \geq 0 \end{cases},$$

while the time shifted output $y(t - t_0)$ is given by

$$\begin{aligned} y(t - t_0) &= \begin{cases} 0 & t - t_0 < 0 \\ x(t - t_0) + x(t - t_0 - 2) & t - t_0 \geq 0 \end{cases} \\ &= \begin{cases} 0 & t < t_0 \\ x(t - t_0) + x(t - t_0 - 2) & t \geq t_0 \end{cases}. \end{aligned}$$

Since these are not the same this system is not time invariant.

Part (e): For this system we can easily see that it is not memoryless, it is causal, and stable. I claim that this system is non-linear. To see this first note that the output to the input $x_1(t) = 1$ is $y_1(t) = 2$ since $x_1(t)$ is everywhere positive. Next note that the output to the input $x_2(t) = -2$ is $y_2(t) = 0$ since $x_2(t)$ is everywhere negative. If the system was linear then the output to the combined system $x_1(t) + x_2(t)$ should be $y_1(t) + y_2(t) = 2$. The input to this combined system $x_1(t) + x_2(t) = -1$, however, maps to the zero function, showing that this system is not linear. To see if this system is time invariant consider the output to the input $x(t - t_0)$, which is

$$y(t; t_0) = \begin{cases} 0 & x(t - t_0) < 0 \\ x(t - t_0) + x(t - t_0 - 2) & x(t - t_0) \geq 0 \end{cases},$$

which is equal to $y(t)$ shifted by t_0 to the right showing that this system is time invariant.

Part (f): It is easy to see that this system is not memoryless, is linear and is stable. The system is not causal since the value of $y(-3)$ depends on the value of $x(-1)$ which indexes a point in time $t = -1$ greater than -3. To see if it is time invariant consider the output of $x(t - t_0)$, which is $x(\frac{t}{3} - t_0)$.

This is to be compare to the shifted output which is $x(\frac{t-t_0}{3})$. Since these two are not equal the system is *not* time invariant.
Part (g): We can see that this system is linear, not necessarily bounded as the input $x(t) = \sqrt{t}$ demonstrates, is memoryless, is causal, and its time invariant.

Problem 1.28 (some properties of discrete systems)

Part (a): This system is not memoryless, not causal, is linear and is stable. To see if it is time invariant consider the output of the input $x[n-n_0]$, which is given by

$$x[-n-n_0]\,.$$

This is to be compared to the time-shifted output which is given by

$$x[-(n-n_0)] = x[-n+n_0]\,,$$

since these two expression are not equal this system is not time invariant.
Part (b): This system is not memoryless, is time invariant, is linear, is causal and is stable. To show that this system is time invariant consider the output to the input $x[n-n_0]$, which would be

$$x[n-2-n_0] - 2x[n-8-n_0]\,,$$

while the time shifted output is

$$x[n-n_0-2] - 2x[n-n_0-8]\,,$$

which is the same showing that they system is time invariant.
Part (c): This system is memoryless, it is not time invariant, it is linear, it is causal, but not stable since

$$|y[n]| \leq Cn\,,$$

grows linearly as $n \to \infty$.
Part (d): For the system $y[n] = \mathcal{E}v\{x[n-1]\} = \frac{1}{2}(x[n-1] + x[-n-1])$, we see that this system is not memoryless, is linear, is stable, and is not causal. To see if it is time invariant consider the output to $x[n-n_0]$, where we get

$$\frac{1}{2}(x[n-1-n_0] + x[-n-1-n_0])\,.$$

The time shifted output $y[n-n_0]$ is given by

$$\frac{1}{2}(x[n-n_0-1]+x[-(n-n_0)-1]) = \frac{1}{2}(x[n-n_0-1]+x[-n+n_0-1])\,.$$

Since these two expressions are not the same we conclude that the system is not time invariant.

Part (e): For the system

$$y[n] = \begin{cases} x[n] & n \geq 1 \\ 0 & n = 0 \\ x[n+1] & n \leq 1 \end{cases},$$

this system is linear, not memoryless since when $n \leq -1$ the output $y[n]$ depends on $x[n+1]$ which is in the future. This system is not time invariant since the input $\delta[n]$ becomes the output 0, while the input $\delta[n-1]$ gives the output

$$S\{\delta[n-1]\} = \begin{cases} 0 & n \geq 1 \\ 1 & n = 1 \\ 0 & n = 0 \\ 1 & n = -1 \\ 0 & n \leq -2 \end{cases},$$

which is not equal to the the zero function (the output of $\delta[n]$ shifted by one). This system is not causal and it is stable.

Part (f): For the function

$$y[n] = \begin{cases} x[n] & n \geq 1 \\ 0 & n = 0 \\ x[n] & n \leq 1 \end{cases}.$$

This system is linear, memoryless, causal, and stable. To see if it is time invariant lets consider the time-shifted input $x[n] = \delta[n-1]$. The output of this system to this signal is

$$S\{\delta[n-1]\} = \begin{cases} 0 & n \geq 2 \\ 1 & n = 1 \\ 0 & n = 0 \\ 0 & n \leq -1 \end{cases},$$

while the time shifted result of an input $x[n] = \delta[n]$ would be

$$S\{\delta[n]\}[n-1] = \begin{cases} \delta[n] & n-1 \geq 1 \\ 0 & n-1 = 0 \\ \delta[n] & n-1 \leq -1 \end{cases} = \begin{cases} 0 & n \geq 2 \\ 0 & n = 1 \\ 1 & n = 0 \\ 0 & n \leq -1 \end{cases},$$

which are not the same functions showing that the system is not time invariant.

Part (g): For the system $y[n] = x[4n+1]$ this system is linear, not memoryless, not causal, and stable. This system is also not time invariant. To show this later fact let the input to the system be $x[n] = \delta[n-1]$. Then the output is

$$S\{\delta[n-1]\} = \delta[4n+1-1] = \delta[4n] = \begin{cases} 0 & n \neq 0 \\ 1 & n = 0 \end{cases}.$$

While the shift of the output to an input of $\delta[n]$ would be

$$S\{\delta[n]\}[n-1] = \delta[4(n-1)+1] = \delta[4n-4+1] = \delta[4n-3] = 0\,,$$

for all n. Since these two are not equal the given system is not time invariant.

Problem 1.29 (linearity and homogeneity)

Part (a): I claim that the system $y[n] = \mathcal{R}e\{e^{j\frac{\pi n}{4}}x[n]\}$ is additive. To show this we want to show that for any two inputs $x_1[n]$ and $x_2[n]$ we have

$$\mathcal{R}e\{e^{j\frac{\pi}{4}n}x_1[n]\} + \mathcal{R}e\{e^{j\frac{\pi}{4}n}x_2[n]\} = \mathcal{R}e\{e^{j\frac{\pi}{4}n}(x_1[n]+x_2[n])\}\,. \quad (4)$$

To show this define $x_3[n] \equiv x_1[n] + x_2[n]$ and consider

$$\begin{aligned} \mathcal{R}e\{e^{j\frac{\pi}{4}n}x_3[n]\} &= \mathcal{R}e\{e^{j\frac{\pi}{4}n}(\mathcal{R}e\{x_3[n]\} + j\mathcal{I}m\{x_3[n]\})\} \\ &= \mathcal{R}e\{(\cos(\frac{\pi}{4}n) + j\sin(\frac{\pi}{4}n)(\mathcal{R}e\{x_3[n]\} + j\mathcal{I}m\{x_3[n]\})\} \\ &= \mathcal{R}e\{\cos(\frac{\pi}{4}n)\mathcal{R}e\{x_3[n]\} - \sin(\frac{\pi}{4}n)\mathcal{I}m\{x_3[n]\} \\ &+ j\sin(\frac{\pi}{4}n)\mathcal{R}e\{x_3[n]\} + j\cos(\frac{\pi}{4}n)\mathcal{I}m\{x_3[n]\}\} \\ &= \cos(\frac{\pi}{4}n)\mathcal{R}e\{x_3[n]\} - \sin(\frac{\pi}{4}n)\mathcal{I}m\{x_3[n]\} \end{aligned}$$

$$
\begin{aligned}
&= \cos(\frac{\pi}{4}n)\mathcal{R}e\{x_1[n]\} - \sin(\frac{\pi}{4}n)\mathcal{I}m\{x_1[n]\} \\
&+ \cos(\frac{\pi}{4}n)\mathcal{R}e\{x_2[n]\} - \sin(\frac{\pi}{4}n)\mathcal{I}m\{x_2[n]\} \\
&= \mathcal{R}e\{e^{j\frac{\pi}{4}n}x_1[n]\} + \mathcal{R}e\{e^{j\frac{\pi}{4}n}x_2[n]\}\,,
\end{aligned}
$$

therefore this system is additive.

Part (b-i): This system can be shown to be homogeneous but it is not additive as the input $x(t) = x_1(t) + x_2(t) = at + bt$ would show.

Part (b-ii): This system can be shown to be homogeneous but it is not additive as the input $x[n] = x_1[n] + x_2[n]$ with $x_1[n] = n + \frac{1}{2}$ and $x_2[n] = n - \frac{1}{2}$ would show.

Problem 1.30 (some system inverses)

Part (a): An inverse for this system is given by $y(t) = x(t+4)$ as can be checked by functional composition.

Part (b): This system is not invertible due to the many-to-one nature of the cosign function. For example, let $x_1(t) = \pi$ and $x_2(t) = -\pi$ be constant functions, then $y_1(t) = -1$ and $y_2(t) = -1$ but $x_1(t) \neq x_2(t)$. This problem is mitigated if the range of $x(t)$ is restricted to be taken from a principal domain of $\cos(\cdot)$ say $0 \leq x(t) \leq \pi$. Then the system is invertible with an inverse given by $y(t) = \arccos(x(t))$.

Part (c): This system is not invertible since the inputs $x[n] = A\delta[n]$ for each value of A all map to the zero function.

Part (d): This system is invertible with the inverse given by the derivative system i.e. $y(t) = \frac{dx}{dt}$.

Part (e): This system is invertible since the inverse to this system can be written as

$$
y[n] = \begin{cases} x[n+1] & n \geq 0 \\ x[n] & n \leq -1 \end{cases} .
$$

Part (f): This is not invertible since $x_1[n] = A\delta[n]$ and $x_2[n] = B\delta[n+1]$ both map to the zero function.

Part (g): This function is invertible with an inverse given by $x[1-n]$ since under the original system and the proposed inverse an input $x[n]$ will be mapped to

$$
x[n] \to x[1-n] \to x[1-(1-n)] = x[n]\,.
$$

Part (h): For the given system we can solve for the function $x(t)$ in terms of $y(t)$ explicitly thereby computing the system inverse. From the given system we see that $y(t)$ is given by

$$y(t) = e^{-t} \int_{-\infty}^{t} e^{\tau} x(\tau) d\tau \,,$$

or

$$\int_{-\infty}^{t} e^{\tau} x(\tau) d\tau = y(t) e^{t} \,.$$

Taking the derivative of this expression and multiplying both sides by e^{-t} we get

$$x(t) = e^{-t} \frac{d}{dt}(y(t)e^{t}) = y(t) + \frac{dy}{dt} \,,$$

as the explicit inverse to the given system.

Part (i): For this system an inverse can be found by considering

$$\begin{aligned} y[n+1] - \frac{1}{2} y[n] &= \sum_{k=-\infty}^{n+1} \left(\frac{1}{2}\right)^{n+1-k} x[k] - \frac{1}{2} \sum_{k=-\infty}^{n} \left(\frac{1}{2}\right)^{n-k} x[k] \\ &= x[n+1] + \sum_{k=-\infty}^{n} \left(\frac{1}{2}\right)^{n+1-k} x[k] - \sum_{k=-\infty}^{n} \left(\frac{1}{2}\right)^{n+1-k} x[k] \\ &= x[n+1] \,. \end{aligned}$$

Thus the system $y[n] \rightarrow x[n]$ given by

$$x[n] = y[n] - \frac{1}{2} y[n-1] \,,$$

is the inverse of the original system.

Part (j): For this system $y(t)$ is given by the integral $y(t) = \int_{-\infty}^{t} x(\tau) d\tau + C$, where C is an arbitrary constant. Since C can be anything this system is not invertible.

Part (k): This system is not invertible since the function $x[n] = A\delta[n]$ is mapped to the zero function irrespective of what the value of A is.

Part (l): This system is invertible with an inverse given by $y(t) = x(t/2)$.

Part (m): This system is not invertible since the domain of $y[n]$ is different than that of $x[n]$. For example this suggested mapping requires these mappings (among others)

$$\begin{aligned} y[-2] &= x[-4] \\ y[-1] &= x[-2] \\ y[+1] &= x[+2] \\ y[+2] &= x[+4] \,, \end{aligned}$$

so we see that the odd values: $x[1], x[3], \cdots$ and $x[-1], x[-3], \cdots$ of $x[n]$ are not observable under this mapping. Thus this system cannot be invertible.
Part (n): This system is invertible because we can determine its inverse as $x[n] = y[2n]$.

Problem 1.31 (using LTI properties to determine system output)

Part (a): We begin by writing the signal $x_2(t)$ in terms of shifts and multiples of $x_1(t)$. We see that

$$x_2(t) = x_1(t) - x_1(t-2)\,.$$

From which the output to this input (since our system is LTI) is given by

$$y_2(t) = y_1(t) - y_1(t-2)\,.$$

Part (b): Again, we begin by writing the signal $x_3(t)$ in terms of shifts and multiples of $x_1(t)$. We see that

$$x_3(t) = x_1(t) + x_1(t+1)\,.$$

From which we see that the output to this input (since our system is LTI) is given by

$$y_3(t) = y_1(t) + y_1(t+1)\,.$$

Problem 1.32 (periodicity of time scaling)

Part (1): This statement is true and the fundamental period of $y_1(t)$ would be the fundamental period of $x(t)$ divided by two.
Part (2): Since $x(t) = y_1(t/2)$ and we are told that $y_1(t)$ is periodic with period T say then we see that $x(t)$ would be periodic with a period of $2T$.
Part (3): This is true and $y_2(t)$ is periodic with a period of $2T$.
Part (4): This is true and $x(t)$ will be periodic with a period of $\frac{T}{2}$.

Problem 1.33 (discrete periodicity of time scaling)

Part (1): This statement is true. The easy case to consider is when the period of $x[n]$ is N with N is an even natural number then the period of $y_1[n]$ is $N/2$. If however N is an odd natural number the period of $y_1[n]$

could not be $N/2$ since this is not a natural number. We could, however, take the period of $y_1[n]$ to be N and obtain a periodic signal.
Part (2): This statement is false. The fact that $y_1[n]$ is periodic (and $y_1[n] = x[2n]$) means that the even components of $x[n]$ are periodic. This however does not tell us what the odd components of $x[n]$ are doing. To find a counter example where $y_1[n]$ is periodic and $x[n]$ is not periodic create a signal $x[n]$ who's odd components are not periodic.
Part (3): This is true. If $x[n]$ is periodic with period N_0 then $y_2[n]$ will be periodic with period $2N_0$.
Part (4): This is true. The signal $y_2[n]$ is the signal $x[n]$, placed on the even values of n and with zeros on the odd values of n. Thus if $y_2[n]$ is periodic then $x[n]$ has to be periodic. If N_0 is the period of $y_2[n]$, the period of $x[n]$ is $N_0/2$ since $y_2[n]$ is $x[n]$ on only the even values of n.

Problem 1.34 (some properties of even/odd functions)

Part (b): Observe that $x_1[-n]x_2[-n] = -x_1[n]x_2[n]$, showing that $x_1[n]x_2[n]$ is an odd function.
Part (c): Note that

$$\begin{aligned}\mathcal{E}v\{x[n]\} + \mathcal{O}d\{x[n]\} &= \frac{1}{2}(x[n] + x[-n]) + \frac{1}{2}(x[n] - x[-n]) \\ &= x[n]\,.\end{aligned}$$

From which we see that when we take $x_e[n] \equiv \mathcal{E}v\{x[n]\}$ and $x_o[n] \equiv \mathcal{O}d\{x[n]\}$ that

$$\begin{aligned}x[n]^2 &= (x_e[n] + x_0[n])^2 \\ &= x_e[n]^2 + 2x_e[n]x_o[n] + x_o[n]^2\,.\end{aligned}$$

When we sum the left hand side of this expression from $n = -\infty$ to $n = +\infty$ because of Part (a) and Part (b) of this problem the middle term above vanishes and we get

$$\sum_{n=-\infty}^{\infty} x[n]^2 = \sum_{n=-\infty}^{\infty} x_e[n]^2 + \sum_{n=-\infty}^{\infty} x_o[n]^2\,,$$

as we were to show.
Part (d): All of the manipulations for the discrete case carry over to the continuous case.

Problem 1.35 (the period of a discrete time signal)

The given discrete system will have a fundamental period N_0 that has to satisfy

$$N_0\left(\frac{m2\pi}{N}\right) = 2\pi k\,,$$

or solving for N_0

$$N_0 = \frac{kN}{m}\,,$$

for some integer k. Consider the expression $\frac{N}{m}$ for various value of N and m. To have a *fundamental* period we want to make N_0 as small as possible which requires that we look for a k that is as small as possible. This in tern requires that we factor any common multiples out of m and N. The largest multiple we could factor out of N and m is defined as $\gcd(N, m)$. If we factor this number of of m and N as

$$\begin{aligned} m &= \gcd(N, m)\hat{m} \\ N &= \gcd(N, m)\hat{N}\,, \end{aligned}$$

where we have introduced the remaining factors of m and N as $\hat{m}$ and $\hat{N}$ respectively, then we get

$$N_0 = \frac{k\hat{N}}{\hat{m}}\,.$$

If we pick $k = \hat{m}$ so that N_0 is an integer then we get

$$N_0 = \hat{N} = \frac{N}{\gcd(m, N)}\,,$$

as we were to show.

Problem 1.36 (the period of a continuous time signal)

Part (a): For $x[n]$ to be periodic requires that there exists a N and k both integer such that

$$\omega_0 T N = 2\pi k\,, \tag{5}$$

or since $T_0 = \frac{2\pi}{\omega_0}$ this requires that

$$\frac{T}{T_0} = \frac{k}{N}\,.$$

From which we see that since the expression $\frac{k}{N}$ is a rational number a requirement for $x[n]$ to be periodic is that $\frac{T}{T_0}$ must be rational.

Part (b): From Equation 5 we see that the fundamental period N of the discrete signal $x[n]$ can be expressed as

$$N = \left(\frac{2\pi}{\omega_0}\right)\left(\frac{1}{T}\right)k = \frac{T_0}{T}k = \frac{q}{p}k\,.$$

Now let p and q be expressed in terms of factors as

$$\begin{aligned} p &= \gcd(p,q)\hat{p} \\ q &= \gcd(p,q)\hat{q}\,. \end{aligned}$$

Then N expressed above becomes

$$N = \frac{\hat{q}}{\hat{p}}k\,.$$

To have N be a fundamental period pick $k = \hat{p}$ so that

$$N = \hat{q} = \frac{q}{\gcd(p,q)}\,,$$

is the discrete fundamental period of $x[n]$. The fundamental frequency of the signal $x[n]$ is given by

$$\begin{aligned} \frac{2\pi}{N} &= \frac{2\pi}{q}\gcd(p,q) = \frac{2\pi}{p}\left(\frac{p}{q}\right)\gcd(p,q) \\ &= \frac{2\pi}{p}\left(\frac{T}{T_0}\right)\gcd(p,q) \\ &= \omega_0\left(\frac{T}{p}\right)\gcd(p,q)\,. \end{aligned}$$

Part (c): The time elapsed to get the first period of $x[n]$ is determined by sampling the function $x(t)$, N times with a sampling period of T for a total time of NT. Writing this expression in terms of the fundamental period of $x(t)$ or T_0 we have

$$\begin{aligned} NT &= \hat{q}T \\ &= \left(\frac{q}{\gcd(p,q)}\right)T\left(\frac{p}{q}T_0\right) \\ &= \frac{p}{\gcd(p,q)}T_0\,, \end{aligned}$$

or $\frac{p}{\gcd(p,q)}$ periods of $x(t)$.

Problem 1.37 (some properties of the cross-correlation function)

Part (a): From the definition of the cross-correlation function $\phi_{xy}(t)$

$$\phi_{xy}(t) = \int_{-\infty}^{\infty} x(t+\tau)y(\tau)d\tau\,, \tag{6}$$

we see that

$$\phi_{yx}(t) = \int_{-\infty}^{\infty} y(t+\tau)x(\tau)d\tau = \int_{-\infty}^{\infty} y(v)x(v-t)dv = \int_{-\infty}^{\infty} x(-t+v)y(v)dv = \phi_{xy}(-t)\,,$$

when we make the substitution of $v = t + \tau$.

Part (b): From Part (a) of this problem $\phi_{xx}(t) = \phi_{xx}(-t)$ when we switch the x and x arguments. Thus the odd part or $\phi_{xx} - \phi xx(-t)$ is zero.

Part (c): From the definition of $\phi_{xy}(t)$ in Equation 6 since $y(\tau) = x(\tau + T)$ we see that $\phi_{xy}(t)$ is given by

$$\phi_{xy}(t) = \int_{-\infty}^{\infty} x(t+\tau)x(\tau+T)d\tau\,.$$

Let $v = \tau + T$ then $\phi_{xy}(t)$ above becomes

$$\phi_{xy}(t) = \int_{-\infty}^{\infty} x(t-T+v)x(v)dv = \phi_{xx}(t-T)\,.$$

Finally, we see $\phi_{yy}(t)$ becomes

$$\begin{aligned}
\phi_{yy}(t) &= \int_{-\infty}^{\infty} y(t+\tau)y(\tau)d\tau \\
&= \int_{-\infty}^{\infty} x(t+\tau+T)x(\tau+T)d\tau \\
&= \int_{-\infty}^{\infty} x(t+T+v-T)x(v)dv = \int_{-\infty}^{\infty} x(t+v)x(v)dv \\
&= \phi_{xx}(t)\,.
\end{aligned}$$

Problem 1.38 (versions of the delta function)

Part (a): Recalling the definition of $u_\Delta(t)$ provided in the book and given by

$$u_\Delta(t) = \begin{cases} 0 & t < 0 \\ \left(\frac{1-0}{\Delta}\right)t & 0 < t < \Delta \\ 1 & \Delta < t \end{cases}\,.$$

We see that the derivative of this expression is given by

$$\delta_\Delta(t) = \frac{d}{dt}u_\Delta(t) = \begin{cases} 0 & t < 0 \\ \frac{1}{\Delta} & 0 < t < \Delta \\ 0 & \Delta < t \end{cases} .$$

If we evaluate the above when $t \to 2t$ we see that

$$\begin{aligned} \delta_\Delta(2t) &= \begin{cases} 0 & 2t < 0 \\ \frac{1}{\Delta} & 0 < 2t < \Delta \\ 0 & \Delta < 2t \end{cases} = \begin{cases} 0 & t < 0 \\ \frac{1}{\Delta} & 0 < t < \frac{\Delta}{2} \\ 0 & \Delta < t \end{cases} \\ &= \frac{1}{2}\begin{cases} 0 & t < 0 \\ \frac{1}{(\frac{\Delta}{2})} & 0 < t < \frac{\Delta}{2} \\ 0 & \frac{\Delta}{2} < t \end{cases} = \frac{1}{2}\delta_{\frac{\Delta}{2}}(t) . \end{aligned}$$

Thus we see that when we take the limit $\Delta \to 0$ we have

$$\lim_{\Delta\to 0} \delta_\Delta(2t) = \lim_{\Delta\to 0} \frac{1}{2}\delta_{\frac{\Delta}{2}}(t) = \frac{1}{2}\delta(t) ,$$

as we were to show.

Part (b): For this part to this problem by using the given expressions for $r^i_\Delta(t)$ we will derive $u^i_\Delta(t)$ from its definition of an integral as

$$u^i_\Delta(t) = \int_{-\infty}^{t} r^i_\Delta(\tau) d\tau . \tag{7}$$

Using the derived expression for $u^i_\Delta(t)$ we will then evaluate $\lim_{\Delta\to 0} u^i_\Delta(t)$ to show that this equals the unit step function $u(t)$. For notational simplicity, in each part of this problem we will drop the i superscript notation and write $r(\tau)$ for $r^i(\tau)$ where the i is understood.

Part (1): For the given expression for $r_\Delta(t)$ we find $u_\Delta(t)$ given by Equation 7 as

$$\begin{aligned} u_\Delta(t) &= \begin{cases} 0 & t < -\frac{\Delta}{2} \\ \int_{-\frac{\Delta}{2}}^{t} r_\Delta(\tau)d\tau & -\frac{\Delta}{2} < t < +\frac{\Delta}{2} \\ \int_{-\frac{\Delta}{2}}^{+\frac{\Delta}{2}} r_\Delta(\tau)d\tau & t > \frac{\Delta}{2} \end{cases} \\ &= \begin{cases} 0 & t < -\frac{\Delta}{2} \\ \int_{-\frac{\Delta}{2}}^{t} \frac{1}{\Delta}d\tau & -\frac{\Delta}{2} < t < +\frac{\Delta}{2} \\ \int_{-\frac{\Delta}{2}}^{+\frac{\Delta}{2}} \frac{1}{\Delta}d\tau & t > \frac{\Delta}{2} \end{cases} \end{aligned}$$

$$= \begin{cases} 0 & t < -\frac{\Delta}{2} \\ \frac{1}{\Delta}(t + \frac{\Delta}{2}) & -\frac{\Delta}{2} < t < +\frac{\Delta}{2} \\ 1 & t > \frac{\Delta}{2} \end{cases} .$$

Using this expression we see that the limit of $u_\Delta(t)$ as Δ goes to 0 is given by

$$\lim_{\Delta \to 0} u_\Delta(t) = \begin{cases} 0 & t < 0 \\ 1 & 0 < t \end{cases} = u(t) .$$

Part (2): For the given expression for $r_\Delta(t)$ we find $u_\Delta(t)$ given by Equation 7 as

$$\begin{aligned} u_\Delta(t) &= \begin{cases} 0 & t < \Delta \\ \int_\Delta^t \frac{1}{\Delta} d\tau & \Delta < t < 2\Delta \\ 1 & t > 2\Delta \end{cases} \\ &= \begin{cases} 0 & t < \Delta \\ \frac{t-\Delta}{\Delta} & \Delta < t < 2\Delta \\ 1 & t > 2\Delta \end{cases} . \end{aligned}$$

Using this expression we see that the limit of $u_\Delta(t)$ as Δ goes to 0 is given by

$$\lim_{\Delta \to 0} u_\Delta(t) = \begin{cases} 0 & t < 0 \\ 1 & 0 < t \end{cases} = u(t) .$$

Part (3): For the given expression for $r_\Delta(t)$ we find $u_\Delta(t)$ given by Equation 7 as

$$u_\Delta(t) = \begin{cases} 0 & t < -\Delta \\ \int_{-\Delta}^t r_\Delta(\tau) d\tau & -\Delta < t < \Delta \\ \int_{-\Delta}^\Delta r_\Delta(\tau) d\tau & t > \Delta \end{cases} .$$

To evaluate the second integral above we need to split the region of integration into two parts as follows

$$\begin{aligned} \int_{-\Delta}^t r_\Delta(\tau) d\tau &= \begin{cases} \int_{-\Delta}^t \frac{1}{\Delta^2}(\tau + \Delta) d\tau & t < 0 \\ \int_{-\Delta}^0 \frac{1}{\Delta^2}(\tau + \Delta) d\tau - \int_0^t \frac{1}{\Delta^2}(\tau - \Delta) d\tau & t > 0 \end{cases} \\ &= \begin{cases} \frac{1}{2\Delta^2}(t + \Delta)^2 & t < 0 \\ \frac{1}{2} - \int_0^t \frac{1}{\Delta^2}(\tau - \Delta) d\tau & t > 0 \end{cases} \\ &= \begin{cases} \frac{1}{2\Delta^2}(t + \Delta)^2 & t < 0 \\ \frac{1}{2} - \frac{1}{\Delta^2}(\frac{t^2}{2} - \Delta t) & t > 0 \end{cases} . \end{aligned}$$

When we put this back into the total expression for $u_\Delta(t)$ we see that

$$u_\Delta(t) = \begin{cases} 0 & t < -\Delta \\ \frac{1}{2\Delta^2}(t-\Delta)^2 & -\Delta < t < 0 \\ \frac{1}{2} + \frac{1}{\Delta^2}(\frac{t^2}{2} - \Delta t) & 0 < t < \Delta \\ 1 & t > \Delta \end{cases} .$$

Using this expression we see that the limit as Δ goes to 0 is given by

$$\lim_{\Delta \to 0} u_\Delta(t) = \begin{cases} 0 & t < 0 \\ 1 & 0 < t \end{cases} = u(t) .$$

Part (4): For this expression for $r_\Delta(t)$ we find $u_\Delta(t)$ given by Equation 7 in the following steps

$$\begin{aligned} u_\Delta(t) &= \begin{cases} 0 & t < -\Delta \\ \int_{-\Delta}^t r_\Delta(\tau) d\tau & -\Delta < t < 0 \\ \int_{-\Delta}^0 r_\Delta(\tau) d\tau + \int_0^t r_\Delta(\tau) d\tau & 0 < t < \Delta \\ \int_{-\Delta}^0 r_\Delta(\tau) d\tau + \int_0^\Delta r_\Delta(\tau) d\tau & \Delta < t \end{cases} \\ &= \begin{cases} 0 & t < -\Delta \\ \int_{-\Delta}^t \left(-\frac{\tau}{\Delta^2}\right) d\tau & -\Delta < t < 0 \\ \frac{-\tau^2}{2\Delta^2}\Big|_{-\Delta}^0 + \int_0^t \left(\frac{\tau}{\Delta^2}\right) d\tau & 0 < t < \Delta \\ \frac{1}{2} + \frac{1}{2} & t > \Delta \end{cases} \\ &= \begin{cases} 0 & t < -\Delta \\ \frac{1}{2\Delta^2}(\Delta^2 - t^2) & -\Delta < t < 0 \\ \frac{1}{2} + \frac{t^2}{2\Delta^2} & 0 < t < \Delta \\ 1 & t > \Delta \end{cases} , \end{aligned}$$

when we simplify some. Using this expression we see that the limit as Δ goes to 0 is given by

$$\lim_{\Delta \to 0} u_\Delta(t) = \begin{cases} 0 & t < 0 \\ 1 & 0 < t \end{cases} = u(t) .$$

Part (5): For this expression for $r_\Delta(t)$ we find $u_\Delta(t)$ given by Equation 7 in the following steps

$$u_\Delta(t) = \begin{cases} 0 & t < -\Delta \\ \int_{-\Delta}^t -\frac{1}{\Delta} d\tau & -\Delta < t < 0 \\ \int_{-\Delta}^0 -\frac{1}{\Delta} d\tau + \int_0^t \frac{2}{\Delta} d\tau & 0 < t < \Delta \\ -\frac{1}{\Delta}(0+\Delta) + \frac{2}{\Delta}(\Delta - 0) & t > \Delta \end{cases}$$

$$= \begin{cases} 0 & t < -\Delta \\ -\frac{1}{\Delta}(t+\Delta) & -\Delta < t < 0 \\ -1 + \frac{2}{\Delta} t & 0 < t < \Delta \\ 1 & t > \Delta \end{cases},$$

when we simplify some. Using this expression we see that the limit as Δ goes to 0 is given by

$$\lim_{\Delta \to 0} u_\Delta(t) = \begin{cases} 0 & t < 0 \\ 1 & 0 < t \end{cases} = u(t)\,.$$

Part (6): For this expression for $r_\Delta(t)$ we find $u_\Delta(t)$ given by Equation 7 in the following steps

$$\begin{aligned} u_\Delta(t) &= \begin{cases} \int_{-\infty}^{t} \frac{1}{2\Delta} e^{\tau/\Delta} d\tau & t < 0 \\ \int_{-\infty}^{0} \frac{1}{2\Delta} e^{\tau/\Delta} d\tau + \int_{0}^{t} \frac{1}{2\Delta} e^{-\tau/\Delta} d\tau & t > 0 \end{cases} \\ &= \begin{cases} \frac{1}{2} e^{t/\Delta} & t < 0 \\ 1 - \frac{1}{2} e^{-t/\Delta} & 0 < t \end{cases}, \end{aligned}$$

when we simplify some. Using this expression we see that the limit as Δ goes to 0 is given by

$$\lim_{\Delta \to 0} u_\Delta(t) = \begin{cases} 0 & t < 0 \\ 1 & 0 < t \end{cases} = u(t)\,.$$

Problem 1.39 (the product of singular functions)

Recall the definition of $u_\Delta(t)$ of

$$u_\Delta(t) = \begin{cases} 0 & t < 0 \\ \frac{1}{\Delta} t & 0 < t < \Delta \\ 1 & t > \Delta \end{cases},$$

we have that $u_\Delta(0) = 0$ so using $u_\Delta(t)\delta(t) = u_\Delta(0)\delta(t)$ we see that

$$\lim_{\Delta \to 0} u_\Delta(t)\delta(t) = \lim_{\Delta \to 0} u_\Delta(0)\delta(t) = \lim_{\Delta \to 0} 0 = 0\,.$$

Next consider the product of $u_\Delta(t)\delta_\Delta(t)$, where $\delta_\Delta(t)$ is given by

$$\delta_\Delta(t) = \frac{d}{dt} u_\Delta(t) = \begin{cases} 0 & t < 0 \\ \frac{1}{\Delta} & 0 < t < \Delta \\ 0 & t > \Delta \end{cases},$$

so that that product above is given by

$$u_\Delta(t)\delta_\Delta(t) = \begin{cases} 0 & t < 0 \\ \frac{1}{\Delta^2}t & 0 < t < \Delta \\ 0 & t > \Delta \end{cases} .$$

It is this expression we will study. If we define this expression as the function $\tilde{r}_\Delta(t)$ as in Problem 1.38 we see that the integral of $\tilde{r}_\Delta(t)$ is given by

$$\begin{aligned} \int_{-\infty}^{t} u_\Delta(\tau)\delta_\Delta(\tau)d\tau &= \begin{cases} 0 & t < 0 \\ \frac{1}{2\Delta^2}t^2 & 0 < t < \Delta \\ \frac{1}{2} & t > 0 \end{cases} \\ &= \frac{1}{2}\begin{cases} 0 & t < 0 \\ \frac{1}{\Delta^2}t^2 & 0 < t < \Delta \\ 1 & t > 0 \end{cases} . \end{aligned}$$

This later expression is $\frac{1}{2}$ times a function that limits to the unit step function $u(t)$ as $\Delta \to 0$. Thus we have shown that

$$\lim_{\Delta \to 0} u_\Delta(t)\delta_\Delta(t) = \frac{1}{2}u(t) .$$

On taking the derivative of this expression with respect to t we see that

$$\lim_{\Delta \to 0} \frac{d}{dt}u_\Delta(t)\delta_\Delta(t) = \frac{1}{2}\delta(t) ,$$

the desired expression.

Problem 1.40 (system additively and homogeneity)

Part (a): If our system is additive since the zero function $x_0[n] \equiv 0$, has the property that $x_0[n] = x_0[n] + x_0[n]$ we see that if $y_0[n]$ is the system output to the input of $x_0[n]$ we see that $y_0[n]$ must satisfy

$$y_0[n] = y_0[n] + y_0[n] .$$

Solving this equation for $y_0[n]$ gives $y_0[n] = 0$. If our system is homogeneous then let $x_2[n] = \alpha x_1[n]$ with $\alpha = 0$. The output of our system to an input of $x_2[n]$ is given by $y_2[n] = \alpha y_1[n]$, since our system is homogeneous where

$y_1[n]$ is the output of the system to the input $x_1[n]$. This later expression is zero since $\alpha = 0$.
Part (b): An example like this is given by $y[n] = x[n]^2$.
Part (c): No, since the system may introduce a phase shift to the output. For example, if the system is given by $y[n] = x[n-2]$ then an input given by

$$x[n] = \begin{cases} 1 & -2 \leq n \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

would have $x[n]$ zero between the times $0 \leq n \leq 2$ but the output to this input would be

$$y[n] = \begin{cases} 1 & 0 \leq n \leq 2 \\ 0 & \text{otherwise} \end{cases},$$

in contradiction to the above.

Problem 1.41 (time invariant systems)

Part (a): If $g[n] = 1$ then $y[n] = 2x[n]$ and this system is time invariant.
Part (b): If $g[n] = n$ then $g[n] + g[n-1] = n + n - 1 = 2n - 1$ so this system becomes

$$y[n] = (2n-1)x[n] .$$

The output of the time shifted input $x[n - n_0]$ is $(2n-1)x[n-n_0]$ while the time shifted output to the input $x[n]$ is $y[n-n_0] = (2(n-n_0)-1)x[n-n_0]$. Since these two expressions are not the same the system is not time invariant.
Part (c): If $g[n] = 1 + (-1)^n$ then

$$g[n] + g[n-1] = 1 + (-1)^n + 1 + (-1)^{n-1} = 2 + (-1)^{n-1}(-1+1) = 2 .$$

From this and by Part (a) above this would be a time invariant system.

Problem 1.42 (series connections)

Part (a): Yes, since a series combination is functional composition, these properties will hold.
Part (b): Not necessarily. If the first and second systems are non-linear inverses of each other the combined system maybe be linear. For example, let $y_1(t) = x(t)^3$ and $y_2(t) = x(t)^{1/3}$, represent the output produced by the two systems then their combined result is the linear system $y(t) = x(t)$.

Part (c): Assume our input signal is $x[n]$ and let $x_1[n]$ be the output from this signal after passing through system one, $x_2[n]$ the output of $x_1[n]$ from system two and finally $x_3[n] = y[n]$ the output from the three combined systems. Now from the given system definitions $x_1[n]$ is given by

$$x_1[n] = \begin{cases} x[n/2] & n \text{ even} \\ 0 & n \text{ odd} \end{cases} .$$

With this $x_2[n]$ is given by

$$\begin{aligned} x_2[n] &= x_1[n] + \frac{1}{2}x_1[n-1] + \frac{1}{4}x_1[n-2] \\ &= \begin{cases} x[n/2] & n \text{ even} \\ 0 & n \text{ odd} \end{cases} \\ &+ \frac{1}{2}\begin{cases} x[(n-1)/2] & n-1 \quad \text{even} \\ 0 & n-1 \quad \text{odd} \end{cases} \\ &+ \frac{1}{4}\begin{cases} x[(n-2)/2] & n-2 \quad \text{even} \\ 0 & n-2 \quad \text{odd} \end{cases} . \end{aligned}$$

When this is fed into system three we get the final output $x_3[n] = y[n]$

$$\begin{aligned} y[n] &= x_2[2n] \\ &= \begin{cases} x[n] & 2n \text{ even} \\ 0 & 2n \text{ odd} \end{cases} \\ &+ \frac{1}{2}\begin{cases} x[(2n-1)/2] & 2n-1 \quad \text{even} \\ 0 & 2n-1 \quad \text{odd} \end{cases} \\ &+ \frac{1}{4}\begin{cases} x[(2n-2)/2] & 2n-2 \quad \text{even} \\ 0 & 2n-2 \quad \text{odd} \end{cases} \\ &= x[n] + \frac{1}{4}x[n-1] . \end{aligned}$$

Thus we see that the combined system is linear and time invariant.

Problem 1.43 (periodic systems)

Part (a): Let the period of $x(t)$ be T. Now if our system is time invariant then the output to $x(t+T)$ must be $y(t+T)$ where $y(t)$ is the output of our system to the input $x(t)$. Thus since $x(t) = x(t+T)$, by taking the system

operation on both sides of this expression gives $y(t) = y(t+T)$ showing that $y(t)$ is periodic with period T as claimed. Nothing that we have done in this problem is specific to continuous systems and can be repeated in the discrete case.
Part (b): Consider the nonlinear system with an output given by $y(t) = \cos(x(t))$. Then this system output is time invariant and the input $x(t) = t$ (which is not periodic) produces the output $y(t) = \cos(t)$ which is periodic of period 2π.

Problem 1.44 (causality in continuous-time systems)

Part (a): The definition of causality means that the output $y(t)$ at any time t depends only on the input at the present time and the past. Thus if the input signal $x(t)$ is such that $x(t) = 0$ for $t < t_0$, then $y(t)$ must be determined from $x(t)$ for $t < t_0$. Because $x(t) = 0$ when $t < t_0$ and because of our system is linear and therefore additive and homogeneous from Problem 1.40 Part (a) the output $y(t)$ must also be zero when $t < t_0$.

To prove the other direction let the statement given in the book be denoted by $\mathcal{C}$. Then to show that causality we can argue as follows. Consider two functions $x_1(t)$ and $x_2(t)$ that are equal for $t < t_0$. Now define $x_3(t) = x_1(t) - x_2(t)$ since $x_1(t) = x_2(t)$ when $t < t_0$ we see that $x_3(t) = 0$ when $t < t_0$. Thus by property $\mathcal{C}$. the output to our system from the input $x_1(t) - x_2(t)$ must be zero for $t < t_0$. Because our system is linear the output to the input $x_1(t) - x_2(t)$ must equal $y_1(t) - y_2(t)$ where $y_1(t)$ and $y_2(t)$ are the system outputs to the inputs $x_1(t)$ and $x_2(t)$ respectively. Thus $y_1(t) = y_2(t)$ when $t < t_0$, and so the output of our system when $t < t_0$ only depends on the input for $t < t_0$. Thus our system is causal.
Part (b): Consider the nonlinear system with an output $y(t) = x(t)x(-t)$. This system will have the property that if the input $x(t) = 0$ for $t < t_0$ then the output $y(t)$ will be zero for $t < t_0$ but this system is not causal when $t < 0$.
Part (c): Consider the system that has an output given by $y(t) = \frac{1}{1+x(t)^2}$. This system is causal but if $x(t) = 0$ the output is $y(t) = 1 \neq 0$.
Part (e): Consider the nonlinear system $y[n] = x[n]^2$. Then the only input that produces the zero output is the zero input. This system is not invertible however since the inputs $x[n] = 1$ and $x[n] = -1$ both map to the same value 1.

Problem 1.45 (correlation with a fixed signal)

Recall the definition of $\phi_{hx}(t)$ from Problem 1.37 where we see that the output from the described system is

$$\phi_{hx}(t) = \int_{-\infty}^{\infty} h(t+\tau)x(\tau)d\tau\,.$$

Part (a): This is certainly a linear system. To see if this system is time invariant consider the output to the time shifted input $x(t-t_0)$ which would be

$$\int_{-\infty}^{\infty} h(t+\tau)x(\tau-t_0)d\tau\,.$$

Let $v = \tau - t_0$ so $\tau = v + t_0$ and $dv = d\tau$ and the above becomes

$$\int_{-\infty}^{\infty} h(t+v+t_0)x(v)dv = \phi_{hx}(t+t_0)\,.$$

since this output is not equal to $\phi_{hx}(t-t_0)$ we can conclude that this system is *not* time invariant. This system is not causal since the output $\phi_{hx}(t)$ depends on integrating the function $x(t)$ for all $-\infty < t < +\infty$.

Part (b): For the output $\phi_{xh}(t)$ since it is given by

$$\phi_{xh}(t) = \int_{-\infty}^{\infty} x(t+\tau)h(\tau)d\tau\,,$$

we see that this system is still linear and cannot be causal since when $\tau > 0$ we are requiring $x(t+\tau)$ which is in the future. To see if this is time invariant consider the output to the input $x(t-t_0)$ which is

$$\int_{-\infty}^{\infty} x(t-t_0+\tau)h(\tau)d\tau\,.$$

Since this is the same as $\phi_{xh}(t-t_0)$ the system is time invariant.

Problem 1.46 (a simple feedback loop)

To solve this problem we will simulate this system one sample at at time using the fact that $y[n] = 0$ for $n < 0$.

Part (a): When $x[n] = \delta[n]$ we find

$$\begin{aligned}
x[-1] &= 0\,, e[-1] = x[-1] - y[-1] = 0\,, y[0] = e[-1] = 0 \\
x[0] &= 1\,, e[0] = x[0] - y[0] = 1\,, y[1] = e[0] = 1 \\
x[1] &= 0\,, e[1] = x[1] - y[1] = 0 - 1 = -1\,, y[2] = e[1] = -1 \\
x[2] &= 0\,, e[2] = 0 - (-1) = +1\,, y[3] = +1 \\
x[3] &= 0\,, e[3] = 0 - 1 = -1\,, y[4] = -1 \\
&\vdots
\end{aligned}$$

Assuming this pattern continues we have

$$y[n] = \begin{cases} 0 & n \le 0 \\ +1 & n > 0 \text{ and odd} \\ -1 & n > 0 \text{ and even} \end{cases}$$

Part (b): When $x[n] = u[n]$ we have

$$\begin{aligned}
x[-1] &= 0\,, e[-1] = x[-1] - y[-1] = 0\,, y[0] = e[-1] = 0 \\
x[0] &= 1\,, e[0] = x[0] - y[0] = 1\,, y[1] = e[0] = 1 \\
x[1] &= 0\,, e[1] = x[1] - y[1] = 1 - 1 = 0\,, y[2] = 0 \\
x[2] &= 1\,, e[2] = 1 - 0 = +1\,, y[3] = +1 \\
x[3] &= 1\,, e[3] = 1 - 1 = 0\,, y[4] = 0 \\
&\vdots
\end{aligned}$$

Assuming this pattern continues we have

$$y[n] = \begin{cases} 0 & n \le 0 \\ 0 & n > 0 \text{ and odd} \\ +1 & n > 0 \text{ and even} \end{cases}\,.$$

Problem 1.47 (some system block diagrams)

Part (c): For this part of the problem we are looking for incrementally linear systems.

- **Part (i):** This is incrementally linear, with the linear system is $x[n] + 2x[n+4]$ with a zero input response of $y_0[n] = n$.

- **Part (ii):** This is incrementally linear with the linear system given by

$$y[n] = \begin{cases} 0 & n \text{ even} \\ \sum_{k=-\infty}^{(n-1)/2} x[k] & n \text{ odd} \end{cases}$$

and a zero input response of

$$y_0[n] = \begin{cases} n/2 & n \text{ even} \\ (n-1)/2 & n \text{ odd} \end{cases} .$$

- **Part (iii):** To be incrementally linear the output of the difference between two signals must be a linear system. Consider the output difference $y_1[n] - y_2[n]$ we have

$$\begin{aligned} y_1[n] - y_2[n] &= \begin{cases} x_1[n] - x_1[n-1] + 3 & x_1[0] \geq 0 \\ x_1[n] - x_1[n-1] + 3 & x_1[0] \geq 0 \end{cases} \\ &- \begin{cases} x_2[n] - x_2[n-1] + 3 & x_2[0] \geq 0 \\ x_2[n] - x_2[n-1] + 3 & x_2[0] \geq 0 \end{cases} . \end{aligned}$$

Now pick signals x_1 and x_2 such that $x_1[0] = 1$ and $x_2[0] = -1$ then the above difference becomes

$$\begin{aligned} y_1[n] - y_2[n] &= x_1[n] - x_1[n-1] + 3 - x_2[n] + x_2[n-1] + 3 \\ &= x_1[n] - x_2[n] - (x_1[n-1] - x_2[n-1]) + 6 . \end{aligned}$$

This later system is not linear due to the constant 6 term.

- **Part (iv):** The output from this system will be

$$y(t) = x(t) + t\frac{dx(t)}{dt} .$$

Note that this output *is* linear in $x(t)$, so we expect to be able to construct an incrementally linear system from it. One such incrementally linear system might have an output given by

$$y(t) = x(t) + t\frac{dx(t)}{dt} + 1 ,$$

with a zero input response given by $y_0(t) = 1$.

- **Part (v):** The output from this system can be given by the following composition

$$\begin{aligned} v[n] &= x[n] + \cos(n\pi) \\ z[n] &= v[n]^2 = (x[n] + \cos(n\pi))^2 \\ w[n] &= x[n]^2 \,, \end{aligned}$$

from which we see that

$$\begin{aligned} y[n] &= (x[n] + \cos(n\pi))^2 - x[n]^2 = 2x[n]\cos(n\pi) + \cos(n\pi)^2 \\ &= 2(-1)^n x[n] + 1 \,. \end{aligned}$$

This is an incrementally linear system with a linear system given by

$$y[n] = 2(-1)^n x[n] \,,$$

and a zero input response given by $y_0[n] = 1$.