HW 21 Soln 11-24-03

(1) $b_1 = 0$; $b_2 \neq 0$

vectors are linearly independent.

only span b_1.

only basis that spanned b_1

(2) $b_1 = (r, 0)^T$ $\quad b_2 = (0, s)^T$

vectors are linearly ~~dependent~~ independent

= span $\mathbb{R}^2$

$\{b_1, b_2\}$ are basis for $\mathbb{R}^2$

(3) $\vec{b}_1 = \binom{1}{2}$ $\quad \vec{b}_2 = \binom{2}{1}$

vectors are linearly independent

span $\mathbb{R}^2$

$\{b_1, b_2\}$ are basis for $\mathbb{R}^2$.

(4) $b_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ $\quad b_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

vectors are linearly independent.

Don't span $\mathbb{R}^3$.

Not basis for $\mathbb{R}^3$.

(5) $\vec{b}_1 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$ $\quad \vec{b}_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ $\quad \vec{b}_3 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$

$$\vec{B} = [b_1\ b_2\ b_3] = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -2 & 1 \\ -2 & 1 & 1 \end{bmatrix} \underset{RRE}{\Rightarrow} \begin{bmatrix} 1 & 1 & -2 \\ 0 & -3 & 3 \\ 0 & 3 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 3 & 3 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ $\bar{b}_i$ are linearly independent.

They span $\mathbb{R}^3$.

+ ~~they~~ they are a basis for $\mathbb{R}^3$.

(6) $b_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $b_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $b_3 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

b_i are ~~not~~ linearly dependent.

They span $\mathbb{R}^2$.

(7) $b_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $b_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $b_3 = \begin{pmatrix} 1 \\ 4 \\ 9 \end{pmatrix}$

$$B = [b_1\ b_2\ b_3] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 3 \\ 0 & 2 & 7 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

b_i's span $\mathbb{R}^3$, they are linearly independent

+ are a basis for $\mathbb{R}^3$.

(1) $M' = \begin{pmatrix} A' & 0 \\ B' & C' \end{pmatrix}$

Then $p_\lambda(M) = p_\lambda(A')\, p_\lambda(C')$

~~if λ_i an eivalue of A' w/ eivector b_i then~~

~~$\begin{pmatrix} b_i \\ \xi_i \end{pmatrix}$ is an e.vector w/ ξ_i given by~~

~~$A' b_i = \lambda_i b_i$~~

~~$B' b_i + C' \xi_i =$~~

If λ_i is an eivalue of C' ~~then~~ w/ eivector b_i then

$\bar{V} = \begin{pmatrix} 0 \\ b_i \end{pmatrix}$ is an eigenvector of M'

If λ_i is an e.value of A' w/ eivector d_i then

$\vec{V} = \begin{pmatrix} d_i \\ \xi_i \end{pmatrix}$ will be an eivector of M' iff

$A' d_i = \lambda_i d_i$ ✓

$B' d_i + C' \xi_i = \lambda_i \xi_i \Rightarrow (C' - I\lambda_i)\xi_i = -B' d_i$

$\Rightarrow (\lambda_i I - C')\xi_i = B' d_i$

If λ_i is not an eivalue of C' then ξ_i exists & an evector of M' can be formed. If λ_i is an evalue of C' then $B' d_i$ may not be in the range of $\lambda_i I - C'$ & M' is not diagonalizable

② $\lambda_1 \cdot \lambda_2 = -25 - 11 = -36$

$\lambda_1 + \lambda_2 = 0$

$\therefore \lambda_1 = 6 \;\&\; \lambda_2 = -6.$

③ $\lambda_1 \lambda_2 = -6 + 6 = 0 \qquad \lambda_1 + \lambda_2 = 1$

$\lambda_1 = 0 \qquad \lambda_2 = 1$

④ $\begin{pmatrix} 2 & 1 \\ 5 & 0 \end{pmatrix} \qquad \lambda_1 + \lambda_2 = 2 \qquad \lambda_1 \lambda_2 = -5$

$$\begin{vmatrix} 2-\lambda & 1 \\ 5 & -\lambda \end{vmatrix} = -2\lambda + \lambda^2 - 5 = 0$$

$$\lambda^2 - 2\lambda - 5 = 0$$

$$\lambda = \frac{2 \pm \sqrt{4 - 4(-5)}}{2} = \frac{2 \pm \sqrt{4(6)}}{2} = 1 \pm \sqrt{6}$$

$(1-\sqrt{6})(1+\sqrt{6}) = 1 - 6$ ✓

⑤ $$\begin{pmatrix} 1005 & 11 \\ 1 & 995 \end{pmatrix} = \begin{pmatrix} 1000+5 & 11 \\ 1 & 1000-5 \end{pmatrix} = \underbrace{\begin{pmatrix} 5 & 11 \\ 1 & -5 \end{pmatrix}}_{A.} + 1000 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\lambda_1 + \lambda_2 = 0 \qquad \lambda_1 \cdot \lambda_2 = -25 - 11 \quad \ldots$ from #2 $\quad \lambda_1 = 6$

$\lambda_2 = -6$

Then $\lambda_1 = 6 + 1000 = 1006$

$\lambda_2 = -6 + 1000 = 994$

(6) $\lambda_1 + \lambda_2 + \lambda_3 = 3$

$\lambda_1 \lambda_2 \cdot \lambda_3 =$

$1\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0. \Rightarrow \lambda_1 = 0$ say

$\Rightarrow \lambda_2 + \lambda_3 = 3$

since sum of the rows is 3 3 is an e.value

$\lambda_1 = 0 ; \lambda_2 = 3, \Rightarrow \lambda_3 = 0$ also.

(7) $\begin{pmatrix} 6 & 2 & 2 \\ 2 & 6 & 2 \\ 2 & 2 & 6 \end{pmatrix}$ $\sum$ each row $= 10 \Rightarrow$ one e.value is 10

$\lambda_1 + \lambda_2 + \lambda_3 = 18$ $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = 6\begin{vmatrix} 6 & 2 \\ 2 & 6 \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 2 & 6 \end{vmatrix}$

$+ 2\begin{vmatrix} 2 & 6 \\ 2 & 2 \end{vmatrix}$

$\lambda_1 \lambda_2 \lambda_3 = 6(36-4) - 2(12-4) + 2(4-12)$

$= 6^3 - 24 - 24 + 8 + 8 - 24 = 6^3 - 3 \cdot 24 + 2 \cdot 8$

$$(L-\lambda I)^p \xi = 0$$

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

$$\begin{vmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} = 0 = \qquad (2-\lambda)^2 = 0$$

$$\lambda = 2$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \vec{0} \quad \Rightarrow \quad v_2 = 0 \quad v_1 \text{ arb.}$$

$$\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

consider $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\cancel{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}}$$

So $\cancel{(L-2I)^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}}$

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$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\begin{vmatrix} 2-\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 1 = 0 \quad \Rightarrow \quad (\lambda - 1)^2 = 0 \quad \checkmark$$

$$\lambda = \frac{2 \pm \sqrt{4-4}}{2} = 1$$

the e.vector for $\lambda = 1$ is

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

$$\Rightarrow \quad \vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\left(\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)^2 = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \checkmark$$

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$$\frac{36}{62} + \frac{15}{62} + \frac{11}{62} = \frac{36+26}{62} = 1$$

$$rA = (1\ 1\ 1)A = r$$

$$rA^n = r$$

$$rA(A^{n-1}) = r(A^{n-1}) = \dots r.$$

$$A^T r^T$$

$$\lambda_1 = 1$$

$$x(0) = \sum_{i=1}^{m} b_i c_i$$

$$x(n) = \sum_{i=1}^{m} c_i \lambda_i^n b_i$$

(2) $\langle x|y\rangle = x_1y_1 + 2x_2y_2 + 3x_3y_3$

$\langle x|$ & $\langle y|$

when $x = (1,2,3)^T$ $y = (-1,2,5)^T$

$\langle x| = x^T G$

w/ $G_{ij} = \langle e_i|e_j\rangle$

$G_{11} = \langle e_1|e_1\rangle = 1$; $G_{12} = \langle e_1|e_2\rangle = 0$; $G_{13} = \langle e_1|e_3\rangle = 0$

$G_{21} = \langle e_2|e_1\rangle = 0$; $G_{22} = \underbrace{2\langle e_2|e_2\rangle}_{=2}$; $G_{23} = \langle e_2|e_3\rangle = 0$

$G_{31} = \langle e_3|e_1\rangle = 0$; $G_{32} = \langle e_3|e_2\rangle = 0$; $G_{33} = \langle e_3|e_3\rangle = 3$

$\therefore$ $G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

So $\langle x| = (1,2,3)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} = (1\ \ 4\ \ 9)$ #

$\langle y| = (-1\ \ 2\ \ 5)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} = (-1\ \ 4\ \ 15)$ #

$\langle x|y\rangle = -1 + 2(4) + 3(15) = -1 + 8 + 45$

$= 7 + 45 = 52$ #

Using inner product.

$\langle x|y\rangle = (1\ 4\ 9)\begin{pmatrix}-1\\2\\5\end{pmatrix} = -1 + 8 + 45 = 52$ ✓

Yes #

(6) $G_{ij} = \langle e_i | e_j \rangle$

$\cancel{G_{11} = \int_0^1}$ $\quad G_{11} = \langle e_1 | e_1 \rangle = \int_0^1 1 \, dx = 1$

$\cancel{G_{12} = \int}$ $\quad G_{12} = \langle e_1 | e_2 \rangle = \int_0^1 x \, dx = \frac{1}{2}$

$G_{13} = \langle e_1 | e_3 \rangle = \int_0^1 x^2 \, dx = \frac{1}{3}$

$G_{21} = G_{12} = \frac{1}{2}$

$G_{22} = \int_0^1 x^2 \, dx = \frac{1}{3}$

$G_{23} = \int_0^1 x^3 \, dx = \frac{1}{4}$

$G_{31} = \langle e_3 | e_1 \rangle = G_{13} = \frac{1}{3}$

$G_{32} = G_{23} = \frac{1}{4}$

$G_{33} = \int_0^1 x^4 \, dx = \frac{1}{5}$

$$\therefore \quad G = \begin{pmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{pmatrix} \quad \#$$

$$\langle 1+x+x^2| = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{pmatrix}$$

$$= \begin{pmatrix} 1+1/2+1/3 & \frac{1}{2}+\frac{1}{3}+\frac{1}{4} & \frac{1}{3}+\frac{1}{4}+\frac{1}{5} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{6}{6}+\frac{3}{6}+\frac{2}{6} & \frac{6}{12}+\frac{4}{12}+\frac{3}{12} & \frac{20}{60}+\frac{15}{60}+\frac{12}{60} \end{pmatrix} \qquad \begin{array}{r} 35 \\ 12 \\ \hline 47 \end{array}$$

$$= \begin{pmatrix} \frac{11}{6} & \frac{13}{12} & \frac{47}{60} \end{pmatrix}$$

#

$$|1+x+x^2\rangle = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

#

$$\langle 1+x+x^2 | 1+x+x^2 \rangle = \frac{11}{6} + \frac{13}{12} + \frac{47}{60}$$

$$= \frac{110}{60} + \frac{5\cdot 13}{60} + \frac{47}{60} = \frac{222}{60} = \frac{111}{30} = \frac{37}{10}$$

Check

$$\| 1+x+x^2 \|^2 = \langle 1+x+x^2 | 1+x+x^2 \rangle = \int_0^1 (1+x+x^2)^2 dx$$

$$= \int_0^1 (1+x+x^2)(1+x+x^2)\,dx = \int_0^1 (1+x+x^2+x+x^2+x^3+x^2+x^3+x^4)\,dx$$

$$= \int_0^1 (1 + 2x + 3x^2 + 2x^3 + x^4)\,dx$$

$$= \left(x + x^2 + x^3 + \tfrac{1}{2}x^4 + \tfrac{x^5}{5}\right)\Big|_0^1 = 1 + 1 + 1 + \tfrac{1}{2} + \tfrac{1}{5}$$

$$= 3 + \frac{5+2}{10} = 3 + \frac{7}{10}$$

$$= \frac{37}{10} \checkmark$$ yes

#

(11) $\langle f|g\rangle = \int_{-1}^{1} f(t)g(t)\,dt$

$\{x_1, x_2, x_3, x_4\} = \{1, t, t^2, t^3\}$

$y_1 = 1$ $\quad\quad \langle y_1|y_1\rangle = \int_{-1}^{+1} dt = 2$

$$y_2 = x_2 - \frac{\langle y_1|x_2\rangle}{\langle y_1|y_1\rangle} y_1 = t - \left(\frac{\int_{-1}^{+1}(1)\cdot t\,dt}{\int_{-1}^{+1} 1^2\,dt}\right)(1)$$

$$= t - \frac{t^2/2\Big|_{-1}^{+1}}{t\Big|_{-1}^{+1}}$$

$$= t - \frac{0}{}(1) = t$$

$\langle y_2|y_2\rangle = \int_{-1}^{+1} t^2\,dt = \frac{t^3}{3}\Big|_{-1}^{+1} = \frac{2}{3}$

$$y_3 = x_3 - \frac{\langle y_1|x_3\rangle}{\langle y_1|y_1\rangle} y_1 - \frac{\langle y_2|x_3\rangle}{\langle y_2|y_2\rangle} y_2$$

$$= t^2 - \frac{\int_{-1}^{+1}(1)t^2\,dt}{2} - \frac{\int_{-1}^{+1} t\cdot t^2\,dt}{\left(\frac{2}{3}\right)}(t)$$

$$= t^2 - \frac{1}{2}\left(\frac{2}{3}\right) - \left(\frac{3}{2}\right)\left(\frac{t^4}{4}\Big|_{-1}^{+1}\right)$$

$$= t^2 - \frac{1}{3} - \frac{3}{8}\cdot 0 = t^2 - \frac{1}{3}$$

Then $\langle \gamma_3 | \gamma_3 \rangle = \int_{-1}^{+1} (t^2 - \frac{1}{3})^2 \, dt$

$$= \int_{-1}^{+1} (t^4 - \frac{2}{3}t^2 + \frac{1}{9}) \, dt$$

$$= \frac{t^5}{5}\Big|_{-1}^{+1} - \frac{2}{3}\frac{t^3}{3}\Big|_{-1}^{+1} + \frac{1}{9}t\Big|_{-1}^{+1}$$

$$= \frac{2}{5} - \frac{2}{9} + \frac{1}{9} = \frac{2}{5} - \frac{1}{9} =$$

(12) $\langle f|g\rangle = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} f(t)g(t)e^{-t^2}\,dt$

$Y_1 = X_1 = 1 \qquad \langle y_1|y_1\rangle = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-t^2}\,dt = 1$ ✓

$$Y_2 = X_2 - \frac{\langle y_1|x_2\rangle}{\langle y_1|y_1\rangle} Y_1 = t - \frac{\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} 1(t)e^{-t^2}\,dt}{1}\cdot 1$$

$$= t - \cancelto{0}{\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} t e^{-t^2}\,dt} = t$$ ✓

$$\langle y_2|y_2\rangle = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} t^2 e^{-t^2}\,dt = \frac{1}{\sqrt{\pi}}\frac{1}{2}\sqrt{\pi} = \frac{1}{2}$$ ✓

$$Y_3 = X_3 - \frac{\langle y_1|x_3\rangle}{\langle y_1|y_1\rangle} Y_1 - \frac{\langle y_2|x_3\rangle}{\langle y_2|y_2\rangle} Y_2$$

$$= X_3 - \frac{\langle y_1|x_3\rangle}{1} Y_1 - 2\langle y_2|x_3\rangle Y_2$$

Compute: $\langle y_1|x_3\rangle = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} 1(t^2)e^{-t^2}\,dt = \frac{1}{2}$ ✓

$$\langle y_2 | x_3 \rangle = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} t\,t^2 e^{-t^2}\,dt = 0$$

$$\therefore \quad y_3 = t^2 - \frac{1}{2}$$

$$\langle y_3 | y_3 \rangle = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} (t^2 - 1/2)^2 e^{-t^2}\,dt$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} (t^4 - t^2 + 1/4) e^{-t^2}\,dt$$

$$= \frac{1}{\sqrt{\pi}} \left[\sqrt{\pi}\left(\frac{3}{4}\right) - \sqrt{\pi}\left(\frac{1}{2}\right) + \frac{1}{4}\sqrt{\pi} \right]$$

$$= \frac{3}{4} - \frac{1}{2} + \frac{1}{4} = 2 - \frac{1}{2} = \frac{3}{2}$$

Now

$$y_4 = x_4 - \frac{\langle y_1 | x_4 \rangle}{\langle y_1 | y_1 \rangle} y_1 - \frac{\langle y_2 | x_4 \rangle}{\langle y_2 | y_2 \rangle} y_2 - \frac{\langle y_3 | x_4 \rangle}{\langle y_3 | y_3 \rangle} y_3$$

This requires the following inner products

$$\langle y_1 | x_4 \rangle = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} 1(t^3) e^{-t^2}\,dt = 0 \quad ; \quad \langle y_2 | x_4 \rangle = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} t\,t^3 e^{-t^2}\,dt = \frac{3}{4}$$

$$\langle y_3 | x_4 \rangle = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} (t^2 - \tfrac{1}{2}) t^3 e^{-t^2}\,dt = 0$$

In summary

$$\therefore \quad y_1 = 1$$

$$y_2 = t$$

$$y_3 = t^2 - \frac{1}{2}$$

$$y_4 = t^3 - \frac{3}{4}(t) = t^3 - \frac{3t}{4}$$

(4) $d_1 = \frac{(1\ 1\ 1)^T}{\sqrt{3}}$ $\quad d_2 = \frac{(-2\ 1\ 1)^T}{\sqrt{6}}$ $\quad d_3 = \frac{(0\ 1\ -1)^T}{\sqrt{2}}$

let $L \equiv \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

Then $A_{ij} = \langle d_i | L d_j \rangle$

& $L = \sum_{i,j=1}^{3} A_{ij} | d_i \rangle\langle d_j |$

So compute A_{ij} $i,j = \{1,2,3\}$

$$|Ld_1\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix} 6 \\ 15 \\ 24 \end{pmatrix}$$

$$|Ld_2\rangle = \frac{1}{\sqrt{6}}\begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} \quad \& \quad |Ld_3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

Thus $A_{11} = \langle d_1 | L d_1 \rangle = \frac{1}{3}(1\ 1\ 1)\begin{pmatrix} 6 \\ 15 \\ 24 \end{pmatrix} = 15$

$A_{12} = \langle d_1 | L d_2 \rangle = \frac{1}{\sqrt{18}} \cdot 9 = \frac{3}{\sqrt{2}}$

$A_{13} = \langle d_1 | L d_3 \rangle = \frac{1}{\sqrt{6}}(-3) = -\frac{3}{\sqrt{6}}$

$A_{21} = \langle d_2 | L d_1 \rangle = \frac{1}{\sqrt{18}}(27) = \frac{9}{\sqrt{2}}$

$$A_{22} = \langle d_2 | L d_2 \rangle = \frac{1}{6} \cdot 0 = 0$$

$$A_{23} = \langle d_2 | L d_3 \rangle = \frac{1}{\sqrt{12}} \cdot 0 = 0$$

$$A_{31} = \langle d_3 | L d_1 \rangle = \frac{1}{\sqrt{6}}(-9) = -\frac{9}{\sqrt{6}}$$

$$A_{32} = \langle d_3 | L d_2 \rangle = \frac{1}{\sqrt{12}}(0) = 0$$

$$A_{33} = \langle d_3 | L d_3 \rangle = 0$$

Now

$$|d_1\rangle\langle d_1| = \frac{1}{3}\begin{pmatrix}1\\1\\1\end{pmatrix}(1\ 1\ 1) = \frac{1}{3}\begin{pmatrix}1 & 1 & 1\\1 & 1 & 1\\1 & 1 & 1\end{pmatrix}$$

$$|d_1\rangle\langle d_2| = \frac{1}{\sqrt{18}}\begin{pmatrix}1\\1\\1\end{pmatrix}(-2\ 1\ 1) = \frac{1}{\sqrt{18}}\begin{pmatrix}-2 & 1 & 1\\-2 & 1 & 1\\-2 & 1 & 1\end{pmatrix}$$

etc.

But to answer the problem we have

$$\begin{pmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{pmatrix} = 15|d_1\rangle\langle d_1| + \frac{3}{\sqrt{2}}|d_1\rangle\langle d_2| + \frac{-3}{\sqrt{6}}|d_1\rangle\langle d_3|$$

$$+ \frac{9}{\sqrt{2}}|d_2\rangle\langle d_1| + 0\cdot|d_2\rangle\langle d_2| + 0|d_2\rangle\langle d_3|$$

$$+ \frac{-9}{\sqrt{6}}|d_3\rangle\langle d_1| + 0\cdot|d_3\rangle\langle d_2| + 0\cdot|d_3\rangle\langle d_3|$$

which is correct

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(2) Note: $(1,1,1,1)^T$ + $(1,2,3,4)^T$ are not orthogonal.

We 1st perform gram schmidt orthogonalization

$$y_1 = (1,1,1,1)^T$$

$$y_2 = (1,2,3,4)^T - \frac{\langle y_1 | x_2 \rangle}{\langle y_1 | y_1 \rangle} y_1$$

$$\langle y_1 | x_2 \rangle = (1\ 1\ 1\ 1)^T \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = 1+2+3+4 = 10$$

$$\langle y_1 | y_1 \rangle = 4$$

So $$y_2 = (1\ 2\ 3\ 4)^T - \frac{10}{4}(1\ 1\ 1\ 1)^T$$

$$= \left(1-\frac{5}{2}\ \ 2-\frac{5}{2}\ \ 3-\frac{5}{2}\ \ 4-\frac{5}{2}\right)^T$$

$$= \left(-\frac{3}{2}\ \ \frac{-1}{2}\ \ \frac{1}{2}\ \ \frac{3}{2}\right)^T = \frac{1}{2}(-3\ -1\ 1\ 3)^T$$

I can drop the normalization of y_2 : let $y_2 = (-3\ -1\ 1\ 3)^T$

Then

$$W = \text{span}\{w_1, w_2\} = \text{span}\{y_1, y_2\}$$

$$\text{ + } P_W = \sum_{i=1}^{2} P_{y_i} = \sum_{i=1}^{2} \frac{|y_i\rangle\langle y_i|}{\langle y_i | y_i \rangle}$$

$$|y_1\rangle\langle y_1| = \begin{pmatrix}1\\1\\1\\1\end{pmatrix}(1\ 1\ 1\ 1) = \begin{pmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{pmatrix}$$

$$|y_2\rangle\langle y_2| = \begin{pmatrix}-3\\-1\\1\\3\end{pmatrix}(-3\ -1\ 1\ 3) = \begin{pmatrix}9&3&-3&-9\\3&1&-1&-3\\-3&-1&1&3\\-9&-3&3&9\end{pmatrix}$$

$+\ \langle y_1|y_1\rangle = 4 \quad + \quad \langle y_2|y_2\rangle = 20$

So

$$P_W = \frac{|y_1\rangle\langle y_1|}{4} + \frac{|y_2\rangle\langle y_2|}{20} = \frac{1}{10}\begin{pmatrix}7&4&1&-2\\4&3&2&1\\1&2&3&4\\-2&1&4&7\end{pmatrix}$$

#

(1) A Basis for the orthogonal complement $W^\perp$ is

Since $(I - P_W)v \in W^\perp$ apply $I - P_W$ to two randomly chosen vectors

$$(I - P_W)\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \frac{1}{10}\begin{pmatrix}3\\-4\\-1\\2\end{pmatrix}$$

$$(I - P_W)\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \frac{1}{10}\begin{pmatrix}-4\\7\\-2\\-1\end{pmatrix}$$

#

Since these two vectors are not linearly dependent they can be used in the gram schmidt process to produce an ~~basis~~ orthonormal basis if required. As given they are a basis for $W^{\perp}$.

Write $x = (1\ 1\ -1\ -1)^T$

$$x = P_W x + x_{W^{\perp}}$$

$$\Rightarrow \quad x_{W^{\perp}} = x - P_W x = \frac{1}{10}\begin{pmatrix} -2 \\ 6 \\ -6 \\ 2 \end{pmatrix} \in W^{\perp}$$

$$\Rightarrow \quad P_W x = \frac{1}{10}\begin{pmatrix} 12 \\ 4 \\ -4 \\ -12 \end{pmatrix} \in W$$

Thus

$$x = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 12 \\ 4 \\ -4 \\ -12 \end{pmatrix} + \frac{1}{10}\begin{pmatrix} \cancel{2} & -2 \\ \cancel{4} & 6 \\ & -6 \\ & 2 \end{pmatrix}$$

#

To evaluate several least squares solutions when

$XS = [()() \cdots ()]$ columns of data to fit x with

$yS = [()() \cdots ()]$ each column we desire a coefficient

In Matlab

```
xSSq = xS.^2          xyS = xS.*yS
ySum = sum(yS,1)
xSum = sum(xS,1)
xSqSum = sum(xS.^2,1)
```

$$\hat{f}(k,t) = \exp(-Dk^2t)\,\hat{f}(k,0)$$

$$f(x,t) = G_t * f_0(x)$$

w/ G_t inverse F.T. of $\exp(-Dk^2t)$ w.r.t. k

$$e^{-k^2/2} \xrightarrow{\text{I.F.T.}} e^{-x^2/2}$$

$$\mathcal{F}\{f(cx)\} = \frac{1}{c}\hat{f}(k/c)$$

$$\frac{1}{c}\hat{f}(k/c) \longrightarrow f(x)$$

$$\mathcal{F}^{-1}\{\hat{f}(ck)\} = \frac{1}{c}f(x/c)$$

$$\exp\left(-(2Dt)\frac{k^2}{2}\right) = \exp\left(-k^2/2\,\frac{1}{\left(\frac{1}{2Dt}\right)}\right) \xrightarrow{\text{I.F.T.}} c\cdot e^{-x^2/2}$$

$$c = \frac{1}{2Dt}$$

$$\cancel{\exp\left(-\left(\sqrt{Dt}\,k\right)^2\right)} \xrightarrow{\cancel{\text{I.F.T.}}} \cancel{1}$$

$$\exp\left(-\frac{\left(\sqrt{Dt}\,k\sqrt{2}\right)^2}{2}\right) \xrightarrow{\text{I.F.T.}} \frac{1}{\sqrt{2Dt}}\exp\left(-\frac{1}{2}\frac{x^2}{2Dt}\right)$$

$$= \frac{\exp(-x^2/4Dt)}{\sqrt{2Dt}} \cdot \cancel{\frac{\sqrt{4}}{\sqrt{2\pi}}}$$

w/ this definition of F.T.

$$\mathcal{F}^{-1}\{\sqrt{2\pi}\,\hat{f}(k)\hat{g}(k)\} = f * g(x)$$

So $\mathcal{F}^{-1}\{\hat{f}(k)\hat{g}(k)\} = \frac{1}{\sqrt{2\pi}} f * g(x)$

$$\mathcal{F}^{-1}\{\exp(-Dk^2t)\hat{f}(k,0)\} = \frac{1}{\sqrt{2\pi}} \cdot \frac{\exp(-x^2/4Dt)}{\sqrt{2Dt}} \cdot f_0(x)$$

✓

$$G_t * f_0(x) = \int_{-\infty}^{\infty} G_t(x-y) f_0(y)\, dx =$$

$x - 1 \cong 2\sqrt{t}$ $\qquad x \gg 1 + 2\sqrt{t}$

or