Solutions to the Problems in Probability Without Tears by Derek Rowntree

John Weatherwax

To my family.

Introduction

This is a solution manual to the problems in the excellent textbook:

Probability Without Tears
by Derek Rowntree

You can always find the most recent "up-to-date" version of this manual at the following location:

`https://waxworksmath.com/Authors/N_Z/Rowntree/Probability_Without_Tears/rowntree.html`

As a final comment, I've worked hard to make these notes as good as I can, but I have no illusions that they are perfect. If you feel that that there is a better way to accomplish or explain an exercise or derivation presented in these notes; or that one or more of the explanations is unclear, incomplete, or misleading, please tell me. If you find an error of any kind – technical, grammatical, typographical, whatever – please tell me that, too. I'll gladly add to the acknowledgments in later printings the name of the first person to bring each problem to my attention.

Probability Without Tears

Chapter 2: Examination Questions

Question 1

Part (i): This is $\left(\frac{1}{5}\right)\left(\frac{1}{4}\right) = \frac{1}{20}$.

Part (ii): This is $1 - \frac{1}{5} - \frac{1}{3} = \frac{7}{15}$ which is different than the answer given in the book.

Part (iii): This is

$$\left(1 - \frac{1}{5}\right)\left(1 - \frac{1}{4}\right) = \frac{3}{5}.$$

Part (iv): Let A be the event that Ann goes to University and B the event that Beryl goes to university. Then the probability we want is

$$P(A)P(B^c) + P(A^c)P(B) = \frac{1}{5}\left(1 - \frac{1}{4}\right) + \left(1 - \frac{1}{5}\right)\frac{1}{4} = \frac{7}{20}.$$

Question 2

Part (i): This is

$$P(A)P(B)P(C) = (0.2)(0.3)(0.1) = 0.006.$$

Part (ii): This would be

$$\begin{aligned} P(\text{exactly one machine breaks down}) &= P(A)P(B^c)P(C^c) + P(A^c)P(B)P(C^c) + P(A^c)P(B^c)P(C) \\ &= 0.2(0.7)(0.9) + 0.8(0.3)(0.9) + 0.8(0.7)(0.1) = 0.398. \end{aligned}$$

Question 3

Part (a)-(i): After the first contestant picks his category the second contestant can then pick the same category with chance $\frac{1}{3}$. The third contestant will pick this same category with a probability of $\frac{1}{3}$. All combined then this would be

$$\frac{1}{3} \times \frac{1}{3} = \frac{1}{9}.$$

Part (a)-(ii): Following the same logic as above after the first contestant picks his category the second contestant can then pick a different category with chance $\frac{2}{3}$. The third contestant will pick a different category with a probability of $\frac{1}{3}$. All combined then this would be

$$\frac{2}{3} \times \frac{1}{3} = \frac{2}{9}.$$

Part (a)-(iii): This is the event that is the complement of the above two. Thus this event has a probability of

$$1 - \frac{1}{9} - \frac{2}{9} = \frac{6}{9} = \frac{2}{3}.$$

We can check that this is correct by running a MC simulation with the following `python` code

```
import numpy as np
from collections import Counter
nmc = 100000
n_desired_result = 0
draws = np.random.choice(['A', 'B', 'C'], [nmc, 3])
for ii in range(nmc):
    cnt = Counter(draws[ii, :])
    if sorted(cnt.values())==[1, 2]:
        n_desired_result += 1
print(f'prob= {n_desired_result/nmc:0.6f}')
```

Running this gives

```
prob= 0.667450
```

Part (b): This is

$$1-P(\text{No card is drawn twice}) = 1-\left(\frac{45}{52}\right)\left(\frac{44}{52}\right)\left(\frac{43}{52}\right)\left(\frac{42}{52}\right)\left(\frac{41}{52}\right)\left(\frac{40}{52}\right)\left(\frac{39}{52}\right) = 0.777532.$$

Question 4

Part (i): Once a coin has been select its probability of heads is "known". Thus conditioning on the coin selected we have

$$p = \frac{1}{3}(1) + \frac{1}{3}\left(\frac{1}{2}\right) + \frac{1}{3}\left(\frac{1}{2}\right) = \frac{2}{3}.$$

With out loss of generality lets assume that biased coin is A. By drawing a tree diagram for this process at the end we have a total of 16 outcomes.

Part (ii): In this tree diagram we find six paths that have a head on both tosses for a probability of $\frac{6}{16} = \frac{3}{8}$.

Part (iii): In this tree diagram we find have two paths that have a tail on both tosses for a probability of $\frac{2}{16} = \frac{1}{8}$.

Question 5

In a given hour local buses come at

$$05\,,20\,,35\,,50\,,$$

minutes after the start of the hour. Express bus come at 10 minutes after the hour. If we look at the periods of time where we would have to wait longer than six minutes we have the intervals of times (working backwards)

$$50 - 59\,,35 - 44\,,20 - 29\,,10 - 14\,.$$

The total time is then

$$9 + 9 + 9 + 4 = 31\,.$$

Thus the probability is $\frac{31}{60}$.

Question 6

We are told that $P(A) = \frac{1}{4}$ and $P(B) = \frac{1}{5}$ and the events A and B are independent.

Part (i): For the game to be a draw both players have to loose which happens with a probability of

$$\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{5}\right) = \frac{3}{5}\,.$$

Part (ii): We can build the events where A and B both win/loose in Table 1. In that table we see that the probability that the game is a draw (bottom right corner) or A wins (top row) is given by

$$\frac{12}{20} + \frac{1}{20} + \frac{1}{5} = \frac{17}{20}\,.$$

If one game is played the probability that B wins is the sum of the first column in Table 1. This is

$$P(B \text{ wins}) = \frac{1}{20} + \frac{3}{20} = \frac{1}{5}\,.$$

If two games are played the probability that B wins both is then $\left(\frac{1}{5}\right)^2 = \frac{1}{25}$.

	$P(B) = 1/5$	$P(B^c) = 4/5$
$P(A) = 1/4$	1/20	1/5
$P(A^c) = 3/4$	3/20	12/20

Table 1: The table for the density $P(A, B)$

Question 7

Part (a): We can compute this table most easily in `R` with the following code

```
hearts = c(seq(5, 10), c(10, 10, 10, 11)) ## the 6 numered cards followed by J, Q, K, Ace
spades = c(seq(5, 10), c(10, 10, 10, 11))
df = data.frame(outer(hearts, spades, FUN='+'))
colnames(df) = c('5', '6', '7', '8', '9', '10', 'J', 'Q', 'K', 'A')
rownames(df) = c('5', '6', '7', '8', '9', '10', 'J', 'Q', 'K', 'A')
print(df)
```

Running this gives

```
    5  6  7  8  9 10  J  Q  K  A
5  10 11 12 13 14 15 15 15 15 16
6  11 12 13 14 15 16 16 16 16 17
7  12 13 14 15 16 17 17 17 17 18
8  13 14 15 16 17 18 18 18 18 19
9  14 15 16 17 18 19 19 19 19 20
10 15 16 17 18 19 20 20 20 20 21
J  15 16 17 18 19 20 20 20 20 21
Q  15 16 17 18 19 20 20 20 20 21
K  15 16 17 18 19 20 20 20 20 21
A  16 17 18 19 20 21 21 21 21 22
```

Note that this is a 10×10 grid which has 100 elements. We can count the number of times we get each "score" using

```
values = as.vector(t(df))
print(as.data.frame(table(values)))
```

which gives

```
   values Freq
1      10    1
2      11    2
```

3	12	3
4	13	4
5	14	5
6	15	12
7	16	13
8	17	12
9	18	11
10	19	10
11	20	18
12	21	8
13	22	1

Part (i): From the above we see that the most probable score is one of 20 which has a frequency of 18.

Part (ii): Picture cards are Jack's, Queen's, and King's which occupy a 3×3 grid in the above "table" of outcomes. This means the probability is then $\frac{9}{100}$.

Part (iii): The "score" 15 happens 12 times and so has a probability of $\frac{12}{100} = \frac{3}{25}$.

Part (iv): A "score" less than or equal to 15 happens $12+5+4+3+2+1 = 27$ times and so has a probability of $\frac{27}{100}$.

Part (b): These sample outcomes correspond to the upper or lower triangular region of the grid above (excluding the diagonal). There are $\frac{100-10}{2} = 45$ of these elements.

Part (i): From the above "$\frac{1}{2}$ grid" we see that the most probable score is one of 20 which has a frequency of 6.

Part (ii): There are three ways this can happen so the probability is $\frac{3}{45} = \frac{1}{15}$.

Part (iii): There are six ways this can happen so the probability is $\frac{6}{45} = \frac{2}{15}$.

Part (iv): There are 12 ways this can happen so the probability is $\frac{12}{45} = \frac{4}{15}$.

Question 8

We are told that $P(H) = 0.4$.

Part (i): This would be $P(T) = 1 - P(H) = 0.6$.

Part (ii): This would be $1 - P(\text{No tails}) = 1 - 0.4^2 = 0.84$.

Part (iii): This would be $E(T) = nP(T) = 50(0.6) = 30$.

Question 9

Part (i): We must first draw a red ball and then another red ball. This will happen with probability of

$$\frac{5}{20}\left(\frac{8}{20}\right) = \frac{1}{10}\,.$$

Part (ii): We might or might not draw a red ball from the first box on the first draw. Conditioning on which we do we find

$$\frac{5}{20}\left(\frac{8}{20}\right) + \frac{15}{20}\left(\frac{7}{20}\right) = 0.3625\,.$$

Part (iii): The complement of this event is that we don't draw a red or a white ball from box B. This means we draw a blue ball from box B. We can compute this by conditioning on whether we draw a blue ball from A as in Part (ii) above. Thus the probability we draw a blue ball from B is

$$\frac{9}{20}\left(\frac{5}{20}\right) + \frac{11}{20}\left(\frac{4}{20}\right) = \frac{89}{400}\,.$$

The probability we don't draw a blue ball from box B is then

$$1 - \frac{89}{400} = 0.7775\,.$$

Question 10

Part (i): We would need all of the men to be out which happens with probability of $\frac{1}{2} \times \frac{1}{4} \times \frac{1}{4} = \frac{1}{32}$.

Part (ii): The call has to come for A, B, or C while they are not out which happens with probability of

$$\frac{2}{5}\left(1 - \frac{1}{2}\right) + \frac{2}{5}\left(1 - \frac{1}{4}\right) + \frac{1}{5}\left(1 - \frac{1}{4}\right) = 0.65\,.$$

Part (iii): We would need to have three calls for A, B, or C for a probability of

$$\left(\frac{2}{5}\right)^3 + \left(\frac{2}{5}\right)^3 + \left(\frac{1}{5}\right)^3 = 0.136\,.$$

Part (iv): Let A, B, and C are the events that a call comes in for man A, B, and C respectively. Under this we have

$$P(A) + P(B) + P(C) = 1\,.$$

When we perform an experiment that can result in only one of A, B, or C multiple times say n times then the probability that we have x_A calls for A, x_B calls for B, and x_C calls for C (when $x_A + x_B + x_C = n$) is given by a multinomial distribution and has a probability density of

$$\left(\begin{array}{c} n \\ x_A\,, x_B\,, x_C \end{array} \right) P(A)^{x_A} P(B)^{x_B} P(C)^{x_C}$$

Here we have $n = 3$ and we want to know the value of the above when $x_A = x_B = x_C = 1$ where we find

$$\frac{3!}{1!1!1!} \left(\frac{2}{5} \right) \left(\frac{2}{5} \right) \left(\frac{1}{5} \right) = \frac{24}{125}\,.$$

Part (v): If we let $p(O_B) = \frac{1}{4}$ be the probability that man B is "out" then the probability that B is "in" is $P(I_B) = 1 - P(O_B) = \frac{3}{4}$. Let N be the number of calls that are needed to be made to reach B. Then we want to compute

$$P(N > 3) = 1 - P(N \leq 3) = 1 - (P(N = 1) + P(N = 2) + P(N = 3))\,.$$

Each of these can be computed from what we know. We have

$$P(N > 3) = 1 - P(I_B) - P(O_B)P(I_B) - P(O_B)^2 P(I_B) = \frac{1}{64}\,,$$

when we evaluate.

Question 11

Part (a): This would be

$$0 \times 24 + 1 \times 14 + 2 \times 10 + 3 \times 2 + 4 \times 0 = 40\,.$$

Part (b): This would be $\frac{40}{200} = \frac{1}{5}$.

Part (c): This would be $\frac{10}{50} = \frac{1}{5}$.

Part (d): In the given sample the proportion of trays with two or more bruised apples is

$$\frac{10}{50} + \frac{2}{50} = \frac{12}{50} = \frac{6}{25}\,.$$

Then in $N = 1000$ trays we would expect to find

$$N \left(\frac{6}{25} \right) = 1000 \left(\frac{6}{25} \right) = 240\,,$$

trays with two or more bruised apples.

Question 12

Part (a): This is $\left(\frac{1}{6}\right)^3 = \frac{1}{216}$.

Part (b): This would be

$$\left(\frac{5}{6}\right)\left(\frac{4}{6}\right)\left(\frac{3}{6}\right) = \frac{5}{18},$$

when we simplify.

Part (c): The probability that we get one even number is $\frac{1}{2}$. The probability we get X even numbers (from $n = 4$ throws) is then given by a binomial distribution and so this probability is

$$\binom{4}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 = \frac{3}{8}.$$

Let T be a random variable denoting the "score" (the number showing on the top of the die) on one throw of the die. Then as each outcome has the same probability the expectation of T is given by

$$E[T] = \frac{1}{6}\sum_{k=1}^{6} k = \frac{1}{6}\left(\frac{6(7)}{2}\right) = \frac{7}{2}.$$

The expected total score from four throws is then four times this or 14.

Question 13

Part (a): If we conditioned on the probability that a given graduate is selected for each of the three types of "programs" and then matriculates from that program to become a senior we have

$$0.1(0.6) + 0.3(0.2) + 0.6(0.05) = 0.15.$$

Part (b): Let M be the event that a graduate has a MA in Management Studies and S the event that a graduate is a senior executive then we seek to compute $P(M|S)$. We have

$$P(M|S) = \frac{P(MS)}{P(S)} = \frac{P(S|M)P(M)}{P(S)} = \frac{0.6(0.1)}{0.15} = \frac{2}{5}.$$

Question 14

Draw a Venn Diagram with three groups: S for sociology, P for politics, and H for history such that we have a representation of all possible combinations. Abusing notation a bit let S, P, and H represent the number of people in each of the groups above (but not in any other group). Let SP, HS, and HP be the number of people in the intersection of the two groups (but not in any other group) and finally let HPS be the number of people in all three groups. Then we are told that

$$\begin{aligned} S &= 16 \\ P &= 21 \\ H &= 20 \\ SP &= 7 \\ HS &= 5 \\ HP &= 8 \\ HPS &= 3\,. \end{aligned}$$

As these "events" are independent the total number of people is the sum of all of these numbers. Performing that summation we find a total of 80 people.

The probability that a person studies only one subject in total is then

$$\frac{16+21+20}{80} = \frac{57}{80}\,.$$

Question 15

Part (a): After the first, second, and third days we have found seven tagged rabbits and two untagged rabbits for a total of at least nine total rabbits on the island.

Part (b): If we start with $N = 12$ rabbits then after the first days catch we have two tagged rabbits and then $N-2 = 10$ untagged rabbits. To get the seconds days catch we must select five untagged rabbits. The number of untagged rabbits X is given by a hypergeometric distribution and so

$$P(X=5) = \frac{\binom{10}{5}\binom{2}{0}}{\binom{12}{5}} = \frac{7}{22} = 0.318182\,,$$

when we expand and simplify.

Part (c): Let S and T be the events of the second and third days catches when $N = 16$ and we are asked to compute

$$P(S,T) = P(T|S)P(S)\,.$$

Using the same logic for Part (b) above but with $N = 16$ we have

$$P(S) = \frac{\binom{14}{5}\binom{2}{0}}{\binom{16}{5}} = \frac{11}{24}\,.$$

Once the event S happens we still have $N = 16$ rabbits but now nine are tagged and seven are untagged. The probability we draw some number of tagged rabbits is again given by a hypergeometric distribution and so we have

$$P(T|S) = \frac{\binom{7}{2}\binom{9}{2}}{\binom{16}{4}} = \frac{27}{65}\,.$$

The probability $P(S,T)$ we seek is then given by

$$P(S,T) = \frac{27}{65}\left(\frac{11}{24}\right) = \frac{99}{520} = 0.190385\,.$$

Chapter 3: Combining and Permuting

Question 1

This would be

$$\begin{aligned} N_B &= 5 \times 6 = 30 \\ N_G &= 4 \times 7 = 28\,. \end{aligned}$$

Question 2

The number of choices is given by $N = 8 \times 6 \times 5 = 240$ so $p = \frac{1}{240}$.

Question 3

The number of entries is given by $N = 12 \times 11 \times 10 \times 9 \times 8 = 95040$ so $2N = 190080$.

Question 4

The number of ways the men can travel on the train is $8 \times 7 \times 6 \times 5 = 1680$. If they do not have to travel in separate coaches the number is $8^4 = 4096$.

Question 5

Problems with combinations are #1 and #2. Problems with permutations are #3 and #4.

Question 6

I think the book must mean 15 horses. Then Hilda has $H = 15 \times 14 \times 13 = 2730$ orderings while Mabel has $M = 7 \times 6 \times 5 \times 4 \times 3 = 2520$ orderings. As $M < H$ Mabel has a better chance of being correct by randomness.

Question 7

The first condition has $10 \times 9 \times 8 = 720$ different choices. The second part has $9 \times 10 \times 10 = 900$ different choices. The total number of registration numbers is then $720 \times 900 = 648000$.

Question 8

Part (a): If we imagine the two books that must be side by side "glued" together then we have five items to arrange which would have $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$ orderings. As there are two ways to order the "book pair" the total number of orderings is $2 \times 120 = 240$.

Part (b): If we care nothing about the ordering then there are $6! = 720$ ways of ordering the books of these 240 have the two books side by side. Thus there are $720 - 240 = 480$ ways in which the two books are *not* side by side.

Question 9

The number of orderings is $2! \times 3! \times 4! = 288$.

Question 10

The number of such registrations is given by $3 \times 2 \times 10 \times 6 = 360$. If each takes 15 minutes this will take $360 \times 15 = 5400$ minutes or 90 hours.

If we learn that the final registration number is F then our final letter search space drops away and we loose the "six" above for a search that takes

$$90 \times \left(\frac{1}{6}\right) = 15\,,$$

hours.

Question 11

Five objects can be arranged in order in $5! = 120$ ways.

Part (a): If the two gentleman sit together it is like we are arranging four objects which can be done in $4! = 24$ ways. There are two orderings of the men so there are $2 \times 24 = 48$ orderings of the type where the men sit together. This will then have a probability of $\frac{48}{120} = \frac{2}{5} = 0.4$ of happening by chance.

Part (b): This ordering must be

$$L\,, G\,, L\,, G\,, L\,.$$

Thus there are two ordering of men and $3! = 6$ ordering of ladies for a total ordering of $2 \times 6 = 12$. The probability of this happening by chance is then $\frac{12}{120} = \frac{1}{10}$.

Question 12

None of these problems deal with combinations and all 6-11 deal with permutations.

Chapter 3: Formulae for Permutations

Question 1

This would be $7C3 = \frac{7!}{3! \cdot 4!} = 35$ as the ordering of the cars selected does not matter.

Question 2

We can select the three boys from the 12 possible in $12C3$ ways and the four girls from the 14 possible in $14C4$ for a total of

$$12C3 \times 14C4\,,$$

ways.

Question 3

Part (a): Here we can select one white cube from the two possible in $2C1 = 2$ ways and then we need to select two red cubes from the three possible in $3C2 = 3$ ways for a total of

$$2 \times 3 = 6\,,$$

ways to select a set with a single white cube.

To select two white cubes from the two possible can be done in $2C2 = 1$ way and then we need to select one red cube from the three possible in $3C1 = 3$ ways for a total of

$$1 \times 3 = 3\,,$$

ways to select a set with two white cubes.

The solution is then the sum $6 + 3 = 9$.

Part (b): Conditioning on selecting one, two, or three red cubes we have the number of sets of each type to be given by

$$\begin{aligned} 3C1 \times 2C2 &= 3 \\ 3C2 \times 2C1 &= 3 \times 2 = 6 \\ 3C3 \times 2C0 &= 1 \times 1 = 1\,, \end{aligned}$$

This gives a total of $3 + 6 + 1 = 10$.

Chapter 4: Probability by Combinations

Question 1

If the probability the man is alive is p then we are told that $p = \frac{3}{4}$. We then want to evaluate

$$\left(\begin{array}{c} 8 \\ 4 \end{array} \right) p^4 (1-p)^4\,.$$

Question 2

We are told that the probability of rain is $p = \frac{3}{4}$ and that I'm going to be in Seattle for $N = 7$ days. The probability that it rains n days (from N) is then given by a binomial random variable and thus $p_n = \left(\begin{array}{c} N \\ n \end{array} \right) p^n q^{N-n}$. To have it rain no more than two days means that we want to consider

$$
\begin{aligned}
p_0 &= \left(\begin{array}{c} 7 \\ 0 \end{array}\right) \left(\frac{1}{4}\right)^7 = \frac{1}{4^7} \\
p_1 &= \left(\begin{array}{c} 7 \\ 1 \end{array}\right) \left(\frac{3}{4}\right) \left(\frac{1}{4}\right)^6 = \frac{21}{4^7} \\
p_2 &= \left(\begin{array}{c} 7 \\ 2 \end{array}\right) \left(\frac{3}{4}\right)^2 \left(\frac{1}{4}\right)^5 = \frac{189}{4^7} \,.
\end{aligned}
$$

Adding these up we get $\frac{1+21+189}{4^7} = \frac{211}{4^7} = 0.0128784$. Note that this is a different number than is given in the back of the book.

Question 3

This would be given by

$$
\sum_{k=9}^{12} \left(\begin{array}{c} 12 \\ k \end{array}\right) 0.7^k 0.3^{12-k} \,.
$$

Question 4

When $p = q = \frac{1}{2}$ the expressions $p^k q^{n-k} = \frac{1}{2}^n$ is the same for all $0 \leq k \leq n$ and thus the relative frequency will be given by the binomial coefficients $\left(\begin{array}{c} 5 \\ k \end{array}\right)$. We can compute these with the `python` code

```
from scipy.special import comb
print(comb(5, range(6)))
```

which gives

```
[ 1.  5. 10. 10.  5.  1.]
```

Question 5

The expected frequency will be given by the binomial probability $\left(\begin{array}{c} 6 \\ k \end{array}\right) p^k q^{6-k}$ for $0 \leq k \leq 6$ multiplied by the number of families $F = 320$. We can compute these with the `python` code

```
from scipy.stats import binom
ks = range(0, 6+1)
pmf = binom.pmf(ks, 6, 0.5)
print(320*pmf)
```

which gives

```
[  5.  30.  75. 100.  75.  30.   5.]
```

Question 6

The expected frequency will be given by the binomial probability $\binom{4}{k} p^k q^{4-k}$ for $0 \leq k \leq 4$ and $p = 0.25$ multiplied by the number of samples $S = 128$. We can compute these with the `python` code

```
from scipy.stats import binom
ks = range(0, 4+1)
pmf = binom.pmf(ks, 4, 0.25)
print(128*pmf)
```

which gives

```
[40.5 54.  27.   6.   0.5]
```

Rounding these we get the results in the book.

Question 7

In the $162 \times 4 = 648$ marbles drawn we found

$$65 + 2(52) + 3(14) + 4 = 215\,,$$

blue ones giving a fraction of $p = \frac{215}{648} = 0.33179$.

The expected frequency will be given by the binomial probability $\binom{4}{k} p^k q^{4-k}$ for $0 \leq k \leq 4$ and $p = 0.25$ multiplied by the number of samples $S = 162$. We can compute these with the `python` code

```
from scipy.stats import binom
ks = range(0, 4+1)
p = 215./648
pmf = binom.pmf(ks, 4, p)
print(162*pmf)
```

which gives

```
[32.29732669 64.14711537 47.77700856 15.81533001  1.96321937]
```

Rounding these we get the results in the book.

Examination Questions

Question 1

We have to attempt the problem (event A) and be successful (event S) which means that the probability we are correct C is

$$P(C) = P(A \cap S) = \frac{9}{10} \times \frac{2}{3} = \frac{3}{5}.$$

The probability we are not correct is then $P(C^c) = \frac{2}{5}$. Finding at least one correct exam is the complement of finding no correct exams and is given by

$$1 - \left(\frac{2}{5}\right)^3 = \frac{117}{125}.$$

Question 2

Let $p = \frac{1}{4}$ then this would be

$$\left(\begin{array}{c} 4 \\ 3 \end{array}\right) p^3(1-p) + \left(\begin{array}{c} 4 \\ 3 \end{array}\right) p^4 = \frac{13}{256},$$

when we evaluate.

Question 3

This would be the number

$$N = \left(\begin{array}{c} 11 \\ 8 \end{array}\right)\left(\begin{array}{c} 4 \\ 2 \end{array}\right)\left(\begin{array}{c} 8 \\ 5 \end{array}\right) = 55440,$$

when we evaluate.

Question 4

We need to evaluate the number of teams *without* these specified players. That is the number

$$N' = \binom{10}{8}\binom{3}{2}\binom{6}{5} = 810\,,$$

when we evaluate. The probability that we don't have to change our original team is then $\frac{N'}{N} = 0.0146104$. This is very different than the number given in the back of the book.

Question 5

Part (i): This would be $\left(\frac{1}{10}\right)^3 = \frac{1}{1000}$.

Part (ii): If $p = \frac{1}{10}$ this would be the number

$$\binom{3}{2} p^2(1-p) + \binom{3}{3} p^3 = \frac{7}{250}\,,$$

when we evaluate.

The probability that a sample will have fewer than two people going to the beach would be $1 - \frac{7}{250} = \frac{243}{250}$. In a 100 samples of four we then expect to find

$$100 \times \left(\frac{243}{250}\right) = 97.2\,.$$

Question 6

The probability that we get just one six is

$$P_1 = \binom{10}{1}\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^9 = \frac{10 \cdot 5^9}{6^{10}}\,.$$

The probability that we get just two sixes is

$$P_2 = \binom{10}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^8 = \frac{\binom{10}{2} \cdot 5^8}{6^9}\,.$$

Evaluating these we find $P_1 > P_2$.

Question 7

Many of my answers differed from the ones in the back of the book. If anyone sees anything wrong with what I have done please contact me.

If we take $p = 0.9$ then for the home team to have enough players will happen if

- The goal keeper shows up and
- From the 11 other players at least 10 other players show up.

Thus this happens with a probability

$$P_H = 0.9\left(\binom{11}{11}p^{11} + \binom{11}{10}p^{10}(1-p)^1\right)$$

Evaluating this I find $P_H = 0.627621$.

For the away team different teams can be formed depending on how many goal keepers show up. We have the following probabilities for none, one, or two goal keepers showing up $G = 0$, $G = 1$, and $G = 2$ as

$$\begin{aligned} P(G=0) &= (1-p)^2 \\ P(G=1) &= 2p(1-p) \\ P(G=2) &= p^2 \,. \end{aligned}$$

The probability that the away team can form a team can be computed by conditioning on the number of goal keepers that show up G as

$$\begin{aligned} P_A &= P(G=1)\left(\binom{11}{11}p^{11} + \binom{11}{10}p^{10}(1-p)^1\right) \\ &+ P(G=2)\left(\binom{11}{11}p^{11} + \binom{11}{10}p^{10}(1-p)^1 + \binom{11}{9}p^{9}(1-p)^2\right) \,. \end{aligned}$$

Evaluating this I compute $P_A = 0.862979$.

As everything is independent the probabilities that these specifications occur is given by

$$\begin{aligned} P(\text{both complete}) &= P_H P_A = 0.541624 \\ P(\text{home complete and away is not}) &= P_H(1-P_A) = 0.085997 \\ P(\text{home is not complete and away is complete}) &= (1-P_H)P_A = 0.321355 \\ P(\text{both not complete}) &= (1-P_H)(1-P_A) = 0.051024 \,. \end{aligned}$$

Question 8

Let $p = 0.2$ then these would be given by

$$\begin{aligned} \binom{5}{0} p^0(1-p)^5 &= 0.32768 \\ \binom{5}{1} p^1(1-p)^4 &= 0.4096 \\ 1 - \binom{5}{1} p^1(1-p)^4 - \binom{5}{0} p^0(1-p)^5 &= 0.26272 \,. \end{aligned}$$

Question 9

Part (a-i): Both die must show six and so the probability of this happening is $\left(\frac{1}{6}\right)^2 = \frac{1}{36}$.

Part (a-ii): There are three "pairs" where this outcome occurs. They are $\{(1,3), (2,2), (3,1)\}$ and thus the probability is then $\frac{3}{36} = \frac{1}{12}$.

Part (a-iii): There are six "pairs" where this outcome occurs and thus the probability is then $\frac{6}{36} = \frac{1}{6}$.

Part (b): At each location a given digit has a $p = \frac{1}{10}$ chance of occurring.

Part (b-i): This would be $p^3 = \frac{1}{1000}$ as the last three spots must match whatever digit is shown in the first spot.

Part (b-ii): Once the first digit is specified we have nine choices for the second digit, eight choices for the third digit, and seven choices for the fourth digit. This gives a probability of

$$\left(\frac{9}{10}\right)\left(\frac{8}{10}\right)\left(\frac{9}{10}\right) = \frac{63}{125} \,.$$

Part (b-iii): We can choose the three common digits in 10 ways and the other "different" digit in nine ways and the location of the different digit in four ways. This gives a probability of

$$\frac{10 \cdot 9 \cdot 4}{10^4} = \frac{36}{1000} = \frac{9}{250} \,.$$

Part (b-iv): The complement of two or more digits are the same is the statement that all digits are different. There are $10 \times 9 \times 8 \times 7 = 5040$ four digit numbers with all digits unique. Thus the probability we want is

$$\frac{10^4 - 5040}{10^4} = \frac{4960}{10000} = \frac{496}{1000} \,,$$

This is different than the number in the back of the book.

Question 10

Part (a): This would be

$$1 - 0.1 - 0.3 - 0.2 = 0.4\,.$$

Part (b): The probability we get one three is $\frac{3}{6} = \frac{1}{2}$ so to get two of them will happen with a probability of $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$.

Part (c): The fraction of all inoculated people infected was $\frac{500}{10500} = 0.047619$ while the fraction of infected people that were not inoculated was $\frac{250}{1250} = 0.2$ quite a difference.

Question 11

Part (a-i): Let B be the outcome from the "backgammon die" and D the outcome from the "regular die" then the following pairs will have a sum of three

$$(B, D) \in \{(2, 1)\}\,.$$

Thus the probability of this even is $\frac{1}{36}$.

Part (a-ii): The following pairs will have a sum of six

$$(B, D) \in \{(2, 4)\,, (4, 2)\}\,.$$

Thus the probability of this even is $\frac{2}{36} = \frac{1}{18}$.

Part (a-iii): As no pairs have a sum of 16 the probability of this event is zero.

Part (a-iv): As B always gives an even number if D gives an even number the sum will be even. This is $6 \times 3 = 18$ pairs and so the probability is $\frac{18}{36} = \frac{1}{2}$.

Part (a-v): The following pairs will have $B < D$

$$(B, D) \in \{(2, 3)\,, (2, 4)\,, (2, 5)\,, (2, 6)\,, (4, 5)\,, (4, 6)\}\,.$$

Thus the probability of this even is $\frac{6}{36} = \frac{1}{6}$.

Part (b-i): Let $p = \frac{1}{12}$ then this is $\left(\begin{array}{c} 3 \\ 0 \end{array}\right) p^0 (1 - p)^3$.

Part (b-ii): This is $\left(\begin{array}{c} 3 \\ 1 \end{array}\right) p^1 (1 - p)^2$.

Part (b-iii): This is $1 - \left(\begin{array}{c} 3 \\ 0 \end{array}\right) p^0 (1 - p)^3$.

Question 12

As the two colors are thoroughly mixed the proportion of red marbles is $p = \frac{1}{2}$ and the number of red marbles found from six is distributed as a binomial random variable with $n = 6$ and $p = \frac{1}{2}$.

Part (i): From the problem statement Fred will give any initial blue marbles that he has to Tom and Tom will give any initial red marbles to Fred. After this exchange they will both have a complete set of the color color they prefer if they start with

- Fred initially gets 6 blue marbles and Tom initially gets 6 red marbles
- Fred initially gets 5 blue marbles and Tom initially gets 5 red marbles
- Fred initially gets 4 blue marbles and Tom initially gets 4 red marbles
- Fred initially gets 3 blue marbles and Tom initially gets 3 red marbles
- Fred initially gets 2 blue marbles and Tom initially gets 2 red marbles
- Fred initially gets 1 blue marbles and Tom initially gets 1 red marbles
- Fred initially gets 0 blue marbles and Tom initially gets 0 red marbles

The sum of the probabilities of each of these events is the desired probability. We can evaluate that with the following

```
from scipy.stats import binom
n = 6
p = 1./2
ks = range(0, n+1)
pmf = binom.pmf(ks, n, p)
sum(pmf**2)
```

This gives 0.225586.

Part (ii): Fred will have more blue than he wants if Tom has no reds to exchange with Tom. This can happen if

- Fred initially gets 3, 4, 5, or 6 blue marbles and Tom initially gets 0 red marbles
- Fred initially gets 4, 5, or 6 blue marbles and Tom initially gets 1 red marbles
- Fred initially gets 5, or 6 blue marbles and Tom initially gets 2 red marbles
- Fred initially gets 6 blue marbles and Tom initially gets 3 red marbles

```
from scipy.stats import binom
n = 6
p = 1./2
pt_1 = ( binom.pmf(range(3, n+1), n, p) * binom.pmf(0, n, p) ).sum()
pt_2 = ( binom.pmf(range(4, n+1), n, p) * binom.pmf(1, n, p) ).sum()
pt_3 = ( binom.pmf(range(5, n+1), n, p) * binom.pmf(2, n, p) ).sum()
pt_4 = ( binom.pmf(range(6, n+1), n, p) * binom.pmf(3, n, p) ).sum()
np.round(pt_1 + pt_2 + pt_3 + pt_4, 6)
```

This gives 0.072998.

Question 13

We can draw $\frac{52!}{(52-4)!} = 52 \cdot 51 \cdot 50 \cdot 49 = 6497400$ such permutations.

Part (a-i): I think this is given by

$$\frac{\binom{4}{1}\binom{52-4}{3}}{\binom{52}{4}} = 0.255551\,,$$

but this does not equal the number in the back of the book.

Part (a-ii): This is given by the complement of the hand where no cards are aces or

$$1 - \frac{\binom{4}{0}\binom{52-4}{4}}{\binom{52}{4}} = 0.281263\,.$$

Another way to derive this is to condition on the probability we keep getting no aces as we draw cards or

$$1 - P(\text{no aces drawn}) = 1 - \left(\frac{48}{52}\right)\left(\frac{47}{51}\right)\left(\frac{46}{50}\right)\left(\frac{45}{49}\right)\,,$$

which gives the same as the above when we evaluate.

We can compute the probability of getting $0, 1, 2, 3, 4$ aces in a hand of four cards using

```
import numpy as np
ks = np.arange(0, 4+1)
np.round( (comb(4, ks) * comb(52-4, 4-ks)) / comb(52, 4), 3)
```

Running this I get

```
array([0.719, 0.256, 0.025, 0.001, 0.   ])
```

Part (a-iii): There are 13 cards in each suit and so the number of hands that have a single card from each is given by

$$\frac{\binom{13}{1}^4}{\binom{52}{4}} = 0.105498\,.$$

Part (a-iv): There are $\binom{13}{4}$ ways to choose the four different denominations of cards. Once the four denominations are chosen the suit of each card can be specified in four different ways. That means that hands of this type have a probability of

$$\frac{4^4\binom{13}{4}}{\binom{52}{4}} = 0.67611\,.$$

Part (b-i): If the first card is a diamond there are $13-1=12$ diamonds left in the pack of cards that have $52-1=51$ cards left. Thus the probability the second card is a diamond is

$$\frac{12}{51}\,.$$

Part (b-ii): In this case we still have 13 diamond cards left thus the probability of drawing a diamond is given by

$$\frac{13}{51}\,.$$

I think the statement "hence or otherwise" means to find the probability that the second card is a diamond without knowing what the suit of the first card is. We can evaluate this probability by conditioning on the suit of the first card from the set clubs, diamonds, hearts, and spades denoted $\{C, D, H, S\}$. We find

$$\begin{aligned}
P(C_2 = D) &= P(C_2 = D|C_1 = C)P(C_1 = C) + P(C_2 = D|C_1 = D)P(C_1 = D) \\
&+ P(C_2 = D|C_1 = H)P(C_1 = H) + P(C_2 = D|C_1 = S)P(C_1 = S) \\
&= \frac{13}{51}\times\left(\frac{13}{52}\right) + \frac{12}{51}\times\left(\frac{13}{52}\right) + \frac{12}{51}\times\left(\frac{13}{52}\right) + \frac{13}{51}\times\left(\frac{13}{52}\right) \\
&= \frac{3(13^2)+12(13)}{51(52)} = \frac{1}{4}\,.
\end{aligned}$$

Question 14

Part (a-i): We have a total of eight children five of which are boys. We can get a group of only boys if we

- Draw four boys from the eight
- Draw one boy and three girls from the eight

The first event happens with a probability of

$$\frac{\binom{5}{4}\binom{3}{0}}{\binom{8}{4}} = \frac{1}{14},$$

and the second with a probability of

$$\frac{\binom{5}{1}\binom{3}{3}}{\binom{8}{4}} = \frac{1}{14}.$$

Adding these together gives $\frac{1}{7}$.

Part (a-ii): We can get a group of two boys and two girls if we

- Draw two boys and two girls from the eight
- Draw three boys and one girl from the eight

The first event happens with a probability of

$$\frac{\binom{5}{2}\binom{3}{2}}{\binom{8}{4}} = \frac{3}{7},$$

and the second with a probability of

$$\frac{\binom{5}{3}\binom{3}{1}}{\binom{8}{4}} = \frac{3}{7}.$$

Adding these together gives $\frac{6}{7}$.

Part (a-iii): In one of the groups we have to select the two brothers *and* the two sisters. This can happen in only one way for a probability of $\frac{1}{\binom{8}{4}} = \frac{1}{70}$. Selecting the complement of this set will also have the same probability for a total probability of

$$\frac{2}{70} = \frac{1}{35}\,.$$

Part (b): This is the ratio of the probabilities in **Part (a-iii)** to that in **Part (a-ii)** or

$$\frac{\frac{1}{35}}{\frac{6}{7}} = \frac{1}{30}\,.$$

The probability that the two bothers are in the same group is given by twice

$$\frac{\binom{6}{2}}{\binom{8}{4}} = \frac{3}{14}\,,$$

or $\frac{3}{7}$. Then the probability we seek is the ratio of the probabilities in **Part (a-iii)** to that number or

$$\frac{\frac{1}{35}}{\frac{3}{14}} = \frac{2}{15}\,,$$

which is different than the answer in the back of the book.

Question 15

Part (a): If $p = 0.1$ this would be $\binom{4}{0} p^0(1-p)^4 = \frac{6561}{10000} = 0.6561$.

Part (b): This would be $\binom{4}{1} p^1(1-p)^3 = \frac{729}{2500} = 0.2916$.

Part (c): Now if a tile has three or four broken corners it will certainly have two adjacent broken corners but I believe this part of the problem is asking about the number of tiles with *only* two (and not more than two) broken corners that are adjacent. To calculate this later percentage if we "hold" one of these broken corners then there are *two* corners where the other broken corner can go that will give us an adjacent corner and *one* corner where the two broken corners would not be adjacent. This means that we should take $\frac{2}{3}$ of $\binom{4}{2} p^2(1-p)^2$ and the total proportion of tiles with an adjacent broken corners is given by

$$\frac{2}{3}\binom{4}{2} p^2(1-p)^2 = 0.0324\,.$$

We want the expected value V under the given assumptions. We have

$$E[V] = 1000(0.1)0.6561 + 1000(0.02)0.2916 + 1000(0.01)0.0324 = 71.766\,.$$

Question 16

We will have to keep rolling if we do not roll a six. Not rolling a six happens with a probability of $\frac{5}{6}$. The probability we will first roll a six on roll r is then given by

$$P(R = r) = \left(\frac{5}{6}\right)^{r-1} \frac{1}{6}\,.$$

The probability we have to roll more than three times would then be

$$1 - \sum_{r=1}^{3} P(r)\,.$$

We can compute this using

```
rs = np.arange(1, 3+1)
P_r = ( (5./6)**(rs-1) )*(1./6)
print(np.round(1 - sum(P_r), 6))
```

This gives 0.578704.

Question 17

Part (a): As these features are independent I would estimate this as

$$\left(\frac{60}{100}\right)\left(\frac{45}{100}\right) = 0.27\,.$$

Part (b): In the same was as above I would estimate this as

$$\left(\frac{60}{100}\right)\left(1 - \frac{45}{100}\right) = 0.33\,.$$

Part (c): In the same was as above I would estimate this as

$$\left(1 - \frac{60}{100}\right)\left(1 - \frac{45}{100}\right) = 0.22\,.$$

Question 18

Part (i): We can evaluate this by conditioning on the value of one of the die. Let D_i be the value shown on the ith die for $1 \le i \le 3$. Then the probability we are looking for can be computed

$$\begin{aligned} P &= \sum_{i=1}^{6} P(D_3 = i) P(D_1 + D_2 = 18 - i) = \frac{1}{6} \sum_{i=1}^{6} P(D_1 + D_2 = 18 - i) \\ &= \frac{1}{6} P(D_1 + D_2 = 12) = \frac{1}{6^3} = \frac{1}{216}, \end{aligned}$$

since $D_1 + D_2$ will be less than or equal to 12.

Part (ii): In the same way as above

$$\begin{aligned} P &= \sum_{i=1}^{6} P(D_3 = i) P(D_1 + D_2 = 5 - i) = \frac{1}{6} \sum_{i=1}^{6} P(D_1 + D_2 = 5 - i) \\ &= \frac{1}{6} \sum_{i=1}^{4} P(D_1 + D_2 = i) = \frac{1}{6} \sum_{i=2}^{4} P(D_1 + D_2 = i) \\ &= \frac{1}{6} \left(\frac{1}{36} + \frac{2}{36} + \frac{3}{36} \right) = \frac{1}{36}. \end{aligned}$$

Part (iii): This would be

$$\left(1 - \frac{1}{6}\right)^3 = \frac{125}{216}.$$

Part (iv): Again by conditioning this would be

$$\begin{aligned} P &= \sum_{i=1}^{6} P(D_3 = i) P\left(D_1 D_2 = \frac{90}{i}\right) = \frac{1}{6} \sum_{i=1}^{6} P\left(D_1 D_2 = \frac{90}{i}\right) \\ &= \frac{1}{6} \sum_{d \in \{90,45,30,22.5,18,15\}} P(D_1 D_2 = d) = \frac{1}{6} \sum_{d \in \{30,22.5,18,15\}} P(D_1 D_2 = d) \\ &= \frac{1}{6} \left(\frac{2}{36} + 0 + \frac{2}{36} + \frac{2}{36} \right) = \frac{1}{36}. \end{aligned}$$

Question 19

Note that there are 900 possible lottery tickets.

Part (a-i): There are nine choices for the first digit and thus nine tickets that have the same digit in all locations thus a probability of $\frac{9}{900} = \frac{1}{100}$.

Part (a-ii): There are $9 \times 9 \times 8 = 648$ numbers with different digits for a probability of $\frac{648}{900} = \frac{18}{25} = 0.72$. There are nine possible numbers for the first digit. There are nine digits (nine minus the one chosen for the first digit plus the digit 0) for the second digit. For the third digit there are eight digits.

Part (a-iii): From the two parts above, from the 900 tickets there will be $N_1 = 9$ tickets with all three digits the same. There will be $N_3 = 648$ tickets with all digits different this leaves

$$900 - 9 - 648 = 243\,,$$

tickets with two unique digits. Thus the probability is $\frac{243}{900} = 0.27$. This is different than the number given in the back of the book.

Part (a-iv): The three digit numbers that have ascending consecutive digits are

$$\{123\,, 234\,, 345\,, 456\,, 567\,, 678\,, 789\,, 890\}\,,$$

for a total of eight numbers. The three digit numbers that have descending consecutive digits are

$$\{987\,, 876\,, 765\,, 654\,, 543\,, 432\,, 321\,, 210\}\,,$$

for another eight numbers. Thus the probability we *don't* get a number of this form is

$$1 - \frac{16}{900} = \frac{221}{225} = 0.982222\,.$$

This is different than the number given in the back of the book.

Part (a-v): We are looking for numbers of the form $\overline{abc}$ where $a = c$. For a we have nine choices $1 - 9$ and then for b we have another nine choices This gives 81 numbers of this form for a probability of

$$\frac{81}{900} = \frac{9}{100} = 0.09\,.$$

This is different than the number given in the back of the book.

Note that we can check some of the above calculations with some simple `python` code

```
tickets = np.arange(100, 999+1)
def n_unique_digits(x, n=1):
    if len(set(str(x)))==n:
        return True
    else:
        return False

N = len(tickets)
N_i = sum(list(map(lambda x: n_unique_digits(x, n=1), tickets)))
N_ii = sum(list(map(lambda x: n_unique_digits(x, n=3), tickets)))
N_iii = sum(list(map(lambda x: n_unique_digits(x, n=2), tickets)))
assert N_iii == (900 - N_i - N_ii)
```

New seat ordering	number of seats that match the original order
ABC	3
ACB	1
BAC	0
BCA	0
CAB	0
CBA	1

Table 2: All possible reseatings and the number of people who are sitting in their original order.

```
P_i = N_i/N
P_ii = N_ii/N
P_iii = N_iii/N
```

Part (b): We can compute this by conditioning on the probability of a wet day R (where R is for rain) as follows

$$P = P(W|R)P(R) + P(W|R^c)P(R^c) = 0.3(0.2) + 0.6(0.8) = 0.54\,.$$

Question 20

Five unlikely objects can be arraigned in $5! = 120$ ways.

Part (a): Thinking of the two men as one unit there are $4! = 24$ ways to order these four "items". There are $2! = 2$ ways to order the men and so our probability is

$$\frac{4! \times 2}{5!} = \frac{2}{5}\,.$$

Part (b): In this case the viewers must sit $LGLGL$ and there are 3! ways to arrange the ladies and 2! ways to arrange the gentlemen. This gives a probability of

$$\frac{2! \times 3!}{5!} = \frac{1}{10}\,.$$

Part (c-f): There are $3! = 6$ ways in which the three people can be sitting when they get up. Lets denote these three people by A, B, and C and assume that they are initially ordered as ABC. Then in Table 2 for each of the possible permutations of these three people we count the number of matching seats. If M is a random variable denoting the number of matches then using this table we find

$$
\begin{aligned}
P(M=3) &= \frac{1}{6} \\
P(M=2) &= 0 \\
P(M=1) &= \frac{2}{6} = \frac{1}{3} \\
P(M=0) &= \frac{3}{6} = \frac{1}{2} \,.
\end{aligned}
$$

For us to win only one of the prizes one of our five must be selected from the two draws from the 100 tickets. The probability this will happen is

$$
\frac{5}{100} \times \left(\frac{95}{99}\right) + \frac{95}{100} \times \left(\frac{5}{99}\right) = \frac{19}{198} = 0.09596 \,.
$$

This is different than the number given in the back of the book.

References

[1] G. Corliss. Which root does the bisection algorithm find? *SIAM Review*, 19(2):325–327, 1977.

[2] W. Ferrar. *A text-book of convergence*. The Clarendon Press, 1938.