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$$-k \frac{d^2T}{dx^2} = Q(x)$$

$$\frac{d^2T}{dx^2} = -\frac{Q(x)}{k}$$

$$\frac{dT}{dx} = -\int_0^x \frac{Q(\xi)}{k}\, d\xi + C_1$$

$$\frac{dT}{dx}(0) = 0 + C_1 = -\frac{q}{k} \Rightarrow C_1 = -\frac{q}{k}$$

$$\frac{dT}{dx}(x) = -\frac{q}{k} - \frac{1}{k}\int_0^x Q(\xi)\, d\xi$$

$$T =$$

$$T(x) = -\frac{q}{k}\int_0^x d\xi - \frac{1}{k}\int_{\eta=0}^{x}\int_0^{\eta} Q(\xi)\, d\xi + C_2$$

$$T(L) = -\frac{q}{k}\int_0^L d\xi - \frac{1}{k}\int_{\eta=0}^{L}\int_0^{\eta} Q(\xi)\, d\xi\, d\eta + C_2 = T_L$$

$$\Rightarrow C_2 = T_L + \frac{q}{k}(L) + \frac{1}{k}\int_0^L\int_0^{\eta} Q(\xi)\, d\xi\, d\eta$$

$$\Rightarrow T(x) = T_L - \frac{q}{k}x + \frac{q}{k}L - \frac{1}{k}\int_{\eta=0}^{x}\int_{0}^{\eta} Q(\xi)\,d\xi\,d\eta + \frac{1}{k}\int_{\eta=0}^{L}\int_{\xi=0}^{\eta} Q(\xi)\,d\xi\,d\eta$$

$$= T_L + \frac{q}{k}(L-x) + \underbrace{\frac{1}{k}\int_{\eta=0}^{L} F(\eta)\,d\eta - \frac{1}{k}\int_{0}^{x} F(\eta)\,d\eta}_{\frac{1}{k}\int_{x}^{L} F(\eta)\,d\eta}$$

$$\Rightarrow T(x) = T_L + \frac{q}{k}(L-x) + \frac{1}{k}\int_{x}^{L}\Big[\underbrace{\int_{0}^{\eta} Q(\xi)\,d\xi}\Big]\,d\eta \qquad \text{eq } 2.4$$

Q constant

$$\underbrace{\int_{x}^{L} Q(\eta-0)\,d\eta}_{Q\frac{\eta^2}{2}\Big|_{x}^{L}}$$

$$\Rightarrow T(x) = T_L + \frac{q}{k}(L-x) + \frac{Q}{2k}(L^2 - x^2) \qquad \text{eq } 2.5$$

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ϕ_j are defined on the nodes & several ϕ_j's go into a given def of solution over an element.

$$R(T,x) \equiv -k\frac{d^2T}{dx^2} - Q$$ = this is a 2nd order, linear, operator & one begins to look at operator theory to see how to handle

$$\int_0^L W(x)\, R(T,x)\, dx = 0.$$

in the approximation $T(x;t) = \sum_{k=1}^{n+1} a_k \phi_k$ cannot require that $T \in C^2$

For if the 2nd derivatives of T are continuous then we cannot

Solve $-k\frac{d^2T}{dx^2} = \delta(x-x_s)$

$$-k\frac{d^2T}{dx^2} = \delta(x-x_s)$$

$$\left.\begin{aligned} -k\frac{d^2T}{dx^2} &= 0 \qquad x < x_s \\ -k\frac{d^2T}{dx^2} &= 0 \qquad x > x_s \end{aligned}\right\} \rightarrow \begin{aligned} T &= ax+b \\ T &= cx+d \end{aligned}$$

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B.C. $-k\frac{dT}{dx} = q \Rightarrow a = -\frac{q}{k}$

$+ \; T(L) = T_L \Rightarrow cL + d = T_L \Rightarrow d = T_L - cL$

$$\therefore \quad T = \begin{cases} -\frac{q}{k}x + b & x < x_s \\ cx - cL + T_L & x > x_s \end{cases}$$

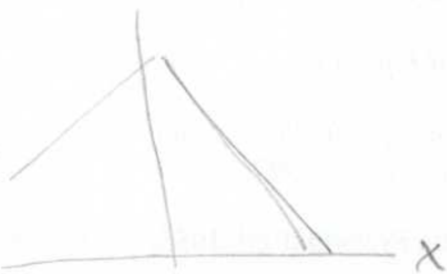
T must be $\in C^0 \;\therefore$

$$-\frac{q}{k}x_s + b = c(x_s - L) + T_L \Rightarrow b = \cancel{\frac{q}{k}x_s - \frac{q}{k}x_s + \frac{q}{k}L + T_L}$$

$+ \quad -\frac{q}{k} = c$ $\rightarrow\leftarrow$ then $f''(x) \sim H(x)$ Heavyside fn

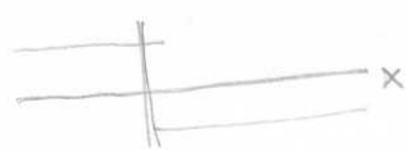
$$\Rightarrow T = \cancel{\begin{cases} -\frac{q}{k}x + \frac{q}{k}L + T_L & x < x_s \\ -\frac{q}{k}(x-L) + T_L & x > x_s \end{cases}}$$

$f(x)$



$$b = +\frac{q}{k}x_s + c(x_s - L) + T_L$$

$f'(x) \sim H(x) \neq \delta(x)$



$$\therefore\ T = \begin{cases} -\frac{q}{k}x + \frac{q}{k}x_s + c(x_s - L) + T_L & x \leq x_s \\ c(x-L) + T_L & x > x_s \end{cases}$$

Have not use Integral property of delta fn

$$-k\frac{d^2T}{dx^2} = \delta(x-x_s)$$

$$\Rightarrow \underbrace{-k\int_0^L \frac{d^2T}{dx^2}\,dx}_{} = \int_0^L \delta(x-x_s)\,dx = 1$$

$$-k\left.\frac{dT}{dx}\right|_0^L = 1$$

$$= -k\left(\frac{dT(L)}{dx} - \frac{dT(0)}{dx}\right) = 1$$

$$\frac{dT(L)}{dx} = c \qquad \frac{dT(0)}{dx} = -\frac{q}{k}$$

$$= -k\left(c + \frac{q}{k}\right) = 1$$

$$-kc - q = 1$$

$$c = (q+1)/(-k)$$

Then

$$T = \begin{cases} -\frac{q}{k}(x - x_s) + \frac{(q+1)}{(-k)}(x_s - L) + T_L & 0 \le x \le x_s \\ -\frac{(q+1)(x-L)}{k} + T_L & x_s < x \le L \end{cases}$$

$$\Rightarrow T = \begin{cases} -\frac{1}{k}\left[q(x - \cancel{x_s}) + \cancel{q x_s} - qL + \underline{x_s - L}\right] + T_L \\ -\frac{1}{k}(q+1)(x-L) + T_L & x_s < x < L \end{cases}$$

$$\Rightarrow T = \begin{cases} -\frac{1}{k}\left[q(x-L) + x_s - L\right] + T_L & \text{''} \\ -\frac{1}{k}(q+1)(x-L) + T_L \end{cases} \quad \text{eq 2.15}$$

$$\int_0^L \phi(x)\left(-k\frac{d^2T}{dx^2}\right)dx = -k\frac{dT}{dx}\phi \Big|_0^L + \int_0^L k\frac{d\phi}{dx}\frac{dT}{dx}\,dx \quad \text{eq 2.16}$$

---

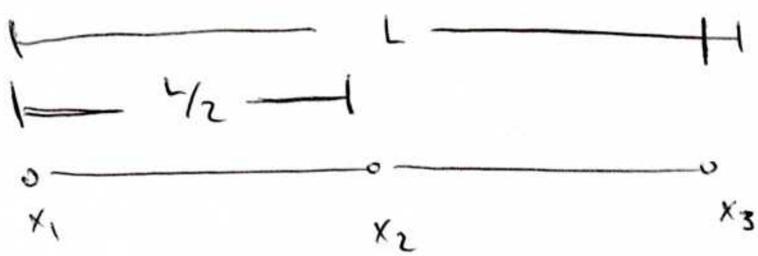
into eq bottom pg 9 let $T = \sum_{i=1}^{n+1} a_i \phi_i$

$$\Rightarrow \int_0^L k \sum_{j=1}^{n+1} \frac{d\phi_i}{dx} a_j \frac{d\phi_j}{dx}\,dx - \int_0^L \phi_i Q\,dx - \sum_{j=1}^{n+1} k\,\phi_i a_j \frac{d\phi_j}{dx}\Big|_0^L = 0$$

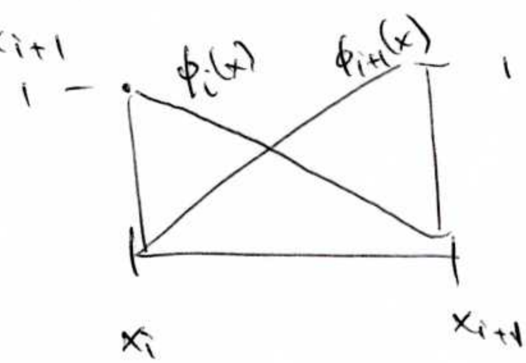
$$\Rightarrow \sum_{j=1}^{n+1} k\left[\int_0^L \frac{d\phi_i}{dx}\frac{d\phi_j}{dx}\,dx\right]a_j - \int_0^L \phi_i Q\,dx + \phi_i\left(-kT(x)\right)\Big|_0^L = 0$$

eq 21B

$i = 1, 2, 3, \ldots, n+1$



linear over each element e_i $\quad x_i < x < x_{i+1}$



$\underline{T(x)} = \phi_i(x)\, a_i + \phi_{i+1}\, a_{i+1}$

knowing T is linear in element e_i

can split any linear fn φ into

$\phi_i = \cancel{x}\, x \qquad \phi_i = \dfrac{x_{i+1} - x}{x_{i+1} - x_i}$

$$\phi_{i+1} = \frac{x - x_i}{x_{i+1} - x_i}$$

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Eq 2.18 is vaild $\forall$ intervals (not just $(0,L)$) thus we can

integrate from $\underbrace{0 \text{ to } L/2}$ & replace ϕ_i w/ ϕ_1

domain of def of element 1.

& ϕ_i w/ ϕ_2

$\Rightarrow$ Then ϕ_3 in j summation is ~~not~~ defined to be 0 in element 1 & drops out of the summation

& we obtain eqs 2.27.

1st eq in 2.27 can be written.

$$k\int_0^{L/2} \frac{d\phi_1}{dx} \sum_{j=1}^{2} \frac{d\phi_j}{dx} a_j\,dx - \int_0^{L/2} \phi_1(x) Q\,dx + \cancel{\phi_1}(L/2)^{0} - \underbrace{\left(-k\frac{dT(0)}{dx}\right)}_{q} 1 = 0$$

$$\& \quad k\int_0^{L/2} \frac{d\phi_2}{dx} \sum_{j=1}^{2} \frac{d\phi_j}{dx} a_j\,dx - \int_0^{L/2} \phi_2 Q\,dx - \underbrace{0}_{?} = 0$$

$$\Rightarrow k\int_0^{L/2} \frac{d\phi_1}{dx} \left(\frac{d\vec{\phi}}{dx}\right)^T \vec{a}\,dx - \qquad = 0$$

$$k\int_0^{L/2} \frac{d\phi_2}{dx} \left(\frac{d\vec{\phi}}{dx}\right)^T \vec{a}\,dx - \qquad = 0$$

Now

$$\left(\frac{d\phi}{dx}\right)^T \left(\frac{d\phi}{dx}\right) a$$

don't see eq 2.28
convention in this book is
ϕ = row vector
ϕ^T = column vector
$\vec{a}$ is a column vector

$$\underbrace{(\phi_1' \quad \phi_2')\begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix}}\begin{pmatrix}a_1 \\ a_2\end{pmatrix}$$

$$\phi_1'\phi_1 + \phi_2'\phi_2$$

$$\phi = \left[\ \frac{L/2 - x}{L/2 - 0} \quad \frac{x - 0}{L/2 - 0}\ \right] = \left[1 - \frac{2}{L}x\ ,\ \frac{2}{L}x\right]$$

$$\frac{d\phi}{dx} = \left[-\frac{2}{L}\ ,\ \frac{2}{L}\right]$$

2.28

Then

$$\left(\frac{d\phi}{dx}\right)^T \left(\frac{d\phi}{dx}\right) a$$

$$\begin{pmatrix}\phi_1' \\ \phi_2'\end{pmatrix}(\phi_1 \quad \phi_2)\begin{pmatrix}a_1 \\ a_2\end{pmatrix}$$

test fn

$$= \begin{pmatrix}\phi_1'\phi_1 & \phi_1'\phi_2 \\ \phi_2'\phi_1 & \phi_2'\phi_2\end{pmatrix}\begin{pmatrix}a_1 \\ a_2\end{pmatrix} = \begin{pmatrix}a_1\phi_1'\phi_1 + a_2\phi_1'\phi_2 \\ a_1\phi_2'\phi_1 + a_2\phi_2'\phi_2\end{pmatrix}$$

grouping of both terms in 2.27

test fn

- Then eq 2.28 becomes

$$\int_0^{L/2} \left(\frac{d\phi}{dx}\right)^T \left(\frac{d\phi}{dx}\right) a$$

$$\|$$

$$k \int_0^{L/2} \begin{pmatrix} -\frac{2}{L} \\ \frac{2}{L} \end{pmatrix} \begin{pmatrix} -\frac{2}{L} & \frac{2}{L} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} dx - Q \int_0^{L/2} \begin{pmatrix} 1-\frac{2x}{L} \\ \frac{2x}{L} \end{pmatrix} dx - \begin{pmatrix} q \\ 0 \end{pmatrix} = \vec{0}$$

$$\Rightarrow k \int_0^{L/2} \begin{pmatrix} \frac{4}{L^2} & -\frac{4}{L^2} \\ -\frac{4}{L^2} & \frac{4}{L^2} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} dx - Q \left.\begin{pmatrix} x - \frac{2x^2}{2L} \\ \frac{2x^2}{2L} \end{pmatrix}\right|_0^{L/2} - \begin{pmatrix} q \\ 0 \end{pmatrix} = 0$$

$$\Rightarrow k \begin{pmatrix} \frac{2}{L} & -\frac{2}{L} \\ -\frac{2}{L} & \frac{2}{L} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} - Q \begin{pmatrix} L/4 \\ L/4 \end{pmatrix} - \begin{pmatrix} q \\ 0 \end{pmatrix} = \vec{0}$$

$$\Rightarrow \frac{2k}{L} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} - \frac{QL}{4} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} q \\ 0 \end{pmatrix} = \vec{0} \quad \text{eq } 2.29$$

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For element #2, restricting eq 2.18 to the 2nd element gives
⇒ range of integration goes from $(L/2, L)$

$$\sum_{j=2}^{3} k \int_{L/2}^{L} \frac{d\phi_2}{dx}\frac{d\phi_j}{dx}\,dx\; a_j - \int_{L/2}^{L} \phi_2 Q\,dx + \phi_2\left(-k\frac{dT}{dx}\right)\Bigg|_{x=L/2}^{L} = 0$$

ϕ_2 in 2nd element is defined only from $L/2$ to L

$\phi_2(L) = 0$ $\phi_2(L/2) = 1$ but if no flux is specified at a node can assume that term vanishes there, Assumption pg 11.

& Also $$\sum_{j=2}^{3} k \int_{L/2}^{L} \frac{d\phi_3}{dx}\frac{d\phi_j}{dx}\,dx\; a_j - \int_{L/2}^{L} \phi_3 Q\,dx + \phi_3\left(-k\frac{dT}{dx}\right)\Bigg|_{L/2}^{L} = 0$$

$\phi_3(L) = 1$ Dirchlet B.C's. $\phi_3(L/2) = 0$.

$$\Rightarrow \begin{cases} k\displaystyle\int_{L/2}^{L}\left[\sum_{j=2}^{3}\frac{d\phi_2}{dx}\frac{d\phi_j}{dx}a_j\right]dx - \int_{L/2}^{L}\phi_2 Q\,dx = 0 \\ k\displaystyle\int_{L/2}^{L}\left[\sum_{j=2}^{3}\frac{d\phi_3}{dx}\frac{d\phi_j}{dx}a_j\right]dx - \int_{L/2}^{L}\phi_3 Q\,dx = 0 \end{cases}$$

$$\sum_{j=2}^{3} \frac{d\phi_j}{dx} a_j = \left(\frac{d\phi_2}{dx} \quad \frac{d\phi_3}{dx} \right) \begin{pmatrix} a_2 \\ a_3 \end{pmatrix}$$

$$\left(\frac{d\phi}{dx}\right)^T (\bar{a})$$

$$\Rightarrow \quad k \int_{L/2}^{L} \frac{d\phi_2}{dx} \left(\frac{d\phi}{dx}\right)^T a \, dx - \int_{L/2}^{L} \phi_2 Q \, dx = 0$$

$$k \int_{L/2}^{L} \frac{d\phi_3}{dx} \left(\frac{d\phi}{dx}\right)^T a \, dx - \int_{L/2}^{L} \phi_3 Q \, dx = 0$$

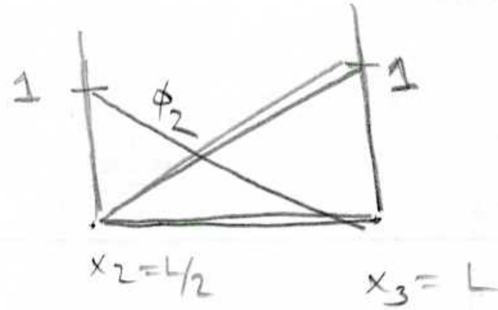
2 eqs

$$\underbrace{\begin{pmatrix} \frac{d\phi_2}{dx} \\ \frac{d\phi_3}{dx} \end{pmatrix} \left(\frac{d\phi}{dx}\right)^T a}$$

$$\underbrace{\left(\frac{d\phi}{dx}\right) \cdot \left(\frac{d\phi}{dx}\right)^T}_{\in \mathbb{R}^{2\times 2}} a \qquad \begin{pmatrix} x \\ x \end{pmatrix} (x \quad x) \in \mathbb{R}^{2\times 2}$$

$$\Rightarrow \quad k \int_{L/2}^{L} \phi \phi^T a \, dx - \int_{L/2}^{L} \phi Q \, dx = 0$$

Now for element 2, $\phi = \cdots$



$$\phi = \begin{pmatrix} \dfrac{L-x}{L-L/2} \\[2ex] \dfrac{x-L/2}{L-L/2} \end{pmatrix} = \begin{pmatrix} \dfrac{2(L-x)}{L} \\[2ex] \dfrac{2(x-L/2)}{L} \end{pmatrix} = \begin{pmatrix} \dfrac{2(L-x)}{L} \\[2ex] \dfrac{2x-L}{L} \end{pmatrix} = \begin{pmatrix} 2(1-\frac{x}{L}) \\[2ex] \frac{2x}{L}-1 \end{pmatrix}$$

Then

$$\cancel{\phi\phi^T = \begin{pmatrix} 2(1-x/L) \\ \frac{2x}{L}-1 \end{pmatrix} \begin{pmatrix} 2(1-x/L) & \frac{2x}{L}-1 \end{pmatrix}}$$

Need to take $\left(\frac{d\phi}{dx}\right)\left(\frac{d\phi}{dx}\right)^T$

$$\frac{d\phi}{dx} = \begin{pmatrix} -\frac{2}{L} \\[1ex] \frac{2}{L} \end{pmatrix} \quad \text{Then } \frac{d\phi}{dx}\left(\frac{d\phi}{dx}\right)^T = \begin{pmatrix} -2/L \\ 2/L \end{pmatrix} \begin{pmatrix} -2/L & 2/L \end{pmatrix}$$

$$= \begin{pmatrix} 4/L^2 & -4/L^2 \\ -4/L^2 & 4/L^2 \end{pmatrix}$$

Then integrating from $L/2$ to L gives

$$= \frac{L}{2} \cdot \frac{4}{L^2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{2}{L} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$+ \int_{L/2}^{L} \phi \, dx = \frac{L}{4}\begin{pmatrix}1\\1\end{pmatrix}$$

+ eqs reduce to

$$\frac{2k}{L}\begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix}\begin{pmatrix}a_2\\a_3\end{pmatrix} - \frac{QL}{4}\begin{pmatrix}1\\1\end{pmatrix} = 0$$

Need to combine ϕ_2 contributions into global nodal contribution at $x_2 \Rightarrow$

Adding 2nd eq in 2.29 + 1st eq in 2.30

$$\Rightarrow \frac{2k}{L}\begin{pmatrix}1 & -1 & 0\\-1 & 2 & -1\\0 & -1 & 1\end{pmatrix}\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix} = \frac{QL}{4}\begin{pmatrix}1\\2\\1\end{pmatrix} + \begin{pmatrix}q\\0\\0\end{pmatrix}$$

know:

$a_3 = T_L$ (Break eqs up into 1st 2 eqs + last eq

$$\frac{2k}{L}\begin{pmatrix}1 & -1\\-1 & 2\end{pmatrix}\begin{pmatrix}a_1\\a_2\end{pmatrix} + \frac{2k}{L}a_3\begin{pmatrix}0\\-1\end{pmatrix} = \frac{QL}{4}\begin{pmatrix}1\\2\end{pmatrix} + \begin{pmatrix}q\\0\end{pmatrix}$$

$$+ \frac{2k}{L}(-a_2 + a_3) = \frac{QL}{4} \Rightarrow -a_2 + a_3 = \frac{QL^2}{8k}$$

If $a_3 = T_L$ Equations for a_1 + a_2 become

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$$\frac{2k}{L}\begin{pmatrix}1 & -1\\ -1 & 2\end{pmatrix}\begin{pmatrix}a_1\\ a_2\end{pmatrix} = \frac{QL}{4}\begin{pmatrix}1\\ 2\end{pmatrix} + \begin{pmatrix}q\\ 0\end{pmatrix} + T_L\begin{pmatrix}0\\ 1\end{pmatrix} \cdot \frac{2k}{L}$$

$$\begin{pmatrix}1 & -1\\ -1 & 2\end{pmatrix}\begin{pmatrix}a_1\\ a_2\end{pmatrix} = \frac{L^2 Q}{8k}\begin{pmatrix}1\\ 2\end{pmatrix} + \frac{L}{2k}\begin{pmatrix}q\\ 0\end{pmatrix} + T_L\begin{pmatrix}0\\ 1\end{pmatrix}$$

$$\begin{pmatrix}a_1\\ a_2\end{pmatrix} = \frac{1}{1}\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}\left\{ - - - - \right\}$$

$$= \frac{QL^2}{8k}\begin{pmatrix}4\\ 3\end{pmatrix} + \frac{L}{2k}\begin{pmatrix}2q\\ q\end{pmatrix} + T_L\begin{pmatrix}1\\ 1\end{pmatrix}$$

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$-\frac{d^2u}{dx^2} = 1$ Has $\underbrace{q = ?}_{\text{don't know flux?}}$, $k = 1$. $q = -k\frac{dT}{dx}\Big|_{x=0} = -\frac{dT}{dx}\Big|_{x=0}$

$=$

Approx w/ 1 term $u(x) = a_1 \sin \pi x$. in eq $\overline{2.18 \text{ w/ } n = 0}$

$i = 1$
$j = 1$

$n+1 \Rightarrow$ # test fns $n = 0 \Rightarrow$ only 1 test fn.

$$\int_0^1 \frac{d\phi_1}{dx}\frac{d\phi_1}{dx} a_1 - \int_0^1 \phi_1 (1) \cdot dx + \phi_1 \cancel{\left(-\frac{dT}{dx}\right)}^{0}\Bigg|_0^1 = 0$$

$\phi_1 = \sin \pi x$

$$\Rightarrow \pi^2 \int_0^1 \cos^2 \pi x \, dx \, a_1 - \underbrace{\int_0^1 \sin \pi x \, dx}_{\frac{2}{\pi}} = 0$$

$$\frac{\pi^2}{2} a_1 - \frac{2}{\pi} = 0$$

$$a_1 = \frac{4}{\pi^3}$$

$\therefore$ Galerkin $u(x) \cong \frac{4}{\pi} \sin(\pi x)$

From 2.5 Exact solution is (That is for B.C of one Dirichlet & Neumann Not 2 Dirichlet !!)

$$u(x) = 0 + \frac{1}{1}(1-x) + \frac{1}{2}(1-x^2)$$

$$u(1/2) = \frac{1}{2} + \frac{1}{2}\frac{3}{4} = \frac{1}{2} + \frac{3}{8} = \frac{7}{8}$$

$$-\frac{d^2u}{dx^2} = 1 \quad \cancel{\Rightarrow} \quad \cancel{u(x) = -\frac{x^3}{6} + \frac{c_1x^2}{2}} + c_2x$$

$$-\frac{du}{dx} = x + c_1$$

$$-u = \frac{x^2}{2} + c_1x + c_2$$

$$\Rightarrow \quad u = -\frac{x^2}{2} + c_1x + c_2$$

$$u(0) = u(1) = 0 \Rightarrow c_2 = 0$$

$$-\frac{1}{2} + c_1 = 0 \Rightarrow c_1 = \frac{1}{2}$$

$$u(x)_{Exact} = -\frac{x^2}{2} + \frac{x}{2} = \frac{x}{2}(1-x)$$

$$u_{Exact}(1/2) = \frac{1}{4}\left(1-\frac{1}{2}\right) = \frac{1}{8}$$

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(2.2) $\frac{d^2u}{dx^2} + u + x = 0 \qquad 0<x<1 \qquad u(0) = u(1) = 0$

$u(x) = a_1 \phi_1 \qquad \phi_1 = x(1-x) = x - x^2$ $\quad \phi$ satisfies both Dirichlet B.C.'s

$R(u,x) = \frac{d^2u}{dx^2} + u + x = -2a_1 + x(1-x)a_1 + x$

Require a_1 $\ni$ $\int_0^1 W(x)R(u,x)\,dx = 0$ w/ $W(x) = \phi_1(x)$

$$\Rightarrow \int_0^1 (x-x^2)(-2a_1 + x(1-x)a_1 + x)\,dx = 0$$

$$\Rightarrow \int_0^1 (-2a_1x + 2a_1x^2 + x^2(1-x)^2a_1 + x^2(1-x))\,dx = 0$$

$$\Rightarrow \int_0^1 (-2a_1x + 2a_1x^2 + x^2(1-2x+x^2)a_1 + x^2 - x^3)\,dx = 0$$

$$\Rightarrow \int_0^1 (-2a_1x + 2a_1x^2 + a_1x^2 - 2a_1x^3 + a_1x^4 + x^2 - x^3)\,dx = 0$$

$$\Rightarrow \int_0^1 (-2a_1x + (3a_1+1)x^2 - (2a_1+1)x^3 + a_1x^4)\,dx = 0$$

$$\Rightarrow \left(-a_1x^2 + (3a_1+1)\frac{x^3}{3} - \frac{(2a_1+1)x^4}{4} + \frac{a_1x^5}{5}\right|_0^1 = 0$$

$$\Rightarrow -a_1 + \frac{1}{3}(3a_1+1) - \frac{1}{4}(2a_1+1) + \frac{a_1}{5} = 0$$

$$\Rightarrow -\cancel{a_1} + \cancel{a_1} + \frac{1}{3} - \frac{a_1}{2} - \frac{1}{4} + \frac{a_1}{5} = 0$$

$$\Rightarrow a_1\left(-\frac{1}{2} + \frac{1}{5}\right) + \frac{1}{3} - \frac{1}{4} = 0$$

$$\Rightarrow a_1\left(-\frac{5}{10} + \frac{2}{10}\right) + \frac{4}{12} - \frac{3}{12} = 0$$

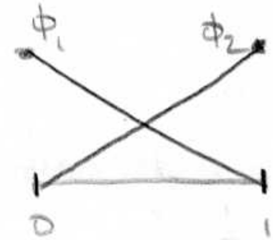
$$a_1\left(-\frac{3}{10}\right) + \frac{1}{12} = 0$$

$$a_1 = \frac{5\cancel{10}}{3}\cdot\frac{1}{\cancel{12}6} = \frac{5}{18}$$

(2.3)

$$\frac{d^2u}{dx^2} = 1 \qquad 0 < x < 1$$

$$u(0) = 0 \qquad \left.\frac{du}{dx}\right|_{x=1} = 1$$



(a) one element

Then approximation to $u(x)$ is $= a_1\phi_1(x) + a_2\phi_2(x)$

Then eq 2.18 becomes w/ $\phi_i = \phi_1$

$$\sum_{j=1}^{2}(-1)\left[\int_0^1 \frac{d\phi_1}{dx}\frac{d\phi_j}{dx}\,dx\right]a_j - \int_0^1 \phi_1\, 1\, dx + \underbrace{\phi_1\left(\frac{dT}{dx}\right|_0^1}_{=0} = 0$$

Only non zero if weight fn is 1 at same point flux is defined.

$$+\ \sum_{j=1}^{2}(-1)\left[\int_0^1 \frac{d\phi_2}{dx}\frac{d\phi_j}{dx}\,dx\right]a_j - \int_0^1 \phi_2\, 1\, dx + \underbrace{\phi_2\left(\frac{dT}{dx}\right|_0^1}_{=1-0} = 0$$

$$\Rightarrow \int_0^1 (-1)\frac{d\phi_1}{dx}\left(\frac{d\phi}{dx}\right)^T \vec{a}\,dx - \int_0^1 \phi_1\,dx + 0 = 0$$

$$\int_0^1 (-1)\frac{d\phi_2}{dx}\left(\frac{d\phi}{dx}\right)^T \vec{a}\,dx - \int_0^1 \phi_2\,dx + 1 = 0$$

$$\Rightarrow \int_0^1 (-1)\left(\frac{d\phi}{dx}\right)\left(\frac{d\phi}{dx}\right)^T \vec{a}\, dx - \int_0^1 \phi\, dx + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \bar{0}$$

Now $\phi = \begin{pmatrix} 1-x \\ x \end{pmatrix}$ $\quad \frac{d\phi}{dx} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$$\Rightarrow \int_0^1 (-1)\begin{pmatrix} -1 \\ 1 \end{pmatrix}\begin{pmatrix} -1 & 1 \end{pmatrix} \vec{a}\, dx - \int_0^1 \begin{pmatrix} 1-x \\ x \end{pmatrix} dx + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$

$$\Rightarrow \int_0^1 (-1)\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} dx\ \vec{a} - \begin{pmatrix} x - x^2/2 \\ x^2/2 \end{pmatrix}\Bigg|_0^1 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$

$$\Rightarrow (-1)\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} x \Big|_0^1 \vec{a} - \begin{pmatrix} 1 - 1/2 \\ 1/2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} - \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$

This matrix $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$ represents the interaction of each element w/ itself & is singular.
1,1, 1,2, 2,1, 2,2

$$\Rightarrow \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} -1/2 \\ 1/2 \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}$$

Jump to next pg

W/ only 1 element on (0,1) How do I do this another way?

Pick a test fn that is linear & satisfys the Dirichlet B.C.s.

(Each element of approximating space must satisfy Homogenous

take $\phi_1 = x$ (Aside this also satisfy's Neumann BC as well)

The $u \cong a_1 x$

Then the coefficient a_1 is chosen ∋

$$\int_0^1 \phi_1 R(u)\, dx = 0 \qquad R(u) = \frac{d^2 u}{dx^2} - 1$$

$$\Rightarrow \int_0^1 x(-1)\, dx = 0 \quad \rightarrow\leftarrow$$

We know however that $a_1 = 0$ & The linear eq becomes:

$$\left.\begin{array}{l} a_2 - \tfrac{1}{2} = 0 \\ \& \; -a_2 - \tfrac{1}{2} + 1 = 0 \Rightarrow -a_2 + \tfrac{1}{2} = 0 \end{array}\right] \Rightarrow a_2 = \tfrac{1}{2}$$

Thus Approximation is $u \simeq \frac{1}{2} x$

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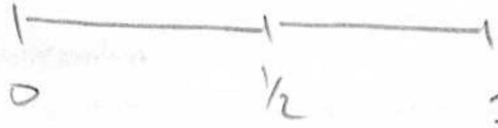
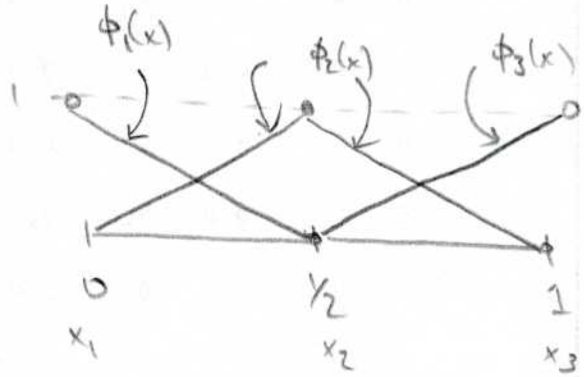
(2.3) (a) two elements

continued



~~As In~~ Procedure is this integrate over an element Here 2 elements exist $(0,½)$ & $(½,1)$ Then test fns can be only those that "overlap" w/ given element i.e. in 2.18 integrate over 1st element & can only test against ϕ_1 & ϕ_2



$$= \sum_{j=1}^{2}(-1)\int_0^{1/2}\frac{d\phi_1}{dx}\frac{d\phi_j}{dx}\,dx\,a_j$$

$$-\int_0^{1/2} 1\,\phi_1\,dx + \underbrace{\phi_1\left(1\,\frac{dT}{dx}\right|_0^{1/2}} = 0$$

$\phi_1(½) = 0 \qquad \frac{dT}{dx}(½) = 0$

$\phi_1(0) = 1 \qquad \frac{dT}{dx}(0) = 0 \Leftarrow$ following rule on pg 11.

$$+ \sum_{j=1}^{2}(-1)\int_0^{1/2}\frac{d\phi_2}{dx}\frac{d\phi_j}{dx}\,dx\,a_j - \int_0^{1/2} 1\,\phi_2\,dx + \underbrace{\phi_2\left(1\,\frac{dT}{dx}\right|_0^{1/2}} = 0$$

$\phi_2(½) = 1 \qquad \frac{dT}{dx}(½) = 0$

$\phi_2(0) = 0 \qquad \frac{dT}{dx}(0) = 0$

$$\Rightarrow (-1)\int_0^{1/2} \left(\frac{d\vec{\phi}}{dx}\right)\left(\frac{d\vec{\phi}}{dx}\right)^T \vec{a}\, dx - \int_0^{1/2} \vec{\phi}\, dx = 0 \qquad \phi = \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \qquad (1)$$

For 2nd element term that would change would be flux terms

$$\phi_2 \left(1 \frac{dT}{dx}\right|_{1/2}^{1} = \cancel{\phi_2(1)}^{0} - \phi_2(1/2)\cancel{\frac{dT}{dx}(1/2)}^{0} = 0$$

$$\phi_3 \left(1 \frac{dT}{dx}\right|_{1/2}^{1} = \phi_3(1)\frac{dT}{dx}(1) - 0 = 1$$

$$\Rightarrow (-1)\int_{1/2}^{1} \left(\frac{d\vec{\phi}}{dx}\right)\left(\frac{d\vec{\phi}}{dx}\right)^T \vec{a}\, dx - \int_{1/2}^{1} \vec{\phi}\, dx + \begin{pmatrix}0\\1\end{pmatrix} = 0 \qquad \vec{\phi} = \begin{pmatrix}\phi_2\\ \phi_3\end{pmatrix} \quad (2)$$

Eqs (1) become: $\phi = \begin{pmatrix}\frac{1/2 - x}{1/2}\\ \\ \frac{x}{1/2}\end{pmatrix} = \begin{pmatrix}1-2x\\ 2x\end{pmatrix}$

$$\phi_x = \begin{pmatrix}-2\\ 2\end{pmatrix} = 2\begin{pmatrix}-1\\ 1\end{pmatrix}$$

$$(-1)4\int_0^{1/2} \begin{pmatrix}-1\\ 1\end{pmatrix}\begin{pmatrix}-1 & 1\end{pmatrix}\begin{pmatrix}a_1\\ a_2\end{pmatrix} dx - \int_0^{1/2}\begin{pmatrix}1-2x\\ 2x\end{pmatrix} dx = 0$$

$$\Rightarrow -4\left(\tfrac{1}{2}\right)\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\begin{pmatrix}a_1\\ a_2\end{pmatrix} - \left.\begin{pmatrix}x - x^2\\ x^2\end{pmatrix}\right|_0^{1/2} = 0$$

$$\Rightarrow \quad -2\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} - \begin{pmatrix} 1/4 \\ 1/4 \end{pmatrix} = 0 \qquad (1')$$

+ Eqs (2) Become: w/ $\phi = \begin{pmatrix} \dfrac{1-x}{1-1/2} \\ \dfrac{x-1/2}{1-1/2} \end{pmatrix} = \begin{pmatrix} 2(1-x) \\ 2x-1 \end{pmatrix}$; $\dfrac{d\phi}{dx} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$

$$(-1)\,4\int_{1/2}^{1}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} a_2 \\ a_3 \end{pmatrix} dx - \int_{1/2}^{1}\begin{pmatrix} 2-2x \\ 2x-1 \end{pmatrix} dx + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$

$$\Rightarrow -2\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} a_2 \\ a_3 \end{pmatrix} - \begin{pmatrix} 2x-x^2 \\ x^2-x \end{pmatrix}\Bigg|_{1/2}^{1} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$

$$\Rightarrow -2\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} a_2 \\ a_3 \end{pmatrix} - \begin{pmatrix} (2-1)-(1-1/4) \\ 0-(1/4-1/2) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$

$$\Rightarrow -2\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} a_2 \\ a_3 \end{pmatrix} - \underbrace{\begin{pmatrix} 1-3/4 \\ 1/4 \end{pmatrix}}_{\begin{pmatrix} 1/4 \\ 1/4 \end{pmatrix}} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$

$$\begin{pmatrix} -2 & 2 & 0 \\ 2 & -2-2 & 2 \\ 0 & 2 & -2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} -1/4 \\ -1/4 - 1/4 \\ -1/4 + 1 \end{pmatrix} = 0$$

But we know $a_1 \equiv 0$ Then 1st eq becomes

$$2a_2 = 1/4 \Rightarrow a_2 = 1/8$$

Then last eq gives $2(1/8) - 2a_3 = -3/4$

$$1/8 - a_3 = -3/8 \Rightarrow a_3 = 4/8 = 1/2$$

$$\therefore \; U(x) \cong 0\phi_1(x) + \frac{1}{8}\phi_2(x) + \frac{1}{2}\phi_3(x)$$

(2.3) (b)

(i) $v(x) = a_1 x \quad \phi_1(x) = W_1(x) = x$

$$\int_0^1 \phi_1(x)\left(\frac{d^2 v}{dx^2} - 1\right) dx = 0$$

$$\Rightarrow \phi_1(x)\left(\frac{dv}{dx} - x\right)\Big|_0^1 - \int_0^1 \frac{d\phi_1}{dx}\left(\frac{dv}{dx} - x\right) dx = 0$$

$$\Rightarrow 1(a_1 - 1) - 0 - \int_0^1 1(a_1 - x)\, dx = 0$$

$$\Rightarrow 1(a_1 - 1) - \left(a_1 x - \frac{x^2}{2}\right)\Big|_0^1 = 0$$

$$\Rightarrow \cancel{a_1} - 1 - \cancel{a_1} + \frac{1}{2} = 0$$

$$-\frac{1}{2} = 0 \quad ?$$

I don't see how to get w/ this approximation?

(ii) $$\int_0^1 (-1)(1)\, a_1\, dx - \int_0^1 (-1) x\, 1 - (-1) x\, a_1 \Big|_0^1 = 0$$

(ii) If $v(x) \cong a_1 x + a_2 x^2$

Then The residual of this approximation is

$$R(v, x) = 0 + 2a_2 - 1$$

Gerlerkin Approx says

Pick $w \ni$ $\int_0^1 w R(u,x)\,dx = 0$

$w = \phi_1, \phi_2$

$\Rightarrow w = \phi_1$ $\int_0^1 x(2a_2 - 1)\,dx = 0$

$= (2a_2 - 1)\frac{x^2}{2}\Big|_0^1 = \frac{1}{2}(2a_2 - 1) = 0$

$a_2 = 1/2$

$W = \phi_2$ $\int_0^1 x^2(2a_2 - 1)\,dx = \frac{x^3}{3}(2a_2 - 1) = 0$

a_1 is arbitrary?

See notes on problem

(c) (i) $u = a_1 \sin\left(\frac{\pi}{2}x\right)$

Residual of an element spanned by this approximation space is

$$R(u) = \frac{d^2u}{dx^2} - 1$$

$$R\left(a_1 \sin\left(\tfrac{\pi}{2}x\right)\right) = a_1\left(\tfrac{\pi}{2}\right)^2(-1)\sin\left(\tfrac{\pi}{2}x\right) - 1$$

Then choose a_1 (components of vector) to be orthogonal to the residual.

$$\int_0^1 \phi_1 R(u)\,dx = 0$$

$$\Rightarrow \int_0^1 \sin\left(\tfrac{\pi}{2}x\right)\left[-a_1\frac{\pi^2}{4}\sin\left(\tfrac{\pi}{2}x\right) - 1\right]dx = 0$$

$\sin^2\theta = \frac{1-\cos(2\theta)}{2}$

Check
$\cos(2\theta) = 1 - 2\sin^2\theta$
$= \cos^2\theta - \sin^2\theta$
$= 1 - s$ ✓.

$$\Rightarrow -a_1\frac{\pi^2}{4}\int_0^1 \left(\frac{1-\sin(\pi x)}{2}\right)dx + \frac{2}{\pi}\cos\left(\tfrac{\pi}{2}x\right)\Big|_0^1 = 0$$

$$\Rightarrow -\frac{\pi^2 a_1}{8}\left[x\Big|_0^1 + \frac{1}{\pi}\cos(\pi x)\Big|_0^1\right] + \frac{2}{\pi}(0-1) = 0$$

$$\Rightarrow -\frac{\pi^2 a_1}{8}\left(1 + \frac{1}{\pi}(-1-1)\right) = \frac{2}{\pi}$$

got $a_1 = \frac{-16}{\pi^3}$ on MMA

$$\Rightarrow a_1 = \frac{\frac{-16}{\pi^3}}{\left(1-\frac{2}{\pi}\right)} = \frac{\frac{-16}{\pi^3}\,\pi}{(\pi-2)} = \frac{-16}{\pi^2(\pi-2)}$$

(ii) $u \simeq a_1 \sin(\frac{\pi}{2}x) + a_2 \sin(\frac{3\pi}{2}x)$

Residual of a fn spanned by this basis would be

$$R(u) = \frac{d^2u}{dx^2} - 1 = -\frac{\pi^2}{4}a_1 \sin(\frac{\pi}{2}x) - \frac{9\pi^2}{4}a_2 \sin(\frac{3\pi}{2}x) - 1$$

Then to obtain a_1 & a_2 insure orthogonality of solution space & Residual (i.e. Residual has no component in the solution space that could be obtained by linear combination)

$$\Rightarrow \int_0^1 \phi_1 R = 0 \quad \& \quad \int_0^1 \phi_2 R = 0$$

$$\Rightarrow \int_0^1 \sin(\frac{\pi}{2}x)\left[-\frac{\pi^2}{4}a_1 \sin(\frac{\pi}{2}x) - \frac{9\pi^2}{4}a_2 \sin(\frac{3\pi}{2}x) - 1\right]dx = 0$$

$$\& \int_0^1 \sin(\frac{3\pi}{2}x)\left[-\frac{\pi^2}{4}a_1 \sin(\frac{\pi}{2}x) - \frac{9\pi^2}{4}a_2 \sin(\frac{3\pi}{2}x) - 1\right]dx = 0$$

Should be solved for a_1 & a_2

1st eq gives

$$-\frac{2}{\pi} - a_1\frac{\pi^2}{8} = 0$$

2nd eq gives

$$-\frac{2}{3\pi} - \frac{9a_2\pi^2}{8} = 0$$

$$\Rightarrow a_1 = \frac{-16}{\pi^3}, \quad a_2 = \frac{-16}{27\pi^3}$$

(2.3) (d) $\frac{d^2u}{dx^2} = 1$

$$\Rightarrow \quad \frac{du}{dx} = x + C_1 \qquad \left.\frac{du}{dx}\right|_{x=1} = 1 = 1 + C_1 \Rightarrow C_1 = 0$$

$$\Rightarrow \quad \frac{du}{dx} = x \qquad 0 < x < 1$$

$$\Rightarrow \quad u = \frac{x^2}{2} + C_2 \qquad 0 < x < 1$$

$$u(0) = C_2 = 0 \Rightarrow \quad u(x) = \frac{x^2}{2} \qquad 0 < x < 1$$

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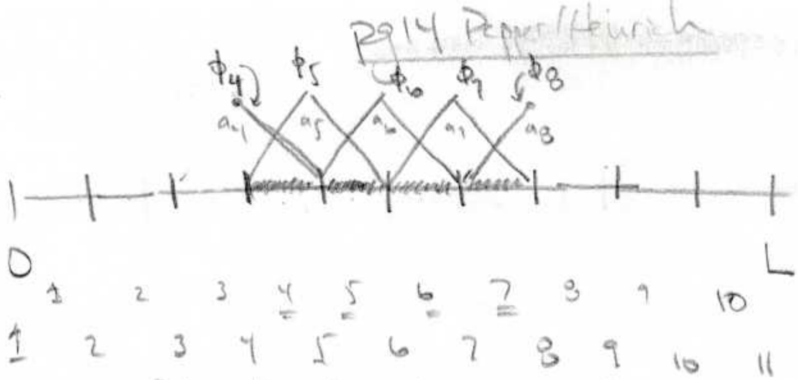
2.4 1-



As these are internal elements, each element

Is constructed following exercise 1 or pg 12 Pepper / Heinrich

& we get assuming eqs like

$$\frac{2k}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{pmatrix}a_4\\ a_5\end{pmatrix} - \frac{QL}{4}\begin{pmatrix}1\\1\end{pmatrix} = \bar{0} \qquad \text{elt } 4$$

$$\frac{2k}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{pmatrix}a_5\\ a_6\end{pmatrix} = \frac{QL}{4}\begin{pmatrix}1\\1\end{pmatrix} \qquad \text{elt } 5$$

$$\frac{2k}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{pmatrix}a_6\\ a_7\end{pmatrix} = \frac{QL}{4}\begin{pmatrix}1\\1\end{pmatrix} \qquad \text{elt } 6$$

$$\frac{2k}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{pmatrix}a_7\\ a_8\end{pmatrix} = \frac{QL}{4}\begin{pmatrix}1\\1\end{pmatrix} \qquad \text{elt } 7$$

combining into 1 eq gives

$$\frac{2k}{L}\underbrace{\begin{pmatrix}1 & -1 & 0 & 0 & 0\\ -1 & 2 & -1 & 0 & 0\\ 0 & -1 & 2 & -1 & 0\\ 0 & 0 & -1 & 2 & -1\\ 0 & 0 & 0 & -1 & 1\end{pmatrix}}_{\text{tridiagonal}}\begin{pmatrix}a_4\\ a_5\\ a_6\\ a_7\\ a_8\end{pmatrix} = \frac{QL}{4}\begin{pmatrix}1\\2\\2\\2\\1\end{pmatrix}$$

(2.5) $-k\dfrac{d^2T}{dx^2} = Q$ multiply by T

$\Rightarrow -k\dfrac{d^2T}{dx^2}T = QT$ integrate from $0, L$, I.B.P. on left

$$-k\int_0^L T\frac{d^2T}{dx^2}\,dx = \int_0^L QT\,dx$$

$$\Rightarrow -k\left(T\frac{dT}{dx}\Big|_0^L - \int_0^L \left(\frac{dT}{dx}\right)^2 dx\right) = \int_0^L QT\,dx$$

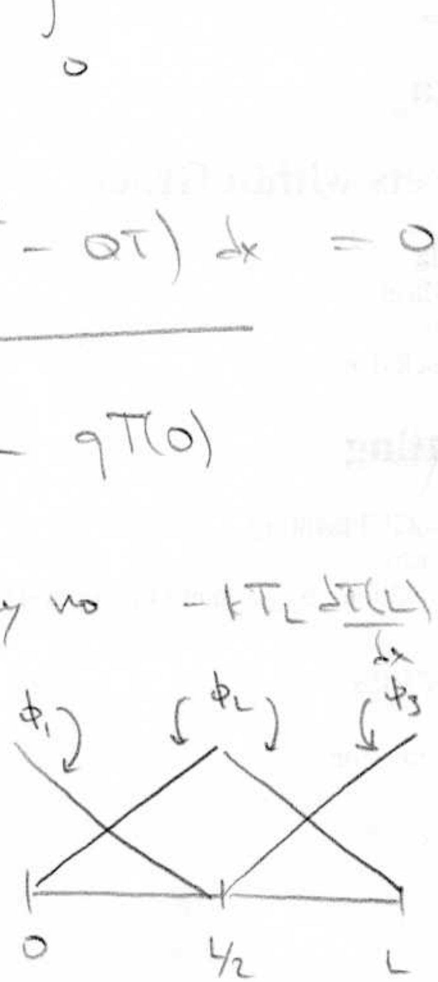
$$\Rightarrow -kT\frac{dT}{dx}\Big|_0^L + k\int_0^L \left(\frac{dT}{dx}\right)^2 dx - \int_0^L QT\,dx = 0$$

$$-kT_L\frac{dT(L)}{dx} - qT(0) + \int_0^L \left(k\left(\frac{dT}{dx}\right)^2 - QT\right)dx = 0$$

Defn $F(T) = \displaystyle\int_0^L \left(\frac{k}{2}\left(\frac{dT}{dx}\right)^2 - QT\right) - qT(0)$

why $\frac{1}{2}$? why no $-kT_L\dfrac{dT(L)}{dx}$?

$T \approx a_1\phi_1 + a_2\phi_2 + a_3\phi_3$



$$\phi_1 = \frac{L/2 - x}{L/2} = \frac{L-2x}{L} \qquad 0 < x < L/2 \qquad 0 \text{ else}$$

$$\phi_2 = \begin{cases} \dfrac{x-0}{\frac{L}{2}} = \dfrac{2}{L}x & 0 < x < L/2 \\[2ex] \dfrac{L-x}{L-L/2} = \dfrac{L-x}{L/2} = \dfrac{2(L-x)}{L} & L/2 < x < L \end{cases}$$

$$\phi_3 = \frac{x - L/2}{L - L/2} = \frac{2x-L}{L} \qquad L/2 < x < L \quad 0 \text{ else}$$

$$\frac{dT}{dx} = a_1\frac{d\phi_1}{dx} + a_2\frac{d\phi_2}{dx} + a_3\frac{d\phi_3}{dx}$$

$$F(T) = \int_0^{L/2} \cdots \; + \int_{L/2}^{L} \cdots$$

$$\frac{d\phi_1}{dx} = -\frac{2}{L} \qquad 0 < x < L/2 \qquad 0 \text{ else}$$

$$\frac{d\phi_2}{dx} = \begin{cases} \dfrac{2}{L} & 0 < x < L/2 \\[2ex] -\dfrac{2}{L} & L/2 < x < L \end{cases}$$

$$\frac{d\phi_3}{dx} = \frac{2}{L} \qquad L/2 < x < L \qquad 0 \text{ else}$$

$$\therefore \; F(T) = \int_0^{L/2} \left(\frac{k}{2}\left(-\frac{2}{L}a_1 + \frac{2}{L}a_2\right)^2 - Q\left(\frac{a_1}{L}(L-2x) + \frac{2a_2}{L}x\right)\right)dx$$

$$+\int_{L/2}^{L}\left(\frac{k}{2}\left(-\frac{2}{L}a_2+\frac{2}{L}a_3\right)^2-Q\left(\frac{2a_2}{L}(L-x)+\frac{a_3}{L}(2x-L)\right)\right)dx$$

$$-q(a_1)$$

$$\Rightarrow F(a_1,a_2,a_3)=\frac{k}{2}\frac{4}{L^2}(a_2-a_1)^2\frac{L}{2}-\frac{Q}{L}\left[a_1(Lx-x^2)+a_2x^2\right]\Big|_0^{L/2}$$

$$+\frac{k}{2}\frac{4}{L^2}\frac{L}{2}(a_3-a_2)^2-\frac{Q}{L}\left[2a_2\left(Lx-\frac{x^2}{2}\right)+a_3(x^2-Lx)\right]\Big|_{L/2}^{L}$$

$$-q\,a_1$$

$$\Rightarrow F(a_1,a_2,a_3)=\frac{k}{L}(a_2-a_1)^2+\frac{k}{L}(a_3-a_2)^2$$

$$-\frac{Q}{L}\left[a_1\underbrace{\left(\frac{L^2}{2}-\frac{L^2}{4}\right)}_{\frac{L^2}{4}}+a_2\frac{L^2}{4}\right]-\frac{Q}{L}\left[2a_2\underbrace{\left(L^2-\frac{L^2}{2}\right)}_{L^2/2}-2a_2\underbrace{\left(L^2/2-\frac{1}{2}\frac{L^2}{4}\right)}\right.$$

$$\left.+a_3(L^2-L^2)-a_3\left(\frac{L^2}{4}-\frac{L^2}{2}\right)\right]$$

$$\Rightarrow F(a_1,a_2,a_3)=\frac{k}{L}(a_2-a_1)^2+\frac{k}{L}(a_3-a_2)^2-\frac{Q}{L}\cdot\frac{L^2}{4}(a_1+a_2)$$

$$-\frac{Q}{L}\left[\underbrace{a_2L^2-\frac{2a_2}{2}\left(\frac{3L^2}{4}\right)}_{\frac{L^2}{4}a_2}+a_3\frac{L^2}{4}\right]$$

$$\Rightarrow F(a_1,a_2,a_3) = \frac{k}{L}(a_2-a_1)^2 + \frac{k}{L}(a_3-a_2)^2 - \frac{QL}{4}(a_1+a_2) - \frac{QL}{4}(a_2+a_3) - a_1 q$$

To minimize This Fn F w.r.t a_1, a_2, a_3 ~~we~~

$\nabla F = 0$ How check Again this is indeed a min/max & not saddle?

$$\Rightarrow \left(\frac{2k}{L}(a_2-a_1)(-1) - \frac{QL}{4} - q\right)\hat{a}_1 + \left(\frac{2k}{L}(a_2-a_1) - \frac{2k}{L}(a_3-a_2) - \frac{QL}{4} - \frac{QL}{4}\right)\hat{a}_2$$

$$+ \left(\frac{2k}{L}(a_3-a_2) - \frac{QL}{4}\right)\hat{a}_3$$

solving $\equiv \vec{0}$ gives

$$\frac{2k}{L}(a_1-a_2) - \frac{QL}{4} - q = 0$$

$$\frac{2k}{L}(a_1 - a_2 - a_2 + a_3) + \frac{QL}{2} = 0 \Rightarrow \frac{2k}{L}(-a_1 + 2a_2 - a_3) - \frac{QL}{2} = 0$$

$$\frac{2k}{L}(-a_2+a_3) - \frac{QL}{4} = 0$$

$$\Rightarrow \frac{2k}{L}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} q \\ 0 \\ 0 \end{pmatrix} + \frac{QL}{4}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

Sam as 2.31 ✓.

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(2.6) Eq 2.11 $\int_0^L w(x) R(\tilde{T},x)\,dx = 0$

$R(u,x) = \frac{d^2 u}{dx^2} - 1$

$R(a_1 x^2, x) = 2a_1 - 1$

$2.11 \Rightarrow \int_0^1 x^2 (2a_1 - 1) = 0 \Rightarrow (2a_1 - 1)\underbrace{\int_0^1 x^2\,dx}_{\text{finite \#}} = 0$

$\Rightarrow a_1 = 1/2$

(2.7) Galerkin Approximations. to $-\frac{d^2 u}{dx^2} = 1 \qquad 0 < x < 1$

$u(0) = u(1) = 0$

(a) $u(x) \simeq a_1 \sin \pi x + a_2 \sin 2\pi x$ (Both satisfy Dirichlet B.C. at $x = 0, 1$)

to determine a_1 & a_2 compute $R = -\frac{d^2 u}{dx^2} - 1$

$\int_0^1 \phi_1 R(u,x)\,dx = 0 \qquad \int_0^1 \phi_2 R(u,x)\,dx = 0$

$R(u) = -a_1 \pi^2 (-1) \sin \pi x - a_2 (\pi 2)^2 (-1) \sin(2\pi x) - 1$

$$\int_0^1 \sin(n\pi x)\sin(m\pi x)\,dx = 0 \quad m \neq n$$

Then eq (1) is

$$\int_0^1 \sin(\pi x)\left[a_1\pi^2 \sin(\pi x) + a_2 4\pi^2 \sin(2\pi x) - 1\right] dx = 0$$

$$+ \int_0^1 \sin(2\pi x)\left[a_1\pi^2 \sin(\pi x) + a_2 4\pi^2 \sin(2\pi x) - 1\right] dx = 0$$

Solve 2 eqs for a_1 + a_2.

1st eq is: $-\frac{2}{\pi} + a_1\frac{\pi^2}{2} = 0$ 2nd eq is: $2a_2\pi^2 = 0 \Rightarrow a_2 = 0$

$$\Rightarrow a_1 = \frac{2}{\pi^2}\frac{\pi}{2} = \frac{1}{\pi}$$

$a_1 = \frac{1}{\pi}$ $a_2 = 0$

(b) $u(x) \simeq a_1 \sin(\pi x) + a_2 \sin(2\pi x) + a_3 \sin(3\pi x)$

$$R(u) = -a_1\pi^2(-1)\sin(\pi x) - a_2 4\pi^2(-1)\sin(2\pi x) - 9\pi^2 a_3(-1)\sin(3\pi x) - 1$$

evaluate $\int_0^1 \phi_1 R\,dx = 0$ $\int_0^1 \phi_2 R\,dx = 0$ $\int_0^1 \phi_3 R\,dx = 0$ + solve for a_1, a_2, a_3.

$$\int_0^1 \sin\pi x\left[\pi^2 a_1 \sin(\pi x) + 4\pi^2 a_2 \sin(2\pi x) + 9\pi^2 a_3 \sin(3\pi x) - 1\right] dx = 0$$

$$\int_0^1 \sin(2\pi x)\left[\pi^2 a_1 \sin(\pi x) + 4\pi^2 a_2 \sin(2\pi x) + 9\pi^2 a_3 \sin(3\pi x) - 1\right] dx = 0$$

$$\int_0^1 \sin(3\pi x)\left[\pi^2 a_1 \sin(\pi x) + 4\pi^2 a_2 \sin(2\pi x) + 9\pi^2 a_3 \sin(3\pi x) - 1\right] dx = 0$$

integrate & solve for a_1, a_2, a_3.

1st eq is same as 1st part $-\frac{2}{\pi} + \frac{a_1\pi^2}{2} = 0 \Rightarrow a_1 = \frac{4}{\pi^3}$

2nd eq is same as 1st part $2a_2\pi^2 = 0 \Rightarrow a_2 = 0$

3rd eq is: $-\frac{2}{3\pi} + \frac{9\pi^2}{2} a_3 = 0 \Rightarrow a_3 = \frac{4}{27\pi^3}$

Note: when ϕ's are also orthogonal (As in $\sin(n\pi x)$ over $(0,1)$ $n \in \mathbb{Z}$)
then to obtain more terms & hence a better approximation we simply
keeps the coefficients already obtained & just computes new ones which
don't depend ~~on~~ on the previous coefficients in any way.

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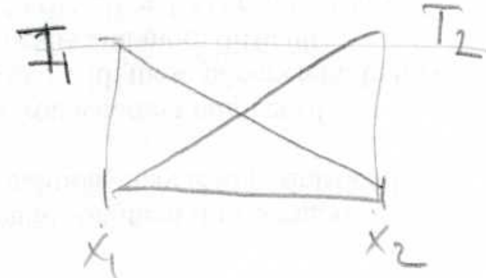
$$T_1 = \alpha_1 + \alpha_2 x_1$$
$$T_2 = \alpha_2 + \alpha_2 x_2$$
$$\Rightarrow \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$$

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \frac{1}{x_2 - x_1} \begin{pmatrix} x_2 & -x_1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$$

$$= \frac{1}{h_1} \begin{pmatrix} T_1 x_2 - T_2 x_1 \\ -T_1 + T_2 \end{pmatrix}$$

one element



$$T(x) = \frac{(T_1 x_2 - T_2 x_1)}{h_1} + \frac{(T_2 - T_1)}{h_1} x$$

$$T(x) = \frac{T_1}{h_1}(x_2 - x) + \frac{T_2}{h_1}(-x_1 + x)$$

$$= \frac{(x_2 - x)}{h_1} T_1 + \frac{(x - x_1)}{h_1} T_2 \qquad \text{eq } 3.12$$

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$$\frac{\alpha_2}{2} h^{(e)} + \frac{\alpha_3}{4}\left(h^{(e)}\right)^2 = T_2^{(e)} - T_1^{(e)}$$

$$\alpha_2 h^{(e)} + \alpha_3 \left(h^{(e)}\right)^2 = T_3^{(e)} - T_1^{(e)}$$

$$\begin{pmatrix} \frac{h^{(e)}}{2} & \frac{\left(h^{(e)}\right)^2}{4} \\ h^{(e)} & \left(h^{(e)}\right)^2 \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} T_2^{(e)} - T_1^{(e)} \\ T_3^{(e)} - T_1^{(e)} \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} \alpha_2 \\ \alpha_3 \end{pmatrix} = \frac{1}{\left(h^{(e)}\right)^3 \left[\frac{1}{2} - \frac{1}{4}\right]} \begin{pmatrix} \left(h^{(e)}\right)^2 & -\frac{\left(h^{(e)}\right)^2}{4} \\ -h^{(e)} & \frac{h^{(e)}}{2} \end{pmatrix} \begin{pmatrix} T_2^{(e)} - T_1^{(e)} \\ T_3^{(e)} - T_1^{(e)} \end{pmatrix}$$

$$= \frac{4}{\left(h^{(e)}\right)^3} \left(h^{(e)}\right) \begin{bmatrix} h^{(e)}\left(T_2^{(e)} - T_1^{(e)}\right) - \frac{\left(h^{(e)}\right)}{4}\left(T_3^{(e)} - T_1^{(e)}\right) \\ -\left(T_2^{(e)} - T_1^{(e)}\right) + \frac{1}{2}\left(T_3^{(e)} - T_1^{(e)}\right) \end{bmatrix}$$

$$= \frac{4}{\left(h^{(e)}\right)^2} \begin{pmatrix} h^{(e)}\left[T_2^{(e)} - T_1 - \frac{1}{4}T_3 + \frac{1}{4}T_1\right] \\ -T_2 + T_1 + \frac{T_3}{2} - \frac{T_1}{2} \end{pmatrix}$$

$$\alpha_2 = \frac{4}{h^{(e)}}\left(T_2 - \frac{3}{4}T_1 - \frac{1}{4}T_3\right) = \frac{1}{h^{(e)}}\left(-3T_1 + 4T_2 - T_3\right)$$

$$\alpha_3 = \frac{4}{\left(h^{(e)}\right)^2}\left(-T_2 + \frac{T_3}{2} + \frac{T_1}{2}\right) = \frac{2}{\left(h^{(e)}\right)^2}\left(T_1 - 2T_2 + T_3\right)$$

into $T^{(e)}(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2$

$$T^{(e)}(x) = T_1^{(e)} + \frac{1}{h^{(e)}}\left(-3T_1 + 4T_2 - T_3\right)x + \frac{2}{\left(h^{(e)}\right)^2}\left(T_1 - 2T_2 + T_3\right)x^2$$

$$T^{(e)}(x) = \left(1 - \frac{3}{h^{(e)}}x + \frac{2x^2}{(h^{(e)})^2}\right)T_1 + \left(\frac{4}{h^{(e)}}x - \frac{4}{h^{(e)2}}x^2\right)T_2$$

$$+ \left(-\frac{1}{h^{(e)}}x + \frac{2}{h^{(e)2}}x^2\right)T_3$$

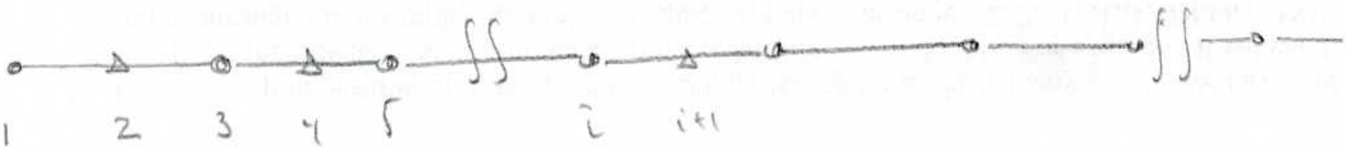
$$= \left(1 - 3\left(\frac{x}{h}\right) + 2\left(\frac{x}{h}\right)^2\right)T_1 + 4\left(\frac{x}{h}\right)\left(1 - \frac{x}{h}\right)T_2$$

$$+ \left(\frac{x}{h}\right)\left(-1 + \frac{2x}{h}\right)T_3$$

$\therefore$ Shape fns are $N_1^{(e)}(x) = 1 - 3\left(\frac{x}{h^{(e)}}\right) + 2\left(\frac{x}{h^{(e)}}\right)^2$

$$N_2^{(e)}(x) = 4\left(\frac{x}{h^{(e)}}\right)\left(1 - \frac{x}{h^{(e)}}\right) \qquad \text{eq } 3.19$$

$$N_3^{(e)}(x) = \frac{x}{h^{(e)}}\left(2\left(\frac{x}{h^{(e)}}\right) - 1\right)$$



elt 1 $\quad 1 < x < 3$

elt 2 $\quad 3 < x < 5$

$\vdots$

elt i $\quad 2i-1 < x < 2(i+1)-1 = 2i+1$

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$$\frac{dT^{(e)}}{dx} = \begin{bmatrix} -\frac{3}{h^{(e)}} + \frac{4x}{h^{(e)2}} & \frac{4}{h}\left(1-\frac{x}{h}\right) + 4\frac{x}{h}\left(\frac{-1}{h}\right) & \frac{1}{h}\left(2\frac{x}{h}-1\right) + \frac{x}{h}\left(\frac{2}{h}\right) \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{h}\left(4\frac{x}{h}-3\right) & \frac{4}{h}\left(1-\frac{x}{h}-\frac{x}{h}\right) & \frac{1}{h}\left[\frac{2x}{h}-1+\frac{2x}{h}\right] \end{bmatrix} \begin{bmatrix} '' \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{h}\left(4\frac{x}{h}-3\right) & \frac{4}{h}\left(1-\frac{2x}{h}\right) & \frac{1}{h}\left(\frac{4x}{h}-1\right) \end{bmatrix} \begin{bmatrix} '' \end{bmatrix} \quad \text{eq } 3.23$$

(3.14) $N_1^{(e)} = 1 - \frac{x}{h^{(e)}}$ w/ $h = L/2$

$N_2^{(e)} = \frac{x}{h^{(e)}}$

(3.34) ⇒ w/ linear elements $\int_0^{L/2} \left\{ k \left(\begin{bmatrix} -\frac{2}{L} \\ \frac{2}{L} \end{bmatrix} \begin{bmatrix} -\frac{2}{L} & \frac{2}{L} \end{bmatrix} \right) \begin{pmatrix} T_1^{(e)} \\ T_2^{(e)} \end{pmatrix} \right.$

$$\left. - \begin{pmatrix} 1 - \frac{2x}{L} \\ \frac{2x}{L} \end{pmatrix} Q \right\} dx + \begin{pmatrix} 0 - \left(-k \frac{dT}{dx}\Big|_0\right) \\ -k \frac{dT}{dx}\Big|_{x=L/2} \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \int_0^{L/2} \left\{ k \begin{bmatrix} \frac{4}{L^2} & -\frac{4}{L^2} \\ -\frac{4}{L^2} & \frac{4}{L^2} \end{bmatrix} \begin{pmatrix} T_1^{(e)} \\ T_2^{(e)} \end{pmatrix} - \begin{pmatrix} 1 - \frac{2x}{L} \\ \frac{2x}{L} \end{pmatrix} Q \right\} dx$$

$$+ \begin{pmatrix} -q \\ -k \frac{dT}{dx}\Big|_{x=L/2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \frac{4k}{L^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \frac{L}{2} \begin{pmatrix} T_1^{(e)} \\ T_2^{(e)} \end{pmatrix} - Q \begin{pmatrix} x - \frac{x^2}{L} \\ \frac{x^2}{L} \end{pmatrix} \Bigg|_0^{L/2} = \begin{bmatrix} q \\ -\left(-k \frac{dT}{dx}\Big|_{x=L/2}\right) \end{bmatrix}$$

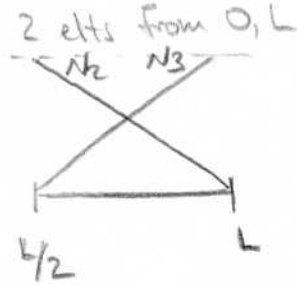
$$\Rightarrow \frac{2k}{L}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} T_1^{(e_1)} \\ T_2^{(e_2)} \end{pmatrix} - \underbrace{Q\begin{pmatrix} \frac{L}{2} - \frac{L^2}{4L} \\ \frac{L}{4} \end{pmatrix}}_{\frac{LQ}{4}\begin{pmatrix} 1 \\ 1 \end{pmatrix}} = \begin{pmatrix} q \\ -\left(-k\frac{dT}{dx}\Big|_{x=L/2}\right) \end{pmatrix}$$

$$\Rightarrow \frac{2k}{L}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} T_1^{(e_1)} \\ T_2^{(e_2)} \end{pmatrix} = \frac{QL}{4}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} q \\ -\left(-k\frac{dT}{dx}\Big|_{x=L/2}\right) \end{pmatrix} \qquad \text{eq 3.35}$$

For element 2:

$$\int_{L/2}^{L} \left\{ k\left[\begin{pmatrix} \frac{dN_2^{(e_2)}}{dx} \\ \frac{dN_3^{(e_2)}}{dx} \end{pmatrix}\begin{pmatrix} \frac{dN_2^{(e_2)}}{dx} & \frac{dN_3^{(e_2)}}{dx} \end{pmatrix}\right]\begin{pmatrix} T_2^{(e_2)} \\ T_3^{(e_2)} \end{pmatrix} - \begin{bmatrix} N_2^{(e_2)} \\ N_3^{(e_2)} \end{bmatrix} Q \right\} dx$$

$$+ \left\{ \begin{bmatrix} N_2^{(e_2)} \\ N_3^{(e_2)} \end{bmatrix}\left(-k\frac{dT}{dx}\right) \right\}_{L/2}^{L} = \vec{0}$$



$$N_2^{(e_2)}(x) = \frac{L-x}{L/2} = \frac{2(L-x)}{L} \qquad \frac{dN_2^{(e_2)}}{dx} = -\frac{2}{L}$$

$$N_3^{(e_2)}(x) = \frac{x - L/2}{L/2} = \frac{2x-L}{L} \qquad \frac{dN_3^{(e_2)}}{dx} = \frac{2}{L}$$

$$\Rightarrow \int_{L/2}^{L} \left\{ k\begin{pmatrix} -\frac{2}{L} \\ \frac{2}{L} \end{pmatrix}\begin{pmatrix} -\frac{2}{L} & \frac{2}{L} \end{pmatrix}\begin{pmatrix} T_2^{(e_2)} \\ T_3^{(e_2)} \end{pmatrix} - \begin{pmatrix} \frac{2(L-x)}{L} \\ \frac{2x-L}{L} \end{pmatrix} Q \right\} dx$$

$$+\begin{pmatrix} 0 - \left(-k \frac{dT}{dx}\Big|_{x=L/2}\right) \\ -k \frac{dT}{dx}\Big|_{x=L} - 0 \end{pmatrix} = \vec{0}$$

$$\Rightarrow \quad k \begin{pmatrix} \frac{4}{L^2} & -\frac{4}{L^2} \\ -\frac{4}{L^2} & \frac{4}{L^2} \end{pmatrix} \frac{L}{2} \begin{pmatrix} T_2^{(e_2)} \\ T_3^{(e_2)} \end{pmatrix} - Q \begin{pmatrix} \frac{2}{L}\left(Lx - \frac{x^2}{2}\right) \\ \frac{x^2 - Lx}{L} \end{pmatrix} \Bigg|_{L/2}^{L}$$

$$+ \begin{pmatrix} -\left(-k \frac{dT}{dx}\Big|_{x=L/2}\right) \\ -k \frac{dT}{dx}\Big|_{x=L} \end{pmatrix} = \vec{0}.$$

$$\Rightarrow \quad \frac{2k}{L}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} T_2^{(e_2)} \\ T_3^{(e_2)} \end{pmatrix} - Q \underbrace{\begin{pmatrix} \frac{2}{L}\left(L^2 - L^2/2\right) - \frac{2}{L}\left(\frac{L^2}{2} - \frac{1}{8}L^2\right) \\ 0 - \frac{1}{L}\left(\frac{L^2}{4} - \frac{L^2}{2}\right) \end{pmatrix}} + \begin{pmatrix} -\left(-k \frac{dT}{dx}\Big|_{x=L/2}\right) \\ -k \frac{dT}{dx}\Big|_{x=L} \end{pmatrix} = 0$$

$$\underbrace{\begin{pmatrix} \frac{2}{L}L^2\frac{1}{2} - \frac{2}{L}L^2\left(\frac{3}{8}\right) \\ -\frac{L^2}{L}\left(-\frac{1}{4}\right) \end{pmatrix}}$$

$$\begin{pmatrix} L - \frac{3}{4}L \\ \frac{L}{4} \end{pmatrix} = \frac{L}{4}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\rightarrow \frac{2k}{L}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} T_2^{(e_2)} \\ T_3^{(e_2)} \end{pmatrix} - \frac{QL}{4}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -k \frac{dT}{dx}\Big|_{x=L/2} \\ -\left(-k \frac{dT}{dx}\Big|_{x=L}\right) \end{pmatrix}$$

$$\rightarrow \frac{2k}{L}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} T_2 \\ T_3 \end{pmatrix} = \frac{QL}{4}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -k\frac{dT}{dx}\Big|_{x=L/2} \\ -\left(-k\frac{dT}{dx}\right)\Big|_{x=L} \end{pmatrix} \qquad \text{eq } 3.36$$

Last eq in 3.37 is

$$\frac{2k}{L}(-T_2 + T_3) = \frac{QL}{4} - \left(-k\frac{dT}{dx}\right)\Big|_{x=L}$$

$$\Rightarrow \left(-k\frac{dT}{dx}\right)\Big|_{x=L} = \frac{QL}{4} + \frac{2k}{L}(T_2 - T_L) \qquad \text{eq } 3.39$$

$$\int_0^L \left\{ k \begin{bmatrix} \frac{1}{L}(4\frac{x}{L}-3) \\ \frac{4}{L}(1-2\frac{x}{L}) \\ \frac{1}{L}(4\frac{x}{L}-1) \end{bmatrix} \begin{pmatrix} \frac{1}{L}(4\frac{x}{L}-3) & \frac{4}{L}(1-\frac{2x}{L}) & \frac{1}{L}(4\frac{x}{L}-1) \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix} \right.$$

$$\left. - \begin{pmatrix} 1-3\frac{x}{L}+2(\frac{x}{L})^2 \\ 4\frac{x}{L}(1-\frac{x}{L}) \\ \frac{x}{L}(2\frac{x}{L}-1) \end{pmatrix} Q \right\} dx + \underbrace{\begin{pmatrix} 0-(-k\frac{dT}{dx}(0)) \\ 0 \\ -k\frac{dT}{dx}(L) \end{pmatrix}}_{\begin{pmatrix} -q \\ 0 \\ -k\frac{dT}{dx}(L) \end{pmatrix}} = \vec{0}$$

eq 3.41

$$\Rightarrow \int_0^L k \begin{pmatrix} \frac{1}{L^2}(4\frac{x}{L}-3)^2 & \frac{4}{L^2}(4\frac{x}{L}-3)(1-\frac{2x}{L}) & \frac{1}{L^2}(4\frac{x}{L}-3)(\frac{4x}{L}-1) \\ \frac{4}{L^2}(1-\frac{2x}{L})(\frac{4x}{L}-3) & \frac{16}{L^2}(1-\frac{2x}{L})^2 & \frac{4}{L^2}(1-\frac{2x}{L})(\frac{4x}{L}-1) \\ \frac{1}{L^2}(\frac{4x}{L}-1)(\frac{4x}{L}-3) & \frac{4}{L^2}(\frac{4x}{L}-1)(1-\frac{2x}{L}) & \frac{1}{L^2}(\frac{4x}{L}-1)^2 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix}$$

$$- \begin{pmatrix} x - \frac{3}{L}\frac{x^2}{2} + \frac{2}{L^2}\frac{x^3}{3} \\ \frac{4}{L}\frac{x^2}{2} - \frac{4}{L^2}\frac{x^3}{3} \\ \frac{2}{L^2}\frac{x^3}{3} - \frac{x^2}{2L} \end{pmatrix} Q \Bigg|_0^L = \begin{pmatrix} q \\ 0 \\ -(-k\frac{dT}{dx}(L)) \end{pmatrix}$$

$$\Rightarrow \int_0^L k \begin{pmatrix} \frac{1}{L^2}\left(\frac{16x^2}{L^2} - \frac{24}{L}x + 9\right) & \frac{4}{L^2}\left(\frac{4x}{L} - \frac{8x^2}{L^2} - 3 + \frac{6x}{L}\right) & \frac{1}{L^2}\left(\frac{16x^2}{L^2} - \frac{4x}{L} - \frac{12x}{L} + 3\right) \end{pmatrix}$$

Integrating w/ MMA gives:

$$\Rightarrow k \begin{pmatrix} \frac{1}{L^2}\frac{7L}{3} & \frac{4}{L^2}\left(\frac{-2L}{3}\right) & \frac{1}{L^2}\frac{L}{3} \\ \frac{4}{L^2}\left(\frac{-2L}{3}\right) & \frac{16}{L^2}\frac{L}{3} & \frac{4}{L^2}\left(\frac{-2L}{3}\right) \\ \frac{1}{3L} & -\frac{8}{3}\frac{1}{L} & \frac{1}{L^2}\left(\frac{7L}{3}\right) \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix}$$

$$- \; Q \begin{pmatrix} L/6 \\ 4\left(\frac{L}{6}\right) \\ \frac{L}{6} \end{pmatrix} = \begin{pmatrix} 9 \\ 0 \\ -\left(-k\frac{dT}{dx}(L)\right) \end{pmatrix}$$

$$\Rightarrow \frac{k}{L} \begin{pmatrix} \frac{7}{3} & -\frac{8}{3} & \frac{1}{3} \\ -\frac{8}{3} & \frac{16}{3} & -\frac{8}{3} \\ \frac{1}{3} & -\frac{8}{3} & \frac{7}{3} \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix} = \frac{QL}{6} \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix} + \begin{pmatrix} 9 \\ 0 \\ -\left(-k\frac{dT}{dx}(L)\right) \end{pmatrix}$$

$$\Rightarrow \frac{k}{6L} \begin{pmatrix} 14 & -16 & 2 \\ -16 & 32 & -16 \\ 2 & -16 & 14 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix} = \frac{QL}{6} \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix} + \begin{pmatrix} 9 \\ 0 \\ -\left(-k\frac{dT}{dx}(L)\right) \end{pmatrix} \qquad \text{eq 3.42}$$

$$\frac{k}{6L}\begin{pmatrix}14 & -16\\ -16 & 32\end{pmatrix}\begin{pmatrix}T_1\\ T_2\end{pmatrix} = \frac{QL}{6}\begin{pmatrix}1\\ 4\end{pmatrix} + \begin{pmatrix}9\\ 0\end{pmatrix} - \begin{pmatrix}2\\ -16\end{pmatrix} T_3 \left(\frac{k}{6L}\right)$$

$$\Rightarrow \frac{k}{6L}\begin{pmatrix}14 & -16\\ -16 & 32\end{pmatrix}\begin{pmatrix}T_1\\ T_2\end{pmatrix} = \frac{QL}{6}\begin{pmatrix}1\\ 4\end{pmatrix} + \begin{pmatrix}9\\ 0\end{pmatrix} + \frac{k}{6L} T_L \begin{pmatrix}-2\\ 16\end{pmatrix} \quad \text{eq } 3.43$$

+ Heat Flux at the Right Border is gr

$$-\left(-k \frac{dT}{dx}(L)\right) = \frac{k}{6L}(2T_1 - 16T_2 + 14T_3) - \frac{QL}{6}(1)$$

$$\Rightarrow -k\frac{dT}{dx}(L) = \frac{k}{6L}(16T_2 - 2T_1 - 14T_3) + \frac{QL}{6} \quad \text{eq } 3.44$$

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$$-\int_0^L w(x)\frac{d}{dx}\left(k(x)\frac{dT}{dx}\right)dx = 0$$

$$\Rightarrow -w(x)\left(k(x)\frac{dT}{dx}\right)\Big|_0^L + \int_0^L k(x)\frac{dw}{dx}\frac{dT}{dx}\,dx = 0$$

$$\int_0^L k(x)\frac{dw}{dx}\cdot\frac{dT}{dx}\,dx - \cancel{w(x)k(x)\frac{dT}{dx}}^{\,0}\Big|_{x=L} + w(x)k(x)\frac{dT}{dx}\Big|_{x=0} = 0 \qquad \text{eq } 3.48$$

Dropping flux terms where temperature is perscribed gives

$$\int_0^L k(x)\frac{dw}{dx}\frac{dT}{dx}\,dx - w\left[-h(T-T_\infty)\right]\Big|_{x=0} = 0$$

$$\int_0^L k(x)\frac{dw}{dx}\cdot\frac{dT}{dx}\cdot dx + wh(T-T_\infty)\Big|_{x=0} = 0 \qquad \text{eq } 3.49$$

- -

$N_i\cancel{h(T-T_\infty)}$ on 1st element

$N_i h\cancel{(T) - N_i h}T_\infty$

x

$$h\cancel{T_\infty}\cancel{\begin{pmatrix} N_1^{(e)} \\ N_2^{(e)} \end{pmatrix}}$$

$N_i h(T - T_\infty)\Big|_{x=0}$ on 1st element $\quad i = 1, 2, 3, \dots, n+1$

$\Rightarrow N_i h(T_1 - T_\infty)$ on 1st element $\Rightarrow i = 1 + 2 \quad N_1(0) = 1$
$N_2(0) = 0$

$$\Rightarrow \begin{bmatrix} h & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_1^{(e_1)} \\ T_2^{(e_1)} \end{bmatrix} - hT_\infty \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

— — — —

$$\begin{pmatrix} N_1(x) \\ N_2(x) \end{pmatrix} h \left(N^1(x) \; N^2(x) \right) \begin{bmatrix} T_1 - T_\infty \\ T_2 - T_\infty \end{bmatrix}\Bigg|_{x=0}$$

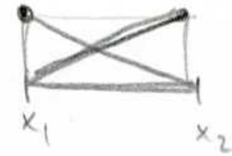
$$= h \begin{bmatrix} N_1(x) \\ N_2(x) \end{bmatrix} \left(N^1(x) \; N^2(x) \right) \left(\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} - T_\infty \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right)\Bigg|_{x=0}$$

$$= h \begin{pmatrix} N_1 \\ N_2 \end{pmatrix} [N^1 \; N^2] \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}\Bigg|_{x=0} - hT_\infty \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} (\cancel{N^1}^{\,1} + \cancel{N^2}^{\,0})\Bigg|_{x=0}$$

$$= h \begin{bmatrix} N_1(x) N^1(x) & N_1(x) N^2(x) \\ N_2(x) N^1(x) & N_2(x) N^2(x) \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}\Bigg|_{x=0} - hT_\infty \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= h \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} - hT_\infty \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Integrating we get: $N_1^{(e_1)} = \dfrac{x_2 - x}{x_2 - x_1}$; $N_2^{(e_1)} = \dfrac{x - x_1}{x_2 - x_1}$



$$\int_{x_1}^{x_2} \begin{bmatrix} \dfrac{x_2 - x}{x_2 - x_1} & \dfrac{x - x_1}{x_2 - x_1} \end{bmatrix} \begin{bmatrix} k_1^{(e_1)} \\ k_2^{(e_1)} \end{bmatrix} \left(\begin{pmatrix} -\dfrac{1}{h^{(e_1)}} \\ \dfrac{1}{h^{(e_1)}} \end{pmatrix} \begin{pmatrix} -\dfrac{1}{h^{(e_1)}} & \dfrac{1}{h^{(e_1)}} \end{pmatrix} \right) dx \begin{pmatrix} T_1^{(e_1)} \\ T_2^{(e_1)} \end{pmatrix}$$

$$+ \begin{pmatrix} h & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} T_1^{(e_1)} \\ T_2^{(e_1)} \end{pmatrix} = h T_\infty \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \cancel{\frac{1}{h^{(e_1)2}} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}$$

$$= \int_{x_1}^{x_2} \begin{bmatrix} \dfrac{x_2 - x}{x_2 - x_1} & \dfrac{x - x_1}{x_2 - x_1} \end{bmatrix} dx \cdot \begin{bmatrix} k_1^{(e_1)} \\ k_2^{(e_1)} \end{bmatrix} \left[\frac{1}{h^{(e_1)2}} \begin{pmatrix} +1 & -1 \\ -1 & +1 \end{pmatrix} \right] \begin{pmatrix} T_1^{(e_1)} \\ T_2^{(e_1)} \end{pmatrix}$$

$$+ \begin{pmatrix} h & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} T_1^{(e_1)} \\ T_2^{(e_1)} \end{pmatrix} = h T_\infty \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \left[\frac{-(x_2 - x)^2}{2(x_2 - x_1)} \quad \frac{(x - x_1)^2}{2(x_2 - x_1)} \right]_{x_1}^{x_2} \begin{bmatrix} k_1^{(e_1)} \\ k_2^{(e_1)} \end{bmatrix} \frac{1}{h^{(e_1)2}} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} T_1^{(e_1)} \\ T_2^{(e_1)} \end{pmatrix}$$

$$+ \begin{pmatrix} h & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} T_1^{(e_1)} \\ T_2^{(e_1)} \end{pmatrix} = h T_\infty \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \left[0 + \frac{(x_2-x_1)}{2} \quad \frac{(x_2-x_1)}{2} \right] \begin{bmatrix} k_1^{(e_1)} \\ k_2^{(e_1)} \end{bmatrix} \frac{1}{h^{(e_1)2}} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} T_1^{(e_1)} \\ T_2^{(e_1)} \end{pmatrix} + '' = ''$$

$$= \frac{1}{2h^{(e_1)}} (+1 \quad 1) \Big(\qquad \Big]$$

$$\Rightarrow \frac{1}{2h^{(e_1)}} \left(+k_1^{(e_1)} + k_2^{(e_1)} \right) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} T_1^{(e_1)} \\ T_2^{(e_1)} \end{bmatrix} + \begin{pmatrix} h & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} T_1^{(e_1)} \\ T_2^{(e_1)} \end{pmatrix} = h T_\infty \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad e_1$$

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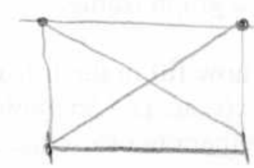
$\forall$ elements $e_i \neq e_1$ we obtain:



$$\int_{x_i}^{x_{i+1}} \left[N_i^{(e_i)} \quad N_{i+1}^{(e_i)} \right] \begin{bmatrix} k_i^{(e_i)} \\ k_{i+1}^{(e_i)} \end{bmatrix} \left(\begin{bmatrix} \frac{dN_i^{(e_i)}}{dx} \\ \frac{dN_{i+1}^{(e_i)}}{dx} \end{bmatrix} \cdot \left[\frac{dN_i^{(e_i)}}{dx} \quad \frac{dN_{i+1}^{(e_i)}}{dx} \right] \right) \begin{pmatrix} T_i^{(e_i)} \\ T_{i+1}^{(e_i)} \end{pmatrix} dx = 0$$

$e_1: x_1 < x < x_2$

$e_i: x_i < x < x_{i+1}$

For Linear elements

w/ $N_i^{(e_i)}(x) = \dfrac{x_{i+1} - x}{x_{i+1} - x_i} \qquad N_{i+1}^{(e_i)}(x) = \dfrac{x - x_i}{x_{i+1} - x_i}$

$$\frac{dN_i^{(e_i)}(x)}{dx} = \frac{-1}{x_{i+1} - x_i} \qquad \frac{dN_{i+1}}{dx} = \frac{1}{x_{i+1} - x_i}$$

$$\Rightarrow \int_{x_i}^{x_{i+1}} \begin{bmatrix} \dfrac{x_{i+1}-x}{x_{i+1}-x_i} & \dfrac{x-x_i}{x_{i+1}-x_i} \end{bmatrix} \begin{bmatrix} k_i^{(ei)} \\ k_{i+1}^{(ei)} \end{bmatrix} \begin{pmatrix} \dfrac{-1}{x_{i+1}-x_i} \\ \dfrac{1}{x_{i+1}-x_i} \end{pmatrix} \begin{pmatrix} \dfrac{-1}{x_{i+1}-x_i} & \dfrac{1}{x_{i+1}-x_i} \end{pmatrix} \cdot \begin{pmatrix} T_i^{(ei)} \\ T_{i+1}^{(ei)} \end{pmatrix} dx = \vec{0}$$

$$\Rightarrow \left(-\frac{(x_{i+1}-x)^2}{2(x_{i+1}-x_i)} \quad \frac{(x-x_i)^2}{2(x_{i+1}-x_i)} \right) \Bigg|_{x_i}^{x_{i+1}} \begin{bmatrix} k_i^{(ei)} \\ k_{i+1}^{(ei)} \end{bmatrix} \begin{pmatrix} \dfrac{1}{(x_{i+1}-x_i)^2} & \dfrac{-1}{(x_{i+1}-x_i)^2} \\ \dfrac{-1}{(x_{i+1}-x_i)^2} & \dfrac{1}{(x_{i+1}-x_i)^2} \end{pmatrix} \cdot \begin{pmatrix} T_i^{(ei)} \\ T_{i+1}^{(ei)} \end{pmatrix} = \vec{0}$$

$$\Rightarrow \left(0 + \frac{h^{(ei)}}{2} \quad \frac{h^{(ei)}}{2} \right) \begin{pmatrix} k_i^{(ei)} \\ k_{i+1}^{(ei)} \end{pmatrix} \frac{1}{h^{(ei)2}} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} T_i^{(ei)} \\ T_{i+1}^{(ei)} \end{pmatrix} = \vec{0}$$

$$\Rightarrow \frac{\left(k_i^{(ei)} + k_{i+1}^{(ei)}\right)}{2h^{(ei)}} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} T_i^{(ei)} \\ T_{i+1}^{(ei)} \end{pmatrix} = \vec{0} \qquad \text{eq 3.53}$$

If $0<x<L$ has been discretized w/ 2 elements : resulting eqs become

$$\frac{1}{L}\begin{pmatrix} k_1^{(e_1)}+k_2^{(e_1)} & -(k_1^{(e_1)}+k_2^{(e_1)}) & 0 \\ -(k_1^{(e_1)}+k_2^{(e_1)}) & +k_1^{(e_1)}+k_2^{(e_1)}+k_1^{(e_2)}+k_2^{(e_2)} & -(k_1^{(e_2)}+k_2^{(e_2)}) \\ 0 & -(k_1^{(e_2)}+k_2^{(e_2)}) & k_1^{(e_2)}+k_2^{(e_2)} \end{pmatrix}\begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix}$$

$$+\begin{pmatrix} h & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix} = hT_\infty\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

w/ $k_1^{(e_1)} \equiv k_1$; $k_2^{(e_1)} = k_1^{(e_2)} = k_2$; $k_2^{(e_2)} = k_3$

$$\Rightarrow \frac{1}{L}\begin{pmatrix} k_1+k_2+Lh & -k_1-k_2 & 0 \\ -k_1-k_2 & k_1+2k_2+k_3 & -k_2+k_3 \\ 0 & -k_2-k_3 & k_2+k_3 \end{pmatrix}\begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix}$$

$$= \begin{pmatrix} hT_\infty \\ 0 \\ 0 \end{pmatrix} \quad \text{eq } 3.54$$

If instead we have a convection Boundary term at $x = L$. formula follows from

3.48 → The Boundary condition is

$$-k\frac{dT}{dx} + h(T - T_\infty) = 0 \qquad \text{at } x = L$$

Then 3.48 Becomes

$$\int_{L/2}^{L} k(x)\frac{dw}{dx}\frac{dT}{dx}\,dx + \left[w\left(-k(x)\frac{dT}{dx}\right)\right]_{x=L} = 0$$

$$\Rightarrow \int_{L/2}^{L} k(x) \begin{bmatrix} \frac{dN_1^{(en)}}{dx} \\ \frac{dN_2^{(en)}}{dx} \end{bmatrix} \begin{bmatrix} \frac{dN_1^{(en)}}{dx} & \frac{dN_2^{(en)}}{dx} \end{bmatrix} \begin{bmatrix} T_1^{(en)} \\ T_2^{(en)} \end{bmatrix} dx + \begin{bmatrix} N_1^{(en)}(x) \\ N_2^{(en)}(x) \end{bmatrix} -k(x) \begin{bmatrix} \frac{dN_1^{(en)}}{dx}(x) & \frac{dN_2^{(en)}}{dx}(x) \end{bmatrix} \begin{bmatrix} T_1^{(en)} \\ T_2^{(en)} \end{bmatrix} \Bigg|_{x=L} = \bar{0}$$

$$\Rightarrow \int_{L/2}^{L} \begin{bmatrix} N_1^{(en)}(x) & N_2^{(en)}(x) \end{bmatrix} \underbrace{\begin{bmatrix} k_1^{(en)} \\ k_2^{(en)} \end{bmatrix}}_{2\times1} \underbrace{\begin{bmatrix} \frac{dN_1^{(en)}}{dx} \\ \frac{dN_2^{(en)}}{dx} \end{bmatrix}}_{2\times1} \begin{bmatrix} \frac{dN_1^{(en)}}{dx} & \frac{dN_2^{(en)}}{dx} \end{bmatrix} \begin{bmatrix} T_1^{(en)} \\ T_2^{(en)} \end{bmatrix} dx$$

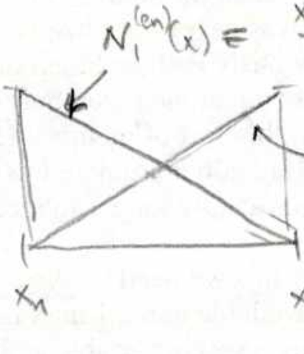
can't take product

$\cancel{\begin{bmatrix} N_1^{(en)}(x) \\ N_2^{(en)}(x) \end{bmatrix}}$ (last term is really an approximation of

$$-k\frac{dT}{dx} = -h(T-T_\infty)$$

By Boundary condition

$$N_1^{(en)}(x) = \frac{x_{n+1}-x}{x_{n+1}-x_n}$$



$N_2^{(en)}(x)$

x_n x_{n+1}

w/ $N_1^{(en)} =$ $=$

$\Rightarrow$

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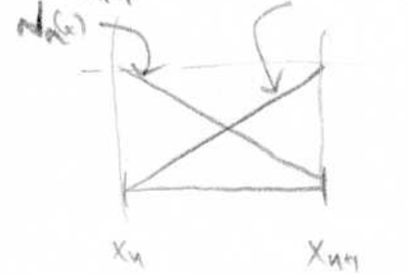
convective BC at $x=L \Rightarrow -k\frac{dT}{dx} + h(T-T_\infty) = 0$ at $x=L$

w/ 3.43 restricted to the last element

$$\int_{x_n}^{x_{n+1}} k(x)\frac{dW}{dx}\frac{dT}{dx}\,dx - \cancel{\left[W\left(-k(x)\frac{dT}{dx}\right)\right]_{x=x_n}}^{\,0} + \left[W\left(-k(x)\frac{dT}{dx}\right)\right]_{x=x_{n+1}} = 0$$



$W = N_i \qquad T(x) \simeq \sum_{j=n}^{n+1} T_j N_j(x)$ in element n.

$$\Rightarrow \int_{x_n}^{x_{n+1}} k(x)\frac{dN_i}{dx}\sum_{j=n}^{n+1} T_j \frac{dN_j(x)}{dx}\,dx + N_i(x)\left(-h(T-T_\infty)\right)\Big|_{x=x_{n+1}} = 0 \qquad i = n \,\&\, n+1.$$

in matrix form, approximating $k(x) \simeq k_1^{(en)} N_n(x) + k_2^{(en)} N_{n+1}(x) \qquad x_n < x < x_{n+1}$

$$\Rightarrow \int_{x_n}^{x_{n+1}} \begin{bmatrix} N_n^{(en)}(x) & N_{n+1}^{(en)}(x) \end{bmatrix} \begin{bmatrix} k_1^{(en)} \\ k_2^{(en)} \end{bmatrix} \begin{bmatrix} \frac{dN_n}{dx} \\ \frac{dN_{n+1}}{dx} \end{bmatrix} \begin{bmatrix} \frac{dN_n}{dx} & \frac{dN_{n+1}}{dx} \end{bmatrix} \begin{bmatrix} T_1^{(en)} \\ T_2^{(en)} \end{bmatrix} dx$$

$$+ \begin{bmatrix} N_n(x) \\ N_{n+1}(x) \end{bmatrix} (-h) \begin{bmatrix} N_n(x) & N_{n+1}(x) \end{bmatrix} \begin{bmatrix} T_1^{(en)} - T_\infty \\ T_2^{(en)} - T_\infty \end{bmatrix} \Bigg|_{x=L=x_{n+1}} = 0$$

Also $N_n(x) = \dfrac{x_{n+1}-x}{x_{n+1}-x_n} \qquad N_{n+1}(x) = \dfrac{x-x_n}{x_{n+1}-x_n}$

$\dfrac{dN_n}{dx} = \dfrac{-1}{x_{n+1}-x_n} \qquad \dfrac{dN_{n+1}}{dx} = \dfrac{1}{x_{n+1}-x_n}$

Then elemental eqs for last element becomes: let $x_{n+1} - x_n = h_n$

$$\int_{x_n}^{x_{n+1}} \begin{bmatrix} \frac{x_{n+1}-x}{h_n} & \frac{x-x_n}{h_n} \end{bmatrix} \begin{bmatrix} k_1^{(en)} \\ k_2^{(en)} \end{bmatrix} \begin{bmatrix} -1/h_n \\ 1/h_n \end{bmatrix} \begin{bmatrix} -1/h_n & 1/h_n \end{bmatrix} \begin{bmatrix} T_1^{(en)} \\ T_2^{(en)} \end{bmatrix} dx$$

$$+ \begin{bmatrix} 0 \\ 1 \end{bmatrix} (-h) \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} T_1^{(en)} - T_\infty \\ T_2^{(en)} - T_\infty \end{bmatrix} = 0$$

$$\Rightarrow \frac{1}{h_n^2} \frac{1}{h_n} \left[\frac{(x_{n+1}-x)^2}{2}(-1) \quad \frac{(x-x_n)^2}{2} \right]_{x_n}^{x_{n+1}} \begin{bmatrix} k_1^{(en)} \\ k_2^{(en)} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} T_1^{(en)} \\ T_2^{(en)} \end{bmatrix}$$

$$+ (-h) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} T_1^{(en)} - T_\infty \\ T_2^{(en)} - T_\infty \end{bmatrix} = 0$$

$$\Rightarrow \frac{1}{h_n^3} \left[0 + \frac{h_n^2}{2} \quad \frac{h_n^2}{2} \right] \left[\quad \cdots \quad \right]$$

$$\Rightarrow \frac{1}{2h_n} \left(k_1^{(en)} + k_2^{(en)} \right) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} T_1^{(en)} \\ T_2^{(en)} \end{bmatrix} - h \begin{bmatrix} 0 \\ T_2^{(en)} - T_\infty \end{bmatrix} = 0$$

$$\Rightarrow \frac{1}{2h^{(en)}}\left(k_1^{(en)} + k_2^{(en)}\right)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} T_1^{(en)} \\ T_2^{(en)} \end{bmatrix} - h\underbrace{\begin{bmatrix} 0 \\ T_2^{(en)} \end{bmatrix}} = +hT_\infty\begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\text{''} \quad + \begin{pmatrix} 0 & 0 \\ 0 & -h \end{pmatrix}\begin{bmatrix} T_1^{(en)} \\ T_2^{(en)} \end{bmatrix} = \text{''} \qquad \text{eq } 3.55$$

Example 3.1

$$k(x) = 40 + \frac{(60-40)}{(10-0)}\frac{W/mC}{\underbrace{cm}} x$$

$$= 40 + 2 \cdot x \cdot \frac{W}{mC}\quad \frac{W}{10^{-2}\,m} = \frac{W}{m^2 C}\cdot 10^2 = 10^2\frac{W}{m^2C}$$

$$= 40 + 200\frac{W}{m^2C}\,x$$

Then exact solution to $-\frac{d}{dx}\left((40+200x)\frac{dT}{dx}\right) = 0$ w/ 3.45

$$-(40+200x)\frac{d^2T}{dx^2} - 200\frac{dT}{dx} = 0$$

$$-(40+200x)\frac{dT}{dx} + 100(T-400) = 0 \qquad x=0 \quad + \quad T(.1) = 39.18$$

$$\Rightarrow \quad -40\frac{dT}{dx} + 100(T-400) = 0 \qquad \text{at } x=0$$

$$\text{eq } 3.45 \Rightarrow -k(x)\frac{dT}{dx} = C_1$$

$$\Rightarrow -(40+200x)\frac{dT}{dx} = C_1$$

$$\Rightarrow \frac{dT}{dx} = \frac{-C_1}{40+200x}$$

$$T = -\frac{C_1}{200}\ln(40+200x) + C_2$$

$$T(10\cdot 10^{-2}m) = -\frac{C_1}{200}\ln(40+200x) + C_2 = 39.18$$
$$= -\frac{C_1}{200}\ln(40+20) + C_2 = 39.18$$
1 B.C.

$$\Rightarrow -\frac{\ln(60)}{200}C_1 + C_2 = 39.18$$

2nd B.C. $\frac{dT}{dx} = \frac{-C_1}{40+200x}$

Then $\left. +40\frac{C_1}{40+200x} + 100\left(\frac{C_1}{200}\ln(40+200x) + C_2 - 400\right) = 0 \right|_{x=0}$

$$\Rightarrow +C_1 + 100\left(\frac{C_1}{200}\ln(40) + C_2 - 400\right) = 0$$

$$\Rightarrow \left(+1 + \frac{1}{2}\ln(40)\right)C_1 + 100C_2 = 40000$$

$\Rightarrow$ 2 eqs to solve for exact sol become:

$$\frac{\ln(60)}{200}C_1 + C_2 = 39.18$$

$$\left(+1 + \frac{\ln(40)}{2}\right)C_1 + 100C_2 = 40{,}000$$

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \frac{1}{\left(\frac{1}{2}\ln(60) + 1 - \frac{1}{2}\ln(40)\right)} \begin{pmatrix} 100 & -1 \\ \left(1 - \frac{\ln(40)}{2}\right) & \frac{\ln(60)}{200} \end{pmatrix} \begin{pmatrix} 39.18 \\ 40000 \end{pmatrix}$$

Then $T(0) = \frac{C_1}{200}\ln(40) + C_2$

$$= \frac{1}{\left(\frac{1}{2}\ln(60) + 1 - \frac{1}{2}\ln(40)\right)}\left((39.18\cdot 100 - 40000)\ln(40) + 39.18\left(1 - \frac{\ln(40)}{2}\right) + \frac{40000}{200}\ln(60)\right)$$

1

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Coarsest possible grid $\Rightarrow$ (one) element.

$$\frac{1}{2(10\cdot 10^{-2})}(40+60)\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{pmatrix}T_1\\ T_2\end{pmatrix} + \begin{pmatrix}100 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}T_1\\ T_2\end{pmatrix} = 100\cdot 400\begin{pmatrix}1\\ 0\end{pmatrix}$$

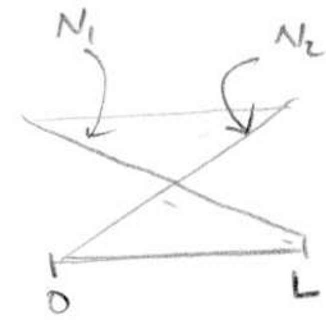
$$\Rightarrow .5\,\frac{100}{10}\cdot 10^2\left[\quad\right] \quad + \quad = \begin{pmatrix}40000\\ 0\end{pmatrix}$$

$$500\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{pmatrix}T_1\\ T_2\end{pmatrix} + \begin{pmatrix}100 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}T_1\\ T_2\end{pmatrix} = \begin{pmatrix}40000\\ 0\end{pmatrix}$$

$$\Rightarrow \begin{bmatrix}5+1 & -5\\ -5 & 5\end{bmatrix}\begin{pmatrix}T_1\\ T_2\end{pmatrix} = \begin{pmatrix}400\\ 0\end{pmatrix} \qquad \text{eq } 3.56$$

$$\begin{pmatrix}T_1\\ T_2\end{pmatrix} = \frac{1}{30-25}\begin{pmatrix}5 & 5\\ 5 & 6\end{pmatrix}\begin{pmatrix}400\\ 0\end{pmatrix} = \begin{pmatrix}5 & 5\\ 5 & 6\end{pmatrix}\begin{pmatrix}80\\ 0\end{pmatrix}$$

$$\begin{pmatrix}T_1\\ T_2\end{pmatrix} = \begin{pmatrix}400\\ 400\end{pmatrix}$$



Then $T(x) = 400\,\underbrace{N_1(x)}_{=\left(\frac{L-x}{L}\right)} + 400\,\underbrace{N_2(x)}_{=\frac{x-}{L}}$

$$= \frac{400}{L}\left(L-x+x\right) = 400$$

For an ~~Adi~~ Adiabatic B.C. at $x = L$ gives

$$-(40 + 200 \cdot x)\left.\frac{dT}{dx}\right|_{x=L=10\cdot10^{-2}} = 0$$

$$\Rightarrow \quad -60 \left.\frac{dT}{dx}\right|_{x=10\cdot10^{-2}} = 0$$

$$\left.\frac{dT}{dx}\right|_{x=10\cdot10^{-2}} = 0$$

From sol $\Rightarrow C_1 = 0 \Rightarrow T = C_2$

w/ $\quad -k(x)\cancelto{0}{\frac{dT}{dx}} + h(T - T_\infty) = 0$

$$\Rightarrow \quad C_2 = T_\infty = 400 \quad \Rightarrow \quad T(x) = 400$$

$$\begin{pmatrix} 6 & -5 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = \begin{pmatrix} 400 \\ 39.18 \end{pmatrix} \Rightarrow \quad T_1 = 99.3167$$

$$\frac{|T_1 - 100|}{100} = .0068 < .7\% \text{ error}$$

$$aT + b\frac{dT}{dx} + c = 0$$

$$\parallel$$

$$hT - k(0)\frac{dT}{dx} - hT_\infty = 0$$

$$-k(0)\frac{dT}{dx} + h(T - T_\infty) = 0$$

- - - - - - -

eq 3.52

$$\frac{1}{2(.05)}(40+50)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{pmatrix} T_1 \\ T_2 \end{pmatrix} + \begin{pmatrix} 100 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = 100 \cdot 400 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \frac{90}{2\left(\frac{.1}{2}\right)}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{pmatrix} T_1 \\ T_2 \end{pmatrix} + \begin{pmatrix} 100 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = \begin{pmatrix} 40000 \\ 0 \end{pmatrix}$$

$$900\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{pmatrix} T_1 \\ T_2 \end{pmatrix} + \begin{pmatrix} 100 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = \begin{pmatrix} 40000 \\ 0 \end{pmatrix}$$

last eq on pg 28.

For element 2.

$$\frac{1}{(.1)}(50+60)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{pmatrix} T_1^{(e2)} \\ T_2^{(e2)} \end{pmatrix} = \vec{0}$$

$$\Rightarrow 1100\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{pmatrix} T_2 \\ T_3 \end{pmatrix} = 0$$

Assembling eqs

$$\Rightarrow \begin{bmatrix} 1000 & -900 & 0 \\ -900 & 900+1100 & -1100 \\ 0 & -1100 & 1100 \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{pmatrix} 40000 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{bmatrix} 10 & -9 & 0 \\ -9 & 20 & -11 \\ 0 & -11 & 11 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{pmatrix} 400 \\ 0 \\ 0 \end{pmatrix}$$

Dirichlet conditions req $T_3 = 39.18$

$$\Rightarrow \begin{bmatrix} 10 & -9 & 0 \\ -9 & 20 & -11 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix} = \begin{pmatrix} 400 \\ 0 \\ 39.18 \end{pmatrix}$$

Solving we get $T_1 = 99.822$, $T2 = 66.4689$, $T_3 = 39.18$

$$\frac{|T_1 - 100|}{100} = .0017 < .2\% \text{ error.}$$

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$$-W\, rk \frac{dT}{dr}\Big|_{r_1}^{r_2} + \int_{r_1}^{r_2} \frac{dw}{dr} \cdot rk \frac{dT}{dr}\, dr = 0$$

$$\Rightarrow \underbrace{-W(r_2)\, r_2 k(r_2) \frac{dT}{dr}(r_2)}_{} + \underbrace{W(r_1)\, r_1 k(r_1) \frac{dT}{dr}(r_1)}_{} + \int_{r_1}^{r_2} \frac{dw}{dr} \cdot rk \frac{dT}{dr}\, dr = 0$$

$= 0$ If Drichlet condition are applied flux is zero. $\quad + W(r_1) r_1 h (T(r_1) - T_\infty)$

$$\Rightarrow \quad rW(r) h (T - T_\infty)\Big|_{r=r_1} + \int_{r_1}^{r_2} r k(r) \frac{dw}{dr} \frac{dT}{dr}\, dr = 0 \qquad \text{eq } 3.63$$

testing $W = N_1^{(ei)}, N_2^{(ei)}$

$$\underbrace{K^{(ei)} \int_{r_1^{(ei)}}^{r_2^{(ei)}} r \begin{bmatrix} \frac{dN_1}{dr}^{(ei)} \\ \frac{dN_2}{dr}^{(ei)} \end{bmatrix} \begin{bmatrix} \frac{dN_1}{dr}^{(ei)} & \frac{dN_2}{dr}^{(ei)} \end{bmatrix} \begin{bmatrix} T_1^{(ei)} \\ T_2^{(ei)} \end{bmatrix} dr}_{}$$

produces term like $\frac{k^{(ei)}}{2h^{(ei)}} (r_1^{(ei)} + r_2^{(ei)}) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

Terms like $\quad rWh(T - T_\infty)\Big|_{r^*} = r^* \begin{bmatrix} N_1^{(ei)} \\ N_2^{(ei)} \end{bmatrix} h \begin{bmatrix} N_1^{(ei)} & N_2^{(ei)} \end{bmatrix} \begin{bmatrix} T_1^{(ei)} - T_\infty \\ T_2^{(ei)} - T_\infty \end{bmatrix}\Big|_{r_1^*}$

Case **1**: $\quad N_1^{(ei)}(r^*) = 1 \qquad N_2^{(ei)}(r^*) = 0$

$\Rightarrow$

$$N_1^{(ei)}(r^*) = 0 \qquad N_2^{(ei)}(r^*) = 1$$

CASE 1:

$$= r^* h \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix}} \begin{bmatrix} T_1^{(e_i)} - T_\infty \\ T_2^{(e_i)} - T_\infty \end{bmatrix} = 0$$

$$r^* h \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_1^{(e_i)} - T_\infty \\ T_2^{(e_i)} - T_\infty \end{bmatrix} = 0$$

CASE 2:

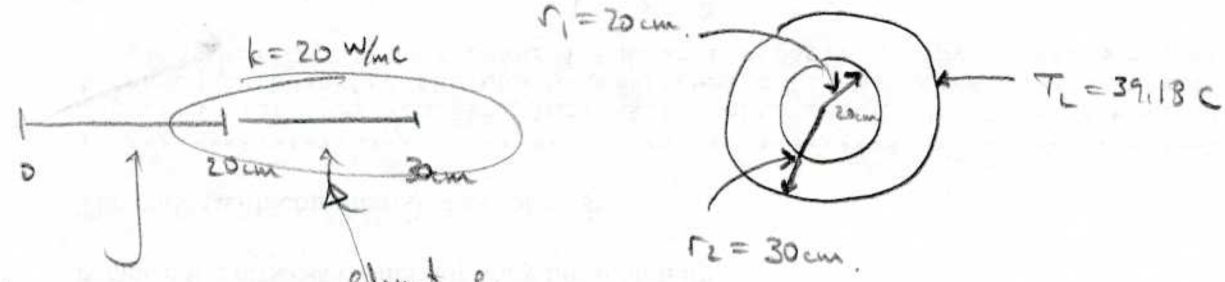
$$r^* h \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix}} \begin{bmatrix} T_1^{(e_i)} - T_\infty \\ T_2^{(e_i)} - T_\infty \end{bmatrix} = 0$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} T_1^{(e_i)} - T_\infty \\ T_2^{(e_i)} - T_\infty \end{bmatrix} = 0$$

$$r^* h \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \\ \end{bmatrix} - r^* h T_\infty \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

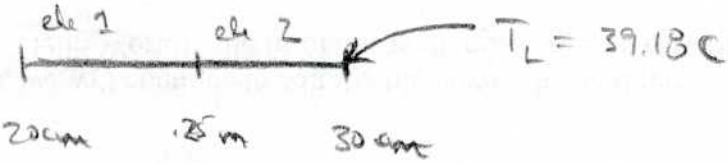
$$+ \; r^* h \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \\ \end{bmatrix} - r^* h T_\infty \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Ex 3.2



element e_1:

$h^{(e_1)} = r_2^{(e_1)} - r_1^{(e_1)} =$ This region is region considered.



$h^{(e_1)} = .25 - .2 = .05$

$k^{(e_1)} = 20$

eq 3.64 gives; for e_1

$$\left[\frac{20}{2(.05)} .45 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + (.2) 50 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right] \begin{bmatrix} T_1^{(e_1)} \\ T_2^{(e_1)} \end{bmatrix}$$

$$= (.2) 50 \cdot 400 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left(\frac{(20)(.45)}{2 \frac{.1}{2}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{1}{5} \cdot 50 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{pmatrix} T_1^{(e_1)} \\ T_2^{(e_1)} \end{pmatrix} = 4000 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left(90 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + 10 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{pmatrix} T_1^{(e_1)} \\ T_2^{(e_1)} \end{pmatrix} = 4000 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The 2nd element becomes

$$\frac{20}{2(\frac{1}{2})}(.55)\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{pmatrix}T_2\\ T_3\end{pmatrix} = 0$$

$$\begin{array}{r} 555 \\ \underline{\times\ \ 5} \\ 110 \end{array}$$

$$\Rightarrow \quad 110\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{pmatrix}T_2\\ T_3\end{pmatrix} = 0$$

Assembling these two elemental matrices gives:

$$\begin{pmatrix}10 & -9 & 0\\ -9 & 9+1 & -1\\ 0 & -1 & 1\end{pmatrix}\begin{pmatrix}T_1\\ T_2\\ T_3\end{pmatrix} = \begin{pmatrix}400\\ 0\\ 0\end{pmatrix}$$ imposing R.H.S B.C gives

$$\begin{pmatrix}10 & -9 & 0\\ -9 & 10 & -1\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}T_1\\ T_2\\ T_3\end{pmatrix} = \begin{pmatrix}400\\ 0\\ 39.18\end{pmatrix} \Rightarrow \begin{array}{l} T_1 = 229.085 \\ T_2 = 210.095 \\ T_3 = 39.18 \end{array}$$

Exact sol to

$$-\frac{1}{r}\frac{d}{dr}\left(rk\frac{dT}{dr}\right) = 0 \Rightarrow rk\frac{dT}{dr} = C_1 \Rightarrow \frac{dT}{dr} = \frac{C_1}{20r} \Rightarrow T = C_2 + \frac{C_1}{20}\ln r$$

$$T(.3) = \frac{C_1}{20}\ln(.3) + C_2 = 39.18$$ outer radius (pipe wall)

$$-20\frac{C_1}{20(.2)} + 50\left(C_2 + \frac{C_1}{20}\ln(.2) - 400\right) = 0$$

$$\Rightarrow \quad \frac{1}{20}\ln(.3)\,C_1 + C_2 = 39.13 \quad \Bigg\} \Rightarrow C_1 =$$

$$\underbrace{\left(-\frac{1}{.2} + \frac{5}{2}\right)}\,C_1 + 50 = 20000 \qquad C_2 =$$

$$\underbrace{-5 + \frac{5}{2}}$$

$$-\frac{5}{2}$$

Exact solution gives:

$$T(.2) = \frac{C_1}{20}\ln(.2) + C_2 = -678,072\ ?$$

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$$\int_{x_i}^{x_{i+1}} N_j(x) N_i(x)\, dx = \int_{-1}^{+1} N_j(\xi) N_k(\xi) \frac{h^{(ci)}}{2}\, d\xi = \quad \text{eq } 3.67a$$

$$\int_{x_i}^{x_{i+1}} \frac{dN_j}{dx} \frac{dN_k}{dx}\, dx = \int_{-1}^{1} \frac{dN_j}{d\xi} \cdot \frac{d\xi}{dx} \cdot \frac{dN_k}{d\xi} \cancel{\frac{d\xi}{dx}} \cancel{\frac{dx}{d\xi}}\, d\xi$$

$$= \frac{2}{h^{(ci)}} \int_{-1}^{1} \frac{dN_j}{d\xi} \frac{dN_k}{d\xi}\, d\xi \quad \text{eq } 3.67b$$

$$\int_{x_i}^{x_{i+1}} \frac{dN_j}{dx} N_k\, dx = \int_{-1}^{1} \frac{dN_j}{d\xi} \cdot \cancel{\frac{d\xi}{dx}} \cdot N_k \cancel{\frac{dx}{d\xi}}\, d\xi = \quad \text{eq } 3.67c$$

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a, b non neg int.

$$\int_{-1}^{1} \left(\frac{1}{2}\right)^a (1-\xi)^a \left(\frac{1}{2}\right)^b (1+\xi)^b\, d\xi = \frac{1}{2^{a+b}} \int_{-1}^{1} (1-\xi)^a (1+\xi)^b\, d\xi$$

let $v = 1+\xi \qquad \xi = v-1$

$dv = d\xi$

$$= \frac{1}{2^{a+b}} \int_0^2 (1-v+1)^a v^b\, dv = \frac{1}{2^{a+b}} \int_0^2 (2-v)^a v^b\, dv$$

looks like B fn look up.

Beta fn

$$B(p,q) = \int_0^1 t^{p-1} (1-t)^{q-1}\, dt$$

let $v = 2w$

Then

$$= \frac{1}{2^{a+b}} \int_0^1 (2-2w)^a\, 2^b w^b\, 2\, dw = \frac{2^{a+b+1}}{2^{a+b}} \int_0^1 (1-w)^a w^b\, dw = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}$$

$$= 2\int_0^1 (1-w)^a w^b \, dw = 2B(a+1, b+1)$$

$$= \frac{2\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)} = \frac{2a!\,b!}{(a+b+1)!} \quad \text{eq } 3.70$$

a & b positive integers.