Some Notes and Solutions from the Book: Matrices and Transformations: by Anthony J. Pettofrezzo

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June 18, 2012

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Chapter 1: Matrices

Special Real Matrices

Problem 9

Consider $(A^2)^T = A^T A^T = (-A)(-A) = A^2$ and so A is symmetric.

Problem 10

Consider

$$A^T = \begin{bmatrix} 0 & -a & -b \\ a & 0 & c \\ b & -c & 0 \end{bmatrix} = - \begin{bmatrix} 0 & a & b \\ -a & 0 & -c \\ -b & c & 0 \end{bmatrix} = -A\,,$$

and

$$A^2 = \begin{bmatrix} 0 & -a & -b \\ a & 0 & c \\ b & -c & 0 \end{bmatrix} \begin{bmatrix} 0 & -a & -b \\ a & 0 & c \\ b & -c & 0 \end{bmatrix} = \begin{bmatrix} -a^2 - b^2 & bc & -ac \\ cb & -a^2 - c^2 & -ab \\ -ca & -ab & -b^2 - c^2 \end{bmatrix}.$$

We see that A^2 is symmetric.

Problem 11

If AB is symmetric then $(AB)^T = AB$, while when we use the fact that A and B are symmetric the left-hand-side is equal to $B^T A^T = BA$. Now to show the other direction we note that if $BA = AB$ then $AB = BA = (-B^T)(-A^T) = B^T A^T = (AB)^T$ showing that AB is symmetric.

Special Complex Matrices

Problem 1

Part (a): $\bar{A} = \begin{bmatrix} 1 & 3 - 2i \\ i & 2 + i \end{bmatrix}$.

Part (b): $A^* = \bar{A}^T = \begin{bmatrix} 1 & i \\ 3 - 2i & 2 + i \end{bmatrix}$.

Problem 2

Part (a): To have A be Hermitian means that $A^* = A$, thus the matrices where that is true are A, C, and D.

Part (b): To have A be skew Hermitian means that $A^* = -A$, thus the matrices where this is true are C, E, and H.

Problem 3

The diagonals of a skew-Hermitian matrix must satisfy $\overline{a_{ii}} = -a_{ii}$. If we write $a_{ii} = A + iB$ this means that

$$A - iB = -A - iB\,,$$

or

$$\begin{aligned} A &= -A \\ -B &= -B\,, \end{aligned}$$

To have $A = -A$ means that $A = 0$ while the second equation implies no constraint. Thus the elements of the diagonal of a skew symmetric matrix must be pure imaginary (or zero).

Problem 7

Since $(AB)^* = B^*A^*$ and $(A^*)^* = A$ we have $(AA^*)^* = (A^*)^*A^* = AA^*$.

Problem 10

Note that

$$A = \frac{1}{2}(A + A^*) + \frac{1}{2}(A - A^*)\,,$$

where $\frac{1}{2}(A + A^*)$ is Hermitian and $\frac{1}{2}(A - A^*)$ is skew-Hermitian.

Problem 11

Let X be Hermitian matrix so that $X^* = X$, then writing the ijth component of this relationship gives $a_{ij} + ib_{ij} = a_{ji} - ib_{ji}$. Equating the real and imaginary parts of that equation gives $a_{ij} = a_{ji}$ and $b_{ij} = -b_{ji}$. Thus the matrix A (defined to be composed of the elements a_{ij}) is real and symmetric and the matrix B made up of the elements b_{ij} is real and skew symmetric.

Problem 12

Let X be skew-Hermitian so that $X^* = -X$. Let the ijth component of X be denoted by $a_{ij} + ib_{ij}$. Then from $X^* = -X$ we see that

$$-a_{ij} - ib_{ij} = a_{ji} - ib_{ji}\,.$$

Equating real and imaginary parts gives

$$\begin{aligned} -a_{ij} &= a_{ji} \\ b_{ij} &= b_{ji}\,, \end{aligned}$$

so the matrix A with elements a_{ij} is real and skew-symmetric and B a matrix with elements b_{ij} is real and symmetric.

Chapter 2: Inverses and Systems of Matrices

Determinants

Problem 1

$$\begin{vmatrix} 5 & 3 \\ 2 & 4 \end{vmatrix} = 20 - 6 = 14\,.$$

Problem 2

$$\begin{vmatrix} 4 & -2 \\ 2 & 1 \end{vmatrix} = 4 + 4 = 8\,.$$

Problem 3

$$\begin{vmatrix} 5 & 2 \\ 1 & 0 \end{vmatrix} = 0 - 2 = -2\,.$$

Problem 4

$$\begin{vmatrix} 8 & 4 \\ 2 & 1 \end{vmatrix} = 0\,.$$

Problem 5

$$\left|\begin{array}{ccc|cc} 2 & 3 & 1 & 2 & 0 \\ 1 & 4 & -3 & 1 & 4 \\ -1 & 2 & 0 & -1 & 2 \end{array}\right. = 0 + 9 + 2 - (-4) - (-12) - 0 = 11 + 4 + 12 = 27\,.$$

Problem 6

$$\left|\begin{array}{ccc|cc} -3 & 1 & 1 & -3 & 1 \\ 1 & -3 & 1 & 1 & -3 \\ 1 & 1 & -3 & 1 & 1 \end{array}\right. = -27 + 1 + 1 - (-3) - (-3) - (-3) = -25 + 9 = -16\,.$$

Problem 7

If two rows are identical we can use the theorem that the value of the determinant remains unchanged if every row is increased by a scalar multiple of the corresponding element of another row. The scalar multiple is -1 and we combine the two rows that are identical. This gives a row of all zeros. A matrix with a row of all zeros must have a determinant since *each* term

$$(-1)^k a_{1i_1} a_{2i_2} \cdots a_{ni_n} \, ,$$

will have a factor from the row that is all zeros. Thus each term is the sum of zero meaning that the total determinant is zero.

Problem 9

Since when we multiply a single row by k that modifies the determinant by multiplying its value by k (this is Theorem 2-4) the matrix kA is k times all four rows of A and thus gives $k^4|A|$. Thus we get $|kA| = k^4 m$.

Problem 10

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$ then

$$|A||B| = (ad - cb)(ps - qr) = adps - adqr - cbps + cbqr \, ,$$

and

$$AB = \begin{bmatrix} ap + br & aq + bs \\ cp + dr & qc + ds \end{bmatrix} .$$

So we have that $|AB|$ given by

$$\begin{aligned} |AB| &= (ap + br)(qc + ds) - (cp + dr)(aq + bs) \\ &= apqc + apds + brqc + brds \\ &- cpaq - cpbs - draq - drbs \\ &= adps - adqr - cpbs + cbqr \, , \end{aligned}$$

the same as $|A||B|$.

Problem 12

To evaluate the determinant of an upper triangular matrix like that discussed we can expand using cofactors on the first column. This gives a_{11} times the determinant of the smaller matrix obtained by deleting the first row and column. Evaluating this matrix using cofactors along the first column of this smaller matrix we get $a22$ times the again smaller matrix. Continuing we get

$$|A| = \prod_{i=1}^{n} a_{ii} \, .$$

Problem 14

Expand the left-hand-side in terms of cofactors of the elements in the first column as

$$\begin{aligned}
a \begin{vmatrix} d & r & s \\ 0 & e & f \\ 0 & g & h \end{vmatrix} - c \begin{vmatrix} b & m & n \\ 0 & e & f \\ 0 & g & h \end{vmatrix} &= ad(eh - fg) - cb(eh - fg) \\
&= (ad - cb)(eh - fg) \\
&= \begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{vmatrix} e & f \\ g & h \end{vmatrix} ,
\end{aligned}$$

as we were to show.

Problem 15

Just evaluating using cofactors of the first column gives

$$\begin{aligned}
\begin{vmatrix} a+b & a & a \\ a & a+b & a \\ a & a & a+b \end{vmatrix} &= (a+b)((a+b)^2 - a^2) - a(a(a+b) - a^2) \\
&+ a(a^2 - a(a+b)) - a(ab) + a(-ab) \\
&= 2a^2b + ab^2 + 2ab^2 + b^3 - a^2b - a^2b \\
&= 3ab^2 + b^3 = b^2(3a+b) \, .
\end{aligned}$$

Problem 16

To make the matrix $\begin{bmatrix} -3 & 1 & 1 \\ 1 & -3 & 1 \\ 1 & 1 & -3 \end{bmatrix}$ match $\begin{bmatrix} a+b & a & a \\ a & a+b & a \\ a & a & a+b \end{bmatrix}$ we take $a = 1$ and $b = -4$ so that we get $16(3-4) = -16$ for the determinant.

Inverse of a Matrix

Problem 1 (a 2×2 inverse)

For 2×2 matrix $\left[\begin{array}{cc} a & b \\ c & d \end{array}\right]$ the inverse is $\frac{1}{ad-bc}\left[\begin{array}{cc} d & -b \\ -c & a \end{array}\right]$ so for $\left[\begin{array}{cc} 3 & 2 \\ 5 & 4 \end{array}\right]$ our inverse is given by

$$\frac{1}{12-10}\left[\begin{array}{cc} 4 & -2 \\ -5 & 3 \end{array}\right] = \frac{1}{2}\left[\begin{array}{cc} 4 & -2 \\ 5 & 3 \end{array}\right] = \left[\begin{array}{cc} 2 & -1 \\ -\frac{5}{2} & \frac{3}{2} \end{array}\right].$$

Problem 5 (a 3×3 inverse)

For the 3×3 matrices we will use the fact that

$$A^{-1} = \frac{(A_{ij})^T}{\det(A)}.$$

Lets first evaluate $\det(A)$.

$$\begin{aligned} \det(A) &= 1(24-25) - 2(12-15) + 3(10-12) \\ &= -1 - 2(-3) + 3(-2) = -1 + 6 - 6 = -1\,. \end{aligned}$$

Then lets evaluate each of the needed cofactors

$$\begin{aligned}
A_{11} &= \left|\begin{array}{cc} 4 & 5 \\ 5 & 6 \end{array}\right| = -1\,, \quad A_{31} = \left|\begin{array}{cc} 2 & 3 \\ 4 & 5 \end{array}\right| = -2 \\
A_{12} &= -\left|\begin{array}{cc} 2 & 5 \\ 3 & 6 \end{array}\right| = +3\,, \quad A_{32} = -\left|\begin{array}{cc} 1 & 3 \\ 2 & 5 \end{array}\right| = +1 \\
A_{13} &= \left|\begin{array}{cc} 2 & 4 \\ 3 & 5 \end{array}\right| = -2\,, \quad A_{33} = \left|\begin{array}{cc} 1 & 2 \\ 2 & 4 \end{array}\right| = 0 \\
A_{21} &= -\left|\begin{array}{cc} 2 & 3 \\ 5 & 6 \end{array}\right| = +3\,, \quad A_{22} = \left|\begin{array}{cc} 1 & 3 \\ 3 & 6 \end{array}\right| = -3 \\
A_{23} &= -\left|\begin{array}{cc} 1 & 2 \\ 3 & 5 \end{array}\right| = -(5-6) = +1\,,
\end{aligned}$$

Thus the cofactor matrix of A is

$$\left[\begin{array}{ccc} -1 & 3 & -2 \\ 3 & -3 & 1 \\ -2 & +1 & 0 \end{array}\right].$$

Thus the inverse of our matrix is

$$A^{-1} = \frac{1}{-1}\left[\begin{array}{ccc} -1 & 3 & -2 \\ +3 & -3 & 1 \\ -2 & +1 & 0 \end{array}\right] = \left[\begin{array}{ccc} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{array}\right].$$

Problem 3 (another 2×2 inverse)

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}^{-1} = \frac{1}{\cos(\theta)^2 + \sin(\theta)^2} \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}.$$

Problem 5

Let $B = \begin{bmatrix} x & t & v \\ y & u & w \end{bmatrix}$, then to be a left inverse of $A = \begin{bmatrix} 1 & 1 \\ 3 & 4 \\ 0 & 0 \end{bmatrix}$ means that $BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Thus in this case we need to have

$$\begin{bmatrix} x & t & v \\ y & u & w \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 3 & 4 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} x + 3t & x + 4t \\ y + 3u & y + 4u \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus

$$\begin{aligned} x + 3t &= 1 \\ x + 4t &= 0\,. \end{aligned}$$

Which give $t = -1$ and $x = 4$, and

$$\begin{aligned} y + 3u &= 0 \\ y + 4u &= 1\,. \end{aligned}$$

So $y = -3u$ thus $u = 1$ and $y = -3$. Thus $B = \begin{bmatrix} 4 & -1 & v \\ -3 & 1 & w \end{bmatrix}$ is a left inverse of A for any values of v and w. For B to be a right inverse of A then

$$B = \begin{bmatrix} x & t & v \\ y & u & w \end{bmatrix},$$

would need to satisfy

$$\begin{bmatrix} 1 & 1 \\ 3 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x & t & v \\ y & u & w \end{bmatrix} = \begin{bmatrix} x + y & t + u & v + w \\ 3x + 4y & 3t + 4u & 3u + 4w \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The bottom row shows that this expression cannot be satisfied.

Problem 10

The fact that $AA^{-1} = I$ means that the inverse of A^{-1} is A. This is also the statement given.

Problem 11

If $AB = 0$ and $B \neq 0$ then A is singular. To show this assume that A was not singular. Then a left inverse of A would exist. Multiply by A^{-1} on the left of $AB = 0$ gives $A^{-1}AB = 0$ or $B = 0$. This is a contradiction to the fact that $B \neq 0$, showing that A must be singular.

Problem 12

Consider that $(B^{-1}A^{-1})(AB) = B^{-1}B = I$ showing that $B^{-1}A^{-1}$ is the left-inverse of AB. The same technique will show that $B^{-1}A^{-1}$ is the right-inverse of AB thus $B^{-1}A^{-1}$ is the inverse of AB. This is the statement we are asked to prove.

Problem 15

Let A be symmetric and let B be its inverse. Then $AB = I$. Taking the transpose of both sides gives $B^T A^T = I$ or $B^T A = I$, showing that B^T is an inverse of A, the unique inverse of A is know to be B thus $B^T = B$ and our inverse is symmetric.

Problem 16

We want to prove that if $AB = BA$ then $A^{-1}B^{-1} = B^{-1}A^{-1}$. Starting with $AB = BA$ and multiplying by $B^{-1}A^{-1}$ on the left to get

$$I = (BA)(B^{-1}A^{-1}) .$$

Thus the inverse of BA is $B^{-1}A^{-1}$ or in symbols this statement is $(AB)^{-1} = B^{-1}A^{-1}$.

Problem 17

Consider $AA^{-1} = I$ then by taking transposes of both sides we get $(A^{-1})^T A^T = I$, showing that the inverse of A^T is $(A^{-1})^T$ or

$$(A^T)^{-1} = (A^{-1})^T .$$

Systems of Matrices

Problem 10

The inverse of the given matrix is

$$\frac{1}{a^2+b^2}\begin{bmatrix} a & -b \\ b & a \end{bmatrix},$$

which has a complex representation of

$$\frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}.$$

While the reciprocal of the complex number $a+ib$ is given by

$$(a+ib)^{-1} = \frac{1}{a+ib} = \frac{1}{a+ib}\frac{a-ib}{a-ib} = \frac{a-ib}{a^2+b^2},$$

the same as the above.

Problem 11

We have

$$A \odot B = AB - BA = -(BA - AB) = -(B \odot A) = -B \odot A.$$

Problem 12

Consider the left-hand-side

$$A \odot (B \odot C) = A \odot (BC - CB) = ABC - ACB - BCA + CBA,$$

while the right-hand-side is given by

$$\begin{aligned}(A \odot B) \odot C &= (AB - BA) \odot C \\ &= (AB - BA)C - C(AB - BA) \\ &= ABC - BAC - CAB + CBA,\end{aligned}$$

which is *not* equal to the earlier expression.

Rank of a Matrix

Problem 1

To be linearly dependent means there exists constants k_1, k_2, and k_3 not all zero such that

$$k_1(2x+z) + k_2(x+y) + k_3(2y-z) = 0,$$

or

$$(2k_1 + k_2)x + (k_2 + 2k_3)y + (k_1 - k_3)z = 0\,.$$

To have this be true for all x, y, and z means that

$$\begin{aligned} 2k_1 + k_2 &= 0 \\ k_2 + 2k_3 &= 0 \\ k_1 - k_3 &= 0\,. \end{aligned}$$

This last equation means that $k_1 = k_3$ so

$$\begin{aligned} 2k_1 + k_2 &= 0 \\ k_2 + 2k_1 &= 0\,. \end{aligned}$$

The solution to this is $k_2 = -2k_1$ where k_1 is arbitrary. Thus we have

$$k_1(2x + z) - 2k_1(x + y) + k_1(2y - z) = 0\,,$$

for any value of k_1.

Problem 2

Lets try to show that the three functions are linearly dependent. This means that there exists constants k_1, k_2, and k_3 such that

$$k_1(x + y - z) + k_2(2x + y + 3z) + k_3(x + 2y + z) = 0\,,$$

or

$$(k_1 + 2k_2 + k_3)x + (k_1 + k_2 + 2k_3)y + (-k_1 + 3k_2 + 4k_3)z = 0\,.$$

For this to be true for all x, y, and z means that

$$\begin{aligned} k_1 + 2k_2 + k_3 &= 0 \\ k_1 + k_2 + 2k_3 &= 0 \\ -k_1 + 3k_2 + 4k_3 &= 0\,. \end{aligned}$$

We would want to solve this system for k_1, k_2, and k_3. Taking the negative of the first equation and adding to the second equation gives

$$-k_2 + k_3 = 0 \quad \Rightarrow \quad k_2 = k_3\,.$$

Then the first equation is given by

$$k_1 + 3k_2 = 0\,, \tag{1}$$

and the third equation is given by

$$-k_1 + 7k_2 = 0 \quad \text{or} \quad k_1 = 7k_2\,.$$

When this is put into Equation 1 gives $10k_2 = 0$ or $k_2 = 0$. Thus all k's are zero. Thus the three functions are not linearly dependent.

Problem 3

The rank of a matrix is the order of the largest submatrix that has a non-zero determinant. For the matrix $\begin{bmatrix} 4 & 2 & 3 \\ 8 & 5 & 2 \\ 12 & -4 & 5 \end{bmatrix}$ we can try to take the determinant directly (then the rank would be 3). We find

$$\begin{aligned} & 4\begin{vmatrix} 5 & 2 \\ -4 & 5 \end{vmatrix} - 2\begin{vmatrix} 8 & 2 \\ 12 & 5 \end{vmatrix} + 3\begin{vmatrix} 8 & 5 \\ 12 & -4 \end{vmatrix} \\ = \; & 4(25+8) - 2(40-24) + 3(-32-60) = -86 \neq 0\,, \end{aligned}$$

thus this system has a rank of 3.

Problem 4

To see if this matrix has rank 3 we try to take the determinant of this matrix. If it is non-zero then the rank is 3. We find

$$\begin{aligned} \begin{vmatrix} 2 & 3 & 3 \\ 3 & 6 & 12 \\ 2 & 4 & 8 \end{vmatrix} &= 2\begin{vmatrix} 6 & 12 \\ 4 & 8 \end{vmatrix} - 3\begin{vmatrix} 3 & 12 \\ 2 & 8 \end{vmatrix} + 3\begin{vmatrix} 3 & 6 \\ 2 & 4 \end{vmatrix} \\ &= 2(48-48) - 3(24-24) + 3(12-12) = 0\,. \end{aligned}$$

To see if this matrix has a rank of 2 consider the possible 2×2 determinants

$$\begin{vmatrix} 2 & 3 \\ 3 & 6 \end{vmatrix}, \begin{vmatrix} 3 & 3 \\ 6 & 12 \end{vmatrix}, \ldots$$

This second determinant is $36 - 18 = 18 \neq 0$ thus this matrix has rank 2.

Problem 5

Since we have to take determinants of submatrices of this matrix the rank can be at most 3. Consider the determinant of

$$\begin{vmatrix} 3 & 4 & 0 \\ 6 & 8 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 1\begin{vmatrix} 3 & 4 \\ 6 & 8 \end{vmatrix} = 0\,.$$

Consider the determinant of

$$\begin{vmatrix} 3 & 4 & 2 \\ 6 & 8 & 4 \\ 1 & 0 & 1 \end{vmatrix} = 1\begin{vmatrix} 4 & 2 \\ 8 & 4 \end{vmatrix} + 1\begin{vmatrix} 3 & 4 \\ 6 & 8 \end{vmatrix} = 16 - 16 = 0\,.$$

Consider the determinant of

$$\begin{vmatrix} 4 & 0 & 2 \\ 8 & 0 & 4 \\ 0 & 1 & 1 \end{vmatrix} = -\begin{vmatrix} 4 & 2 \\ 8 & 4 \end{vmatrix} = 0\,.$$

Thus the rank cannot be 2. Consider the 2×2 determinant. Many of them are zero for example

$$\begin{vmatrix} 3 & 4 \\ 6 & 8 \end{vmatrix} = 0\,,$$

but

$$\begin{vmatrix} 8 & 0 \\ 0 & 1 \end{vmatrix} = 8 \neq 0\,,$$

Thus this matrix has a rank of 2.

Problem 6 (the rank of a skew symmetric matrix cannot be 1)

Let the matrix A be skew symmetric and assume that the rank of this matrix was 1. This means that there exists a 1 rowed minor of the matrix and that no larger r-rowed minor can exist namely all 2-rowed minors must have a zero determinant. Consider the 2-rowed minor consisting of the submatrix with elements $(1,1)$, $(1,2)$, $(2,1)$, and $(2,2)$. Since the diagonal elements must be zero the determinant of this minor is given by

$$a_{11}a_{22} - a_{12}a_{21} = -a_{12}a_{21}\,.$$

Since A is skew symmetric so we get that $a_{12} = -a_{21}$ and the above condition becomes $a_{12}^{@} = 0$. Thus we have just shown that a_{12} must be zero. In the same way, consider the 2-rowed minor of the elements $(2,2)$, $(2,3)$, $(3,2)$ and $(3,3)$. We can then show that $a_{23} = 0 = a_{32}$. Following this we can show that all elements on the sub and super diagonal of A are zero.

Problem 7

The maximum number of entries to change a full matrix A into a row echelon form we need to eliminate all elements below the diagonal, thus we need $n-1$ elementary row operations to eliminate the elements in the first column, $n-2$ operations to eliminate the elements in the second column, etc. When we get to the $n-1$st column we need one elementary row operation to eliminate the elements in the $n-1$st column. In general the kth row needs $n-k$ elementary row operations to make a column of zeros below it. Thus we need

$$\sum_{k=1}^{n-1} n - k = \sum_{k=1}^{n-1} k = \left.\frac{k(k+1)}{2}\right|_1^n = \frac{n(n+1)}{2}\,.$$

Problem 8

The elementary row transformations matrices for a 2×2 matrix are given by exchanging rows

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Scalar multiplication of a row

$$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}.$$

and row addition and multiplication

$$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}.$$

Problem 9

Consider the following sequence of matrix manipulations

$$P_1 = \begin{bmatrix} 1 & 0 \\ -1/3 & 1 \end{bmatrix} \begin{bmatrix} 6 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 0 & 1/3 \end{bmatrix}.$$

$$P_2 = \begin{bmatrix} 1/6 & 0 \\ 0 & 1 \end{bmatrix} P_1 = \begin{bmatrix} 6 & 2 \\ 0 & 1/3 \end{bmatrix}.$$

$$P_3 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} P_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1/3 \end{bmatrix}.$$

$$P_4 = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus

$$A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/6 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/3 & 1 \end{bmatrix}.$$

Problem 10

Consider the following sequence of matrix manipulations

$$P_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 7/2 \end{bmatrix}.$$

$$P_2 = \begin{bmatrix} 1 & 0 \\ 0 & 2/7 \end{bmatrix} P_1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

$$P_3 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} P_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus

$$A^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2/7 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1/2 \end{bmatrix}.$$

Problem 11

Consider the following sequence of matrix manipulations

$$P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 \\ -2 & 3 & 1 \\ -5 & 9 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 0 \\ 0 & -1 & 1 \\ -5 & 9 & 3 \end{bmatrix}.$$

$$P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix} P_1 = \begin{bmatrix} 1 & -2 & 0 \\ 0 & -1 & 1 \\ -5 & 9 & 3 \end{bmatrix}.$$

$$P_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} P_2 = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 3 \end{bmatrix}.$$

$$P_4 = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} P_3 = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{bmatrix}.$$

$$P_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} P_4 = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}.$$

$$P_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} P_5 = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$P_7 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} P_6 = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$P_8 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} P_7 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We can get the inverse of A by reversing the order of the elementary matrix products above.

Problem 12

From exercise 9 we can write A^{-1} as the product of invertible elementary row transformations. Then A can be computed by inverting and reversing the order of all product matrices in A. Thus since

$$A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/6 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/3 & 1 \end{bmatrix},$$

we get

$$A = \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1/3 \end{bmatrix}.$$

Now note that elementary row transformation matrices are easily inverted and we have

$$\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

With these two results we get

$$A = \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/3 \end{bmatrix}.$$

System of Linear Equations

Problem 1

The augmented system is given by

$$\left[\begin{array}{ccc|c} 2 & -4 & 5 & 10 \\ 2 & -11 & 10 & 36 \\ 4 & -1 & 5 & -6 \end{array}\right].$$

Dividing the first row by 2 gives

$$\left[\begin{array}{ccc|c} 1 & -2 & 5/2 & 5 \\ 2 & -11 & 10 & 36 \\ 4 & -1 & 5 & -6 \end{array}\right].$$

Multiply the first row by -2 and add it to the second row. Multiply the first row by -4 and add it to the third row and we get

$$\left[\begin{array}{ccc|c} 1 & -2 & 5/2 & 5 \\ 0 & -7 & 5 & 26 \\ 0 & 7 & -5 & -26 \end{array}\right].$$

Adding the second row to the third row gives

$$\left[\begin{array}{ccc|c} 1 & -2 & 5/2 & 5 \\ 0 & -7 & 5 & 26 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Divide the second row by -7 to get

$$\left[\begin{array}{ccc|c} 1 & -2 & 5/2 & 5 \\ 0 & 1 & -5/7 & -26/7 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Multiply the second row by 2 and add to the first to get

$$\left[\begin{array}{ccc|c} 1 & 0 & -5/7 & 28/7 \\ 0 & 1 & -5/7 & -26/7 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus if we take z as arbitrary and once it is specified we have x and y given by

$$\begin{aligned} x &= \frac{28}{7} + \frac{5}{7}z \\ y &= -\frac{26}{7} + \frac{5}{7}z\,. \end{aligned}$$

Problem 2

The augmented system is given by

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 1 & -2 & -1 & 5 \\ 2 & 4 & -2 & 7 \end{array}\right].$$

Multiply the first row by -1 and add it to the second row. Multiply the first row by -2 and add it to the third row. We then get

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -4 & 0 & 3 \\ 0 & 0 & 0 & 3 \end{array}\right].$$

Since $0 \neq 3$ this system has no solution.

Problem 3

The augmented system is given by

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 12 \\ 1 & -1 & -2 & 3 \\ 0 & 3 & 3 & k \end{array}\right].$$

If we change the order of rows 1 and 2 we get

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & 3 \\ 2 & 1 & -1 & 12 \\ 0 & 3 & 3 & k \end{array}\right].$$

Multiply row 1 by -2 and add it to row 2 to get

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & 3 \\ 0 & 3 & 3 & 18 \\ 0 & 3 & 3 & k \end{array}\right].$$

Problem 5

To have a non trivial solution if and only if the rank of the matrix of coefficients is less than n. The matrix of in this case is (and transforms as)

$$\begin{bmatrix} 3 & 8 & 2 \\ 2 & 1 & 3 \\ -5 & -1 & 1 \end{bmatrix} \Rightarrow$$

The rank is the size of the largest nonzero determinant

$$\begin{aligned} \begin{vmatrix} 3 & 8 & 2 \\ 2 & 1 & 3 \\ -5 & -1 & 1 \end{vmatrix} &= 3\begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix} - 2\begin{vmatrix} 8 & 2 \\ -1 & 1 \end{vmatrix} - 5\begin{vmatrix} 8 & 2 \\ 1 & 3 \end{vmatrix} \\ &= 3(1+3) - 2(* + 2) - 5(24-2) = -128 \neq 0\,. \end{aligned}$$

Thus this matrix has rank 3. This system has no nontrivial solution.

Problem 6

Consider the determinant of the given matrix

$$\begin{aligned} \begin{vmatrix} 1 & 3 & -5 \\ 3 & -1 & 5 \\ 3 & 2 & -1 \end{vmatrix} &= 1\begin{vmatrix} -1 & 5 \\ 2 & -1 \end{vmatrix} - 3\begin{vmatrix} 3 & -5 \\ 2 & -1 \end{vmatrix} + 3\begin{vmatrix} 3 & -5 \\ -1 & 5 \end{vmatrix} \\ &= 1 - 10 - 3(-3+10) + 3(15-5) = 0\,. \end{aligned}$$

Thus the rank cannot be three so it must be less than three and there can be nontrivial solutions to the linear homogeneous equations.

Chapter 3: Transformation of the Plane

Mappings

Problem 1

Part (a): Yes, it is a one-to-one mapping.

Part (b): No, this cannot be onto.

Part (c): Yes, this is a one-to-one mapping.

Part (d): Yes, but this is not one-to-one since $T(1) = 0 = T(-1)$.

Rotations

Problem 7

Consider the equation $x^2 - xy + y^2 = 16$, but in a coordinate system rotated by $\pi/4$. In that case we have

$$
\begin{aligned}
x &= \frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{2}}y' \\
y &= -\frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{2}}y' \,,
\end{aligned}
$$

and we get

$$\left(\frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{2}}y'\right)^2 - \left(\frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{2}}y'\right)\left(-\frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{2}}y'\right) + \left(-\frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{2}}y'\right)^2 = 16$$

or when we expand this and simplify we get

$$\frac{3}{2}x'^2 + \frac{1}{2}y'^2 = 16 \,.$$

the same as in the back of the book.

Problem 8

If $\theta = \arctan(\frac{4}{3})$, then $\cos(\theta) = \frac{3}{5}$ and $\sin(\theta) = \frac{4}{5}$ and we get

$$
\begin{aligned}
x &= \cos(\theta)x' + \sin(\theta)y' = \frac{3}{5}x' + \frac{4}{5}y' \\
y &= -\sin(\theta)x' + \cos(\theta)y' = -\frac{4}{5}x' + \frac{3}{5}y' \,.
\end{aligned}
$$

Problem 9

Since the equation $x^2+y^2=r^2$ is a circle centered at $(0,0)$ with a radius r we expect it to be invariant under rotations of the plane through any angle θ i.e. to have a mapped equation given by $x'^2+y'^2=r^2$. Lets see if this is indeed true. Since

$$\begin{aligned} x &= \cos(\theta)x' + \sin(\theta)y' \\ y &= -\sin(\theta)x' + \cos(\theta)y' . \end{aligned}$$

The equation $x^2+y^2=r^2$ becomes

$$(\cos(\theta)x' + \sin(\theta)y')^2 + (-\sin(\theta)x' + \cos(\theta)y')^2 = r^2 ,$$

or

$$\cos(\theta)^2x'^2 + 2\cos(\theta)\sin(\theta)x'y' + \sin(\theta)^2y'^2 + \sin(\theta)^2x'^2 - 2\sin(\theta)\cos(\theta)x'y' + \cos(\theta)^2y'^2 = r^2 ,$$

or when we simplify

$$x'^2 + y'^2 = r^2 ,$$

as claimed.

Problem 10 (can this matrix be a rotation?)

To be a rotation about the origin would require that the given matrix be of the form

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} .$$

So the $(1,2)$ and $(2,1)$ elements need to be negatives of each other. This given matrix cannot represent a rotation about the origin.

Problem 11 (multiplication of rotation matrices)

Part (b): Is given by $T_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$ to be closed means that if T_{θ_1} and T_{θ_2} are both clockwise rotations then the product $T_{\theta_1}T_{\theta_2}$ is also a clockwise rotation. Consider that product

$$\begin{aligned} T_{\theta_1}T_{\theta_2} &= \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) \\ \sin(\theta_1) & \cos(\theta_1) \end{bmatrix} \begin{bmatrix} \cos(\theta_2) & -\sin(\theta_2) \\ \sin(\theta_2) & \cos(\theta_2) \end{bmatrix} \\ &= \begin{bmatrix} \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) & -\sin(\theta_2)\cos(\theta_1) - \sin(\theta_1)\cos(\theta_2) \\ \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2) & -\sin(\theta_1)\sin(\theta_2) + \cos(\theta_1)\cos(\theta_2) \end{bmatrix} \\ &= \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{bmatrix} . \end{aligned}$$

Part (b): To commute means that $T_{\theta_1}T_{\theta_2} = T_{\theta_2}T_{\theta_1}$, which we can see is true from the above formula.

Problem 12 (determine a rotation matrix)

What is the angle between $A = (3, 4)$ and $A' = (-4, 3)$?

$$||A||||A'|| \cos(\theta) = A \cdot A' = -12 + 12 = 0\,.$$

Thus $\cos(\theta) = 0$ or $\theta = \frac{\pi}{2}$. Is this the same angle between B and B'?

$$||B||||B'|| \cos(\hat{\theta}) = B \cdot B' = 2 - 2 = 0\,.$$

Yes.

Problem 13 (the distance between two points is rotation invariant)

The distance, d, between two points is given by $d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$ but since

$$\begin{aligned} x_1 &= \cos(\theta)x'_1 + \sin(\theta)y'_1 \\ y_1 &= -\sin(\theta)x'_1 + \cos(\theta)y'_1\,. \end{aligned}$$

We have

$$\begin{aligned} x_1 - x_2 &= \cos(\theta)(x'_1 - x'_2) + \sin(\theta)(y'_1 - y'_2) \\ y_1 - y_2 &= -\sin(\theta)(x'_1 - x'_2) + \cos(\theta)(y'_1 - y'_2)\,. \end{aligned}$$

Thus

$$\begin{aligned} (x_1 - x_2)^2 + (y_1 - y_2)^2 &= \cos(\theta)^2(x'_1 - x'_2)^2 \\ &+ 2\cos(\theta)\sin(\theta)(x'_1 - x'_2)(y'_1 - y'_2) \\ &+ \sin(\theta)^2(y'_1 - y'_2)^2 \\ &+ \sin(\theta)^2(x'_1 - x'_2)^2 \\ &- 2\cos(\theta)\sin(\theta)(x'_1 - x'_2)(y'_1 - y'_2) \\ &+ \cos(\theta)^2(y'_1 - y'_2)^2 \\ &= (x'_1 - x'_2)^2 + (y'_1 - y'_2)^2 = d'^2\,. \end{aligned}$$

Showing that the distance is the same independent of the frame we measure it in.

Reflections, Dilations, and Magnifications

Problem 1 (transformations represented by matrices)

This matrix represents expansion of each dimension by two, i.e. dilation of the plane.

Problem 2 (transformations represented by matrices)

Applying this transformation each point (x, y) is mapped to $(-x, -y)$ which is a reflection through the origin. This is a one-to-one mapping of the plane onto itself.

Problem 3 (transformations represented by matrices)

Applying this transformation matrix the point $(x, y) \to (2y, 2x) = 2(y, x)$ looks like a composite mapping. First note that the mapping $(x, y) \to (y, x)$ is a rotation about the line $y = x$ and the factor of 2 is a dilation.

Problem 4 (the product of reflection matrices)

For the two reflection matrices $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ their product is given by

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

which is a rotation of the plane by the angle $\pi = 180\,\text{deg}$. In other words the rotation matrix $\begin{bmatrix} \cos(\pi) & \sin(\pi) \\ -\sin(\pi) & \cos(\pi) \end{bmatrix}$. For some other products of refection matrices we find

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

both are rotations about the angle $\theta = 0$.

Problem 5 (distance is invariant under reflection)

Part (a): For reflection about the x axis the transformation matrix T is $T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and we have the new point (x', y') is given by

$$\begin{aligned} x' &= x \quad \Rightarrow \quad x = x' \\ y' &= -y \quad \Rightarrow \quad y = -y', \end{aligned}$$

so we see that the distance between two points (x_1, y_1) and (x_2, y_2) in the original space is given by

$$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 = (x_1' - x_2')^2 + (y_1' - y_2')^2 = d'^2.$$

Part (b): For a reflection about the y axis the transformation matrix T is $T = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ and the same proof above applies.

Problem 6 (the reflection matrix that maps across the line $3x - 4y = 0$)

To find this reflection matrix F we perform this mapping in steps and then combined the steps into one procedure.

1. Rotate the given line to the x or y axis.
2. Reflect all points across that coordinate axis.
3. Rotate all points back to the original line orientation i.e. undo the first transformation.

To perform the first step we will rotate the given line to the x-axis. To do this requires a clockwise rotation of by an amount $\theta = \tan^{-1}\left(\frac{3}{4}\right)$, where

$$\cos(\theta) = \frac{4}{5} \quad \text{and} \quad \sin(\theta) = \frac{3}{5},$$

This ration is then given by the matrix $\begin{bmatrix} 4/5 & 3/5 \\ -3/5 & 4/5 \end{bmatrix}$. To check this result observe that under this transformation the point $(4, 3)$ maps to

$$\begin{bmatrix} \frac{16}{5} + \frac{9}{5} \\ -\frac{12}{5} + \frac{12}{5} \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix},$$

as it should. Next note that the matrix representing a reflection across the x-axis is given by $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Finally we need the inverse of the first rotation matrix which is the transpose of the original matrix or

$$\begin{bmatrix} 4/5 & -3/5 \\ 3/5 & 4/5 \end{bmatrix}.$$

Thus in total applying all three steps we have the transformation T, given by

$$\begin{aligned} T &= \begin{bmatrix} 4/5 & -3/5 \\ 3/5 & 4/5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 4/5 & 3/5 \\ -3/5 & 4/5 \end{bmatrix} = \begin{bmatrix} 4/5 & -3/5 \\ 3/5 & 4/5 \end{bmatrix} \begin{bmatrix} 4/5 & 3/5 \\ 3/5 & -4/5 \end{bmatrix} \\ &= \begin{bmatrix} 7/25 & 24/25 \\ 24/25 & -7/25 \end{bmatrix}. \end{aligned}$$

Problem 7 (dilation and rotation matrices are commutative)

A dilation matrix in the plane has the form $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ for some $k > 0$ while a rotation counterclockwise by the angle θ has the form $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$. With these two we have

$$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} = \begin{bmatrix} k\cos(\theta) & -k\sin(\theta) \\ k\sin(\theta) & k\cos(\theta) \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}.$$

showing that these two matrices commute.

Problem 8

The reflection of the plane with respect to the line $y = x$ is $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$. The reflection of the plane with respect to the line $y = \sqrt{3}x$ is $\begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$. Note that both of these were transformations we discussed in this section of the book. The transformation we want then is the product

$$\begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix},$$

this is a rotation counterclockwise through the angle $\theta = \frac{5\pi}{6}$.

Problem 9

Part (a): If $4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ then $x' = 4x$ and $y' = 4y$ so $x^2 + y^2 = 1$ becomes $x'^2 + y'^2 = 16$. In the same way we have

Part (b): $\frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Part (c): $r\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Problem 9

The triangles new vertexes become $(0, 0)$, $(3, 6)$ and $(9, 3)$.

Problem 10

From this magnification of the plane we have $x' = \frac{1}{5}x$ and $y' = \frac{1}{2}y$ so our line $2x + 5y = 10$ becomes

$$2(5x') + 5(2y') = 10$$

or $x' + y' = 1$.

Problem 11

The squares vertexes are mapped to $(0,0)$, $(2,0)$, $(2,1)$, and $(0,1)$.

Other Transformation

Problem 1

The point (x,y) is mapped to $(0,y)$ or we are projecting all points onto the y-axis.

Problem 2

The point (x,y) is mapped to $(x+y,0)$ or all points are mapped to the x-axis to the location $x+y$.

Problem 3

The point (x,y) is mapped to the point $(x-2y,y)$. This transformation is a shear parallel to the x-axis of an amount $2y$ to the "left". This is like the example given in the book but with shearing in the negative x direction.

Problem 4

The new points are $(-1-2,-1)$, $(3-2,-1)$, $(3+4,2)$, and $(-1+4,2)$ or $(-3,-1)$, $(1,-1)$, $(7,2)$ and $(3,2)$ this is a shear parallel to the x-axis.

Problem 5 (shear parallel to the y-axis)

Part (a): Under the given shear parallel to the y-axis we have $x' = x$ and $y' = 2x + y$. So the circle $x^2 + y^2 = 1$ becomes

$$x'^2 + (y' - 2x')^2 = 1\,.$$

Expanding this we get

$$5x'^2 - 4x'y' + y'^2 = 1\,,$$

which is an ellipse.

Part (b): The new points are $(0,0)$, $(2,2k)$, $(2,2k+1)$, and $(0,1)$ for $k > 0$. The points $(2,0)$ and $(2,1)$ go to the points $(2,2k)$ and $(2,2k+1)$ and the rectangle becomes a parallelogram.

Problem 6

The transformation that is a shear parallel to the x-axis looks like $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ and the transformation that is a dilation of the plane looks like $\begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$. Together they form the transformation

$$\begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} c & ck \\ 0 & c \end{bmatrix}.$$

Thus

$$\begin{aligned} x' &= cx + cky \\ y' &= cy\,. \end{aligned}$$

Solving for x and y we get

$$\begin{aligned} x &= \frac{1}{c}(x' - ky') \\ y &= \frac{y'}{c}\,. \end{aligned}$$

Under this transformation the line $y = 3x - 2$ goes to

$$\frac{y'}{c} = \frac{3}{c}(x' - ky') - 2\,.$$

or

$$(1 + 3k)y' - 3x' = -2c\,.$$

If we take $k = -\frac{1}{3}$ we get $-3x' = -2c$ or $x' = \frac{2}{3}c$. Taking $c = 9$ we have specified the parameters of the transformation.

Problem 7

If T must map the given points we would have to have $T = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$.

Problem 8

Consider the product

$$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} = \begin{bmatrix} \cos(\theta) + k\sin(\theta) & -\sin(\theta) + k\cos(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix},$$

vs.

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & k\cos(\theta) - \sin(\theta) \\ \sin(\theta) & k\sin(\theta) + \cos(\theta) \end{bmatrix}.$$

Since these two products are not equal a shear parallel to the x-axis does not commute with a rotation of the plane.

Linear Homogeneous Transformations

Problem 1

$(0, 0)$ times any matrix is the point $(0, 0)$ back again.

Problem 2

We have

$$\begin{aligned} x' &= ax + by \\ y' &= cx + dy\,, \end{aligned}$$

since the transformation T is $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. This transformation is nonsingular and so an inverse exists so that we can solve for x and y in terms of x' and y' as

$$\begin{aligned} x &= \tilde{a}x' + \tilde{b}y' \\ y &= \tilde{c}x' + \tilde{d}y'\,. \end{aligned} \tag{2}$$

In this case the line $Ax + By = C$ becomes

$$A(\tilde{a}x' + \tilde{b}y') + B(\tilde{c}x' + \tilde{d}y') = C\,,$$

or

$$(A\tilde{a} + B\tilde{c})x' + (A\tilde{b} + B\tilde{d})y' = C\,,$$

or another line.

Problem 3 (parallel lines are parallel lines)

Let our lines be written as $y = mx + b_1$ and $y = mx + b_2$. Note: this parametrization will not handle vertical line (which would have $m = \pm\infty$) but lines of this type could be considered separately with a different parametrization and we would find that the same statement holds. Using the two Equations 2 we have these lines transforming as

$$\begin{aligned}\tilde{c}x' + \tilde{d}y' &= m(\tilde{a}x' + \tilde{b}y') + b_1 \\ \tilde{c}x' + \tilde{d}y' &= m(\tilde{a}x' + \tilde{b}y') + b_2\,,\end{aligned}$$

or

$$\begin{aligned}(\tilde{d} - m\tilde{b})y' + (\tilde{c} - m\tilde{a})x' &= b_1 \\ (\tilde{d} - m\tilde{b})y' + (\tilde{c} - m\tilde{a})x' &= b_2\,.\end{aligned}$$

As the coefficients of x' and y' are equal these two lines are parallel.

Problem 4

Consider the square with vertexes given by the points $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$. These are mapped to the points $(0,0)$, (a,c), $(a+b, c+d)$ and (b,d). If we plot these new points in the x-y axis when a, b, c, and d are positive the resulting structure has the appearance of a parallelogram.

Problem 5

Note that $\begin{vmatrix} 6 & 3 \\ 2 & 1 \end{vmatrix} = 6 - 6 = 0$, thus this matrix is singular. A point (x,y) is mapped to the new point

$$\begin{aligned}x' &= 6x + 3y = 3(2x + y) \\ y' &= 2x + y\,.\end{aligned}$$

Thus all points (x,y) are mapped to the line $x' = 3y'$.

Problem 6

Note that

$$\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4x + 2y \\ 2x + y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Both equations represent the expression $2x + y = 0$ and points on this line represent the null space of the matrix operator. Now points on this line are the ones that satisfy $y = -2x$ and thus

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -2x \end{bmatrix} = x \begin{bmatrix} 1 \\ -2 \end{bmatrix},$$

and so points on the line $y = -2x$ are represented by the span of the vector $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

Orthogonal Matrices

For these problems we need to recall some definitions. For a matrix to be orthogonal means that $AA^T = I$. To be a *proper* orthogonal matrix means that $\det(A) = +1$. To be an improper orthogonal matrix means that $\det(A) = -1$.

Problem 1

We compute

$$\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and

$$\det\left(\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}\right) = \frac{2}{4} + \frac{2}{4} = 1\,.$$

This is a proper orthogonal matrix.

Problem 2

We compute

$$\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and

$$\det\left(\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}\right) = -\frac{3}{4} - \frac{1}{4} = -1\,.$$

This is a improper orthogonal matrix.

Problem 3

We compute

$$\begin{bmatrix} 2 & 0 \\ 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 4 & x \\ x & x \end{bmatrix}.$$

Since the $(1,1)$ component is not 1, this matrix is not orthogonal.

Problem 4

The $(1,1)$ element of AA^T is $3^2 + 5^2 \neq 1$. Thus A is not orthogonal.

Problem 7

We have

$$A + I = \begin{bmatrix} \frac{25}{13} & \frac{5}{13} \\ \frac{5}{13} & \frac{1}{13} \end{bmatrix}$$

Thus

$$\det(A + I) = \frac{25}{13^2} - \frac{25}{13^2} = 0\,.$$

Problem 8

If A and B are improper orthogonal matrices where $\det(A) = -1 = \det(B)$, $AA^T = I$, and $BB^T = I$. We know that the product AB is an orthogonal matrix (from this section in the book) and we have that

$$\det(AB) = \det(A)\det(B) = (-1)^2 = 1\,,$$

showing that AB is a proper orthogonal matrix.

Problem 9

For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ to be an orthogonal matrix we must have

$$AA^T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a^2 + b^2 & ac + bd \\ ac + bd & c^2 + d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus we must have

$$a^2 + b^2 = 1\,.$$

There are two ways this can be true $a = 1$ and $b = 0$ or $a = 0$ and $b = 1$. We also must have

$$c^2 + d^2 = 1\,,$$

There are two ways this can be true $c = 1$ and $d = 0$ or $c = 0$ and $d = 1$. From this logic the possible choices for (a, b, c, d) are

$$(a, b, c, d) \in \{(1, 0, 1, 0)\,, (1, 0, 0, 1)\,, (0, 1, 1, 0)\,, (0, 1, 0, 1)\}\,.$$

We need to ask ourselves which of these combinations satisfy $ac + db = 0$. Since only the second and third cases above do the two orthogonal matrices of order 2 are

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Problem 10

Consider

$$C^T AC(C^T BC) = C^T ABC = C^T BAC = C^T BCC^T AC = (C^T BC)(C^T AC)\,.$$

Translations

Problem 1

With k a nonzero constant we have

Part (a): $(k, -2k, k)$.

Part (b): $(3k, 0, k)$.

Part (c): $(0, 0, k)$.

Part (d): $(3k, 4k, k)$.

Problem 2

Part (a): $(1, 3)$.

Part (b): $(2, 0)$.

Part (c): $(0, 2)$.

Part (d): $(1, -2)$.

Problem 3

Part (a): $\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$.

Part (b): $\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$.

Problem 4

The inverse matrix is $\begin{bmatrix} 1 & 0 & -x_0 \\ 0 & 1 & -y_0 \\ 0 & 0 & 1 \end{bmatrix}$, since when we multiply by this

$$\begin{bmatrix} 1 & 0 & -x_0 \\ 0 & 1 & -y_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & x_0 \\ 0 & 1 & y_0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

we get the identity. In addition, this matrix is the translation $x' = x - x_0$ and $y' = y - y_0$, expressed in homogeneous coordinates.

Problem 5

We have

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \\ 1 \end{bmatrix},$$

or the inhomogeneous point $(8, -5)$.

Problem 8

The transition to consider is $x' = x - 3$ and $y' = y + 4$. Thus $x = x' + 3$ and $y = y' - 4$ into

$$xy + 4x - 3y - 13 = 0\,,$$

gives

$$(x' + 3)(y' - 4) + 4(x' + 3) - 3(y' - 4) - 13 = 0\,,$$

or

$$x'y' - 1 = 0\,,$$

when we simplify.

Problem 10

The books equation 3-19 is $\begin{bmatrix} 1 & 0 & x_0 \\ 0 & 1 & y_0 \\ 0 & 0 & 1 \end{bmatrix}$.

Part (a): We have

$$\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & d \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & a+c \\ 0 & 1 & b+d \\ 0 & 0 & 1 \end{bmatrix},$$

showing that the operation is closed.

Part (b): Take the product in the other order shows that the product operation is commutative.

Problem 11

Since d^2 is given by $d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$ if we perform a translation of the plane via $x_1 \to x_1 + x_0$ and $x_2 \to x_2 + x_0$ then since both x_1 and x_2 are incremented by the same amount their difference does not change (similarly for y_1 and y_2).

0.1 Rigid Motion Transformations

Problem 1

We first map the point to the origin (with T_1), then perform the desired rotation (with T_2), and then map the origin back to the original point $(1, \sqrt{3}, 1)$ (with T_3). These are done with the mappings

$$T_1 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -\sqrt{3} \\ 0 & 0 & 1 \end{bmatrix}, \quad T_2 = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad T_3 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & \sqrt{3} \\ 0 & 0 & 1 \end{bmatrix}.$$

Then the total transformation is $R = T_3 T_2 T_1$ and is expressed as

$$\begin{aligned} R &= T_3 \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & \sqrt{3} \\ 0 & 0 & 1 \end{bmatrix} = T_3 \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 1 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & -\sqrt{3} \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & \sqrt{3} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 1 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & -\sqrt{3} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 2 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

The point $(0, 0, 1)$ is mapped to the point $(2, 0, 1)$.

Problem 2

This problem is just like #1. We create a transformation that moves the given point to the origin, a transformation to perform the rotation (about the origin), and a transformation to move the point back to the original location. The total transformation is then the product of these three.

Problem 3

From the matrix in #1 we can check that $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \pm 1$ using

$$\begin{vmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{4} + \frac{3}{4} = +1 \,.$$

We also check that $a_{11}^2 + a_{21}^2 = 1$ since $\frac{1}{4} + \frac{3}{4} = 1$ and $a_{12}^2 + a_{22}^2 = 1$ since $\frac{3}{4} + \frac{1}{4} = 1$. Finally, we check that $a_{11}a_{12} + a_{21}a_{22} = 0$ since $\frac{1}{2}\left(-\frac{\sqrt{3}}{2}\right) + \frac{\sqrt{3}}{2}\left(\frac{1}{2}\right) = 0$.

Problem 5

Part (a): To construct the total transformation matrix we combine several transformations. First move the line $x = 2$ to the y-axis (using T_1), then reflect all points across the y-axis (using T_2), finally move points back to the line $x = 3$ (using T_3). The matrices T_1, $_2$, and T_3 are given by

$$T_1 = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} , \quad T_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} , \quad T_3 = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} .$$

Then the total transformation is $R = T_3 T_2 T_1$ and is expressed as

$$\begin{aligned} R &= \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 6 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \end{aligned}$$

Part (b): The transformation $T_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ maps the line $x - \sqrt{3}y + 1 = 0$ to the line $x + \sqrt{3}y = 0$. If we plot this line we see that it is the x-axis rotated by an angle $\theta = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right)$ clockwise. Thus we can map points on this line to the x-axis by applying a counterclockwise rotation through the angle

$$\tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \tan^{-1}\left(\frac{\sqrt{3}/2}{1/2}\right) = \frac{\pi}{3} \,.$$

This happens with the transformation matrix T_2 given by

$$T_2 = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} .$$

The reflection across the line $y = 0$ is given by $T_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Finally we need to invert the rotation T_2 where we find

$$T_2^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and invert the translation T_1 where we find

$$T_1^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The total transformation is then given by $T_1^{-1}T_2^{-1}T_3T_2T_1$.

Problem 6

Let T_1 be given by

$$T_1 = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and T_2 by

$$T_2 = \begin{bmatrix} 1 & 0 & x_0 \\ 0 & 1 & y_0 \\ 0 & 0 & 1 \end{bmatrix},$$

and construct T_2T_1 where we find

$$T_2T_1 = \begin{bmatrix} 1 & 0 & x_0 \\ 0 & 1 & y_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & x_0 \\ \sin(\theta) & \cos(\theta) & y_0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This is a matrix of the form of 3-20. Lets check the needed requirements. We can check that $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \pm 1$, $a_{11}^2 + a_{21}^2 = 1$, $a_{12}^2 + a_{22}^2 = 1$, and $a_{11}a_{12} + a_{21}a_{22} = 0$ all hold true for this matrix. For the matrix product taken in the other order we have

$$T_1T_2 = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & x_0 \\ 0 & 1 & y_0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & x_0\cos(\theta) - y_0\sin(\theta) \\ \sin(\theta) & \cos(\theta) & x_0\sin(\theta) - y_0\cos(\theta) \\ 0 & 0 & 1 \end{bmatrix}.$$

This again is a matrix of the form of 3-20 and again all of the needed conditions on the coefficients a_{ij} hold true.

Problem 7

The rotation of the plane about the origin and a matrix representing the reflection of the plane about the x-axis is the combined transformation

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ -\sin(\theta) & -\cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This is a matrix of the form of 3-20. Lets check the needed requirements. We can check that $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \pm 1$, $a_{11}^2 + a_{21}^2 = 1$, $a_{12}^2 + a_{22}^2 = 1$, and $a_{11}a_{12} + a_{21}a_{22} = 0$ all hold true for this matrix.

Chapter 4: Eigenvalues and Eigenvectors

Characteristic Functions

We perform many of the more routine calculations (matrices powers, traces, etc) for this chapter in the python script `chap_4_problems.py`. In the first few problems in these notes we could compute the characteristic equation using the standard expression $\det(A - \lambda I) = 0$ but rather than do that I will use fact that an expression for the characteristic equation of a $n \times n$ square matrix A exists that involves traces of powers of the matrix A. Namely the characteristic equation

$$c_0\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_{n-2}\lambda^2 + c_{n-1}\lambda + c_n = 0\,, \tag{3}$$

where the coefficients c_k for $1 \le k \le n$ are given by

$$\begin{aligned}
c_0 &= 1 \\
c_1 &= -t_1 \\
c_2 &= -\frac{1}{2}(c_1 t_1 + t_2) \\
c_3 &= -\frac{1}{3}(c_2 t_1 + c_1 t_2 + t_3) \\
c_4 &= -\frac{1}{4}(c_3 t_1 + c_2 t_2 + c_1 t_3 + c_4) \\
&\vdots \\
c_{n-1} &= -\frac{1}{n-1}(c_{n-2}t_1 + c_{n-3}t_2 + \cdots + c_2 t_{n-3} + c_1 t_{n-2} + t_{n-1}) \\
c_n &= -\frac{1}{n}(c_{n-1}t_1 + c_{n-2}t_2 + \cdots + c_2 t_{n-2} + c_1 t_{n-1} + t_n)\,.
\end{aligned} \tag{4}$$

with $c_k = \text{trace}(A^k)$ for $1 \le k \le n$.

Problem 1

For the given A we find that $c_0 = 1$, $c_1 = -9$, and $c_2 = 14$ thus the characteristic equation is given by

$$c_0\lambda^2 + c_1\lambda + c_2 = 0 \quad \Rightarrow \quad \lambda^2 - 9\lambda + 14 = 0\,.$$

Problem 2

For the given A we find that $c_0 = 1$, $c_1 = -2$, and $c_2 = -5$ thus the characteristic equation is given by

$$c_0\lambda^2 + c_1\lambda + c_2 = 0 \quad \Rightarrow \quad \lambda^2 - 2\lambda - 5 = 0\,.$$

Problem 3

For the given A we find that $c_0 = 1$, $c_1 = -2$, and $c_2 = 0$ thus the characteristic equation is given by

$$c_0\lambda^2 + c_1\lambda + c_2 = 0 \quad \Rightarrow \quad \lambda^2 - 2\lambda = 0\,.$$

Thus we have eigenvalues given by $\lambda = 0$ and $\lambda = 2$.

Problem 4

For the given A we find that $c_0 = 1$, $c_1 = -6$, $c_2 = 11$, $c_3 = -6$. Thus the characteristic equation is given by

$$c_0\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3 = 0 \quad \Rightarrow \quad \lambda^3 - 6\lambda^2 - \lambda - 6 = 0\,.$$

Problem 5

For the given A we find that $c_0 = 1$, $c_1 = -2$, $c_2 = -5$, $c_3 = 6$. Thus the characteristic equation is given by

$$c_0\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3 = 0 \quad \Rightarrow \quad \lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0\,.$$

Problem 6

For the given A we find that $c_0 = 1$, $c_1 = -6$, $c_2 = 3$, $c_3 = 10$. Thus the characteristic equation is given by

$$c_0\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3 = 0 \quad \Rightarrow \quad \lambda^3 - 6\lambda^2 + 3\lambda + 10 = 0\,.$$

Problem 9

An eigenvalue λ is a number such that $\det(A - \lambda I) = 0$. If $\lambda = 0$ then we have $\det(A) = 0$ or the matrix is singular.

Problem 10

A symmetric matrix of order 2 looks like $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$. A matrix like this will have eigenvalues given by the solution to

$$\det(A - \lambda I) = \begin{vmatrix} a-\lambda & b \\ b & c-\lambda \end{vmatrix} = 0\,,$$

or

$$(a-\lambda)(c-\lambda) - b^2 = 0\,.$$

Expanding we get

$$\lambda^2 - (a+c)\lambda + (ac - b^2) = 0\,.$$

This quadratic equation has a solution for λ given by

$$\begin{aligned} \lambda &= \frac{(a+c) \pm \sqrt{(a+c)^2 - 4(ac-b^2)}}{2} = \frac{a+c \pm \sqrt{a^2 + 2ac + c^2 - 4ac + 4b^2}}{2} \\ &= \frac{a+c \pm \sqrt{(a-c)^2 + 4b^2}}{2}\,. \end{aligned}$$

To have distinct eigenvalues we must have $(a-c)^2 + 4b^2 \neq 0$. This expression will only be zero if $b = 0$ and $a = c$. If $b = 0$ we have a diagonal matrix and if $a = c$ then this diagonal matrix has equal elements.

Problem 11

Part (a): We know that λ_1, λ_2, and λ_3 all solve $\det(A - \lambda I) = 0$. Then $\det(kA - \lambda I) = 0$ is solved by $k\lambda_1$, $k\lambda_2$, and $k\lambda_3$ since, for example

$$\det(kA - k\lambda_1) = k^n \det(A - \lambda_1 I) = 0\,.$$

Thus if λ is an eigenvalue of A then $k\lambda$ is an eigenvalue of kA.

Part (b): We claim that the eigenvalues needed are $\lambda_1 - k$, $\lambda_2 - k$, and $\lambda_3 - k$ are the eigenvalues of $A - kI$. This is because for example

$$\det(A - kI - \lambda_1 I0 = \det(A - (k + \lambda_1)I) = 0\,.$$

Problem 12

Using the fact that $\det(A) = \det(A^T)$ on the expression $\det(A - \lambda I)$ we have

$$\det((A - \lambda I)^T) = \det(A^T - \lambda I) = 0\,.$$

Problem 13

Let D be a diagonal matrix with diagonal elements d_{ii}. The the matrix $D - \lambda I$ is a diagonal matrix with diagonal elements given by $d_{ii} - \lambda$. Thus

$$\det(D - \lambda I) = \prod_{i=1}^{n} (d_{ii} - \lambda) = 0 \,.$$

This last expression has the solutions $\lambda = d_{ii}$.

A Geometric Interpretation of Eigenvectors

Problem 1-2

The invariant vectors are $k \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $k \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Problem 3

We have $\lambda = 1$ as a repeated eigenvalue. The eigenvectors must then solve

$$\begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} .$$

Thus the eigenvector components must have $v_1 = 0$ and v_2 arbitrary. Thus the invariant subspace is given by $k \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Problem 4

The eigenvalues are given by solving $\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0$. This becomes $\lambda^2 - 1 = 0$ where $\lambda = \pm 1$. For $\lambda = 1$ the eigenvector is given by

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} .$$

Thus $v_1 = v_2$ or the vector $v_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. For $\lambda = -1$ the eigenvector is given by

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} .$$

Thus $v_1 = -v_2$ or the vector $v_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Thus the invariant subspaces are given by $k \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $k \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Problem 5

The eigenvalues are given by solving $\begin{vmatrix} 1-\lambda & 0 \\ 0 & -\lambda \end{vmatrix} = 0$. This becomes $-\lambda(\lambda - 1) = 0$ where $\lambda = 0$ and $\lambda = 1$. For $\lambda = 0$ the eigenvector is given by

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} .$$

Thus $v_1 = 0$ and v_2 is arbitrary. This is the vector $v_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. For $\lambda = 1$ the eigenvector is given by

$$\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} .$$

This gives the vector $v_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Thus the invariant subspaces are given by $k \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $k \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Problem 6

As this is a multiple of the identity matrix we then have that the invariant subspaces are given by $k \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $k \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Problem 7

A rotation (about the origin) is expressed by the matrix $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$. The eigenvalues for this matrix are given by

$$\begin{vmatrix} \cos(\theta) - \lambda & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) - \lambda \end{vmatrix} = 0 .$$

Expanding we get

$$(\cos(\theta) - \lambda)^2 + \sin(\theta)^2 = 0 ,$$

or

$$\cos(\theta)^2 - 2\cos(\theta)\lambda + \lambda^2 + \sin(\theta)^2 = 0 ,$$

or

$$\lambda^2 - 2\cos(\theta)\lambda + 1 = 0 .$$

Using the quadratic equation to solve for λ we get

$$\lambda = \frac{-2\cos(\theta) \pm \sqrt{4\cos(\theta)^2 - 4}}{2} = -\cos(\theta) \pm \sqrt{\cos(\theta)^2 - 1}\,.$$

Since $|\cos(\theta)| \leq 1$ we have that $\cos(\theta)^2 - 1 \leq 0$ unless $|\cos(\theta)| = 1$ which happens when $\theta = \pi k$ for k an integer. Thus as our eigenvalues are not real we don't have real eigenvectors and there is no invariant subspaces.

Some Theorems

Notes on Theorems 4-1

The proof in the book requires that the matrices $A - \lambda_i I$ and $A - \lambda_j I$ commute for all i and j. We can show that that these two matrices commute by explicitly evaluating the multiplication. We have

$$(A - \lambda_i I)(A - \lambda_j I) = A^2 + (\lambda_i + \lambda_j)A + \lambda_i \lambda_j I = (A - \lambda_j I)(A - \lambda_i I)\,.$$

As this holds for all i and j we can change the order of the terms in the product

$$(A - \lambda_1 I)(A - \lambda_2 I) \cdots (A - \lambda_{n-1} I)(A - \lambda_n I)\,,$$

so that the needed factor (say $A - \theta_j I$) will operate on the jth eigenvector X_j. We can then use the fact that $(A - \lambda_j I)X_j = 0$ to show that many terms vanish.

Problem 1

The eigenvalues are given by $\begin{vmatrix} 3-\lambda & 5 \\ 4 & 4-\lambda \end{vmatrix} = 0$ which when we expand and factor gives

$$(\lambda - 8)(\lambda + 1) = 0\,.$$

Since $\lambda = 8$ and $\lambda = -1$ are distinct we will have eigenvectors that are linearly independent. The eigenvectors for $\lambda = 8$ are given by

$$\begin{bmatrix} -5 & 5 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus $v_1 = v_2$ so the eigenvector is $k \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The eigenvectors for $\lambda = -1$ are given by

$$\begin{bmatrix} 4 & 5 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus $4v_1 = -5v_2$ so the eigenvector is $k\begin{bmatrix} -5/4 \\ 1 \end{bmatrix} \propto \begin{bmatrix} -5 \\ 4 \end{bmatrix}$. We ask if these two vectors are linearly independent. Consider them as columns of a 2×2 matrix and take the determinant of that matrix. We have

$$\begin{vmatrix} 1 & -5 \\ 1 & 4 \end{vmatrix} = 4 + 5 = 9 \neq 0$$

thus the vectors are linearly independent.

Problem 2

Note that

$$A^* = \begin{bmatrix} 0 & 1+i \\ 1-i & 1 \end{bmatrix} = A\,,$$

and A is Hermitian. To get the eigenvalues we have to solve for λ in

$$\begin{vmatrix} -\lambda & 1+i \\ 1-i & 1-\lambda \end{vmatrix} = 0\,.$$

Expanding this we get $\lambda^2 - \lambda - 2 = 0$ or that

$$\lambda = \frac{-1 \pm \sqrt{1 - 4(-2)}}{2} = \frac{-1 \pm 3}{2}\,,$$

which has two solutions given by -2 and 1. As these eigenvalues are distinct the eigenvectors will be orthogonal. We now verify this. The eigenvectors for $\lambda = -2$ are given by

$$\begin{bmatrix} 2 & 1+i \\ 1-i & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Since we know that $\lambda = -2$ is an eigenvalue of A the second row *must* be a multiple of the first row. Thus we will only satisfy the requirement enforced by the first row

$$2v_1 + (1+i)v_2 = 0 \quad \Rightarrow \quad v_1 = -\frac{1+i}{2}v_2\,.$$

Thus the eigenvector is $k\begin{bmatrix} -(1+i) \\ 2 \end{bmatrix}$. The eigenvectors for $\lambda = 1$ are given by

$$\begin{bmatrix} 1 & 1+i \\ 1-i & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Again since we know that $\lambda = 1$ is an eigenvalue of A the second row *must* be a multiple of the first row (namely $-(1-i)$). Thus we will only satisfy the requirement enforced by the first row

$$-v_1 + (1+i)v_2 = 0 \quad \Rightarrow \quad v_1 = (1+i)v_2\,.$$

Thus the eigenvector is $k\begin{bmatrix} 1+i \\ 1 \end{bmatrix}$. We now verify that these two eigenvectors are orthogonal. Consider their complex inner product

$$\mathbf{v}_1^*\mathbf{v}_2 = \begin{bmatrix} -(1-i) & 2 \end{bmatrix} \begin{bmatrix} 1+i \\ 1 \end{bmatrix} = -(1+1) + 2 = 0\,.$$

Problem 4

We have $X_i^T A X_i = \lambda_i X_i^T X_i = \lambda_i$ since $X_i^T X_i = 1$.

Problem 5

By the previous exercise we have $X_i^T A X_i = \lambda_i$. If we take the complex transpose of both sides of this expression we get

$$X_i^* A^* \overline{X_i} = \bar{\lambda}\,,$$

or

$$(\overline{X}_i)^T A^* \overline{X}_i = \bar{\lambda}\,,$$

If we assume that the converse of the previous exercise holds true then we have that $\bar{\lambda}_i$ is an eigenvalue of A^*.

Diagonalization of Matrices

Notes on Diagonalization of Matrices

The linear system from the expression $AC = CB$ is

$$\begin{array}{rcrcrcrcl} 2a & - & 3b & - & 2c & & & = & 0 \\ 2a & & & + & 7c & - & 3d & = & 0 \\ 6a & - & 7b & & & - & 2d & = & 0 \\ & & 2b & + & 6c & - & 2d & = & 0\,. \end{array}$$

To compute the equivalent system (and find all solutions) to these equations we write the above in reduced row-echelon form. We have

$$\begin{bmatrix} 2 & -3 & -2 & 0 \\ 2 & 0 & 7 & -3 \\ 6 & -7 & 0 & -2 \\ 0 & 2 & 6 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & -3 & -2 & 0 \\ 0 & 3 & 9 & -3 \\ 6 & 2 & 6 & -2 \\ 0 & 2 & 6 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & -3 & -2 & 0 \\ 0 & 1 & 3 & -1 \\ 0 & 1 & 3 & -1 \\ 0 & 1 & 3 & -1 \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} 2 & -3 & -2 & 0 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 0 & 7 & -3 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus we can let c and d be arbitrary and then a and b must satisfy

$$\begin{aligned} 2a &= -7c + 3d \\ b &= -3c + d\,. \end{aligned}$$

Thus we can make A similar to B with any matrix of the form

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -\frac{7}{2}c + \frac{3}{2}d & -3c + d \\ c & d \end{bmatrix}.$$

If we let $d \to 2d'$ and $c \to 2c'$ then the matrix above becomes

$$\begin{bmatrix} 3d' - 7c' & 2d' - 6c' \\ 2c' & 2d' \end{bmatrix}.$$

We must have the determinant of this matrix nonzero so that it is invertible. This requires that

$$(3d' - 7c')(2d') - 2c'(2d' - 6c') \neq 0.$$

When we simplify some this is equal to

$$2(3d' - 3c')(d' - 2c') \neq 0.$$

Thus $d' \neq c'$ and $d' \neq 2c'$.

Problem 1 (are A and B similar)

For the given A and B we ask if A is similar to B. In other words, does there exist a matrix C such that $A^{-1}AC = B$ or $AC = BC$. Following the book, for A and B to be similar we must have

$$\begin{bmatrix} -2 & -1 \\ 0 & 11 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

When we multiply we get

$$\begin{bmatrix} -2a - c & -2b - d \\ 11c & 11d \end{bmatrix} = \begin{bmatrix} c & d \\ 2a + 3c & 2b + 3d \end{bmatrix}$$

Thus as a linear system for a, b, c, and d this is

$$\begin{bmatrix} -2 & 0 & -2 & 0 \\ 0 & -2 & 0 & -2 \\ -2 & 0 & 8 & 0 \\ 0 & -2 & 0 & 8 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0.$$

When we start to the coefficient matrix into row echelon form we find it can be written as $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$. Thus we must have $c = d = 0$ and then $a = b = 0$ thus A and B are *not* similar.

Problem 3

Assuming that A and B are similar (this could be checked as the examples in the book), note that we have $|A| = 2$ and $|B| = -4 + 6 = 2$. The eigenvalues of A are $\{1, 2\}$. The eigenvalues of B are the solutions λ to

$$\begin{vmatrix} 4-\lambda & 3 \\ -2 & -1-\lambda \end{vmatrix} = 0\,.$$

This last expression becomes

$$(4-\lambda)(-1-\lambda) + 6 = 0\,,$$

or $(\lambda - 1)(\lambda - 2) = 0$. Thus the eigenvalues of B are $\{1, 2\}$ also.

Problem 4

The eigenvalues of A are 2 with algebraic multiplicity of 2. The eigenvalues of B are also 2 with algebraic multiplicity of 2. We ask if A and B are similar. Does there exists an invertible matrix C such that $C^{-1}AC = B$ or equivalently $AC = CB$. For this to be true we would have to have

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}.$$

If we multiply we get

$$\begin{bmatrix} 2a & 2b \\ 2c & 2d \end{bmatrix} = \begin{bmatrix} 2a & 3a+2b \\ 2c & 3c+2d \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 3a \\ 0 & 3c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

As a linear system for a, b, c, and d we see that $a = 0 = c$ but b and d can be arbitrary. Thus C must have the form $\begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix}$. The problem with this is that C in this case is not invertible. Thus A and B are *not* similar.

Problem 10 (similar to a diagonal matrix)

If A is similar to a diagonal matrix D, then there exists and invertible matrix C such that $C^{-1}AC = D$. Taking the transpose of this relationship gives

$$C^T A^T C^{-T} = D\,.$$

If we let $\tilde{C}^{-1} = C^T$ or $\tilde{C} = C^{-T}$ then $\tilde{C}^{-1}A^T\tilde{C} = D$ and we have the statement that A^T is similar to a diagonal matrix.

Problem 11 (similar to a multiple of the identity matrix)

We are told that $C^{-1}AC = kI$. Solving for A we see that $A = C(kI)C^{-1} = kI$ as we were to show.

The Hamilton-Cayley Theorem

Notes on the inverse of A using the Hamilton-Cayley Theorem

As discussed in the book one can get the inverse of an invertible matrix A, using the Hamilton-Cayley Theorem where we find

$$A^{-1} = -\frac{1}{c_n}(c_0 A^{n-1} + c_1 A^{n-2} + c_2 A^{n-3} + \cdots + c_{n-3}A^2 + c_{n-2}A + c_{n-1}I)\,. \tag{5}$$

The coefficients c_0, c_1, c_2 etc are functions of the trace of powers of the matrix A and are given by Equation 4. If we consider the expression on the right-hand-side of Equation 5 as a function of the values of c_k for $0 \leq k \leq n$ we might try to compute A^{-1} using a multidimensional minimization routine where we search over the values of c_k, compute the right-hand-side of Equation 5, and pick c_k to minimize a cost function like

$$\left|\left|A\left(-\frac{1}{c_n}(c_0 A^{n-1} + c_1 A^{n-2} + c_2 A^{n-3} + \cdots + c_{n-3}A^2 + c_{n-2}A + c_{n-1}I)\right) - I\right|\right|_F^2\,.$$

This would require us to precompute A^k for each k. If we have already done the work compute these powers then we could just use Equations 4 to compute c_k exactly.

Notes on Example 3: Using the Hamilton-Cayley Theorem

We have

$$\begin{aligned} A^{-2} &= \frac{1}{2^2}(A - 3I)^2 = \frac{1}{4}(A^2 - 6A + 9I) \\ &= \frac{1}{4}(3A + 2I - 6A + 9I) \\ &= \frac{1}{4}(-3A + 11I) = \frac{11}{4}I - \frac{3}{4}A\,. \end{aligned}$$

Where we have used the Hamilton-Cayley Theorem to replace A^2 with lower powers in going from the first line to the second in the above.

Problem 1

The Hamilton-Cayley Theorem states that a matrix A satisfies its own characteristic equation $|A - \lambda I| = 0$. For this matrix the characteristic equation is given by is given by solving

$$\left| \begin{array}{cc} 2-\lambda & -1 \\ 3 & 4-\lambda \end{array} \right| = 0\,.$$

Expanding we get

$$\lambda^2 - 6\lambda + 11 = 0\,.$$

Thus to show the Hamilton-Cayley Theorem we must show that for the A given we have

$$A^2 - 6A + 11I = 0\,,$$

or

$$\left[\begin{array}{cc} 2 & -1 \\ 3 & 4 \end{array} \right]^2 - 6 \left[\begin{array}{cc} 2 & -1 \\ 3 & 4 \end{array} \right] + \left[\begin{array}{cc} 11 & 0 \\ 0 & 11 \end{array} \right] = 0\,,$$

which is true.

Problem 2

The characteristic equation for A is given by

$$|A - \lambda I| = \left| \begin{array}{cc} 7-\lambda & 9 \\ 2 & 1-\lambda \end{array} \right| = 0\,,$$

or

$$(7-\lambda)(1-\lambda) - 18 = 0\,.$$

When we simplify we get

$$\lambda^2 - 8\lambda - 11 = 0\,.$$

When we convert this to a polynomial in A we see that it vanishes.

Problem 3

For this matrix the characteristic equation $|A - \lambda I| = 0$ is $(5-\lambda)(2-\lambda) = 0$ or $\lambda^2 - 7\lambda + 10 = 0$. For the matrix A this would be

$$A^2 - 7A + 10 = \left[\begin{array}{cc} 25 & 0 \\ 0 & 4 \end{array} \right] - 7 \left[\begin{array}{cc} 5 & 0 \\ 0 & 2 \end{array} \right] + \left[\begin{array}{cc} 10 & 0 \\ 0 & 10 \end{array} \right] = \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right]\,,$$

when we simplify.

Problem 4

For this matrix the characteristic equation is

$$\begin{vmatrix} 1-\lambda & 1 & -2 \\ 1 & -\lambda & 3 \\ -2 & 3 & 2-\lambda \end{vmatrix} = 0\,.$$

Some steps in the minor expansion give

$$\begin{aligned} |A-\lambda I| &= (1-\lambda)\begin{vmatrix} -\lambda & 3 \\ 3 & 2-\lambda \end{vmatrix} - \begin{vmatrix} 1 & -2 \\ 3 & 2-\lambda \end{vmatrix} - 2\begin{vmatrix} 1 & -2 \\ -\lambda & 3 \end{vmatrix} \\ &= (1-\lambda)(-\lambda(2-\lambda)-9) - (2-\lambda+6) - 2(3-2\lambda) \\ &= -\lambda^3 + 3\lambda^2 + 12\lambda - 23\,. \end{aligned}$$

When we convert this to a polynomial in A and evaluate we get the zero matrix as we should.

Problem 5 (using the Hamilton-Cayley Theorem to find inverses)

For this matrix we have

$$A^2 = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 29 & 12 \\ 12 & 5 \end{bmatrix}\,.$$

Thus $t_1 = \mathrm{tr}(A) = 6$ and $t_2 = \mathrm{tr}(A^2) = 34$. Thus $c_0 = 1$, $c_2 = -t_1 = -6$ and $c_2 = -\frac{1}{2}(c_1 t_1 + t_2) = -\frac{1}{2}(-6(6)+34) = 1$. Thus the characteristic equation is

$$\lambda^2 - 6\lambda + 1 = 0\,.$$

By the Hamilton-Cayley Theorem we have that A must satisfy $A^2 - 6A + I = 0$. On multiplying by A^{-1} we get

$$A^{-1} = -A + 6I\,.$$

Thus in components we have

$$A^{-1} = -\begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}\,.$$

Problem 6 (using the Hamilton-Cayley Theorem to find inverses)

Using the "trace version" to compute the characteristic equation (rather than the determinant version) we first calculate

$$\begin{aligned} t_1 &= \mathrm{tr}(A) = 16 \\ t_2 &= \mathrm{tr}(A^2) = 214 \\ t_3 &= \mathrm{tr}(A^3) = 3097\,. \end{aligned}$$

Then with these we have

$$
\begin{aligned}
c_0 &= 1 \\
c_1 &= -t_1 = -16 \\
c_2 &= -\frac{1}{2}(c_1 t_1 + t_2) = 21 \\
c_3 &= -\frac{1}{3}(c_2 t_1 + c_1 t_2 + t_3) = -3\,.
\end{aligned}
$$

Thus the characteristic polynomial for A looks like

$$c_0\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3 = 0 \quad \Rightarrow \quad \lambda^3 - 16\lambda^2 + 21\lambda - 3 = 0\,.$$

When we convert this into a polynomial over A and multiply by A^{-1} we get

$$A^{-1} = \frac{1}{3}(A^2 - 16A + 21I)\,.$$

Problem 7 (inverse powers using the Hamilton-Cayley Theorem)

For A given by $A = \left[\begin{array}{cc} 7 & 2 \\ 5 & 1 \end{array}\right]$ we have

$$A^2 = \left[\begin{array}{cc} 7 & 2 \\ 5 & 1 \end{array}\right]\left[\begin{array}{cc} 7 & 2 \\ 5 & 1 \end{array}\right] = \left[\begin{array}{cc} 59 & 16 \\ 30 & 11 \end{array}\right]\,.$$

Thus $t_1 = \text{tr}(A) = 8$ and $t_2 = \text{tr}(A^2) = 70$ so our characteristic equation is given by $c_0\lambda^2 + c_1\lambda + c_2 = 0$ where $c_0 = 1$, $c_1 = -t_1 = -8$, and $c_2 = -\frac{1}{2}(c_1 t_1 + t_2) = -3$ or

$$\lambda^2 - 8\lambda - 3 = 0\,.$$

Thus by the Hamilton-Cayley Theorem we have that A must satisfy $A^2 - 8A - 3I = 0$. Multiply this expression by A^{-1} to get $A - 8I - 3A^{-1} = 0$ or

$$A^{-1} = \frac{1}{3}(A - 8I)\,.$$

Using this we can compute

$$A^{-2} = A^{-1}(A^{-1}) = A^{-1}\left(\frac{1}{3}(A - 8I)\right) = \frac{1}{3}(I - 8A^{-1}) = \frac{1}{3}I - \frac{9}{3}\left(\frac{1}{3}(A - 8I)\right) = \frac{67}{9}I - \frac{8}{9}A\,.$$

For A^{-3} we have

$$
\begin{aligned}
A^{-3} = A^{-1}A^{-2} &= A^{-1}\left(\frac{67}{9}I - \frac{8}{9}A\right) \\
&= \frac{67}{9}A^{-1} - \frac{8}{9}I = \frac{67}{9}\left(\frac{1}{3}(A - 8I)\right) - \frac{8}{9}I\,.
\end{aligned}
$$

Problem 8 (proof of the Hamilton-Cayley Theorem for matrices of order 2)

Let A be given by

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

so that

$$A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix}$$

Now for this A we have the first two "traces" given by $t_1 = \text{tr}(A) = a + d$ and $t_2 = \text{tr}(A^2) = a^2 + 2bc + d^2$. The characteristic polynomial $c_0\lambda^2 + c_1\lambda + c_2 = 0$ has coefficients given by

$$\begin{aligned} c_0 &= 1 \\ c_1 &= -t_1 = -(a+d) \\ c_2 &= -\frac{1}{2}(c_1t_1 + t_2) = -\frac{1}{2}(-(a+d)(a+d) + a^2 + 2bc + d^2) = ad - bc\,. \end{aligned}$$

Therefor the Hamilton-Cayley Theorem for 2×2 matrices is that A must satisfy

$$A^2 - (a+d)A + (ad-bc)I = 0\,.$$

Lets compute the left-hand-side of this expression and show that it vanishes. We have

$$\begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix} - \begin{bmatrix} a^2+ad & ab+bd \\ ac+cd & ad+d^2 \end{bmatrix} + \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

as it should

Quadratic Forms

Notes on quadratic forms

Consider the rotation product, P, aimed at diagonalizing the matrix $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ or

$$P = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}.$$

This product is computed as

$$\begin{aligned} P &= \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} a\cos(\theta) - b\sin(\theta) & a\sin(\theta) + b\cos(\theta) \\ b\cos(\theta) - c\sin(\theta) & b\sin(\theta) + c\cos(\theta) \end{bmatrix} \\ &= \begin{bmatrix} a\cos^2(\theta) - 2b\sin(\theta)\cos(\theta) + c\sin^2(\theta) & (a-c)\sin(\theta)\cos(\theta + b\cos^2(\theta) - b\sin^2(\theta) \\ (a-c)\sin(\theta)\cos(\theta) - b\sin^2(\theta) + b\cos^2(\theta) & a\sin^2(\theta) + 2b\sin(\theta)\cos(\theta) + c\cos^2(\theta) \end{bmatrix}. \quad (6) \end{aligned}$$

For this to be diagonal it needs to be equal to a diagonal matrix. Thus we must have

$$\begin{bmatrix} a\cos^2(\theta) - 2b\sin(\theta)\cos(\theta) + c\sin^2(\theta) & (a-c)\sin(\theta)\cos(\theta + b\cos^2(\theta) - b\sin^2(\theta) \\ (a-c)\sin(\theta)\cos(\theta) - b\sin^2(\theta) + b\cos^2(\theta) & a\sin^2(\theta) + 2b\sin(\theta)\cos(\theta) + c\cos^2(\theta) \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix},$$

for some values of λ_1 and λ_2. In the above, consider the resulting equations for the $(1, 2)$ and the $(2, 1)$ elements in the above two matrices. When we add these two equations (and divide by 2) we see that θ must satisfy

$$(a - c)\sin(\theta)\cos(\theta) + b(\cos^2(\theta) - \sin^2(\theta)) = 0\,.$$

Manipulating some we see that

$$\frac{b}{c-a} = \frac{\sin(\theta)\cos(\theta)}{\cos^2(\theta) - \sin^2(\theta)} = \frac{\frac{1}{2}(2\sin(\theta)\cos(\theta))}{\cos^2(\theta) - \sin^2(\theta)} = \frac{\frac{1}{2}\sin(2\theta)}{\cos(2\theta)} = \frac{1}{2}\tan(2\theta)\,.$$

Thus solving for θ in the above we find

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2b}{c-a}\right)\,. \tag{7}$$

Once we have a value for θ we see that λ_1 and λ_2 are given by

$$\lambda_1 = a\cos^2(\theta) - 2b\sin(\theta)\cos(\theta) + c\sin^2(\theta) \tag{8}$$
$$\lambda_2 = a\sin^2(\theta) + 2b\sin(\theta)\cos(\theta) + c\cos^2(\theta)\,. \tag{9}$$

Problem 1

$$3x^2 + 10xy + 3y^2 = \begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 5 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}\,.$$

Problem 2

$$x^2 - 2xy + y^2 = \begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}\,.$$

Problem 3

$$2x^2 + 2\sqrt{2}xy + y^2 = \begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}\,.$$

Problem 4-6 Overview

The orthogonal matrix to transform a given quadratic form $\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$ into the canonical form $\lambda_1 x'^2 + \lambda_2 y'^2 = 0$ is given by

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

with θ given by $\theta = \frac{1}{2}\tan^{-1}\left(\frac{2b}{c-a}\right)$.

Problem 4

For $a = 3$, $b = 5$, and $c = 3$ we have

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2(5)}{3-3}\right) = \frac{1}{2}\tan^{-1}(\pm\infty) = \pm\frac{\pi}{4}\,.$$

Lets use the positive sign for our rotation. Thus our transformation is $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$. Using this we find

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0\,,$$

becomes

$$\begin{bmatrix} x' & y' \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = 0\,,$$

or

$$\begin{bmatrix} x' & y' \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{8}{\sqrt{2}} & \frac{2}{\sqrt{2}} \\ \frac{8}{\sqrt{2}} & -\frac{2}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = 0\,,$$

or

$$\begin{bmatrix} x' & y' \end{bmatrix} \begin{bmatrix} 8 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = 0\,,$$

or $8x'^2 - 2y'^2 = 0$.

Problem 5

For $a = 1$, $b = -1$, and $c = 1$ we have

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{-2}{0}\right) = \frac{1}{2}\tan^{-1}(\pm\infty) = \pm\frac{\pi}{4}\,.$$

Lets use the positive sign for our rotation. Thus our transformation is $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$. Using this we find

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0\,,$$

becomes

$$\begin{bmatrix} x' & y' \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = 0\,,$$

or

$$\begin{bmatrix} x' & y' \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{2}} & 0 \\ -\frac{2}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = 0\,,$$

or

$$\begin{bmatrix} x' & y' \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = 0\,,$$

or $2x'^2 = 0$.

Classification of the Conics

Notes on classification of the conics

From the book when we start with a conic given by an expression of the following form

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \alpha \\ 0 & \lambda_2 & \beta \\ \alpha & \beta & f \end{bmatrix} \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = 0\,, \tag{10}$$

and then perform the translation

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\frac{\alpha}{\lambda_1} \\ 0 & 1 & -\frac{\beta}{\lambda_2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x'' \\ y'' \\ 1 \end{bmatrix}\,, \tag{11}$$

doing the algebra we have a new conic with a representation given by

$$\begin{bmatrix} x'' \\ y'' \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{\alpha}{\lambda_1} & -\frac{\beta}{\lambda_2} & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \alpha \\ 0 & \lambda_2 & \beta \\ \alpha & \beta & f \end{bmatrix} \begin{bmatrix} 1 & 0 & -\frac{\alpha}{\lambda_1} \\ 0 & 1 & -\frac{\beta}{\lambda_2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x'' \\ y'' \\ 1 \end{bmatrix} = 0\,,$$

or multiplying the two rightmost matrices

$$\begin{bmatrix} x'' \\ y'' \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{\alpha}{\lambda_1} & -\frac{\beta}{\lambda_2} & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ \alpha & \beta & -\frac{\alpha^2}{\lambda_1} - \frac{\beta^2}{\lambda_2} + f \end{bmatrix} \begin{bmatrix} x'' \\ y'' \\ 1 \end{bmatrix} = 0\,,$$

or finishing the matrix multiplication we finally end with

$$\begin{bmatrix} x'' \\ y'' \\ 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & f - \frac{\alpha^2}{\lambda_1} - \frac{\beta^2}{\lambda_2} \end{bmatrix} \begin{bmatrix} x'' \\ y'' \\ 1 \end{bmatrix} = 0\,, \tag{12}$$

which is the equation given in the book.

Problem 1 (transformation of conics)

The given second degree expression can be written in matrix form as

$$\begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & 1 \\ \frac{1}{2} & 1 & -\frac{3}{2} \\ 1 & -\frac{3}{2} & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0\,.$$

This expression shows the matrix of the conic section. The rotation needed to turn this matrix into its canonical form is given by the eigenvalues of the matrix $F = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$. This F has eigenvalues given by

$$\begin{vmatrix} 1-\lambda & \frac{1}{2} \\ \frac{1}{2} & 1-\lambda \end{vmatrix} = 0\,,$$

or

$$(1-\lambda)^2 - \frac{1}{4} = 0\,.$$

This gives $\lambda = \frac{1}{2}$ or $\lambda = \frac{3}{2}$. The eigenvectors of F are given by $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$ and $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$. The rotation in the plane is then defined by

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix},$$

and results in a conic given by

$$\begin{bmatrix} x' & y' & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & 1 \\ \frac{1}{2} & 1 & -\frac{3}{2} \\ 1 & -\frac{3}{2} & 5 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = 0\,.$$

On multiplying the two right most matrices we get

$$\begin{bmatrix} x' & y' & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2\sqrt{2}} & \frac{3}{2\sqrt{2}} & 1 \\ -\frac{1}{2\sqrt{2}} & \frac{3}{2\sqrt{2}} & -\frac{3}{2} \\ \frac{5}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & 5 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = 0\,,$$

or multiplying the last two matrices we have

$$\begin{bmatrix} x' & y' & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & \frac{5}{2\sqrt{2}} \\ 0 & \frac{3}{2} & -\frac{1}{2\sqrt{2}} \\ \frac{5}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & 5 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = 0\,.$$

We now perform the translation of the coordinate axis represented by Equation 11 which in this case is given by

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\frac{5}{2\sqrt{2}} \\ 0 & 1 & \frac{1}{3\sqrt{2}} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x'' \\ y'' \\ 1 \end{bmatrix}.$$

Using this the representation of our conic becomes

$$\begin{bmatrix} x'' & y'' & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{5}{2\sqrt{2}} & \frac{1}{3\sqrt{2}} & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & \frac{5}{2\sqrt{2}} \\ 0 & \frac{3}{2} & -\frac{1}{2\sqrt{2}} \\ \frac{5}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 & -\frac{5}{2\sqrt{2}} \\ 0 & 1 & \frac{1}{3\sqrt{2}} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x'' \\ y'' \\ 1 \end{bmatrix} = 0\,.$$

When we multiply the matrices we get

$$\begin{bmatrix} x'' & y'' & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & -\frac{4}{3} \end{bmatrix} \begin{bmatrix} x'' \\ y'' \\ 1 \end{bmatrix} = 0\,.$$

This expression is the conic $\frac{1}{2}x''^2 + \frac{3}{2}y''^2 - \frac{4}{3} = 0$. If we multiply by 6 we get $3x''^2 + 9y''^2 - 8 = 0$. Since λ_1 and λ_2 are nonzero we have an ellipse. Since f' is of the opposite sign as λ_1 and λ_2 this is a *real* ellipse.

Problem 2 (transformation of conics)

We can write the quadratic expression $x^2 - 2xy + y^2 + 8x + 8y = 0$ as

$$\begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 4 \\ -1 & 1 & 4 \\ 4 & 4 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0\,.$$

In this case $F = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ which has eigenvalues given by $\begin{vmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} = 0$ or

$$(1-\lambda)^2 - 1 = 0\,,$$

which has solutions $\lambda = 0$ and $\lambda = 2$. The rotation needed is given by

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2b}{c-a}\right) = \frac{1}{2}\tan^{-1}\left(\frac{-2}{0}\right) = \frac{1}{2}\tan^{-1}(\pm\infty) = \pm\frac{\pi}{4}\,.$$

If we take the positive sign then our rotation matrix is given by

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix}.$$

With this our conic transformations as

$$\begin{bmatrix} x' & y' & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 4 \\ -1 & 1 & 4 \\ 4 & 4 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = 0\,.$$

When we multiply these three matrices together we get

$$\begin{bmatrix} x' & y' & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & \frac{8}{\sqrt{2}} \\ 0 & \frac{8}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = 0\,.$$

This is the conic $2x'^2 + \frac{16}{\sqrt{2}}y' = 0$ or a parabola.

Problem 3 (transformation of conics)

We can write the quadratic expression $7x^2 - 48xy - 7y^2 + 60x - 170y + 225 = 0$ as

$$\begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} 7 & -24 & 30 \\ -24 & -7 & -85 \\ 30 & -85 & 225 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0\,.$$

In this case $F = \begin{bmatrix} 7 & -24 \\ -24 & -7 \end{bmatrix}$ which has eigenvalues given by $\begin{vmatrix} 7-\lambda & -24 \\ -24 & -7-\lambda \end{vmatrix} = 0$ or

$$(7-\lambda)(-7-\lambda) - 24^2 = 0\,,$$

which has solutions $\lambda = \pm 25$. The rotation needed has a θ given by

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2b}{c-a}\right) = \frac{1}{2}\tan^{-1}\left(\frac{-2(24)}{-7-7}\right) = \frac{1}{2}\tan^{-1}\left(\frac{24}{7}\right).$$

By drawing the triangle associated with this angle 2θ (note the two) we see that

$$\sin(2\theta) = \frac{24}{25} \quad \text{and} \quad \cos(2\theta) = \frac{7}{25}.$$

This is a nice observation (that we can compute $\sin(2\theta)$ and $\cos(2\theta)$ in a closed form) but does not help directly with the problem at hand. To continue, we will compute θ numerically. We find $\theta \approx 0.643501$ and using this that

$$\sin(\theta) = 0.6 \quad \text{and} \quad \cos(\theta) = 0.8\,.$$

The fact that these numbers are so "clean" (and that this is a textbook problem) indicates that there probably is a algebraic transformation that would allow us to solve for $\sin(\theta)$ and $\cos(\theta)$ without calculating anything numerical. In any case, using these two values for $\sin(\theta)$ and $\cos(\theta)$ we desire to perform the rotation given by

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix}.$$

With this our conic transformations to

$$\begin{bmatrix} x' & y' & 1 \end{bmatrix} \begin{bmatrix} 0.6 & -0.8 & 0 \\ 0.8 & 0.6 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 7 & -24 & 30 \\ -24 & -7 & -85 \\ 30 & -85 & 225 \end{bmatrix} \begin{bmatrix} 0.6 & 0.8 & 0 \\ -0.8 & 0.6 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = 0\,.$$

When we multiply these three matrices together we get

$$\begin{bmatrix} x' & y' & 1 \end{bmatrix} \begin{bmatrix} 25 & 0 & 75 \\ 0 & -25 & -50 \\ 75 & -50 & 225 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = 0\,.$$

We now perform the translation of the coordinate axis represented by Equation 11 which in this case is given by

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x'' \\ y'' \\ 1 \end{bmatrix}.$$

This would then give the conic

$$\begin{bmatrix} x'' & y'' & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & -2 & 1 \end{bmatrix} \begin{bmatrix} 25 & 0 & 75 \\ 0 & -25 & -50 \\ 75 & -50 & 225 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x'' \\ y'' \\ 1 \end{bmatrix} = 0\,.$$

When we multiply the matrices this becomes

$$\begin{bmatrix} x'' & y'' & 1 \end{bmatrix} \begin{bmatrix} 25 & 0 & 0 \\ 0 & -25 & 0 \\ 0 & 0 & 100 \end{bmatrix} \begin{bmatrix} x'' \\ y'' \\ 1 \end{bmatrix} = 0\,.$$

This expression is the conic $25x''^2 - 25y''^2 + 100 = 0$. If we divide by 25 we get $x''^2 - y''^2 + 4 = 0$. Since λ_1 and λ_2 are nonzero but of opposite sign and $f' \neq 0$ we have a hyperbola. Note that I have a different sign for the term 4 than the solution given in the back of the book. This could be a typo or an error in my calculation. If anyone sees anything wrong with my algebra for this problem let me know.

Problem 4 (transformation of conics)

We can write the quadratic expression $x^2 - 2xy + y^2 - 8 = 0$ as

$$\begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0 \,.$$

For this problem we have $F = \begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, which has eigenvalues given by $\lambda = 0$ and $\lambda = 2$. The angle of rotation to transform this conic into standard form is given by

$$\theta = \frac{1}{2} \tan^{-1}\left(\frac{2b}{c-a} \right) = \frac{1}{2} \tan^{-1}\left(\frac{-2}{1-1} \right) = \pm \frac{\pi}{4} \,.$$

Thus the rotation in the plane is then defined by

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} ,$$

so the new conic is given by

$$\begin{bmatrix} x' & y' & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -8 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = 0 \,.$$

When we multiply the three matrices above we get the conic

$$\begin{bmatrix} x' & y' & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -8 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = 0 \,.$$

This becomes $2x'^2 - 8 = 0$ or $x' = \pm 2$ which are two vertical (parallel) lines in the transformed space. Looking at the original expression we see that it can be written as $(x - y)^2 - 8 = 0$ or $x - y = \pm 2\sqrt{2}$. Again we see that this expression represents two parallel lines.

Problems 5-8 (classifying conics)

We won't explicitly compute the canonical form each of the conics given but we will instead derive the needed numbers to determine the conics type. The type of a conic can be determined from the three numbers λ_1, λ_2, and f' where when λ_1 and λ_2 are nonzero f' is given by

$$f' = f - \frac{\alpha^2}{\lambda_1} - \frac{\beta^2}{\lambda_2}\,. \tag{13}$$

We now discuss how to compute each of the needed expressions. First we define the numbers a, b, c, d, e, and f to be the numbers obtained when we write our "algebraic" conic

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0\,,$$

in the "matrix" form

$$\begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} a & b & d \\ b & c & e \\ d & e & f \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0\,.$$

The coefficient matrix Δ defined as $\Delta \equiv \begin{bmatrix} a & b & d \\ b & c & e \\ d & e & f \end{bmatrix}$ is called the **matrix of the conic section**. Then we compute an angle θ given by Equation 7 or

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2b}{c-a}\right)\,.$$

With this value of θ the values for λ_1 and λ_2 are given by the expressions Equation 8 and 9 respectively. Next the values of α and β are given by

$$\begin{aligned} \alpha &= d\cos(\theta) - e\sin(\theta) \\ \beta &= d\sin(\theta) + e\cos(\theta)\,, \end{aligned} \tag{14}$$

All of these calculations are done for a given in the python function `conic_invariants.py`. Once the numbers λ_1, λ_2, and f' are determined the book gives expressions/arguments as to what type of conic we have depending on whether these numbers are 0 and their signs.

Problems 5

For this conic we write it as

$$5x^2 + 4xy + 8y^2 - 36 = \begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} 5 & 2 & 0 \\ 2 & 8 & 0 \\ 0 & 0 & 36 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0\,.$$

With this representation we find $\lambda_1 = 4$, $\lambda_2 = 7.2$, and $f' = -36$. Then since λ_1 and λ_2 are nonzero and of the same sign we have an ellipse. Since f' is a different sign than λ_1 and λ_2 we have a real ellipse.

Problems 6

For this conic we write it as

$$2x^2 + \sqrt{3}xy + y^2 + 14 = \begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} 2 & \sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1 & 0 \\ 0 & 0 & 14 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0\,.$$

With this representation we find $\lambda_1 = \frac{1}{2}$, $\lambda_2 = 2$, and $f' = 15$. Then since λ_1 and λ_2 are nonzero and of the same sign we have an ellipse. Since f' is of the same sign as both λ_1 and λ_2 we have a imaginary ellipse.

Problems 7

For this conic we write it as

$$x^2 - 12xy - 4y^2 = \left[\begin{array}{ccc} x & y & 1 \end{array} \right] \left[\begin{array}{ccc} 1 & -6 & 0 \\ -6 & -4 & 0 \\ 0 & 0 & 0 \end{array} \right] \left[\begin{array}{c} x \\ y \\ 1 \end{array} \right] = 0\,.$$

With this representation we find $\lambda_1 = -8$, $\lambda_2 = 3.076$, and $f' = 0$. Then since λ_1 and λ_2 are nonzero and of the opposite sign we can have a hyperbola ($f' \neq 0$) or two intersecting lines ($f' = 0$). Since $f' = 0$ we have two intersecting lines.

Problems 8

For this conic we write it as

$$3x^2 + 3xy + 3y^2 - 18x + 15y + 91 = \left[\begin{array}{ccc} x & y & 1 \end{array} \right] \left[\begin{array}{ccc} 3 & 3/2 & -9 \\ 3/2 & 3 & 15/2 \\ -9 & 15/2 & 91 \end{array} \right] \left[\begin{array}{c} x \\ y \\ 1 \end{array} \right] = 0\,.$$

With this representation we find $\lambda_1 = 1.5$, $\lambda_2 = 4.5$, and $f' = 0$. Then since λ_1 and λ_2 are nonzero and of the same sign we have a type of ellipse. Since $f' = 0$ this ellipse is a real point.

Problems 9

The fact that for conics that are ellipses we can reduce the expression to

$$\lambda_1 x''^2 + \lambda_2 y''^2 + f' = 0\,,$$

or

$$\frac{x''^2}{\frac{1}{\lambda_1}} + \frac{y''^2}{\frac{1}{\lambda_2}} + f' = 0\,.$$

From this we see that λ_1 and λ_2 are inversely proportional to the squares of the semi-axis for an ellipse.

Invariants for Conics

Problems 1-4

To find the rank of each conic we can construct the matrix of the conic section, Δ, and find the rank of the resulting matrix. If the rank is three we have a proper conic that is an ellipse, hyperbola, or a parabola. If the rank is less than three we have a degenerate conic.

Problems 5-8

To show this we would look at the expressions $a+c$ and $ac-b^2$ in the expression for Δ before we transformation the conic to standard form and after its transformation to standard form where the matrix of the conic take the form $\Delta_1 = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & f' \end{bmatrix}$ or $\Delta_2 = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & 0 & \beta \\ 0 & \beta & f' \end{bmatrix}$. In these cases the value of $a+c$ and $ac-b^2$ should be the same before and after.

Problems 9

This result follows from the fact that in the original representation $F = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ and in the canonical representation we have $F = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ are similar and thus have the same characteristic equation. The characteristic equation in the first case is

$$\lambda^2 - (a+c)\lambda + (ac - b^2) = 0\,,$$

while the characteristic equation of the second expression is

$$(\lambda_1 - \lambda)(\lambda_2 - \lambda) = 0 \quad \text{or} \quad \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2 = 0\,.$$

Equating the coefficients of $O(\lambda)$ and $O(1)$ gives the desired result.

Problems 10

If $|\Delta| \neq 0$ then Δ must be of rank 3 and we have a proper conic.