The Contest Problem Books: Annual High School Mathematics Contests

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To the future problem solvers.

Introduction

I've been trying to work through all of these exams. As I do that I've been writing up many of my solutions. If anyone finds errors (typo's) please let me know and I'll try to correct them. I've also collected a set of "useful facts" at the beginning of these solutions that make solving some of the problems much easier. A serious student of these type of tests should memorize these formulas.

A Catalog of Useful Facts

Facts from Geometry

Regular Polygons

The area A of a polygon with n sides, side length s, a circumradius r, an in-radius a (apothem), and a perimeter $p = ns$ can be computed with any of the following formulas

$$\begin{aligned} A &= \frac{s^2 n}{4\tan\left(\frac{\pi}{n}\right)} && (1) \\ &= \frac{1}{2} r^2 n \sin\left(\frac{2\pi}{n}\right) && (2) \\ &= a^2 n \tan\left(\frac{\pi}{n}\right) && (3) \\ &= \frac{1}{2} pa \,. && (4) \end{aligned}$$

The **interior angles** of a polygon is then angle formed between each adjacent sides. A *regular* polygon has all equal sides and all equal interior angles. The *sum* of the interior angles of any polygon adds to a constant that depends on the number of sides of the polygon. Namely if we let this polygon have n sides and the constant sum by S (in degrees) then we have

$$S = 180(n-2)\,. \tag{5}$$

To help in remembering this formula we note that it duplicates the more familiar results for a triangle and a quadrilateral. Taking $n = 3$ (a triangle) we have $S = 180$ and taking $n = 4$ we have $S = 360$, both of which are the expected results.

If a polygon is regular (it has equal length sides and equal interior angles) then each interior angles α is equal to this sum S divided by n or

$$\alpha = \frac{180(n-2)}{n}\,. \tag{6}$$

Note that considered as a function of the number of sides n the interior angle α increases as the number of sides increases. The limiting value of α is 180.

Parallelograms

Parallelograms have a number of properties. Ones that can be helpful for working contest problems include

- Each diagonal divides the quadrilateral into two congruent triangles.

- The sum of the x coordinates of the two points diagonally opposite each other equals the sum of the x coordinates of the complementary diagonal pair. The same property holds for the sum of the y coordinates.

Rhomboids

A rhombus is a special parallelogram with four equal sides (i.e. it's a regular parallelogram). A rhombus has diagonals that intersect at right angles to each other. The area A of a rhombus with a base b, and height a (altitude), diagonals of length d_1 and d_2, side length s, and interior angle a can be given by any of the following expressions

$$\begin{aligned} A &= ab && (7)\\ &= \frac{1}{2}d_1 d_2 && (8)\\ &= s^2 \sin(a)\,. && (9) \end{aligned}$$

Chords of a Circle

A chord of a circle is a geometric line segment whose endpoints both lie on the circumference of the circle. Some properties of a chord of a circle include

- A chord's perpendicular bisector passes through the center of the circle.

Triangles

Heron's Formula gives the area of a triangle in terms of its three sides a, b, and c as

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}\,, \tag{10}$$

with s the "semiperimeter" or half the triangles perimeter given by

$$s = \frac{1}{2}(a+b+c)\,. \tag{11}$$

Given the Cartesian coordinates of the three points A, B, and C of a triangle we can compute the area of it by evaluating the *absolute value* of the following determinant

$$\text{Area} = \frac{1}{2}\begin{vmatrix} A_x & A_y & 1\\ B_x & B_y & 1\\ C_x & C_y & 1 \end{vmatrix}\,. \tag{12}$$

The **Angle Bisector Theorem** is concerned with the relative lengths of the two segments that a triangles side is divided by a line that bisects the opposite angle. It equates their relative lengths to the relative lengths of the other two sides of the triangle.

Consider $\triangle ABC$ and let the angle bisector of A intersect side BC at a point D, then the angle bisector theorem is the statement that

$$\frac{BD}{CD} = \frac{AB}{AC}\,.$$

The medians of a triangle intersect at a point called the **centroid**[1]. This point divides each median into two parts in the ratio of 1:2.

The circle that passes though all three vertices of a triangle is known as the **circumcircle**. The center of this circle is known as the circumcenter and can be a point inside or outside of the triangle. The location of the circumcenter is where all three of the perpendicular bisectors of all three sides meet. For equilateral triangles with a side length s the radius of the circumcircle is given by

$$\frac{s}{\sqrt{3}}\,. \tag{13}$$

If the triangle is a general triangle with side lengths a, b, and c then the radius of the circumcircle is given by

$$\frac{abc}{4K}\,, \tag{14}$$

with K the area of the triangle given by Heron's formula Equation 10. In the case of a right triangle, the hypotenuse is a diameter of the circumcircle, and its center is exactly at the midpoint of the hypotenuse. If you know an angle (say A) and the length of the opposite side (say a) then the radius of the circumcircle is given by

$$\frac{a}{2\sin(A)}\,. \tag{15}$$

The **incircle** or inscribed circle of a triangle is the largest circle contained in the triangle; it touches (is tangent to) the three sides. The center of the incircle is called the triangle's incenter. The radius of the incircle can be written as

$$r = \frac{K}{s}\,, \tag{16}$$

where K is the triangles area and s is the semiperimeter given by Equation 11.

If we extend segments from A, B, and C to the opposite sides of their triangle all throught the same point such that $BC = a + b$, $AC = c + d$, and $AB = e + f$ then **Ceva's** Theorem is

$$\frac{a}{b} \cdot \frac{c}{d} \cdot \frac{e}{f} = 1\,.$$

Arithmetic Identities

An **arithmetic sequence** (or **arithmetic progression**) is defined to have its terms a_n given by the formula

$$a_n = a_1 + (n-1)d \quad \text{for} \quad n \geq 1\,. \tag{17}$$

[1] http://en.wikipedia.org/wiki/Median_(geometry)

The sum of the first N of these terms is given by

$$S_N \equiv \sum_{n=1}^{N} a_n = \frac{1}{2} N(a_1 + a_N) = \frac{N}{2}(2a_1 + (N-1)d)\,. \tag{18}$$

A geometric sequence is defined to have terms a_n given by

$$a_n = a_1 d^{n-1} \quad \text{for} \quad n \geq 1\,. \tag{19}$$

The sum of the first N of these terms is given by

$$S_N \equiv \sum_{n=1}^{N} a_n = a_1 \frac{1 - d^N}{1 - d}\,. \tag{20}$$

If we sum starting from a different index, say m, then we get

$$\sum_{n=m}^{N} a_n = a_1 \sum_{n=m}^{N} d^{n-1} = a_1 \left(\frac{d^{m-1} - d^N}{1 - d} \right)\,.$$

Another important sum that we often need is given by

$$\sum_{n=1}^{N} n d^{n-1} = \frac{1 - d^{N+1}}{(1-d)^2} - \frac{(N+1)d^N}{1-d}\,. \tag{21}$$

Expressions for Finite Sums

The following sums can be helpful at times

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \tag{22}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \tag{23}$$

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2} \right)^2\,. \tag{24}$$

Divisibility Facts

For Nine (9): The number nine will divide a number if and only if it divides the sum of that numbers digits. The remainder of this division is the same at the remainder when we divide the digit sum by nine. For example, the sum of the digits in the number 123 is 6. This is also the remainder of 123 when divided by nine. As another example, the sum of the digits in the number 783 is 18. As 18 is divisible by nine so is the original number 783.

For Eleven (11): To determine divisibility of a number by the number 11 we sum the digits but with alternating signs. We start positive, then negative, then positive until we have exhausted the digits in our number. If the remainder when we divide the summed digits by 11 is zero then we can divide the original number by 11 (see [3]). As an example consider the number 123. We sum

$$1 - 2 + 3 = 2\,,$$

and this number is not divisible by 11. As another example consider the larger number 968. When we perform the required digit sum we get

$$9 - 6 + 8 = 11\,,$$

which is divisible by 11. Thus the original number is divisible by 11.

The 1950 Examination

Problem 1

The sum of these three numbers is 12. Based on that, the parts of 64 are then

$$\frac{2}{12}64\,,\,\frac{4}{12}64\,,\,\frac{6}{12}64\,.$$

The smallest of these is

$$\frac{2}{12}(64) = \frac{1}{6}(64) = \frac{32}{3} = 10\frac{2}{3}\,.$$

Problem 2

From the relationship and what we are told we have

$$16 = 8g - 4 \quad \text{so} \quad g = \frac{5}{2}\,.$$

Thus

$$R = \frac{5}{2}S - 4\,.$$

If $S = 10$ we compute that $R = 5(5) - 4 = 21$.

Problem 3

Write this as

$$(x - r_1)(x - r_2) = x^2 - 2x + \frac{5}{2} = 0\,.$$

Then $-(r_1 + r_2) = -2$.

Problem 4

Call this expression E. Then we have

$$\begin{aligned} E &= \frac{(a^2 - b^2)(ab - a^2) - ab(ab - b^2)}{ab(ab - a^2)} \\ &= \frac{a^3b - a^4 - ab^3 + a^2b^2 - a^2b^2 + ab^3}{ab(ab - a^2)} \\ &= \frac{a^3(b - a)}{a^2b(b - a)} = \frac{a}{b}\,. \end{aligned}$$

Problem 5

The terms of a geometric sequence are given by $a_n = a_0 r^n$ for $n \geq 0$. For there to be *five* geometric means between the two given numbers means that

$$\begin{aligned} a_0 &= 8 \\ a_1 &= a_0 r \\ a_2 &= a_0 r^2 \\ a_3 &= a_0 r^3 \\ a_4 &= a_0 r^4 \\ a_5 &= a_0 r^5 \\ a_6 &= a_0 r^6 = 8r^6 = 5862\,. \end{aligned}$$

Solving this last equation for r we find $r = 3$. This means that the fifth term is $a_4 = 648$.

Problem 6

From the second equation we have

$$x = -\frac{1}{2}(y+3)\,.$$

If we put this into the first equation we would have

$$2\left(\frac{1}{4}\right)(y+3)^2 - 3(y+3) + 5y + 1 = 0\,.$$

If we simplify this we get

$$y^2 + 10y - 7 = 0\,.$$

Problem 7

This would be the number

$$(tu1) = 100t + 10u + 1\,.$$

Problem 8

The new radius is $r' = 2r$. The new area is then

$$A' = \pi r'^2 = \pi(4r^2) = 4\pi r^2 = 4A\,,$$

which is a change

$$\frac{A' - A}{A} = \frac{3A}{A} = 3\,,$$

or 300%.

Problem 9

This would be the triangle with the largest altitude so it must have its vertex at the “top” of the circle. This means that its base is $2r$ and its height is r for an area of

$$\frac{1}{2}(2r)r = r^2 .$$

Problem 10

Call this expression E. Then we have

$$E = \frac{(\sqrt{3}-\sqrt{2})(\sqrt{3}+\sqrt{2})}{\sqrt{3}(\sqrt{3}+\sqrt{2})} = \frac{1}{\sqrt{3}(\sqrt{3}+\sqrt{2})} = \frac{1}{3+\sqrt{6}} .$$

Problem 11

We write C as

$$C = \frac{e}{r}\left(\frac{rn}{R+nr}\right) = \frac{e}{r}\left(\frac{R+rn-R}{R+nr}\right) = \frac{e}{r}\left(1-\frac{R}{R+nr}\right) .$$

From this if n increases then $\frac{1}{R+nr}$ decreases so C increases. This is assuming that $R > 0$ and $r > 0$.

Problem 12

Consider a regular polygon. Then the sum of the interior angles is given by

$$180(n-2) ,$$

so the angle measure of one interior angle is then

$$\frac{180(n-2)}{n} .$$

The measure of an exterior angle is then

$$180 - \frac{180(n-2)}{n} = \frac{360}{n} .$$

As we have n of these when we sum all n of them we get 360 a constant.

Problem 13

If we factor the quadratic we see that we have

$$(x-1)(x-2)x(x-4)=0\,.$$

Thus the roots are $x \in \{0,1,2,4\}$.

Problem 14

If we multiply the first equation by two we get

$$4x-6y=16\,.$$

If we add this to the second equation we get

$$0=16+9=25\,.$$

This is a contradiction and there can be no solutions to this system of equations.

Problem 15

The factors of this are

$$(x+2i)(x-2i)=0\,,$$

none of which are real.

Problem 16

Write this expression as

$$((a^2-9b^2)^2)^2=(a^2-9b^2)^4\,.$$

The binomial expansion then tells us that this has $4+1=5$ terms.

Problem 17

If we evaluate these expressions for some of the given x we see that (A) is not correct for $x=2$, (B) is not correct for $x=1$, (C) is correct for all x, and (D) is not correct for $x=0$.

Problem 18

Only (1) is true in general.

Problem 19

One man working for one day does $\frac{1}{md}$ amount of work (so that md man-days gets the job done). If we have $m+r$ men and they work for D days they will do

$$(m+r)D\frac{1}{md}\,,$$

amount of work. As we want the amount of work equal to one job we have

$$(m+r)D\frac{1}{md}=1 \quad \text{so} \quad D=\frac{md}{m+r}\,.$$

Problem 20

If we recall that $x-1$ is a factor of $x^{13}-1$ as

$$(x-1)(x^{12}+x^{11}+x^{10}+x^9+\cdots+x+1)=x^{13}-1\,,$$

so that

$$\frac{x^{13}-1}{x-1}=x^{12}+x^{11}+x^{10}+x^9+\cdots+x+1\,.$$

Then we have that

$$\frac{x^{13}-1+2-2}{x-1}=x^{12}+x^{11}+x^{10}+x^9+\cdots+x+1\,,$$

or

$$\frac{x^{13}+1}{x-1}-\frac{2}{x-1}=x^{12}+x^{11}+x^{10}+x^9+\cdots+x+1\,.$$

From this expression by moving the second term on the left to the right-hand-side we see that two is the remainder of the given division.

Problem 21

If we let this rectangular box have base dimensions $w\times d$ (width by depth) and a height of h. Then the three areas given are

$$\begin{aligned} S &= hd = 12 \\ F &= hw = 8 \\ B &= wd = 6\,. \end{aligned}$$

Taking the product of these three areas we get

$$w^2d^2h^2=576\,.$$

Thus the volume is $wdh=24$.

Problem 22

If the value of the item starts at V_0 then the first discount gives its value of $0.8V_0$. The second discount gives its value at $0.9 \times 0.8V_0 = 0.72V_0$ for a net

$$1 - 0.72 = 0.28\,,$$

i.e. a 28% discount.

Problem 23

Let M be the monthly rent. Then in one year his income from rent set equal to 5.5% of his initial investment is

$$12M - 12(0.125)M - 325 = 0.055(10000)\,.$$

Solving this for M gives $M = 83.33333$.

Problem 24

Write this expression as

$$(x-2) + \sqrt{x-2} = 2\,.$$

Then if we let $v = \sqrt{x-2}$ this is

$$v^2 + v - 2 = 0\,,$$

or

$$(v+2)(v-1) = 0\,.$$

Thus $v = -2$ or $v = 1$. As $v \geq 0$ the first solution is spurious. The second solution is

$$\sqrt{x-2} = 1 \quad \text{or} \quad x - 2 = 1\,,$$

or $x = 3$. There is one real solution.

Problem 25

Call this expression E. Then we have

$$E = \log_5(125) + \log_5(625) - \log_5(25) = 3 + 4 - 2 = 5\,.$$

Problem 26

Write this as

$$\log_{10}(mn) = b\,,$$

or

$$mn = 10^b \quad \text{so} \quad m = \frac{10^b}{n}\,.$$

Problem 27

The cars average speed is the total distance traveled divided by the total time. Here that is

$$\frac{D}{T} = \frac{120 + 120}{\frac{120}{30} + \frac{120}{40}} = \frac{240}{7} = 34.28571\,.$$

Problem 28

At the time (T hours from the start) they meet B has traveled $60 + 12 = 72$ miles and A has traveled $60 - 12 = 48$ miles thus we have

$$T = \frac{72}{v_B} = \frac{48}{v_A}\,.$$

In addition we know that $v_A = v_B - 4$. Solving for v_B and putting that in the above we get a single equation for v_A. Solving that we get $v_A = 8$.

Problem 29

The "rate" of the first machine in envelopes per minute is given by

$$r_1 = \frac{500}{8}\,.$$

We don't know the rate of the second machine r_2 but we know that in two minutes we want to have 500 envelopes so

$$2(r_1 + r_2) = 500\,. \tag{25}$$

If x is the time for the second machine to address 500 envelopes alone then

$$xr_2 = 500\,.$$

Solving that for r_2 in terms of x and putting it into Equation 25 we get

$$2\left(\frac{500}{8} + \frac{500}{x}\right) = 500\,.$$

If we divide by 1000 we get the answer (B).

Problem 30

Let the number of boys and girls initially be given by B and G respectively. Then after fifteen girls leave we are left with B boys and $G - 15$ girls. We are told at this stage that

$$G - 15 = \frac{B}{2}.$$

Next after 45 boys leave we are left with $B - 45$ boys and $G - 15$ girls. We are told at this stage that

$$B - 45 = \frac{G - 15}{5}.$$

If we solve these two equations for G we get $G = 40$.

Problem 31

Originally we order 4 black pairs and b blue pairs. If k is the price for a black pair of socks then this original order should cost

$$4k + b(2k) = 4k + 2bk.$$

When the order has the two pair colors exchanged it would then cost

$$bk + 4(2k) = bk + 8k.$$

As we are told that this is 50% more than the original order we know that

$$bk + 8k = 1.5(4k + 2bk).$$

If we divide by k we can solve for b to find $b = 1$. Thus the ratio of blue to black is $b : 4 = 1 : 4$.

Problem 32

If we draw this figure the latter becomes the hypotenuse of a right triangle with a base leg of seven and a height leg of h so that using the Pythagorean theorem we have

$$h^2 + 7^2 = 25^2 \quad \text{so} \quad h = 24.$$

If the top of the ladder now goes to $h - 4 = 20$ feet above the ground then new base leg length (denoted l') is

$$l'^2 = 25^2 - 20^2 = 225 \quad \text{so} \quad l' = 15.$$

Thus the change is $15 - 7 = 8$.

Problem 33

The six inch diameter pipe will "carry" an amount of water proportional to

$$\pi 3^2 = 9\pi\,.$$

Each one inch diameter pipe will "carry" an amount of water proportional to

$$\pi \left(\frac{1}{2}\right)^2 = \frac{\pi}{4}\,.$$

We want to know n such that

$$\frac{\pi n}{4} = 9\pi\,.$$

Solving we get $n = 36$.

Problem 34

The circumference of a circle is given by $C = 2\pi r$. The changes of the circumference of a circle are related to the changes of the circle radius by

$$\Delta C = C_1 - C_0 = 25 - 20 = 5 = 2\pi(r_1 - r_0) = 2\pi\Delta r\,.$$

Thus we find

$$\Delta r = \frac{5}{2\pi}\,.$$

Problem 35

As we are given the three side lengths we can use Heron's formula to compute the triangles area. We find

$$\begin{aligned} s &= \frac{1}{2}(24 + 10 + 26) = 30 \\ A &= \sqrt{s(s - AB)(s - BC)(s - AC)} = 120\,, \end{aligned}$$

when we simplify.

Let the triangle have vertices A, B, and C with AB on the x-axis and C above the x-axis. Let the tangent points across the triangle from a vertex be denoted with a "tick". Thus A' is the tangent point of the circle on the side BC across from the vertex A. Let the point O be the center of the incircle.

Now if we connect each vertex of the triangle to the center of the inscribed circle we introduce six triangles that are equal in pairs and all have the same height r the radius of the incircle.

These triangles are

$$\triangle AC'O \cong \triangle AB'O$$
$$\triangle BC'O \cong \triangle BA'O$$
$$\triangle CB'O \cong \triangle CA'O\,.$$

We can thus write the area of the full triangle $\triangle ABC$ as the sum of these smaller triangles as

$$A = 2\left(\frac{1}{2}AC'r\right) + 2\left(\frac{1}{2}BA'r\right) + 2\left(\frac{1}{2}CB'r\right)\,.$$

This is equal to

$$A = r(AC' + BA' + CB')\,.$$

Next by the two-tangent theorem we have that

$$AC' = AB'$$
$$BC' = BA'$$
$$CB' = CA'\,.$$

Using this we can show that

$$60 = AB + BC + CA = AC' + C'B + BA' + A'C + CB' + B'A = 2AC' + 2BA' + 2CB'\,.$$

Thus we have

$$AC' + BA' + CB' = 30\,.$$

Thus

$$A = 30r\,.$$

Setting this to the area from Heron's formula we find $r = 4$.

Problem 36

Let L be the list price. The the merchant buys the goods for B or

$$B = (1 - 0.25)L = 0.75L\,. \tag{26}$$

Let M be the marked price and S the sale price. Then we want to find S and M such that

$$S = (1 - 0.2)M = 0.8M\,, \tag{27}$$

i.e. the sale price is 20% of the marked price and so that

$$S - B = 0.25S\,,$$

i.e. the sale price represents a 25% profit from the "buy" price. The above is equivalent to

$$0.75S = B\,. \tag{28}$$

To solve this problem we want to write the mark price M in terms of the list price L.

Starting with Equation 26 and replacing B using Equation 28 and then using Equation 27 to replace S with M we get

$$0.75(0.8M) = 0.75L\,.$$

This means that

$$M = \frac{1}{0.8}L = \frac{10}{8}L = \frac{5}{4}L = 1.25L\,.$$

Thus the mark price is 125% of the list price.

Problem 37

As *all* statements are true thus (E) is the only incorrect statement.

Problem 38

From the problem we have that

$$\left|\begin{array}{cc} 2x & 1 \\ x & x \end{array}\right| = 2x^2 - x = 3\,,$$

or

$$2x^2 - x - 3 = 0\,.$$

We can solve this with the quadratic equation and find

$$x = \frac{1 \pm \sqrt{1 - 4(2)(-3)}}{2(2)} = \frac{1 \pm 5}{4}\,.$$

Taking the negative and the positive sign we find the *two* roots of

$$\left\{-1\,, \frac{3}{2}\right\}\,.$$

Problem 39

Forming the given sum we have

$$2 + 1 + \frac{1}{2} + \frac{1}{4} + \cdots = 2 + \sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k = 2 + \frac{1}{1 - \frac{1}{2}} = 4\,.$$

Thus (1) and (2) are not true. For (3) the sum above has the term 2 which is not "close" to zero. Both (4) and (5) are true.

Problem 40

Write the given expression as

$$\frac{(x+1)(x-1)}{x-1} = x+1\,.$$

Thus as $x \to 1$ this expression goes towards two.

Problem 41

Write this quadratic as

$$\begin{aligned} ax^2 + bx + c &= a\left(x^2 + \frac{b}{a}x\right) + c \\ &= a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2\right) - a\left(\frac{b}{2a}\right)^2 + c \\ &= a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}\,. \end{aligned}$$

The smallest value of this expression is when the quadratic is zero and takes the value

$$c - \frac{b^2}{4a} = \frac{4ac - b^2}{4a}\,.$$

Problem 42

Let v be the value of the left-hand-side of the given expression. Then as this expression continues forever our expression is

$$v = x^v = 2\,.$$

Taking the logarithm of this is

$$v\log(x) = \log(2)\,.$$

But we know that $v = 2$ and thus we have

$$2\log(x) = \log(2)\,,$$

or

$$\log(x) = \frac{1}{2}\log(2) = \log\left(2^{1/2}\right)\,.$$

This means that $x = \sqrt{2}$.

Problem 43

We can write this sum S as

$$\begin{aligned} S &= \sum_{k=0}^{\infty} \left(\frac{1}{7}\right)^{2k+1} + 2\sum_{k=1}^{\infty} \left(\frac{1}{7}\right)^{2k} \\ &= \frac{1}{7}\left(\frac{1}{1-\left(\frac{1}{7}\right)^2}\right) + 2\left[\left(\frac{1}{1-\left(\frac{1}{7}\right)^2}\right) - 1\right] = \frac{3}{16}\,, \end{aligned}$$

when we simplify.

Problem 44

If we remember the curve $y = \log(x)$ you will note that it cuts the x-axis at the point $x = 1$ where $y = 0$.

Problem 45

We can select two of the 100 vertices in

$$\binom{100}{2} = \frac{100!}{2!98!} = 4950\,,$$

ways. A diagonal must not include any of the 100 sides and thus the number of diagonals is

$$4950 - 100 = 4850\,.$$

Problem 46

If we double the specific sides of the triangle as suggested we would get a triangle with sides of length

$$24\,, 14\,, 10\,.$$

Now if this is to be a valid triangle it must satisfy the triangle inequality. This means that

$$2AC + BC = 24\,,$$

must be larger than $2AB = 24$ which it is not. As the new triangle does not satisfy the triangle inequality its area must be zero.

Problem 47

Once you have drawn the rectangle sharing a base with the triangle one can recognize that the "top" triangle is similar to the full triangle. As the height of the "top" triangle is $h - x$ and its base is $2x$ using this symmetry we have

$$\frac{h-x}{h} = \frac{2x}{b}.$$

Solving for x we get

$$x = \frac{bh}{2h+b}.$$

Problem 48

Draw this triangle and connect the internal point to the three corners of the triangle. This produces three internal triangles. Draw the three perpendiculars from the internal point to the three sides of the triangle. Call these perpendiculars h_1, h_2, and h_3 as they are heights of the three internal triangles formed. Let the side of the equilateral triangle be s.

The area of the original triangle in terms of the three internal triangles (and their heights is given by)

$$A = \frac{1}{2}sh_1 + \frac{1}{2}sh_2 + \frac{1}{2}sh_3 = \frac{s}{2}(h_1 + h_2 + h_3).$$

Using trigonometry the area of the equilateral triangle is given by

$$A = \frac{1}{2} \times s \times (s\sin(60^\circ)) = \frac{s^2\sqrt{3}}{4}.$$

Setting these two equal to each other we get

$$h_1 + h_2 + h_3 = \frac{s\sqrt{3}}{2}.$$

Note that the right-hand-side is the altitude of our equilateral triangle.

Problem 49

Let the points $A = (0,0)$, $B = (2,0)$ and $C = (x,y)$. Then we can compute that that the median from A to BC is located at

$$M = \left(\frac{x+2}{2}, \frac{y}{2}\right).$$

Expressing the fact that M is a distance of 1.5 from A we have

$$\left(\frac{x+2}{2} - 0\right)^2 + \left(\frac{y}{2} - 0\right)^2 = \frac{9}{4}.$$

Simplifying this we get

$$(x + 2)^2 + y^2 = 9\,,$$

which is a circle centered at $(-2, 0)$ of radius three.

Problem 50

The speed of the privateer (with the wind) is 11 mph while the speed of the merchantman is 8 mph. That means that in one hour the privateer is $11 - 8 = 3$ miles closer to the merchantman. In two hours the privateer is 6 miles closer to the merchantman and is thus $10 - 6 = 4$ miles away from him. At this point it is 1:45 PM.

If the two ships meet t hours later then the merchantman has traveled

$$8t = D\,.$$

The problem statement "17 miles while the merchantman makes 15" I think means that the privateer's velocity is now $\frac{17}{15}$ of the velocity of the merchantman's. Thus the privateer will have traveled

$$\frac{17}{15} \times 8t\,.$$

As the privateer has reached the merchantman the above must equal $D + 4$ since the merchantman is four miles from the privateer. Thus we have

$$\frac{17}{15} \times 8t = D + 4 = 8t + 4\,.$$

Solving for t we get $t = \frac{15}{4} = 3.75$ hours. Adding this to 1:30 PM we find that the time of overtaking is then 5:30 PM.

The 1951 Examination

Problem 1

This would be

$$\frac{100(M-N)}{N}\,.$$

Problem 2

Let the length of the field be l then the width is $\frac{l}{2}$. We are told that the perimeter of the field is x or

$$2l + 2w = 2l + l = 3l = x \quad \text{so} \quad l = \frac{x}{3}\,.$$

Then $w = \frac{x}{6}$. The area is then

$$\frac{x}{3} \cdot \frac{x}{6} = \frac{x^2}{18}\,.$$

Problem 3

If a square has a diagonal of length $a+b$ then using the Pythagorean theorem its side length s must satisfy

$$s^2 + s^2 = (a+b)^2\,.$$

This means that $s = \frac{a+b}{\sqrt{2}}$. The area of this square is then

$$s^2 = \frac{(a+b)^2}{2}\,.$$

Problem 4

If you draw the barn with the given measurements you can compute the area of the faces that need painting. I find this area to be

$$2 \times 2 \times (10 \times 5) + 2 \times 2 \times (13 \times 5) + 10 \times 13 = 590\,.$$

Problem 5

When A sells to B at a 10% profit it means that he sells for \$11000. Next B sells back to A at a 10% loss so he sells something worth \$11000 for $0.9 \times 11000 = 9900$. At this point A had \$11000 cash and then pays \$9900 for his house and a total profit of

$$11000 - 9900 = 1100\,.$$

Problem 6

Let this rectangular box have base dimensions $w \times d$ (width by depth) and a height of h. Then the three areas given are

$$\begin{aligned} B &= wd \\ S &= hd \\ F &= hw \,. \end{aligned}$$

The product of these three is

$$w^2 d^2 h^2 \,,$$

which is the volume squared.

Problem 7

These two relative errors are

$$\begin{aligned} \text{re}_1 &= \frac{0.02}{10} = 0.002 \\ \text{re}_2 &= \frac{0.2}{100} = 0.002 \,, \end{aligned}$$

which are the same.

Problem 8

Let P be the original price. Then we are told that the cut price C is given by

$$C = (1 - 0.1)P = 0.9P \,.$$

To get this back to the original price it needs to *increase* by a fraction x such that

$$(1 + x)C = P \,.$$

Solving for x this means that

$$x = \frac{1}{0.9} - 1 = 0.111111 \,,$$

which is an $11\frac{1}{9}\%$ increase.

Problem 9

As the midpoints of each side divide the side of length a into two segments of length $\frac{a}{2}$ this must also be the side of the first smaller equilateral triangle. The next equilateral triangle will have a side of length $\frac{a}{4}$. Thus the sum of the infinite number of perimeters is given by

$$3a + \frac{3a}{2} + \frac{3a}{4} + \cdots = 3a \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = \frac{3a}{1 - \frac{1}{2}} = 6a\,.$$

Problem 10

The area of a circle is πr^2 which is not linear in r.

Problem 11

If we square every term and then sum we would get

$$\frac{a^2}{1 - r^2}\,.$$

Problem 12

Note that the "sector angle" between any two adjacent numbers on the clock has a degree measure of

$$\frac{360}{12} = 30^\circ\,.$$

At 2:00 the minute hand is pointing at 12 and the hour hand is pointing at 2. Thus the initial location of the minute hand is

$$m_0 = 0^\circ\,,$$

measured from vertical and the initial location of the hour hand is

$$h_0 = 2 \times 30^\circ = 60^\circ\,.$$

In 15 minutes the minute hand will "move"

$$360 \times \frac{1}{4} = 90^\circ\,,$$

since in 15 minutes the minute hand moves one quarter of the way around the clock. This places it at

$$m_{15} = 0 + 90 = 90\,.$$

In the same 15 minutes the hour hand will move one quarter of a "sector angle" or

$$\frac{1}{4} \times 30 = \frac{15}{2} = 7.5^\circ \,.$$

This places it at

$$h_{15} = 60 + 7.5 = 67.5 \,.$$

The angle between these two is then

$$m_{15} - h_{15} = 90 - 67.5 = 22.5^\circ \,.$$

Problem 13

The rate of As work is

$$r_A = \frac{1}{9} \,,$$

jobs-per-day. The rate of B work is

$$r_B = 1.5 r_A = \frac{1}{6} \,,$$

jobs-per-day. Thus B can do one job in $\frac{1}{r_B} = 6$ days.

Problem 15

Write this give number as $n(n^2 - 1) = n(n-1)(n+1)$. For n integer this is the product of three consecutive integers. For any three consecutive integers there will be at least one even number and at least one number of the form $3k$ for some k. Thus this product is divisible by $2 \times 3 = 6$.

Problem 16

The roots of this quadratic are given by

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{b^2 - b^2}}{2a} = -\frac{b}{2a} \,,$$

which is a single number and thus $f(x) = 0$ has only one solution. This means that the function $f(x)$ will be tangent to the x-axis.

Problem 17

All of the expressions are of the form $y = kx$ or $xy = k$ except (D).

Problem 18

Let the factors of the given quadratic be given by $Ax + B$ and $Cx + D$. This means that

$$
\begin{aligned}
21x^2 + ax + 21 &= (Ax + B)(Cx + D) \\
&= ACx^2 + (AD + BC)x + BD\,.
\end{aligned}
$$

Equating coefficients this means that $AC = 21$, $a = AD + BC$, and $BD = 21$. Since the only factors of 21 are 1, 3, 7, and 21 we see that both A and C (and then B and D) must also be odd. From the form of a as the product of two odd numbers is odd and the sum of two odd numbers is even we see that a must be a certain even number.

Problem 19

For the example with 256 we can write this number $N = 256256$ as

$$
N = 25600 + 256 = 256 \cdot 10^3 + 256 = 256(10^3 + 1) = 256 \times 1001\,.
$$

Thus if x is any three digit number then our six digit number N is equal to

$$
N = 1001x\,.
$$

This will always be divisible by 1001.

Problem 20

Call this expression E. Then we can write E as

$$
\begin{aligned}
E = (x + y)^{-1}(x^{-1} + y^{-1}) &= \frac{1}{x + y}\left(\frac{1}{x} + \frac{1}{y}\right) \\
= \frac{x + y}{(x + y)xy} = \frac{1}{xy} &= x^{-1}y^{-1}\,.
\end{aligned}
$$

Problem 21

If $x > 0$, $y > 0$, and $x > y$ but $z < 0$ then $xz < yz$ thus (C) is not correct in that case.

Problem 22

From what we are given we have

$$
a^2 - 15a = 10^2\,,
$$

or

$$a^2 - 15a - 100 = 0\,,$$

or

$$(a-20)(a+5) = 0\,,$$

thus $a = 20$ or $a = -5$.

Problem 23

The volume of a cylinder is given by $V = \pi r^2 h$. If we change the radius r to $r + \Delta r$ the change in volume is given by

$$\begin{aligned} \Delta V &= \pi(r+\Delta r)^2 h - \pi r^2 h \\ &= \pi(r^2 + 2r\Delta r + \Delta r^2) - \pi r^2 h \\ &= 2\pi r h \Delta r + \pi h \Delta r^2\,. \end{aligned}$$

If we change the height h to $h + \Delta h$the change in volume is given by

$$\Delta V = \pi r^2 (h + \Delta h) - \pi r^2 h = \pi r^2 \Delta h\,.$$

For these two things to be equal means that

$$\pi r^2 \Delta h = 2\pi r h \Delta r + \pi h \Delta r^2\,.$$

If $\Delta h = \Delta r = x$ then this

$$\pi r^2 x = 2\pi r h x + \pi h x^2\,.$$

If we divide by πx we get

$$r^2 = 2rh + hx \quad \text{or} \quad x = \frac{r(r-2h)}{h}\,.$$

Using the numbers given in the problem we have

$$x = \frac{8(8-2(3))}{3} = \frac{16}{3} = 5\,\frac{1}{3}\,.$$

Problem 24

Call this expression E. Then we can write E as

$$E = \frac{2^{n+4} - 2^{n+1}}{2^{n+4}} = \frac{2^3 - 1}{2^3} = 1 - \frac{1}{8} = \frac{7}{8}\,.$$

Problem 25

To solve this problem we need to recall that the **apothem** is a line segment from the center of a figure to the midpoint of one side. Let a_s and a_t be the length of the apothem of the square and the triangle respectively.

For the square with side s if $A = s^2 = P = 4s$ then $s = 4$. In this case the center of the square is a distance $\frac{s}{2} = 2$ from the midpoint of a side and thus for the length of the apothem of the square we find $a_s = 2$.

For the equilateral triangle with side s if $A = \frac{\sqrt{3}}{4}s^2 = P = 3s$ then $s = 4\sqrt{3}$. The apothem of the triangle will start at the intersection of the three medians to each of the sides (by the fact that this is an equilateral triangle these are also the three altitudes and three angle bisectors). All of this means that the apothem of the triangle is one leg of a right triangle with opposite angle 30° and the other leg of length $\frac{s}{2} = 2\sqrt{3}$. From the definition of tangent we have

$$\tan(30°) = \frac{1}{\sqrt{3}} = \frac{a_t}{2\sqrt{3}}.$$

Thus $a_t = 2$. Thus these two lengths are equal.

Problem 26

If we multiply this expression by $m - 1$ we get

$$\frac{x(x-1) - (m+1)}{x-1} = \frac{x}{m}(m-1),$$

or

$$x - \frac{m+1}{x-1} = x - \frac{x}{m},$$

or

$$\frac{m+1}{x-1} = \frac{x}{m},$$

or

$$m(m+1) = x(x-1),$$

or

$$x^2 - x - m(m+1) = 0.$$

We can factor this as

$$(x - (m+1))(x+m) = 0.$$

Thus the two roots are $x = m + 1$ or $x = -m$. These will be equal when

$$-m = m + 1 \quad \text{or} \quad m = -\frac{1}{2}.$$

Problem 27

None of these relationships are true thus (E) is the correct answer. One way to think about this is that if one of these were true it would have certainly have been discussed in your geometry class.

Problem 28

We are told that the pressure P can be related to the area A and the velocity V via

$$P = kAV^2 \,,$$

for some constant k. We know that $P = 1$ pound when $A = 1$ square foot and $V = 16$ mph. Putting these in the above gives

$$1 = k \cdot 1 \cdot 16^2 \quad \text{so} \quad k = \frac{1}{256} \,,$$

Thus the formula is

$$P = \frac{AV^2}{256} \,.$$

We want to know V when $P = 36$ pounds and $A = 3^2 = 9$ square feet. This means that

$$36 = \frac{9V^2}{256} \quad \text{so} \quad V = 32 \,,$$

miles-per-hour.

Problem 30

Draw these poles in the Cartesian coordinate plane with the short one on the left (on $x = 0$) and the tall one on the right (on $x = 100$). Then a line from the top of the short one to the base of the long one must connect the two points $(0, 20)$ and $(100, 0)$. This line takes the form

$$y - 20 = \left(\frac{0 - 20}{100 - 0} \right) (x - 0) \quad \text{so} \quad y = 20 - \frac{1}{5} x \,.$$

Next a line from the base of the short one to the top of the long one must connect the two points $(0, 0)$ and $(100, 80)$. This line takes the form

$$y = \frac{80}{100} x = \frac{4}{5} x \,.$$

The point of intersection of these two lines has $x = 20$ where $y = \frac{4}{5}(20) = 16$.

Problem 31

The number of handshakes between n people is given by $\binom{n}{2} = \frac{n(n-1)}{2}$. Setting this equal to 28 and solving for n gives $n = 8$.

Problem 32

In drawing this figure we place A and B on the x axis of a Cartesian coordinate system with the center of the circle O at $(r, 0)$ so that A is at $(0, 0)$ and B is at $(2r, 0)$. Then C is on the upper semicircle. For the sake of something specific let C be "to the left of" the circle center. Then drawing a segment from C to the circle center O (of length r) introduce the angle $\angle COA = \theta$ so that $\angle BOC = \pi - \theta$.

Then using the law of cosigns in the two triangles $\triangle COA$ and $\triangle COB$ we have

$$
\begin{aligned}
AC^2 &= r^2 + r^2 - 2r^2\cos(\theta) = 2r^2(1 - \cos(\theta)) \\
BC^2 &= r^2 + r^2 - 2r^2\cos(\pi - \theta) = 2r^2(1 + \cos(\theta))\,,
\end{aligned}
$$

where in the second expression we have used the fact that

$$
\cos(\pi - \theta) = \cos(\pi)\cos(\theta) - \sin(\pi)\sin(\theta) = -\cos(\theta)\,.
$$

From this we see that

$$
\begin{aligned}
AC + BC &= \sqrt{2r^2(1 - \cos(\theta))} + \sqrt{2r^2(1 + \cos(\theta))} \\
&= \sqrt{2}r\sqrt{1 - \cos(\theta)} + \sqrt{2}r\sqrt{1 + \cos(\theta)} \\
&= \sqrt{2}r(\sqrt{1 - \cos(\theta)} + \sqrt{1 + \cos(\theta)})\,.
\end{aligned}
$$

To evaluate this recall that

$$
\begin{aligned}
(1 + x)^{1/2} &= 1 + \frac{x}{2} + \binom{1/2}{2}x^2 + \binom{1/2}{3}x^3 + \cdots \\
&= 1 + \frac{x}{2} - \frac{x^2}{8} + \cdots\,.
\end{aligned}
$$

Where we have evaluated

$$
\binom{1/2}{2} = \frac{(1/2)(1/2 - 1)}{2!} = \frac{(1/2)(-1/2)}{2} = -\frac{1}{8}\,.
$$

Using this we find that

$$
\begin{aligned}
AC + BC &\approx \sqrt{2}r\left(1 - \frac{1}{2}\cos(\theta) - \frac{1}{8}\cos(\theta)^2 + 1 + \frac{1}{2}\cos(\theta) - \frac{1}{8}\cos(\theta)^2\right) \\
&= \sqrt{2}r\left(2 - \frac{1}{4}\cos(\theta)^2\right) \\
&\leq 2\sqrt{2}r = \sqrt{2}AB\,.
\end{aligned}
$$

Problem 33

Looking at (C) we see that these lines don't intersect and cannot be a solution to the given equation.

Problem 34

The meaning of $\log_{10}(7)$ is the power we "put on" 10 to get seven. Thus $10^{\log_{10}(7)} = 7$.

Problem 35

If we take the logarithms of these two equations we get

$$x\ln(a) = q\ln(c)$$
$$y\ln(c) = z\ln(a)\,.$$

If we then take the ratio of these two we find

$$\frac{x\ln(a)}{z\ln(a)} = \frac{q\ln(c)}{y\ln(c)} \quad \text{so} \quad \frac{x}{z} = \frac{q}{y}\,.$$

This is equivalent to $xy = qz$.

Problem 37

For each of the number choices given we will compute the desired divisions and determine if their remainders satisfy the requirements of the problem. Let N be our unknown number. As

$$N \bmod 10 = 9\,,$$

we know that N must end in a nine. All of the given numbers satisfy this.

Note that

$$59 \bmod 9 = 5\,,$$

which is not what it should be. Thus $N \neq 59$. Next note that

$$419 = 46 \cdot 9 + 5 \quad \text{so} \quad 419 \bmod 9 = 5\,,$$

which is not what it should be. Thus $N \neq 419$. Next note that

$$1259 = 157 \cdot 8 + 3 \quad \text{so} \quad 1259 \bmod 8 = 3\,,$$

which is not what it should be. Thus $N \neq 1259$. We can check the remainders if we divide $N = 2519$ by the numbers $9, 8, \ldots, 3, 2$ and find that all of the needed conditions are satisfied. Thus $N = 2519$.

Problem 38

Let $h = 600$ feet be the "height" of the mountain and w_0 the length of the horizontal "run" needed to get over it at a gradient of 3%. This means that

$$\frac{h}{w_0} = 0.03 \quad \text{or} \quad \frac{600}{w_0} = 0.03\,,$$

so $w_0 = 20000$ feet. To reduce this gradient to 2% means that wen need a "run" w_1 that satisfies

$$\frac{600}{w_1} = 0.02\,,$$

so $w_1 = 30000$ feet. The additional length is $w_1 - w_0 = 10000$ feet.

Problem 39

Let the well be at a depth of d. The rock falls for a time t_d "downwards" until it hits the surface of the water releasing a sound wave travels for a time t_u "upwards" until it is heard. We are told the total time

$$t_d + t_u = 7.7\,.$$

From the speed of sound we know that

$$t_u = \frac{d}{1120}\,.$$

From the amount of distance a stone falls in t time we have

$$d = 16t_d^2\,.$$

These give three equations and three unknowns t_d, t_u, and d. Solving the previous two equations for t_u and t_d in terms of d and putting them into the first equation gives

$$\sqrt{\frac{d}{16}} + \frac{d}{1120} = 7.7\,.$$

We can write this as

$$d + 280\sqrt{d} - 8624\,.$$

This is a quadratic equation for $\sqrt{d}$. Solving for $\sqrt{d}$ the only positive solution is $\sqrt{d} = 28$, so $d = 784$ feet.

Problem 40

Note that

$$\frac{(x+1)^2}{(x^3+1)^2} = \frac{(x+1)^2}{(x+1)^2(x^2-x+1)^2} = \frac{1}{(x^2-x+1)^2}\,,$$

and thus the first factor simplifies to one. By similar logic the second factor simplifies to one and the total product is one.

Problem 41

Choice (A) does not work for the point $(4, 6)$. Choice (E) does not work for the point $(3, 2)$. Choice (D) does not work for the point $(2, 0)$. Choice (C) does not work for the point $(3, 2)$. Choice (B) works for all points.

Problem 42

If we square the given expression and recall what x is defined as we get

$$x^2 = 1 + x\,,$$

or

$$x^2 - x - 1 = 0\,.$$

Solving this we find

$$x = \frac{1 \pm \sqrt{1 - 4(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}\,.$$

From the original expression we see that the solution $x > 1$ and thus we take the positive root above. Thus we have that

$$x = \frac{1 + \sqrt{5}}{2}\,.$$

We can write this as

$$2x - 1 = \sqrt{5}\,,$$

and then if we bound $\sqrt{5}$ as

$$2 = \sqrt{4} < \sqrt{5} < \sqrt{9} = 3\,,$$

we have

$$2 < 2x - 1 < 3\,.$$

Solving for x we get

$$\frac{3}{2} < x < 2\,.$$

Problem 43

Choice (A) is true.

Choice (B) is the statement that

$$\sqrt{ab} < \frac{1}{2}(a + b)\,,$$

which is the arithmetic-geometric inequality and is true.

For choice (C) let our two numbers be a and b and the target sum t. Thus we have that

$$a + b = t\,,$$

and we want to maximize $ab = a(t-a)$. Considered as a function of a the first derivative of this is

$$f'(a) = t - 2a\,.$$

Setting this equal to zero and solving gives $a = \frac{t}{2}$ (so that $b = t - a = \frac{t}{2}$). As the second derivative of $f(a)$ is given by

$$f''(a) = -2 < 0\,,$$

we have found a maximum of our product ab. Thus (C) is true.

If (D) were true it would be equivalent to the statement

$$\frac{1}{2}(a^2+b^2) > \frac{1}{4}(a+b)^2\,,$$

or

$$2(a^2+b^2) > a^2 + 2ab + b^2\,,$$

or

$$a^2 - 2ab + b^2 > 0\,,$$

or

$$(a-b)^2 > 0\,,$$

which is true.

For choice (D) let our two numbers be a and b and the target product be t. Thus we have that

$$ab = t\,,$$

and we want to maximize $a + b = a + \frac{t}{a}$. Considered as a function of a the first derivative of this is

$$f'(a) = 1 - \frac{t}{a^2}\,.$$

Setting this equal to zero and solving gives $a = \sqrt{t}$ (so that $b = \sqrt{t}$). As the second derivative of $f(a)$ is given by

$$f''(a) = \frac{t}{a^3} > 0\,,$$

we have found a *minimum* of our sum $a + b$. Thus (D) is *not* true.

Problem 44

If we "flip" each of these equations we get

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{a} \quad (29)$$
$$\frac{1}{x} + \frac{1}{z} = \frac{1}{b} \quad (30)$$
$$\frac{1}{y} + \frac{1}{z} = \frac{1}{c}\,. \quad (31)$$

These are three linear equations in the three "variables" $\frac{1}{x}$, $\frac{1}{y}$, and $\frac{1}{z}$. If we add Equations 29 and 30 we get

$$\frac{2}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{a} + \frac{1}{b} \,.$$

If we then replace $\frac{1}{y} + \frac{1}{z}$ using Equations 31 we get

$$\frac{2}{x} + \frac{1}{c} = \frac{1}{a} + \frac{1}{b} \,.$$

If we solve this for x we get

$$x = \frac{2abc}{ac + bc - ab} \,.$$

Problem 45

From the given expressions we have

$$\begin{aligned} 3\log(2) &= 0.9031 \\ 2\log(3) &= 0.9542 \,, \end{aligned}$$

so we can compute $\log(2)$ and $\log(3)$ using the numbers given. The given expressions (except the first) can all be written in terms of $\log(2)$, $\log(3)$, and $\log(5)$ as

$$\begin{aligned} \log(5/4) &= \log(5) - 2\log(2) \\ \log(15) &= \log(3) + \log(5) \\ \log(600) &= \log\left(2^3 \cdot 3 \cdot 5^2\right) = 3\log(2) + \log(3) + 2\log(5) \\ \log(0.4) &= \log\left(\frac{2}{5}\right) = \log(2) - \log(5) \,. \end{aligned}$$

If we can compute $\log(5)$ we can evaluate all of these. As we also know $\log(10) = 1$ we can write

$$\log(2) + \log(5) = 1 \quad \text{so} \quad \log(5) = 1 - \log(2) \,.$$

The only expression we cannot evaluate is thus $\log(17)$.

Problem 47

Denote this expression by E. Then we have

$$E = \frac{s^2 + r^2}{s^2 r^2} = \frac{s^2 + r^2}{(sr)^2} \,. \tag{32}$$

As r and s are roots they must satisfy

$$\begin{aligned} as^2 + bs + c &= 0 \\ ar^2 + br + c &= 0 \,. \end{aligned}$$

If we add these together we get

$$s^2 + r^2 = \frac{-2c - b(s+r)}{a} \,. \tag{33}$$

If we write the quadratic we are given as

$$x^2 + \frac{b}{a}x + \frac{c}{a} = (x-r)(x-s) = 0 \,,$$

and then expand and equate coefficients of x we see that

$$\begin{aligned} s + r &= -\frac{b}{a} \\ sr &= \frac{c}{a} \,. \end{aligned}$$

Using the first expression in Equations 33 gives

$$s^2 + r^2 = \frac{-2c - b\left(-\frac{b}{a}\right)}{a} = \frac{b^2 - 2ac}{a^2} \,.$$

Using the above in 32 gives

$$E = \frac{b^2 - 2ac}{a^2} \cdot \frac{1}{\left(\frac{c}{a}\right)^2} = \frac{b^2 - 2ac}{c^2} \,.$$

Problem 48

In the square inscribed in the circle the length of the segments from the center to each corner is r the radius of the circle. This means that the diagonal of the square has a length $2r$. By the Pythagorean theorem the side of the square a then satisfies

$$2a^2 = (2r)^2 \quad \text{so} \quad a = \sqrt{2}r \,.$$

Thus this squares area is $a^2 = 2r^2$.

In the square inscribed in the semicircle the length of the segments from the center to the corners on the circle is r the radius of the circle. This length/segment forms the hypotenuse of a right triangle with legs b and $\frac{b}{2}$. If this square has a side of length b then by the Pythagorean theorem we have

$$r^2 = b^2 + \left(\frac{b}{2}\right)^2 \,.$$

Solving this for b we get

$$b = \frac{2r}{\sqrt{5}} \,,$$

so this square has an area of $b^2 = \frac{4r^2}{5}$.

The ratio of these two areas is

$$\frac{\frac{4r^2}{5}}{2r^2} = \frac{2}{5} \,.$$

Problem 49

Lets draw the right triangle with right angle B at the origin of a Cartesian coordinate plane. Let the leg BC be along the x-axis and the leg BA be along the y-axis. Draw the median from A to the side BC and intersecting BC at a point A'. Draw the median from C and intersecting AB at a point C'. We are told that $AA' = 5$ and $CC' = \sqrt{40}$.

Let $BA' = A'C = x$ and $BC' = C'A = y$. Then from the right triangle $\triangle ABA'$ the Pythagorean theorem gives

$$AB^2 + BA'^2 = AA'^2 \quad \text{or} \quad (2y)^2 + x^2 = 25\,,$$

or

$$4y^2 + x^2 = 25\,. \tag{34}$$

From the right triangle $\triangle C'BC$ the Pythagorean theorem gives

$$C'B^2 + BC^2 = CC'^2 \quad \text{or} \quad y^2 + (2x)^2 = 40\,,$$

or

$$y^2 + 4x^2 = 40\,. \tag{35}$$

From Equation 35 we have $y^2 = 40 - 4x^2$ which if we put into Equation 34 we get a single equation for x that solving gives $x = 3$. Putting this into Equation 35 we get $y = 2$. This means that

$$\begin{aligned} AB &= 2y = 4 \\ BC &= 2x = 6\,. \end{aligned}$$

Another use of the Pythagorean theorem in the triangle $\triangle ABC$ gives

$$AC^2 = AB^2 + BC^2 = 16 + 36 = 52\,.$$

Thus $AC = \sqrt{52} = 2\sqrt{13}$.

Problem 50

If you know the velocity then "time" and "distance" are "equivalent" so we will consider the *times* when things happen and take note of the positions where each person is at these times.

Thus if we draw a time number line along the top of our paper the times where events happen are

- $T = 0$ Journey starts
- $T = T_1$ Harry gets off

- $T = T_2$ Tom picks up Dick
- $T = T_3$ Harry (and others) get to the finish line

Then depending on how each person is traveling the *distances* of each of the people at each of these three times are given by

For Tom:

- At $T = T_1$ he is at $25T_1$
- At $T = T_2$ he is at $25T_1 - 25(T_2 - T_1)$
- At $T = T_3$ he is at $25T_1 - 25(T_2 - T_1) + 25(T_3 - T_2)$

For Dick:

- At $T = T_1$ he is at $5T_1$
- At $T = T_2$ he is at $5T_1 + 5(T_2 - T_1)$
- At $T = T_3$ he is at $5T_1 + 5(T_2 - T_1) + 25(T_3 - T_2)$

For Harry:

- At $T = T_1$ he is at $25T_1$
- At $T = T_2$ he is at $25T_1 + 5(T_2 - T_1)$
- At $T = T_3$ he is at $25T_1 + 5(T_2 - T_1) + 5(T_3 - T_2)$

If we simplify these we get

For Tom:

- At $T = T_1$ he is at $25T_1$
- At $T = T_2$ he is at $50T_1 - 25T_2$
- At $T = T_3$ he is at $50T_1 - 50T_2 + 25T_3$

For Dick:

- At $T = T_1$ he is at $5T_1$
- At $T = T_2$ he is at $5T_2$
- At $T = T_3$ he is at $-20T_2 + 25T_3$

For Harry:

- At $T = T_1$ he is at $25T_1$
- At $T = T_2$ he is at $20T_1 + 5T_2$
- At $T = T_3$ he is at $20T_1 + 5T_3$

The problem asks us to find T_3. When Tom picks up Dick their location must be same so we have

$$50T_1 - 25T_2 = 5T_2 \quad \text{so} \quad T_1 = \frac{3}{5}T_2\,. \tag{36}$$

At the time T_3 Dick and Harry are at the finish line so we have

$$-20T_2 + 25T_3 = 100 \tag{37}$$

$$20T_1 + 5T_3 = 100\,. \tag{38}$$

If we replace T_1 in the Equation 38 with the expression from Equation 36 we get

$$12T_2 + 5T_3 = 100\,. \tag{39}$$

Solving Equations 37 and 39 give $T_2 = 5$ and $T_3 = 8$ so the trip takes eight hours.

The 1952 Examination

Problem 1

If r is rational then πr^2 is irrational (since π is).

Problem 2

This would be

$$\frac{20(0.8)+30(0.7)}{20+30}=\frac{37}{50}=0.74\,.$$

Problem 3

Call this expression E. Then we have

$$\begin{aligned}E&=a^3-a^{-3}=(a-a^{-1})(a^2+aa^{-1}+a^{-2})\\&=(a-a^{-1})(a^2+1+a^{-2})\\&=\left(a-\frac{1}{a}\right)\left(a^2+1+\frac{1}{a^2}\right)\,.\end{aligned}$$

Problem 4

This would be

$$C=10+3(P-1)=7+3P\,.$$

Problem 5

This line is given by

$$y-(-6)=\left(\frac{-6-12}{0-6}\right)(x-0)\,,$$

or

$$y=-6+3x\,.$$

If $x=3$ then in the above we get $y=3$ which is (A).

Problem 6

The two roots are given by

$$x = \frac{7 \pm \sqrt{49 - 4(-9)}}{2} = \frac{7 \pm \sqrt{83}}{2}.$$

Thus their difference would be

$$x_+ - x_- = \frac{7 + \sqrt{83}}{2} - \frac{7 - \sqrt{83}}{2} = \sqrt{85},$$

or its negative. Only (E) is correct.

Problem 7

For this we have

$$(x^{-1} + y^{-1})^{-1} = \left(\frac{1}{x} + \frac{1}{y}\right)^{-1} = \left(\frac{x+y}{xy}\right)^{-1} = \frac{xy}{x+y}.$$

Problem 8

If the circles are "far apart" then they will have four common tangents i.e. two external and two internal tangents. If the circles move closer at the point they touch they will have three common tangents i.e. two external and a single internal tangent as the two internal tangents from before collapse into one internal tangent. If the circles move together still further they loose the internal tangent and we only have two external tangents. Thus it is impossible to have a single common tangent.

Problem 9

Write the given expression as

$$m = \frac{cab}{a-b} = \frac{ca}{\frac{a}{b} - 1}.$$

Then solving for $\frac{1}{b}$ (by first solving for $\frac{a}{b}$) in this we get

$$\frac{1}{b} = \frac{m + ca}{ma} \quad \text{so} \quad b = \frac{ma}{m + ca}.$$

Problem 10

The average speed $\bar{v}$ is the total distance traveled divided by the total time taken. If D is the length of the "hill" (in miles) then the time going up and the time going down are given by

$$t_{\text{up}} = \frac{D}{10}$$
$$t_{\text{down}} = \frac{D}{20}\,.$$

From this we find that

$$\bar{v} = \frac{D+D}{\frac{D}{10}+\frac{D}{20}} = \frac{40}{3} = 13\,\frac{1}{3}\,,$$

in miles an hour.

Problem 11

For this function $f(x)$ we have that $f(1)$ is undefined and thus C is incorrect.

Problem 12

In this problem we are told that

$$\sum_{k=0}^{\infty} a_0 r^k = 6$$
$$a_0 + a_0 r = 4\,\frac{1}{2} = \frac{9}{2}\,.$$

As this first expression can be written as

$$\frac{a_0}{1-r} = 6\,,$$

and the second expression as

$$a_0 = \frac{9}{2(1+r)}\,. \tag{40}$$

Combining these we get

$$\frac{9}{2(1+r)(1-r)} = 6\,.$$

Solving this for r gives $r \in \left\{-\frac{1}{2}\,,\frac{1}{2}\right\}$. This means that

$$1 + r \in \left\{\frac{1}{2}\,,\frac{3}{2}\right\}\,.$$

If we put these into Equation 40 we get

$$a_0 \in \{9\,,3\}\,.$$

Problem 13

If we call this function $f(x)$ then we find

$$f'(x) = 2x + p = 0 \quad \text{when} \quad x = -\frac{p}{2},$$

and that

$$f''(x) = 2 > 0,$$

so $-\frac{p}{2}$ is a minimum of the function $f(x)$.

Another way to see this is to "complete the square" by writing $f(x)$ as

$$\begin{aligned} x^2 + px + q &= x^2 + px + \left(\frac{p}{2}\right)^2 - \frac{p^2}{4} + q \\ &= \left(x + \frac{p}{2}\right)^2 + q - \frac{p^2}{4}. \end{aligned}$$

The smallest the right-hand-side will be is again when $x = -\frac{p}{2}$.

Problem 14

Let C_H and C_S be the costs (paid) for the house and the store respectively. Then we are told that

$$C_H(0.8) = 12000,$$

i.e. the house was sold at a 20% loss and that

$$C_S(1.2) = 12000,$$

i.e. the store was sold at a 20% profit. These mean that

$$\begin{aligned} C_H &= 15000 \\ C_S &= 10000, \end{aligned}$$

The total then paid initially was

$$C_H + C_S = 25000,$$

and the total amount sold was $12000 + 12000 = 24000$ resulting in a loss of \$1000.

Problem 15

As

$$6^2 + 8^2 = 36 + 64 = 100 > 9^2 = 81,$$

the angle between the two sides that are in the ratio $6 : 8$ must be *less* than 90° and thus the triangle is acute.

Problem 16

The original area can be written as $A = bh$. If the base increases by 10% then the new base b' is related to the old base b via $b' = 1.1b$. If the area is unchanged then the new height h' must satisfy

$$A = bh = b'h' = 1.1bh' \,.$$

This means that

$$h' = \frac{1}{1.1} h \,.$$

We can write this as

$$h' = \frac{10}{11} h = \left(1 - \frac{1}{11}\right) h \,.$$

Now

$$\frac{1}{11} = 0.09090909 \,,$$

Thus

$$h' = (1 - 0.09090909) h \,,$$

which is a drop of 9.090909% or $9\frac{1}{11}$%.

Problem 17

To start, we let B be the bought price and L the list price. Then we are told that $B = 0.8L$. The merchant sells at a price S and we want S to be

$$S = B + 0.2S \,,$$

to make a profit of 20% on the sales price S. Note that if you thought the expression was

$$S = 1.2B \,,$$

this is wrong as this is a 20% profit on the bought price (and not the sales price as it should be). Finally the marked price to the sales price should be

$$0.8M = S \,,$$

so that the sales price is a 20% discount to the marked price M. Then for this problem we want to know M in terms of L. We have

$$M = \frac{1}{0.8} S = \frac{1}{0.8} \left(\frac{B}{0.8}\right) = \frac{1}{0.8^2}(0.8L) = \frac{1}{0.8} L = \frac{5}{4} L = 1.25L \,.$$

This means that we should mark the item up 125% from the list price.

Problem 18

Write the expression we are given as

$$\log(pq) = \log(p+q)\,,$$

or

$$pq = p + q\,,$$

or

$$p(q-1) = q\,,$$

or

$$p = \frac{q}{q-1}\,.$$

Problem 19

Draw the triangle $\triangle ABC$ with the segment AC on the x-axis and the point B above the segment AC. Then with the angle at B trisected, draw the segments BD, and BE so that the points on the segment AC are in order A, D, E and then C. Let each of these three pieces of the angle at B have an angle measure of ϕ. Thus

$$\angle ABD = \angle DBE = \angle EBC = \phi\,.$$

Let the angle $\angle BAD = \psi$. Then in triangle $\triangle ABD$ we know two of the angles and thus can compute

$$\angle ADB = \pi - \psi - \phi\,.$$

Then we can use supplemental angles, the fact that the three angles in a triangle sum to 180°, or the exterior angle theorem to express all of the angles in this figure in terms of ϕ and ψ. For the angles we have not yet specified we have

$$\begin{aligned}
\angle BDE &= \psi + \phi \\
\angle BED &= \pi - \psi - 2\phi \\
\angle BEC &= \psi + 2\phi \\
\angle BCE &= \pi - \psi - 3\phi\,.
\end{aligned}$$

Now using the law-of-sines in the triangles $\triangle ABD$ gives

$$\frac{\sin(\phi)}{AD} = \frac{\sin(\pi - \psi - \phi)}{AB} = \frac{\sin(\psi)}{BD}\,, \tag{41}$$

in the triangle $\triangle DBE$ gives

$$\frac{\sin(\phi)}{DE} = \frac{\sin(\pi - \psi - 2\phi)}{BD} = \frac{\sin(\psi + \phi)}{BE}\,, \tag{42}$$

and in the triangle $\triangle EBC$ gives

$$\frac{\sin(\phi)}{EC} = \frac{\sin(\pi - \psi - 3\phi)}{BE} = \frac{\sin(\psi + 2\phi)}{BC}\,. \tag{43}$$

If we divided the first and the second equations found in Equations 43 by those found in Equations 41 we get

$$\frac{AD}{EC} = \frac{AB}{BE} \times \frac{\sin(\pi - \psi - 3\phi)}{\sin(\pi - \psi - \phi)} . \tag{44}$$

Recall that $\sin(\pi - x) = \sin(x)$ this becomes

$$\frac{AD}{EC} = \frac{AB}{BE} \times \frac{\sin(\psi + 3\phi)}{\sin(\psi + \phi)} . \tag{45}$$

From Equation 43 we have that

$$\sin(\psi + 3\phi) = \frac{BE}{BC} \sin(\psi + 2\phi) ,$$

and from Equation 42 we have that

$$\sin(\psi + \phi) = \frac{BE}{BD} \sin(\psi + 2\phi) .$$

This means that

$$\frac{\sin(\psi + 3\phi)}{\sin(\psi + \phi)} = \frac{BE}{BC} \times \frac{BD}{BE} = \frac{BD}{BC} ,$$

so that Equation 45 becomes

$$\frac{AD}{EC} = \frac{AB}{BE} \cdot \frac{BD}{BC} ,$$

which is choice (D).

Problem 20

From what we are given we know that

$$x = \frac{3}{4} y ,$$

If we "put this in" for x in each of the choices for (A) we get

$$\frac{\frac{3}{4}y + y}{y} = \frac{7}{4} ,$$

which is true. For (B) we get

$$\frac{y}{y - \frac{3}{y}y} = \frac{y}{\frac{y}{4}} = 4 ,$$

which is true. For (C) we get

$$\frac{\frac{3}{4}y + 2y}{\frac{3}{4}y} = 1 + \frac{8}{3} = \frac{11}{3} ,$$

which is true. For (D) we get

$$\frac{\frac{3}{4}y}{2y} = \frac{3}{8} ,$$

which is true. For (E) we get

$$\frac{\frac{3}{4}y - y}{y} = -\frac{1}{4} ,$$

which is not true.

Problem 21

To start we draw a regular polygon with $n > 4$ (say a regular hexagon) and extend the sides to form a star. Next, recall that the interior angles I of a regular polygon are given by

$$I_{\text{regular polygon}} = \frac{180(n-2)}{n} \,. \tag{46}$$

Then from the drawing the exterior angles have a measure

$$180 - \frac{180(n-2)}{n} = \frac{360}{n} \,.$$

From the drawing we note that this is the base angle of the isosceles triangle that form the triangles that are attached to each face of the regular polygon i.e. these are the "points" of the star. From this fact the angle measure of each point of the star is then

$$180 - 2\left(\frac{360}{n}\right) = 180\left(1 - \frac{4}{n}\right) = 180\left(\frac{n-4}{n}\right) \,.$$

Problem 22

Draw the triangle $\triangle ABC$ with A on the y-axis C at the origin and B on the x-axis. Then draw D so that AD is a leg of the right triangle $\triangle ADB$ with right angle at D. Then using the Pythagorean theorem in the right triangle $\triangle ABC$ gives

$$AB^2 = b^2 + 1^2 = 1 + b^2 \,,$$

and in $\triangle ADB$ the Pythagorean theorem gives

$$AD^2 + BD^2 = AB^2 \,.$$

Using the first expression for AB we can write this as

$$4 + BD^2 = 1 + b^2 \,,$$

or

$$BD = \sqrt{b^2 - 3} \,.$$

Problem 23

Lets define r as

$$r \equiv \frac{m-1}{m+1} \,,$$

then our expression is

$$x^2 - bx = arx - cr \,,$$

or

$$x^2 - (b + ar)x + cr = 0\,.$$

Now if we factor the left-hand-side into $(x - x_-)(x - x_+)$ where x_- and x_+ are the two roots such that $x_+ = -x_-$ and $x_- < 0$. When we expand this and equate coefficients to the expression above we would get

$$-(x_+ + x_-) = -(b + ar) \tag{47}$$

$$x_+ x_- = cr\,. \tag{48}$$

Now using the fact that $x_- = -x_+$ in Equation 47 we see that

$$b + ar = 0 \quad \text{so} \quad r = -\frac{b}{a}\,.$$

If we recall what r is defined as we have

$$\frac{m-1}{m+1} = -\frac{b}{a}\,.$$

We can solve this for m to find that

$$m = \frac{a-b}{a+b}\,.$$

Problem 24

Method 1: As $AB = 20$ and $AD = DB$ we have that $AD = DB = \frac{20}{2} = 10$. Using the Pythagorean theorem in $\triangle ACB$ gives

$$12^2 + BC^2 = 20^2 \quad \text{so} \quad BC = 16\,.$$

From the two similar triangles $\triangle BDE \sim \triangle BCA$ we have that

$$\frac{BD}{BE} = \frac{BC}{AB} = \frac{16}{20} = \frac{4}{5}\,.$$

As we know that $BD = 10$ we get

$$\frac{10}{BE} = \frac{4}{5} \quad \text{so} \quad BE = \frac{50}{4} = 12.5\,.$$

Next using the Pythagorean theorem in $\triangle EDB$ we have

$$EB^2 = ED^2 + BD^2\,,$$

or using what we know

$$12.5^2 = ED^2 + 10^2 \quad \text{so} \quad ED = 7.5\,.$$

Thus the area of the quadrilateral $ADEC$ is the area of the triangle $\triangle ACB$ minus the area of the triangle $\triangle EDB$. This means that the area we want (called $\mathcal{Q}$) is

$$\begin{aligned}\mathcal{Q} &= \frac{1}{2}AC \cdot CB - \frac{1}{2}ED \cdot DB \\ &= \frac{1}{2}(12)(16) - \frac{1}{2}(7.5)(10) = 58.5\,.\end{aligned}$$

Method 2: If we draw the segment AE we can decompose the quadrilateral $ADEC$ into two right triangles $\triangle ACE$ and $\triangle ADE$. From the calculations above we know what we need to compute their two areas. Thus we find

$$\begin{aligned}\mathcal{Q} &= \frac{1}{2}AC \cdot CE + \frac{1}{2}AD \cdot ED \\ &= \frac{1}{2}(12)(16 - 12.5) + \frac{1}{2}(10)(7.5) = 58.5\,,\end{aligned}$$

the same as we had before.

Problem 25

The powderman runs for 30 seconds while the fuse burns. During this time he runs a distance of $8 \cdot 30 = 240$ yards. After this he needs an additional amount of time for the noise of the blast to reach him. This will be

$$\frac{3 \cdot 240}{1080}\,,$$

seconds and so

$$\frac{3 \cdot 240}{1080}(8)\,,$$

additional yards. Thus when he hears the blast he has run

$$\begin{aligned}D &= 8 \cdot 30 + \frac{3 \cdot 8 \cdot 30 \cdot 8}{1080} \\ &= 240\left(1 + \frac{24}{1080}\right) \\ &= 240(1 + 0.02222) = 245.33\,.\end{aligned}$$

Problem 26

From the given expression we get

$$\left(r + \frac{1}{r}\right)^3 = 3^{3/2}\,.$$

If we expand the left-hand-side of this we get

$$r^3 + 3r^2\left(\frac{1}{r}\right) + 3r\left(\frac{1}{r}\right)^2 + \left(\frac{1}{r}\right)^3 = 3^{3/2}\,,$$

or

$$r^3 + 3r + \frac{3}{r} + \frac{1}{r^3} = 3^{3/2}\,,$$

or

$$r^3 + 3\left(r + \frac{1}{r}\right) + \frac{1}{r^3} = 3^{3/2}\,,$$

or

$$r^3 + 3(\sqrt{3}) + \frac{1}{r^3} = 3^{3/2}\,,$$

or finally that

$$r^3 + \frac{1}{r^3} = 3^{3/2} - 3(\sqrt{3}) = 0\,.$$

Problem 27

Let r be the radius of the circle that is the altitude of the first equilateral triangle and that the second equilateral triangle is inscribed into. For the first triangle (the one where r is the altitude) we have

$$\tan(60^\circ) = \frac{r}{s_1/2}\,,$$

where s_1 is the first triangles side length. As $\tan(60^\circ) = \sqrt{3}$ this gives

$$s_1 = \frac{2}{\sqrt{3}} r\,,$$

so the perimeter of this triangle is $3s_1 = 2\sqrt{3}r$.

For the second triangle (the one inscribed in this circle) when you look at that triangle and draw the radius to a corner of the inscribed triangle we have

$$\frac{s_2/2}{r} = \cos(30^\circ) = \frac{\sqrt{3}}{2}\,.$$

This means that

$$s_2 = \sqrt{3}r\,,$$

so the perimeter is $3\sqrt{3}r$.

These together mean that the requested ratio is

$$2\sqrt{3}r : 3\sqrt{3}r = 2 : 3\,.$$

Problem 28

For notational ease we will call the right-hand-side of y for part (A) as $f_A(x)$, the right-hand-side for (B) as $f_B(x)$ etc.

For (A) the point $(3, 13)$ is not a solution.

For (B) we have

$$f_B(1) = 1 - 1 + 1 + 2 = 3$$
$$f_B(2) = 8 - 4 + 2 + 2 = 8\,,$$

the last of which does not match the number in the table.

For (C) we have

$$f_C(1) = 1 + 1 + 1 = 3$$
$$f_C(2) = 4 + 2 + 1 = 7$$
$$f_C(3) = 9 + 3 + 1 = 13$$
$$f_C(4) = 16 + 4 + 1 = 21$$
$$f_C(5) = 25 + 5 + 1 = 31\,.$$

As this function matches every entry in the table it must be the answer.

Problem 29

Draw a circle in the x-y plane with its center at $O = (0, 0)$ and AB along the x-axis and CD along the y-axis. In this configuration the points A, B, C, and D have the following Cartesian representation

$$A = (-5, 0)$$
$$B = (5, 0)$$
$$C = (0, -5)$$
$$D = (0, 5)\,.$$

If we place the point $H = (h_x, h_y)$ in the first quadrant and on the arch between the points D and B. We want to find the (x, y) location where $CH = 8$. Note that

$$CH = \sqrt{(0 - h_x)^2 + (-5 - h_y)^2} = \sqrt{h_x^2 + 25 + 10h_y + h_y^2}\,.$$

As $h_x^2 + h_y^2 = 5^2$ since H is on the circle we get that

$$CH = \sqrt{50 + 10h_y}\,.$$

We want to find h_y such that $CH = 8$. Solving for h_y we get

$$h_y = \frac{7}{5} = 1.4\,.$$

This means that

$$h_x^2 = 5^2 - h_y^2 = 25 - \frac{7}{25} = \frac{576}{25} \quad \text{so} \quad h_x = \frac{24}{5} = 4.8\,.$$

What we want to solve this problem is $AK : KB$. From the similar triangles $\triangle CKO$ and $\triangle CHH'$ (where $H' = (0, h_y)$) we have that

$$\frac{5}{5+h_y} = \frac{OK}{h_x} \Rightarrow OK = \frac{5h_x}{5+h_y} = \frac{15}{4}\,.$$

This means that

$$AK = 5 + OK = \frac{35}{4} = 8.75\,,$$

and

$$KB = 5 - OK = 5 - \frac{15}{4} = 1.25\,.$$

Problem 30

Let the terms of the sequence be $a_k = a_0 + dk$ for $k \geq 0$. Then we are told that

$$\sum_{k=0}^{9} a_k = 4\sum_{k=0}^{4} a_k\,,$$

or

$$10a_0 + d\sum_{k=0}^{9} k = 4 \cdot 5a_0 + 4d\sum_{k=0}^{4} k\,,$$

or evaluating the sum we get

$$10a_0 + d\frac{9(9+1)}{2} = 20a_0 + 4d\frac{4 \cdot 5}{2}\,.$$

This simplifies to

$$2a_0 = d \quad \text{so} \quad \frac{a_0}{d} = \frac{1}{2}\,,$$

or $a_0 : d = 1 : 2$.

Problem 31

We have $\binom{12}{2}$ pairs of points. Evaluating this I find

$$\binom{12}{2} = \frac{12!}{10! \cdot 2!} = \frac{12 \cdot 11}{2} = 66\,.$$

Problem 32

As x is K's speed the time it takes K to go 30 miles is

$$\frac{30}{x}\,.$$

Problem 33

Let r be the radius of the circle and s the side of the square. Then we know that the perimeters are given by

$$\begin{aligned} P_{\text{circle}} &= 2\pi r \\ P_{\text{square}} &= 4s\,. \end{aligned}$$

We are told that these are equal so we know that

$$r = \frac{2s}{\pi}\,,$$

The area of the circle is

$$A_{\text{circle}} = \pi r^2 = \pi\left(\frac{4s^2}{\pi^2}\right) = \frac{4}{\pi}s^2\,,$$

and the area of the square is

$$A_{\text{square}} = s^2\,.$$

As

$$\frac{4}{\pi} > 1\,,$$

we see that $A_{\text{circle}} > A_{\text{square}}$.

Problem 34

Let P_0 be the initial price. Then we are told that

$$P_0\left(1+\frac{p}{100}\right)\left(1-\frac{p}{100}\right) = 1\,.$$

Solving for P_0 we get

$$P_0 = \frac{1}{\left(1+\frac{p}{100}\right)\left(1-\frac{p}{100}\right)} = \frac{100^2}{(100+p)(100-p)} = \frac{10^4}{10^4-p^2}\,.$$

Problem 35

Call this expression E. Then we have

$$\begin{aligned} E &= \frac{\sqrt{2}}{\sqrt{2}+\sqrt{3}-\sqrt{5}} \times \left(\frac{\sqrt{2}+\sqrt{3}+\sqrt{5}}{\sqrt{2}+\sqrt{3}+\sqrt{5}}\right) \\ &= \frac{\sqrt{2}(\sqrt{2}+\sqrt{3}+\sqrt{5})}{(\sqrt{2}+\sqrt{3})^2-5} = \frac{\sqrt{2}(\sqrt{2}+\sqrt{3}+\sqrt{5})}{2+2\sqrt{6}+3-5} \\ &= \frac{\sqrt{2}(\sqrt{2}+\sqrt{3}+\sqrt{5})}{2\sqrt{6}}\,. \end{aligned}$$

If we multiply this by the fraction $\frac{\sqrt{6}}{\sqrt{6}}$ we get

$$E = \frac{\sqrt{12}(\sqrt{2}+\sqrt{3}+\sqrt{5})}{2\cdot 6} = \frac{2(\sqrt{6}+\sqrt{9}+\sqrt{15})}{12} = \frac{\sqrt{6}+3\sqrt{15}}{6}\,.$$

Problem 36

Call this expression E. Then we can write

$$E = \frac{(x+1)(x^2+x+1)}{(x-1)(x+1)} = \frac{x^2+x+1}{x-1}\,.$$

At $x = -1$ this takes the value $\frac{1+1+1}{-2} = -\frac{3}{2}$.

Problem 37

Draw this circle in the x-y plane with its center at the origin. Then draw the two parallel chords at $y = +4$ and $y = -4$. We then ask where do these two chords intersect the circle. If we put $y = +4$ into the equation for the circle

$$x^2 + y^2 = 8^2\,,$$

and solve for x we get $x = \pm 4\sqrt{3}$. Let the "cap" of the circle be the region above the line $y = +4$. This "cap" sits over a triangle that is two triangles when split into two by the y axis. These two smaller triangles are right triangles with legs of length 4 and $4\sqrt{3}$ and thus an acute angle of

$$\theta = \tan^{-1}\left(\frac{4\sqrt{3}}{4}\right) = \tan^{-1}(\sqrt{3}) = 60^\circ = \frac{\pi}{3}\,.$$

Thus the central arc of the cap covers and angular range of twice this or

$$\frac{2\pi}{3}\,.$$

This sector of the circle will have an area of

$$\left(\frac{\frac{2\pi}{3}}{2\pi}\right) A_{\text{circle}} = \frac{1}{3} A_{\text{circle}} = \frac{1}{3}\pi r^2 = \frac{64\pi}{3}\,.$$

As this is the area of the "cap" and the triangle it sits on top of a triangle with an area of

$$\frac{1}{2}(8\sqrt{3})4 = 16\sqrt{3}\,.$$

This means that the area of the cap is

$$\frac{64\pi}{3} - 16\sqrt{3}\,.$$

Using this the area of the section we want to measure is given by

$$\pi r^2 - 2\left(\frac{64\pi}{3} - 16\sqrt{3}\right) = 32\sqrt{3} + \frac{64\pi}{3}\,,$$

when we simplify.

Problem 38

The area of a trapezoid is given by

$$A = \frac{1}{2}h(b_1 + b_2)\,,$$

where b_1 and b_2 are the lengths of the two bases. From what we are told this means that

$$1400 = \frac{1}{2}(50)(b_1 + b_2)\,,$$

which we can write as

$$b_1 + b_2 = 56\,.$$

If both bases are multiples of eight we can write $b_1 = 8n_1$ and $b_2 = 8n_2$ so that the above is equivalent to

$$n_1 + n_2 = 7\,.$$

The integer solutions that satisfy this (and give unique bases) are

$$(n_1, n_2) \in \{(1, 6), (2, 5), (3, 4)\}\,.$$

Thus there are three solutions.

Problem 39

If l is the length and w is the width of the rectangle then we are told that

$$2l + 2w = p \quad (49)$$

$$l^2 + w^2 = d^2\,. \quad (50)$$

For this problem we want $l - w$ in terms of p and d. From Equation 49 we have

$$l = \frac{p}{2} - w\,,$$

which we can put into Equation 50 to get

$$\left(\frac{p}{2} - w\right)^2 + w^2 = d^2\,.$$

We can expand the above to get a quadratic in w which is

$$w^2 - \frac{p}{2}w - \frac{d^2}{2} + \frac{p^2}{8} = 0\,.$$

Solving this I find

$$w = \frac{1}{4}(p \pm \sqrt{8d^2 - p^2})\,. \quad (51)$$

This means that $l = \frac{p}{2} - w$ is given by

$$l = \frac{p}{4} \mp \frac{1}{4}\sqrt{8d^2 - p^2}\,.$$

We seek to evaluate the argument of the square root above i.e. $8d^2 - p^2$. Now as $d^2 = l^2 + w^2$ we have $8d^2 = 8l^2 + 8w^2$ so that

$$\begin{aligned} 8d^2 - p^2 &= 8l^2 + 8w^2 - (2l + ww)^2 \\ &= 4(l - w)^2\,, \end{aligned}$$

when we simplify. Without loss of generality we can take $l > w$ so that

$$\sqrt{8d^2 - p^2} = 2(l - w)\,.$$

Note that if we take the plus sign the right-hand-side of Equation 51 gives

$$\frac{1}{4}(2l + 2w + 2(l - w)) = l\,,$$

which is a contradiction while if we take the minus sign we get

$$\frac{1}{4}(2l + 2w - 2(l - w)) = w\,,$$

which is consistent with the right-hand-side. Thus we now know the consistent expressions are

$$\begin{aligned} w &= \frac{1}{4}(p - \sqrt{8d^2 - p^2}) \\ l &= \frac{1}{4}(p + \sqrt{8d^2 - p^2})\,. \end{aligned}$$

This means that

$$l - w = \frac{1}{2}\sqrt{8d^2 - p^2}\,.$$

Problem 40

If $f(x)$ "sampled" at adjacent points x_i and x_{i+1} then the *difference* in these two samples is

$$\begin{aligned} f(x_{i+1}) - f(x_i) &= (ax_{i+1}^2 + bx_{i+1} + c) - (ax_i^2 + bx_i + c) \\ &= a(x_{i+1}^2 - x_i^2) + b(x_{i+1} - x_i) \\ &= a(x_{i+1} + x_i)(x_{i+1} - x_i) + b(x_{i+1} - x_i)\,. \end{aligned}$$

As the two points are the same distance apart for each i we have

$$x_{i+1} - x_i = h\,,$$

and the above becomes

$$f(x_{i+1}) - f(x_i) = ah(x_{i+1} + x_i) + bh\,.$$

Again with $x_{i+1} = x_i + h$ this is

$$f(x_{i+1}) - f(x_i) = 2ahx_i + ah^2 + bh\,.$$

This is linear function of x_i. This means that this difference must increase by the *same* amount as we increase i. Taking the differences of the numbers given we get

```
[1] 125 127 131 129 133 135 137
```

From the first two and the last four numbers it looks like these differences should increase by two as we "move to the right". Notice that if the third and fourth differences were

```
  129 131
```

all of the differences would increase by two at each step. Thus the differences

$$f(x_4) - f(x_3) = 131$$
$$f(x_5) - f(x_4) = 129\,,$$

must be wrong. This means that $f(x_4)$ is "too large" by two and we should have had

$$4227 \to 4225\,.$$

Problem 41

Let r be the initial radius and h the initial altitude (height) so that the original cylinder has a volume of

$$V = \pi r^2 h\,.$$

From the problem statement we are told that

$$\pi(r+6)^2 h = \pi r^2 h + y \quad (52)$$
$$\pi r^2 (h+6) = \pi r^2 h + y\,. \quad (53)$$

If we expand and simplify Equation 52 we get

$$12\pi r h + 36\pi h = y\,, \quad (54)$$

while simplifying Equation 53 gives

$$6\pi r^2 = y\,. \quad (55)$$

If we subtract Equation 54 from Equation 55 we get

$$6\pi r^2 - 24r - 36\pi h = 0\,.$$

As we know that $h = 2$ this becomes

$$r^2 - 4r - 12 = 0\,.$$

The only positive solution is $r = 6$.

Problem 42

(A) is correct and

$$10^r D = P.QQQ \cdots ,$$

so (B) is correct. In the same way we see that (C) and (E) are correct. For (D) note that the left-hand-side is

$$(10^{r+s} - 10^r)D = PQ.QQQ \cdots - P.QQQ \cdots = PQ - P = P(Q-1) ,$$

since the fractional parts cancel in the above subtraction. Note that this is not the expression given in the problem statement.

Problem 43

Let the original diameter be d. The diameter of each semi-circle is $\frac{d}{n}$ (and so the radius is $r = \frac{d}{2n}$). One of the small semi-circles has an arc length of

$$\pi \left(\frac{d}{2n} \right) .$$

The sum of these n semi-circles is then

$$\frac{\pi d}{2} .$$

Note that this is equal to the semi-circumference of the original circle.

Problem 44

Let $N = ab$ where the digits a and b are such that $1 \le a \le 9$ and $0 \le b \le 9$. Then we are told that

$$N = 10a + b = k(a+b) . \tag{56}$$

We want to write $N' = ba$ in terms of $a + b$ where

$$N' = ba = 10b + a . \tag{57}$$

From Equation 56 we have

$$9a + (a+b) = k(a+b) \quad \text{so} \quad a = \frac{(k-1)(a+b)}{9} . \tag{58}$$

Also from Equation 56 we have

$$(10-k)a = (k-1)b \quad \text{or} \quad b = \frac{(10-k)a}{9} .$$

Using Equation 58 for a in this we get

$$b = \frac{(10-k)(a+b)}{9} .$$

We now have both a and b in terms of $a+b$. Then from Equation 57 we have

$$\begin{aligned} N' = 10b + a &= 10\left(\frac{(10-k)(a+b)}{9}\right) + \frac{(a+b)(k-1)}{9} \\ &= (11-k)(a+b)\,, \end{aligned}$$

when we simplify.

Problem 45

From the arithmetic-geometric mean inequality one should already know that

$$\sqrt{ab} \le \frac{a+b}{2}\,.$$

This eliminates (A) and (B). To decide where $\frac{2ab}{a+b}$ falls let $a=1$ and $b=4$ so that

$$\begin{aligned} \frac{a+b}{2} &= \frac{5}{2} = 2.5 \\ \sqrt{ab} &= 2 \\ \frac{2ab}{a+b} &= \frac{8}{5} = 1.8\,. \end{aligned}$$

Thus the order should be

$$\frac{2ab}{a+b} < \sqrt{ab} < \frac{a+b}{2}\,.$$

Problem 46

Let the original rectangle have length l and width w with $l > w$. Then the length and width of the "new" rectangle are given by

$$\begin{aligned} l' &= \sqrt{l^2+w^2} + l \\ w' &= \sqrt{l^2+w^2} - l\,. \end{aligned}$$

The new area is then

$$A' = l'w' = (\sqrt{l^2+w^2}+l)(\sqrt{l^2+w^2}-l) = (l^2+w^2) - l^2 = w^2\,,$$

which is (C).

Problem 47

First we take the $1/x$ power of the expression $z^x = y^{2x}$ to get $z = y^2$. Next write the second equation as

$$2^z = 2^{2x+1}\,,$$

or taking the $\log_2(\cdot)$ of both sides we get

$$z = 2x + 1\,,$$

or

$$x = \frac{z-1}{2} = \frac{y^2-1}{2}\,.$$

If we put these two into $x + y + z = 16$ we can get a single equation for y as

$$\frac{y^2-1}{2} + y + y^2 = 16\,.$$

We can simplify that to

$$3y^2 + 2y - 33 = 0\,,$$

or

$$(3y+11)(y-3) = 0\,.$$

Thus the only integer solution is $y = 3$. This means that

$$x = \frac{y^2-1}{2} = \frac{9-1}{2} = 4\,,$$

and $z = y^2 = 9$. Thus $(x, y, z) = (4, 3, 9)$.

Problem 48

Without loss of generality let $v_1 > v_2 > 0$ so the first cyclist is the faster one. If we start with the first cyclist at $x = 0$ and the second cyclist at $x = k$ then when both move in the same direction their positions are given by

$$x_1(s) = v_1 s + 0$$
$$x_2(s) = v_2 s + k\,.$$

We are told that if $s = r$ then $x_1 = x_2$ or

$$v_1 r = v_2 r + k\,. \tag{59}$$

If they move in opposite directions then we can take the position of x_2 as

$$x_2(s) = -v_2 s + k\,.$$

Then $x_1 = x_2$ when $s = t$ means that

$$v_1 t = -v_2 t + k\,. \tag{60}$$

From Equation 59 we get that

$$v_1 = v_2 + \frac{k}{r}\,. \tag{61}$$

If we put that into Equation 60 we get that

$$v_2 t + \frac{kt}{r} = -v_2 t + k\,.$$

Solving for v_2 we find

$$v_2 = \frac{k(r-t)}{2tr}\,.$$

Then for v_1 using Equation 61 we get

$$v_1 = \frac{k(r+t)}{2tr}\,,$$

when we simplify. These mean that

$$\frac{v_1}{v_2} = \frac{r+t}{r-t}\,.$$

Problem 50

Call this expression L. Then we can write

$$L = 1 + \sqrt{2}\sum_{k=1}^{\infty}\left(\frac{1}{4}\right)^k + \sum_{k=1}^{\infty}\left(\frac{1}{4}\right)^k\,.$$

Lets now evaluate

$$\begin{aligned} S &= \sum_{k=1}^{\infty}\left(\frac{1}{4}\right)^k = \sum_{k=0}^{\infty}\left(\frac{1}{4}\right)^{k+1} = \frac{1}{4}\sum_{k=0}^{\infty}\left(\frac{1}{4}\right)^k \\ &= \frac{1}{4}\left(\frac{1}{1-\frac{1}{4}}\right) = \frac{1}{4-1} = \frac{1}{3}\,. \end{aligned}$$

This means that

$$L = 1 + \frac{\sqrt{2}}{3} + \frac{1}{3} = \frac{1}{3}(4+\sqrt{2})\,.$$

The 1953 Examination

Problem 1

From the problem the boy pays

$$b = \frac{10}{3},$$

cents per orange. He then sells them for

$$s = \frac{20}{5} = 4,$$

cents per orange. The profit per orange is then

$$4 - \frac{10}{3} = \frac{2}{3}.$$

To have a total profit of 1.0 means that he needs to buy and sell N oranges where

$$\frac{2}{3}N = 100 \quad \text{or} \quad N = 150.$$

Problem 2

From what we are told the sales price of the refrigerator is given by

$$\begin{aligned} P &= 250(1 - 0.2)(1 - 0.15) = 250(1 - 0.15 - 0.2 + 0.03) \\ &= 250(1 - 0.32) = 250(0.68). \end{aligned}$$

This is 68% of the original price.

Problem 3

This would be $(x + iy)(x - iy)$.

Problem 4

Two roots are $x = 0$ and $x = 4$. The other roots must solve $x^2 + 8x + 16 = 0$. We can write this as

$$(x + 4)^2 = 0,$$

so $x = -4$ with algebraic multiplicity of two.

Problem 5

Taking $f(x) = 6^x$ to both sides we can write this as $x = 6^{2.5} = 36\sqrt{6}$.

Problem 6

Each quarter is $2.5 = \frac{5}{2}$ dimes. This means that the difference is

$$\frac{5}{2}(5q + 1) - \frac{5}{2}(q + 5) = 10(q - 1)\,.$$

Problem 7

If we multiply this expression by $\frac{\sqrt{a^2+x^2}}{\sqrt{a^2+x^2}}$ we get

$$\frac{a^2 + x^2 - (x^2 - a^2)}{(a^2 + x^2)\sqrt{a^2 + x^2}} = \frac{2a^2}{(a^2 + x^2)^{3/2}}\,.$$

Problem 8

The intersection of these two curves happens when

$$\frac{8}{x^2 + 4} = 2 - x\,,$$

or

$$8 = (2 - x)(x^2 + 4)\,,$$

or

$$x(x^2 - 2x + 4) = 0\,.$$

This has solutions $x = 0$ or

$$x = \frac{2 \pm 2\sqrt{-3}}{2} = 1 \pm \sqrt{3}i\,.$$

Problem 9

Nine ounces of lotion at 50% alcohol has

$$9\left(\frac{1}{2}\right) = \frac{9}{2}\,,$$

ounces of alcohol and $\frac{9}{2}$ ounces of water. If we add x ounces of water we will get a lotion with a

$$\frac{9/2}{9/2+9/2+x} = \frac{9/2}{9+x},$$

fraction of alcohol. To have this equal 30% = 0.3 will happen when $x = 6$.

Problem 10

Each revolution covers

$$2\pi r = \pi d = 6\pi,$$

feet. To revolve 5280 feet we need

$$\frac{5280}{6\pi} = \frac{880}{\pi},$$

revolutions.

Problem 11

Let r_0 the the inner radius and r_1 the outer radius. Then to have its width be 10 feet wide we need to have $r_1 = r_0 + 10$. The difference in circumferences is then

$$\begin{aligned} C_1 - C_0 &= 2\pi r_1 - 2\pi r_0 = 2\pi(r_1 - r_0) = 2\pi(10) = 20\pi \\ &\approx 20(3.14) = 62.8\,. \end{aligned}$$

Problem 12

This ratio would be

$$\frac{\pi\left(\frac{8}{2}\right)^2}{\pi\left(\frac{12}{2}\right)^2} = \frac{4^2}{6^2} = \frac{4}{9}\,.$$

Problem 13

Let h be the common altitude. Then the area of the triangle is

$$\frac{1}{2}bh = \frac{1}{2}(18)h = 9h\,.$$

The area of the trapezoid is

$$\frac{h}{2}(b_1 + b_2)\,,$$

where b_1 and b_2 are the two bases. Setting these two expressions equal (and dividing by h) we get

$$\frac{1}{2}(b_1 + b_2) = 9\,.$$

The expression on the left is the median of the trapezoid.

Problem 14

I imagine P and Q on the x-axis of a Cartesian coordinate plane. Then (A) can be true if we take $p = 2$ and $q = 1$ with P at the origin $(0, 0)$ and Q at $(1, 0)$.

Now (B) can be true if P and Q are externally tangent i.e. P has $p = 2$ located at $(0, 0)$ and Q has $q = 1$ located at $(3, 0)$.

Now (C) can be true if P and Q are "separated" i.e. P has $p = 2$ located at $(0, 0)$ and Q has $q = 1$ located at $(5, 0)$.

Now (D) can be true if Q is "inside" Q i.e. P has $p = 2$ located at $(0, 0)$ and Q has $q = 1$ located at $(2, 0)$.

Problem 15

Let the original square have a side of length s. Then the radius of the inscribed circle is $\frac{s}{2}$ and so the amount of "waste" when we cut out the first circle is

$$s^2 - \pi\left(\frac{s}{2}\right)^2\,.$$

Now we have a circle of radius $\frac{s}{2}$. From this we cut out a square. The diagonal of the square that we cut out will have a length equal to the diameter of the circle or $2\left(\frac{s}{2}\right) = s$. Let the sides of this square be l. Then by the Pythagorean theorem we must have

$$l^2 + l^2 = s^2 \quad \text{so} \quad l = \frac{s}{\sqrt{2}}\,.$$

Thus the square we cut out has an area of

$$\frac{s^2}{2}\,.$$

This gives an amount of "waste" of

$$\pi\left(\frac{s}{2}\right)^2 - \frac{s^2}{2}\,.$$

The total "waste" removed from these two procedures is then

$$\left(s^2 - \pi\left(\frac{s}{2}\right)^2\right) + \left(\pi\left(\frac{s}{2}\right)^2 - \frac{s^2}{2}\right) = s^2 - \frac{s^2}{2} = \frac{s^2}{2}\,.$$

This is half of the area of the original square.

Problem 16

Let S be the sales price and B be the price that Adams buys the article for. Then the expenses E are $E = 0.15S$ and the profit should be $0.1S$. Adam's profit is the difference between the sales price S and what he pays for the article (plus his expenses). This means that

$$S - (B + E) = \text{Profit}\,.$$

From the above this means that

$$S - (B + 0.15S) = 0.1S\,.$$

This is equivalent to $B = 0.75S$. Then the mark-up to the sales price is

$$\frac{S - B}{B} = \frac{S - 0.75S}{0.75S} = \frac{1}{3}\,.$$

This is the percent $33\frac{1}{3}\%$.

Problem 17

Let x be the fraction of \$4500 that is invested at 4% so that $1 - x$ is the fraction of \$4500 that is invested at 6%. As we are told that both investments yield the same amount we have

$$4500x(0.04) = 4500(1 - x)(0.06)\,.$$

If we solve this for x we find $x = 0.6$ or 60%. This means that each investment made $4500(0.6)(0.04) = 108$ for a total of $2 \times 108 = 216$. This is a total return of

$$\frac{216}{4500} = 0.048\,,$$

or 4.8%.

Problem 18

We can write this as

$$\begin{aligned} x^4 + 4 &= (x^4 + 4x^2 + 4) - 4x^2 \\ &= (x^2 + 2)^2 - 4x^2 = (x^2 + 2 - 2x)(x^2 + 2 + 2x) \\ &= (x^2 - 2x + 2)(x^2 + 2x + 2)\,. \end{aligned}$$

Problem 19

Lets define $E = xy^2$. Then if we assume that

$$\begin{aligned} x &\to 0.75x = \frac{3}{4}x \\ y &\to 0.75y = \frac{3}{4}y\,, \end{aligned}$$

so that E becomes

$$\frac{3}{4}x\left(\frac{9}{16}y^2\right) = \frac{27}{64}xy^2\,.$$

This means that E has been decreased by

$$xy^2 - \frac{27}{64}xy^2 = \frac{37}{64}xy^2\,.$$

Problem 20

From the expression for y we have

$$y^2 = x^2 + 2 + \frac{1}{x^2}\,.$$

Thus

$$x^2y^2 = x^4 + 2x^2 + 1\,. \tag{62}$$

Also from the definition of y we have

$$xy = x^3 + x\,. \tag{63}$$

Adding Equations 62 and 63 we get

$$x^2(y^2 + y) = x^4 + 2x^2 + 1 + x^3 + x = x^4 + x^3 + 2x^2 + x + 1\,.$$

This is almost the polynomial we are told equals zero. If we subtract $-6x^2$ from both sides we *do* get that polynomial or

$$x^2(y^2 + y) - 6x^2 = x^4 + x^3 - 4x^2 + x + 1 = 0\,.$$

This means that

$$x^2(y^2 + y - 6) = 0\,.$$

Problem 21

Lets take 10^x of both sides to get

$$x^2 - 3x + 6 = 10\,,$$

or

$$x^2 - 3x - 4 = 0\,,$$

or

$$(x-4)(x+1) = 0\,.$$

This has solutions $x = 4$ or $x = -1$.

Problem 22

Call this expression E. Then we can write E as

$$E = 27 \cdot \sqrt[4]{9} \cdot \sqrt[3]{9} = 3^3 \cdot 3^{2/4} \cdot 3^{2/3}\,.$$

Now as

$$3 + \frac{1}{2} + \frac{2}{3} = \frac{25}{6}\,,$$

we have

$$\log_3(E) = \frac{25}{6} = 4\,\frac{1}{6}\,.$$

Problem 23

Let $v = \sqrt{x+10}$ then this expression is

$$v - \frac{6}{v} = 5\,.$$

If we multiply by v and bring everything to one side we get

$$v^2 - 5v - 6 = 0\,.$$

We can factor this to write

$$(v-6)(v+1) = 0\,.$$

This means that $v = 6$ or $v = -1$. As $v > 0$ only $v = 6$ is a solution. In this case then

$$\sqrt{x+10} = 6 \quad \text{so} \quad x = 26\,.$$

If you "thought" that $v = -1$ was a possible solution then x would need to satisfy

$$\sqrt{x+10} = -1\,.$$

If we square this we get

$$x + 10 = 1 \quad \text{so} \quad x = -9\,.$$

This is an extraneous root.

Problem 24

We are told that

$$(10a+b)(10a+c) = 100a(a+1) + bc\,.$$

If we expand both sides we get

$$100a^2 + 10ac + 10ab + bc = 100a^2 + 100a + bc\,,$$

or

$$10(ac+ab) = 100a\,,$$

or

$$c + b = 10\,.$$

Problem 25

The terms of this sequence take the form $a_0 r^{k-1}$ for $k \geq 1$. We are told that

$$a_0 r^{k-1} = a_0 r^k + a_0 r^{k+1}\,.$$

We can divide by $a_0 r^{k-1} \neq 0$ to get

$$1 = r + r^2 \quad \text{or} \quad r^2 + r - 1 = 0\,.$$

This has solutions

$$r = \frac{-1 \pm \sqrt{5}}{2}\,.$$

As $r > 0$ we must take the positive sign above.

Problem 27

We are told that the radii of each circle is given by

$$r_n = \left(\frac{1}{2}\right)^{n-1}\,,$$

for $n \geq 1$. The area for the nth circle is then

$$A_n = \pi r_n^2 = \pi \left(\frac{1}{2}\right)^{2(n-1)} = \pi \left(\frac{1}{4}\right)^{n-1}\,.$$

The sum we are asked about is

$$\begin{aligned}\sum_{n=1}^{\infty} A_n = \sum_{n=1}^{\infty} \pi \left(\frac{1}{4}\right)^{n-1} &= \pi \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n \\ &= \pi \frac{1}{1-\frac{1}{4}} = \frac{4\pi}{3}\,.\end{aligned}$$

Problem 28

I draw the triangle $\triangle ABC$ with the segment BC on an x-axis and A above this axis in the first quadrant of a Cartesian coordinate plane and use the suggested labeling to label the sides.

As AD is a bisector of $\angle A$ then from the *angle bisector theorem* we have

$$\frac{c}{y} = \frac{b}{x} \quad \text{or} \quad \frac{x}{y} = \frac{b}{c} \,. \tag{64}$$

From this we see that (E) is wrong.

Lets use the above to solve for x in terms of a, b, and c. Using Equation 64 and the fact that $x + y = a$ we have that

$$x = \frac{b}{c} y = \frac{b}{c}(a - x) \,.$$

Solving this for x gives

$$x = \frac{ab}{c + b} \,.$$

Thus (A) and (B) are wrong.

Next lets use the above to solve for y in terms of a, b, and c. Using Equation 64 and the fact that $x + y = a$ we have that

$$y = \frac{c}{b} x = \frac{c}{b}(a - y) \,.$$

Solving this for y gives

$$y = \frac{ac}{c + b} \,.$$

This is (D).

Problem 29

Let s be the length of the side of the square. Then we would like to know for what value of s is $s^2 = 1.1025$. Thus

$$s = (1 + 0.1025)^{1/2} \,.$$

To evaluate this we recall the Taylor series for $(1 + x)^{1/2}$ we have

$$\begin{aligned}
(1 + x)^{1/2} &= 1 + \frac{x}{2} + \binom{1/2}{2} x^2 + \binom{1/2}{3} x^3 + \cdots \\
&= 1 + \frac{x}{2} + \frac{1/2(1/2 - 1)}{2} x^2 + \frac{(1/2)(1/2 - 1)(1/2 - 2)}{6} x^3 + \cdots \\
&= 1 + \frac{x}{2} + \frac{1}{4}\left(-\frac{1}{2}\right) x^2 + \frac{1}{4 \cdot 6}\left(\frac{3}{2}\right) x^3 + \cdots \\
&= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{1}{16} x^3 + \cdots \,.
\end{aligned}$$

Taking $x = 0.1025$ we get

$$(1 + 0.1025)^{1/2} = 1 + 0.05125 - 0.001313281 + O(10^{-5})\,.$$

If we take the first *two* terms in the above series we have $s = 1.05125$ so that $s^2 = 1.105127$ which is larger than the goal of 1.1025. Thus we need to make s smaller. If we take the first *three* terms in the above series we have $s = 1.049937$ so that $s^2 = 1.102367$ which is larger than the target of 1.1025. Thus we need to make s larger. Taking the first *four* terms we get $s = 1.050004$ which gives $s^2 = 1.102508$ and is slightly too large.

Based on this we might try $s = 1.05$ for which we find $s^2 = 1.1025$ as we desire.

Problem 30

The price the house is sold to Mr. B is

$$9000(0.9) = 8100\,.$$

This is a 10% loss for Mr. A. Mr. B then sells this property back to Mr. A at a price of

$$8100(1.1) = 8910\,.$$

This means that Mr. B makes a profit of

$$8910 - 8100 = 810\,,$$

on that trade. Mr. A made a profit of

$$8100 - 8910 = -810\,,$$

which is negative so Mr. A made a loss of \$810 dollars.

Problem 31

Each rail is

$$30\left(\frac{1}{5280}\right) = \frac{3}{528}\,,$$

miles long. The time (in hours) it takes to hear n clicks (where we transverse $n - 1$ rails) is

$$T_h = \frac{3(n-1)}{528v}\,,$$

where v is the velocity of the train in miles-per-hour. This means that

$$v = \frac{3(n-1)}{528T_h}\,.$$

If T_h is in hours then

$$T_m = 60T_h\,,$$

is the time in minutes. This means that

$$v = \frac{3(n-1)}{528\left(\frac{T_m}{60}\right)}\,,$$

where now T_m is in minutes. The above is equivalent to

$$v = \frac{180(n-1)}{528T_m}\,.$$

If we want $v \approx n$ then from the above we need

$$\frac{180}{528T_m} \approx 1\,.$$

This means that

$$T_m = \frac{180}{528} = 0.34\,,$$

minutes or $0.34(60) \approx 20$ seconds.

Problem 33

Let our isosceles right triangle have legs of length s. Then the hypotenuse has a length of $\sqrt{2}s$. The perimeter is then of length $2s + \sqrt{2}s$ so

$$2p = 2s + \sqrt{2}s \quad \text{so} \quad s = \left(\frac{2}{2+\sqrt{2}}\right)p\,.$$

Next we will lie the triangle with its hypotenuse along an x-axis (so that the right angle is "above" the hypotenuse) and drop an altitude from the right angle onto the hypotenuse. Then as the acute angles of the isosceles right triangle are 45° this altitude will have a length given by

$$s\sin\left(\frac{\pi}{4}\right) = \frac{s}{\sqrt{2}}\,.$$

Using this the area of this triangle is given by

$$\begin{aligned} A &= \frac{1}{2}bh = \frac{1}{2}(\sqrt{2}s)\left(\frac{s}{\sqrt{2}}\right) = \frac{s^2}{2} \\ &= \frac{2p^2}{(2+\sqrt{2})^2} = \frac{2p^2(2-\sqrt{2})^2}{(2+\sqrt{2})^2(2-\sqrt{2})^2} \\ &= \frac{(2-\sqrt{2})^2p^2}{2} = (3-2\sqrt{2})p^2\,, \end{aligned}$$

when we expand and simply.

Problem 34

Let the triangle be $\triangle ABC$ with the length of AC be twelve. Draw the circumscribed circle and note that it passes though A, B, and C. Connect the points C and A to the origin of the circumscribed circle denoted O each of these segments has a length r the radius of the circle.

Now as $\angle ABC = 30°$ we know that the arc length of AC is 60. From this arc length we have that $\angle AOC = 60$. Then from the fact that triangle AOC has $AO = OC = r$ we have that $\angle OAC = \angle ACO = \angle AOC = 60°$ and thus $\triangle AOC$ is equilateral so that $AO = OC = r = AC = 12$. This means that the diameter is $2r = 24$.

Problem 35

We have

$$f(x+2) = \frac{(x+2)(x+1)}{2} = \frac{(x+2)(x+1)x}{2x} = \frac{(x+2)f(x+1)}{x}.$$

Problem 36

Method 1: Synthetic division of our polynomial with respect to $x = 3$ gives

$$\begin{array}{c|ccc} 3 & 4 & -6 & m \\ & & 12 & 18 \\ \hline & 4 & 6 & m+18 \end{array}$$

To be divisible by $x - 3$ means that $m + 18 = 0$ so $m = -18$.

Method 2: Evaluating this polynomial at $x = 3$ we must get zero. This means that

$$4(9) - 18 + m = 0 \quad \text{so} \quad m = -18.$$

Of the choices given only 36 has -18 as an exact divisor.

Problem 37

Draw the triangle with the six inch "base" on the horizontal x-axis and the twelve inch "sides" above it. Drop an altitude from the vertex to the base through the center of the circumscribing circle. This forms a right triangle with a hypotenuse of twelve, a leg of length $\frac{6}{2} = 3$ and another leg the altitude of the triangle h. Using the Pythagorean theorem we can write this as

$$h = \sqrt{12^2 - 3^2} = \sqrt{135}.$$

As this altitude passes though the center of the circumscribing circle it is of length $r + x$ where r is the radius of the circle and x is the remaining distance we have

$$\sqrt{135} = r + x \,.$$

Drawing another radius from the center of the circle to the base of the triangle we form another right triangle with a hypotenuse of length r and legs of length 3 and x. Thus again using the Pythagorean theorem we get

$$r^2 = 3^2 + x^2 \,.$$

This gives two equations and two unknowns which we can solve. If we square the first expression and put it into the second we find

$$x = \frac{7\sqrt{15}}{5} \,.$$

Putting this into the above we get

$$r = \sqrt{135} - x = \frac{8\sqrt{15}}{5} \,.$$

Problem 38

We have $f(4) = 4 - 2 = 2$ so that

$$F[3, f(4)] = F[3, 2] = 2^2 + 3 = 7 \,.$$

Problem 39

We can write this in terms of a common logarithm as

$$\frac{\log(b)}{\log(a)} \times \frac{\log(a)}{\log(b)} = 1 \,.$$

Problem 40

The negation of this statement is "some men are dishonest".

Problem 41

Let O be the origin of a Cartesian coordinate system. Let the x-axis be the straight road so that the girls' camp G is located on the y axis at the point $(0, 300)$. The boys' camp B

is located along this road at a point $(b, 0)$ such that $GB = 500$ (rods). We want to put the canteen at $C = (c, 0)$ such that $GC = CB$.

Now as $\triangle GOB$ is a right triangle we know that

$$OB = \sqrt{GB^2 - OG^2} = \sqrt{500^2 - 300^2} = 400\,.$$

Let $OC = x$ then $CB = 400 - x$ and using the right triangle $\triangle GOC$ we have

$$GC = \sqrt{300^2 + x^2}\,.$$

To have $GC = CB$ means that

$$\sqrt{300^2 + x^2} = 400 - x\,.$$

If we square this we get

$$300^2 + x^2 = 400^2 - 800x + x^2 \quad \text{so} \quad x = 87.5\,.$$

This means that $CB = 400 - x = 312.5$.

Problem 42

Draw the two circles on the x-axis. Let the first circle with a radius of four have its center at the point $A = (0, 0)$ and the circle with a radius of five have it center at $B = (41, 0)$. I drew the common internal tangent in the North-East direction and let the points of tangency for circle A and circle B be denoted as T_A and T_B respectively. Note that there would be another common internal tangent going in the South-East direction. Next draw the two lines AT_A and BT_B. Because the segment T_AT_B is tangent to both circle A and circle B this segment T_AT_B is perpendicular to the lines AT_A and BT_B. Finally, if we extend the segment AT_A five units further (the radius of the B circle) we get a point C.

With this diagram note that in $\triangle ACB$ we have $\angle ACB = 90°$ and so using the Pythagorean theorem we have that

$$AC^2 + CB^2 = AB^2\,,$$

or as $AC = 4 + 5 = 9$ and $AB = 41$ we have

$$9^2 + CB^2 = 41^2 \quad \text{so} \quad CB = 40\,.$$

As $CB = T_AT_B = 40$ the length of the common internal tangent is forty.

Problem 43

Assume p and d are numbers given as fractions i.e. 30% is $p = 0.3$. Then of A is the original price of the article the new sales price (after the increase) is

$$S = A(1 + p)\,.$$

If the number of sales N decreases by d then the number of sales is now

$$N(1-d)\,.$$

The income now is $A(1+p)N(1-d)$ and we want this to be equal to the original income AN. Thus

$$AN(1+p)(1-d) = AN\,.$$

Solving this for d we get

$$d = \frac{p}{1+p}\,.$$

The right-hand-side of the above is the value that d cannot exceed.

Problem 44

The first wrong equation must be

$$(x-8)(x-2) = 0\,,$$

or

$$x^2 - 10x + 16 = 0\,,$$

where here the 16 is wrong. The second wrong equation must be

$$(x+9)(x+1) = 0\,,$$

or

$$x^2 + 10x + 9 = 0\,,$$

where here the 10 is wrong. Thus the correct equation must be

$$x^2 - 10x + 9 = 0\,.$$

Problem 45

This is the Arithmetic-Geometric mean inequality which is

$$\sqrt{ab} \le \frac{1}{2}(a+b)\,. \qquad (65)$$

If you ever forget the direction of the inequality you can "check" it by taking specific numbers. For example if we take $a = 1$ and $b = 4$ the above is

$$2 \le \frac{1}{2}(5) = 2.5\,.$$

Problem 46

Let the length of the long side be l, the length of the short side be w, and the length of the diagonal be d. Then we are told that

$$l + w - d = \frac{1}{2} l \,.$$

From the Pythagorean theorem we have $d = \sqrt{l^2 + w^2}$ so that the above is

$$\frac{l}{2} + w = \sqrt{l^2 + w^2} \,. \tag{66}$$

We want to know the value of $\frac{w}{l}$. If we divide the above equation by l we get

$$\frac{1}{2} + \frac{w}{l} = \sqrt{1 + \left(\frac{w}{l}\right)^2} \,.$$

Squaring this we get

$$1 + \left(\frac{w}{l}\right)^2 = \frac{1}{4} + \frac{w}{l} + \left(\frac{w}{l}\right)^2 \,,$$

from which we can solve to find $\frac{w}{l} = \frac{3}{4}$.

Problem 47

By plotting both $\log(1 + x)$ and x we see that $x > \log(1 + x)$. The Taylor series of $\log(1 + x)$ is

$$\log(1 + x) = \sum_{k \geq 1} \frac{(-1)^{k+1} x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

Taylor series with a remainder would say that

$$\log(1 + x) = x - \frac{\xi^2}{2} \,,$$

for $\xi \in [0, x]$. Thus we see that $\log(1 + x) < x$.

Problem 48

Let the short side be "on top" with a length of s and the long side be "on bottom" with a length t. Next drop two verticals from the corners of the short side onto the long side. These two verticals cut off two congruent right triangles with one leg the altitude of the trapezoid. Because this is an isosceles trapezoid the length of the other leg of the two right triangles is

$$\frac{t - s}{2} \,.$$

From the fact that the longer base length equals the diagonal by using the Pythagorean theorem we have

$$\text{diagonal}^2 = t^2 = s^2 + \left(s + \frac{t-s}{2}\right)^2 .$$

If we expand this and simplify we get

$$\frac{3}{4}t^2 = \frac{5}{4}s^2 + \frac{st}{2} .$$

As we want to know $\frac{s}{t}$ we divide the above by t^2 to get an equation for this ratio. That equation is

$$5\left(\frac{s}{t}\right)^2 + 2\left(\frac{s}{t}\right) - 3 = 0 .$$

Using the quadratic formula we find

$$\frac{s}{t} \in \left\{-1, \frac{3}{5}\right\} .$$

As this ratio must be positive we have $\frac{s}{t} = \frac{3}{5}$.

Problem 49

Let $f(k)$ be the total distance $AC + BC$ which we can write as

$$f(k) = \sqrt{(5-0)^2 + (5-k)^2} + \sqrt{(2-0)^2 + (1-k)^2} = \sqrt{25 + (5-k)^2} + \sqrt{4 + (1-k)^2} .$$

We can compute the extreme values of $f(k)$ by taking the derivative, setting it equal to zero, and solving for k. I find

$$f'(k) = \frac{2(5-k)(-1)}{2\sqrt{25 + (5-k)^2}} + \frac{2(1-k)(-1)}{2\sqrt{4 + (1-k)^2}} .$$

Setting this equal to zero we can write

$$\frac{5-k}{\sqrt{25 + (5-k)^2}} = \frac{-(1-k)}{\sqrt{4 + (1-k)^2}} .$$

If we square this and cross multiply we get

$$(5-k)^2(4 + (1-k)^2) = (1-k)^2(25 + (5-k)^2) .$$

If we expand everything and simplify we get the quadratic equation

$$21k^2 - 10k - 75 = 0 .$$

This has solutions $k = -\frac{35}{21}$ or $k = \frac{15}{7}$. As $k > 0$ the second one is the solution we seek.

Problem 50

Draw AC along an x-axis and B "above" the segment AC. Draw the inscribed circle and three radii (of length four) to the three sides of the triangle. Note that each of these radii will be perpendicular to the side of the triangle that it intersects. Let the segment AB be divided up into two lengths of eight and six. Then from the fact that two tangents to a circle from the same exterior point have that the same length we conclude that

- The segment AC is broken up into two segments of length eight and then some unknown length x.
- The segment BC is broken up into two segments of length six and then some unknown length x.

We will use Heron's formula to evaluate the area of this triangle and then equate it to the sum of the six right triangles that make it up.

To use Heron's formula we need the semi-perimeter

$$s = \frac{1}{2}(14 + (6 + x) + (8 + x)) = 14 + x\,.$$

Then we have

$$A^2 = s(s - 14)(s - (6 + x))(s - (8 + x)) = (14 + x)x(8)(6) = 48x(14 + x)\,.$$

Each of the six right triangles have an area given by one of

$$\frac{1}{2}8(4) = 16$$
$$\frac{1}{2}(4)x = 2x$$
$$\frac{1}{2}6(4) = 12\,.$$

Thus the area of the triangle as the sum of the six right triangles can be written as

$$2(16) + 2(2x) + 2(12) = 56 + 4x\,.$$

If we equate these two expressions we have

$$\sqrt{48x(x + 14)} = 56 + 4x\,.$$

If we square this and simplify we get

$$x^2 + 7x - 98 = 0\,,$$

which has roots $x = 7$ or $x = -14$. As $x > 0$ we have that $x = 7$. This means that the three sides of the triangle are of lengths 14, 15, and 13. Thus the smallest side is of length 13.

The 1954 Examination

Problem 1

Squaring this we find

$$(5 = \sqrt{y^2 - 25})^2 = 25 - 2 \cdot 5\sqrt{y^2 - 25} + (y^2 - 25)$$
$$= y^2 - 10\sqrt{y^2 - 25}\,.$$

Problem 2

Note that $x = 1$ is not a point where we can evaluate the given expression at because of the denominator $x - 1$ in many of the fractions. Thus the roots of the first equation can only be $x = 4$.

Problem 3

We are told that

$$x \propto y^3$$
$$y \propto z^{1/5}\,,$$

so that we have

$$x \propto z^{3/5}\,.$$

Problem 4

I find that

$$132 = 2^2 \cdot 3 \cdot 11\,,$$

and that

$$6432 = 2^5 \cdot 3 \cdot 67\,.$$

This means that the highest common divisor of these two numbers is

$$2^2 \cdot 3 = 12\,.$$

Reducing this by eight gives four.

Problem 5

A regular hexagon has six faces and connecting the center of the circumscribed circle to each of the "corners" of the regular hexagon we get six equilateral triangles each with a side of length $s = 10$. The area of a single equilateral triangle with this side length is

$$\frac{1}{2}(10)(5\sqrt{3}) = 25\sqrt{3}\,.$$

Thus six of these have an area of $150\sqrt{3}$.

Problem 6

Call this expression E. Then we have

$$E = \frac{1}{16} + 1 - \frac{1}{\sqrt{64}} - \frac{1}{32^{4/5}}\,.$$

Now $32^{1/5} = 2$ so $32^{4/5} = 16$ and we have

$$E = \frac{1}{16} + 1 - \frac{1}{8} - \frac{1}{16} = \frac{7}{8}\,.$$

Problem 7

The original price of the dress is $25 + 2.5 = 27.5$. Thus the percent savings is

$$\frac{2.5}{27.5} = \frac{25}{275} = \frac{1}{11} = 0.0909091\,,$$

about 9%.

Problem 8

The area of the square is s^2 with s the length of the side. The area of the triangle is

$$\frac{1}{2}bh = \frac{1}{2}(2s)h\,.$$

Setting these two equal we get $s = h$.

Problem 9

I drew the circle with its center O at the origin and the segment OP (of length thirteen) along the x-axis. Then from P I draw the chord RP which intersects the circle at R and also Q. We are told that $PQ = 9$ and $QR = 7$. First using the law of cosigns in the triangle $\triangle POQ$ we have

$$r^2 = 13^2 + 9^2 - 2(9)(13)\cos(\angle OPR)\,.$$

Next using the law of cosigns in the triangle $\triangle POR$ we have

$$r^2 = 13^2 + 16^2 - 2(13)(16)\cos(\angle OPR)\,.$$

These are two equations for the two unknowns r and $\cos(\angle OPR)$. Solving each for $\cos(\angle OPR)$ we have

$$\cos(\angle OPR) = \frac{r^2 - 250}{234} = \frac{r^2 - 425}{416}\,.$$

Solving this for r we get $r = 5$.

Problem 10

Recall that

$$(a+b)^6 = \sum_{k=0}^{6} \binom{6}{k} a^k b^{6-k}\,,$$

so if we take $a = b = 1$ we get

$$2^6 = \sum_{k=0}^{6} \binom{6}{k}\,.$$

Thus the sum is $2^6 = 64$.

Problem 11

Let the sales price be S, the marked price be M, and the cost be C. From the sign he posted the sales price will be

$$S = M - \frac{1}{3}M = \frac{2}{3}M\,.$$

We are told that

$$C = \frac{3}{4}S = \frac{3}{4}\left(\frac{2}{3}M\right) = \frac{1}{2}M\,.$$

Thus $\frac{C}{M} = \frac{1}{2}$.

Problem 12

This has no solution as the left-hand-side of the second equation implies

$$2(2x - 3y) = 2(7) = 14 \neq 20\,.$$

Problem 13

To solve this problem I first drew a quadrilateral with vertices A, B, C, and D inscribed in (i.e. on) a circle. I then drew the "angles" in each of the four arcs cut off by the sides of the quadrilateral. That is I drew a point N on the circle and on the arc between the two vertices A and B. In the same way we have

- O on the circle and on the arc between the two vertices B and C
- P on the circle and on the arc between the two vertices C and D
- M on the circle and on the arc between the two vertices D and A

Then we are asked to evaluate the angle sum

$$\angle M + \angle N + \angle O + \angle P\,.$$

If we relate each of these angles to the arcs that they "cut off" we have

$$\begin{aligned}
\angle M + \angle N + \angle O + \angle P &= \frac{1}{2}\widehat{ABCD} + \frac{1}{2}\widehat{ADCB} + \frac{1}{2}\widehat{BADC} + \frac{1}{2}\widehat{DABC} \\
&= \frac{1}{2}(360 - \widehat{AD}) + \frac{1}{2}(360 - \widehat{AB}) + \frac{1}{2}(360 - \widehat{BC}) + \frac{1}{2}(360 - \widehat{DC}) \\
&= \frac{4}{2}(360) - \frac{1}{2}(\widehat{AD} + \widehat{DC} + \widehat{BC} + \widehat{AB}) \\
&= 2(360) - \frac{1}{2}(360) = 540\,.
\end{aligned}$$

Problem 14

Call this expression E. Then we have

$$\begin{aligned}
E &= \sqrt{\frac{4x^4 + (x^4 - 1)^2}{(2x^2)^2}} = \frac{\sqrt{4x^4 + x^8 - 2x^4 + 1}}{2x^2} = \frac{\sqrt{8x^4 + 2x^4 + 1}}{2x^2} \\
&= \frac{\sqrt{(x^4 + 1)^2}}{2x^2} = \frac{x^4 + 1}{2x^2} = \frac{x^2}{2} + \frac{1}{2x^2}\,.
\end{aligned}$$

Problem 15

For this we have

$$\begin{aligned}
\log(125) &= \log\left(5^3\right) = 3\log(5) = 3\log\left(\frac{10}{2}\right) \\
&= 3(\log(10) - \log(2)) \\
&= 3(1 - \log(2)) = 3 - 3\log(2)\,,
\end{aligned}$$

assuming $\log(x)$ is the base ten logarithm.

Problem 16

For this we find

$$\begin{aligned} f(x+h) - f(x) &= 5(x+h)^2 - 2(x+h) - 1 - 5x^2 + 2x + 1 \\ &= 5(x^2 + 2xh + h^2) - 2x - 2h - 5x^2 + 2x = 10xh + 5h^2 - 2h \\ &= h(10x + 5h - 2)\,. \end{aligned}$$

Problem 17

This is a shift downwards (by seven) of the function $2x^3$ which goes from $-\infty$ on the left to $+\infty$ on the right.

Problem 18

We have

$$2x - 3 > 7 - x\,,$$

or

$$3x > 10\,,$$

or $x > \frac{10}{3}$.

Problem 19

Draw the triangle $\triangle ABC$ with AB along the x-axis of a Cartesian coordinate plane with the point C "above" the segment AB. Let the points of contact of the inscribed circle with the three sides be named A', B', and C' where A' is on the triangles side opposite the point A, B' is on the triangles side opposite the point B, and C' is on the triangles side opposite the point C. Note that these three points A', B', and C' cut the inscribed circle into three arcs which we will denote a, b, and c. The arc a is opposite the point A', the arc b is opposite the point B', and finally the arc c is opposite the point C'.

Then from the fact that A' is on a circle forming an angle that cuts an arc of length a we have

$$\angle A' = \frac{a}{2}\,.$$

From the fact that A is an angle with its vertex outside of a circle we have

$$\angle A = \frac{1}{2}(c + b - a)\,.$$

As all arc lengths in a circle must sum to 360 we have $c + b = 360 - a$ so the above becomes

$$\angle A = \frac{1}{2}(360 - 2a) = 180 - a\,,$$

or

$$a = 180 - \angle A\,.$$

We have $\angle A'$ and $\angle A$ related via a and thus

$$\angle A' = \frac{a}{2} = \frac{1}{2}(180 - \angle A) = 90 - \frac{1}{2}\angle A\,.$$

From this we see that $\angle A' < 90$. Similar reasoning holds for $\angle B'$ and $\angle C'$. Thus the triangle $\triangle A'B'C'$ is acute.

Problem 20

Now that if $x > 0$ then the left-hand-side of this expression is positive and cannot equal zero. In general, the rational roots of this polynomial are of the form $\frac{p}{q}$ where p is a factor of the constant term i.e. six and q is a factor of the coefficient of x^3 i.e. one. This means that

$$x \in \{\pm 1, \pm 2, \pm 3, \pm 6\}\,.$$

As we know that $x < 0$ lets try $x = -1$. The polynomial evaluated there gives

$$-1 + 6 - 11 + 6 = 0\,.$$

Thus $x + 1$ is a factor of this cubic. If we factor this using synthetic division we find

$$\begin{array}{r|rrrr} & 1 & 6 & 11 & 6 \\ -1 & & -1 & -5 & -6 \\ \hline & 1 & 5 & 6 & 0 \end{array}$$

This means that

$$x^3 + 6x^2 + 11x + 6 = (x+1)(x^2 + 5x + 6)\,.$$

This later quadratic can be factored to get

$$(x+1)(x+2)(x+3)\,,$$

showing that the roots are $x \in \{-1, -2, -3\}$.

Problem 21

If we *square* this expression we get

$$2^2 x + \frac{2^2}{x} + 8 = 25\,.$$

If we multiply this by x we get

$$4x^2 + 4 + 8x = 25x\,,$$

or

$$4x^2 - 17x + 4 = 0\,.$$

Problem 22

Call this expression E. Then we have

$$\begin{aligned} E &= \frac{2x^2 - x - 4 - x}{(x+1)(x-2)} = \frac{2x^2 - 2x - 4}{(x+1)(x-2)} \\ &= \frac{2(x^2 - x - 2)}{(x+1)(x-2)} = \frac{2(x-2)(x+1)}{(x+1)(x-2)} = 2\,. \end{aligned}$$

Problem 23

The sales price is the cost plus the margin or

$$S = C + M = C + \frac{1}{n}C = \frac{n+1}{n}C\,.$$

We are told that $M = \frac{1}{n}C$ or $C = nM$ so that the above is

$$S = \frac{n+1}{n}(nM) = (n+1)M\,.$$

This means that

$$M = \frac{1}{n+1}S\,.$$

Problem 24

This equation is

$$2x^2 + (1-k)x + 8 = 0\,,$$

the discriminate is then given by

$$(1-k)^2 - 4(2)(8) = (1-k)^2 - 64\,.$$

To have real and equal roots means that this is zero or

$$1 - k = \pm 8\,,$$

or

$$k - 1 = \pm 8\,,$$

or $k \in \{-7, 9\}$.

Problem 25

Lets perform synthetic division by one. We find

$$\begin{array}{c|ccc} 1 & a(b-c) & b(c-a) & c(a-b) \\ & & a(b-c) & c(b-a) \\ \hline & a(b-c) & bc-ac & 0 \end{array}$$

This means we can write our expression as

$$(x-1)(a(b-c)x + c(b-a)) = 0\,.$$

From this we see that the other root is

$$x = -\frac{c(b-a)}{a(b-c)} = \frac{c(a-b)}{a(b-c)}\,.$$

Problem 26

Let the segment AB be drawn on an x-axis in a Cartesian coordinate plane with C a point on AB with proportions as given. Let the "left-most" circle have a center of P (midway between A and C) and a radius denoted r_L. Let the "right-most" circle have a center of Q (midway between C and B) and a radius denoted r_R. Let the common tangent be tangent to the "left-most" circle at a point P' and to the right most circle at a point Q'. Recall that tangents to circle are at a right angle to their radii. Finally, let x be the distance from B to D.

Lets let $CB = b$ so that $AC = 3b$. Then from the above we will have $r_R = \frac{b}{2}$ and $r_L = \frac{3b}{2}$. Next as $\triangle DQQ' \sim \triangle DPP'$ we have

$$\frac{DQ}{QQ'} = \frac{DP}{PP'}\,,$$

or

$$\frac{x + b/2}{r_R} = \frac{x + b + 3b/2}{r_L}\,,$$

or

$$\frac{x + b/2}{\frac{b}{2}} = \frac{x + b + 3b/2}{\frac{3b}{2}}\,.$$

Solving this for x we get $x = BD = \frac{b}{2}$ the radius of the smaller circle.

Problem 27

Let r be the radius of the circle that is the base of the cone and also the radius of the sphere. Let h be the height of the right circular cone. Then we have

$$V_{\text{cone}} = \frac{1}{3}\pi r^2 h$$
$$V_{\text{sphere}} = \frac{4}{3}\pi r^3 \,.$$

As we are told that

$$V_{\text{cone}} = \frac{1}{2} V_{\text{sphere}} \,,$$

we have

$$\frac{\pi r^2 h}{3} = \frac{1}{2}\left(\frac{4\pi r^3}{3}\right) \,.$$

This simplifies to $\frac{h}{r} = 2$.

Problem 28

Call this expression E. Then we can write

$$E = \frac{3mr - nt}{4nt - 7mr} = \frac{3 - \frac{nt}{mr}}{4\left(\frac{nt}{mr}\right) - 7} \,.$$

What is the value of $\frac{nt}{mr}$. From the ratios given we have

$$\frac{nt}{mr} = \frac{3 \cdot 14}{4 \cdot 9} = \frac{7}{6} \,.$$

This means that E is given by

$$E = \frac{3 - \frac{7}{6}}{4\left(\frac{7}{6}\right) - 7} = -\frac{11}{14} \,,$$

when we simplify.

Problem 29

Draw this triangle with the AB leg along an x-axis and the BC leg sticking straight up into the "air" so that the hypotenuse is angled in the North-East direction. Without loss of generality let $AB = 1$ so that $BC = 2$. Then the length of the hypotenuse AC is given by

$$AC = \sqrt{1 + 4} = \sqrt{5} \,.$$

Next draw a segment from B perpendicular the hypotenuse. Let the intersection of this segment with the segment AC be denoted as B'. Let the length $AB' = x$ so that $B'C = \sqrt{5} - x$. From the right triangle $\triangle AB'B$ we get

$$BB'^2 = AB^2 - AB'^2 = 1 - x^2 \, .$$

From the right triangle $\triangle CB'B$ we have

$$BB'^2 = BC^2 - B'C^2 = 4 - (\sqrt{5} - x)^2 \, .$$

Setting these equal to each other and solving for x we find $x = \frac{1}{\sqrt{5}}$. This means that

$$\sqrt{5} - x = \frac{4}{\sqrt{5}} \, ,$$

so that

$$\frac{AB'}{B'C} = \frac{x}{\sqrt{5} - x} = \frac{1}{4} \, ,$$

when we simplify.

Problem 30

Let r_A, r_B, and r_C be the *rate* at which A, B, and C can do a job (measured in jobs per day). Then we are told that

$$\begin{aligned} 2(r_A + r_B) &= 1 \\ 4(r_B + r_C) &= 1 \\ \frac{12}{5}(r_A + r_C) &= 1 \, . \end{aligned}$$

These are three equations and three unknowns and we would like to know the value of r_A. If we use the bottom two equations to eliminate r_C and then put the expression we get for r_B in the first equation we can solve for r_A and find that $r_A = \frac{1}{3}$. This means that $3r_A = 1$ or it takes A three days to do the job by him/herself.

Problem 31

Let the segment BC be along the x-axis of a Cartesian coordinate plane with A "above" BC. Let point O be drawn inside $\triangle ABC$. Connect this point to B and C with the segments BO and CO.

Because $AB = AC$ we have

$$\angle B = \angle C = \frac{180 - 40}{2} = 70 \, .$$

Let θ be equal to $\angle OBC = \angle OCA$ and let γ be defined as $\angle ABO$. Then as $\angle ABC = 70$ we have

$$\gamma + \theta = 70\,.$$

At the same time as $\angle ACB = 70$ we have

$$\theta + \angle BCO = 70\,.$$

Taken together these mean that

$$\angle ABO = \angle BCO = \gamma\,.$$

In triangle $\triangle BOC$ we have

$$\theta + \gamma + \angle BOC = 180\,,$$

or

$$\angle BOC = 180 - (\theta + \gamma) = 180 - 70 = 110\,,$$

using what we know about the value of $\gamma + \theta$.

Problem 32

We have

$$\begin{aligned} x^4 + 64 &= x^4 + 8^2 \\ &= (x^2+8)^2 - 16x^2 = (x^2+8-4x)(x^2+8+4x) \\ &= (x^2-4x+8)(x^2+4x+8)\,. \end{aligned}$$

Problem 34

Recall that $\frac{1}{3} = 0.\bar{3}$ where the bar over the three means that we repeat that argument (here the digit three) forever. This means that (A) is not correct. Let $e = 0.33333333$ with eight threes. This means that

$$\frac{1}{3} - e = 0.00000000\bar{3}\,.$$

There are eight zeros after the decimal place in the above. This means that

$$10^8\left(\frac{1}{3} - e\right) = 0.\bar{3} = \frac{1}{3}\,,$$

or

$$\frac{1}{3} - e = \frac{1}{3\cdot 10^8}\,.$$

This means that $\frac{1}{3}$ is larger than e by $\frac{1}{3\cdot 10^8}$.

Problem 35

We are told that

$$BM + MA = BC + CA\,.$$

Using the Pythagorean theorem and the notation given in this problem this means that

$$x + \sqrt{(x+h)^2 + d^2} = h + d\,,$$

or

$$\sqrt{(x+h)^2 + d^2} = h + d - x\,.$$

If we square this and expand we can get a linear equation in x. When we solve that we get

$$x = \frac{hd}{2h+d}\,.$$

Problem 36

Let $v_0 = 15$ be the boats speed in still water. Let D be the distance traveled. Now average speed is total distance divided by total time which in this case is

$$\frac{D+D}{\frac{D}{v_0+5} + \frac{D}{v_0-5}} = \frac{v_0^2 - 25}{v_0}\,,$$

when we simplify. The ratio we want is this over v_0 or

$$\frac{v_0^2 - 25}{v_0^2} = \frac{8}{9}\,,$$

when we simplify.

Problem 37

As $\angle n = 90^\circ$ and RS bisects angle R in the upper left triangle with angles m, n, and vertex R we have

$$m + 90^\circ + \frac{R}{2} = 180^\circ \quad \text{or} \quad \frac{R}{2} = 90^\circ - m\,.$$

Next in triangle $\triangle RPQ$ we have

$$R + p + q = 180^\circ\,,$$

or

$$2\left(\frac{R}{2}\right) + p + q = 180^\circ\,.$$

Using what we have above for $\frac{R}{2}$ the above is given by

$$2\,(90^\circ - m) + p + q = 180^\circ\,,$$

which we can write as

$$m = \frac{1}{2}(p+q)\,.$$

Problem 38

Start with what we are given and write it as

$$3^3 \cdot 3^x = 135 = 5 \cdot 3^3 ,$$

thus $3^x = 5$. From this we take the base 10 logarithm to get

$$x \log(3) = \log\left(\frac{10}{5}\right) = \log(10) - \log(2) = 1 - \log(2) .$$

Solving for x we get

$$x = \frac{1-\log(2)}{\log(3)} \approx \frac{1-0.3010}{0.4771} = 1.4651 .$$

Problem 39

Let the circle have a center at $(0, 0)$ with a radius r and so in the Cartesian coordinate plane is denoted as $x^2 + y^2 = r^2$. Let $P = (x^*, y^*)$ be the external point. The midpoint of a point on the circle and P is located at

$$(m_x, m_y) = \left(\frac{1}{2}(x + x^*), \frac{1}{2}(y + y^*)\right) .$$

From this we see that

$$\left(m_x - \frac{1}{2}x^*\right)^2 + \left(m_y - \frac{1}{2}y^*\right)^2 = \left(\frac{1}{2}x\right)^2 + \left(\frac{1}{2}y\right)^2 = \frac{1}{4}(x^2 + y^2) = \frac{1}{4}r^2 .$$

In terms of the variables (m_x, m_y) this is the equation of a circle with a center at the point $\left(\frac{1}{2}x^*, \frac{1}{2}y^*\right)$ and a radius of $\frac{r}{2}$.

Problem 40

Taking the square root of this we get

$$a + \frac{1}{a} = \sqrt{3} .$$

If we then take the cube of both sides we get

$$\begin{aligned}\left(a + \frac{1}{a}\right)^3 &= a^3 + 3a^2\left(\frac{1}{a}\right) + 3a\left(\frac{1}{a}\right)^2 + \left(\frac{1}{a}\right)^3 \\ &= a^3 + \frac{1}{a^3} + 3a + 3\left(\frac{1}{a}\right) = a^3 + \frac{1}{a^3} + 3\left(a + \frac{1}{a}\right) \\ &= a^3 + \frac{1}{a^3} + 3\sqrt{3} .\end{aligned}$$

We know this equal $3^{3/2}$. This means that what we want can be written as

$$a^3 + \frac{1}{a^3} = \left(a + \frac{1}{a}\right)^3 - 3\sqrt{3} = 3^{3/2} - 3^{3/2} = 0 .$$

Problem 41

The rational roots of this polynomial of the form $\frac{p}{q}$ will have p a factor of the constant term i.e. -9 and q a factor of the highest power term i.e. 4. This means that

$$p \in \{\pm 1\,, \pm 3\,, \pm 9\}\,,$$

and

$$q \in \{\pm 1\,, \pm 2\,, \pm 4\}\,.$$

This means that

$$\frac{p}{q} \in \left\{\pm 1\,, \pm\frac{1}{2}\,, \pm\frac{1}{4}\,, \pm 3\,, \pm\frac{3}{2}\,, \pm\frac{3}{4}\,, \pm 9\,, \pm\frac{9}{2}\,, \pm\frac{9}{4}\right\}\,.$$

Lets try some of these to see if we can "guess" a root easily. We find $x = 1$, $x = -1$, and $x = +3$ do not work but $x = -3$ makes the left-hand-side zero and is a root. This means that $x + 3$ is a factor of this polynomial. To find the other roots lets use synthetic division to factor $x + 3$ out. We have

$$\begin{array}{r|rrrr} & 4 & -8 & -63 & -9 \\ -3 & & -12 & 60 & 9 \\ \hline & 4 & -20 & -3 & 0 \end{array}$$

This means that we can write our polynomial as

$$(x + 3)(4x^2 - 20x - 3) = 4(x + 3)\left(x^2 - 5x - \frac{3}{4}\right)\,.$$

We can next use Vieta's formula[2]

`https://en.wikipedia.org/wiki/Vieta's_formulas`

to note that the sum of the roots of the quadratic is equal to $-(-5) = 5$. This means that the sum of all the roots of the cubic polynomial are given by $-3 + 5 = 2$.

Problem 42

We will "complete the square" to write these two functions as

$$\begin{aligned} y_1(x) &= x^2 - \frac{1}{2}x + \frac{1}{16} - \frac{1}{16} + 2 \\ &= \left(x - \frac{1}{4}\right)^2 + \frac{31}{16}\,, \end{aligned}$$

[2]we actually could have used it directly from the beginning

and

$$y_2(x) = x^2 + \frac{1}{2}x + \frac{1}{16} - \frac{1}{16} + 2$$
$$= \left(x + \frac{1}{4}\right)^2 + \frac{31}{16}.$$

In this form it is easier to see that $y_1(x)$ is "to the right" of $y_2(x)$.

Problem 43

Draw our right triangle with AB on the x-axis and AC on the y-axis of a of a Cartesian coordinate plane. Draw the the inscribed circle and denote the three points of tangency by A', B', and C' where A' is the point of tangency opposite the point A and similarly for the others. Draw segments from the center of the inscribed circle to the three points of tangency. These segments are perpendicular to the side they intersect. As the radius of the inscribed circle is one we have $AC' = AB' = 1$.

By the fact that common external tangents are of equal lengths we have $CB' = CA'$ and $BA' = BC'$. Let $CB' = CA' = y$ and $BA' = BC' = x$. Then

$$CA' + A'B = y + x = CB = 10\,. \tag{67}$$

In addition

$$AB = AC' + C'B = 1 + x$$
$$AC = AB' + B'C = 1 + y\,.$$

As we have a right triangle the Pythagorean theorem gives

$$AB^2 + AC^2 = BC^2\,,$$

or using the above we have

$$(1 + x)^2 + (1 + y)^2 = 10\,. \tag{68}$$

From Equation 67 we have $y = 10 - x$ which when we put in the above gives

$$(1 + x)^2 + (11 - x)^2 = 100\,.$$

Expanding and simplifying gives $x^2 - 10x + 11 = 0$. Using the quadratic formula this has solutions $x = 5 \pm \sqrt{14}$. Now as $\sqrt{14} < \sqrt{16} = 4$ both signs give positive solutions and we can't eliminate any. We have

$$y = 10 - x = 10 - (5 \pm \sqrt{14}) = 5 \mp \sqrt{14}\,.$$

As this problem is symmetrical we need to take one sign for x and the other sign for y. Because of that, lets take $x = 5 + \sqrt{15}$ so that we have $y = 5 - \sqrt{14}$. This means that the perimeter of the triangle is given by

$$(1 + y) + 10 + (1 + x) = 12 + x + y = 12 + 10 = 22\,.$$

Problem 44

The statement "born in the first half of the century" means that the man was born in the years

$$\{1800\,, 1801\,, 1802\,, \ldots\,, 1849\}\,.$$

Say the man was born in year y. Then he would be

- One year old in year $y + 1$
- Two years old in year $y + 2$
- etc.

Thus he would be x years old in year $y + x$. In the problem we are told that $y + x = x^2$ so $y = x(x-1)$. This means that as x is an integer y is the product of two consecutive integers and thus y is *even*. Therefore choices (A) and (B) are not true. Now by prime factorizing the choices given we see that

$$1812 = 2^2 \cdot 3 \cdot 151\,,$$

which is not the product of two consecutive integers. In the same way

$$1836 = 2^2 \cdot 3^3 \cdot 17\,,$$

which is also not the product of two consecutive integers. Finally

$$1806 = 2 \cdot 3 \cdot 7 \cdot 43 = 42 \cdot 43\,,$$

which is the product of two consecutive integers.

Problem 45

Recall that a rhombus has four equal sides, equal opposite acute angles, and equal opposite obtuse angles. We draw our rhombus with AB along the x-axis, DC parallel to AB above it and shifted to the "right". Next we draw the segment running parallel to BD. Let the d be the distance to this segment from the point A. Let the length of this segment be l. The by imagining this segment starting at A and increasing its distance from A, its length will initially increase until a maximum of $l = BD$, and then decrease back down to zero when it goes thought the point C. This is the choice (D).

Problem 46

Let O be the center of the circle. Note that we have $DA = DB$ (because they are both external tangents from the same point) and the radius of the circle is $\frac{3}{16}$. From the diagram we have

$$DC = DO + OC = DO + \frac{3}{16}\,.$$

We now ask what is the length of DO. As $\angle ADB = 60°$ and $\angle OBD = 90°$ then by symmetry we have

$$\sin(30°) = \frac{3/16}{DO} \quad \text{so} \quad DO = \frac{3}{8} \,.$$

Then using this we have

$$DC = \frac{3}{8} + \frac{3}{16} = \frac{9}{16} \,.$$

Now as

$$x = DC - \frac{1}{2} \,,$$

we have

$$x = \frac{9}{16} - \frac{1}{2} = \frac{1}{16} \,.$$

Problem 47

From the problem we are told that $AM = \frac{p}{2}$ and

$$MT = \sqrt{\left(\frac{p}{2}\right)^2 - q^2} \,.$$

These mean that

$$AT = \frac{p}{2} + MT$$
$$TB = p - AT = \frac{p}{2} - MT \,.$$

Recall that if we have a quadratic polynomial with roots $r_1 = AT$ and $r_2 = TB$ we can write it as

$$(x - r_1)(x - r_2) = x^2 - (r_1 + r_2)x + r_1 r_2 = 0 \,.$$

Thus the coefficients are the sum of the roots and the product of the roots (this is Vieta's theorem). In this case the sum of the roots is $AT + TB = p$ and the product of the roots is

$$\begin{aligned} AT \cdot TB &= \left(\frac{p}{2} + \sqrt{\left(\frac{p}{2}\right)^2 - q^2}\right)\left(\frac{p}{2} - \sqrt{\left(\frac{p}{2}\right)^2 - q^2}\right) \\ &= \left(\frac{p}{2}\right)^2 - \left(\left(\frac{p}{2}\right)^2 - q^2\right) = q^2 \,. \end{aligned}$$

This means that the equation we seek is

$$x^2 - px + q^2 = 0 \,.$$

Problem 48

Let D_1 be the distance form the start where the accident occurred and v_1 be the trains normal velocity. Let D_2 be the distance from the accident to the trains final destination. Then in the first case the total time taken by the trip would be

$$T_1 \equiv 1 + \frac{1}{2} + \frac{D_2}{\frac{3}{4}v_1}\,.$$

The $\frac{1}{2}$ is the half-hour delay. In the second case (where the accident had happened 90 miles further downstream) the total time taken would be

$$T_2 \equiv 1 + \frac{90}{v_1} + \frac{1}{2} + \frac{D_2 - 90}{\frac{3}{4}v_1}\,.$$

We are told that $T_1 - T_2 = \frac{1}{2}$ hours. This means that

$$\begin{aligned} T_1 - T_2 &= \left(1 + \frac{1}{2} + \frac{D_2}{\frac{3}{4}v_1}\right) - \left(1 + \frac{90}{v_1} + \frac{1}{2} + \frac{D_2 - 90}{\frac{3}{4}v_1}\right) \\ &= -\frac{90}{v_1} + \frac{90}{\frac{3}{4}v_1} = \frac{30}{v_1}\,. \end{aligned}$$

Solving for v_1 we get $v_1 = 60$. As D_1 is located one hour into the trains trip we have $D_1 = 60(1) = 60$ miles. Now the total time taken under the first case is 3.5 hours too long so

$$T_1 - \left(1 + \frac{D_2}{60}\right) = 3.5\,,$$

or

$$\left(1 + \frac{1}{2} + \frac{D_2}{\frac{3}{4}(60)}\right) - \left(1 + \frac{D_2}{60}\right) = 3.5\,.$$

Solving this for D_2 gives $D_2 = 540$ miles so the total distance traveled is

$$D_1 + D_2 = 60 + 540 = 600\,,$$

miles.

Problem 49

Lets write this difference D as

$$\begin{aligned} D &= (2a+1)^2 - (2b+1)^2 \\ &= 4a^2 + 4a + 1 - 4b^2 - 4b - 1 \\ &= 4(a^2 - b^2) + 4(a - b) = 4(a-b)(a+b+1)\,. \end{aligned}$$

Now if a and b are odd then $a - b$ is even and D is divisible by eight. If a and b are odd then $a - b$ is even and D is divisible by eight. If a is odd and b is even then $a - b$ is odd while $a + b + 1$ is even and D is divisible by eight. If a is even and b is odd then $a - b$ is odd while $a + b + 1$ is even and D is divisible by eight. In all cases for a and b we have that D is divisible by eight.

Problem 50

Let the *top* of the clock i.e. the vertical 12 o'clock position be the "zero" angle and an increasing angle corresponding to clockwise movement (i.e. normal) of the hands of the clock. Then if t is the number of minutes after 7 o'clock the angle the minute hand makes with this vertical is given by

$$\theta_{\text{minute}} = 0 + 360\left(\frac{t}{60}\right) = 6t\,,$$

and the angle the hour hand makes with this vertical is given by

$$\theta_{\text{hour}} = 180 + \left(\frac{360}{12}\right) + 360\left(\frac{t}{12\cdot 60}\right) = 210 + \frac{t}{2}\,.$$

We want to know for what values of t we have

$$|\theta_{\text{minute}} - \theta_{\text{hour}}| = \left|\frac{11t}{2} - 210\right| = 84\,.$$

These are the two equations

$$\begin{aligned}\frac{11t}{2} - 210 = 84 \quad \text{where} \quad t = 53.45\\ \frac{11t}{2} - 210 = -84 \quad \text{where} \quad t = 22.90\,,\end{aligned}$$

both in minutes. These are closest to the times 7:53 and 7:23.

The 1955 Examination

Problem 1

Call this number E. Then E has six zeros followed by 375. Thus

$$\begin{aligned} E &= 0.375 \cdot 10^{-6} \\ &= 375 \cdot 10^{-9} \quad \text{which is (C)} \\ &= 3.75 \cdot 10^{-7} \quad \text{which is (A)} \\ &= 3\frac{3}{4} \cdot 10^{-7} \quad \text{which is (B)} \,. \end{aligned}$$

Now

$$\frac{3}{8} = \frac{3}{2}\left(\frac{1}{4}\right) = \frac{3}{2}(0.25) = \frac{1}{2}(0.75) = 0.375 \,,$$

so we can also write E as

$$E = \frac{3}{8} \cdot 10^{-6} \,,$$

which is (E). None of these are (D).

Problem 2

Let the *top* of the clock i.e. the vertical 12 o'clock position be the "zero" angle and an increasing angle corresponding to clockwise movement (i.e. normal) of the hands of the clock. Then if t is the number of minutes after 12 o'clock the angle the minute hand makes with this vertical is given by

$$\theta_{\text{minute}}(t) = 360\left(\frac{t}{60}\right) = 6t \,,$$

and the angle the hour hand makes with this vertical is given by

$$\theta_{\text{hour}}(t) = 360\left(\frac{t}{12 \cdot 60}\right) = \frac{t}{2} \,.$$

Using these we compute

$$\begin{aligned} \theta_{\text{minute}}(25) &= 150^\circ \\ \theta_{\text{hour}}(25) &= 12.5^\circ \,. \end{aligned}$$

This means that

$$\theta_{\text{minute}}(25) - \theta_{\text{hour}}(25) = 137.5 \,.$$

This is 137 and 0.5 degrees. As one minute is

$$1' = \left(\frac{1}{60}\right)^\circ \,,$$

we see that multiplying by 30 we get that

$$30' = \left(\frac{1}{2}\right)^\circ ,$$

Thus this angle is $137.30'$.

Problem 3

The average is increased by 20.

Problem 4

"Cross multiply" to write this as

$$x - 2 = 2(x - 1) ,$$

or

$$x - 2 = 2x - 2 \quad \text{so} \quad x = 0 .$$

Problem 5

We are told that

$$y = \frac{c}{x^2} ,$$

and that when $x = 1$ we have

$$16 = \frac{c}{1} ,$$

so that $c = 16$. This means that $y = \frac{16}{x^2}$. If $x = 8$ we have

$$y = \frac{16}{8^2} = \frac{4^2}{2^2 \cdot 4^2} = \frac{1}{4} .$$

Problem 6

The first price paid for N oranges is $P_1 = \frac{10}{3}$ while the second price payed for N oranges is $P_2 = \frac{20}{5}$ both in units of cents-per-orange. To "break-even" he must sell $2N$ of them at the sales price S such that

$$NP_1 + NP_2 = (2N)S .$$

Solving for S we have

$$S = \frac{1}{2}\left(\frac{10}{3} + \frac{20}{5}\right) = \frac{5}{3} + \frac{10}{5} = \frac{25}{15} + \frac{30}{15} = \frac{55}{15} = \frac{11}{3} .$$

Problem 7

After the cut the workers new wages w' from his original wages w are

$$w' = 0.8w \,.$$

We want to know r such that after a r "percent increase" that

$$w'' = (1+r)w' = w \,.$$

This means that $(1+r)0.8w = w$ or

$$1 + r = \frac{1}{0.8} = \frac{1}{\frac{8}{10}} = \frac{10}{8} = \frac{5}{4} = 1.25 \,.$$

Thus $r = 0.25$ or a 25% increase.

Problem 8

We can write the equation for the graph as

$$(x - 2y)(x + 2y) = 0 \,.$$

This has solutions $x = 2y$ or $x = -2y$. These are two straight lines.

Problem 9

Draw our triangle $\triangle ABC$ with one base AB along the x axis and the point C "above" the x-axis. Draw the inscribed circle with a center in the triangle and tangent to the three sides. Drawing segments from each vertex to the center of that circle decomposes the triangle into six right triangles where the radius of the inscribed circle is the one of the legs. We will seek to determine the lengths of other legs of these right triangles. Let the lengths from A to the two tangent points be x, the lengths from C to the two tangent points be y, and the lengths from B to the two tangents points be z. Then given the total lengths of the three sides of the triangle we can write

$$\begin{aligned} x + y &= 8 \\ y + z &= 15 \\ x + z &= 17 \,. \end{aligned}$$

As a matrix system this is

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 15 \\ 17 \end{bmatrix} \,.$$

If we solve this system we get

$$
\begin{aligned}
x &= 5 \\
y &= 3 \\
z &= 12\,.
\end{aligned}
$$

This gives us the lengths of the "parts" of each side. These "parts" are legs of the six right triangles that the center of the inscribed circle divides it up into.

One way to evaluate this area of the original triangle is to sum the area of these six right triangles. Towards that direction let r be the radius of this circle. Then using the lengths x, y, and z we have

$$
\begin{aligned}
A &= 2\left(\frac{1}{2}xr\right) + 2\left(\frac{1}{2}zr\right) + 2\left(\frac{1}{2}yr\right) \\
&= 5r + 12r + 3r = 20r\,.
\end{aligned}
$$

We can also evaluate this area using Heron's formula. We have the semiperimeter s given by

$$s = \frac{1}{2}(8 + 15 + 17) = 20\,.$$

So that Heron's formula for the area A gives

$$A = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{20(12)(5)(3)} = \sqrt{3600} = 60\,.$$

If we set this equal to the expression above we find $r = \frac{60}{20} = 3$.

Problem 10

The time the train is "traveling" is $\frac{a}{40}$ hours. The time the train is waiting is $n \cdot m$ minutes or $\frac{nm}{60}$ hours. The total time in hours is then

$$\frac{a}{40} + \frac{mn}{60} = \frac{3a + 2mn}{120}\,.$$

Problem 11

If "no slow learners attend this school" is *not* true then *some* slow learners must attend this school.

Problem 12

We start with

$$\sqrt{5x-1}+\sqrt{x-1}=2\,. \tag{69}$$

If we square both sides and simplify we get

$$\sqrt{(5x-1)(x-1)}=3(1-x)\,.$$

Squaring again to get

$$(5x-1)(x-1)=9(x-1)^2\,.$$

One solution to the above is $x=1$. If $x\neq 1$ then we have

$$5x-1=9(x-1)\,,$$

which has a solution of $x=2$. Now in Equation 69 if we take $x=1$ we get

$$\sqrt{4}+0=2\,,$$

which is true. In Equation 69 if we take $x=2$ we get

$$\sqrt{9}+\sqrt{1}=4\neq 2\,.$$

So the only solution to the original equation is $x=1$.

Problem 13

Call this expression E. Then we have

$$E=\frac{a^{-4}-b^{-4}}{a^{-2}-b^{-2}}=\frac{(a^{-2}-b^{-2})(a^{-2}+b^{-2})}{a^{-2}-b^{-2}}=a^{-2}+a^{-2}\,.$$

Problem 14

Let s be the length of the side of the square. We are told that $l=1.1s$ and $w=0.9s$. This means that

$$R:S=lw:s^2=1.1(0.9)s^2:s^2=0.99:1=99:100\,.$$

Problem 15

Let A and R be the area and radius of the larger circle and a and r be the area and radius of the smaller circle. Then we are told that

$$a:A=\pi r^2:\pi R^2=1:3\,.$$

In other wards

$$\frac{r^2}{R^2} = \frac{1}{3} \quad \text{so} \quad R = \sqrt{3}r\,.$$

This means that

$$R - r = (\sqrt{3} - 1)r = 0.732051r\,.$$

Problem 16

This would be (E) meaningless/undefined.

Problem 17

We are given

$$\log(x) - 5\log(3) = -2\,,$$

or

$$\log(x) - \log\left(3^5\right) = -2\,,$$

or

$$\log\left(\frac{x}{3^5}\right) = -2\,.$$

Take the 10^x of both sides of this we get

$$\frac{x}{3^5} = 10^{-2}\,,$$

so

$$x = \frac{3^5}{100} = \frac{243}{100} = 2.43\,.$$

Problem 18

If the discriminant is D then the roots of quadratic equation $ax^2 + bx + c = 0$ are

$$\frac{-b \pm \sqrt{D}}{2a}\,.$$

As we are told that $D = 0$ and we have $a = 1$ with $b = 2\sqrt{3}$ the above becomes

$$\frac{-2\sqrt{3}}{2} = -\sqrt{3}\,.$$

Thus we have equal, real, and irrational roots.

Problem 19

Let a and b be the roots. Then we are told that

$$a + b = 6\,,$$

and

$$|a - b| = 8\,.$$

This last expression means that $a - b = \pm 8$. If you take the positive sign and add the two equations you get $2a = 14$ so $a = 7$ and then $b = -1$. These roots are the solution to the quadratic

$$(x - 7)(x + 1) = x^2 - 6x - 7 = 0\,.$$

If you take the negative sign and add the two equations you get $2a = -2$ so $a = -1$ and then $b = 7$. These roots are the solution to the quadratic

$$(x + 1)(x - 7) = x^2 - 6x - 7 = 0\,.$$

As a slightly "different" method of solving this we can use Vieta's formula which states that the coefficient of the x term in the quadratic is $-(a + b) = -6$ (in both cases above) and that the constant term must be $ab = -7$ (in both cases above). This is the polynomial

$$x^2 - 6x - 7 = 0\,,$$

the same as the above conclusion.

Problem 20

We want to know when

$$\sqrt{25 - t^2} + 5 = 0\,.$$

As $\sqrt{25 - t^2} \geq 0$ and $5 > 0$ we have

$$\sqrt{25 - t^2} + 5 > 0\,.$$

Therefore no real values of t will be a solution.

Problem 21

I drew my right triangle $\triangle CAB$ in a Cartesian x-y coordinate plane with the point A at the origin, the segment AB along the x-axis and AC along the y-axis. Now we take $h = AC$, $b = AB$, $c = BC$ and the area of this triangle is

$$A = \frac{1}{2} bh\,.$$

Let the altitude to the hypotenuse be the segment AA' where A' is on BC such that AA' is perpendicular to BC and a is the length of AA'. For this problem we are asked for a in terms of the area A and c. Note that

$$\triangle ABC \sim \triangle A'BC\,,$$

which means that

$$\frac{AA'}{AC} = \frac{AB}{BC} \quad \text{or} \quad \frac{a}{h} = \frac{b}{c}\,.$$

This means that

$$a = \frac{hb}{c} = \frac{2A}{c}\,.$$

Problem 22

This problem is asking which is smaller. The first discount is

$$d_1 = (1 - 0.2)^2(1 - 0.1) = 0.8^2(0.9) = 0.576\,,$$

While the second discount is

$$d_2 = (1 - 0.4)(1 - 0.05)^2 = 0.6(0.95)^2 = 0.5415\,.$$

In the first case he would pay $10000d_1 = 5760$ while in the second case he would pay $10000d_2 = 5415$. This is a difference of $5760 - 5415 = 345$.

Problem 23

The value V of the petty cash "initially" is given by

$$V = 25q + 10d + 5n + 1c\,,$$

in cents. This was found to be in error in that

- he should have added $5x$ but instead added $25x$
- he should have added $10x$ but instead added x.

This means that to the total was added $25x + x = 26x$ when it should have been $5x + 10x = 15x$. To get the correct value from V we subtract $26x$ and then add $15x$. This means that the correct total is

$$V - 26x + 15x = V - 11x\,.$$

This means subtract $11x$ cents.

Problem 24

Lets write this function as

$$\begin{aligned} 4x^2 - 12x - 1 &= 4(x^2 - 3x) - 1 \\ &= 4\left(x^2 - 3x + \frac{9}{4} - \frac{9}{4}\right) - 1 \\ &= 4\left(x - \frac{3}{2}\right)^2 - 9 - 1 = 4\left(x - \frac{3}{2}\right)^2 - 10\,. \end{aligned}$$

This function then has a minimum value of -10 when $x = \frac{3}{2}$.

Problem 25

Method 1: Now $x^2 + 3 = 0$ if and only if $x = \pm\sqrt{3}i$. We now check if the value $x = \pm\sqrt{3}i$ is a zero of the original quartic equation. We find in that case that

$$x^4 + 2x^2 + 9 = 9^2 + 2(3)(-1) + 9 \neq 0\,,$$

and thus the answer is no and thus (A) is not correct. For (B) to be correct $x = -1$ would need to be a root of the original quartic but

$$x^4 + 2x^2 + 9 = 1^2 + 2(1) + 9 \neq 0\,,$$

thus (B) cannot be correct. Now $x^2 - 3 = 0$ if and only if $x = \pm\sqrt{3}$. We can then show that putting this number into the original quartic does not give zero and (C) cannot be correct. Note that for (D) the roots of this quadratic equation are given by

$$x^2 - 2x - 3 = (x - 3)(x + 1) = 0\,.$$

But the original quartic polynomial does not have $x = 3$ as a root and thus (D) cannot be correct.

Method 2: We can factor the given expression by "completing the square with respect to the constant term" as follows

$$\begin{aligned} x^4 + 2x^2 + 9 &= x^4 + 6x^2 + 9 - 4x^2 \\ &= (x^2 + 3)^2 - 4x^2 = (x^2 + 3 - 2x)(x^2 + 3 + 2x) \\ &= (x^2 - 2x + 3)(x^2 + 2x + 3)\,. \end{aligned}$$

From the above we can "read off" a factor directly.

Problem 26

Mr. A sells his house at

$$1.1(10000) = 11000\,,$$

to Mr. B. Mr. B then sells the house back to Mr. A for $0.9(11000) = 9900$.

During these transactions Mr. A received 11000 and then "lost" 9900 for a net profit of

$$11000 - 9900 = 1100\,.$$

Mr. B "lost" 11000 and then received 9900 for a net profit of

$$-11000 + 9900 = -1100\,.$$

Thus (E) is the correct solution.

Problem 27

Note that by Vieta's formula we must have $r + s = p$ and $rs = q$. Then we can evaluate

$$r^2 + s^2 = (r+s)^2 - 2rs = p^2 - 2q\,.$$

Problem 28

For the original polynomial $y_1(x) = ax^2 + bx + c$ we can write

$$\begin{aligned} y_1(x) &= ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x\right) + c = a\left(x^2 + \frac{b}{a}x + \frac{b}{4a^2} - \frac{b}{4a^2}\right) + c \\ &= a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b}{4a}\,. \end{aligned}$$

This is a parabola with its minimum at $x = -\frac{b}{2a}$.

For the second polynomial $y_2(x) = ax^2 - bx + c$ in the same way we can write

$$y_2 = ax^2 - bx + c = a\left(-x + \frac{b}{2a}\right)^2 + c - \frac{b}{4a} = a\left(x - \frac{b}{2a}\right)^2 + c - \frac{b}{4a}\,.$$

This is a parabola with its minimum at $x = \frac{b}{2a}$ and the same minimum *value* as the parabola $y_1(x)$ above.

These two curves will intersect when $y_1(x) - y_2(x) = 0$ which we find is equivalent to

$$y_1(x) - y_2(x) = bx + c - (-bx + c) = 2bx = 0\,.$$

This has the solution $x = 0$ so the intersection point is a point on the y-axis.

Problem 29

Using the "secant-tangent angle" theorem we have that

$$\begin{aligned}\angle RPB &= \frac{1}{2}(b + d + d - (b - x)) \\ &= \frac{1}{2}(2d + x)\,,\end{aligned}$$

and

$$\begin{aligned}\angle APN &= \frac{1}{2}(a + c - x + c - a) \\ &= \frac{1}{2}(2c - x)\,.\end{aligned}$$

Now the angle we want can be written as the sum of the above two so

$$\angle APB = \angle APN + \angle RPB = c + d\,.$$

Now in circle O we have $c + a = 180$ or

$$c = 180 - a\,. \tag{70}$$

In circle O' we have $x + b - x + d = 180$ or $b + d = 180$ or

$$d = 180 - b\,. \tag{71}$$

Using these we have

$$\angle APB = 180 - a + 180 - b = 360 - (a + b)\,.$$

The solution in the book seems to be for the *external* angle at $\angle APB$ which would be $360 - \angle APB$ (as expressed above).

Problem 30

For the first equation we have $3x^2 = 27$ or $x^2 = 9$ or $x = \pm 3$.

For the second equation we have

$$(2x - 1)^2 - (x - 1)^2 = 0\,,$$

or

$$(2x - 1 - x + 1)(2x - 1 + x - 1) = 0\,,$$

or

$$x(3x - 2) = 0\,.$$

This has solutions $x = 0$ or $x = \frac{3}{2}$.

For the third equation we have

$$\sqrt{x^2-7} = \sqrt{x-1}\,,$$

which if we square gives

$$x^2 - 7 = x^{-}1\,,$$

or

$$x^2 - x - 6 = 0\,,$$

or

$$(x+2)(x-3) = 0\,.$$

This has the solutions $x = -2$ or $x = 3$. If we put $x = -2$ in the original equation we get $\sqrt{4-7} = \sqrt{-3}$ which means that $x = -2$ is not a solution in the real numbers. If we put $x = 3$ in the original equation we get $\sqrt{2} = \sqrt{2}$ which means $x = 3$ is a solution in the real numbers.

From these solutions we see that (B) is the correct choice.

Problem 31

Let the "base" BC be along the x-axis and the point A "above" the segment BC. Let the side of this equilateral triangle be s. Then we have

$$[ABC] = \frac{\sqrt{3}}{4}s^2\,.$$

Let segment DE be draw parallel to BC. As DE is parallel to BC we know the angles $\angle ADC = \angle AED = 60°$. Then as $\angle BAC = 60$ we know that $\triangle ADE$ is also an equilateral triangle. Let the length of the sides of that triangle be s' so that

$$[ADE] = \frac{\sqrt{3}}{4}s'^2\,.$$

As we are told

$$[DBCE] = \frac{1}{2}[ABC]\,,$$

we have

$$[ADE] = \frac{1}{2}[ABC] = \frac{\sqrt{3}}{8}s^2 = \frac{\sqrt{3}}{4}s'^2\,.$$

Solving for s' we have $s' = \frac{s}{\sqrt{2}}$. This means that $DE = \frac{s}{\sqrt{2}}$. The median of the trapezoid is then

$$\frac{1}{2}(BC + DE) = \frac{1}{2}\left(s + \frac{s}{\sqrt{2}}\right)\,.$$

If $s = 2$ this is

$$1 + \frac{1}{\sqrt{2}} = \frac{2+\sqrt{2}}{\sqrt{2}}\,.$$

Problem 32

For this quadratic the discriminant is

$$D = (2b)^2 - 4ac = 4b^2 - 4ac\,.$$

If this equals zero then we have $b^2 = ac$ or $b = \sqrt{ac}$. This means that b is the geometric mean of a and c. We can see that a, b, and c form three terms in a geometric progress by letting $a = a_0$, $b = a_0 r$, and $c = a_0 r^2$. Then the above relationship $b^2 = ac$ becomes

$$(a_0 r)^2 = (a_0)(a_0 r^2)\,,$$

which is true.

Problem 33

Let θ_{min} be the angle of the minute hand in degrees from the vertical (the 12 o'clock position) measured t time from 8 AM. Then

$$\theta_{\text{min}} = 0 + 360t = 360t\,,$$

where here t in hours since 8 AM or

$$\theta_{\text{min}} = 360\left(\frac{t}{60}\right) = 6t\,,$$

where here t in minutes since 8 AM.

Let θ_{hour} be the angle of the hour hand in degrees from the vertical (the 12 o'clock position) measured t time from 8 AM. Then

$$\theta_{\text{hour}} = \left(\frac{360}{12}\right)8 + \frac{360}{12}t\,,$$

where here t in hours since 8 AM or

$$\theta_{\text{hour}} = 240 + 30\left(\frac{t}{60}\right) = 240 + \frac{t}{2}\,,$$

where here t in minutes since 8 AM.

At the start of Henry's trip These will be equal when

$$6t = 240 + \frac{t}{2}\,.$$

Solving for t we get $t = \frac{480}{11}$ minutes since 8 AM.

Using the same logic for the end of Henry's trip, if t is now the time from 2 PM we have

$$\theta_{\text{min}} = 6t\,,$$

where here t is minutes from 2 PM and

$$\theta_{\text{hour}} = \left(\frac{360}{12}\right) 2 + \frac{t}{2} \,,$$

where here t is in minutes from 2 PM.

At the end of Henry's trip We are told that

$$\theta_{\text{min}} - \theta_{\text{hour}} = 180 \,.$$

Using the above expressions we get an equation for t, the time in minutes from 2 PM. Solving this give $t = \frac{480}{11}$.

Thus this trip starts at 8 AM plus $\frac{480}{11}$ minutes and runs to 2 PM plus $\frac{480}{11}$. This trip then covers the hour ranges

$$8 - 9 \,, 9 - 10 \,, 10 - 11 \,, 11 - 12 \,, 12 - 1 \,, 1 - 2 \,,$$

or six hours total.

Problem 34

Place the two circles in a Cartesian x-y plane with the smaller circle centered at the origin (denoted O) and the larger circle to the right of the smaller circle on the x-axis and centered at the point denoted O'. From the given diameters the radius of the smaller and larger circle are $r = 3$ and $R = 9$ respectively. Then the point O' is located with $x = 3 + 9 = 12$.

Let the upper external tangent to these two circle be denoted AB where A is tangent to the smaller circle and B is tangent to the larger circle. For part of this problem, we would like to determine the length AB. Note that the segments OA and $O'B$ both form right angles to the segment AB. From the point B walk $r = 3$ units along BO' (to a point denoted C) and draw a segment CO parallel to AB. Note that $OC = AB$.

Then we notice that $\triangle OCO'$ is a right triangle with a hypotenuse of length $OO' = 12$ and a leg length $O'B - BC = 9 - 3 = 6$. Thus

$$OC^2 = OO'^2 - O'C^2 = 12^2 - 6^2 = 108 = 2^2 \cdot 3^3 \,.$$

Thus

$$OC = 2 \cdot 3 \cdot \sqrt{3} = 6\sqrt{3} \,.$$

Next we need to determine what fraction of each circles is encircled by the wire (the other parts of the circle are "hidden" by the wire that is tangent to the circles). From the diagram we have

$$\cos(\angle CO'O) = \frac{6}{12} = \frac{1}{2} \,,$$

and thus $\angle CO'O = 60°$. This means that $\angle COO' = 90° - \angle CO'O = 30°$. By the symmetry in this problem there is a symmetric triangle "below" this one. This means that we can compute the angle measure in each circle where the wire is "tight" to the circle. For the smaller circle on the left it is

$$360° - 2(30° + 90°) = 120°\,,$$

where we have removed the angles $\angle AOC = 90°$ and the angle $\angle COO' = 30°$. The fraction of the circumference of the smaller circle that is encircled by wire is thus

$$\frac{120}{360} = \frac{1}{3}\,.$$

For the larger circle a similar calculation gives

$$360° - 2(60°) = 240°\,.$$

Thus the fraction of the circumference of the larger circle that is encircled by wire is thus

$$\frac{240}{360} = \frac{2}{3}\,.$$

Thus the total distance covered by the wire is

$$\frac{1}{3}(2\pi(3)) + AB + \frac{2}{3}(2\pi(9)) + AB\,.$$

Using the value of $AB = OC$ found above this can be simplified to

$$14\pi + 12\sqrt{3}\,.$$

Problem 35

Let n be the total number of marbles in the bag in the beginning. Let b_i be the number of marbles boy i has. Then we are told that

$$\begin{aligned} b_1 &= \frac{n}{2} + 1 \\ b_2 &= \frac{1}{3}\left(n - \left(\frac{n}{2} + 1\right)\right) = \frac{n-2}{6} \\ b_3 &= n - b_1 - b_2 = \frac{n-2}{3}\,. \end{aligned}$$

Note that $b_3 = 2b_2$ as expected. Now we know that b_i are all positive integers. This means that n must be even (for b_1 to be an integer) and that $n-2$ must be divisible by six (for b_2) and that $n-2$ must be divisible by three (for b_3). These are all true if $n - 2 = 6k$ for some $k \geq 1$. This means that $n = 6k + 2$ and thus that

$$\begin{aligned} b_1 &= 3k + 1 \\ b_2 &= k \\ b_3 &= 2k\,. \end{aligned}$$

Note that

- If $k = 1$ then $n = 8$ with $b_1 = 5$, $b_2 = 1$, and $b_3 = 2$
- If $k = 6$ then $n = 38$ with $b_1 = 19$, $b_2 = 6$, and $b_3 = 12$,

but other values for k are possible also.

Problem 36

The surface of the oil will form a rectangle the length of which will be the length of the cylinder or $l = 10$. This means that to have an area of 40 means that the width of the oil surface must then be $w = \frac{40}{10} = 4$. If we imagine viewing the cylinder from one end, this width will be represented as a horizontal line above or below the diameter of the cylinder. For illustration we will draw this level below the diameter. Then from the center of the circle we will draw a segment the length of the radius $r = 3$ to where the surface of the oil meets the circle on the left and another segment of length $r = 3$ vertically bisecting the width of the oil surface. Let d be the depth of the oil measured at the center of the width (at its deepest point). Then these segments form a right triangle with

$$r^2 = \left(\frac{w}{2}\right)^2 + (r - d)^2 .$$

If we put in the known values in the above we get

$$3^2 = 2^2 + (3 - d)^2 .$$

We can solve this for d and find $d = 3 \pm \sqrt{5}$. One of these numbers is a surface area that lies below the mid-level of the cylinder and the other lies above it.

Problem 37

Our two numbers can be written as

$$\begin{aligned} x &= htu \\ y &= uth \, , \end{aligned}$$

or

$$\begin{aligned} x &= 100h + 10t + u \\ y &= 100u + 10t + h \, . \end{aligned}$$

If we subtract these two numbers we get

$$x - y = 100(h - u) + (u - h) = (h - u)(100 - 1) = 99(h - u) \, .$$

As we are told that $h > u$ we can evaluate a number of different values for $h - u$ and see what the units digit is. We find

- If $h - u = 1$ the units digit of $x - y$ is 9
- If $h - u = 2$ the units digit of $x - y$ is 8
- If $h - u = 3$ the units digit of $x - y$ is 7
- If $h - u = 4$ the units digit of $x - y$ is 6
- If $h - u = 5$ the units digit of $x - y$ is 5
- If $h - u = 6$ the units digit of $x - y$ is 4
- If $h - u = 7$ the units digit of $x - y$ is 3
- If $h - u = 8$ the units digit of $x - y$ is 2
- If $h - u = 9$ the units digit of $x - y$ is 1

Thus we know that $h - u = 6$. This means that $x - y = 99 \times 6 = 594$. Thus the next two digits from right to left are nine and then five.

Problem 38

Let a, b, c, and d be the numbers. Then we are told that

$$\begin{aligned}
\frac{1}{3}(a+b+c)+d &= 29\\
\frac{1}{3}(a+b+d)+c &= 23\\
\frac{1}{3}(a+c+d)+b &= 21\\
\frac{1}{3}(b+c+d)+a &= 17\,.
\end{aligned}$$

We can write these as

$$\begin{aligned}
a+b+c+3d &= 87\\
a+b+3c+d &= 69\\
a+3b+c+d &= 63\\
3a+b+c+d &= 51\,.
\end{aligned}$$

Solving this we find $a = 3$, $b = 9$, $c = 12$, and $d = 21$.

Problem 39

If we "complete the square" we find

$$\begin{aligned} y = x^2 + px + q &= x^2 + px + \frac{p^2}{4} + q - \frac{p^2}{4} \\ &= \left(x + \frac{p}{2}\right)^2 + q - \frac{p^2}{4} \,. \end{aligned}$$

As we are told that the smallest value for y is zero then from the above that happens when $x = -\frac{p}{2}$ and we must then have

$$q - \frac{p^2}{4} = 0 \quad \text{so} \quad q = \frac{p^2}{4} \,.$$

Problem 40

These two fractions will be equal if

$$\frac{ax + b}{cx + d} = \frac{b}{d} \,,$$

or

$$(ax + b)d = b(cx + d) \,,$$

or

$$adx + bd = cbx + bd \,,$$

or

$$(ad - cb)x = bd - bd = 0 \,.$$

Thus these two fractions are *equal* if $x = 0$ or $ad - cb = 0$. These show that (D) and (E) are not solutions to this problem.

If we assume (B) is true then by substitution we see that the two fractions are equal and this is not a solution to this problem.

If we assume (C) is true then by substitution we see that the two fractions are equal and this is not a solution to this problem.

This means that (A) must be the solution. If we assume (A) is true then the left-hand-side fraction is

$$\frac{ax + b}{cx + d} = \frac{x + b}{x + d} \,,$$

which will not equal $\frac{b}{d}$ unless $x = 0$.

Problem 41

Let L be the distance between Aytown and Beetown, v_0 the usual train rate, and T the time needed to run the route normally so that

$$T = \frac{L}{v_0} \,. \tag{72}$$

The time to run the route with the delay is

$$1 + \frac{1}{2} + \frac{V - 1v_0}{\frac{4}{5}v_0} = T + 2 \,. \tag{73}$$

If the train had covered 80 more miles the time to run with the delay would be

$$\left(\frac{1v_0 + 80}{v_0}\right) + \frac{1}{2} + \frac{L - 1v_0 - 80}{\frac{4}{5}v_0} = T + 1 \,. \tag{74}$$

We would like to determine v_0. These are three equations for the three unknowns L, v_0, and T. If we subtract Equation 73 from 74 we get

$$\frac{80}{v_0} - \frac{80}{\frac{4}{5}v_0} = -1 \,.$$

We can solve this for v_0 and find $v_0 = 20$ mph.

Problem 42

For these two expressions to be equal means that

$$\sqrt{a + \frac{b}{c}} = a\sqrt{\frac{b}{c}} \,. \tag{75}$$

If we divide both sides by $\sqrt{\frac{b}{c}}$ we get

$$\sqrt{\left(a + \frac{b}{c}\right)\left(\frac{c}{b}\right)} = a \,,$$

or

$$\sqrt{\frac{ac}{b} + 1} = a \,.$$

If we square this we have

$$\frac{ac}{b} + 1 = a^2 \,.$$

Solving the above for c gives

$$c = \frac{b}{a}(a^2 - 1) \,.$$

This is choice (C).

Now choice (A) in Equation 75 reduces to $\sqrt{2} = 1$ which is false.

Now choice (B) would imply that $a = b = c = 1$ also and so is false.

Now choice (D) in Equation 75 reduces to $\sqrt{a + \frac{a}{c}} = a\sqrt{\frac{a}{c}}$. Taking $c = 1$ in that expression shows it to be false.

Now choice (E) in Equation 75 reduces to

$$\sqrt{a + \frac{a}{a-1}} = a\sqrt{\frac{a}{a-1}}\,.$$

Taking $a = 2$ in that expression we get $\sqrt{2+2} = 2\sqrt{2}$ which is false.

Problem 43

If we put $y = (x+1)^2$ into $(x+1)y = 1$ we get

$$(x+1)^3 = 1\,.$$

Write the one above as $e^{2\pi i k}$ for $k \in \{0, 1, 2, \dots\}$. This means that

$$x = e^{\frac{2\pi i}{3}k} - 1\,.$$

From the above we know $x + 1$ so that using $y = (x+1)^2$ we have

$$y = e^{\frac{4\pi i}{3}k}\,.$$

We will get three distinct roots by taking $k \in \{0, 1, 2\}$.

If we take $k = 0$ we get $x = 0$ and $y = 1$.

If we take $k = 1$ we get

$$\begin{aligned} x &= -\frac{1}{2} + \frac{1}{2}i - 1 = -\frac{3}{2} + \frac{i}{2} \\ y &= e^{\frac{4\pi}{3}i} = -\frac{1}{2} - \frac{i}{2}\,. \end{aligned}$$

If we take $k = 2$ we get

$$\begin{aligned} x &= e^{\frac{4\pi}{3}i} - 1 = -\frac{1}{2} - \frac{1}{2}i - 1 = -\frac{3}{2} - \frac{i}{2} \\ y &= e^{\frac{8\pi}{3}i} = e^{3\pi i - \frac{\pi}{3}i} = -\frac{1}{2} + \frac{i}{2}\,. \end{aligned}$$

Thus we see that there are one real pair and two imaginary pairs of solutions.

Problem 44

Let the circles radius be denoted R. Then draw the segment OB (of length R) and note that the triangle $\triangle OBC$ is isosceles so

$$\angle BOC = \angle BCO = y\,.$$

Now

$$x + \angle AOB + y = 180 \quad \text{so} \quad \angle AOB = 180 - x - y\,.$$

Now $\widehat{AD} = x$ (as $\angle AOD = x$) and $\widehat{BD} = y$ (as $\angle BOC = y$). Thus

$$y = \frac{1}{2}(\widehat{AD} - \widehat{BD}) = \frac{1}{2}(x - y)\,.$$

We can write this last expression as $3y = x$.

Problem 45

Let the first series have terms

$$a_0\,, a_0r\,, a_0r^2\,, \ldots\,,$$

and the second series have terms

$$0\,, d\,, 2d\,, \ldots\,.$$

By adding we get the sequence

$$a_0\,, a_0r + d\,, a_0r^2 + 2d\,, \ldots\,.$$

We are told what a few terms in this series are. From the first term we have $a_0 = 1$. From the second we have

$$a_0r + d = 1 \quad \text{so} \quad r + d = 1\,.$$

From the third term we have

$$a_0r^2 + 2d = 2 \quad \text{so} \quad r^2 + 2d = 2\,.$$

From the first we have $d = 1 - r$ which if we put this into the second equation we get

$$r^2 + 2(1 - r) = 2\,.$$

We can solve this to get $r = 0$ or $r = 2$. We are told that $r \neq 0$ so $r = 2$. This means that $d = -1$. Now the nth terms of the third series is

$$a_0r^{n-1} + (n-1)d = 2^{n-1} - (n-1)\,.$$

Thus the sum S we want to evaluate is

$$\begin{aligned} S &= \sum_{n=1}^{10}(2^{n-1} - (n-1)) = \sum_{n=0}^{9} 2^n - \sum_{n=0}^{n} n \\ &= \frac{2^{10} - 1}{2 - 1} - \frac{10(9)}{2} = 1024 - 1 - 45 = 978\,. \end{aligned}$$

Problem 46

If we look at where the line $2x + 3y = 6$ and $4x - 3y = 6$ intersect we find they intersect when $x = 2$ and $y = \frac{2}{3}$. The vertical line $x = 2$ and the horizontal line $y = \frac{2}{3}$ *also* intersects this point. Thus there is only *one* intersection point for these four lines.

Problem 47

Let E be the difference between these two expressions. Then

$$\begin{aligned} E &= (a+b)(a+c) - (a+bc) = a^2 + ac + ab + bc - a - bc \\ &= a^2 + a(b+c-1) = a(a+b+c-1)\,. \end{aligned}$$

If this is to equal zero then we must have $a = 0$ or $a + b + c = 1$.

Problem 48

The choice (A) is true by how the segment FH and EH are constructed.

By the "parallel projection theorem" the segment FE is parallel the segment AB and $FE = \frac{1}{2}AB$. If we imagine drawing a line through H and parallel to AB. Then as $AF = EH$ we have $HG = GB$. This means that the segment FG is a median of the triangle $\triangle HFB$ so choice (E) is true.

The extension of HE back to AB (because HE is parallel to AC) will meet AB at a point D' where the length AD' will equal the length FE. As $AD = FE$ we know that $D' = D$ and thus get that $HD = AC$. As $AC \parallel DH$ we have that $\angle CAD = \angle HDB$ thus

$$\triangle CAD \cong \triangle HDB\,,$$

by "side-angle-side" and thus we can conclude that $CD = HB$ so the choice (C) is correct.

Now as CD is a median of $\triangle ACB$ on the side AB we have that it will bisect any line segment in the triangle $\triangle ABC$ parallel to AB, specifically it will bisect the segment FE. Let E' be the point (the point of bisection) on the intersection of CD and FE. This means that

$$FE' = E'E = \frac{1}{2}FE = \frac{1}{2}\left(\frac{1}{2}AB\right) = \frac{1}{4}AB\,.$$

Because $\triangle CAD \cong \triangle HDB$ we have

$$E'E = EG\,.$$

This means that

$$FG = FE + EG = \frac{1}{2}AB + \frac{1}{4}AB = \frac{3}{4}AB\,,$$

thus choice (D) is correct.

All of these leave choice (B) as the statement that is not necessarily correct.

Problem 49

Note that we can write the first graph as

$$y_1(x) = \frac{(x-2)(x+2)}{x-2} = x + 2\,,$$

when $x \neq 2$ with the addition that $y_1(x)$ is not defined at $x = 2$. Thus to plot $y_1(x)$ we could plot $x + 2$ but exclude the point $(2, 4)$ from that graph.

If we then look for the intersection of $y_1(x)$ and $y_2(x) = 2x$ we find that these two lines would intersect only at the point $(2, 4)$. Because this point is now allowed in the definition of $y_1(x)$ there are no points of intersection of the two graphs.

Problem 50

We assume that A increases his speed by v (in mph) so that his new speed is $50 + v$. Now if T is the time to pass then A must be going enough faster than B so that in T time A has passed by 30 feet or

$$(50 + v - 40)T = 30\,. \tag{76}$$

In addition as now A and C are headed towards each other to make sure that they don't collide

$$(50 + v + 50)T < 210\,. \tag{77}$$

From the first equation we have

$$T = \frac{30}{10 + v}\,.$$

If we put this into Equation 77 we get

$$30\left(\frac{100 + v}{10 + v}\right) < 210\,.$$

Solving this we find $v > 5$ mph for the smallest amount by which A can increase his speed and still pass B.

The 1956 Examination

Problem 1

When we take $x = 2$ we find

$$2 + 2(2^2) = 2 + 2 * 4 = 10\,.$$

Problem 2

Let C_1 and C_2 be the costs of the two pipes. Then we are told that

$$\begin{aligned}1.2C_1 &= 1.20\\ 0.8C_2 &= 1.20\,,\end{aligned}$$

so we have $C_1 = 1.0$ and $C_2 = 1.5$. Then the profit on this sale is

$$2(1.2) - (C_1 + C_2) = 2.4 - 2.5 = -0.1\,,$$

or a loss of ten cents.

Problem 3

The distance (in miles) light travels is $d = 587 \times 10^{10}$ per year. In 100 years light would then travel 587×10^{12} miles.

Problem 4

We must make a rate r such that

$$4000(0.05) + 3500(0.04) + (10000 - 7500)r = 500\,.$$

Solving this for r I find $r = 0.064$.

Problem 5

If you draw/imagine a circle (with a radius r) surrounded by other circles and connect the center of the central circle with the center of each surrounding circle. Those segments have a length $2r$. The distance between the center any two adjacent "surrounding" circles is also $2r$. This means that the triangle connecting the center of the central circle and two adjacent surrounding circles is an equilateral triangle with a side length of $2r$. This triangle has a vertex angle of $60°$. We can place

$$\frac{360}{60} = 6\,,$$

such triangles around the center of the central circle. Counting the number of surrounding vertexes means that we have six surrounding circles.

Problem 6

Let x be the number of cows and y be the number of chickens. Then if l is the number of legs and h the number of heads we have

$$\begin{aligned} l &= 4x + 2y \\ h &= x + y\,. \end{aligned}$$

We are told that

$$l = 2h + 14\,,$$

or

$$4x + 2y = 2(x + y) + 14\,.$$

Solving this for x gives $x = 7$.

Problem 7

Write this as

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0\,,$$

or factoring based on the two roots r_1 and r_2 would give

$$(x - r_1)(x - r_2) = 0\,.$$

If we expand this and equate coefficients in the above expansion we get

$$r_1 r_2 = \frac{c}{a}\,.$$

If we are told that $r_1 r_2 = 1$ we see that $c = a$.

Problem 8

Write this as

$$2^3 \times 2^x = 5^0 = 1 = 2^0\,,$$

so $3 + x = 0$ or $x = -3$.

Problem 9

We have

$$((a^{9/6})^{1/3})^4(a9/3)^{4/6} = ((a^{3/2})^{4/3})(a^{12/6}) = (a^2)(a^2) = a^4\,.$$

Problem 10

If C is the center of the circle of radius 10 then when we extend from C to D the length of that segment is also 10. This means that in triangle $\triangle BCD$ we have two equal sides $CB = CD = 10$ and thus is isosceles with a vertex angle of $\angle BCD$. As each angle in the equilateral triangle $\triangle ABC$ is 60 degrees we have

$$\angle BCD = 180 - \angle ACB = 180 - 60 = 120\,.$$

Then since $\triangle BCD$ is isosceles we have

$$\angle CDB = \frac{1}{2}(180 - \angle BCD) = 30\,.$$

Problem 11

Call this expression E. Then we have

$$\begin{aligned} E &= 1 - \frac{(1-\sqrt{3})}{1-3} + \frac{(1+\sqrt{3})}{1-3} \\ &= 1 + \frac{1-\sqrt{3}}{2} - \frac{1+\sqrt{3}}{2} \\ &= 1 + \frac{1}{2} - \frac{\sqrt{3}}{2} - \frac{1}{2} - \frac{\sqrt{3}}{2} = 1 - \sqrt{3} \end{aligned}$$

Problem 12

We can write

$$\frac{x^{-1}-1}{x-1} = \frac{1-x}{x(x-1)} = -\frac{1}{x}\,.$$

Problem 13

This would be

$$\frac{(x-y)(-100)}{y} = \frac{100(y-x)}{y}\,.$$

Problem 14

Draw the circle in a Cartesian coordinate plane with its center O at $(0, 0)$. Have the point A at the value $(0, R)$ where R is the radius of the circle. The tangent line at A is then parallel to the x-axis of our coordinate plane. Let the point B be on the circle and in the fourth quadrant. Let the point C be on the circle and in the third quadrant. Then the secant line goes North-East and intersects the tangent line at P. Let the midpoint of the segment CB be denoted O'. Next draw segments from O to C (of length R) and form O to O' and from O to the point P.

The Pythagorean theorem in $\triangle OO'C$ we have

$$OO'^2 + O'C^2 = R^2 \quad \text{or} \quad OO'^2 + 10^2 = R^2 . \tag{78}$$

The Pythagorean theorem in $\triangle OAP$ we have

$$AP^2 + AO^2 = OP^2 \quad \text{or} \quad 300 + R^2 = OP^2 . \tag{79}$$

The Pythagorean theorem in $\triangle OO'P$ we have

$$O'P^2 + OO'^2 = OP^2 \quad \text{or} \quad (10 + BP)^2 + OO'^2 = OP^2 . \tag{80}$$

Setting Equation 79 and 80 equal gives

$$300 + R^2 = (10 + BP)^2 + OO'^2 .$$

Replacing OO'^2 using Equation 78 gives

$$300 + R^2 = (10 + BP)^2 + (R^2 - 100) .$$

We can solve this for BP to find $BP = 10$.

Problem 15

Multiply this by $x^2 - 4$ to get

$$15 - 2(x + 2) = x^2 - 4 ,$$

which we can write as

$$x^2 + 2x - 15 = 0 ,$$

or

$$(x + 5)(x - 3) = 0 .$$

This has solutions $x = -5$ and $x = 3$.

Problem 16

We are told that

$$x + y + z = 98\,, \tag{81}$$

and that

$$\begin{aligned} \frac{x}{y} &= \frac{2}{3} \quad \text{or} \quad x = \frac{2}{3}y \\ \frac{y}{z} &= \frac{5}{8} \quad \text{or} \quad y = \frac{5}{8}z\,. \end{aligned}$$

The second of these into the first means that

$$x = \frac{2}{3}\left(\frac{5}{8}z\right) = \frac{5}{12}z\,.$$

If we use these to write everything in terms of the variable z Equation 81 gives

$$\frac{5}{12}z + \frac{5}{8}z + z = 98\,.$$

Solving this gives $z = 48$. This means that $x = 20$ and $y = 30$.

Problem 17

If we add these two fractions we would get

$$\frac{(2x-3)A + B(x+2)}{(2x-3)(x+2)} = \frac{(2A+B)x + (2B-3A)}{(2x-2)(x+2)}\,.$$

If we set this equal to the given fraction we must have

$$\begin{aligned} 2A + B &= 5 \\ 2B - 3A &= -11 \end{aligned}$$

Solving these for A and B gives $A = 3$ and $B = -1$.

Problem 18

By "flipping" this we can write this expression as

$$10^{-2y} = 25^{-1}\,.$$

Taking the square root of this gives

$$10^{-y} = 5^{-1} = \frac{1}{5}\,.$$

Problem 19

The "rate" of the first and second candle's are given by

$$r_1 = \frac{L}{4}$$
$$r_2 = \frac{L}{3}.$$

Here we assume that the initial candle lengths are L. Note that $r_1 < r_2$ meaning that candle one burns "slower" than candle two. The lengths of each candle at the time t is given by

$$L_1(t) = L - r_1 t \quad \text{for} \quad 0 \le t \le 4$$
$$L_2(t) = L - r_2 t \quad \text{for} \quad 0 \le t \le 3.$$

We are asked about the time t when

$$L_1(t) = 2L_2(t).$$

Using the above expressions this is

$$L - r_1 t = 2(L - r_2 t).$$

If we solve this for t we find $t = \frac{12}{5} = 2\,\frac{2}{5}$ hours.

Problem 20

Write this as

$$\left(\frac{2}{10}\right)^x = 2.$$

Taking the logarithm base 10 of this gives

$$x(\log(2) - \log(10)) = \log(2),$$

or

$$x = \frac{\log(2)}{\log(2) - 1} = \frac{0.3010}{0.3010 - 1} = -0.43,$$

when we evaluate.

Problem 21

If one "draws" this situation and counts the number of intersections with the hyperbola I can count that we can have two, three, or four intersections.

Problem 22

Let v_0 be Jones' initial velocity. Then we are told that on his first trip we have

$$v_0 T_0 = 50\,,$$

miles. This means that $T_0 = \frac{50}{v_0}$. On his next trip Jones' traveled with a velocity of $3v_0$ so that

$$3v_0 T_1 = 300 \quad \text{so} \quad T_1 = \frac{100}{v_0} = 2\left(\frac{50}{v_0}\right) = 2T_0$$

Problem 23

The roots of this quadratic take the form

$$\frac{2\sqrt{2} \pm \sqrt{8 - 4ac}}{2a} = \frac{\sqrt{2} \pm \sqrt{2 - ac}}{a}\,.$$

We are told that $2 - ac = 0$. This means that the above becomes

$$\frac{\sqrt{2}}{a}\,.$$

These roots are equal and real.

Problem 24

Defining some angles and using the properties of isosceles triangles we have

$$\theta \equiv \angle ABC = \angle ACB$$
$$\alpha \equiv \angle ADE = \angle AED\,.$$

Now in triangle $\triangle ADE$ we have

$$\angle DAE = 180 - 2\alpha\,.$$

Now in triangle $\triangle ABC$ we have

$$2\theta + (30 + \angle DAE) = 180\,,$$

using the above we get

$$2\theta + (180 - 2\alpha) + 30 = 180\,,$$

which we can simplify to

$$\theta - \alpha = -15\,.$$

We can evaluate $\angle ADB$ in two ways

$$\angle ADB = 180 - (\theta + 30)\,,$$

or

$$\angle ADB = 180 - \angle ADC = 180 - (\alpha + x)\,.$$

Setting these equal we get

$$\alpha + x = \theta + 30\,,$$

or

$$\theta - \alpha = x - 30\,.$$

From the above we know that $\theta - \alpha = -15$ so the above is $-15 = x - 30$ and $x = 15$.

Problem 25

This sum can be evaluated as

$$\begin{aligned} S &= \sum_{k=1}^{n} (2k+1) = 2\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 = 2\left(\frac{n(n+1)}{2}\right) + n \\ &= n(n+2)\,, \end{aligned}$$

when we simplify.

Problem 26

Given the base angle and the vertex angle in an isosceles triangle we have determined all three angles. As there are many triangles that are similar to this configuration we have not uniquely determined the triangle.

Problem 27

Place the triangle so that the constant angle "opens" to the right and has one of its edges on the x axis of a Cartesian coordinate system. Let the length of that side of the original triangle be b and the length of the other side of that angle (again in the original triangle) have length l. Then the area of the original triangle is

$$A = \frac{1}{2} b \times (l \sin(\theta))\,.$$

Here θ is the radian measure of the angle and $l \sin(\theta)$ is the height of the triangle onto the base b. If l and b are doubled while θ does not change the above shows that the area will be multiplied by four.

Problem 28

Let W, D, S, and C stand for the amount (in dollars) received by the wife, daughter, son, and cook respectively. Let E be the value of the entire estate (again in dollars). Then in the problem statement we are told that

$$W + D + S + C = E \tag{82}$$
$$D + S = \frac{E}{2} \tag{83}$$
$$\frac{D}{S} = \frac{4}{3} \tag{84}$$
$$W = 2S \tag{85}$$
$$C = 500\,.$$

Equation 84 indicates that $D = \frac{4}{3}S$. If we put that into Equation 83 we get

$$\frac{4}{3}S + S = \frac{E}{2}\,.$$

This gives $S = \frac{3}{14}E$ so that $D = \frac{4}{3}S = \frac{2}{7}E$ and

$$W = 2S = \frac{3}{7}E\,.$$

Using these expressions for W, D, and S in terms of E in Equation 82 we get

$$\frac{3}{7}E + \frac{2}{7}E + \frac{3}{14}E + 500 = E\,.$$

Solving this for E gives $E = 7000$.

Problem 29

From the first equation we have $y = \frac{12}{x}$. If we put that into the second equation we get

$$x^2 + \left(\frac{12}{x}\right)^2 = 25\,,$$

which we can write as

$$x^4 - 25x^2 + 144 = 0\,.$$

We can factor this to get

$$(x^2 - 9)(x^2 - 16) = 0\,.$$

This means that $x = \pm 3$ (so $y = \pm 4$) or $x = \pm 4$ (so $y = \pm 3$). This means the intersection points are given by

$$(4, 3)\,, (3, 4)\,, (-3, -4)\,, (-4, -3)\,.$$

One can check that the slope of the lines connecting the two points $(-3, -4) \leftrightarrow (4, 3)$ and the two points $(-4, -3) \leftrightarrow (3, 4)$ is $+1$ while the slope connecting the two points $(-4, -3) \leftrightarrow (-3, -4)$ and the two points $(3, 4) \leftrightarrow (4, 3)$ is -1. As consecutive edges have slopes that have a product of -1 the edges are at right angles. Thus the connected points are a rectangle.

Problem 30

Dropping the altitude to the base of the triangle if the side is of length s then we must have

$$s\sin(60^\circ) = \sqrt{6}\,.$$

This gives $s = 2\sqrt{2}$. Then using the formula for the area of an equilateral triangle given its side length given in Equation 286 we get

$$\frac{\sqrt{3}}{4}(2\sqrt{2})^2 = 2\sqrt{3}\,.$$

Problem 31

The twentieth number would need to equal the number 20 in base ten. The largest two digit base four number would be 33 which is

$$3 \times 4 + 3 = 15\,.$$

The next base four number is 100 which equals 16. We need four more so the base four representation of twenty is 110.

Problem 32

As the two swimmers meet at exactly the same distance from where they started (during the same time) they are traveling the same speed and will travel one-half of the length of the pool in $T_0 = 1.5$ minutes. This means that they will both reach the opposite side of the pool in $2T_0$ and will meet again (at the center) at $3T_0 = 4.5$ minutes.

Problem 33

The $\sqrt{2}$ is an irrational number and thus has a infinite non-repeating decimal representation.

Problem 34

Call this expression E. Then let $n = 2$ and we find $E = 12$. Then from this specific case the only choice that is not eliminated is (A). To show that (A) is true in general we write this expression as

$$E = n^2(n-1)(n+1) = n[(n-1)n(n+1)]\,.$$

Now the term in brackets (i.e. $[\cdot]$) is the product of three consecutive integers and is thus divisible by three.

Now if n is even then the term n^2 will be divisible by four. If n is odd then $n-1$ and $n+1$ are even and the product $(n-1)(n+1)$ is divisible by four.

Thus for all values of n the expression E is divisible by $3 \times 4 = 12$.

Problem 36

This sum is equivalent to

$$\frac{K(K+1)}{2}.$$

If this is equal to a perfect square N^2 we can consider the K values given as solutions an see if when we put these into the above the value we obtain is a perfect square. For example if $K = 1$ the above is one which is a perfect square. If $K = 8$ the above is 36 which is also perfect square. If $K = 49$ the above is 1225 which is 35^2.

Problem 37

If d_1 and d_2 are the lengths of the diagonals of a rhombus then the area is given by $A = \frac{d_1 d_2}{2}$. We are told the length of one diagonal. The other diagonal will cut the 60° angle in $1/2$ forming a right triangle (diagonals of rhomboids are perpendicular) with an angle 30° and a leg length of $\frac{1}{2} \times \frac{3}{16} = \frac{3}{32}$. This means that one-half the other diagonal (call this x) must satisfy

$$\tan(30^\circ) = \frac{3/32}{x} \quad \text{so} \quad x = \frac{3\sqrt{3}}{32},$$

to give a second diagonal length of $2x = \frac{3\sqrt{3}}{16}$. The area is then

$$A_i = \frac{1}{2}\left(\frac{3}{16}\right)\left(\frac{3\sqrt{3}}{16}\right) = \frac{9\sqrt{3}}{2 \cdot 16^2},$$

inches squared. Now we are told that

$$\left(\frac{3}{2}\right)^2 = \frac{9}{4} = 400^2,$$

is the conversion between inches squared and miles squared. This means that our area in miles squared is

$$A_m = \frac{400^2\sqrt{3}}{2 \cdot 4^3} = 1250\sqrt{3},$$

when I simplify.

Problem 38

I drew my right triangle $\triangle ACB$ with one leg of length a along the x-axis of a Cartesian coordinate plane and another leg of length b along the y-axis. Lets specify the points A, B, and C at the locations $A = (0, b)$, $B = (a, 0)$ and $C = (0, 0)$. Then the altitude from C to the hypotenuse will intersect the hypotenuse at a point C'. Let the distances $AC' \equiv m$ and $C'B \equiv n$ so that

$$AC' + C'B = m + n = AB = c\,.$$

Now the altitude to the hypotenuse breaks the original triangle up into two parts a "upper" triangle and a "lower" triangle. The area of the full triangle is the sum of these two triangles which can be written (upper plus lower) as

$$\frac{1}{2}xm + \frac{1}{2}xn = \frac{x}{2}(m + n) = \frac{cx}{2}\,.$$

This is because in each of these subtriangles x is a height.

We can also evaluate the area of this triangle as $\frac{1}{2}ab$. Setting both of these equal to each other gives

$$xc = ab\,.$$

The expression we want in in terms of x, and a, and b so if we square this we get

$$x^2c^2 = a^2b^2\,,$$

using $c^2 = a^2 + b^2$ this is

$$(a^2 + b^2)x^2 = a^2b^2\,.$$

If we divide both sides by $x^2a^2b^2$ we get

$$\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{x^2}\,.$$

Problem 39

Let $a = n$ and $c = n + 1$ where n is an integer. Then the other leg is given by

$$b^2 = c^2 - a^2 = (n + 1)^2 - n^2 = 2n + 1 = n + n + 1 = a + c\,.$$

Problem 40

From what we are given we have $gt = V - V_0$. Using this in the second expression as

$$S = t\left(\frac{1}{2}gt + V_0\right) = t\left(\frac{V - V_0}{2} + V_0\right) = t\left(\frac{V + V_0}{2}\right)\,.$$

Solving this for t gives

$$t = \frac{2S}{V + V_0}\,.$$

Problem 41

We start with

$$3y^2 + y + 4 = 2(6x^2 + y + 2)\,,$$

or moving the y on the right-hand-side to the left-hand-side we have

$$3y^2 - y = 12x^2\,.$$

If we then take $y = 2x$ this then is

$$3(4x^2) - 2x = 12x^2\,.$$

Simplifying this we get $x = 0$. Thus there is only one solution.

Problem 42

Write this equation as

$$\sqrt{x+4} + 1 = \sqrt{x-3}\,.$$

If we square this we get

$$x + 4 + 2\sqrt{x+4} + 1 = x - 3\,,$$

or simplifying

$$2\sqrt{x+4} = -8\,.$$

As the left-hand-side is positive and the right-hand-side is negative we can have no solution.

Problem 43

A scalene triangle has sides of all different lengths. Without loss of generality let the three sides be $a < b < c$. Then we must have

$$a + b + c \leq 12\,.$$

From the triangle inequality we have $c < a + b$ so that

$$a + b + c > 2c\,,$$

thus

$$2c < 12 \quad \text{so} \quad c < 6\,.$$

As c is an integer this means that $c \leq 5$.

Lets consider the possible choices that we might be able to have. We have

- $c = 5$, $b = 4$, $a = 3$ which works.

- $c = 5$, $b = 4$, $a = 2$ which works.
- $c = 5$, $b = 4$, $a = 1$ which does not work because it does not satisfy the triangle inequality $a + b > c$.
- $c = 5$, $b = 3$, $a = 2$ which does not work because it does not satisfy the triangle inequality $a + b > c$.
- $c = 5$, $b = 3$, $a = 1$ which does not work because it does not satisfy the triangle inequality $a + b > c$.
- $c = 5$, $b = 2$, $a = 1$ which does not work because it does not satisfy the triangle inequality $a + b > c$.
- $c = 4$, $b = 3$, $a = 2$ which works.
- $c = 4$, $b = 3$, $a = 1$ which does not work because it does not satisfy the triangle inequality $a + b > c$.
- $c = 4$, $b = 2$, $a = 1$ which does not work because it does not satisfy the triangle inequality $a + b > c$.

Thus there are three such triangles.

Problem 44

If we multiply $x < a$ by x (which is negative) we get

$$x^2 > ax\,.$$

If we multiply $x < a$ by a (which is also negative) we get

$$ax > a^2\,.$$

Combining both of these expressions we have

$$x^2 > ax > a^2\,.$$

Problem 45

The number of revolutions in a mile with the initial radius is given by

$$N_0 = \frac{M}{2\pi r_0}\,,$$

where M is the length of a mile measured in inches (and r_0 measured in inches). Then if the radius changes to r_1 the change in the number of revolutions will be

$$N_1 - N_0 = \frac{M}{2\pi r_1} - \frac{M}{2\pi r_0} = \frac{M}{2\pi}\left(\frac{r_0 - r_1}{r_0 r_1}\right)\,,$$

or

$$\frac{N_1 - N_0}{N_0} = \frac{r_0 - r_1}{r_1} .$$

Now as $r_0 - r_1 = \frac{1}{4}$ inches and $r_0 = \frac{25}{2}$ inches (so that $r_1 = r_0 - \frac{1}{4}$) we have

$$\frac{N_1 - N_0}{N_0} = \frac{\frac{1}{4}}{\frac{25}{2} - \frac{1}{4}} = \frac{1}{49} \approx \frac{1}{50} = 0.02 .$$

This is an increase of 2%.

Problem 46

Let $f = \frac{N+1}{N} = 1 + \frac{1}{N}$ then the given equation is

$$1 + x = f(1 - x) .$$

solving this for x gives

$$x = \frac{f-1}{f+1} = \frac{\frac{1}{N}}{2 + \frac{1}{N}} = \frac{1}{1+2N} .$$

As we are told that $N > 0$ we have $\frac{1}{2N+1} < 1$ so that $x < 1$.

Problem 47

Let v be the rate of work for one machine measured in "jobs" per days. Let n be the number of machines the engineer has currently. Then we are told that

$$3nv = 1 , \tag{86}$$

meaning that in three days with the current number of machines we can get one job finished. We are also told that

$$2(n+3)v = 1 .$$

From Equation 86 we have $nv = \frac{1}{3}$. Putting that into the above gives

$$2\left(\frac{1}{3} + 3v\right) = 1 .$$

Solving for v gives $v = \frac{1}{18}$. We are asking for d where $dv = 1$. From the v before we see that this is $d = 18$ or eighteen days to do the job with one machine.

Problem 48

We want

$$\frac{3p+25}{2p-5} = n , \tag{87}$$

for n a positive integer. If we solve the above for p we get

$$p = \frac{5n+25}{2n-3} = \frac{6n-9-n+34}{2n-3} = 3 + \frac{34-n}{2n-3}\,.$$

Now when $n = 1$ we get

$$p = -10\,,$$

which is not a positive integer. If $n = 2$ we get

$$p = 35\,,$$

which is one. If $n = 3$ we get

$$p = \frac{40}{3} = 13.\bar{3}\,,$$

which is not a positive integer. If $n = 4$ we get

$$p = \frac{20+25}{5} = 9\,,$$

which is a positive integer. Notice that these numbers are decreasing as n increases. Continuing this pattern if we take $n \to \infty$ we see that

$$p \to \frac{5}{2} < 3\,.$$

If we take $p = 3$ in Equation 87 we get

$$n = \frac{9+25}{1} = 34\,,$$

an integer. Thus from this pattern we will get integer values of n when $3 \leq p \leq 35$.

Problem 50

I drew this triangle with AB along the x-axis and C "above" the segment AB. Then the segment DE is parallel to CB and forms the square $CDEB$. Now as $AC = CB$ we have $\angle CAB = \angle CBA$ and

$$\angle ACB = 180 - 2\angle CAB\,. \tag{88}$$

As $AC = CD$ (and using the above) we have

$$\begin{aligned}
\angle CAD &= \frac{1}{2}(180 - \angle ACD) = \frac{1}{2}(180 - (\angle ACB + 90)) \\
&= \frac{1}{2}(90 - \angle ACB) = \frac{1}{2}(90 - 180 + 2\angle CAB) \\
&= -45 + \angle CAB\,.
\end{aligned}$$

Now

$$\begin{aligned}
x &= \angle CAB - \angle CAD \\
&= \angle CAB - (-45 + \angle CAB) = 45\,.
\end{aligned}$$

The 1957 Examination

Problem 1

If we draw the isosceles triangle $\triangle ABC$ in the x-y plane with the point $B = (0, 0)$ the point $C = (c, 0)$ (where $c > 0$), and the point $A = (x_A, y_A)$ in the first quadrant with $y_A > 0$. Then the altitude, median, and angle bisector from A to BC are all the same line segment. These three segments from B to AC and from C to AB may not all be the same segment. Thus we have

$$1 + 3 + 3 = 7\,,$$

distinct lines.

Problem 2

If we write the given quadratic as

$$x^2 - \frac{h}{2}x + k = 0\,, \tag{89}$$

and if we denote the two roots as x_1 and x_2 we can write this as

$$(x - x_1)(x - x_2) = 0\,.$$

If we expand this out we see that it is equal to

$$x^2 - (x_1 + x_2)x + x_1x_2 = 0\,.$$

Equating coefficients of x in this expression with Equation 89 and using what we are told about the roots we see that

$$\begin{aligned} x_1 + x_2 &= 4 = \frac{h}{2} \\ x_1x_2 &= -3 = k\,. \end{aligned}$$

Thus we see that $h = 8$ and $k = -3$.

Problem 3

Let the given expression be denoted E. Then we can simplify E as

$$\begin{aligned} E &= 1 - \frac{1}{1 + \frac{a}{1-a}} = 1 - \frac{1}{\frac{1-a+a}{1-a}} \\ &= 1 - \frac{1}{\frac{1}{1-a}} = 1 - (1 - a) = a\,. \end{aligned}$$

Of course we can't perform these manipulations unless $a \neq 1$.

Problem 4

This would be C as that is $a = 3x + 2$ times the two terms in the other factor $x - 5$.

Problem 5

Write this expression as

$$\log\left(\frac{a}{b}\right) + \log\left(\frac{b}{c}\right) + \log\left(\frac{c}{d}\right) - \log\left(\frac{ay}{dx}\right) = \log\left(\frac{a}{b} \cdot \frac{b}{c} \cdot \frac{c}{d} \cdot \frac{dx}{ay}\right) = \log\left(\frac{x}{y}\right).$$

Problem 6

Performing the given folding, the volume V would be

$$x(10 - 2x)(14 - 2x) = x(140 - 20x - 28x + 4x^2) = x(4x^2 - 48x + 140).$$

Problem 7

From the area information the inscribed circle will have a radius r given by

$$\pi r^2 = 48\pi,$$

which means that $r = 4\sqrt{3}$.

This radius is "on" the perpendicular bisectors of each of the sides. This means that from the center of the circle the dropped perpendicular form a right triangle with one leg of length r, another leg of length $s/2$ (where s is the length of the side of the triangle) and a hypotenuse with an acute angle with the base of $\frac{1}{2}60° = 30°$. This means that

$$\tan(30°) = \frac{r}{s/2}.$$

From what we know about the tangent we have

$$\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{r}{s/2},$$

so that

$$s = 2\sqrt{3}r.$$

Thus the side length is $s = 8 \cdot 3 = 24$ so the perimeter is

$$3 \cdot 24 = 72.$$

Problem 8

We are told that

$$x : y : z = 2 : 3 : 5\,.$$

This means that

$$\begin{aligned} \frac{x}{z} &= \frac{2}{5} \quad \text{so} \quad x = \frac{2}{5}z \\ \frac{y}{z} &= \frac{3}{5} \quad \text{so} \quad y = \frac{3}{5}z\,. \end{aligned}$$

We are also told that

$$x + y + z = 100\,.$$

Replacing x and y with the relationships above in terms of z gives

$$x + y + z = \frac{2}{5}z + \frac{3}{5}z + z = 100\,.$$

Solving for z we get $z = 50$. This then means that

$$\begin{aligned} x &= 20 \\ y &= 30\,. \end{aligned}$$

Replacing x and y in $y = ax - 10$ with the above gives

$$30 = a(20) - 10\,.$$

Solving for a then gives

$$a = 2\,.$$

Problem 9

This would be

$$2 - (-2)^{2+2} = 2 - 2^4 = 2 - 16 = -14\,.$$

Problem 10

Write this as

$$\begin{aligned} y &= 2(x^2 + 2x) + 3 = 2(x^2 + 2x + 1 - 1) + 3 \\ &= 2(x + 1)^2 - 2 + 3 = 2(x + 1)^2 + 1\,. \end{aligned}$$

Thus the lowest point is when $x = -1$ where $y = 1$.

Problem 11

Starting at 2:00 as the "zero time" the angular position of the hands of the clock with zero degrees the location where the hands point straight up are given by

$$\theta_{\text{mh}}(t) = 0 + \frac{360}{60}t = 6t$$
$$\theta_{\text{hh}}(t) = 2\left(\frac{360}{12}\right) + \frac{360}{60(12)}t = 60 + \frac{1}{2}t\,.$$

Here θ_{mh} is the angular location of the minute hand, θ_{hh} is the angular location of the hour hand and t is the time after 2:00 in minutes. Thus using the above when $t = 15$ we have

$$\theta_{\text{mh}}(15) = 90$$
$$\theta_{\text{hh}}(15) = 60 + \frac{15}{2}\,.$$

This means that

$$\theta_{\text{mh}}(15) - \theta_{\text{hh}}(15) = \frac{45}{2} = 22.5^\circ\,.$$

Problem 12

Let $a = 10^{-49}$ and $b = 2 \cdot 10^{-50}$ then $5b = a$ which is the "opposite" of the statement (D). Next we have

$$a - b = 10^{-49}\left(1 - \frac{2}{10}\right) = 10^{-49}\left(\frac{4}{5}\right) = 0.8 \cdot 10^{-49} = 8 \cdot 10^{-50}\,,$$

which is (C).

Problem 13

The first two choices are not rational numbers. If r is a rational number between the two given numbers we must have

$$\sqrt{2} < r < \sqrt{3}\,.$$

Squaring this gives

$$2 < r^2 < 3\,. \tag{90}$$

For $r = 1.5$ we find $r^2 = 2.25$. For $r = 1.8$ we find $r^2 = 3.24$. For $r = 1.4$ we find $r^2 = 1.96$. The only value for r^2 that satisfies Equation 90 is when $r = 1.5$.

Problem 14

We can write y as

$$y = \sqrt{(x-1)^2} + \sqrt{(x+1)^2} = |x - 1| + |x + 1|\,.$$

Problem 15

We expect a function for $s(t)$ of the form $s(t) = s_0 + vt + \frac{a}{2}t^2$ for constants s_0, v, and a. Using the fact that $s(0) = 0$ we get $s_0 = 0$. Thus we now have

$$s(t) = vt + \frac{a}{2}t^2 \,.$$

From the table we should have $s(1) = 10$ and $s(2) = 40$ so

$$\begin{aligned} v + \frac{a}{2} &= 10 \\ 2v + \frac{a}{2}(4) &= 40 \,. \end{aligned}$$

Solving these give $v = 0$ and $a = 20$ so

$$s(t) = 10t^2 \,.$$

From this we compute $s(2.5) = 10\left(\frac{5}{2}\right)^2 = 62.5$.

Problem 16

As one can't buy a fraction of a goldfish the cost of n goldfish is $0.15n$ and a plot of cost as a function of n would be a set of distinct points i.e. $(1, 0.15)$, $(2, 0.3)$, $(3, 0.45)$, etc. up to $(12, 1.8)$.

Problem 17

Lets draw our cube in the x-y-z Cartesian coordinate system with the vertices as

$$\begin{aligned} A &= (3, 0, 0) \\ B &= (3, 0, 3) \\ C &= (3, 3, 3) \\ D &= (3, 3, 0) \\ E &= (0, 3, 0) \\ F &= (0, 3, 3) \\ G &= (0, 0, 3) \\ H &= (0, 0, 0) \,. \end{aligned}$$

Then the path $A \to H$ in order of the letters alphabetically and then back to A will have eight legs for a total length of $8 \times 3 = 24$.

Problem 18

Draw the segment MB. Then as AB is a diameter we have that $\angle AMB = 90°$. With this triangle we have that

$$\triangle AOP \sim \triangle AMB\,.$$

This means that we can write

$$\frac{AP}{AB} = \frac{AO}{AM} \quad \text{or} \quad AP \cdot AM = AO \cdot AB\,.$$

Problem 19

This would be

$$10011 = 1 + 1 \cdot 2^1 + 0 \cdot 2^2 + 0 \cdot 2^3 + 1 \cdot 2^4 = 1 + 2 + 16 = 19\,.$$

Problem 20

The average velocity is the total distance divided by the total time. If the length of the distance driven (in miles) is L then the total distance driven is $2L$ and the total time driven is

$$T = \frac{L}{50} + \frac{L}{45}\,.$$

This means that the average velocity is

$$\bar{v} = \frac{2L}{\frac{L}{50} + \frac{L}{45}} = \frac{2}{\frac{1}{50} + \frac{1}{45}}\,.$$

As the least common multiple of 45 and 50 is 450 if we multiply $\bar{v}$ by $\frac{450}{450}$ we get

$$\bar{v} = \frac{900}{19} = 47\,\frac{7}{19}\,.$$

Problem 22

Write this as

$$\sqrt{x-1} + 1 = \sqrt{x+1}\,,$$

then square both sides to get

$$(x-1) + 2\sqrt{x-1} + 1 = x + 1\,,$$

which simplifies to

$$2\sqrt{x-1} = 1\,.$$

If we square both sides again we get

$$4(x-1)=1 \quad \text{or} \quad x=\frac{5}{4}.$$

If we put this value back into the original expression we get

$$\sqrt{\frac{1}{4}}-\sqrt{\frac{5}{4}}+1=\frac{1}{2}-\frac{\sqrt{5}}{2}+1=\frac{3-\sqrt{5}}{2}\neq 0.$$

Thus there are no solutions to this equation.

Problem 23

If we subtract these two equations we get

$$x^2-x=0,$$

which has the solutions $x=0$ or $x=1$. If $x=0$ then $y=10$. If $x=1$ then $y=9$. The distance between these two points is

$$\sqrt{(0-1)^2+(10-9)^2}=\sqrt{2}.$$

Problem 24

Let x and y be digits so that our two numbers are xy and yx. If we subtract their squares we have

$$\begin{aligned}(xy)^2-(yx)^2 &= (10x+y)^2-(10y+x)^2 \\ &= 10^2x^2+20xy+y^2-(10^2y^2+20xy+x^2) \\ &= 100(x^2-y^2)+(y^2-x^2)=(100-1)(x^2-y^2) \\ &= 99(x^2-y^2)=9\cdot 11\cdot(x-y)(x+y).\end{aligned}$$

This will be divisible by nine, eleven, the sum of the digits, and the difference of the digits. This will not be divisible by the product of the digits. To see this consider $x=2$ and $y=1$ then we have

$$(21)^2-(12)^2=441-144=297=9\cdot 11\cdot 3,$$

which is not divisible by $xy=2$.

Problem 25

In the x-y Cartesian coordinate plane the area of a triangle denoted by the three points R, P, and Q is given by

$$\frac{1}{2}\left|\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right|\right|,$$

which is the norm of the vector cross product of the two vectors in the plane. In this problem the location of the points are given. Thus the vectors needed are

$$\begin{aligned}\overrightarrow{PQ} &= (b-0, 0-a) = (b, -a)\\ \overrightarrow{PR} &= (c-0, d-a) = (c, d-a)\,.\end{aligned}$$

Here I have "constructed" the cross product so that its value will be positive (pointing out of the page). We next compute

$$\overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ b & -a & 0\\ c & d-a & 0\end{vmatrix} = \hat{k}(bd-ab+ac)\,.$$

Thus the area is given

$$\frac{bd-ab+ac}{2}\,.$$

Problem 26

The intersection of the medians of a triangle is a point called the "centroid" upon which we are able to balance the entire triangle if we were to place it on the point of a pin. This "means" that the areas of the three triangles must be equal for if one was different we would expect our triangle to not balance.

Problem 27

Using Vieta's formula

https://en.wikipedia.org/wiki/Vieta's_formulas

if r_1 and r_2 are roots of the given polynomial then

$$\begin{aligned}r_1 + r_2 &= -p\\ r_1 r_2 &= q\,.\end{aligned}$$

We want to evaluate

$$\frac{1}{r_1}+\frac{1}{r_2} = \frac{r_1+r_2}{r_1 r_2} = -\frac{p}{q}\,.$$

Problem 28

This is the identity $b^{\log_b(a)} = a$.

Problem 29

If $x = 0$ then this inequality is satisfied. If $x \neq 0$ then it is equivalent to

$$x^2 - 1 \geq 0 \,,$$

or

$$(x+1)(x-1) \geq 0 \,.$$

If $x = \pm 1$ then this inequality is satisfied. If $x < -1$ then $x - 1 < 0$ and $x + 1 < 0$ so their product is positive. If $-1 < x < 1$ then $x-1 < 0$ and $x+1 > 0$ so their product is negative. If $x > 1$ then $x - 1 > 0$ and $x + 1 > 0$ so their product is positive. The the total region where this inequality is true is $x = 0$ or $x \leq -1$ or $x \geq +1$.

Problem 30

If we take $n = 1$ this formula would be

$$\sum_{k=1}^{1} k^2 = 1 = \frac{(1+c)(2+k)}{6} \,. \tag{91}$$

If we take $n = 2$ this formula would be

$$\sum_{k=1}^{2} k^2 = 1 + 2^2 = 5 = \frac{2(2+c)(4+k)}{6} \,. \tag{92}$$

From Equation 91 we have

$$k = \frac{6}{1+c} - 2 \,. \tag{93}$$

If we put that into Equation 92 we have

$$15 = (2+c)\left(\frac{6}{1+c} + 2\right) \,.$$

Solving this for c gives $c \in \left\{1, \frac{1}{2}\right\}$. If $c = \frac{1}{2}$ then Equation 93 gives $k = 2$ and the formula becomes

$$\frac{n\,(n+1/2)\,(2n+2)}{6} \,.$$

If $c = 1$ then Equation 93 gives $k = 1$ and the formula becomes

$$\frac{n(n+1)(2n+1)}{6} \,.$$

As these two formulas are the same we have a consistent solution. One of which is $(c, k) = (1, 1)$.

Problem 31

Let the leg length of the isosceles right triangles be l. Then the length of the hypotenuse of these triangles is then $\sqrt{l^2+l^2} = \sqrt{2}l$. Now a given side of the original square, will have two lengths of l "cut off" when we remove these triangles. The length of what remains is $1-2l$ and is one side of the octagon. Another side of the octagon is the hypotenuse of the removed triangles. If these two lengths are equal we must have

$$1 - 2l = \sqrt{2}l\,.$$

Solving for l I find

$$l = \frac{1}{2+\sqrt{2}} = \frac{1}{2+\sqrt{2}} \times \left(\frac{2-\sqrt{2}}{2-\sqrt{2}}\right) = \frac{2-\sqrt{2}}{2}\,.$$

Problem 32

Notice that we can write

$$n^5 - n = n(n^4-1) = n(n^2-1)(n^2+1) = (n-1)n(n+1)(n^2+1)\,.$$

Now $n-1$, n, $n+1$ is a sequence of three consecutive integers and thus the product will be divisible by both two and three and thus by $2 \times 3 = 6$. I claim that $n^5 - n$ is also divisible by five. To show that let $n = 5k + r$ where $0 \le r \le 4$ and then using the binomial theorem we have

$$n^5 = (5k+r)^5 = \sum_{l=0}^{5} \left(\begin{array}{c} 5 \\ l \end{array}\right) (5k)^l r^{5-l}\,.$$

This means that

$$\begin{aligned} n^5 - n &= \sum_{l=0}^{5} \left(\begin{array}{c} 5 \\ l \end{array}\right) (5k)^l r^{5-l} - 5k - r \\ &= \sum_{l=2}^{5} \left(\begin{array}{c} 5 \\ l \end{array}\right) (5k)^l r^{5-l} + \left(\begin{array}{c} 5 \\ 1 \end{array}\right) (5k) r^4 + \left(\begin{array}{c} 5 \\ 0 \end{array}\right) r^5 - 5k - r \\ &= \sum_{l=2}^{5} \left(\begin{array}{c} 5 \\ l \end{array}\right) (5k)^l r^{5-l} + 5k(5r^4 - 1) + r^5 - r\,. \end{aligned}$$

Now the sum and the term after it are divisible by five and it remains to be shown that $r^5 - r$ is divisible by five. We find

- If $r = 0$ then $r^5 - r = 0$ which is divisible by five.
- If $r = 1$ then $r^5 - r = 0$ which is divisible by five.
- If $r = 2$ then $r^5 - r = 30$ which is divisible by five.

- If $r = 3$ then $r^5 - r = 240$ which is divisible by five.
- If $r = 4$ then $r^5 - r = 1020$ which is divisible by five.

Thus in all cases $n^5 - n$ is divisible by five. Thus the entire expression is divisible by $6 \times 5 = 30$.

Problem 33

Write this equation as

$$9^2 \cdot 9^x - 9^x = 240\,,$$

or

$$(81 - 1)9^x = 240 \quad \text{or} \quad 9^x = \frac{240}{80} = 3\,.$$

This means that $x = \frac{1}{2}$.

Problem 34

The points (x, y) that satisfy the inequality $x^2 + y^2 < 25$ are the ones in a circle in the x-y Cartesian plane centered at zero with a radius of five and not including the circles boundary. The points that satisfy the equality $x + y = 1$ are the points on a line. The intersection of these two sets will then be a straight line segment but not including its two end-points.

Problem 35

Break the segment AC into eight equal parts at the points $C_1\,, C_2\,, C_3\,, \ldots\,, C_7\,, C_8$ where $C_8 = C$ in the original diagram. Then

$$AC_1 = C_1C_2 = C_2C_3 = \cdots = C_6C_7 = C_7C = \frac{AC}{8}\,.$$

Vertically above each C_i place the point B_i on the segment AB. Then each of the triangles $\triangle AC_iB_i$ is similar to the original triangle $\triangle ACB$. This means that

$$\frac{C_iB_i}{AC_i} = \frac{CB}{AC}\,,$$

for $i = 1, 2, \ldots, 6, 7$. Solving for C_iB_i we get

$$C_iB_i = \frac{CB}{AC} AC_i\,.$$

The length AC_i is given by

$$AC_i = \left(\frac{AC}{8}\right) i\,.$$

Using this in the above we get that

$$C_iB_i = \frac{CB}{AC} \times \left(\frac{AC}{8}\right) i = \frac{CB}{8} i\,.$$

As $CB = 10$ we get $C_iB_i = \frac{5}{4}i$. Thus the sum we want to evaluate is given by

$$\sum_{i=1}^{7} C_iB_i = \frac{5}{4}\sum_{i=1}^{7} i = \frac{5}{4}\left(\frac{8(7)}{2}\right) = 35\,.$$

Problem 36

We want to maximize xy subject to $x+y=1$. The later is $y = 1-x$ so we want to maximize

$$x(1-x) = -x^2 + x\,.$$

This is a quadratic. We can optimize it by completing the square or by using calculus. The later would require setting the first derivative equal to zero. This is

$$-2x + 1 = 0\,.$$

This has the solution $x = \frac{1}{2}$. The second derivative is $-2 < 0$ indicating that we have found a maximum. This maximum has a value of $-\frac{1}{4} + \frac{1}{2} = \frac{1}{4}$.

Completing this square we can write the above as

$$\begin{aligned} -x^2 + x = -(x^2 - x) &= -\left(x^2 - x + \frac{1}{4} - \frac{1}{4}\right) \\ &= -\left(x - \frac{1}{2}\right)^2 + \frac{1}{4}\,. \end{aligned}$$

This functional form has the same conclusions about its maximum as before.

Problem 37

We have $MC = AC - AM = 12 - x = NP$. From how y is defined as $y = MN + NP$ we have

$$MN = y - NP = y - (12 - x) = x + y - 12\,.$$

From the similar triangles $\triangle ACB \sim \triangle AMN$ we have

$$\frac{BC}{MN} = \frac{AC}{AM}\,,$$

or

$$\frac{5}{x+y-12} = \frac{12}{x}\,.$$

If we solve the above for y we get

$$y = \frac{144 - 7x}{12}\,.$$

a	b
3	0
4	1
5	2
6	3
7	4
8	5
9	6

Table 1: Choices of a and b where $a - b = 3$.

Problem 38

Let the original two digit number be ab where $1 \le a \le 9$ and $0 \le b \le 9$ then the subtraction requested is

$$\begin{aligned} ab - ba &= a \cdot 10 + b - (b \cdot 10 + a) \\ &= 10(a-b) + (b-a) = (10-1)(a-b) = 9(a-b)\,. \end{aligned}$$

We are told that

$$9(a-b) = n^3\,,$$

or

$$3^2(a-b) = n^3\,,$$

for some positive integer n. Now for the ranges of a and b denoted above we have that $-8 \le a - b \le 9$. If $a - b < 0$ then $3^2(a-b) < 0$ and n^3 would not be positive (as we are told it should be). This means that we need to have $1 \le a - b \le 9$. Now if $a - b \in \{1, 2, 4, 5, 6, 7, 8, 9\}$ then $3^2(a-b)$ will not be a perfect cube. If $a - b = 3$ it will be. A table of the possible values for a and b where $a - b = 3$ is given in Table 1. There are seven of these choices.

Problem 39

The total distance covered by the first man after t hours is $4t$ miles. The second man will cover a *total* distance of

$$\begin{cases} \frac{4}{2}t & \text{when} \quad 0 < t < 1 \\ \frac{4}{2} + \frac{5}{2}(t-1) & \text{when} \quad 1 < t < 2 \\ \frac{9}{2} + \frac{6}{2}(t-2) & \text{when} \quad 2 < t < 3 \\ \frac{15}{2} + \frac{7}{2}(t-3) & \text{when} \quad 3 < t < 4 \end{cases}$$

This pattern will continue. If we let the first interval (the one where $0 < t < 1$) be denoted $n = 0$ then the general pattern for n seems to have an "offset" of

$$\sum_{k=1}^{n} \frac{k+3}{2} = \frac{3}{2}n + \frac{1}{2}\sum_{k=1}^{n} k = \frac{3n}{2} + \frac{n(n+1)}{4}\,,$$

and a "slope" of
$$\frac{4+n}{2}.$$
Thus the total distance the second man will have traveled is given by
$$\frac{3n}{2}+\frac{n(n+1)}{4}+\left(\frac{4+n}{2}\right)(t-n) \quad \text{when} \quad n<t<n+1.$$
We would like to find t and n such that the total distance traveled by both men is 72 miles or
$$\frac{3n}{2}+\frac{n(n+1)}{4}+\left(\frac{4+n}{2}\right)(t-n)+4t=72.$$
Lets assume that this happens for $t=n$ for some n. Then we need to solve for n in
$$\frac{3n}{2}+\frac{n(n+1)}{4}+4n=72.$$
This is a quadratic in n and has solutions $n=9$ or $n=-32$. Thus $t=n=9$ *is* a solution! In nine hours the first man has traveled $4(9)=36$ miles and the second man has then traveled $72-36=36$ miles. Thus they meet midway between the points M and N.

Problem 40

Write y as
$$\begin{aligned} y &= -(x^2-bx)-8 \\ &= -\left(x^2-bx+\left(\frac{b}{2}\right)^2-\left(\frac{b}{2}\right)^2\right)-8 \\ &= -\left(x-\frac{b}{2}\right)^2+\frac{b^2}{4}-8. \end{aligned}$$
To have our vertex on the x-axis means that when $x=\frac{b}{2}$ we have $y=0$ so
$$\frac{b^2}{4}-8=0,$$
so $b=\pm 4\sqrt{2}$ which is a positive or a negative irrational number.

Problem 41

This is a linear system for x and y and thus will *not* have a unique solution if
$$\left|\begin{array}{cc} a & a-1 \\ a+1 & -a \end{array}\right| = -a^2-(a-1)(a+1) = -a^2-(a^2-1) = -2a^2+1=0.$$
Solving this for a I find $a=\pm\frac{\sqrt{2}}{2}$.

Problem 42

As $i^4 = 1$ we have $i^{4n} = 1$. Thus lets write

$$i^{-n} = \frac{1}{i^n}\left(\frac{i^{3n}}{i^{3n}}\right) = \frac{i^{3n}}{1} = i^{3n}\,.$$

This means that we can write S (called S_n here) as

$$S_n = i^n + i^{3n} = i^n + (i^3)^n = i^n + (-i)^n\,.$$

From this we see that

- If $n = 0$ we have $S_0 = 2$
- If $n = 1$ we have $S_1 = i - i = 0$.
- If $n = 2$ we have $S_2 = -1 + i^2 = -2$
- If $n = 3$ we have $S_3 = -i - i^3 = -i + i = 0$
- If $n = 4$ we have $S_4 = i^4 + (-i)^4 = 1 + 1 = 2$
- If $n = 5$ we have $S_5 = i^5 + (-i)^5 = -i - (-i) = 0$

and the pattern of values seems to repeat. We see that

$$\begin{aligned}
S_{4k} &= i^{4k} + (-i)^{4k} = 1 + 1 = 2\\
S_{4k+1} &= i^{4k+1} + (-i)^{4k+1} = i + (-i) = 0\\
S_{4k+2} &= i^{4k+2} + (-i)^{4k+2} = i^2 + (-i)^2 = -1 - 1 = -2\\
S_{4k+3} &= i^{4k+3} + (-i)^{4k+3} = i^3 + (-i)^3 = -i - (-i) = 0\,.
\end{aligned}$$

Thus there are *three* distinct values for S_n.

Problem 43

If we "draw" this region in the x-y plane with the idea of "counting" the number of lattice points in this region.

- Notice that if $x = 0$ then we can have $y = 0$ and we have one lattice point in this region with $x = 0$.
- If $x = 1$ then $0 \le y \le 1$ and we have two lattice points in this region with $x = 1$.
- If $x = 2$ then $0 \le y \le 4$ and we have five lattice points in this region with $x = 2$.
- If $x = 3$ then $0 \le y \le 9$ and we have ten lattice points in this region with $x = 3$.
- If $x = 4$ then $0 \le y \le 17$ and we have 17 lattice points in this region with $x = 4$.

This is a total of $1 + 2 + 5 + 10 + 17 = 35$ total lattice points.

Problem 44

Let $\angle CAD = \theta$ and $\angle DAB = \phi$. Then as $AC = CD$ we have $\angle CDA = \angle CAD = \theta$. In triangle $\triangle ACD$ this means that

$$\angle ACD = \pi - 2\theta\,.$$

Along the line CDB by supplementary angles we have

$$\angle ADB = \pi - \angle ADC = \pi - \theta\,.$$

From the triangle $\triangle ABC$ we have that

$$\begin{aligned}\angle ABC &= \pi - \angle BAC - \angle ACB = \pi - (\angle CAD + \angle DAB) - (\pi - 2\theta)\\ &= 2\theta - (\theta - \phi) = \theta - \phi\,.\end{aligned}$$

If we compute the given angle difference we find

$$\angle CAB - \angle ABC = (\angle CAD + \angle DAB) - (\theta - \phi) = (\theta + \phi) - (\theta - \phi) = 2\phi = 30\,.$$

This means that $\phi = 15$ and is also the measure of the angle we seek.

Problem 45

From the given expression if we take $y = 1$ we have

$$x = x - 1\,,$$

which is a contradiction so $y \neq 1$. Given that if we solve

$$\frac{x}{y} = x - y\,, \tag{94}$$

for x in terms of y we get

$$x = \frac{y^2}{y-1}\,. \tag{95}$$

In the above if $y = 2$ then $x = 4$ and we have two integer solutions. If $y = \pi$ then x will be irrational.

To solve for y in terms of x we write the given expression as

$$y^2 - xy + x = 0\,,$$

which has solutions

$$y = \frac{x \pm \sqrt{x^2 - 4x}}{2}\,.$$

As we are told that y is real this means that

$$x^2 - 4x = x(x-4) \geq 0\,.$$

This means that $x \leq 0$ or $x \geq 4$.

From all of these facts we have shown that B, C, D, and E are false and A must be true.

Problem 46

If you can recall that the center of the circle will be at the intersection of the perpendicular bisectors of the two chords you can draw these bisectors and find their intersection. This intersection will be at a distance of $\frac{2+6}{2} = 4$ from the $(2, 6)$ chord and at a distance of $\frac{3+4}{2} = \frac{7}{2}$ from the end of the $(3, 4)$ chord. If we draw this point then the center of the circle has a radius that is given by the hypotenuse of a right triangle with a side of length $\frac{1}{2}$ and four. Thus

$$R^2 = \frac{1}{4} + 16 = \frac{65}{4},$$

so $R = \frac{\sqrt{65}}{2}$ and a diameter of the circle of $2R = \sqrt{65}$.

Problem 47

The segment XY is on the perpendicular bisector of AB. This means that $BM = AM$. Next as $MQ = QB$ (and $MQ = QA$) we have that $\angle ABM = \angle BAM = 45°$. This means that $\widehat{AD} = 2\angle ABM = 90°$. Now draw the segments $AO = DO = r$. Then $\angle AOD = \widehat{AD} = 90°$ and triangle $\triangle AOD$ is a right triangle. This means that

$$AD^2 = AO^2 + DO^2 = r^2 + r^2 = 2r^2 \quad \text{so} \quad AD = \sqrt{2}r.$$

Problem 48

Note that if $M = C$ then $AM = AC = 0 + CB = CM + MB$ and we have equality between the two expressions. The same holds if $M = B$ and we again have equality of the length AM and the sum $BM + CM$. To show that this is true in general, along the segment AM move from M towards A a distance of CM and call that point P. We will now show that the two triangles $\triangle APC$ and $\triangle BMC$ are congruent.

As $CM = MP$ we see that the triangle $\triangle CMP$ is isosceles. As the triangle $\triangle ABC$ is equilateral each internal angle is $60°$. This means that

$$\widehat{AC} = \widehat{CB} = \widehat{AB} = 2(60) = 120.$$

Now $\angle CMP = \angle CMA = \frac{1}{2}\widehat{AC} = 60°$. This means that $\triangle CMP$ is actually an equilateral triangle and thus

$$CP = CM. \tag{96}$$

Also

$$AC = CB, \tag{97}$$

as both sides are sides in the equilateral triangle $\triangle ABC$. Note that

$$\angle CAM = \frac{1}{2}\widehat{CM} = \angle CBM. \tag{98}$$

Note that $\angle AMB = \frac{1}{2}\widehat{AB} = \frac{1}{2}(120) = 60^\circ$ and as $\triangle CMP$ is an equilateral triangle we have

$$\angle CMB = \angle CMP + \angle AMB = 60^\circ + 60^\circ = 120^\circ\,.$$

Using supplementary angles we have

$$\angle CPA = 180^\circ - \angle CPM = 120^\circ\,.$$

Thus we have just shown that

$$\angle CPA = 120^\circ = \angle CMB\,. \tag{99}$$

Now Equations 98 and 99 have shown equivalence of two of the corresponding angles between the triangles $\triangle APC$ and $\triangle BMC$ thus the third angle must be equal also. Using Equations 96 and 97 with the above angle equivalence we can apply the theorem "Side-Angle-Side" to show the congruence

$$\triangle APC \cong \triangle BMC\,.$$

This means that $AP = MB$. Using this with we have that

$$AM = MP + PA = MC + MB\,,$$

showing that equality is always true.

Problem 49

Let the "bottom" trapezoid have the length of its left-most slanted side be a so that the length of the "top" trapezoid's left-most slanted side is $6 - a$. Let the "bottom" trapezoid have the length of its right-most slanted side be b so that the length of the "top" trapezoid's right-most slanted side is $4 - b$. Let the length of the "top" of the "bottom" trapezoid be h (this is also the "bottom" of the "top" trapezoid). The fact that the "top" and the "bottom" trapezoid have the same perimeter means that

$$3 + (6 - a) + h + (4 - b) = h + a + 9 + b\,,$$

or simplifying this is

$$a + b = 2\,. \tag{100}$$

We are asked about the values of the fractions

$$\frac{6-a}{a} \quad \text{and} \quad \frac{4-b}{b}\,,$$

and told that they are equal. Setting these two expressions equal we get $b = \frac{2}{3}a$. Putting this into Equation 100 we can solve for a to find $a = \frac{6}{5}$. This then means that $b = \frac{4}{5}$. We can check that

$$\frac{6-a}{a} = 4$$
$$\frac{4-b}{b} = 4\,.$$

Thus the ratio we seek is 4:1.

Problem 50

Draw our circle in the x-y Cartesian coordinate plane with its center at the origin $(0,0)$ and a radius R. Let the diameter AB be the segment on the x-axis from $A = (-R, 0)$ to $B = (+R, 0)$. Then if x is a location on the x-axis such that $-R \leq x \leq +R$ from the construction of the points A' and B' we have that $A' = (-R, x - R)$ and $B' = (R, R - x)$. The midpoint of the segment $A'B'$ is located at the point

$$\left(\frac{1}{2}(-R + R), \frac{1}{2}(x - R + R - x)\right) = (0, 0)\,,$$

which is a fixed point independent of x.

The 1958 Examination

Inverses of square roots

Squaring and “square rooting” are not exactly inverses of each other. Recall that

$$(\sqrt{x})^2 = x\,, \tag{101}$$

but that

$$\sqrt{x^2} = |x|\,, \tag{102}$$

note the absolute value. See if you can find where these identities are used in this test.

Problem 1

We have

$$[2-3(2-3)^{-1}]^{-1} = [2-3(-1)^{-1}]^{-1} = (2+3)^{-1} = \frac{1}{5}\,.$$

Problem 2

We are given

$$\frac{1}{x} - \frac{1}{y} = \frac{1}{z}\,,$$

from which we have

$$\frac{1}{z} = \frac{y-x}{xy} \Rightarrow z = \frac{xy}{y-x}\,.$$

Problem 3

We can write the given expression as

$$\frac{a^{-1}b^{-1}}{a^{-3}-b^{-3}}\left(\frac{a^3b^3}{a^3b^3}\right) = \frac{a^2b^2}{b^3-a^3}\,.$$

Problem 4

Replacing each x in the original expression we get

$$\frac{(\frac{x+1}{x-1})+1}{(\frac{x+1}{x-1})-1}\,,$$

which we could simplify if needed. Another way to solve this problem is to recognize that when $x = \frac{1}{2}$ then

$$\frac{x+1}{x-1} = \frac{\frac{3}{2}}{-\frac{1}{2}} = -3\,.$$

If we put this in for x into the expression we get

$$\frac{x+1}{x-1} = \frac{-2}{-4} = \frac{1}{2}\,.$$

Problem 5

Call this expression E. Then we have

$$\begin{aligned}
E &= 2+\sqrt{2}+\frac{1}{2+\sqrt{2}}+\frac{1}{\sqrt{2}-2} \\
&= 2+\sqrt{2}+\frac{1}{2+\sqrt{2}}-\frac{1}{2-\sqrt{2}} \\
&= 2+\sqrt{2}+\frac{(2-\sqrt{2})-(2+\sqrt{2})}{(2+\sqrt{2})(2-\sqrt{2})} \\
&= 2+\sqrt{2}+\frac{-2\sqrt{2}}{4-2} = 2+\sqrt{2}-\sqrt{2} = 2\,.
\end{aligned}$$

Problem 6

Computing the arithmetic mean of these two numbers we find

$$\frac{1}{2}\left(\frac{x+a}{x}+\frac{x-a}{x}\right) = \frac{1}{2}\left(\frac{2x}{x}\right) = 1\,.$$

Problem 7

The line that joins these two points takes the form

$$y-1 = \left(\frac{1-9}{-1-3}\right)(x+1) = \frac{-8}{-4}(x+1) = 2(x+1)\,,$$

or

$$y = 1+2(x+1)\,.$$

The x-intercept of this line is the x value when $y = 0$. Taking $y = 0$ in the above we get

$$\frac{-1}{2} = x+1 \Rightarrow x = -\frac{3}{2}\,.$$

Problem 8

Many of these expressions are known to be irrational. Normally taking cube roots gives irrational numbers but for perfect cubes that statement is not true. Looking at the numbers given we can simplify them some to decide if they are rational or irrational. For the second number we have

$$\sqrt[3]{\frac{8}{10}} = \frac{2}{\sqrt[3]{10}}\,,$$

which is irrational. For the third number we have

$$\sqrt[4]{\frac{16}{10^5}} = \frac{2}{10^{5/4}}\,,$$

which is irrational. For the fourth number we have

$$\sqrt[3]{-1}\sqrt{(0.09)^{-1}} = -\frac{1}{\sqrt{0.09}} = -\frac{1}{\sqrt{\frac{9}{100}}} = -\sqrt{\frac{100}{9}} = -\frac{10}{3}\,,$$

which is rational.

Problem 9

We want to solve

$$x^2 + b^2 = (a - x)^2 = a^2 - 2ax + x^2\,,$$

or

$$b^2 - a^2 = -2ax\,.$$

This has the solution

$$x = \frac{a^2 - b^2}{2a}\,.$$

Problem 10

We want to know when

$$x^2 + kx + k^2 = 0\,,$$

has real roots. Using the quadratic equation we find the solutions to the above given by

$$x = \frac{-k \pm \sqrt{k^2 - 4k^2}}{2} = -\frac{k \pm \sqrt{-3k^2}}{2} = \frac{-k \pm i\sqrt{3}|k|}{2}\,.$$

For no real value for k (but zero) will this expression be real.

Problem 11

We want to find the roots of

$$\sqrt{5-x} = x\sqrt{5-x}\,.$$

Now if $\sqrt{5-x} \neq 0$ we can divide by it to get $x = 1$. If $\sqrt{5-x} = 0$ then we have the solution that $x = 5$. Thus there are two real roots to this equation.

Problem 12

We want to solve

$$P = \frac{s}{(1+k)^n}\,,$$

for n. From the above we have

$$(1+k)^n = \frac{s}{P}\,,$$

or taking logarithms this gives

$$n\log(1+k) = \log\left(\frac{s}{P}\right)\,.$$

Solving for n we get

$$n = \frac{\log\left(\frac{s}{P}\right)}{\log(1+k)}\,.$$

Problem 13

We want to evaluate

$$\frac{1}{x} + \frac{1}{y} = \frac{y+x}{xy} = \frac{10}{20} = \frac{1}{2}\,.$$

Problem 14

If we work "backwards" note that the last boy will dance with all g girls. The next-to-last boy will dance with $g-1$ girls. The next-to-next-to-last boy will dance with $g-2$ girls and so on. Until we get to the first boy who will dance with five girls. This means that there are more girls than boys and that

$$b = g - 4\,.$$

We can "check" this expression satisfies the problem statement by imagining the smallest numbers which satisfy the above namely $g = 5$ and $b = 1$. Then the last (and first) boy dances with all (and five) girls. As another "check" if $g = 6$ and $b = 2$ then the last boy dances with all the girls (six of them) and the next to last boy dances with five girls.

Problem 16

The radius of the inscribed circle must satisfy

$$\pi r^2 = 100\pi\,,$$

so $r = 10$. As a hexagon has six sides any triangle with a vertex at the circle center and two edges to the corners of the hexagon will have a angle

$$\frac{360}{6} = 60\,,$$

degrees. This means that the height of this triangle will have $h = r = 10$ and will be the leg of a right triangle with one acute angle given by

$$\frac{60}{2} = 30\,,$$

degrees. This means that the other leg of this triangle will have a length given by

$$r\tan(30^\circ) = \frac{r}{\sqrt{3}} = \frac{10}{\sqrt{3}}\,.$$

This triangle then has an area of

$$\frac{1}{2} \times \left(\frac{10}{\sqrt{3}}\right) 10 = \frac{50}{\sqrt{3}}\,.$$

There are a total of $6 \times 2 = 12$ such triangles in this hexagon giving a total area of

$$12 \times \left(\frac{50}{\sqrt{3}}\right) = \frac{600}{\sqrt{3}} = 200\sqrt{3}\,.$$

Problem 17

From the given expression

$$\log(x) \geq \log 2 + \frac{1}{2}\log(x)\,,$$

we can solve for $\log(x)$ to get

$$\log(x) \geq \log 2^2\,.$$

This means that $x \geq 4$.

Problem 18

The area of a circle is $A = \pi r^2$ and we are told that

$$2A = \pi(r+n)^2\,.$$

This means that

$$2\pi r^2 = \pi(r+n)^2 \,.$$

Dividing by π and expanding the right-hand-side gives

$$2r^2 = r^2 + 2rn + n^2 \,,$$

or

$$0 = r^2 - 2nr - n^2 \,.$$

Solving for r we get

$$r = \frac{2n \pm \sqrt{4n^2 - 4(-n^2)}}{2} \,.$$

This simplifies to

$$r = n(1 \pm \sqrt{2}) \,.$$

Now r must be positive so $r = n(1+\sqrt{2})$.

Problem 19

To start, we draw this triangle with the leg of length a on a vertical (i.e. y axis) and the leg of length b on a horizontal (i.e. x axis). We then draw a segment from the origin to the hypotenuse as described. Let this segments length be h then it splits the original triangle into two other right triangles.

- the bottom one has legs s, h, and a hypotenuse b
- the top one has legs h, r, and a hypotenuse a

From the original larger triangle as we are told that $a : b = 1 : 3$ we have that

$$a : b : c = 1 : 3 : \sqrt{10} \,.$$

Next we note that the bottom smaller right triangle is similar to the original right triangle with the correspondence

$$h : s : b = a : b : c = 1 : 3 : \sqrt{10} \,. \tag{103}$$

The top smaller right triangle is similar to the original right triangle with the correspondence

$$r : h : a = a : b : c = 1 : 3 : \sqrt{10} \,. \tag{104}$$

From Equation 104 we have

$$\frac{r}{h} = \frac{1}{3} \,.$$

From Equation 103 we have

$$\frac{h}{s} = \frac{1}{3} \,.$$

If we multiply these two we get

$$\frac{r}{s} = \frac{1}{9} \,,$$

and thus we have $r : s = 1 : 9$.

Problem 20

Write this as

$$4^x - 4^{x-1} = 24\,,$$

or

$$4^x(1 - 4^{-1}) = 24\,,$$

or

$$4^x\left(\frac{3}{4}\right) = 24\,,$$

or

$$4^{x-1} = 8\,,$$

or

$$2^{2(x-1)} = 2^3\,.$$

This means that $2(x-1) = 3$ or that $x = \frac{5}{2}$. From that we find $2x = 5$ so that

$$(2x)^x = 5^{5/2} = 5^{2+\frac{1}{2}} = 25\sqrt{5}\,.$$

Problem 21

The distance from O to A, B, C, D, and E is the radius of the circle r. If COD is "straight" and EO is perpendicular to COD then the area of $\triangle CED$ is

$$\frac{1}{2}bh = \frac{1}{2}(2r)r = r^2\,.$$

Now from the arc AB we have that

$$\angle AOB = \frac{360}{4} = 90^\circ\,.$$

Thus $\triangle AOB$ is a right triangle. This means that the area of triangle AOB is

$$\frac{1}{2}r^2\,.$$

This means that the ratio of the area of $\triangle CED$ to that of $\triangle AOB$ is

$$\frac{r^2}{\frac{1}{2}r^2} = 2\,.$$

Problem 22

The particle starts at the point P which has an x location where $y = 6$. The x value for this value of y is then given by

$$x^2 - x - 6 = 6\,,$$

which can be written as

$$(x+3)(x-4) = 0\,.$$

Thus the two values of x where $y = 6$ are

$$x = -3 \quad \text{and} \quad x = 4\,.$$

This means that $P = (-3, 6)$ or $P = (4, 6)$. Now the point Q has $y = -6$ which means that x can be

$$x^2 - x - 6 = -6\,,$$

which can be written as

$$x^2 - x = x(x-1) = 0\,.$$

Thus there are two solutions where $y = -6$ are $x = 0$ and $x = 1$. This means that the point Q can be $Q = (0, -6)$ or $Q = (1, -6)$.

For $P = (-3, 6)$ the nearest of these two points is $Q = (0, 6)$ at a distance of three. For $P = (4, 6)$ the nearest of these two points is $Q = (1, -6)$ also at a distance of three.

Problem 23

In the given expression we will replace

$$x \to x \pm a\,,$$

to get

$$(x \pm a)^2 - 3 = x^2 \pm 2ax + a^2 - 3\,.$$

The change between this and the original expression is then

$$(x \pm a)^2 - 3 - (x^2 - 3) = x^2 \pm 2ax + a^2 - x^2 = \pm 2ax + a^2\,.$$

Problem 24

Let f be the fraction that converts from feet to miles when you multiply by it. Thus

$$f = \frac{1}{5260}\,\frac{\text{miles}}{\text{feet}}\,.$$

Now the man travels a distance mf miles North and then back South.

His velocity North (in miles-per-hour) is given by

$$v_{\text{North}} = \frac{1\,\text{miles}}{2\,\text{minutes}} \times \left(\frac{60\,\text{minutes}}{1\,\text{hour}}\right) = 30\frac{\text{miles}}{\text{hour}}\,.$$

On his return, his velocity South is

$$v_{\text{South}} = 2\,\frac{\text{miles}}{\text{minute}} \times \left(\frac{60\,\text{minutes}}{1\,\text{hour}}\right) = 120\frac{\text{miles}}{\text{hour}}\,.$$

The time he travels North (in hours) is then given by

$$\frac{mf}{30}\,.$$

The time he travels South (in hours) is then given by

$$\frac{mf}{120}\,.$$

Now the average velocity $\bar{v}$ is the total distance traveled divided by the total time. For this problem that is given by

$$\bar{v} = \frac{2mf}{\frac{mf}{30} + \frac{mf}{120}} = \frac{2}{\frac{1}{30} + \frac{1}{120}} = \frac{2(120)}{4+1} = \frac{2(120)}{5} = 2(4+20) = 2(24) = 48\,.$$

Problem 25

We are told that

$$\log_k x \cdot \log_5 k = 3\,,$$

or

$$\frac{\log x}{\log k} \cdot \frac{\log k}{\log 5} = 3\,,$$

or

$$\log x = 3 \log 5 = \log 5^3\,.$$

This means that $x = 5^3 = 125$.

Problem 26

The sum is given by

$$s = \sum_{i=1}^{n} x_i\,.$$

If we change the values of x_i into x_i' given by

$$x_i' = 5(x_i + 20) - 20 = 5x_i + 80\,.$$

Thus the sum of these new numbers is then given by

$$s' = \sum_{i=1}^{n} x_i' = 5\sum_{i=1}^{n} x_i + 80n = 5s + 80n\,.$$

Problem 27

Equating the slope between the first two points and the second two points gives

$$\frac{3-(-3)}{4-2} = \frac{\frac{k}{2}-3}{5-4}\,,$$

or

$$\frac{6}{2} = \frac{k}{2} - 3\,.$$

Solving for k we get $k = 12$.

Problem 28

Let f_i be fraction of water after each of the ith removal of liquid and replacement with pure antifreeze. Then $f_0 = 1$ as we start with pure antifreeze. Next for f_1 we have

$$f_1 = \frac{0+3(4)}{16} = \frac{12}{16} = \frac{3}{4}\,.$$

Now for f_2 this will be

$$\begin{aligned} f_2 &= \frac{\text{\# of quarts of water}}{\text{16 quarts}} \\ &= \frac{\frac{3}{4}(16) - \frac{3}{4}(4)}{16} = \left(\frac{3}{4}\right)^2\,. \end{aligned}$$

Now $f_1 \cdot 16$ is the amount of water in the radiator and $f_1 \cdot 4$ is the amount of water removed. The difference is the amount of water remaining. Computing f_2 using the method we see that

$$f_2 = \frac{f_1 \cdot 16 - f_1 \cdot 4}{16} = f_1 - \frac{4}{16} \cdot f_1 = \left(1 - \frac{1}{4}\right) f_1 = \frac{3}{4} f_1\,. \tag{105}$$

This means that

$$f_2 = \frac{3}{4} \cdot \frac{3}{4} = \frac{9}{16}\,.$$

Following Equation 105 for another step we see that

$$f_3 = \frac{3^3}{4^3}\,,$$

Following Equation 105 for another step we see that

$$f_4 = \frac{3^4}{4^4} = \frac{81}{256}\,.$$

Problem 29

A hint for this problem is that we expect that the correct solution will be symmetric in the variables in the drawing. To me, only the solution E has that property and thus we will try to prove that E is correct. From $\triangle AEB$ we have

$$x + y = 180 - z\,.$$

But $z = 180 - m$ so that we have

$$x + y = m\,.$$

This of course means that

$$x + y - m = 0\,.$$

If we do the same thing for the "other side" of the triangle we would get

$$a + b - n = 0\,.$$

If we set these two expressions equal to each other we get

$$x + y - m = a + b - n\,,$$

which is equivalent to

$$x + y + n = a + b + m\,.$$

This is E.

Problem 30

We are told that $xy = b$ and

$$\frac{1}{x^2} + \frac{1}{y^2} = a\,.$$

From the second expression we get

$$\frac{x^2 + y^2}{x^2 y^2} = a\,.$$

Using $xy = b$ this is $x^2 + y^2 = ab^2$. Next using

$$(x + y)^2 = x^2 + 2xy + y^2 = x^2 + 2b + y^2\,.$$

Taken together this means that

$$(x + y)^2 = ab^2 + 2b = b(ab + 2)\,.$$

Problem 31

Draw this isosceles triangle with a base of b along the x-axis and two equal sides of length s and a height of length h. Then in terms of these variables the perimeter can be written as

$$2s + b = 32 \Rightarrow b = 32 - 2s\,. \tag{106}$$

From the right triangle obtained when we drop the height onto the base by using the Pythagorean theorem we have

$$s^2 = h^2 + \left(\frac{b}{2}\right)^2\,.$$

In this if we replace b using Equation 106 and $h = 8$ we get

$$s^2 = 8^2 + (16 - s)^2\,.$$

If we expand and simplify this we get $s = 10$ and using Equation 106 we get $b = 12$. This means that the area of the original triangle is

$$\frac{1}{2}bh = \frac{1}{2}(12)(8) = 48\,.$$

Problem 32

Let s be the number of steers bought and c the number of cows bought. Then based on the cost of each we are told that

$$1000 = 25s + 26c\,. \tag{107}$$

As we are looking for integer solutions for s and c we will solve for s (which has a coefficient that is a divisor of the left-hand-side of 1000) and we get

$$s = \frac{1000}{25} - \frac{26}{25}c = 40 - \frac{26}{25}c\,.$$

For s to be an integer means that $c = 25n$ for n an integer with $n \geq 1$. In that case

$$s = 40 - 26n\,.$$

We now ask how large can n be in the above equation and still have a positive solution for s? If we take $n = 1$ we find $s = 14$. If we take $n = 2$ however we find $s < 0$. Thus we have only one solution to this problem and it is given by taking $n = 1$ where we find that $c = 25$ and $s = 14$.

Problem 33

Let the given root be denoted r and then by dividing by the leading coefficient a we can write the given quadratic as

$$x^2 + \frac{b}{a}x + \frac{c}{a} = (x - r)(x - 2r)\,.$$

If we expand the right-hand-side of the above we get

$$x^2 - (r + 2r)x + 2r^2 = x^2 - 3rx + 2r^2 \,.$$

Equating coefficient of x with the original quadratic then gives

$$\begin{aligned} -3r &= \frac{b}{a} \\ 2r^2 &= \frac{c}{a} \,. \end{aligned}$$

From the first equation we have

$$r = -\frac{b}{3a} \,.$$

If we put this into the second equation we gets

$$2\left(-\frac{b}{3a}\right)^2 = \frac{c}{a} \,.$$

We can simplify this to

$$2b^2 = 9ca \,.$$

Problem 34

For the numerator to be larger than the denominator when

$$6x + 1 > 7 - 4x \,,$$

or

$$x > \frac{3}{5} = 0.6 \,.$$

Intersecting this range with the original domain of $-2 \leq x \leq 2$ we get

$$\frac{3}{5} < x \leq 2 \,.$$

Problem 35

Draw the given triangle with its vertex C at the origin of an x-y coordinate plane and the A vertex "on-the-left" and the B vertex "on-the-right". Let the vertex A have coordinates (a, c) where $c > 0$. Let the vertex B have coordinates (b, d) where $d > 0$. Now we can break down the area of $\triangle OAB$ into the area of a parallelogram minus two triangles as

$$\text{Area}(\triangle OAB) = \text{Area}(DCBA) - \text{Area}(\triangle OCB) - \text{Area}(\triangle OAD) \,.$$

Each of these areas can be then be computed as

$$\text{Area}(DCBA) = \frac{1}{2}(b-a)(c+d)$$
$$\text{Area}(\triangle OCB) = \frac{1}{2}ca$$
$$\text{Area}(\triangle OAD) = \frac{1}{2}db\,.$$

As each of a, b, c, and d are integers all of the above are rational. Because all of these expressions are rational so must be Area($\triangle OAB$).

Problem 36

If we draw this triangle and let the altitude be of length h and the side of length 80 be broken up into a length x_1 (below the side 30) and a length x_2 (below the side 70). If we use the Pythagorean theorem for the left and right triangles that result we get

$$x_1^2 + h^2 = 30^2 \tag{108}$$
$$x_2^2 + h^2 = 70^2 \tag{109}$$
$$x_1 + x_2 = 80\,. \tag{110}$$

If we subtract the first two of these equations we get

$$x_1^2 - x_2^2 = 30^2 - 70^2 = 900 - 4900 = -4000\,,$$

or

$$x_2^2 - x_1^2 = 4000\,,$$

or

$$(x_2 - x_1)(x_2 + x_1) = 4000\,.$$

Now using Equation 110 in the above we get

$$x_2 - x_1 = 50\,. \tag{111}$$

Adding this to Equation 110 to get

$$2x_2 = 130 \Rightarrow x_2 = 65\,.$$

So from Equation 111

$$x_1 = 80 - 65 = 15\,.$$

The larger of the two size is $x_2 = 65$.

Problem 37

Let this sum be denoted as S. Then for S we have

$$\begin{aligned} S &= \sum_{l=0}^{2k}(k^2+1+l) = (k^2+1)(2k+1) + \sum_{l=0}^{2k} l = (k^2+1)(2k+1) + \sum_{l=1}^{2k} l \\ &= (k^2+1)(2k+1) + \frac{2k(2k+1)}{2} = (2k+1)(k^2+k+1) \\ &= k^2+k+1+2k^3+2k^2+2k = 2k^3+3k^2+3k+1 \\ &= k^3+k^3+3k^2+3k+1 = k^3+(k+1)^3 \,. \end{aligned}$$

Problem 38

We are told that

$$s = \frac{y}{r} \quad \text{and} \quad c = \frac{x}{r} \,.$$

These look like the definition of the trigonometric functions $\sin(x)$ and $\cos(x)$. Thus we have that

$$s^2 + c^2 = 1 \,,$$

so that

$$s^2 - c^2 = s^2 - (1 - s^2) = 2s^2 - 1 \,.$$

Now following the analogy with the trigonometric function we know that

$$0 \le s^2 \le 1 \,,$$

which means that

$$0 \le 2s^2 \le 2 \,,$$

and so

$$-1 \le 2s^2 - 1 \le 1 \,.$$

Problem 39

Let $v = |x|$ then

$$v^2 + v - 6 = 0 \quad \text{or} \quad (v+3)(v-2) = 0 \,.$$

This means that $v = -3$ or $v = 2$. We know that $v \ge 0$ so the only solution is $v = |x| = 2$. Thus $x = \pm 2$ and the sum is zero.

Problem 40

We want to know the value of a_3. We know that $a_0 = 1$ and $a_4 = 3$ and that

$$a_n^2 - a_{n-1}a_{n+1} = (-1)^n\,.$$

This means that

$$a_{n+1} = \frac{a_n^2 - (-1)^n}{a_{n-1}}\,.$$

So if we take $n = 1$ we get

$$a_2 = \frac{a_1^2 - (-1)^1}{a_0} = \frac{9+1}{1} = 10\,,$$

and if we take $n = 2$ we get

$$a_3 = \frac{a_2^2 - (-1)^2}{a_1} = \frac{100-1}{3} = 33\,.$$

Problem 41

As we are told the roots of

$$x^2 + px + q = 0\,,$$

are s^2 and r^2 we can use that information to write the above as

$$(x - s^2)(x - r^2) = 0\,.$$

Expanding the left-hand-side of this and equating powers of x this means that

$$\begin{aligned} q &= s^2 r^2 \\ p &= -(s^2 + r^2)\,. \end{aligned} \tag{112}$$

Next for $Ax^2 + Bx + C = 0$ to have roots of r and s means that

$$\begin{aligned} r &= \frac{-B - \sqrt{B^2 - 4AC}}{2A} \\ s &= \frac{-B + \sqrt{B^2 - 4AC}}{2A}\,. \end{aligned}$$

From these expressions this means that

$$\begin{aligned} s^2 + r^2 &= \frac{1}{4A^2}\left[(-B - \sqrt{B^2 - 4AC})^2 + (-B + \sqrt{B^2 - 4AC})^2\right] \\ &= \frac{1}{4A^2}\left[B^2 + 2B\sqrt{B^2 - 4AC} + (B^2 - 4AC) + B^2 - 2B\sqrt{B^2 - 4AC} + (B^2 - 4AC)\right] \\ &= \frac{1}{4A^2}\left(2B^2 + 2(B^2 - 4AC)\right) = \frac{1}{A^2}(B^2 - 2AC)\,. \end{aligned}$$

Thus

$$p = -(s^2 + r^2) = \frac{2AC - B^2}{A^2}\,.$$

Note that in the above we have used the fact that $(\sqrt{x})^2 = x$.

Problem 43

We start by drawing the given triangle $\triangle ABC$ and the two medians AD and BE (these segments go from a vertex to the midpoint of the opposite side).

In the right triangle ECB the Pythagorean theorem gives

$$EC^2 + BC^2 = EB^2 \, ,$$

or

$$EC^2 + (2CD)^2 = 16 \, ,$$

or

$$EC^2 + 4CD^2 = 16 \, . \tag{113}$$

Next in the right triangle ACD the Pythagorean theorem gives

$$(2EC)^2 + CD^2 = AD^2 \, ,$$

or

$$4EC^2 + CD^2 = 49 \, . \tag{114}$$

These are two equations in two unknowns. Put EC^2 from Equation 113 into Equation 114 to get

$$4(16 - 4CD^2) + CD^2 = 49 \, .$$

We can solve this for CD and find $CD = 1$. Putting this into Equation 113 gives

$$EC^2 = 16 - 4 = 12 \, ,$$

so

$$EC = 2\sqrt{3} \, .$$

Thus using what we know we have

$$\begin{aligned} AC &= 2EC = 4\sqrt{3} \\ BC &= 2CD = 2 \, . \end{aligned}$$

The Pythagorean theorem in the triangle ACB gives

$$AB^2 = AC^2 + BC^2 = 16 \cdot 3 + 4 = 48 + 4 = 52 = 2^2 \cdot 13 \, ,$$

which means that

$$AB = 2\sqrt{13} \, .$$

Problem 44

I find that none of these expressions is provably true given the hypothesis.

Problem 45

In terms of x and y the correct amount of the check is given by

$$c = x + \frac{y}{100},$$

while the incorrect amount of the check is given by

$$i = y + \frac{x}{100}.$$

We are told that $i - c = 17.82$ or

$$y - x + \frac{x}{100} - \frac{y}{100} = 17.82.$$

We can simplify and write this as

$$y - x = \frac{1782}{99} = 18. \tag{115}$$

Now we know that x and y are two digit numbers so

$$10 \le x \le 99 \quad \text{and} \quad 10 \le y \le 99.$$

To test B (which seems the easiest one to check) if we let $y = 2x$ in Equation 115 we get $x = 18$ and then $y = 2x = 36$ which is a valid solution.

Note that if y is two digits $y \le 99$ which means that

$$x + 18 \le 99 \quad \text{or} \quad x \le 81,$$

which allows x to be larger than 70 and thus test A is false.

Problem 46

We are told that

$$-4 < x < 1.$$

Write the given expression $E(x)$ as

$$E(x) \equiv \frac{x^2 - 2x + 2}{2(x-1)} = \frac{(x-1)^2 + 1}{2(x-1)} = \frac{1}{2}(x-1) + \frac{1}{2(x-1)}. \tag{116}$$

From the range above we have

$$-5 < x - 1 < 0.$$

One thing this means is that

$$-\frac{5}{2} < \frac{x-1}{2} < 0.$$

Another is that

$$-\infty < \frac{1}{x-1} < -\frac{1}{5},$$

so

$$-\infty < \frac{1}{2(x-1)} < -\frac{1}{10}.$$

If we add these two we see that

$$-\infty < \frac{x-1}{2} + \frac{1}{2(x-1)} < -\frac{1}{10}.$$

Thus this expression has no minimum value but has a maximum value.

To find the x where E is largest we computer $\frac{dE}{dx}$ using Equation 116 and find

$$\frac{dE}{dx} = \frac{1}{2}\left(1 - \frac{1}{(x-1)^2}\right).$$

Setting this equal to zero and solving for x we get

$$x^2 - 2x = 0.$$

Thus the maximum occurs when $x = 0$ or $x = 2$. Only $x = 0$ is in the domain $-4 < x < 1$ and we find

$$E(0) = \frac{2}{2} = 1.$$

Problem 48

Let the center of the circle be located at the center of a Cartesian coordinate system at $(0, 0)$. Then in this coordinate system we can take $A = (-5, 0)$, $B = (0, 5)$, $C = (-1, 0)$ and $D = (1, 0)$. Let a point (x, y) be on the circle. Based on the location of these points the distance $CP + PD$ can be written as

$$CP + PD = \sqrt{(x+1)^2 + y^2} + \sqrt{(x-1)^2 + y^2}.$$

But as (x, y) is on the circle we have $x^2 + y^2 = 5^2$ which means that we can write the above as

$$CP + PD = \sqrt{2x + 1 + 5^2} + \sqrt{-2x + 1 + 5^2} = \sqrt{26 + 2x} + \sqrt{26 - 2x}. \qquad (117)$$

Now if we are at the location $(x, y) = (-5, 0)$ then

$$CP + PD = 4 + 6 = 10,$$

and if we are at the location $(x, y) = (5, 0)$ then we have

$$CP + PD = 10,$$

also. If $x = 0$ and $y = 5$ this is

$$CP + PD = 2\sqrt{26} = 2\sqrt{25 + 1} > 10.$$

This indicates that as a hypothesis we should consider

$$CP + PD \geq 10\,,$$

for all x.

The extremes of the sum $CP + PD$ as a function of x will happen when the derivative of $CP + PD$ with respect to x is equal to zero. Based on Equation 117 this expression is

$$\begin{aligned}\frac{d}{dx}(CP + PD) &= \frac{(2)}{2\sqrt{26+2x}} + \frac{(2)}{2\sqrt{26-2x}} \\ &= \frac{1}{\sqrt{26+2x}} + \frac{1}{\sqrt{26-2x}} = 0\,.\end{aligned}$$

We can write the above as

$$\frac{1}{\sqrt{26+2x}} = -\frac{1}{\sqrt{26-2x}}\,.$$

If we square each side of this and "flip" we get

$$26 - 2x = 26 + 2x\,.$$

This has the solution $x = 0$. To see if $x = 0$ is a minimum or a maximum we consider the second derivative of $CP + PD$ with respect to x. We find

$$\frac{d^2}{dx^2}(CP + PD) = -\frac{1}{2}(26+2x)^{-\frac{3}{2}}(2) + \frac{1}{2}(26-2x)^{-\frac{3}{2}}(-2)\,.$$

If we evaluate this at $x = 0$ we find

$$\left.\frac{d^2}{dx^2}(CP + PD)\right|_{x=0} = -(26)^{-\frac{3}{2}} - (26)^{-\frac{3}{2}} < 0\,.$$

This means that $x = 0$ is a maximum. This means that $CP + PD$ is longest when $P = (x, y) = (0, 5)$. Note that this point P is equidistant between C and D.

Problem 49

We can *compute* this expansion and then count how many terms there are. We first have

$$\begin{aligned}(a+b+c)^{10} &= \sum_{i=0}^{10}\binom{10}{i}(a+b)^i c^{10-i} \\ &= \sum_{i=0}^{10}\binom{10}{i}\left(\sum_{j=0}^{i}\binom{i}{j}a^j b^{i-j}\right)c^{10-i} \\ &= \sum_{i=0}^{10}\sum_{j=0}^{i}\binom{10}{i}\binom{i}{j}a^j b^{i-j} c^{10-i}\,.\end{aligned}$$

In this expression, the *inner* sum has $i+1$ terms. This means that the total number of terms T is given by

$$T = \sum_{i=0}^{10}(i+1) = \sum_{i=1}^{10} i + 11 = \frac{10(10+1)}{2} + 11 = 55 + 11 = 66\,,$$

terms.

Problem 50

From the diagram given, for this problem we note that to have $A \to A'$ we need to have $x = 3$ mapped to $y = 5$. to have $B \to B'$ we need to have $x = 4$ mapped to $y = 1$. The line that passes though the points $(3, 5)$ and $(4, 1)$ has a slope given by

$$m = \frac{5 - 1}{3 - 4} = \frac{4}{-1} = -4\,,$$

and thus looks like

$$y - 5 = -4(x - 3)\,.$$

We can write this as

$$y = 17 - 4x\,.$$

Thus if $x = a$ then $y = 17 - 4a$ so

$$x + y = 17 - 3a\,.$$

The 1959 Examination

Problem 1

If s is the side length of the side of the cube then we are told that $s \to 1.5s = \frac{3}{2}s$. This means that the surface area of a cube is given by

$$\text{SA} = 6s^2\,,$$

so the new surface area is

$$\text{SA}' = 6(1.5s)^2 = 6\left(\frac{3}{2}s\right)^2 = \frac{9}{4}\cdot 6s^2\,.$$

This means that the percent increase in surface area is given by

$$\frac{\text{SA}' - \text{SA}}{\text{SA}} = \frac{\text{SA}'}{\text{SA}} - 1 = \frac{9}{4} - 1 = \frac{5}{4} = 1.25\,.$$

Which is 125 percent.

Problem 2

Drawing this triangle with a base AB on the x-axis and the vertex C above the base AB. We then draw a line segment $A'B'$ parallel to AB and though the point P.

Let the altitude from vertex C onto the segment AB intersect AB at the point D and intersect the segment $A'B'$ at the point D'. Then we are told that $CD = 1$ and we want to know the distance $PD = D'D$.

The area of the "top" triangle $CA'B'$ is given by

$$\frac{1}{2}(A'B')(CD')\,.$$

The area of the full triangle ABC is given by

$$\frac{1}{2}(AB)(CD)\,.$$

As the parallel segment $A'B'$ creates a "top" triangle ($\triangle CA'B'$) that is similar to the full triangle ($\triangle ABC$) we know that

$$\begin{aligned} A'B' &= \alpha AB \\ CD' &= \alpha CD\,, \end{aligned}$$

for some $0 \le \alpha \le 1$. This means that the area of the top triangle is α^2 the area of the full triangle. As we are told that

$$\text{Area}\triangle CA'B' = \frac{1}{2}\text{Area}\triangle CAB\,.$$

This means that

$$\alpha^2 = \frac{1}{2} \quad \text{so} \quad \alpha = \frac{1}{\sqrt{2}}\,.$$

As we are told that $CD = 1$ we then know that $CD' = \alpha CD = \alpha$. Thus

$$D'D = CD - CD' = 1 - \alpha = 1 - \frac{1}{\sqrt{2}} = \frac{\sqrt{2}-1}{\sqrt{2}} = \frac{2-\sqrt{2}}{2}\,.$$

Problem 4

We are told that

$$x + \frac{x}{3} + \frac{x}{6} = 78\,.$$

As

$$1 + \frac{1}{3} + \frac{1}{6} = \frac{6}{6} + \frac{2}{6} + \frac{1}{6} = \frac{9}{6} = \frac{3}{2}\,,$$

this means that

$$x = \frac{2}{3}(78) = 52\,.$$

Thus the middle part is

$$\frac{x}{3} = \frac{52}{3} = 17\,\frac{1}{3}\,.$$

Problem 5

If we use $256 = 2^8$ we can write this value as

$$(256)^{0.16}(256)^{0.09} = (2^8)^{\frac{16}{100}}(2^8)^{\frac{9}{100}} = (2^8)^{\frac{25}{100}} = (2^8)^{\frac{1}{4}} = 2^{\frac{8}{4}} = 2^2 = 4\,.$$

Problem 6

The converse of this statement is "if Q is a rectangle then Q is a square". This is false. The inverse of this statement is "if Q is not a square then Q is not a rectangle" which is also false.

Problem 7

The hypothesis must be the larger length. Using the Pythagorean theorem we get

$$a^2 + (a+d)^2 = (a+2d)^2\,,$$

or expanding we get

$$a^2 + a^2 + 2ad + d^2 = a^2 + 4ad + 4d^2\,,$$

which we can simplify to

$$a^2 - 2ad - 3d^2 = 0\,.$$

If we divide this by d^2 we get

$$\left(\frac{a}{d}\right)^2 - 2\left(\frac{a}{d}\right) - 3 = 0\,.$$

Using the quadratic equation we find that $\frac{a}{d}$ is equal to

$$\frac{a}{d} = \frac{-(-2) \pm \sqrt{4 - 4(-3)}}{2} = \frac{2 \pm \sqrt{4(1+3)}}{2} = \frac{2 \pm 4}{2}\,.$$

As the ratio must be positive so that we must take the positive sign so that we get

$$\frac{a}{d} = \frac{2+4}{2} = 3\,.$$

This means that $a : d = 3 : 1$.

Problem 8

We write this expression as

$$x^2 - 6x + 13 = (x^2 - 6x + 9) + 4 = (x-3)^2 + 4\,.$$

Thus we see that this expression is never less than four.

Problem 9

The fraction of the herd fourth son gets would be

$$1 - \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{5}\right) = 1 - \left(\frac{10}{20} + \frac{5}{20} + \frac{4}{20}\right) = 1 - \left(\frac{19}{20}\right) = \frac{1}{20}\,.$$

We are told that

$$\frac{n}{20} = 7 \Rightarrow n = 140\,.$$

Problem 10

Here I draw the triangle ABC with the BC along the x-axis and the segment BA in a North-Easterly direction. Stepping from A "backwards" 1.2 we get point D on AB. Finally point E is drawn on the extension AC.

As the area of $\triangle AED$ is equal to the area of $\triangle ABC$ and that the "base" of $\triangle AED$ (i.e. the segment DA) is

$$\frac{1.2}{3.6} = \frac{1}{3}\,,$$

of the base of ABC (i.e. the segment BA). For these two triangles to have equal areas means that the height of AED must be three times the height of ABC. The height of each of these triangles is related to the segments AC and AE via a trigonometric function $\frac{1}{\sin(\angle BAC)}$ involving the angle $\angle BAC$. This means that

$$\begin{aligned} AE &= \frac{1}{\sin(\angle BAC)} h_{\triangle AED} = \frac{1}{\sin(\angle BAC)} (3h_{\triangle ABC}) \\ &= 3\left(\frac{1}{\sin(\angle BAC)} h_{\triangle ABC}\right) = 3AC = 3(3.6) = 10.8 \,. \end{aligned}$$

Problem 11

We want to evaluate

$$x = \log_2(0.0625) \,.$$

Note that

$$\begin{aligned} \frac{1}{4} &= 0.25 \\ \frac{1}{8} &= 0.125 \\ \frac{1}{16} &= 0.0625 \,, \end{aligned}$$

Thus

$$x = \log_2\left(\frac{1}{16}\right) = -\log_2(16) = -\log_2(2^4) = -4 \,.$$

Problem 12

Adding c to each we would get the numbers

$$20 + c \,, 50 + c \,, 100 + c \,.$$

As this is a geometric progression this means that

$$\frac{50 + c}{20 + c} = r \,,$$

and

$$\frac{100 + c}{50 + c} = r \,.$$

If we set these two expression equal to each other we get

$$\frac{50 + c}{20 + c} = \frac{100 + c}{50 + c} \,.$$

To solve for c we first have

$$(50 + c)^2 = (100 + c)(20 + c) \,,$$

or

$$2500 + 100c + c^2 = 2000 + 120c + c^2\,,$$

or

$$500 = 20c \quad \text{or} \quad c = 25\,.$$

This means that

$$r = \frac{50+25}{20+25} = \frac{75}{45} = \frac{15}{9} = \frac{5}{3}\,.$$

Problem 13

We are told that

$$\frac{1}{n}\sum_{i=1}^{n} x_i = \frac{1}{50}\sum_{i=1}^{50} x_i = 38\,.$$

This means that

$$\sum_{i=1}^{50} x_i = 50(38) = 1900\,.$$

If we drop the two numbers we have

$$\sum_{i=1}^{48} x_i = 1900 - 45 - 55 = 1800\,.$$

The new mean is then

$$\frac{1}{48}\sum_{i=1}^{48} x_i = \frac{1}{48}(1800) = 37.5\,.$$

Problem 14

This set S is

$$S = \{0, \pm 2, \pm 4, \pm 6, \cdots\}\,.$$

This set is closed under addition, subtraction, and multiplication and not division and averaging.

Problem 15

If we let a and b be the two legs of the triangle and c the hypotenuse then we are told that

$$c^2 = 2ab\,.$$

We also have the Pythagorean theorem that states that

$$c^2 = a^2 + b^2\,.$$

Using the first equation we have

$$a^2 + b^2 = 2ab\,,$$

or

$$a^2 - 2ab + b^2 = (a-b)^2 = 0\,.$$

This means that $a = b$ and this triangle is an isosceles and each acute angle is $\frac{\pi}{4}$.

Problem 16

By factoring all polynomials we can write this division as

$$\frac{(x-1)(x-2)}{(x-2)(x-3)} \cdot \frac{(x-3)(x-4)}{(x-1)(x-4)} = 1\,,$$

when we cancel common terms.

Problem 17

When $y = 1$ we are told that $x = -1$. If we put these values in the given expression we have

$$1 = a + \frac{b}{-1} \Rightarrow 1 = a - b\,. \tag{118}$$

When $y = 5$ we are told that $x = -5$. If we put these values in the given expression we have

$$5 = a - \frac{b}{5}\,. \tag{119}$$

From Equation 118 we have $a = 1 + b$. If we put that Equation 119 we have

$$5 = 1 + b - \frac{b}{5}\,.$$

Solving for b we find $b = 5$. If we put this into Equation 118 we get

$$a = 1 + b = 6\,.$$

This means that

$$a + b = 11\,.$$

Problem 18

What we want to compute is

$$\frac{1}{n}\sum_{k=1}^{n} k = \frac{1}{n}\left(\frac{n(n+1)}{2}\right) = \frac{n+1}{2}\,.$$

Problem 19

If we have weights x, y, and z where $x < y < z$ (and other conditions like $x + y \neq z$ so that no two weighs sum to a third) then we can imagine putting some of these three weights on one of the two scales. This would create an imbalance and the scale should "tip" to the heavier side. We could then perfectly weigh an object that offset this "tip". For example if we put all of the weights on one scale we could weigh an object with weight $x + y + z$.

Thus to count the number of possible weights we can distinguish we need to count up the number of ways we can place the weights on the two scales. This can be done in several ways

- We can place one of the three weights on one scale in $\left(\begin{array}{c} 3 \\ 1 \end{array} \right) = 3$ ways. This produces the weights x, y, and z.
- We can place two of the three weights on one scale in $\left(\begin{array}{c} 3 \\ 2 \end{array} \right) = 3$ ways. This produces the weights $x + y$, $x + z$, and $y + z$.
- We can place three of the three weights on one scale in $\left(\begin{array}{c} 3 \\ 3 \end{array} \right) = 1$ ways. This produces the weight $x + y + z$.
- We can place two of the three weights on two scales in $\left(\begin{array}{c} 3 \\ 2 \end{array} \right) = 3$ ways. This produces the weights $y - x$, $z - x$, and $z - y$.
- Finally, we can place three of the three weights on two scales in $\left(\begin{array}{c} 3 \\ 2 \end{array} \right) = 3$ ways since we just have to decide which scale (from two) each weight will go on. This produces the weights that are the differences $z - x - y$ etc.

This gives a total of $3 + 3 + 1 + 3 + 1 + 3 = 13$.

Problem 20

We are told that

$$x = \frac{Cy}{z^2}\,.$$

We know that $(x, y, z) = (10, 4, 14)$ this means that

$$10 = \frac{4C}{14^2} \quad \text{so} \quad C = 490\,.$$

If $(y, z) = (16, 7)$ then we have

$$x = \frac{490(16)}{7^2} = 160\,.$$

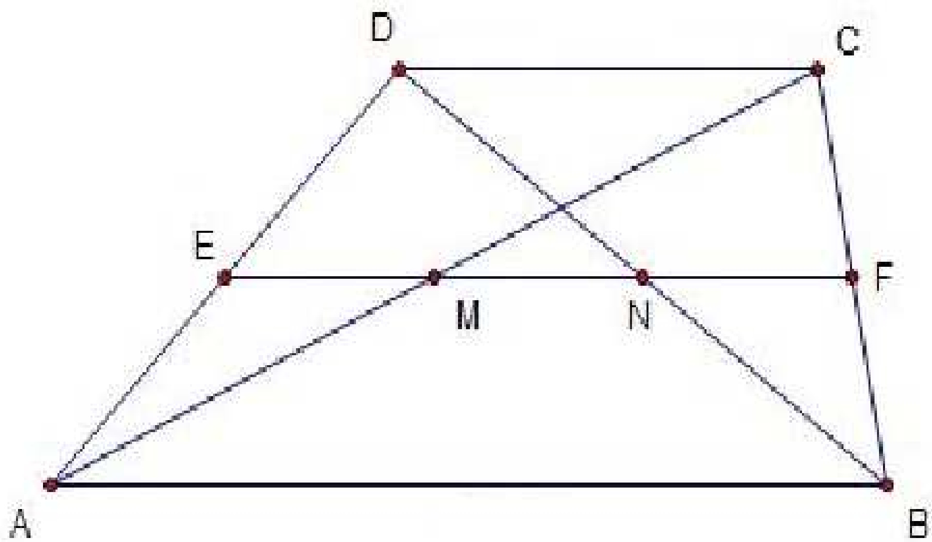

Figure 1: Our trapezoid and its median.

Problem 21

Let the length of the side of the equilateral triangle be s. Then we have $s = \frac{p}{3}$.

Next draw the equilateral triangle inside the circle. Then draw segments from the center of the circle to each of the triangle's vertices. This gives three isosceles triangles with legs of length r (the radius of the circle). The vertex angle of each isosceles triangle is given by

$$\frac{360}{3} = 120^\circ\,.$$

This means that the "leg angles" are

$$\frac{180 - 120}{2} = 30^\circ\,.$$

The altitude of one of the isosceles splits the base of length s into two right triangles each with a hypotenuse of r. As one of the angles in that right triangle is 30° we have

$$\frac{s}{2} = r\cos(30^\circ) = \frac{r\sqrt{3}}{2}\,.$$

Solving for r we get

$$r = \frac{s}{\sqrt{3}} = \frac{p}{3\sqrt{3}}\,.$$

The area of the circle is then

$$\pi r^2 = \frac{\pi p^2}{27}\,.$$

Problem 22

See Figure 1 for a picture of our trapezoid and a line connecting the midpoints of AC and BD. From the problem we are told that $AB = 97$ and that $MN = 3$. Recall that in a trapezoid, the line that connects the midpoints of the diagonals is on the *median* of the triangle and thus from properties of the median of a trapezoid we have

$$\frac{1}{2}(DC + AB) = \frac{1}{2}(DC + 97) = EF\,. \tag{120}$$

We also know that the median bisects the segments AD and BC. Next note that

$$EF = EM + MN + NF = EM + 3 + NF\,. \tag{121}$$

As $\triangle DCA \sim \triangle EMA$ so that we have

$$\frac{EM}{DC} = \frac{EA}{DA} = \frac{1}{2} \quad \text{so} \quad EM = \frac{1}{2}DC\,.$$

In the same way as $\triangle DCB \sim \triangle NFB$ so that we have

$$\frac{NF}{DC} = \frac{FB}{CB} = \frac{1}{2} \quad \text{so} \quad NF = \frac{1}{2}DC\,.$$

If we put these two expressions into Equation 121 we get

$$EF = DC + 3\,.$$

If we put this into Equation 120 we get

$$\frac{1}{2}(DC + 97) = DC + 3\,.$$

We can solve this for DC to get $DC = 91$.

Problem 23

We want to solve

$$\log_{10}(a^2 - 15a) = 2\,.$$

This is equivalent to

$$a^2 - 15a = 100\,,$$

or

$$a^2 - 15a - 100 = 0\,.$$

Using the quadratic equation we find that a is given by

$$a = \frac{15 \pm \sqrt{255 + 400}}{2} = \frac{15 \pm \sqrt{625}}{2} = \frac{15 \pm 25}{2}\,.$$

These two roots are $a \in \{-5, 20\}$. These are two integers.

Problem 24

Now his m ounces of salt water at $m\%$ solution has $m\left(\frac{m}{100}\right)$ ounces of salt in m ounces of liquid. If we then add x ounces of pure salt will have a solution with

$$\frac{\frac{m^2}{100}+x}{m+x}\,,$$

fraction of salt. If this is to equal $\frac{2m}{100}$ then we must have

$$\frac{\frac{m^2}{100}+x}{m+x}=\frac{2m^2}{100}\,,$$

or

$$\frac{m^2}{100}+x=\frac{2m^2}{100}+\frac{2m}{100}x\,,$$

or

$$\left(1-\frac{2m}{100}\right)x=\frac{m^2}{100}\,,$$

so that we need to add

$$x=\frac{m^2}{100-2m}\,,$$

ounces of pure salt.

Problem 25

This is equivalent to

$$-4<3-x<4\,,$$

or

$$-4<x-3<4\,,$$

or

$$-1<x<7\,.$$

Problem 26

I drew this triangle with the side AC alone the x-axis of a Cartesian coordinate plane and the point B “above” this segment. Let the median from A intersect the side BC at A'. Let the median from C intersect the side AB at C'. Let the median from B intersect the side AC at B'. Finally, let the centroid of the triangle be denoted M. We are told that $\angle AMC = 90°$ and so $\triangle AMC$ is a right triangle. This means that

$$AM^2+MC^2=AC^2=(\sqrt{2})^2=2\,.$$

By symmetry we expect $AM = MC$. Thus using the above we find $AM = MC = 1$.

Now the medians of a triangle intersect in such a way that $AM : MA' = 2 : 1$ and thus

$$\frac{AM}{MA'} = 2 = \frac{1}{MA'} \quad \text{so} \quad MA' = \frac{1}{2}\,.$$

Now $\triangle CMA'$ is a right triangle so it has an area of

$$\frac{1}{2} MA' \times MC = \frac{1}{2} \times \frac{1}{2} \times 1 = \frac{1}{4}\,.$$

As the medians of a triangle divide the triangle into six smaller triangles all with equal area the area of the full triangle is then

$$6[CMA'] = 6 \times \frac{1}{4} = \frac{3}{2}\,.$$

Problem 27

If we divide everything by i and note that $\frac{1}{i} = -i$ we can write this expression as

$$x^2 + ix + 2 = 0\,.$$

Then using Vieta's formula (see the test in 1954) if r_1 and r_2 are the roots of the above quadratic we have that

$$-(r_1 + r_2) = i \quad \text{and} \quad r_1 r_2 = 2\,.$$

Thus the sum of the roots is $-i$.

Problem 28

Draw this triangle with the point $A = (0, 0)$ the point B "to the right" of A on a Cartesian x-axis and the point C "above" the segment AB. From the problem statement the angle bisector at A intersects BC at L and the angle bisector at C intersects the segment AB at M.

Using the "angle bisector theorem" at A we have that

$$\frac{AC}{CL} = \frac{AB}{LB}\,.$$

Using the "angle bisector theorem" at C we have that

$$\frac{AC}{AM} = \frac{CB}{BM}\,.$$

Using the fact that $BC = a$, $AC = b$, and $AB = c$ we can write these as

$$\frac{b}{CL} = \frac{c}{LB} \tag{122}$$

$$\frac{b}{AM} = \frac{a}{BM}\,. \tag{123}$$

From Equation 122 we get

$$\frac{CL}{LB} = \frac{b}{c}\,.$$

From Equation 123 we get

$$\frac{AM}{BM} = \frac{b}{a}\,.$$

Together these mean that

$$\frac{\frac{AM}{MB}}{\frac{CL}{LB}} = \frac{b}{a} \cdot \frac{c}{b} = \frac{c}{a}\,.$$

As this is k the answer is (E).

Problem 29

The number of correct answered questions C is

$$C = 15 + \frac{1}{3}(n - 20)\,.$$

We are told that

$$\frac{C}{n} = 0.5\,.$$

This gives a linear equation for n with the solution $n = 50$ or only one solution.

Problem 30

Let C be the circumference of the track, v_A the velocity of A, and v_B the velocity of B. Then we are told that

$$v_A = \frac{C}{40}\,. \tag{124}$$

The other piece of information we are given is that at a velocity of $v_A + v_B$ we can run the track in 15 seconds or

$$v_A + v_B = \frac{C}{15}\,. \tag{125}$$

We want to know the value of $\frac{C}{v_B}$. If we divide Equation 125 by v_B we get

$$\frac{v_A}{v_B} + 1 = \frac{1}{15}\left(\frac{C}{v_B}\right)\,,$$

or using Equation 124 in this we get

$$\frac{1}{40}\left(\frac{C}{v_B}\right) + 1 = \frac{1}{15}\left(\frac{C}{v_B}\right)\,.$$

This is a linear equation in $\frac{C}{v_B}$. Solving this we find $\frac{C}{v_B} = 24$ seconds.

Problem 31

A square with an area of 40 will have a side length $s = \sqrt{40} = 2\sqrt{10}$.

When a square with side s is inscribed in a semicircle the radius of the circle is the hypotenuse of a right triangle with legs of length s and $\frac{s}{2}$. Thus

$$r^2 = s^2 + \left(\frac{s}{2}\right)^2 = 40 + 10 = 50\,,$$

when we use the value of s found above. This means that $r = 5\sqrt{2}$.

If a square is placed in a full circle of radius r then r would be the hypotenuse of a right triangle with side lengths $\frac{s'}{2}$ and $\frac{s'}{2}$ for some new side length s'. This means that

$$50 = r^2 = \left(\frac{s'}{2}\right)^2 + \left(\frac{s'}{2}\right)^2 .$$

Solving this we get $s' = 10$ so the area of this square is $10^2 = 100$.

Problem 32

Draw our circle with a center of O and the tangent from A to the circle with the point of tangency T. The the distance OT is r the radius of the circle. We are told that $TA = \frac{4}{3}r$. As the angle $\angle OTA = 90°$ by using the Pythagorean theorem we have

$$OA^2 = r^2 + \left(\frac{4}{3}r\right)^2 = \frac{25}{9}r^2 .$$

This means that $OA = \frac{5}{3}r$. The distance from the circle to A is $OA - r = \frac{2}{3}r$. As we are told that $l = \frac{4}{3}r$ we have $r = \frac{3}{4}l$ which means that the distance from the circle to A is

$$OA - r = \frac{2}{3}r = \frac{2}{3}\cdot\frac{3}{4}l = \frac{l}{2} .$$

Problem 33

The harmonic progression $3\,,4\,,6$ has the corresponding arithmetic progression given by $\frac{1}{3}\,,\frac{1}{4}\,,\frac{1}{6}$. This arithmetic progression has the common difference d of

$$d = \frac{1}{4} - \frac{1}{3} = \frac{1}{6} - \frac{1}{4} = -\frac{1}{12} .$$

The next few terms of the arithmetic progression are given by

$$\frac{1}{6} - \frac{1}{12} = \frac{1}{12}\,, 0\,, -\frac{1}{12}\,, \ldots$$

Thus the fourth number in the harmonic progression is $\frac{1}{\frac{1}{12}} = 12$ and the sum $S_4 = 3 + 4 + 6 + 12 = 25$.

Problem 34

Writing this expression as

$$(x - r)(x - s) = 0\,,$$

we see that

$$r + s = 3 \tag{126}$$
$$rs = 1\,. \tag{127}$$

If we square $r + s$ (and then expand) we would get

$$r^2 + 2rs + s^2 = 9\,,$$

or using what we know about rs above we have

$$r^2 + s^2 = 9 - 2 = 7\,,$$

a positive integer.

Problem 35

By expanding we see that the left-hand-side of the given expression is given by

$$x^2 - 2mx + m^2 - (x^2 - 2nx + n^2) = -2mx + m^2 + 2nx - n^2 = -2(m - n)x + (m^2 - n^2)\,.$$

Factoring and setting this equal to the right-hand-side gives

$$-2(m - n)x + (m - n)(m + n) = (m - n)^2\,.$$

As we know that $m \neq n$ we can divide by $m - n$ to get

$$-2x + m + n = m - n \quad \text{or} \quad x = n\,,$$

which is one solution.

Problem 36

Draw our triangle $\triangle ABC$ with its base BA along the x axis and the point C "above" BA. We can take $B = (0, 0)$ and $A = (80, 0)$. Let the angle $\angle ABC = 60°$. Let $BC = a$ and $AC = b$. Then we are told that $a + b = 90$. From C we can drop a perpendicular vertical to the segment BA of length h and that intersects BA at the point D. Let $x = BD$ and $y = DA$. Then we have

$$x = a\cos(60°) = \frac{a}{2}$$
$$h = a\sin(60°) = \frac{a\sqrt{3}}{2}$$
$$y = 80 - x = 80 - \frac{a}{2}\,.$$

In the right triangle $\triangle CDA$ we have

$$b^2 = h^2 + y^2 = \frac{3a^2}{4} + \left(80 - \frac{a}{2}\right)^2 = a^2 - 80a + 6400\,,$$

when we expand and simplify. We also know that $b = 90 - a$ which when we put into the above gives

$$(90 - a)^2 = a^2 - 80a + 6400\,.$$

Expanding and simplifying this we find $a = 17$. Thus $b = 90 - a = 73$ and the three sides of the triangle are

$$17\,, 73\,, 80\,.$$

The shortest side is thus 17.

Problem 37

This is the product

$$\prod_{k=3}^{n}\left(1 - \frac{1}{k}\right) = \prod_{k=3}^{n}\left(\frac{k-1}{k}\right) = \frac{2}{3}\cdot\frac{3}{4}\cdot\frac{4}{5}\cdot\frac{5}{6}\cdots\frac{n-1}{n} = \frac{2}{n}\,.$$

Problem 38

If we take $v = \sqrt{2x}$ then $v^2 = 2x$ so that the expression we are given can be written as

$$2v^2 + v - 1 = 0\,.$$

We can write the above as

$$(2v - 1)(v + 1) = 0\,.$$

This means that $v = -1$ or $v = \frac{1}{2}$. As we must have $v \geq 0$ the only valid solution is $v = \frac{1}{2}$. Thus

$$\sqrt{2x} = \frac{1}{2} \quad \text{or} \quad x = \frac{1}{8}\,,$$

so x is a fraction.

Problem 39

We have

$$\begin{aligned} S = \sum_{k=1}^{9} x^k + \sum_{k=1}^{9} ka &= \frac{1 - x^{10}}{1 - x} - 1 + a\left(\frac{10(9)}{2}\right) \\ &= \frac{1 - x^{10} - 1 + x}{1 - x} + 45a \\ &= \frac{x - x^{10}}{1 - x} + 45a\,. \end{aligned}$$

Problem 40

I draw the triangle with AB along the x-axis of a Cartesian coordinate system with C "above" AB. I then drew the segments BD and CF as specified in the problem (where $AD = DC$ and $DE = EB$). From the point E in the direction of C place the point G such that $EG = EF$. Then by the "side-angle-side" theorem we have $\triangle FEB \cong \triangle GED$. Thus $DG = FB = 5$.

Now D is the midpoint of AC and DG is parallel to AB thus

$$DG = \frac{1}{2}AF \quad \text{or} \quad AF = 2DG = 2(5) = 10\,.$$

Thus $AB = AF + FB = 10 + 5 = 15$.

Problem 41

To draw the given figure I needed to place two "large" circles tangent to each other and externally tangent to a horizontal line. A smaller circle was then placed "in the gap" between the two larger circles. This smaller circle is tangent to the horizontal line and to each of the larger circles.

If we imagine placing the y-axis of an x-y Cartesian coordinate system through the center of the smaller circle we can place its center (denoted by o) at $(0, r)$ where r is the smaller circles radius. The two larger circles will then have their centers at $(\pm R, R)$ for R the radius of the larger circles. Let the center of the "right-most" larger circle be denoted by O. Let the point $(+R, r)$ be denoted at the point A. Then the triangle $\triangle oAO$ is a right triangle and we have the lengths

$$\begin{aligned} oA &= R \\ AO &= R - r \\ oO &= R + r\,. \end{aligned}$$

Then the Pythagorean theorem in this triangle gives

$$oO^2 = oA^2 + AO^2 \quad \text{or} \quad (R + r)^2 = R^2 + (R - r)^2\,.$$

Expanding and simplifying the above becomes

$$R(R - 4r) = 0\,.$$

Thus $R = 0$ or $R = 4r = 16$.

Problem 42

Consider the prime factorization of a, b, and c

$$
\begin{aligned}
a &= 2^{a_2} 3^{a_3} 5^{a_5} \cdots \\
b &= 2^{b_2} 3^{b_3} 5^{b_5} \cdots \\
c &= 2^{c_2} 3^{c_3} 5^{c_5} \cdots .
\end{aligned}
$$

Then using these we have

$$
\begin{aligned}
D &= 2^{\min(a_2,b_2,c_2)} 3^{\min(a_3,b_3,c_3)} 5^{\min(a_5,b_5,c_5)} \cdots \\
M &= 2^{\max(a_2,b_2,c_2)} 3^{\max(a_3,b_3,c_3)} 5^{\max(a_5,b_5,c_5)} \cdots ,
\end{aligned}
$$

while

$$
abc = 2^{a_2+b_2+c_2} 3^{a_3+b_3+c_3} 5^{a_5+b_5+c_5} \cdots .
$$

The only times abc will equal DM is when

$$
a_p + b_p + c_p = \min(a_p, b_p, c_p) + \max(a_p, b_p, c_p) , \tag{128}
$$

for all primes p.

Note that if we have a situation where all three of a_p, b_p, and c_p are positive then

$$
\min(a_p, b_p, c_p) + \max(a_p, b_p, c_p) < a_p + b_p + c_p ,
$$

and $DM < abc$. Thus it is impossible for $DM > abc$ and choice (2) is true.

If a, b, and c are relatively prime then one of a_p, b_p, or c_p is zero for each p. This means that

$$
\min(a_p, b_p, c_p) + \max(a_p, b_p, c_p) = a_p + b_p + c_p ,
$$

and $DM = abc$ in this case and choice (4) is true.

Problem 43

To solve this problem we will use the following theorem[3]. If a, b, and c are the three sides of a triangle and A is its area, then the measure of the circumradius of the triangle is given by

$$
R = \frac{abc}{4A} . \tag{129}
$$

We can easily compute its area using **Heron's** formula given by Equation 10. Using that I find $s = 52$ and $A = 468$. This then gives

$$
R = \frac{25 \cdot 39 \cdot 40}{4(468)} = \frac{125}{6} .
$$

This means the diameter is given by $\frac{125}{3}$.

[3]https://artofproblemsolving.com/wiki/index.php/Circumradius

Problem 44

If the two roots are r_1 and r_2 then we have

$$(x - r_1)(x - r_2) = x^2 - (r_1 + r_2)x + r_1 r_2 = x^2 + bx + c = 0\,,$$

thus $b = -(r_1 + r_2)$ and $c = r_1 r_2$. If we consider the given expression for s we have

$$\begin{aligned} s &= b + c + 1 = -(r_1 + r_2) + r_1 r_2 + 1 \\ &= -r_1 - r_2 + r_1 r_2 + 1 = (r_1 - 1)(r_2 - 1)\,. \end{aligned}$$

As we are told that $r_i - 1 > 0$ for $i \in \{1, 2\}$ we see that $s > 0$.

Problem 45

We want to solve for y in

$$(\log_3 x)(\log_x(2x))(\log_{2x} y) = \log_x(x^2)\,.$$

We can write the above as

$$\left(\frac{\ln x}{\ln 3}\right)\left(\frac{\ln(2x)}{\ln x}\right)\left(\frac{\ln y}{\ln(2x)}\right) = 2\,,$$

simplifying this gives

$$\ln y = 2\ln(3) = \ln 3^2\,.$$

This means that $y = 9$.

Problem 46

Create a "grid" like that shown in Table 2 where the rows correspond to whether or not it rained in the morning and the columns correspond to whether or not it rained in the afternoon. From the facts given in this problem and the variables in this grid condition (1) means that

$$2a + b + c = 7\,.$$

Condition (2) means that $a = 0$. Condition (3) means that $b + e = 5$ and Condition (4) means that $c + e = 6$. Using $a = 0$ these become the three equations and three unknowns

$$\begin{aligned} b + c &= 7 \\ b + e &= 5 \\ c + e &= 6\,. \end{aligned}$$

Solving these we find $b = 3$, $c = 4$, and $e = 2$. Then what we want d is the sum of the column sums (or the sum of the row sums) in Table 2 or

$$\begin{aligned} d &= (a + c) + (b + e) = (0 + 4) + (3 + 2) = 9 \quad \text{or} \\ d &= (a + b) + (c + e) = (0 + 3) + (4 + 2) = 9\,. \end{aligned}$$

	Yes	No
Yes	a	b
No	c	e

Table 2: Rain in the morning (rows) or afternoon (columns).

Problem 47

As all students are human and some students think. The thinking students are thinking humans and we have that some humans think which is (2).

Problem 48

We must have $n \geq 0$ and $a_0 > 0$ an integer.

- If we take $n = 3$ then to have $h = 3$ we must have $a_0 = 0$ which is not a valid polynomial. Thus we must have $n \leq 2$.
- If $n = 2$ then we can have $a_0 = 1$ and the polynomial is x^2.
- If $n = 1$ then we can have $a_0 = 1$ and $a_1 = \pm 1$ *or* $a_0 = 2$ with $a_1 = 0$ and the polynomials are $x \pm 1$ or $2x$.
- If $n = 0$ then we can have $a_0 = 3$ polynomial is 3.

Counting these up we see that we have $1 + 2 + 1 + 1 = 5$ polynomials of this form.

Problem 49

The seventh number in this problem is wrong and it should be $\frac{1}{64}$ and not $\frac{1}{84}$. Consider the sum of the first three terms

$$1 - \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$$

Consider the sum of the next three terms

$$\frac{1}{8} - \frac{1}{16} - \frac{1}{32} = \frac{1}{32}.$$

Finally, the sum of the next three terms (which should be)

$$\frac{1}{64} - \frac{1}{128} - \frac{1}{256} = \frac{1}{256}.$$

Thus our sum S is

$$S = \frac{1}{4} + \frac{1}{32} + \frac{1}{256} = \frac{1}{4}\left(1 + \frac{1}{8} + \frac{1}{8^2} + \cdots\right) = \frac{1}{4}\left(\frac{1}{1 - \frac{1}{8}}\right) = \frac{2}{7}.$$

The 1960 Examination

Problem 1

If $x = 2$ is a root we must have

$$8 + 2h + 10 = 0 \quad \text{or} \quad h = -9\,.$$

Problem 2

From the problem we are told that it takes five seconds to do a total of six chimes. Thus there is one second between chimes.

When the clock turns to 12:00 there will be one chime immediately. Then there will need to be eleven more chimes to perform a total of 12 chimes. As each of the additional chimes takes one second in total this will take eleven seconds.

Problem 3

A discount of 40% means we pay

$$0.6(10000) = 6000\,.$$

Discounts of 36% followed by 4% require us to pay

$$(1 - 0.04)(1 - 0.36)(10000) = (1 - 0.04)6400 = 6144\,.$$

The difference is 144.

Problem 4

The third angle in the triangle must be $180 - 2(60) = 60$ meaning that all angles are equal and we have an equilateral triangle with all sides of equal length. The area of such a triangle (given its side length s) is given by

$$\frac{\sqrt{3}}{4}s^2 = \frac{\sqrt{3}}{4}(4^2) = 4\sqrt{3}\,.$$

Problem 5

If $y^2 = 9$ we have that $y = \pm 3$. Putting $y^2 = 9$ into the second equation gives $x^2 + 9 = 9$ so that $x^2 = 0$ or $x = 0$. Thus there are two distinct points $(0, \pm 3)$.

Problem 6

If the circumference is 100 inches than the radius r must be

$$2\pi r = 100 \quad \text{so} \quad r = \frac{50}{\pi}\,.$$

The inscribed square will have a diagonal that is $2r$ and so a side length s of length that will satisfy

$$s^2 + s^2 = (2r)^2 = 4\left(\frac{50}{\pi}\right)^2\,.$$

Solving this for s gives

$$s = \frac{50\sqrt{2}}{\pi}\,.$$

Problem 7

From the area of circle one we have

$$A_1 = 4 = \pi r_1^2 \quad \text{so} \quad r_1 = \frac{2}{\sqrt{\pi}}\,.$$

From the description of circle one relative to circle two we see that the radius of circle two is given by $r_2 = 2r_1$ so the area of circle two is given by

$$A_2 = \pi r_2^2 = 4\pi r_1^2 = 4(4) = 16\,.$$

Problem 8

Let x be the number given then we have $x = 2.\overline{52}$. If we multiply x by 100 we get

$$100x = 252.\overline{52}\,,$$

so

$$100x - x = 252.\overline{52} - 2.\overline{52} = 250\,.$$

Solving for x we get $x = \frac{250}{99}$ so the sum of the numerator and denominator is $250+99 = 349$.

Problem 9

Write this expression as

$$\begin{aligned} E &= \frac{a^2+b^2-c^2+2ab}{a^2+c^2-b^2+2ac} = \frac{(a+b)^2-c^2}{(a+c)^2-b^2} \\ &= \frac{(a+b-c)(a+b+c)}{(a+c-b)(a+c+b)} = \frac{a+b-c}{a-b+c}\,. \end{aligned}$$

Problem 10

The negative of statement (6) is "there is a man that is not a good driver" or "at least one man is a bad driver".

Problem 11

Write the given equation

$$x^2 - 3kx + 2k^2 - 1 = 0\,,$$

in root factored form as

$$(x - r_1)(x - r_2) = 0\,.$$

If we expand this we get

$$x^2 - (r_1 + r_2)x + r_1 r_2 = 0\,.$$

From the quadratic given and what we are told this means that

$$r_1 r_2 = 2k^2 - 1 = 7\,,$$

so $k = \pm 2$.

Taking the plus sign our quadratic is given by

$$x^2 - 6x + 7 = 0\,,$$

This has roots given by

$$x = \frac{6 \pm \sqrt{36 - 4(7)}}{2} = 3 \pm \sqrt{2}\,.$$

Taking the minus sign our quadratic is given by

$$x^2 + 6x + 7 = 0\,.$$

This has roots given by

$$x = \frac{-6 \pm \sqrt{36 - 4(7)}}{2} = -3 \pm \sqrt{2}\,.$$

From this we see that the roots are irrational.

Problem 12

Let the fixed point that all circles pass though be denoted by (p, q). Consider a circle with a center (x_0, y_0) with a radius a which will have the equation

$$(x - x_0)^2 + (y - y_0)^2 = a^2\,.$$

Then if the point (p, q) is on this circle we must have

$$(p - x_0)^2 + (q - y_0)^2 = a^2 .$$

Thus all centers (x_0, y_0) are on a circle centered at (p, q) with a radius of a.

Problem 13

Drawing the line $y = -3x + 2$ we see that it goes through the points $(0, 2)$ and $\left(\frac{2}{3}, 0\right)$. Drawing the line $y = 3x + 2$ we see that it goes through the points $(0, 2)$ and $\left(-\frac{2}{3}, 0\right)$. The line $y = -2$ is a horizontal line and provides a "base" for a triangle that is the intersection of the previous two lines. The line $y = -3x + 2$ intersects this horizontal line $y = -2$ at the point $B = \left(\frac{4}{3}, -2\right)$. By symmetry The line $y = 3x + 2$ intersects this horizontal line $y = -2$ at the point $A = \left(-\frac{4}{3}, -2\right)$. Let C be the point $C = (0, 2)$. Again by symmetry this will be an isosceles triangle as $AC = BC$. It will be an equilateral triangle if $AB = AC$. We compute

$$AB = 4 ,$$

and

$$BC^2 = \left(\frac{4}{3}\right)^2 + 4^2 > 16 .$$

Thus $AB \neq BC$ and this triangle is isosceles.

Problem 14

Write this as

$$(3 - b)x = 6 - a .$$

Then if $b \neq 3$ we have

$$x = \frac{6 - a}{3 - b} .$$

Problem 15

For equilateral triangle I with side A, perimeter P, area K, and circumradius R by dropping a vertical from the top most vertex and using symmetry we can show that

$$\begin{aligned}
P &= 3A \\
K &= \frac{1}{2}A(A\sin(60^\circ)) = \frac{A^2}{2} \cdot \left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{4}A^2 \\
R &= \frac{A}{\sqrt{3}} .
\end{aligned}$$

This means that in comparing triangle I with triangle II we have

$$P : p = A : a$$
$$R : r = A : a$$
$$K : k = A^2 : a^2$$

Thus we see that $P : p = R : r$ always.

Problem 16

From the number 69 we can "remove" at most two 25s to get

$$69 - 2(5^2) = 19\,.$$

From 19 we can "remove" at most three fives to get

$$19 - 3(5) = 4\,.$$

This means that we can write

$$69 = 2 \times 5^2 + 3 \times 5 + 4\,,$$

which is the statement that $69 = (234)_5$.

Problem 17

We want to know the x value when

$$800 = 8 \cdot 10^8 x^{-3/2}\,.$$

Solving this I find $x = 10000$.

Problem 18

Recalling that $81 = 3^4$ these two equations are

$$3^{x+y} = 3^4$$
$$3^{4(x-y)} = 3^1\,,$$

so that

$$x + y = 4$$
$$4(x - y) = 1\,.$$

Solving these we find $x = \frac{17}{8} = 2\frac{1}{8}$ and $y = \frac{9}{8} = 1\frac{1}{8}$.

Problem 19

If we consider (A) we might let $x = n$, $y = n + 1$, and $z = n + 2$ then I is

$$x + y + z = 3n + 3 = 46 \quad \text{or} \quad 3n = 43\,,$$

which has no solution for n in the integers.

If we consider (B) we might let $x = n$, $y = n + 2$, and $z = n + 4$ then I is

$$x + y + z = 3n + 6 = 46 \quad \text{or} \quad 3n = 40\,,$$

which has no solution for n in the integers.

If we consider (C) we might let $x = n$, $y = n + 1$, $z = n + 2$, and $w = n + 3$ then II is

$$x + y + z + w = 4n + 6 = 46 \quad \text{or} \quad n = 10\,,$$

an integer!

If we consider (D)/(E) we might let $x = n$, $y = n+2$, $z = n+4$, and $w = n+6$. To have (D) be true we would need n and even integer. To have (E) be true we would need n an odd integer. Now in this case II is given by

$$x + y + z + w = 4n + 12 = 46 \quad \text{or} \quad 4n = 43\,,$$

which has no solution for n in the integers (even or odd).

Thus the only choice that is true is (C).

Problem 20

For this we have

$$\begin{aligned}
\left(\frac{x^2}{2} - \frac{2}{x}\right)^8 &= \sum_{k=0}^{8} \left(\begin{array}{c} 8 \\ k \end{array}\right) \left(\frac{x^2}{2}\right)^k \left(-\frac{2}{x}\right)^{8-k} \\
&= \sum_{k=0}^{8} \left(\begin{array}{c} 8 \\ k \end{array}\right) \frac{x^{2k} \cdot x^{-(8-k)}}{2^k} (-2)^{8-k} \\
&= \sum_{k=0}^{8} \left(\begin{array}{c} 8 \\ k \end{array}\right) (-1)^{8-k} 2^{8-k-k} x^{3k-8} \\
&= \sum_{k=0}^{8} \left(\begin{array}{c} 8 \\ k \end{array}\right) (-1)^{k} 2^{8-2k} x^{3k-8}
\end{aligned}$$

We want the coefficient when $3k - 8 = 7$ or $k = 5$. This is the value

$$\left(\begin{array}{c} 8 \\ 5 \end{array}\right) (-1)^5 2^{8-10} = \frac{8!}{3!5!}(-1)2^{-2} = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2} \cdot \frac{-1}{4} = -14\,.$$

Problem 21

Let the side of the square be s. Then we have that

$$s^2 + s^2 = (a+b)^2 \quad \text{so} \quad s = \frac{a+b}{\sqrt{2}}\,.$$

If the new square has twice the area the new squares side length must be $\sqrt{2}s$ or

$$a + b\,.$$

The perimeter of this square is $4(a+b)$.

Problem 22

Expand the left-hand-side to get

$$x^2 + 2mx + m^2 - (x^2 + 2nx + n^2) = 2(m-n)x + m^2 - n^2 = (m-n)^2\,.$$

Notice from this that if $m = n$ both sides are equal for all x. If $m \neq n$ then we can write the above as

$$2x + m + n = m - n \quad \text{or} \quad x = -n\,.$$

This will be of the requested form if $a = 0$ and $b = -1$ which is (A).

Problem 23

We want

$$\pi(R+x)^2 H = \pi R^2 (H+x)\,.$$

when $R = 8$ and $H = 3$. In that case the above is equivalent to

$$3(8+x)^2 = 64(3+x)\,,$$

or expanding and simplifying we get

$$3x^2 + 16x = 0\,.$$

Thus $x = 0$ or $x = \frac{16}{3}$. This is a rational value for x.

Problem 24

Apply the function $f(y) = (2x)^y$ to both sides of this expression to get

$$216 = (2x)^x\,.$$

Now since $216 = 2^3 \cdot 3^3$ we get

$$(2 \cdot 3)^3 = (2x)^x\,.$$

Inspecting this we see that $x = 3$ is a solution.

m'	n'	$m' - n'$	$m' + n' + 1$
even	even	even	odd
even	odd	odd	even
odd	even	odd	even
odd	odd	even	odd

Table 3: Choices for the evenness or oddness of m' and n'.

Problem 25

If m and n are odd numbers we can write them as $m = 2m' + 1$ and $n = 2n' + 1$ for some m' and n'. To have $n < m$ means that we must have $n' < m'$. Consider then the difference

$$\begin{aligned} m^2 - n^2 &= (m-n)(m+n) = (2m' - 2n')(2m' + 2n' + 2) \\ &= 4(m' - n')(m' + n' + 1) \,. \end{aligned}$$

This is obviously divisible by four. Now depending on if m' is even or odd (and if n' is even or odd) we can determine the even or oddness of the factors $m' - n'$ and $m' + n' + 1$. See Table 3. Notice in that in all cases the product $(m' - n')(m' + n' + 1)$ is even and is divisible by two. Thus the given difference is divisible by $4 \times 2 = 8$.

Problem 26

For this we have

$$-2 < \frac{5-x}{3} < 2 \,,$$

or

$$-6 < 5 - x < 6 \,,$$

or

$$-11 < -x < 1 \,,$$

or

$$-1 < x < 11 \,.$$

Problem 27

Let θ_i and θ_e be the interior and exterior angle at a vertex of our polygon. Then we are told that $\theta_i = 7.5\theta_e$. In addition we must have

$$\theta_i + \theta_e = 180^\circ \,.$$

These together give $\theta_e = \frac{360}{17}$ and $\theta_i = \frac{2700}{17}$. In addition, we must have that the sum of the n exterior angles of our polygon sum to 360° or

$$\sum_{i=1}^{n} \theta_e = 360^\circ \,,$$

As each of θ_e is a constant we have that $n\theta_e = 360°$ or using the known value of θ_e we get $n = 17$. Then using $S = 180(n-2)$ we find that $S = 2700$. The polygon may or may not be regular.

Problem 28

Multiply by $x - 3$ to get

$$x(x-3) - 7 = 3(x-3) - 7 \, .$$

Simplify this to get

$$x^2 - 6x + 9 = 0 \quad \text{or} \quad (x-3)^2 = 0 \, .$$

This would say that $x = 3$ but attempting to place that value into the initial equation we started with gives terms that are undefined and is not allowed. Thus there are no solutions to this equation.

Problem 29

We are told that

$$5a + b > 51 \tag{130}$$

$$3a - b = 21 \, . \tag{131}$$

From Equation 131 we have that $b = 3a - 21$ which if we put into Equation 130 gives

$$5a + (3a - 21) > 51 \, ,$$

which can be simplified to

$$a > 9 \, .$$

From Equation 131 we also get that $a = \frac{21+b}{3}$ which if we put into $a > 9$ gives

$$\frac{21+b}{3} > 9 \, .$$

This can be simplified to

$$b > 6 \, .$$

Problem 30

The set of points that are equidistant from the coordinate axis are the ones on the lines $y = x$ *or* $y = -x$. The point on $y = x$ and the given line will satisfy

$$3x + 5x = 15 \quad \text{or} \quad x = \frac{15}{8} \, ,$$

so $y = \frac{15}{8}$ also. This point is in the first quadrant.

The point on $y = -x$ and the given line will satisfy

$$3x - 5x = 15 \quad \text{or} \quad x = -\frac{15}{2}\,,$$

so $y = \frac{15}{2}$ also. This point is in the second quadrant.

Problem 31

If we perform "long-division" we find

			x^2	$-2x$	$p-1$
x^2+2x+5	x^4	$+0x^3$	$+px^2$	$+0x$	$+9$
	x^4	$+2x^3$	$+5x^2$		
		$-2x^3$	$(p-5)x^2$	$+0x$	$+9$
		$-2x^3$	$-4x^2$	$-10x$	
			$(p-1)x^2$	$10x$	$+9$
			$(p-1)x^2$	$2(p-1)x$	$5(p-1)$
				$2(6-p)x$	$+q-5p+5$

This means that

$$\frac{x^4 + px^2 + 9}{x^2 + 2x + 5} = x^2 - 2x + p - 1 + \frac{2(6-p)x + q - 5p + 5}{x^2 + 2x + 5}\,.$$

For this polynomial to be a factor means that the remainder (fraction above) must be zero or that

$$2(6-p) = 0 \quad \text{or} \quad p = 6\,,$$

and that

$$q - 5p + 5 = 0 \quad \text{or} \quad q = 25\,.$$

Problem 32

Let r be the radius of the circle. Then we are told that $AB = 2r$. Now as $\triangle ABO$ is a right triangle we have

$$AB^2 + BO^2 = AO^2\,,$$

or

$$4r^2 + r^2 = AO^2 \quad \text{so} \quad AO = \sqrt{5}r\,.$$

We can compute other distances using some of the above. We find

$$AD = AO - OD = \sqrt{5}r - r = AP$$
$$PB = AB - AP = 2r - AP = 2r - (\sqrt{5}r - r) = 3r - \sqrt{5}r\,.$$

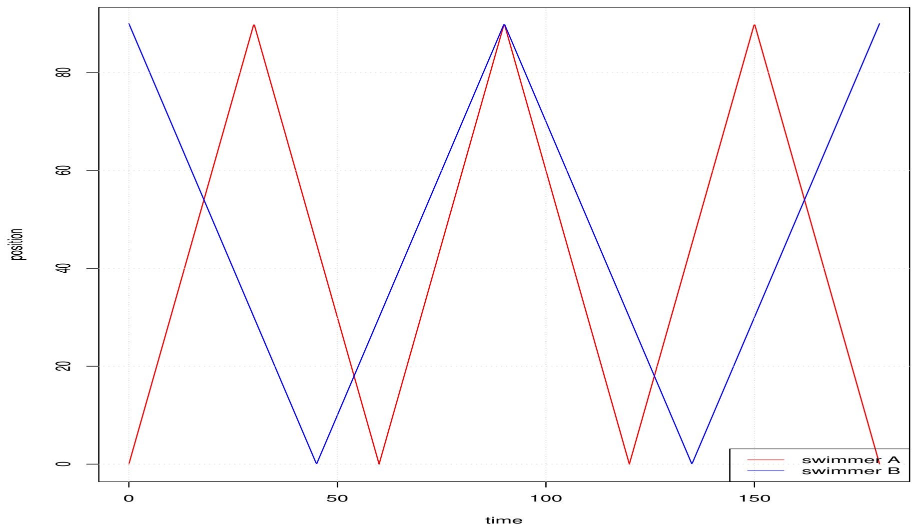

Figure 2: The paths of swimmer A and B for the times $0 \leq t \leq 180$.

Notice that we have all distances in terms of r. Using these expressions we compute

$$AP^2 = (\sqrt{5} - 1)^2 r^2 = (5 - 2\sqrt{5} + 1)r^2 = (6 - 2\sqrt{5})r^2 = 2(3 - \sqrt{5})r^2 \,,$$

and

$$PB \cdot AB = (3 - \sqrt{5})r(2r) \,.$$

Note that these two expression are equal.

Problem 33

To start we will get a list of primes less than or equal to 61. Such a list can be constructed using a technique like the "Sieve of Eratosthenes". Doing that we find this list given by

$$2\,,3\,,5\,,7\,,11\,,13\,,17\,,19\,,23\,,29\,,31\,,37\,,41\,,43\,,47\,,53\,,59\,,61\,.$$

The number P is the product of the above integers. Now a number of the form $P + n$ can only be prime if n has no factors in common with P, but every number suggested for n is either a prime less than 61 or a composite number that will have primes (in its factorization) in common with P. Thus there can be no primes in this list.

Problem 34

Imagine the pool running from left-to-right with A be the "faster" swimmer and starting "on the left going rightwards" at $x = 0$ and B starting "on the right going leftwards" at $x = 90$. Notice that A will be back at their "start" in

$$\frac{180}{3} = 60 \,,$$

seconds while B will be back at their "start" in

$$\frac{180}{2} = 90\,,$$

seconds. This means that in a time equal to the least common multiple of 60 and 90 seconds (or 180 seconds) both swimmers will return to their starting positions. In the twelve minutes that they are swimming they will "reset"

$$\frac{12 \times 60}{180} = 4\,,$$

times. We now need to determine how many times they will pass in one period of 180 seconds. This is perhaps easiest if we graph the location of each swimmer with respect to the point of $x = 0$ over time and look for intersections. Swimmer A is at the position $x_A(t)$ given by

$$x_A(t) = \begin{cases} 3t & 0 < t < \frac{90}{3} = 30 \\ 90 - 3(t-30) & 30 < t < 60 \\ 3(t-60) & 60 < t < 90 \\ 90 - 3(t-90) & 90 < t < 120 \\ 3(t-120) & 120 < t < 150 \\ 90 - 3(t-150) & 150 < t < 180 \end{cases}.$$

The functional form of this simplifies to

$$x_A(t) = \begin{cases} 3t & 0 < t < 30 \\ 180 - 3t & 30 < t < 60 \\ 3t - 180 & 60 < t < 90 \\ 360 - 3t & 90 < t < 120 \\ 3t - 360 & 120 < t < 150 \\ 540 - 3t & 150 < t < 180 \end{cases}.$$

In the same way, swimmer B is at the position $x_B(t)$ given by

$$x_B(t) = \begin{cases} 90 - 2t & 0 < t < \frac{90}{2} = 45 \\ 2(t-45) & 45 < t < 90 \\ 90 - 2(t-90) & 90 < t < 135 \\ 2(t-135) & 135 < t < 180 \end{cases}.$$

This simplifies to

$$x_B(t) = \begin{cases} 90 - 2t & 0 < t < 45 \\ 2t - 90 & 45 < t < 90 \\ 270 - 2t & 90 < t < 135 \\ 2t - 270 & 135 < t < 180 \end{cases}.$$

In the R code `1960_prob_34.R` we plot $x_A(t)$ and $x_B(t)$ as a function of time t (see Figure 2) we see that the two curves cross *five* times in one period $T = 180$. This means that in thirteen minutes we will have $5 \times 4 = 20$ total intersections.

Problem 35

With out loss of generality let $m < n$ then for t to be the mean proportional between m and n means that $m < t < n$ and we have

$$\frac{m}{t} = \frac{t}{n} \quad \text{or} \quad t^2 = mn\,.$$

We are told that $m + n = 10$ thus we have

$$t^2 = m(10 - m)\,.$$

For m to be an integer we see that the possible choices for m are $m \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. We need to have $m < n$ or $m < 10 - m$ which simplifies to

$$m < 5\,.$$

Thus the possible choices for m reduce to $m \in \{1, 2, 3, 4\}$. We also need t to be an integer. For each of these m we find

- If $m = 1$ then $t^2 = 9$ so $t = 3$ an integer.
- If $m = 2$ then $t^2 = 16$ so $t = 4$ an integer.
- If $m = 3$ then $t^2 = 21$ and t will not be an integer.
- If $m = 4$ then $t^2 = 24$ and t will not be an integer.

Thus there are two sets of numbers that t can have.

Problem 36

We can evaluate s_1 as

$$s_1 = \sum_{i=0}^{n-1} (a + di) = an + d \sum_{i=0}^{n-1} i = an + d\left(\frac{n(n-1)}{2}\right) = an + \frac{d}{2} n(n-1)\,.$$

Replacing n with $2n$ and $3n$ we find

$$\begin{aligned} s_2 &= a(2n) + \frac{d}{2}(2n)(2n-1) \\ s_3 &= a(3n) + \frac{d}{2}(3n)(3n-1)\,. \end{aligned}$$

If we use these we see that

$$\begin{aligned} R &= s_3 - s_2 - s_1 \\ &= \frac{d}{2}\left(3n(3n-1) - 2n(2n-1) - n(n-1)\right) = 2dn^2\,, \end{aligned}$$

when we simplify. This depends on only d and n.

Problem 38

As the triangle $\triangle DEF$ is equilateral we know that all angles in the triangle $\triangle DEF$ are 60°. Summing all of the angles in $\triangle ADE$ we have

$$\angle BAC + b + \angle DEA = 180^\circ\,. \tag{132}$$

Now $\angle DEA$ is along the line AC so

$$\angle DEA = 180 - 60 - c = 120 - c\,. \tag{133}$$

As the triangle $\triangle ABC$ is isosceles we know that $\angle ABC = \angle ACB$ so that

$$\angle BAC + 2\angle ABC = 180^\circ\,.$$

Using the sum of the angles in the triangle $\triangle BDF$ to evaluate $\angle ABC$ we have

$$\angle ABC = 180 - (\angle BDF + a) = 180 - (180 - 60 - b) - a = 60 - b - a\,.$$

This means that

$$\angle BAC = 180 - 2\angle ABC = 180 - 2(60 + b - a) = 60 - 2b + 2a\,.$$

Using this with Equation 133 into Equation 132 gives

$$(60 - 2b + 2a) + b + (120 - c) = 180\,,$$

which simplifies to

$$a = \frac{1}{2}(b + c)\,.$$

Problem 39

We can write the given expression as

$$1 + \frac{b}{a} = \frac{1}{\frac{a}{b} + 1}\,.$$

If we let $x = \frac{a}{b}$ then this is

$$1 + \frac{1}{x} = \frac{1}{x + 1}\,.$$

This can be simplified to

$$x^2 + x + 1 = 0\,.$$

If we use the quadratic equation to solve this we find

$$x = \frac{a}{b} = \frac{-1 \pm i\sqrt{3}}{2}\,.$$

Now if a and b are both real the ratio $\frac{a}{b}$ will not be complex (as required by the above). Thus choices (A) and (B) are not correct. If one of a or b is real because of the above relationship the other one will not be. In addition, a and b can both be complex and have a ratio given by a complex number. These together mean that (E) is the correct choice.

Problem 40

I will draw our right triangle in an x-y Cartesian coordinate plane with $A = (0, 4)$, $B = (3, 0)$, and $C = (0, 0)$. Now the angle $\angle ACB = 90°$ so the two angle trisector cuts this angle into segments of measure $30°$. Let the two angle trisectors from C intersect the segment AB at the points D and E such that

$$\begin{aligned}\angle DCB &= 30° + 30° = 60° \\ \angle ECB &= 30° \,.\end{aligned}$$

The slope of the segment AB is given by

$$m_{AB} = \frac{0-4}{3-0} = -\frac{4}{3} \,.$$

This means that the line AB is given by

$$y - 0 = -\frac{4}{3}(x-3) \,. \tag{134}$$

The lines CE and CD are given by

$$y = \tan(30°)x = \frac{1}{\sqrt{3}}x \tag{135}$$

$$y = \tan(60°)x = \sqrt{3}x \,. \tag{136}$$

The location of the point E is then given by the joint solutions to Equations 134 and 135. Solving these two equations we find

$$(x, y) = \left(\frac{12}{4+\sqrt{3}}, \frac{4\sqrt{3}}{4+\sqrt{3}}\right) \,.$$

This means that the length CE can be computed from

$$CE^2 = \frac{12^2}{(4+\sqrt{3})^2} + \frac{4^2 \cdot 3}{(4+\sqrt{3})^2} = \frac{12 \cdot 16}{(4+\sqrt{3})^2} = \frac{12 \cdot 16}{25+8\sqrt{3}} \,. \tag{137}$$

The location of the point D is then given by the joint solution to Equations 134 and 136. Solving these two equations we find

$$(x, y) = \left(\frac{12}{4+3\sqrt{3}}, \frac{12\sqrt{3}}{4+3\sqrt{3}}\right) \,.$$

This means that the length CD can be computed from

$$CD^2 = \frac{12^2}{(4+3\sqrt{3})^2} + \frac{12^2 \cdot 3}{(4+3\sqrt{3})^2} = \frac{12^2 \cdot 4}{43+24\sqrt{3}} \,. \tag{138}$$

The question asks for the smaller of CE or CD. We can determine which one is smaller by considering the ratio $\frac{CE^2}{CD^2}$ where we find

$$\frac{CE^2}{CD^2} = \frac{12 \cdot 16}{25 + 8\sqrt{3}} \times \frac{43 + 24\sqrt{3}}{12^2 \cdot 4} = \frac{43 + 24\sqrt{3}}{75 + 24\sqrt{3}} < 1 \,.$$

This means that the length CE is smaller. We find

$$CE = \frac{4 \cdot 2\sqrt{3}}{4 + \sqrt{3}} \times \frac{4 - \sqrt{3}}{4 - \sqrt{3}} = \frac{32\sqrt{3} - 24}{13} \,,$$

when we simplify.

The 1961 Examination

Problem 1

We have

$$\left(-\frac{1}{125}\right)^{-2/3} = (-125)^{2/3} = (125)^{2/3} = 5^2 = 25\,.$$

Problem 2

We travel $\frac{a}{6}$ feet in r seconds at a speed of $v = \frac{a}{6r}$ feet per second. If we do this for three minuets or $3(60) = 180$ seconds the we travel

$$\frac{a}{6r}(180) = \frac{30a}{r}\,,$$

feet. To convert this to yards we divide by 3 to get at total length of

$$\frac{10a}{r}\,,$$

yards.

Problem 3

One way to solve this problem is the following. Let m_1 be the slope of $2y + x + 3 = 0$ or $m_1 = -\frac{1}{2}$. Let m_2 be the slope of $3y + ax + 2 = 0$ or $m_2 = -\frac{a}{3}$. For the two lines to be orthogonal requires that $m_1 m_2 = -1$ or

$$\frac{a}{6} = -1 \quad \text{or} \quad a = -6\,.$$

Another way to solve this problem is the following. A normal vector to the line $2y+x+3 = 0$ is given by the vector $(-2, 1)$. A normal vector to the line $3y + ax + 2 = 0$ is given by the vector $(-3, a)$. The for these two lines to be orthogonal means that the two normal vectors are orthogonal or their dot product equals zero. This means

$$(-2, 1) \cdot (-3, a) = 6 + a = 0 \quad \text{or} \quad a = -6\,,$$

the same as before.

Problem 4

Each element of the set is of the form x^2. Thus the product of two elements of this set are of the form $x^2y^2 = (xy)^2$ and our set is closed under multiplication.

Problem 5

Notice that

$$4 = \binom{4}{1}$$
$$6 = \binom{4}{2}$$
$$4 = \binom{4}{3},$$

thus we can write S as

$$\sum_{k=0}^{4} \binom{4}{k} (x-1)^{4-k} 1^k = (x-1+1)^4 = x^4 .$$

Problem 6

We have

$$\frac{\log(8)}{\log(1/8)} = -1 .$$

Problem 7

We have

$$\left(\frac{a}{\sqrt{z}} - \frac{\sqrt{z}}{a^2}\right)^6 = \sum_{k=0}^{6} \binom{6}{k} \left(\frac{a}{\sqrt{z}}\right)^k \left(-\frac{\sqrt{z}}{a^2}\right)^{6-k}$$

The when you write out these terms the third one will be either the $k = 2$ or the $k = 4$ term depending on the direction you count from. The $k = 2$ is given by

$$\binom{6}{2} \left(\frac{a}{\sqrt{z}}\right)^2 \left(-\frac{\sqrt{z}}{a^2}\right)^4 = \frac{15z}{a^6} .$$

The $k = 4$ term is given by

$$\binom{6}{4} \left(\frac{a}{\sqrt{z}}\right)^4 \left(-\frac{\sqrt{z}}{a^2}\right)^2 = \frac{15}{z} .$$

Problem 8

When we draw the triangle in question we see that $B + C_2 = 90$ and $C_1 + A = 90$ thus setting these two expressions equal to each other (since they both equal 90) we have

$$B + C_2 = C_1 + A ,$$

or

$$C_1 - C_2 = B - A\,.$$

Problem 9

We have r equal to

$$r = (2a)^{2b} = (2a)^b (2a)^b = 2^b a^b 2^b a^b = (2^{2b} a^b)(a^b) = (2^2 a)^b a^b\,,$$

Thus $x = 2^2 a = 4a$.

Problem 10

For this problem we draw an equilateral triangle and drop a perpendicular from A to the midpoint of the side BC. This line segment has height AD. Forming the right triangle from ABD with hypotenuse AB we see that the length of AD is given by

$$AD^2 + 6^2 = 12^2 \quad \text{so} \quad AD = \sqrt{108}\,.$$

The the length of ED where E is the midpoint of the line AD is one-half of this length. Forming the right triangle EBD with hypotenuse EB we have that EB must satisfy Pythagoras theorem or

$$EB^2 = \left(\frac{1}{2}\sqrt{108}\right)^2 + 6^2 = 63\,.$$

Thus $EB = \sqrt{63}$.

Problem 11

When we draw the given figure we see that the perimeter we are looking to compute is given by $AP + PR + RA$. Breaking PR up into two parts as $PR = PQ + QR$ we see that this perimeter is given by

$$AP + PQ + QR + RA\,.$$

From the figure we have that $PQ = BP$ and $QR = CR$ thus this sum is

$$AP + BP + CR + RA = AB + AC = 2(20) = 40\,.$$

Problem 12

In a geometric progression each term is a multiple r of the previous term. Computing r from the first and second terms give

$$r = \frac{2^{1/3}}{2^{1/2}} = 2^{-1/6}\,.$$

Computing r from the second and third terms give

$$r = \frac{2^{1/6}}{2^{1/3}} = 2^{-1/6}\,,$$

agreeing with the first calculation. Thus the fourth term is then given by

$$2^{1/6}2^{-1/6} = 1\,.$$

Problem 13

We have

$$\sqrt{t^4 + t^2} = |t|\sqrt{t^2 + 1}\,.$$

Problem 14

A rhombus has four equal sides and its diagonals bisect each other at right angles. The area can be written as $\frac{1}{2}$ the product of the diagonals. From the problem we have that

$$A = k = \frac{1}{2}d(2d)\,,$$

or $d = \sqrt{k}$. Now extract out one of the triangles from the rhombus, say the one with a base given by the longer diagonal of length $2d$ and with a height of $\frac{d}{2}$. The using the Pythagorean theorem we have that the side of the rhombus (denoted s) of length

$$s^2 = d^2 + \frac{d^2}{4} = \frac{5}{4}d^2 = \frac{5}{4}k\,.$$

so

$$s = \frac{\sqrt{5k}}{2}\,.$$

Problem 15

To make things easier to understand we will assume that we have m men working for h hours a day for each of d days produce a articles. Then we have mhd hours used to produce a articles or

$$\frac{a}{mhd}\,,$$

articles per hour per man. If we have y men working y hours a day for y days we have $yyy = y^3$ man hours of work. Thus we should get

$$\frac{a}{mhd}y^3\,,$$

articles in that time. Since we are told that $m = h = d = a = x$ this becomes

$$\frac{y^3}{x^2}$$

Problem 16

If we increase the height by m and change the base to b' we will have a new area A' given by

$$A' = \frac{1}{2}b'(h+m)\,.$$

To have this new area equal $\frac{1}{2}$ of the original area we must have

$$A' = \frac{1}{2}b'(h+m) = \frac{1}{2}(\frac{1}{2}bh)\,,$$

so $b' = \frac{1}{2}\frac{bh}{h+m}$. The question then is how much do we have to take (say d) from b get this new length. We then solve for d in

$$b - d = b' = \frac{1}{2}\frac{bh}{h+m}\,.$$

This gives

$$d = b - \frac{1}{2}\frac{bh}{h+m} = \frac{b}{2}\left(\frac{h+2m}{h+m}\right)\,,$$

when we simplify.

Problem 17

Lets write the number 1000, 440 and 340 in terms of an unknown base r. Then we have

$$\begin{aligned} 1000 &= 1r^3 + 0r^2 + 0r^1 + 0r^0 \\ 440 &= 4r^2 + 4r^1 + 0r^0 \\ 340 &= 3r^2 + 4r^1 + 0r^0\,. \end{aligned}$$

Then we are told that subtracting 440 m.u. from 1000 m.u. gives 340 m.u. or when we represent the subtraction in terms of base r

$$r^3 - 4r^2 - 4r = 3r^2 + 4r\,.$$

Or

$$r^3 - 7r^2 - 8r = 0\,.$$

Since $r \neq 0$ we have $r^2 - 7r - 8 = 0$ or $(r+1)(r-8) = 0$. Since r must be positive we must take $r = 8$.

Problem 18

The changes in percent $+25\%$, -25%, $+25\%$, -25% will result in the number x being transformed to

$$(1.25)(0.75)(1.25)(0.75)x = \frac{225}{256}x = 0.8789x\,.$$

As a percentage change this is $1 - \frac{225}{256} = -\frac{31}{256} = -0.12$.

Problem 19

If we consider a possible intersection of the two curves we would have

$$2\log(x) = \log(2x)\,,$$

or

$$x^2 = 2x\,,$$

or since $x \neq 0$ this would be $x = 2$. Thus these two curves intersect at one point.

Problem 20

If we plot the two regions specified we see that the intersection of the two regions is in the first and second quadrants.

Problem 21

We have that

$$\begin{aligned}
\text{Area}(\triangle MNE) &= \frac{1}{2}\text{Area}(\triangle MAE) \\
&= \frac{1}{2}\cdot\frac{1}{3}\text{Area}(\triangle CAE) \\
&= \frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{2}\text{Area}(\triangle ABC) \\
&= \frac{1}{12}\text{Area}(\triangle ABC)\,.
\end{aligned}$$

Problem 22

If we know that we can divide the polynomial $3x^3 - 9x^2 + kx - 12$ by $x - 3$ we have that we can write

$$3x^3 - 9x^2 + kx - 12 = (x-3)(3x^2 + bx + c)\,,$$

for some values of b and c. By expanding the right-hand-side of the above we see that

$$\begin{aligned}
(x-3)(3x^2 + bx + c) &= 3x^3 + bx^2 + cx - 9x^2 - 3bx - 3c \\
&= 3x^2 + (b-9)x^2 + (c-3b)x - 3c\,.
\end{aligned}$$

When we equate this to the left-hand-side above we see that from the x^2 coefficient that $b - 9 = -9$ or $b = 0$. From the constant term that $-3c = -12$ or $c = 4$. Thus equating the x coefficients gives $k = c - 3b = 4$. Thus the polynomial that also divides $3x^3 - 9x^2 + kx - 12$ is

$$3x^2 + bx + c = 3x^2 + 4\,.$$

Problem 23

From the problem statement, we are told that

$$\frac{AP}{AB} = \frac{2}{2+3} = \frac{2}{5} \quad \text{and} \quad \frac{AQ}{AB} = \frac{3}{3+4} = \frac{3}{7}\,.$$

The point Q is at a distance of $\frac{3}{7}AB$ from the point A. The point P is at a distance of $\frac{2}{5}AB$ from the point A. Thus the distance between the points QP is then

$$\frac{3}{7}AB - \frac{2}{5}AB = \frac{1}{35}AB\,.$$

We are told that this length is 2 thus $AB = 70$.

Problem 24

Let x be the price of the first book, then each book after has a price of

$$x\,, x+2\,, x+4\,, x+6\,, \cdots\,,$$

Let k be the index of the book, then the price of the first book is x, the price of the second book is $x+2$, the price of the third book is $x+2(2) = x+4$. Thus the kth book has a price of

$$x + 2(k-1) \quad \text{for} \quad 1 \le k \le 31\,.$$

The price of the book at the far right is $x+60$ and the middle book is at index $k = \frac{31+1}{2} = 16$ so has a price of $x+30$. The book to the left of the middle book has price $x+28$ and the book to the right of the middle book as a price $x+32$. Thus in the problem statement we can assume that the price of the last book equals the price of the middle book plus the left middle book *or* the right middle book. Thus we have two possible equalities

$$\begin{aligned} x+60 &= x+30+x+28 \quad \text{so} \quad x = 2 \quad \text{or} \\ x+60 &= x+30+x+32 \quad \text{so} \quad x = -2\,. \end{aligned}$$

Since $x > 0$ we have $x = 2$ and the adjacent book is the one to the *left* of the middle book.

Problem 25

Lets start with the problem statement and define the angle at the corner where the vertex B is located to be α. We will then use the given information to determine all other angles in the given figure in terms of α. Then since the sides BQ and QP are of the same length the angle BPQ is also α. Then since angles in a triangle sum to 180 degrees we have that the angle BQP is $180 - 2\alpha$. As supplementary angles must add to 180 we have that angle AQP is 2α. As the length QP is equal to AP the angle PAQ is also 2α. As angle of a triangle sum to 180 degrees the angle QPA must sum to $180 - 4\alpha$. Supplementary angles along the line segment BPC then give

$$\alpha + 180 - 4\alpha + \angle APC = 180 \quad \text{so} \quad \angle APC = 3\alpha\,.$$

The lengths of the sides AP and AC begin equal give that $\angle APC = \angle PCA = 3\alpha$. That the angles in a triangle must sum to 180 give us that $\angle PAC = 180 - 6\alpha$. Finally, since we are told that the sides AB and BC are equal we have that

$$\angle BAC = \angle BCA\,.$$

In terms of α and what we have computed above this is

$$3\alpha = 2\alpha + (180 - 6\alpha) = 180 - 4\alpha\,.$$

Solving for α gives that $\alpha = \frac{180}{7} = 25\,\frac{5}{7}$

Problem 26

An arithmetic series has terms given by $a_1 + (k-1)d$ from what we are given in the problem statement we have that

$$\sum_{k=1}^{50}(a_1 + (k-1)d) = 200$$
$$\sum_{k=1}^{100}(a_1 + (k-1)d) = 2700 + 200 = 2900$$

Using Equation 18 these sums on the left-hand-side are given by

$$25(2a_1 + 49d) = 200$$
$$50(2a_1 + 99d) = 2900\,.$$

Subtract the first from the second to get $(99 - 49)d = 50$ so $d = 1$. Then the first equation gives

$$a_1 = \frac{1}{2}(8 - 49) = -20.5\,.$$

Problem 27

If x is the internal angle of the polygon, then there are $N_1 = \frac{360}{x}$ sides to the polygon P_1. If P_2 has an internal angle of size kx then the number of sides if $N_2 = \frac{360}{kx} = \frac{N_1}{k}$. Now N_2 must be a positive integer so $k = N_1$ is one possibility, but all others give non integer numbers for N_2. Thus there is only one possibility.

Problem 28

Consider powers of the given number. The number 2137^0 ends in a 1. The number 2137^1 ends in a 7. The number 2137^2 ends in a 9. The number 2137^3 ends in a 3 and finally the number 2137^4 ends in a 1. The pattern of the ending digit repeats from this point onward. Thus we can evaluate the last digit of 2137^{753} by writing 753 in multiples of 4 as

$$753 = 700 + 40 + 12 + 1 = 752 + 1\,.$$

From which we see that

$$2137^{753} = 2137^{752}\, 2137\,,$$

the product of a number that ends in a 1 and a number that ends in a 7. Thus this number ends in a 7.

Problem 29

Consider the quadratic $ax^2 + bx + c = 0$ as (since $a \neq 0$)

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0\,. \tag{139}$$

Then since r and s are roots of this the above can be factored into

$$(x - r)(x - s) = 0\,.$$

Expanding this out and comparing to Equation 139 we see that

$$\begin{aligned} rs &= \frac{c}{a} \\ -(r + s) &= \frac{b}{a}\,. \end{aligned}$$

The equation that has roots $ar + b$ and $as + b$ looks like

$$(x - (ar + b))(x - (as + b)) = 0\,.$$

or expanding this we get

$$x^2 - (ar + b + as + b)x + (ar + b)(as + b) = 0\,.$$

or grouping terms

$$x^2 - (a(r + s) + 2b)x + (a^2 rs + abr + abs + b^2) = 0\,.$$

If we use what we know about $r + s$ and rs we can write the above as

$$x^2 - \left(a\left(-\frac{b}{a}\right) + 2b\right)x + \left(a^2\left(\frac{c}{a}\right) + ab\left(-\frac{b}{a}\right) + b^2\right) = 0\,,$$

or

$$x^2 - bx + ac = 0\,,$$

when we simplify.

Problem 30

For this problem we are told that $\log_{10}(2) = a$ and $\log_{10}(3) = b$. Then to begin write the desired expression as

$$\log_5(12) = \frac{\log_{10}(12)}{\log_{10}(5)}\,.$$

We need to compute $\log_{10}(5)$. To do this recall that $2 \cdot 5 = 10$. Taking the logarithm to the base 10 of both sides of that identity gives

$$\log_{10}(2) + \log_{10}(5) = 1\,,$$

so that since $\log_{10}(2) = a$ we get $\log_{10}(5) = 1 - a$. Now to evaluate $\log_{10}(12)$ we write

$$\log_{10}(12) = \log_{10}(3 \cdot 2^2) = \log_{10}(3) + 2\log_{10}(2) = b + 2a\,.$$

Thus we finally find

$$\log_5(12) = \frac{b + 2a}{1 - a}\,.$$

Problem 32

The area of a polygon with n sides is given by Equation 2 so that we can write

$$A = \frac{r^2 n \sin\left(\frac{2\pi}{n}\right)}{2} = 3r^2\,,$$

or

$$n \sin\left(\frac{2\pi}{n}\right) = 6\,.$$

or

$$\sin\left(\frac{2\pi}{n}\right) = \frac{6}{n}\,.$$

We now try some of the given values for n. If $n = 8$ is the following proposed equality

$$\sin\left(\frac{2\pi}{8}\right) = \frac{6}{8}\,,$$

which is not true. If $n = 10$ the answer is still no. If $n = 12$ however we get

$$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}\,,$$

which is true.

Problem 33

Write the equation $2^{2x} - 3^{2y} = 55$ in a factored form as

$$(2^x - 3^y)(2^x + 3^y) = 55\,.$$

Lets look at the factors of 55. Note that only $55 = 1 \cdot 55$ and $55 = 5 \cdot 11$. Thus for there to be a solution (x, y) to this equation since

$$2^x + 3^y < 2^x - 3^y\,.$$

requires that $2^x - 3^y$ equals the smaller of the two factors or $2^x - 3^y = 1$ or $2^x - 3^y = 5$. We next ask, can we solve the system

$$\begin{aligned} 2^x - 3^y &= 1 \\ 2^x + 3^y &= 55\,, \end{aligned}$$

for integers x and y? Adding the two equations gives $2^{x+1} = 56$ so $x + 1 = \log_2(56)$. Since 56 is not a power of 2 x will not be an integer. Lets try the other possibility

$$\begin{aligned} 2^x - 3^y &= 5 \\ 2^x + 3^y &= 11\,, \end{aligned}$$

When we again add we get $2^{x+1} = 16$ or $x + 1 = \log_2(16) = 4$, so $x = 3$. Then the first equation is $2^3 - 3^y = 5$ or $3^y = 3$ so $y = +1$ and one solution.

Problem 34

We write the given fraction as

$$\frac{2x+3}{x+2} = \frac{2(x+2-2)+3}{x+2} = \frac{2(x+2)-1}{x+2} = 2 - \frac{1}{x+2}\,.$$

Thus the upper bound for this set is when $x \to +\infty$, which we cannot get directly from elements in the set. The lower bound of this set is when $x =$ and is the value of 1. Thus m is in S and M is not in S.

Problem 35

We need to write

$$695 = a_1 + a_2 2! + a_3 3! + \cdots + a_n n!\,,$$

where $a_1, a_2, \cdots a_n$ are integers and $0 \le a_k \le k$. Recall that $3! = 6$, $4! = 24$, $5! = 120$, and $6! = 720$. Thus we need to find a_i such that

$$695 = a_5(120) + a_4(24) + a_3(6) + a_2(2) + a_1(1)\,.$$

We should take $a_5 = 5$ and then we have to find a_i such that

$$695 = 5(120) + a_4(24) + a_3(6) + a_2(2) + a_1(1)\,,$$

or

$$95 = a_4(24) + a_3(6) + a_2(2) + a_1(1)\,.$$

We need to take $a_4 = 3$ then $24a_4 = 72$ and we then need to satisfy

$$23 = a_3 6 + 2a_2 + a_1\,.$$

Lets take $a_3 = 3$ and then we need to satisfy

$$5 = 2a_2 + a_1\,,$$

Lets take $a_2 = 2$ and $a_1 = 1$. Thus we have

$$695 = 5(5!) + 3(4!) + 3(3!) + 2(2!) + 1(1!)\,.$$

Problem 36

When we draw the triangle ABC we have segments AD and BE meeting at right angles at the point O. Then we will denote the length BO as $2b$, the length EO as b, the length of AO as $2a$, and the length of DO as a. Since the length of BD is $3.5 = \frac{7}{2}$ we can use Pythagorean theorem in the right triangle BOD to get

$$4b^2 + a^2 = \frac{49}{4}\,.$$

The Pythagorean theorem applied to triangle AOE gives

$$4a^2 + b^2 = 9\,.$$

Multiplying the first equation by 4 to get $4a^2 + 16b^2 = 49$, from which we can subtract the second equation above to get

$$15b^2 = 40 \quad \text{so} \quad b^2 = \frac{8}{3}\,.$$

Then we have

$$4a^2 = 9 - \frac{8}{3} = \frac{19}{3} \quad \text{so} \quad a^2 = \frac{19}{12}\,.$$

The length of AB is then given from

$$AB^2 = 4a^2 + 4b^2 = 4\left(\frac{19}{12}\right) + 4\left(\frac{8}{3}\right) = 17\,.$$

Thus $AB = \sqrt{17}$.

Problem 37

Let v_A, v_B, and v_C be the speeds of A, B, and C respectively. Then the problem statement states:

- If A and B ran a race then A will cross the finish line at a time $T_1 = \frac{d}{v_A}$ and B at that same time T_1 will be located at $v_B T_1 = d - 20$.
- If B and C race then B will cross the finish line at a time T_2 given by $T_2 = \frac{d}{v_B}$ and C will be located at $d - 10 = v_c T_2$ at that time.
- If A and C race then A will cross the finish line at a time $T_3 = \frac{d}{v_A}$ and C will be located at $d - 28 = v_C T_3$.

These three equations give us that

$$v_B \left(\frac{d}{v_A}\right) = d - 20$$
$$v_C \left(\frac{d}{v_B}\right) = d - 10$$
$$v_C \left(\frac{d}{v_A}\right) = d - 28\,.$$

From the first and second equation we get $\frac{v_B}{v_A} = \frac{d-20}{d}$ and $\frac{v_C}{v_B} = \frac{d-10}{d}$. Then write the third equation as $\frac{v_C}{v_A} = \frac{d-28}{d}$. If we multiply all of these fractions of v_i together we get

$$\frac{v_B}{v_A} \cdot \frac{v_C}{v_B} \cdot \frac{v_A}{v_C} = \left(\frac{d-20}{d}\right)\left(\frac{d-10}{d}\right)\left(\frac{d}{d-28}\right) = 1\,.$$

This means that

$$(d-20)(d-10) = d(d-28)\,,$$

or

$$d^2 - 30d + 200 = d^2 - 28d\,,$$

or $-2d + 200 = 0$ so $d = 100$.

Problem 40

Method 1: Since we are told that $5x + 12y = 60$ we have that $y = 5 - \frac{5}{12}x$ and we are trying to optimize

$$\sqrt{x^2 + y^2} = \sqrt{x^2 + \left(5 - \frac{5}{12}x\right)^2} = \sqrt{x^2 + 25 - \frac{25}{6}x + \frac{25}{144}x^2}$$
$$= \sqrt{\frac{169}{144}x^2 - \frac{25}{6}x + 25} = \sqrt{\left(\frac{13}{12}x\right)^2 - \frac{25}{6}x + 25}\,.$$

Insert a $-v$ inside the parenthesis with the $\frac{13}{12}x$ term and then to cancel that effect we have to add the terms $\frac{13}{6}xv - v^2$ to get

$$\sqrt{\left(\frac{13}{12}x - v\right)^2 + \frac{13}{6}xv - v^2 - \frac{25}{6}x + 25}\,.$$

To make the terms linear in x vanish we have to take

$$\frac{13v}{6} = \frac{25}{6} \quad \text{or} \quad v = \frac{25}{13}\,.$$

Then in that case we have a constant term given by

$$-\frac{25^2}{13^2} + 25 = 25\left(-\frac{25}{13^2} + 1\right) = \frac{12^2(25)}{13^2}\,.$$

Thus we have

$$\sqrt{\left(\frac{13}{12}x - \frac{25}{13}\right)^2 + \frac{5^2 12^2}{13^2}}\,.$$

Thus we pick x such that $\frac{13}{12}x - \frac{25}{13} = 0$ and we get a smallest value for our objective function of

$$\frac{5 \cdot 12}{13} = \frac{60}{13}\,.$$

Method 2: The smallest point of the function $\sqrt{x^2 + y^2}$ on the line $\mathcal{L} \equiv 5x + 12y = 60$ is a point on $\mathcal{L}$ that is also on a perpendicular to the origin. Let the point A be $(0, 5)$, the point B be $(12, 0)$, and the origin be O. Let the minimum point be P. Then $\triangle AOB$ is similar to $\triangle APO$. Thus

$$\frac{OP}{OB} = \frac{AO}{AB}\,.$$

Or given what we know

$$\frac{OP}{12} = \frac{5}{\sqrt{5^2 + 12^2}}\,,$$

thus OP is

$$OP = \frac{60}{\sqrt{169}} = \frac{60}{13}\,,$$

the same as before.

The 1962 Examination

Problem 1

We have

$$\frac{1^{4y-1}}{5^{-1}+3^{-1}} = \frac{1}{5^{-1}+3^{-1}} = \frac{1}{\frac{3+5}{15}} = \frac{15}{8}\,.$$

Problem 2

We have

$$\sqrt{\frac{4}{3}} - \sqrt{\frac{3}{4}} = \frac{2}{\sqrt{3}} - \frac{\sqrt{3}}{2} = \frac{4}{2\sqrt{3}} - \frac{3}{2\sqrt{3}} = \frac{1}{2\sqrt{3}} = \frac{\sqrt{3}}{6}\,.$$

Problem 3

An arithmetic sequence is given by Equation 17, so letting $n = 1, 2, 3$ gives

$$\begin{aligned} x - 1 &= a_1 \\ x + 1 &= a_1 + d \\ 2x + 3 &= a_1 + 2d\,. \end{aligned}$$

The first and second equations mean that $x + 1 = x - 1 + d$ or $d = 2$. Then the first and third equation give

$$2x + 3 = x - 1 + 2(2)\,,$$

so $x = -4 + 4 = 0$.

Problem 4

From $8^x = 32$ we have $2^{3x} = 2^5$ so $x = \frac{5}{3}$.

Problem 5

Let r be the old radius, then the new circumference is $2\pi(r+1)$ and the new diameter is $2(r+1)$. Thus the ratio is $\frac{2\pi}{2} = \pi$.

Problem 6

The area of an equilateral triangle with a side of length l is given by

$$A = \frac{1}{2}l\left(\frac{\sqrt{3}}{2}l\right) = \frac{\sqrt{3}}{4}l^2 .$$

We are told that this area is $9\sqrt{3}$ so

$$\frac{\sqrt{3}}{4}l^2 = 9\sqrt{3} \quad \text{so} \quad l^2 = 4 \cdot 9 \quad \text{so} \quad l = 6 .$$

The perimeter of this triangle is $3l = 18$ and we are told that this equals the perimeter of a square with side a so $4a = 18$ or $a = \frac{9}{2}$. Then the diagonal d of this square has a length

$$d^2 = \frac{9^2}{4^2} + \frac{9^2}{4^2} = \frac{9^2}{2} ,$$

so $d = \frac{9}{\sqrt{2}} = \frac{9\sqrt{2}}{2}$.

Problem 7

When we recall the definition of the bisectors of the exterior angles of a triangle, and connect the exterior bisectors of B and C at D we see that from the original triangle $A+B+C = 180$ and then from the supplementary angles along the line segments AB we have

$$B + 2B_e = 180 \quad \text{and} \quad C + 2C_e = 180 .$$

Where A, B, and C stand for the internal angular measurements of the angles A, B, and C in degrees, and B_e and C_e stand for the exterior bisector angle in degrees. Thus we can compute the angle $\angle BDC$ as

$$\begin{aligned}\angle BDC = 180 - B_e - C_e &= 180 - \left(90 - \frac{1}{2}B\right) - \left(90 - \frac{1}{2}C\right) \\ &= \frac{1}{2}(180 - A) .\end{aligned}$$

Problem 8

We have n numbers a_k, where one (say a_{k^*}) is $1-\frac{1}{n}$ and the others are 1. Then the arithmetic mean is $\frac{1}{n}\sum_{k=1}^{n} a_k$ or

$$\frac{1}{n}\left[\sum_{k=1;k\neq k^*} a_k + \left(1 - \frac{1}{n}\right)\right] = \frac{1}{n}\left[n - 1 + 1 - \frac{1}{n}\right] = \frac{1}{n}\left(n - \frac{1}{n}\right) = 1 - \frac{1}{n^2} .$$

Problem 9

We have

$x^9-x = x(x^8-1) = x(x^4-1)(x^4+1) = x(x^2-1)(x^2+1)(x^4+1) = x(x-1)(x-1)(x^2+1)(x^4+1)\,,$

giving 5 factors.

Problem 10

The average rate r for the entire trip (in miles per hour) is

$$r = \frac{150+150}{3\frac{1}{3}+4\frac{1}{6}} = 40\,,$$

when we simplify. The average rate going is $\frac{150}{3\frac{1}{3}} = 45$. Thus the average rate going exceed the average rate of the entire trip by 5 miles per hour.

Problem 11

For the equation

$$x^2 - px + \frac{1}{4}(p^2-1) = 0\,,$$

we have roots given by

$$x = \frac{-(-p) \pm \sqrt{p^2 - 4\left(\frac{p^2-1}{4}\right)}}{2} = \frac{p \pm \sqrt{p^2-(p^2-1)}}{2} = \frac{p \pm \sqrt{1}}{2} = \frac{p \pm 1}{2}\,.$$

So the difference between the larger root and the smaller root is

$$\frac{p+1}{2} - \frac{p-1}{2} = 1\,.$$

Problem 12

For the given expression, the binomial expansion gives

$$\begin{aligned}\left(1-\frac{1}{a}\right)^6 &= \sum_{k=0}^{6} \binom{6}{k} 1^k \left(-\frac{1}{a}\right)^{6-k} \\ &= 1 - \frac{6}{a} + \frac{15}{a^2} - \frac{20}{a^3} + \frac{15}{a^4} - \frac{6}{a^5} + \frac{1}{a^6}\,.\end{aligned}$$

Adding the coefficients of the last three terms gives

$$15 - 6 + 1 = 10\,.$$

Problem 13

From the description we have that $R = a\frac{S}{T}$ where a is a constant. From the given information about R when $T = \frac{9}{14}$ and $S = \frac{3}{7}$ we have that

$$\frac{4}{3} = a\frac{(3/7)}{(9/14)} \quad \text{so} \quad a = 2\,.$$

Thus $R = 2\frac{S}{T}$. If $R = \sqrt{48}$ and $T = \sqrt{75}$ we have

$$\sqrt{48} = 2\frac{S}{\sqrt{75}} \quad \text{so} \quad S = 30\,.$$

Problem 14

For the given sum we have

$$4 - \frac{8}{3} + \frac{16}{9} - \cdots = \frac{4}{3^0} - \frac{2^2 2}{3^1} + \frac{2^2 2^2}{3^2} - \cdots = 4\left(1 - \frac{2}{3} + \frac{2^2}{3^2} - \frac{2^3}{3^3} + \cdots\right)$$
$$= 4\sum_{k=0}^{\infty}\left(-\frac{2}{3}\right)^k = 4\left(\frac{1}{1+\frac{2}{3}}\right) = \frac{12}{5}\,.$$

Problem 16

Let the first rectangle R_1 have sides x and 2, and an area $A_1 = 12 = 2x$, so $x = 6$ inches. The diagonal of this rectangle is of length $\sqrt{2^2+6^2} = \sqrt{40} = 2\sqrt{10}$. The similarity between the rectangles R_1 and R_2 mean that

$$\frac{2\sqrt{10}}{15} = \frac{2}{h}\,, \quad \text{so} \quad h = \frac{15}{\sqrt{10}}\,,$$

and

$$\frac{2\sqrt{10}}{15} = \frac{6}{b}\,, \quad \text{so} \quad b = \frac{3(15)}{\sqrt{10}}\,.$$

Then the area of the second rectangle is given by

$$bh = \frac{135}{2}\,,$$

when we do the multiplications.

Problem 17

This question is when we define $a = \log_8(225)$ and $b = \log_2(15)$ what is a in terms of b. Recall that $225 = 15^2$ so from the definition of a we have

$$a = 2\log_8(15) = 2\frac{\log_2(15)}{\log_2(8)} = \frac{2}{3}b\,.$$

Problem 18

Given we know r and $n = 12$ for this polygon we can use Equation 2 to write its area as

$$A = \frac{1}{2}r^2(12)\sin\left(\frac{2\pi}{12}\right) = 3r^2\,.$$

Problem 19

When our function is given by $y = ax^2 + bx + c$ we can determine a, b, and c by putting the points into this expression and solving for them. We have

$$\begin{aligned} 12 &= a - b + c \\ 5 &= c \\ -3 &= 4a + 2b + c\,. \end{aligned}$$

Next put $c = 5$ into the first and third equation to get the system

$$\begin{aligned} 7 &= a - b \\ -8 &= 4a + 2b\,. \end{aligned}$$

Multiply the first equation by 2 and add it to the second to get $a = 1$. Put what we know about a into $7 = a - b$ to get that $b = -6$. Thus we see that $a + b + c = 1 - 6 + 5 = 0$.

Problem 21

We are told at $3 + 2i$ is a root for $2x^2 + rx + s = 0$. When we put that number into this polynomial we have

$$2(4 + 2i)^2 + r(3 + 2i) + s = 0\,,$$

or expanding and simplifying we get

$$10 + 3r + s + (24 + 2r)i = 0\,.$$

Thus $r = -12$ and $10 - 36 + s = 0$ or $s = 26$.

Problem 22

We have

$$121_b = b^2 + 2b + 1 = (b+1)^2 \,.$$

Problem 23

The area of the triangle can be computed in two ways

$$A = \frac{1}{2} AB \cdot CD = \frac{1}{2} BC \cdot AE \,.$$

Thus solving for BC we have that

$$BC = \frac{AB \cdot CD}{AE} \,.$$

Showing that we *know* the value of BC since everything on the right-hand-side of the above expression is known. Since $\triangle CDB$ is a right triangle with hypotenuse BC we can use the Pythagorean theorem to conclude that

$$BC^2 = BD^2 + CD^2 \,.$$

Thus we have

$$\begin{aligned} BD^2 &= BC^2 - CD^2 \\ &= \frac{AB^2 CD^2}{AE^2} - CD^2 = \frac{(AB^2 - AE^2)CD^2}{AE^2} \,. \end{aligned}$$

Thus $BD = \sqrt{AB^2 - AE^2} \left(\frac{CD}{AE}\right)$.

Problem 24

Let p, q, and r be the rates for P, Q, and R respectively. Then

$$\frac{1}{p+q+r} = x \,, \tag{140}$$

and from the problem statement we have

$$\frac{1}{p} = x + b \,, \quad \frac{1}{q} = x + 1 \,, \quad \frac{1}{r} = 2x \,.$$

Thus

$$r = \frac{1}{2x} \,, \quad q = \frac{1}{1+x} \,, \quad p = \frac{1}{x+6} \,,$$

so from Equation 140 we have

$$1 = x(p+q+r) = \frac{1}{2} + \frac{x}{1+x} + \frac{x}{x+6}\,,$$

or

$$\frac{1}{2}(x+1)(x+6) = x(x+6) + x(x+1)\,.$$

Expanding and grouping we get

$$3x^2 + 7x - 6 = 0\,,$$

or if we factor $(3x-2)(x+3) = 0$ so $x = -3$ or $x = \frac{2}{3}$.

Problem 26

To find the maximum of $8x - 3x^2$ write it like the following

$$\begin{aligned} 8x - 3x^2 &= -3\left(x^2 - \frac{8}{3}x\right) = -3\left(x^2 - \frac{8}{3}x + \frac{16}{9}\right) + \frac{16}{4} \\ &= -3\left(x - \frac{4}{3}\right)^2 + \frac{16}{3}\,. \end{aligned}$$

Thus the maximum value is when $x = \frac{4}{3}$ and our objective function has a maximum value of $\frac{16}{3}$.

Problem 27

We can easily see that #1 and #2 are true. The question now becomes if

$$\min(a, \max(b, c)) = \max(\min(a, b), \min(a, c))\,, \tag{141}$$

or #3 is true. One can try several values for a, b, and c to try and find a case where the two expressions are not equivalent in hopes of ruling that expression false. When working this problem, I was not able to find any combination of a, b, and c that made this expression false. To determine if this expression is true for all possible values of a, b, and c we can consider all of the possible orderings of three numbers a, b, and c. We could have any of

$$\begin{aligned} &a < b < c, \\ &a < c < b \\ &b < c < a \\ &b < a < c \\ &c < a < b \\ &c < b < a\,. \end{aligned}$$

We can then explicitly expand the left-hand-side and the right-hand-side of Equation 141 for each of these options and verify that both sides are equal. We do that in Table 4. From this we see that this equation is indeed true.

left-hand-side	right-hand-side
a	$\max(a, a) = a$
$\min(a, b) = a$	$\max(a, a) = a$
$\min(a, c) = c$	$\max(b, c) = c$
$\min(a, c) = a$	$\max(b, a) = a$
$\min(a, b) = a$	$\max(a, c) = a$
$\min(a, b) = b$	$\max(b, c) = b$

Table 4: A comparison between the left-hand-side and right-hand-side of Equation 141 for all possible orderings of a, b, and c.

Problem 28

To solve for x in

$$x^{\log_{10}(x)} = \frac{x^3}{100},$$

we can take $\log_{10}$ of both sides to get

$$(\log_{10}(x))^2 = \log_{10}(x^3) - \log_{10}(10^2) = 3\log_{10}(x) - 2.$$

Let $v = \log_{10}(x)$ then the above is given by

$$v^2 - 3v + 2 = 0,$$

or when we factor

$$(v-2)(v-1) = 0.$$

Thus $v = 2$ or $v = 1$. In the first case $\log_{10}(x) = 2$ so $x = 100$ and in the second case $\log_{10}(x) = 1$ so $x = 10$.

Problem 29

For the given inequality we have $2x^2 + x - 6 < 0$ or $(2x-3)(x+2) < 0$. The points x that make the left-hand-side equal to zero are given by $x = \frac{3}{2}$ and $x = -2$. In between these points we find that $2x^2 + x - 6 = -6 < 0$. Thus the quadratic is below zero in the range $-2 < x < \frac{3}{2}$.

Problem 30 (some logic)

Using De Morgan's law to the statement $\sim (p \wedge q)$ is *equal* to $(\sim p) \vee (\sim q)$. This later event means that $\sim (p \wedge q)$ is true when:

- p is false and q is true

- p is true and q is false
- p is false and q is false

The first statement $p \vee q$ is true when p and q are both true. This does not match any of the three conditions above. For the second statement $p \wedge \sim q$, to be true means that p is true and q is false. By the second of above statements this implies that $\sim (p \wedge q)$ is true. For the third statement $\sim p \wedge q$, to be true means that p must be false and q must be true. By the third of the above statements this implies that $\sim (p \wedge q)$ is true. Thus three statements are true.

Problem 31 (the ratio of angles in polygons)

From Equation 6 the interior angle for a n sided regular polygon is

$$\frac{n-2}{n} 180 \, .$$

Thus if we denote the number of sides of the first regular polygon by n and the number of sides in the second regular polygon as N then from the problem statement we have

$$\frac{n-2}{n} 180 = \frac{N-2}{N} 180 \left(\frac{3}{2} \right) \, .$$

Written this way we have that the interior angle of the polygon with N sides is smaller than the polygon with n sides and thus $N > n$. Solving for N as a function of n we find

$$N = \frac{4n}{6-n} \, .$$

Since N must be a positive integer we need only let $n = 3, 4, 5$ and see for which values of n if this is true. We find

$$\begin{aligned} N(3) &= \frac{4(3)}{3} = 4 \\ N(4) &= \frac{4(4)}{2} = 8 \\ N(5) &= \frac{4(5)}{1} = 20 \, . \end{aligned}$$

Thus there are three regular polygons for which the given statement is true.

Problem 32 (a recursion relationship)

For the recursion $x_{k+1} = x_k + \frac{1}{2}$, when $k = 1, 2, 3, \cdots n-1$ we find

$$\begin{aligned}
x_1 &= 1 \\
x_2 &= 1 + \frac{1}{2} \\
x_3 &= 1 + \frac{1}{2} + \frac{1}{2} = 1 + \frac{2}{2} \\
x_4 &= 1 + \frac{2}{2} + \frac{1}{2} = 1 + \frac{3}{2} \\
x_5 &= 1 + \frac{3}{2} + \frac{1}{2} = 1 + \frac{4}{2} \\
&\vdots \\
x_k &= 1 + \frac{k-1}{2} \,.
\end{aligned}$$

Then to evaluate $x_1 + x_2 + x_3 + \cdots + x_n$ we have

$$\begin{aligned}
\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} \left(1 + \frac{i-1}{2} \right) &= n + \frac{1}{2} \sum_{i=1}^{n} i - 1 \\
&= n + \frac{1}{2} \sum_{i=0}^{n-1} i = n + \frac{1}{2} \left(\frac{n(n-1)}{2} \right) = \frac{n}{4}(n+3) \,.
\end{aligned}$$

Problem 33

For the inequality $2 \le |x-1| \le 5$ we have, if $x - 1 > 0$ then this is given by

$$2 \le x - 1 \le 5 \quad \Rightarrow \quad 3 \le x \le 6 \,,$$

while if $x - 1 < 0$ then this is

$$2 \le -(x-1) \le 5 \quad \Rightarrow \quad 1 \le -x \le 4 \quad \Rightarrow \quad -4 \le x \le -1 \,,$$

Thus the possible values for x are

$$-4 \le x \le -1 \quad \text{or} \quad 3 \le x \le 6 \,.$$

Problem 34

For the equation $x = k^2(x-1)(x-2)$ write this expression as

$$\frac{x}{(x-1)(x-2)} = k^2 \,.$$

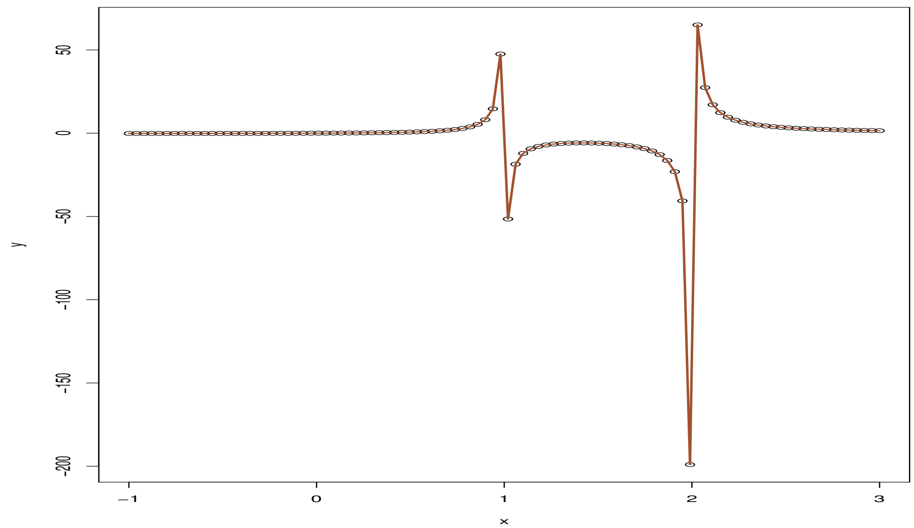

Figure 3: A plot of the function $\frac{x}{(x-1)(x-2)}$ vs. x.

Then since k^2 is positive in order for there to be a real root x we must have the left-hand-side of this expression also positive. If we plot the function $\frac{x}{(x-1)(x-2)}$ we get Figure 3, where the asymptotes at $x = 1$ and $x = 2$ are only indicated notionally. From this plot we see that if we plot k^2 as a horizontal line above the x axis it will intersect the graph of $\frac{x}{(x-1)(x-2)}$ at two places once for $-\infty < x < 1$ and another time for $2 < x < +\infty$. Thus since for any value of k the expression k^2 is positive and there will be two real roots for x.

Problem 35 (a man on his way to dinner)

To begin we should draw (approximate) pictures of the two hands on the clock at the two hypothetical positions. Then assume that the initial angle of the minute hand is at α degrees from twelve o'clock. In this case the hour hand is at $\alpha + 110$ degrees from twelve o'clock. Lets assume that an amount of time, say T in hours has passed. Then the minute hand will be located at

$$h_{\text{minute}}(T) = \alpha + 360T\,,$$

degrees from twelve o'clock, while the hour hand will be located at

$$h_{\text{hour}}(T) = \alpha + 110 + \frac{360}{12}T\,,$$

degrees from twelve o'clock. We are told that when we look again at our watch the hands are separated by 110 degrees. Thus at this new time

$$h_{\text{minute}}(T) - h_{\text{hour}}(T) = \alpha + 360T - \left(\alpha + 110 + \frac{360}{12}T\right) = 110\,.$$

When we solve for T we get $T = \frac{2}{3}$ of an hour or 40 minutes.

Problem 36 (solutions to $(x-8)(x-10) = 2^y$)

Consider the equation $(x-8)(x-10) = 2^y$. Then for all possible values of y the value of 2^y is positive. Thus in order for $(x-8)(x-10)$ to be positive we must have both factors positive or both factors negative. To have both factors positive means that $x > 10$ and to have both factors negative means that $x < 8$. To begin lets assume that both factors are positive and $x > 10$. Then taking the logarithm to the base 2 of both expressions gives

$$\log_2(x-8) + \log_2(x-10) = y\,.$$

Since we want y to be an integer then *both* $\log_2(x-8)$ and $\log_2(x-10)$ must be integers. This will only happen if both $x-8$ and $x-10$ are powers of two. As $x-10 < x-8$ and their difference is $x-8-(x-10) = 2$ the smaller power of two must be assigned to the term $x-10$ and they must be sequential powers of two. Thus one way to look for solutions is to simply assign sequential powers of two to $x-8$ and $x-10$ looking for a consistent system. Since to have both factors positive means that $x > 10$ or $x-10 > 0$. Thus the smallest value for $x-10$ would be 1. We are looking for a consistent solution to the following equations

$$x - 10 = 2^p \quad \text{and} \quad x - 8 = 2^{p+1} \quad \text{for} \quad p \geq 0\,.$$

Solving for x in both equations and setting them equal gives

$$x = 10 + 2^p = 8 + 2^{p+1}\,.$$

This equation is equivalent to $1 + 2^{p-1} = 2^p$, which only holds for $p = 1$. Then $x = 12$ and $y = \log_2(4) + \log_2(2) = 3$.

Now we consider the case where the two factors are negative or $x < 8$. Then we introduce a negative sign and take the base 2 logarithm of both expressions gives

$$\log_2(-(x-8)) + \log_2(-(x-10)) = y\,.$$

Then in the same way as the first part of this problem both terms $-(x-10)$ and $-(x-8)$ must be powers of two, they are related by $-(x-10) > -(x-8)$, and their difference is $-(x-10)+(x-8) = 2$ so they must be sequential powers of two. In addition since $x < 8$ we have that $-(x-8) = 2^p$ for $p \geq 0$. Thus we have the two equations

$$-(x-8) = 2^p \quad \text{and} \quad -(x-10) = 2^{p+1}\,,$$

or

$$x = 8 - 2^p \quad \text{and} \quad x = 10 - 2^{p+1}\,.$$

Setting these two equal to each other gives the relationship $2^{p+1} = 2 + 2^p$, which has the solution when $p \geq 0$ given by $p = 1$. Then we have $x = 8 - 2 = 6$ and $y = \log_2(-(6-8)) + \log_2(-(6-10)) = 3$.

Problem 37 (maximum area)

Place the points A, B, C, and D at the Cartesian coordinates $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$ respectively. We want to pick the points E and F such that $CDFE$ has a maximal area.

Let E be at the point $(x,0)$ and F the point $(0,x)$. Then the are of $CDFE$ is the area of the full square minus the area of the two triangles FAE and CBE. These two triangles have areas $\frac{1}{2}x^2$ and $\frac{1}{2}1(1-x)$ respectively. Thus the area of the quadrilateral (denoted as A) is

$$\begin{aligned} A &= 1 - \frac{1}{2}x^2 - \frac{1}{2}(1-x) = \frac{1}{2} + \frac{x}{2} - \frac{1}{2}x^2 \\ &= -\frac{1}{2}(x^2 - x) + \frac{1}{2} = -\frac{1}{2}(x^2 - x + \frac{1}{4}) + \frac{1}{8} + \frac{1}{2} = -\frac{1}{2}\left(x - \frac{1}{2}\right)^2 + \frac{5}{8}. \end{aligned}$$

The largest we can make this is when we take $x = \frac{1}{2}$ where we get the value of $\frac{5}{8}$.

Problem 38 (the population of Nosuch Junction)

From the problem statement if we let p be the initial population then we are told that

$$\begin{aligned} p &= x^2 \\ p + 100 &= y^2 + 1 \\ p + 200 &= z^2 , \end{aligned}$$

where we don't know the values of p, x, y, or z. Substitute the first equation into the second equation to get

$$100 = y^2 - x^2 + 1 \quad \text{or} \quad y^2 - x^2 = 99 .$$

We can factor this to get

$$(y-x)(y+x) = 99 .$$

Now 99 has the factors $1 \cdot 99$, $3 \cdot 33$, $9 \cdot 11$, so since $x + y > y - x$ we have some potential systems to consider. One is where we consider the factors $1 \cdot 99$ where we would then have

$$\begin{aligned} y - x &= 1 \\ y + x &= 99 . \end{aligned}$$

This has a solution of $(x,y) = (49,50)$, thus $p = 49^2 = 2401$. Using these values we can look at the third equation where we find $p + 200 = 2601$. To be consistent this must equal the square z^2 we see that $z = 51$. Thus these numbers are consistent with what is given in the problem. Using the factor $3 \cdot 33$ of 99 we would find the system

$$\begin{aligned} y - x &= 3 \\ y + x &= 33 . \end{aligned}$$

This has a solution of $(x,y) = (15,18)$, thus $p = 15^2 = 255$. Then $p + 200 = 425$, which is not a perfect square. Finally the factor $9 \cdot 11$ of 99 we would find the system

$$\begin{aligned} y - x &= 9 \\ y + x &= 11 . \end{aligned}$$

This has a solution of $(x,y) = (1,10)$, thus $p = 1^2 = 1$. Then $p + 200 = 201$ which is not a perfect square. Thus only the initial situation is consistent and we have $p = x^2 = 2401$. Since $2401 = 7 \cdot 343$ we have that the original population is a multiple of 7.

Problem 40 (a limiting sum)

We are asked to evaluate

$$\frac{1}{10} + \frac{2}{10^2} + \frac{3}{10^3} + \cdots = \sum_{n \geq 1} \frac{n}{10^n} .$$

Define S to be the sum

$$S = \sum_{n=0}^{\infty} r^n = \frac{1}{1-r} .$$

Then the r derivative of the above is given by

$$\frac{dS}{dr} = \sum_{n=0}^{\infty} n r^{n-1} = \frac{1}{(1-r)^2} .$$

Thus to evaluate the sum needed in this problem we manipulate the above relationship

$$\sum_{n=1}^{\infty} n r^{n-1} = \frac{1}{r} \sum_{n=1}^{\infty} n r^n = \frac{1}{(1-r)^2} .$$

Thus we find

$$\sum_{n=1}^{\infty} n r^n = \frac{r}{(1-r)^2} . \tag{142}$$

Thus we find

$$\sum_{n=1}^{\infty} n \left(\frac{1}{10} \right)^n = \frac{1/10}{(9/10)^2} = \frac{10^2}{10(81)} = \frac{10}{81} .$$

The 1963 Examination

Problem 1 (not on the graph)

The answer is D.

Problem 2 (find n)

For n given by

$$n = x - y^{x-y}\,.$$

Let $x = 2$ and $y = -2$ then we get $n = 2 - (-2)^{2+2} = -14$.

Problem 3 (given the reciprocal what is x)

We are told that

$$\frac{1}{x+1} = x - 1\,.$$

or $1 = x^2 - 1$ or $x = \pm\sqrt{2}$.

Problem 4 (For what values of k do we have the same solutions)

When $y = x^2$ and $y = 3x + k$ have identical solutions means that

$$x^2 - 3x - k = 0\,,$$

so that

$$x = \frac{3 \pm \sqrt{9 + 4k}}{2}\,.$$

One solution for x can be obtained if we take $k = -\frac{9}{4}$.

Problem 5 (where is x)

Take 10^x of both sides of the inequality $\log_{10}(x) < 0$ to get $x < 1$. But as x cannot be negative this means that $0 < x < 1$ and the answer is E.

Problem 6 (finding an angle)

Let the angle we want to know $\angle DAB$ be denoted by θ. As BC is the medium to the hypotenuse AD is is equal in length to one-half the hypotenuse AD. Thus $AC = BC = BA$ and the triangle $\triangle ABC$ is an equilateral triangle. Thus $\angle BAC = \frac{180}{3} = 60$.

Problem 7 (perpendicular lines)

Writing each equation in the form $y = mx + b$ where we recognize that m is the slope we have

$$\begin{aligned} y = \frac{2}{3}x + 4\,, \quad y = -\frac{2}{3}x - \frac{10}{3} \\ y = -\frac{2}{3}x + 4\,, \quad y = -\frac{3}{2}x + 5\,. \end{aligned}$$

To be perpendicular the two lines must have slopes m_1 and m_2 that satisfy $m_1 = -\frac{1}{m_2}$. Thus only 1 and 4 have this property and are therefore perpendicular.

Problem 8 (small positive integers)

We are told $1260x = N^3$ where N is an integer. We factor $1260 = 2(630) = 2^2 \cdot 5 \cdot 63 = 2^2 \cdot 3^2 \cdot 5 \cdot 7$. Thus we are looking for

$$2^2 \cdot 3^2 \cdot 5 \cdot 7x = N^3\,.$$

To make this true we can take $x = 2 \cdot 3 \cdot 5^2 \cdot 7^2 = 7350$. Then $N = 2 \cdot 3 \cdot 5 \cdot 7 = 210$.

Problem 9 (an expansion)

Using the binomial theorem we have

$$\begin{aligned} \left(a - \frac{1}{\sqrt{a}}\right)^7 &= \sum_{k=0}^{7} \left(\begin{array}{c} 7 \\ k \end{array}\right) a^k \left(-\frac{1}{\sqrt{a}}\right)^{7-k} \\ &= \sum_{k=0}^{7} \left(\begin{array}{c} 7 \\ k \end{array}\right) (-1)(-1)^k a^{k-\left(\frac{7-k}{2}\right)} = -\sum_{k=0}^{7} \left(\begin{array}{c} 7 \\ k \end{array}\right) (-1)^k a^{\frac{3k-7}{2}}\,. \end{aligned}$$

The coefficient with the power of a of $-\frac{1}{2}$ is when k satisfies

$$\frac{3k-7}{2} = -\frac{1}{2}\,.$$

Solving we find $k = 2$. Thus we have a coefficient of $a^{-1/2}$ then given by

$$-\left(\begin{array}{c} 7 \\ 2 \end{array}\right)(-1)^2 = -\left(\frac{7 \cdot 6}{2}\right) = -21\,.$$

Problem 10 (the interior of a square)

We begin by drawing a rectangle with corners at the points $(0, 0)$, $(a, 0)$, (a, a) and $(0, a)$. Then introduce an interior point in the center (x, y) given by. From the problem statement the three distances that must be equal can be expressed as

$$d^2 = x^2 + y^2 \tag{143}$$

$$d^2 = (x - a)^2 + y^2 \tag{144}$$

$$d^2 = (a - y)^2 \,. \tag{145}$$

The first two equations expresses the fact that the point (x, y) is equidistant to the vertices's at $(0, 0)$ and $(a, 0)$. The third equation states that (x, y) is equidistant to the line $y = a$. Using the third equation gives $d = a - y$ (since we know that $y < a$) or $y = a - d$. Putting this into Equation 143 and 144 gives

$$d^2 = x^2 + (a - d)^2 \quad \text{and} \quad d^2 = (x - a)^2 + (a - d)^2 \,.$$

When we expand all quadratics and cancel the common d^2 on both sides we get

$$0 = x^2 + a^2 - 2ad \quad \text{and} \quad 0 = x^2 - 2ax + a^2 + a^2 - 2ad = x^2 - 2a(x + d) + 2a^2 \,.$$

Solve for the first equation for x^2 to get

$$x^2 = 2ad - a^2 \,. \tag{146}$$

Put this into the second equation to get

$$0 = 2ad - a^2 - 2a(x + d) + 2a^2 \,,$$

or

$$0 = a^2 - 2ax \quad \text{so} \quad x = \frac{a}{2} \,.$$

Then using this value of x in Equation 146 we have

$$\frac{a^2}{4} = 2ad - a^2 \,.$$

So $2ad = \frac{5a^2}{4}$ so $d = \frac{5}{8}a$.

Problem 11 (the arithmetic mean)

We are told that

$$\frac{1}{50}\left(\sum_{i=1}^{50} x_i\right) = 38 \,.$$

Thus the sum of the x_i's is

$$\sum_{i=1}^{50} x_i = 38(50) = 1900 \,.$$

If we remove the numbers 45 and 55 we get for the new average

$$\frac{1}{48}\left(\sum_{i=1}^{50} x_i - 45 - 55\right) = \frac{1}{48}(1900 - 45 - 55) = 37.5 \,.$$

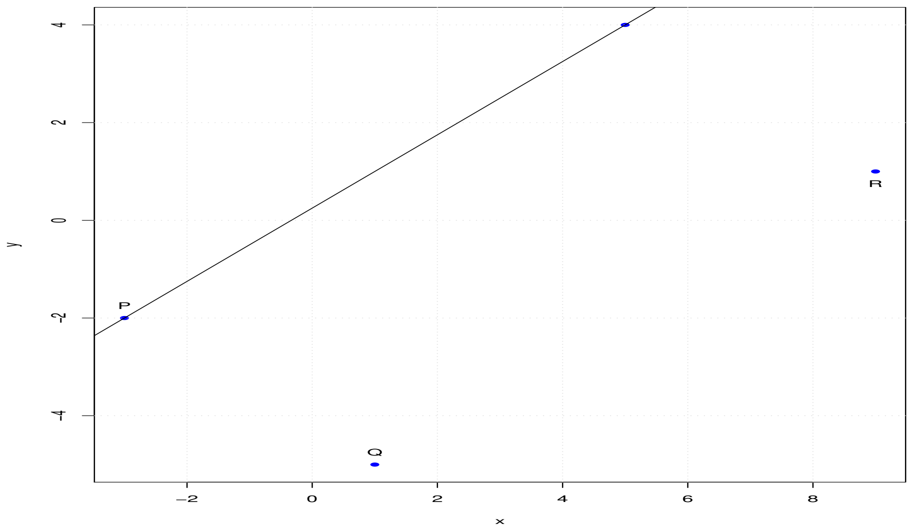

Figure 4: The points specified in Problem 12. The line parallel to QR and through the point P is also drawn.

Problem 12 (the sum of the coordinates of S)

We first draw the points $P = (-3, -2)$, $Q = (1, -5)$, and $R = (9, 1)$ in the xy plane as is done in Figure 4. Then we want to put the point $S = (s_x, s_y)$ such that properties of the parallelogram hold. We first compute

$$d_{QR}^2 = 8^2 + (1+5)^2 = 100 \quad \text{so} \quad d_{QR} = 10\,.$$

We need a line from P parallel to QR of length 10. The slope of the segment QR is

$$m_{QR} = \frac{1+5}{9-1} = \frac{3}{4}\,.$$

The line from the point P with slope the same slope as above (i.e. $\frac{3}{4}$) has the form

$$y - (-2) = \frac{3}{4}(x - (-3))\,,$$

or $y = -2 + \frac{3}{4}x + \frac{9}{4} = \frac{1}{4} + \frac{3}{4}x$. The point S must be on this line and we must pick the point S such that the line segments PS is the same as QR (which we know is 10). This means that $S = (s_x, s_y)$ must satisfy

$$10^2 = (s_x + 3)^2 + (s_y + 2)^2\,.$$

Since S must be on the line above then $s_y = \frac{1}{4} + \frac{3}{4}s_x$ and when we put this into the above

$$\begin{aligned}
10^2 &= (s_x + 3)^2 + \left(\frac{9}{4} + \frac{3}{4}s_x\right)^2 \\
&= s_x^2 + 6s_x + 9 + \frac{1}{16}\left(81 + 54s_x + 9s_x^2\right) \\
&= \left(1 + \frac{9}{16}\right)s_x^2 + \left(6 + \frac{54}{16}\right)s_x + 9 + \frac{81}{16} = \left(\frac{25}{16}\right)s_x^2 + \frac{75}{8}s_x + \frac{225}{16}\,.
\end{aligned}$$

We can solve this using the quadratic equation to get two solutions $s_x \in \{5, -11\}$. Using these values for s_x we have that s_y is given by

$$s_y = \frac{1}{4} + \frac{3}{4}\{5, -11\} = \{4, -8\}\,.$$

The value of (s_x, s_y) we want to use will be the one that makes the slope of the two line segments PQ equal to that of SR. The slope PQ is given by

$$m_{PQ} = \frac{-5+2}{1+3} = -\frac{3}{4}\,,$$

while the two points above give slopes with R of

$$-\frac{3}{4} \quad \text{and} \quad \frac{9}{20}\,,$$

showing that the first solution is the correct one. You can see this solution as the unlabeled point in Figure 4. The sum of the coordinates of S is then computed to be $5 + 4 = 9$. See also the `R` code `1963_prob_12.R`.

Problem 13 (number of equations)

While not a complete solution to this problem we know that we cannot have only one of a, b, c, and d negative (excluding the answer D) since in that case one side of the equation would be a fractional expression while the other side would be a whole number.

Problem 14 (the value of k)

Each equation has roots given by

$$x^2 + kx + 6 = 0 \quad \Rightarrow \quad x = \frac{-k \pm \sqrt{k^2 - 24}}{2}\,,$$

and

$$x^2 - kx + 6 = 0 \quad \Rightarrow \quad x = \frac{+k \pm \sqrt{k^2 - 24}}{2}\,.$$

If each root of the second equation is 5 more than the corresponding root of the first equation then

$$\left(\frac{+k \pm \sqrt{k^2 - 24}}{2}\right) - \left(\frac{-k \pm \sqrt{k^2 - 24}}{2}\right) = 5\,.$$

Thus $k = 5$.

Problem 15 (inscribing a circle in a triangle)

Once we have drawn the given figure we see that the the square has corners that touch the circle. Assume that the circle is of radius r. Then the square that has a diagonal length given by $2r$ will have a side length s where s satisfies

$$(2r)^2 = s^2 + s^2 \quad \text{so} \quad s = \sqrt{2}r\,.$$

Thus the area of the square is $s^2 = 2r^2$. Now we need to calculate the area of the original equilateral triangle. Dividing this triangle up into three equal pieces based on the medians of the triangle we can evaluate the area of the entire triangle in terms of three times the area of a single median triangle. The bottom median triangle is an isosceles triangle that has a height of r and a non base length of $2r$. Thus by the Pythagorean theorem it has a base length b that must satisfy

$$\left(\frac{1}{2}b\right)^2 + r^2 = 4r^2 \quad \text{so} \quad b = 2\sqrt{3r^2} = 2\sqrt{3}r\,.$$

The area of this median triangle is then

$$\frac{1}{2}br = \frac{1}{2}(2\sqrt{3}r)r = \sqrt{3}r^2\,.$$

The area of the original triangle is 3 times this or $3\sqrt{3}r^2$. The ratio of the area of the triangle to that of the square is then $3\sqrt{3} : 2$.

Problem 16 (sequences)

An arithmetic sequence means that the terms a_n are given by Equation 17 for some values of a_1 and d. From what we are told in the problem we have that

$$\begin{aligned}
a &= a_1 \\
b &= a_1 + d \\
c &= a_1 + 2d\,.
\end{aligned}$$

Thus $b = a + d$ or $d = b - a$. Thus one relationship between a, b, and c is given by

$$c = a + 2(b - a) = 2b - a\,.$$

Now a geometric sequence means the terms a_n are given by Equation 19 for some (possibly different) values a_1 and d. We are told that by incrementing a by one we have a geometric sequence or

$$\begin{aligned}
a + 1 &= \hat{a}_1 \\
b &= \hat{a}_1\hat{d} \\
c &= \hat{a}_1\hat{d}^2\,.
\end{aligned}$$

Thus $b = (a+1)\hat{d}$ or $\hat{d} = \frac{b}{a+1}$. Using this another relationship between a, b, and c is given by

$$c = (a+1)\left[\frac{b^2}{(a+1)^2}\right] = \frac{b^2}{a+1}\,.$$

Finally our third statement is that when we increment c by 2 get another geometric sequence. This means

$$\begin{aligned} a &= \tilde{a}_1 \\ b &= \tilde{a}_1\tilde{d} \\ c+2 &= \tilde{a}_1\tilde{d}^2\,. \end{aligned}$$

This case a relationship between a, b, and c is given by $c+2 = \frac{b^2}{a}$. The three relationships for a, b, and c we have thus derived are

$$c = 2b - a\,, \quad c = \frac{b^2}{a+1}\,, \quad c+2 = \frac{b^2}{a}\,.$$

From the last two equations b^2 can be shown equal to

$$b^2 = c(a+1) = a(c+2)\,,$$

These imply $c = 2a$. Then from $c = 2b - a$ we have that a and b are related as

$$2a = 2b - a \quad \text{so} \quad b = \frac{3}{2}a\,.$$

When we put this (with $c = 2a$) into $c+2 = \frac{b^2}{a}$ we get

$$2a + 2 = \frac{9}{4}\left(\frac{a^2}{a}\right) \quad \text{so} \quad a = 8\,.$$

Then $b = \frac{3}{2}a = 12$ and $c = 2a = 16$.

Problem 17 (an expression)

Set the equation equal to -1 and multiply by the denominator, where we get

$$\frac{a}{a+y} + \frac{y}{a-y} = -\frac{y}{a+y} + \frac{a}{a-y}\,,$$

or

$$\frac{a+y}{a+y} + \frac{y-a}{a-y} = 0\,.$$

Which is always true unless one the denominators is zero which happens when $y = \pm a$.

Problem 19 (red and black balls)

Let say we had to count 7 out of every 8 balls red k times. Then after counting all of these we have

$$49 + 7k\,,$$

red balls. The fraction of red balls is $\frac{49+7k}{n}$ which we know must be larger than 0.9 or

$$\frac{49 + 7k}{n} > 0.9\,.$$

Now k and n are related in that

$$k = \frac{n - 50}{8}\,.$$

Putting this expression into the above gives

$$\frac{49 + \frac{7}{8}(n - 50)}{n} > 0.9\,.$$

Solving the above for n we find $n < 210$.

Problem 20 (meeting in the middle)

The position traveled (in miles) of the person starting at R as a function of t (measured in hours) is given by $R(t) = 4.5t = \frac{9}{2}t$. The position traveled of the person starting at S as a function of t is given by

$$S(t) = \sum_{i=1}^{t} \left(\frac{13}{4} + \frac{1}{2}(i - 1) \right) = \frac{t}{2} \left(2 \cdot \frac{13}{4} + (t - 1)\frac{1}{2} \right)\,.$$

Here we have used the fact that we are told that the time t (in hours) when the two men meet is an integer (i.e. not fractional) and the expression 18 to evaluate the sum. The two people meet at a time t when their total distance traveled is 76 miles or

$$\frac{9}{2}t + \frac{13}{4}t + t(t - 1)\frac{1}{4} = 76\,.$$

In the form we can use in the quadratic formula on this is

$$t^2 + 30t - 304 = 0\,,$$

which has solutions $t \in \{-38, 8\}$. As t must be positive we have $t = 8$ and the distance from S where they meet is then given by $\frac{9}{2}t = \frac{9}{2}(8) = 36$ miles.

Problem 21 (can we factor?)

Consider the expression $-y^2 + 2yz$ in the given expression. We write this as

$$-(y^2 - 2yz) = -((y - z)^2 - z^2)\,.$$

When we put that in the expression given we have

$$\begin{aligned} x^2 - y^2 - z^2 + 2yz + x + y - z &= x^2 - z^2 - ((y-z)^2 - z^2) + x + y - z \\ &= x^2 - (y-z)^2 + x + y - z \\ &= (x-y+z)(x+y-z) + x + y - z \\ &= (x+y-z)(x-y+z+1)\,. \end{aligned}$$

Thus the factor is $x - y + z + 1$

Problem 23 (giving coins)

If we assume that A, B, and C start with a, b, and c cents initially. Now we will keep track of the number of cents each has after each transaction. After the first transaction we get that

$$\begin{array}{lll} A & \text{now has} & a-b-c \\ B & \text{now has} & 2b \\ C & \text{now has} & 2c\,. \end{array}$$

After the second transaction

$$\begin{array}{lll} A & \text{now has} & 2(a-b-c) \\ B & \text{now has} & 2b-(a-b-c)-2c = -a+3b-c \\ C & \text{now has} & 4c\,. \end{array}$$

After the third transaction

$$\begin{array}{lll} A & \text{now has} & 4(a-b-c) \\ B & \text{now has} & 2(-a+3b-c) \\ C & \text{now has} & 4c-2(a-b-c)-(-a+3b-c) = -a-b+7c\,. \end{array}$$

As each of these expressions is to equal the value of 16 we have the linear system

$$\begin{aligned} 4a + 4b - 4c &= 16 \\ -2a + 6b - 2c &= 16 \\ -a - b + 7c &= 16\,. \end{aligned}$$

Solving this for a, b, and c we find $a = 26$, $b = 14$, and $c = 8$.

Problem 24 (real roots)

The roots are given by $x = \frac{-b \pm \sqrt{b^2-4c}}{2}$. To be real requires that $b^2 - 4c \geq 0$ or $b^2 \geq 4c$. When we pick b and c from the set $\{1, 2, 3, 4, 5, 6\}$ then

$$\begin{aligned}
b &= 1 \quad \text{no } c \text{ exist} \\
b &= 2 \quad c = 1 \text{ is the only solution} \\
b &= 3 \quad \text{means } 9 \geq 4c \text{ so } c = 1, 2 \text{ are the only solutions} \\
b &= 4 \quad \text{means } 16 \geq 4c \text{ so } c = 1, 2, 3, 4 \text{ are the only solutions} \\
b &= 5 \quad \text{means } 25 \geq 4c \text{ so } c = 1, 2, 3, 4, 5, 6 \text{ are the only solutions} \\
b &= 6 \quad \text{all } c \text{ are valid}\,.
\end{aligned}$$

Thus we have

$$1 + 2 + 4 + 6 + 6 = 19\,,$$

solutions.

Problem 25 (the length of BE)

As a first observation because the area of the square $ABCD$ is 256 we know that the length of any side is $\sqrt{256} = 16$. By construction the angle $\angle DCF$ is equal to the angle $\angle BCE$ and since $CD = CB$ (they are two sides of the square $ABCD$) we can conclude that $FC = CE$. As we are told the area of the right triangle CEF with two equal sides FC and CE we can compute

$$200 = \frac{1}{2}\text{base} \times \text{height} = \frac{1}{2}FC \times CE = \frac{1}{2}CE^2 \quad \text{so} \quad CE = 20\,.$$

Now in the right triangle BCE we know two of the three sides namely $BC = 16$ and $CE = 20$. Using the Pythagorean theorem we then have

$$BE = \sqrt{20^2 - 16^2} = \sqrt{144} = 12\,.$$

Problem 26 (implying the truth)

To solve this problem we must recall the truth table for the "implication" (or $\rightarrow$) operator. This is presented in Table 5 We then consider each statement in tern:

- Here p and r are true and q is false. In this case $p \rightarrow q$ is false so $(p \rightarrow q) \rightarrow r$ is then true.
- Here p and q are false and r is true. In this case $p \rightarrow q$ is true so $(p \rightarrow q) \rightarrow r$ is then true.
- Here p is are true and q and r is false. In this case $p \rightarrow q$ is the statement $T \rightarrow F$ which is false so $(p \rightarrow q) \rightarrow r$ is the statement $F \rightarrow F$ which is true.

P	Q	$P \to Q$
F	F	T
F	T	T
T	F	F
T	T	T

Table 5: The truth table for the implication $\to$ operator. Note that the "answer" is always true, *except* in the cases when the input P is true and the output Q is false.

- Here p is false while q and r are true. In this case $p \to q$ is the statement $F \to T$ which is true so $(p \to q) \to r$ is the statement $F \to T$ which is true.

Thus each of the given statements implies the truth of $(p \to q) \to r$.

Problem 28 (the maximum product of the roots)

First write the given equation as

$$x^2 - \frac{4}{3}x + \frac{k}{3} = 0 .$$

If the roots of this are denoted as r and s then we should be able to write the above as $(x - r)(x - s) = 0$. Expanding this we see that

$$\begin{aligned} -(r + s) &= -\frac{4}{3} \\ rs &= \frac{k}{3} . \end{aligned}$$

From the above we see that the product of the two roots is given by the expression $\frac{k}{3}$. To maximize this product as a function of k we take k as large as we can subject to the constraint that the roots of the quadratic are real. These roots will be real if the discriminant is positive or

$$\frac{16}{9} - 4\left(\frac{k}{3}\right) \geq 0 \quad \text{so} \quad k \leq \frac{4}{3} .$$

Looking at the choices given we would take the value of $k = \frac{4}{3}$.

Problem 29 (the largest value)

Write $s(t)$ as

$$\begin{aligned} s(t) &= 160t - 16t^2 = -16(t^2 - 10t) = -16(t^2 - 10t + 25) + 16(25) \\ &= -16(t - 5)^2 + 400 . \end{aligned}$$

The maximum value of this is 400.

Problem 30 (transformation of F)

With F given by

$$F(x) = \log\left(\frac{1+x}{1-x}\right) .$$

then to evaluate G given by

$$G = F\left(\frac{3x+x^3}{1+3x^2}\right) .$$

we need to evaluate two subexpression

$$\begin{aligned} 1 + \frac{3x+x^3}{1+3x^2} &= \frac{1+3x^2+3x+x^3}{1+3x^2} = \frac{(x+1)^3}{1+3x^2} \\ 1 - \frac{3x+x^3}{1+3x^2} &= \frac{1+3x^2-3x-x^3}{1+3x^2} = -\frac{(x-1)^3}{1+3x^2} . \end{aligned}$$

Thus

$$\frac{1+\frac{3x+x^3}{1+3x^2}}{1-\frac{3x+x^3}{1+3x^2}} = \frac{(x+1)^3}{(1-x)^3} .$$

From this we see that

$$G = \log\left(\left(\frac{x+1}{1-x}\right)^3\right) = 3\log\left(\frac{x+1}{1-x}\right) = 3F .$$

Problem 31 (the number of solutions)

We start by finding *one* solution for x and y by writing 763 as

$$\begin{aligned} 2x + 3y = 763 &= 700 + 7\cdot 9 \\ &= 7\cdot 10^2 + 7\cdot 3^2 \\ &= 7\cdot 2^2\cdot 5^2 + 7\cdot 3^2 \\ &= 2(2\cdot 5^2\cdot 7) + 3(3\cdot 7) . \end{aligned}$$

Thus one solution is $x = 2\cdot 5^2\cdot 7 = 350$ and $y = 3\cdot 7 = 21$. We now ask can we find additional solutions by "shifting" from this base solution $x = 350$ and $y = 21$. That is, can we write 763 as

$$763 = 2(350+n) + 3(21+m) ,$$

for some integers n and m. To have the resulting values of $x = 350+n$ and $y = 21+m$ be positive (and a solution to our equation) we must have

$$350 + n > 0 \quad \text{so} \quad n > -350 \tag{147}$$

$$21 + m > 0 \quad \text{so} \quad m > -21 \tag{148}$$

$$2n + 3m = 0 \quad \text{so} \quad n = -\frac{3}{2}m . \tag{149}$$

Using Equation 149 in 147 we get

$$-\frac{3}{2}m > -350 \quad \text{so} \quad m < \frac{700}{3} = 233.33 \, .$$

This taken with Equation 148 means that m must be an integer such that satisfies

$$-21 < m < 233.33 \, .$$

In order that n also be an integer Equation 149 means that m must be even. Lets count how many solutions do we have that satisfy the constraint that $-21 < m < 233.33$. When m is even and negative or

$$m \in \{-20, -18, -16, \cdots - 4, -2\} \, ,$$

we have 10 solutions. When m is even and positive we have

$$m \in \{0, 2, 4, \cdots, 230, 232\} \, ,$$

we have $\frac{232}{2} + 1 = 117$ solutions. In total then we would have $10 + 117 = 127$ solutions.

Problem 32 (the number of smaller rectangles)

The conditions given require that

$$2x + 2y = \frac{1}{3}(2a + 2b) \tag{150}$$

$$xy = \frac{1}{3}ab \, , \tag{151}$$

with $x < a$ and $y < a$. Dividing Equation 150 by Equation 151 (and canceling the common 2) gives

$$\frac{1}{y} + \frac{1}{x} = \frac{1}{b} + \frac{1}{a} \, .$$

Since we must have $x < a$ and $y < a$ the left-hand-side of this expression is larger than $\frac{2}{a}$, while the right-hand-side is less than $\frac{2}{a}$ (since $b > a$). Thus these two equations cannot be simultaneously satisfied.

Problem 33 (parallel lines)

To be parallel to the given line our new line must have the same slope as the one given or $\frac{3}{4}$. This also means that then angle between each line and the x axis is given by $\theta = \tan^{-1}(\frac{3}{4})$. What is to be determined in this problem is the y intercept of the new line. Draw the given line and the new line parallel to the first and above the original. We are told that the distance between the two lines is 4. Drop a perpendicular from the upper line to the original line. Then the angle this perpendicular makes with the y axis is the same as the angle θ. Thus we have a triangle with an acute angle θ such that $\tan(\theta) = \frac{3}{4}$ and the adjacent side of length 4. This means that the side opposite θ must have a length 3 and the hypotenuse (which is the distance along the y axis) has a length of 5. Thus the new line is five units above or below the old line. This gives the two lines

$$y = \frac{3}{4}x + 1 \quad \text{or} \quad y = \frac{3}{4}x + 11 \,.$$

Problem 34 (some geometry)

As this triangle has two equal sides (by definition it is an isosceles triangle) and the two angles adjacent to the common side and equal sides are equal denote this angle by α and let the third unknown angle be denoted as θ. Dropping a perpendicular from θ to the common side of length c and using the definition of the cosign we have that $c = 2\sqrt{3}\cos(\alpha)$. As the angles in the triangle must add to π we have

$$2\alpha + \theta = \pi \,.$$

Thus

$$c = 2\sqrt{3}\cos\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = 2\sqrt{3}\sin\left(\frac{\theta}{2}\right) \,.$$

As we are told that $c > \sqrt{3}$ this means that

$$\sin\left(\frac{\theta}{2}\right) > \frac{\sqrt{3}}{2} \,.$$

Thus

$$\frac{\theta}{2} > \frac{\pi}{3} \quad \text{so} \quad \theta > \frac{2\pi}{3} \,.$$

In degrees this is $\theta > 120$ degrees.

Problem 35 (the area of a triangle)

Let a be the length of the shortest side of the triangle, let $b = 21$, and since the perimeter of the triangle is 48 we have

$$a + b + c = 48 \quad \text{so} \quad a + c = 27 \,.$$

The area of a triangle in terms of its three sides is given by Heron's formula given by Equation 10. Since we know that A is an integer we can consider each of the choices for a, compute $c = 27 - a$, compute the area A and see what we get. When we do this we get

```
as  8.00000   10 12.00000 14.00000 16.00000
cs 19.00000   17 15.00000 13.00000 11.00000
A  75.89466   84 88.18163 88.99438 86.53323
```

In this table we see that the value of $a = 10$ and $c = 17$ give $A = 84$ and is the solution.

Problem 36 (making bets)

Given we start with x cents and we win the bet we will end up with $x + \frac{x}{2} = \frac{3}{2}x$ cents. If we loose the bet we will end up with $x - \frac{1}{2}x = \frac{1}{2}x$ cents. Then the amount of money we have after the three wins and three losses is then

$$\left(\frac{3}{2}\right)^3 \left(\frac{1}{2}\right)^3 x = \frac{27}{65} x \, .$$

If $x = 64$ the final amount we have is 27 cents. This represents a loss of $64 - 27 = 37$ cents.

Problem 37 (an L_1 optimization)

Given a set of n numbers $\{P_i\}_{i=1}^n$ the median $\tilde{x}$ is the solution that minimizes the sum of the absolute difference to all of the points i.e.

$$\min_{\tilde{x}} \sum_{i=1}^{n} |P_i - \tilde{x}| \, .$$

As this is what the problem asks we need to find the median. With seven numbers the median is the number "in the middle" which in this case is P_4.

Problem 38 (more geometry)

Let the point G divide the segment into the two segments DG (of length y) and GC (of length $b-y$). Let the length of the unknown segment BE be denoted as x. From the problem statement the length of FG is 24 and the length of GE is 8. As $\triangle BEA \sim \triangle GEC$ we have that

$$\frac{x}{b} = \frac{8}{b - y} \, .$$

Solving for y we get

$$y = \left(1 - \frac{8}{x}\right) b \, . \tag{152}$$

Now to relate the length b to the total length of FB we note that $\triangle FDG \sim \triangle FAB$ and thus we have

$$\frac{y}{b} = \frac{24}{24 + 8 + x} \quad \text{or} \quad y = \frac{24b}{32 + x} \, .$$

If put what we know from Equation 152 into the above we get

$$\frac{24}{32 + x} = 1 - \frac{8}{x} \, .$$

Solving for x we get $x = \sqrt{256} = 16$.

In general the similarity relationship $\triangle BEA \sim \triangle GEC$ gives us that

$$\frac{BE}{b} = \frac{BG - BE}{b - y},$$

or solving for y we get

$$y = b - b\frac{BG}{BE} + b = 2b - b\frac{BG}{BE}.$$

Also using $\triangle FDG \sim \triangle FAB$ we get

$$\frac{y}{b} = \frac{BF - BG}{BF},$$

or solving for y we get

$$y = \frac{BF - BG}{BF} b.$$

Equating these two expressions for y we get

$$\frac{BF - BG}{BF} = 2 - \frac{BG}{BE}.$$

The above we can write in the form

$$\frac{1}{BG} + \frac{1}{BF} = \frac{1}{BE}.$$

Problem 40 (some cube roots)

We start by taking the cube of both sides of the given expression by using the identity

$$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3, \tag{153}$$

we get

$$(x + 9) - 3(x + 9)^{2/3}(x - 9)^{1/3} + 3(x + 9)^{1/3}(x - 9)^{2/3} - (x - 9) = 27.$$

Simplifying some this gives

$$-3(x + 9)^{2/3}(x - 9)^{1/3} + 3(x + 9)^{1/3}(x - 9)^{2/3} = 27 - 18 = 9,$$

or

$$-(x + 9)^{1/3}(x^2 - 81)^{1/3} + (x - 9)^{1/3}(x^2 - 81)^{1/3} = 3,$$

or

$$3 = (x^2 - 81)^{1/3}\left[(x - 9)^{1/3} - (x + 9)^{1/3}\right].$$

Now from the problem statement the expression in brackets above is *exactly* equal to -3 and we get

$$3 = (x^2 - 81)^{1/3}(-3).$$

Solving this gives $x^2 = 80$. As we know the value of x^2 we can determine where it falls.

The 1964 Examination

Problem 1

$$(\log_{10}(5\log_{10}(100)))^2 = (\log_{10}(5\cdot 2))^2 = 1\,.$$

Problem 2

$$x^2 - 4y^2 = (x-2y)(x+2y) = 0\,.$$

Which is a pair of straight lines.

Problem 3

The given statement is equivalent to $x = uy + v$. In addition we have

$$\frac{x+2uy}{y} = \frac{x}{y} + 2u\,.$$

Note that since u is an integer so is $2u$ and it contributes to the quotient in the division of $\frac{x}{y}$. Thus the remainder of the requested fraction is the same as the remainder of $\frac{x}{y}$ or v.

Problem 4

If $P = x + y$ and $Q = x - y$ then

$$\begin{aligned} P+Q &= 2x \\ P-Q &= 2y\,. \end{aligned}$$

From this we have

$$\frac{P+Q}{P-Q} - \frac{P-Q}{P+Q} = \frac{x}{y} - \frac{y}{x} = \frac{x^2-y^2}{xy}\,.$$

Problem 5

We have $y = kx$. When $x = 4$ this means that $4k = 8$ so $k = 2$. Thus $y(-8) = k(-8) = -16$.

Problem 6

A geometric progression has terms that satisfy Equation 19 for some values for a_1 and d. Since $\frac{a_2}{a_1} = d = \frac{a_3}{a_2}$, we can set up an equation for x in that

$$\frac{2x+2}{x} = \frac{3x+3}{2x+2}.$$

This equation has two solutions $x = -1$ or $x = -4$. If we assume that $x = -1$ our geometric sequence would be given by $a_1 = -1$, $a_2 = 2(-1) + 2 = 0$, and $a_3 = 3(-1) + 3 = 0$. As this is a degenerate sequence we must have $x = -4$. In that case

$$\begin{aligned} a_1 &= -4 \\ a_2 &= 2(-4) + 2 = -6 \\ a_3 &= 3(-4) + 3 = -9. \end{aligned}$$

Thus we see that $d = \frac{3}{2}$ and $a_1 = -4$. The fourth term is given by $a_4 = -9(3/2) = -\frac{27}{2}$.

Problem 7

For the two roots of a quadratic to be equal means that the discriminant must vanish or for this problem

$$(-p)^2 - 4p = 0 \quad \text{so} \quad p(p-4) = 0.$$

Thus $p = 0$ or $p = 4$ and $n = 2$.

Problem 8 (smaller roots)

Write the given expression as

$$\left(x - \frac{3}{4}\right)\left(x - \frac{3}{4} + x - \frac{1}{2}\right) = 0,$$

or

$$\left(x - \frac{3}{4}\right)\left(2x - \frac{5}{4}\right) = 0.$$

The two roots are then $\frac{3}{4}$ and $\frac{5}{8}$. This second number is the smaller of the two.

Problem 9 (a jobber)

Our jobber will pay $(1 - 0.125)24$ for the item. For him to make a 33% profit means that he must sell if for $(1 + 0.33333)(1 - 0.125)24 = 28$. He must then mark the price higher so that when sold at 20% of that price the price is 28. Thus to have the sale price be 28 dollars let x be the mark price then we must have

$$(1 - 0.2)x = 28 \quad \text{so} \quad x = 1.25(28) = 35\,.$$

None of the given answers are correct.

Problem 10

The area of the triangle is given by

$$A = \frac{1}{2}\text{base} \times \text{height} = \frac{1}{2}(\sqrt{2}s)h = s^2\,.$$

Thus solving for h we get $h = \sqrt{2}s$.

Problem 11

The equation $2^x = 8^{y+1}$ can be written as $2^x = 2^{3y+3}$ and so $x = 3y+3$. The second equation $9^y = 3^{x-9}$ can be written as $3^{2y} = 3^{x-9}$ and so $2y = x-9$. Thus we have to solve the system

$$\begin{aligned} x - 3y &= 3 \\ -x + 2y &= -9\,. \end{aligned}$$

Solving these for x and y we get $x = 21$ and $y = 6$, so that $x + y = 27$.

Problem 12

Negating the given expression is the statement, there exists an x such that $x^2 \leq 0$.

Problem 14

Let x be the price originally paid per sheep. The total amount paided is then $749x$. The farmer then sold the initial 700 sheep for $749x$ (which is $\frac{749x}{700}$ per sheep), and then sold the remaining 49 at a price of

$$\left(\frac{749x}{700}\right) 49\,.$$

The total amount all the sheep were sold for is then the sum of these two numbers or

$$749x + \left(\frac{749x}{700}\right) 49\,.$$

The percent made on this transaction is then given by $\frac{\text{price sold}-\text{price bought}}{\text{price bought}}$ or

$$\frac{\left(749x + \left(\frac{749x}{700}\right)49\right) - 749x}{749x} = \frac{49}{700} = \frac{7}{100} = 0.07\,,$$

or 7%.

Problem 15

Let our line intersect the y-axis at the point $(0, b)$ then we get that the area of the triangle T can be written as

$$T = \frac{1}{2}ab \quad \text{so} \quad b = \frac{2T}{a}\,.$$

For the line the slope $m = \frac{b}{a} = \frac{2T}{a^2}$ and the y-intercept is given by $b = \frac{2T}{a}$ thus the line is

$$y = mx + b = \frac{2T}{a^2}x + \frac{2T}{a}\,.$$

This is the same as

$$a^2 y = 2Tx + 2Ta\,,$$

which is solution B.

Problem 16

If the function $f(x)$ is divisible by 6 means that

$$x^2 + 3x + 2 = 6k\,,$$

for some integer k. This means that

$$x^2 + 3x + 2 - 6k = 0\,.$$

The roots x of the above equation are given by

$$x = \frac{-3 \pm \sqrt{9 - 4(2 - 6k)}}{2} = \frac{-3 \pm \sqrt{1 + 24k}}{2}\,.$$

For values of $k \geq 1$ the negative sign will always give a negative value for x and thus will be a point not in the original set S. We can now let $k = 1, 2, 3, \cdots$, compute the value of x using the above formula, and see which values of x are in the set S. To determine how large we need to look for values of k we must pick k such that

$$\frac{-3 \pm \sqrt{1 + 24k}}{2} \geq 25\,.$$

Solving the above for k we find

$$k \geq \frac{53^2 - 1}{24} = 117\,.$$

Thus we sample $k = 1, 2, 3, \cdots, 117$. In the R code `prob_16_1964.R` we do this and find the following values of k give elements x that are in the set S

```
      valid ks integer xs
 [1,]        1          1
 [2,]        2          2
 [3,]        5          4
 [4,]        7          5
 [5,]       12          7
 [6,]       15          8
 [7,]       22         10
 [8,]       26         11
 [9,]       35         13
[10,]       40         14
[11,]       51         16
[12,]       57         17
[13,]       70         19
[14,]       77         20
[15,]       92         22
[16,]      100         23
[17,]      117         25
```

Thus we have 17 elements with this property.

Problem 18

If the two equations

$$\begin{aligned} 3x + by + c &= 0 \\ cx - 2y + 12 &= 0 \,, \end{aligned}$$

are to have the same graph means that when we try to use one equation to eliminate variables from the other equation we get an identity (i.e. a trivially true expression like $0 = 0$). If we take $-\frac{c}{3}$ times the first equation and add it to the second equation we get

$$(-c + c)x + \left(-\frac{bc}{3} - 2\right) y + \left(-\frac{c^2}{3} + 12\right) = 0 \,.$$

In order for this to be the trivial expression $0 = 0$ for all x and y we must have

$$-\frac{c^2}{3} + 12 = 0 \quad \text{so} \quad c = \pm 6 \,.$$

For each value of c the value of b must satisfy

$$-\frac{bc}{3} - 2 = 0 \,.$$

Thus we have two values of b, and c that give the same graph.

Problem 19

Consider solving the two given expression for x and y in terms of the unknown value for z we need to solve

$$\begin{aligned} 2x - 3y &= z \\ x + 3y &= 14z\,. \end{aligned}$$

If we add these two equations we get $3x = 15z$ so $x = 5z$. Putting this into the second equation gives

$$5z + 3y = 14z \quad \text{so} \quad y = 3z\,.$$

Then the expression we are to evaluate becomes

$$\frac{25z^2 + 3(5z)(3z)}{9z^2 + z^2} = \frac{25 + 45}{10} = 7\,.$$

Problem 20

Using the binomial expansion we have that

$$(x - 2y)^{18} = \sum_{k=0}^{18} \binom{18}{k} x^k (-2)^{18-k} y^{18-k}\,.$$

Thus the sum of the coefficients in the above expansion is

$$\sum_{k=0}^{18} \binom{18}{k} (-2)^{18-k}\,.$$

We can obtain this expression if we let $x = y = 1$ in the binomial expansion (or equivalently the original expression). Thus the value of the sum of the coefficients is then $(1 - 2)^{18} = 1$.

Problem 21

Write the given equation as

$$\frac{\ln(x)}{\ln(b^2)} + \frac{\ln(b)}{\ln(x^2)} = 1\,,$$

or

$$\frac{\ln(x)}{2\ln(b)} + \frac{\ln(b)}{2\ln(x)} = 1\,,$$

or

$$\ln(x)^2 - 2\ln(b)\ln(x) + \ln(b)^2 = 0\,.$$

If we let $v = \ln(x)$ then the above is a quadratic equation in v. Using the quadratic formula we get that

$$v = \frac{2\ln(b) \pm \sqrt{4\ln(b)^2 - 4\ln(b)^2}}{2} = \ln(b)\,.$$

This is the same as $\ln(x) = \ln(b)$ so $x = b$.

Problem 23

Assume that the two numbers are x and y with $y > x$, then the given statements imply that

$$\frac{x+y}{y-x} = 7 \quad \text{and} \quad \frac{xy}{x+y} = \frac{24}{7}\,.$$

The first of these is the same as $y = \frac{4}{3}x$. Putting this into the second equation we get

$$x\left(\frac{4}{3}x\right) = \frac{24}{7}\left(x + \frac{4}{3}x\right)\,.$$

The two solutions to this last equation are $x = 0$ and $x = 6$. This gives $y = 0$ and $y = 8$. As $x = 0$ is not the solution we are looking for we would have $xy = 6(8) = 48$.

Problem 24

We can do this problem by completing the square or with calculus. Using the later we find

$$y' = 2(x-a) + 2(x-b) = 0 \quad \text{so} \quad x = \frac{1}{2}(a+b)\,,$$

for the minimum.

Problem 25

Assume the two factors are of the form $x + Ay + B$ and $x + Cy + D$. Then their product is given by

$$x^2 + (C+A)xy + (D+B)x + ACy^2 + (AD+BC)y + BD\,.$$

In order for this to equal $x^2 + 3xy + x + my - m$ we must have

$$\begin{aligned} C + A &= 3 \\ D + B &= 1 \\ AC &= 0 \\ AD + BC &= m \\ BD &= -m\,. \end{aligned}$$

This is a set of five equations and four unknowns. From the equation $AC = 0$we must have $A = 0$ or $C = 0$. If we assume that $A = 0$ then $C = 3$ and the equations above simplify to

$$\begin{aligned} D + B &= 1 \\ 3B &= m \\ BD &= -m\,. \end{aligned}$$

The last two equations give $3B = -BD$ so $B = 0$ or $D = -3$. If $B = 0$ then $D = 1$ and $m = 0$. If $B = -3$ then $B = 4$ and $m = 12$. This gives two possible solutions. If we go back to our first assumption on A and instead make the assumption that $C = 0$ then $A = 3$ and we get the equations

$$\begin{aligned} D + B &= 1 \\ 3D &= m \\ BD &= -m \,. \end{aligned}$$

This is the same system as before and thus we will get two solutions also $m = 0$ or $m = 12$.

Problem 26

Let v_1, v_2, and v_3 be the velocities of the three runners with $v_1 > v_2 > v_3$. Let the first place runner cross the finish line at time t_1. When this happens we are told that

$$v_1 t_1 - v_2 t_1 = 2 \quad (154)$$

$$v_1 t_1 - v_3 t_1 = 4 \,. \quad (155)$$

Since the race is 10 miles long we know that $t_1 = \frac{10}{v_1}$. We want to evaluate

$$v_2 t_2 - v_3 t_2 \,,$$

where t_2 is the time when the second place runner crosses the finish line. This is given by $t_2 = \frac{10}{v_2}$. Using this the expression we want to evaluate is given by

$$v_2 \left(\frac{10}{v_2} \right) - v_3 \left(\frac{10}{v_2} \right) = 10 \left(1 - \frac{v_3}{v_2} \right) \,.$$

We now try to compute the fraction $\frac{v_3}{v_2}$. If we consider Equation 154 and 155 written in the form

$$\begin{aligned} v_3 t_1 &= v_1 t_1 - 4 \\ v_2 t_1 &= v_1 t_1 - 2 \,, \end{aligned}$$

then when we take the ratio we get

$$\frac{v_3}{v_2} = \frac{v_1 t_1 - 4}{v_1 t - 2} = \frac{10 - 4}{10 - 2} = \frac{3}{4} \,.$$

Using this value we now find

$$v_2 t_2 - v_3 t_2 = 10 \left(1 - \frac{3}{4} \right) = \frac{15}{2} \,.$$

Problem 27

We can first plot each of the two expressions $|x-3|$ and $|x-4|$ on the x axis. The sum of these two plots is the expression we are attempting to bound by a. The plot of $|x-3|$ and $|x-4|$ are shifted "v" functions. Notice that for any x value such that $3 < x < 4$ the sum of the above two functions is a constant as

$$|x-3| + |x-4| = x - 3 - (x-4) = 1\,.$$

When $x = 3$ and $x = 4$ the value of $|x-3| + |x-4|$ one also. For $x < 3$ or $x > 4$ the plot of $|x-3|+|x-4|$ is linearly increasing to infinity. Thus the *smallest* value that $|x-3|+|x-4|$ ever takes is the value of 1. Thus in order for the given inequality to have values of x for which it is true we must have $a > 1$.

Problem 28

We are told that $d = 2$ and thus using Equation 18 the problem statement implys

$$\sum_{i=1}^{n} a_i = na_1 + n(n-1) = 153\,.$$

Lets solve for a_1 since it must be an integer and see what conditions that imposes on n. We find

$$a_1 = \frac{153}{n} - (n-1)\,.$$

As we know that $n > 1$ to find the possible values for n such that a_1 is an integer means that n must be a product of the factors of 153. Since

$$153 = 3^3 17\,,$$

we have that n can take the values

$$3\,, 3^2 = 9\,, 17\,, 3(17) = 51\,, 153\,,$$

or five values.

Problem 29 (the length of the opposite side)

Using the law of cosigns we have

$$\text{RF}^2 = \text{RD}^2 + \text{DF}^2 - 2\text{RD} \cdot \text{DF} \cos(\theta)\,.$$

Using the numbers given this becomes

$$25 = 16 + 36 - 2(4)(6)\cos(\theta)\,,$$

or

$$\cos(\theta) = \frac{27}{2(4)(6)} = \frac{9}{16}.$$

Again using the law of cosigns on the second triangle we get

$$\text{RS}^2 = \text{RF}^2 + \text{FS}^2 - 2\text{RF} \cdot \text{FS} \cos(\theta).$$

Using the numbers given in the problem this becomes

$$\text{RS}^2 = \left(\frac{15}{2}\right)^2 + 25 - 2(5)\left(\frac{15}{2}\right)\left(\frac{9}{16}\right).$$

This simplifies to

$$\text{RS}^2 = \frac{625}{16},$$

so $\text{RS} = \frac{25}{4} = 6.25$.

Problem 30

Write the given quadratic as

$$x^2 + \left(\frac{2+\sqrt{3}}{7+4\sqrt{3}}\right)x - \frac{2}{7+4\sqrt{3}} = 0.$$

Note that the number $7 + 4\sqrt{3}$ is special in that is has a very nice inverse

$$\frac{1}{7+4\sqrt{3}} = \frac{1}{7+4\sqrt{3}}\left(\frac{7-4\sqrt{3}}{7-4\sqrt{3}}\right) = \frac{7-4\sqrt{3}}{49-16(3)} = 7-4\sqrt{3}.$$

Thus the above quadratic is given by

$$x^2 + (2+\sqrt{3}(7-4\sqrt{3})x - 2(7-4\sqrt{3}) = 0.$$

Since the roots of the quadratic equation are given by $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$ the largest root minus the smallest root is given by the expression

$$\sqrt{b^2-4ac}.$$

As $a = 1$ the above is given by

$$\begin{aligned}\sqrt{(2+\sqrt{3})^2(7-4\sqrt{3})^2 + 8(7-4\sqrt{3})} &= \sqrt{(7-4\sqrt{3})[(4+4\sqrt{3}+3)(7-4\sqrt{3})+8]} \\ &= \sqrt{(7-4\sqrt{3})(1+8)} = 3\sqrt{7-4\sqrt{3}}.\end{aligned}$$

We now need to evaluate the above expression. Let x be a number of the form $a + b\sqrt{3}$ such that when I square this number I get $7 - 4\sqrt{3}$. Actually squaring $a + b\sqrt{3}$ we get

$$a^2 + 3b^2 + 2ab\sqrt{3} = 7 - 4\sqrt{3}.$$

Thus $2ab = -4$ and $a^2 + 3b^2 = 7$. We can satisfy these by letting $a = 2$ and $b = -1$ thus

$$\sqrt{7 - 4\sqrt{3}} = 2 - \sqrt{3}\,.$$

From this the difference we seek is given by

$$3(2 - \sqrt{3}) = 6 - 3\sqrt{3}\,.$$

Alternatively, if one didn't like the above derivation one could simply square each of the proposed solutions given and see which one equals $9(7 - 4\sqrt{3})$.

Problem 31

Let $r_1 = \frac{1-\sqrt{5}}{2}$ and $r_2 = \frac{1+\sqrt{5}}{2}$ and then our function $f(n)$ is given by

$$f(n) = \frac{5 + 3\sqrt{5}}{10} r_2^n + \frac{5 - 3\sqrt{5}}{10} r_1^n\,.$$

Thus the difference we want to compute is given by

$$\begin{aligned}
f(n+1) - f(n-1) &= \frac{5 + 3\sqrt{5}}{10} r_2 r_2^n + \frac{5 - 3\sqrt{5}}{10} r_1 r_1^n \\
&- \frac{5 + 3\sqrt{5}}{10} r_2^{-1} r_2^n - \frac{5 - 3\sqrt{5}}{10} r_1^{-1} r_1^n \\
&= \left(\frac{5 + 3\sqrt{5}}{10} r_2 - \frac{5 + 3\sqrt{5}}{10} r_2^{-1} \right) r_2^n \\
&+ \left(\frac{5 - 3\sqrt{5}}{10} r_1 - \frac{5 - 3\sqrt{5}}{10} r_1^{-1} \right) r_1^n\,.
\end{aligned}$$

Lets now try to evaluate each coefficient. For the coefficient of r_2^n we have

$$\begin{aligned}
\left(\frac{5 + 3\sqrt{5}}{10} \left(\frac{1 + \sqrt{5}}{2} \right) - \frac{5 + 3\sqrt{5}}{10} \left(\frac{2}{1 + \sqrt{5}} \right) \right) &= \frac{5 + 3\sqrt{5}}{10} \left(\frac{(1 + \sqrt{5})(1 + \sqrt{5}) - 4}{2(1 + \sqrt{5})} \right) \\
&= \frac{5 + 3\sqrt{5}}{10}\,,
\end{aligned}$$

when we simplify. In the same way for the coefficient of r_1^n we have

$$\begin{aligned}
\left(\frac{5 - 3\sqrt{5}}{10} \right) \left[\frac{1 - \sqrt{5}}{2} - \frac{2}{1 - \sqrt{5}} \right] &= \frac{5 - 3\sqrt{5}}{10} \left(\frac{1 - 2\sqrt{5} + 5 - 4}{2(1 - \sqrt{5})} \right) \\
&= \frac{5 - 3\sqrt{5}}{10}\,,
\end{aligned}$$

when we simplify. Thus we see that

$$f(n+1) - f(n-1) = f(n)\,.$$

Problem 32

From the expression $\frac{a+b}{b+c} = \frac{c+d}{d+a}$ we get

$$(a + b)(d + b) = (b + c)(c + d)\,,$$

or

$$a^2 + (b + d)a + bd = c^2 + (b + d)c + bd\,,$$

or

$$a^2 + (b + d)a = c^2 + (b + d)c\,.$$

Bringing everything to one side we get

$$a^2 + (b + d)a - (b + d)c - c^2 = 0\,,$$

or

$$a^2 + (b + d)(a - c) - c^2 = 0\,,$$

or

$$(a - c)(a + c) + (b + d)(a - c) = 0\,,$$

or

$$(a - c)(a + b + c + d) = 0\,.$$

Problem 33

Let the point P be denotes with the coordinates (P_x, P_y), and let W and H be the width and height H of the rectangle. Then given that we know the lengths of the line segments AP, DP, and CP we can write down this information in terms of the unknowns P_x, P_y, W, and H as

$$9 = P_x^2 + P_y^2 \quad (156)$$
$$16 = P_x^2 + (H - P_y)^2 \quad (157)$$
$$25 = (W - P_X)^2 + (H - P_y)^2\,. \quad (158)$$

Expanding first term on the right-hand-side of this last equation gives

$$25 = W^2 - 2P_xW + P_x^2 + (H - P_y)^2\,.$$

From Equation 157 we see that this is

$$25 = W^2 - 2P_xW + 16 \quad \text{or} \quad 9 = W^2 - 2P_xW\,. \quad (159)$$

The distance (squared) we want to compute is given by

$$(W - P_x)^2 + P_y^2 = W^2 - 2WP_x + P_x^2 + P_y^2\,.$$

Using Equation 159 for the first two terms and Equation 156 for the second two terms we see that the above distance squared is equal to $9 + 9 = 18$. Thus the distance is $3\sqrt{2}$.

Problem 34

The sum we want to evaluate is

$$S \equiv \sum_{k=0}^{n} (k+1) i^k .$$

We write this sum in "groups of four" as

$$\begin{aligned}
S &= \sum_{k\in\{0,4,8,12,\cdots,n-8,n-4,n\}} (k+1)i^k + \sum_{k\in\{1,5,9,13,\cdots,n-9,n-5,n-1\}} (k+1)i^k \\
&+ \sum_{k\in\{2,6,10,14,\cdots,n-10,n-6,n-2\}} (k+1)i^k + \sum_{k\in\{3,7,11,15,\cdots,n-11,n-7,n-3\}} (k+1)i^k \\
&= \sum_{j=0}^{\frac{n}{4}} (4j+1)i^{4j} + \sum_{j=0}^{\frac{n}{4}-1} (4j+2)i^{4j+1} + \sum_{j=0}^{\frac{n}{4}-1} (4j+3)i^{4j+2} + \sum_{j=0}^{\frac{n}{4}-1} (4j+4)i^{4j+3}
\end{aligned}$$

One can check that all of the subscripts are correct in the above summations by considering an example say $n = 16$ and evaluating the sums above. From what we know about the powers of i we can write the above as

$$\begin{aligned}
S &= \sum_{j=0}^{\frac{n}{4}} (4j+1) + i\sum_{j=0}^{\frac{n}{4}-1} (4j+2) - \sum_{j=0}^{\frac{n}{4}-1} (4j+3) - i\sum_{j=0}^{\frac{n}{4}-1} (4j+4) \\
&= 4\left(\frac{n}{4}\right) + 1 + \sum_{j=0}^{\frac{n}{4}-1} (-2) + i\sum_{j=0}^{\frac{n}{4}-1} (-2) \\
&= n + 1 + \frac{n}{4}(-2) + i\frac{n}{4}(-2) = \frac{1}{2}(n + 2 - ni) ,
\end{aligned}$$

when we simplify.

Problem 37

From the arithmetic-geometric inequality we have that

$$\frac{1}{2}(a+b) \geq \sqrt{ab} .$$

Subtracting a on both sides gives

$$\frac{1}{2}(a+b) - a \geq \sqrt{ab} - a ,$$

or

$$\frac{1}{2}(b-a) \geq \sqrt{a}(\sqrt{b} - \sqrt{a}) .$$

Squaring both sides gives

$$\frac{1}{4}(b-a)^2 \geq a(b + a - 2\sqrt{ab}) ,$$

or

$$\frac{1}{4}(b-a)^2 \geq 2a\left(\frac{a+b}{2} - \sqrt{ab}\right),$$

Thus we see that the A.M. minus the G.M. is smaller than

$$\frac{a+b}{2} - \sqrt{ab} \leq \frac{(b-a)^2}{8a}.$$

Problem 38 (the length of QR)

We will solve this geometric problem by placing the points into a Cartesian coordinate system and then using *algebra* to determine any unknown expressions. To do this, let the point Q be located at the origin $(0,0)$, let the point R be located at the point $(2x,0)$, and let the point M be located at $(x,0)$. Here x is an unknown distance. The point P must be at the intersection of three circles

- One from Q of radius 4.
- One from M of radius 3.5.
- One from R of radius 7.

Let the point P be given by (P_x, P_y) then these three equations are given by

$$P_x^2 + P_y^2 = 4^2$$
$$(P_x - x)^2 + P_y^2 = 3.5^2$$
$$(P_x - 2x)^2 + P_y^2 = 7^2.$$

This is three equations and three unknowns P_x, P_y, and x. Expanding the second two equations we get

$$P_x^2 - 2xP_x + x^2 + P_y^2 = \left(\frac{7}{2}\right)^2 = \frac{49}{4}$$
$$P_x^2 - 4xP_x + 4x^2 + P_y^2 = 49.$$

As we know that $P_x^2 + P_y^2 = 16$ these two equations become

$$-2xP_x + x^2 = \frac{49}{4} - 16 = -\frac{15}{4} \tag{160}$$
$$-4xP_x + 4x^2 = 49 - 16 = 33.$$

If we divide these two equations we get

$$\frac{2P_x - x}{4P_x - 4x} = -\frac{5}{44},$$

or

$$2P_x - x = -\frac{5}{11}(P_x - x).$$

The above can be simplified to $P_x = \frac{16}{27}x$. If we put this equation into Equation 160 and simplify we get $x^2 = \frac{81}{4}$ thus the positive solution is $x = \frac{9}{2}$ and the length of QR is twice this or 9.

Problem 39

Note that by considering the triangle given by $BB'C$ we see that $BB' < a$. By considering the triangle $BC'C$ we see that $CC' < a$. Finally by considering the triangle $AA'C$ we have that $AA' < b$. Thus

$$s = AA' + BB' + CC' < b + a + a = b + 2a\,.$$

The 1965 Examination

Problem 1

The given expression is equal to

$$2x^2 - 7x + 5 = 0\,.$$

This has solutions given by

$$x = \frac{7 \pm \sqrt{49 - 4(2)(5)}}{2(2)} = \frac{7 \pm \sqrt{49 - 40}}{4}\,.$$

as there are two real solutions to this equation there must be two real solutions to the original equation.

Problem 2

In a regular hexagon the angle cut by one side is

$$\frac{2\pi}{6} = \frac{\pi}{3}\,.$$

Thus the arc length is $\frac{1}{6}(2\pi r) = \frac{\pi}{3}r$. Since one side of the hexagon forms an equilateral triangle with the two circumradii that meet at the given sides two end points we can conclude that the hexagons side is also equal to r the circle's radius. Thus the ratio of the side length to the arch length is

$$r : \frac{\pi}{3}r \quad \text{or} \quad 3 : \pi\,.$$

Problem 3

We have

$$(81)^{-(2^{-2})} = \frac{1}{(81)^{2^{-2}}} = \frac{1}{81^{1/4}} = \frac{1}{3}\,.$$

Problem 4

The set of points of equal distance to the two two lines l_1 and l_3 must be a line itself running between them. Then every point on this parallel line is a fixed distance (say d) from both l_1 and l_3. As l_2 intersects l_1 the points that we seek are equidistant between all three are on this parallel line and are the same distance d from l_2. There are two points (one on each side of l_2) that satisfy this property.

Problem 5

Let $x = 0.363636\ldots$, then $100x = 36.363636\ldots$. Then

$$100x - x = 36\,.$$

Thus $x = \frac{36}{99} = \frac{4}{11}$ so the sum requested is 15.

Problem 6

The given equation

$$10^{\log_{10}(9)} = 8x + 5\,,$$

is equivalent to

$$8x + 5 = 9 \quad \text{so} \quad x = \frac{1}{2}\,.$$

Problem 7

Start by writing the given quadratic as

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0\,.$$

Then if r and s are the roots of the above quadratic then by factoring above can be written as $(x-r)(x-s) = 0$. Expanding this expression and equating it to the first we have that

$$\begin{aligned} -(r+s) &= \frac{b}{a} \\ rs &= \frac{c}{a}\,. \end{aligned}$$

If we negate the first equation and then divide it by the second we get

$$\frac{1}{s} + \frac{1}{r} = -\frac{b}{a}\left(\frac{a}{c}\right) = -\frac{b}{c}\,.$$

Problem 8

For this problem we drop a perpendicular to the side of the triangle of length 18. Let this perpendicular intersect the line drawn parallel to the base of length 18 at a length of h_1, let the length of this parallel line be given by b_1 and let the "height" of the trapezoid be given by h_2. Then with these definitions given that the area of then trapezoid is one third that of the full triangle means that

$$\frac{1}{2}h_2(18 + b_1) = \frac{1}{3}\left(\frac{1}{2}(18)(h_1 + h_2)\right)\,.$$

We want to know the value of b_1. If we divide the above by h_2 we get

$$18 + b_1 = 6\left(\frac{h_1}{h_2} + 1\right) . \tag{161}$$

To evaluate b_1 we need to determine the value of $\frac{h_1}{h_2}$. From similar triangles we get

$$\frac{h_1}{h_1 + h_2} = \frac{b_1}{b_2} = \frac{b_1}{18} .$$

If we solve this for $\frac{h_1}{h_2}$ in terms of b_1 we get

$$\frac{h_1}{h_2} = \frac{b_1}{18}\frac{1}{\left(1 - \frac{b_1}{18}\right)} = \frac{1}{\frac{18}{b_1} - 1} .$$

If we put this into Equation 161 we get

$$18 + b_1 = 6\left(\frac{1}{\frac{18}{b_1} + 1} + 1\right) .$$

If we solve this for b_1 we find that $b_1^2 = 216 = 2^3 3^3$ so $b_1 = 6\sqrt{6}$.

Problem 9

Write y like

$$y = x^2 - 8x + c = x^2 - 8x + 16 - 16 + c = (x - 4)^2 + c - 16 .$$

Then we want when $x = 4$ to have $y = 0$ so that $c = 16$.

Problem 10

We have

$$x^2 - x - 6 = (x - 3)(x + 2) .$$

By plotting the points where the above factored expression is negative we see that the inequality $x^2 - x - 6 < 0$ is equivalent to $-2 < x < 3$.

Problem 11

Expression I should be evaluated as

$$\sqrt{-4}\sqrt{-16} = 2i(4i) = 8i^2 = -8 .$$

The others are correct.

Problem 12 (an inscribed rhombus)

If we let l be the distance between A and the point E, then the distance between A and the point D is also l since $ADEF$ is a rhombus. Thus we have $BE = 6 - l$ and $CD = 12 - l$. Let x be the distance between B and F so that $CF = 8 - x$. Note that AB is parallel to DF we have $\angle CAB = \angle CDF$ and thus triangles CDF and CAB are similar. With the notation introduced this means that we can write

$$\frac{12-l}{12} = \frac{8-x}{8}\,.$$

As AC is parallel to EF we have $\angle CAB = \angle FEB$ and the triangles ABC and EBF are similar. This means that

$$\frac{6-l}{6} = \frac{x}{8}\,.$$

This is a set of two equations and two unknowns. The second equation gives $x = \frac{4}{3}(6-l)$, which when we put this into the first equation gives

$$\frac{12-l}{12} = 1 - \frac{x}{8} = 1 - \frac{1}{6}(6-l)\,.$$

This gives $l = 4$.

Problem 13

The line can either not intersect, intersect only once (if it is tangent) or intersect with an infinite number of points. For this problem if we plot the disk $x^2 + y^2 \leq 16$ and the given line we see that the two intersect over an infinite number of points.

Problem 14

We can get the desired sum by letting $x = y = 1$. This gives that $(1 - 2 + 1)^7 = 0$.

Problem 15

The statement given means that

$$2(2b+5) = 5b + 2 \quad \text{or} \quad b = 8\,.$$

Problem 16

From the given description of the problem the area of the triangle we want to compute can be written as

$$A_{\triangle DEF} = \frac{1}{2}bh = \frac{1}{2}(15)h\,,$$

thus we need to determine the height of $\triangle DEF$. We will solve this problem by specifying it in Cartesian coordinate, which will turn it from a geometric problem into an algebraic problem. To do that let point C be located at $(0,0)$, let point D be located at $(15,0)$, let point E be located at $(30,0)$, let point B be located at $(0,15)$, and let point A be located at $(0,30)$. Then the line AD is

$$y - 0 = \left(\frac{30-0}{0-15}\right)(x-15) \quad \text{or} \quad y = -2(x-15)\,.$$

The line BE is

$$y - 0 = \left(\frac{15-0}{0-30}\right)(x-30) \quad \text{or} \quad y = -\frac{1}{2}(x-30)\,.$$

These two lines intersect when

$$-2(x-15) = -\frac{1}{2}(x-30)\,.$$

When we solve this we get $x = 10$ and $y = -2(-5) = +10$. Thus the height of $\triangle DEF$ is 10 so the area is $\frac{1}{2}(15)(10) = 75$.

Problem 17

A true statement from a statement like this can be obtained from its contrapositive. The contrapositive of the statement given is: if the weather is fair the picnic will be held.

Problem 18

We need to evaluate

$$\frac{\frac{1}{1+y} - (1-y)}{\frac{1}{1+y}} = 1 - (1-y)(1+y) = 1 - (1-y^2) = y^2\,.$$

Problem 19

We would perform long division of the two given polynomials. When we do that we find that the division results in

$$\frac{x^4 + 4x^3 + 6px^2 + 4qx + r}{x^3 + 3x^2 + 9x + 3} = x + 1 + \frac{(6p-12)x^2 + (4q-12)x + (r-3)}{x^3 + 3x^2 + 9x + 3}\,.$$

Thus to make the two polynomials divisible we must have

$$\begin{aligned} 6p - 12 &= 0 \quad \text{so} \quad p = 2 \\ 4q - 12 &= 0 \quad \text{so} \quad q = 3 \\ r - 3 &= 0 \quad \text{so} \quad r = 3\,. \end{aligned}$$

Then with these we have that $(p+q)r = 15$.

Problem 20

Given the general expression for the sum of an arithmetic sequence from Equation 18 we have the general term given by

$$S_n - S_{n-1} = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n-1} a_k = a_n \,.$$

From the given expression for the arithmetic sum we get

$$\begin{aligned} S_n - S_{n-1} &= 2n + 3n^2 - (2(n-1) + 3(n-1)^2) \\ &= 2n + 3n^2 - (2n - 2 + 3(n^2 - 2n + 1)) \\ &= 6n - 1 \,, \end{aligned}$$

which is the general term.

Problem 21

Denote the given expression by $f(x)$ so

$$f(x) \equiv \log_{10}(x^2 + 3) - 2\log_{10}(x) \,.$$

Then we see that

$$f(x) = \log_{10}(x^2+3) - \log_{10}(x^2) = \log_{10}\left(\frac{x^2+3}{x^2}\right) = \log_{10}\left(1 + \frac{3}{x^2}\right) \,.$$

As $1 + \frac{3}{x^3} > 1$ we have that $\log_{10}\left(1 + \frac{3}{x^2}\right) > 0$ for all x. As $x \to +\infty$ we have that $\log_{10}\left(1 + \frac{3}{x^2}\right) \to \log_{10}(1) = 0$ approaching 0 from above. As $x \to 0$ we have that $1 + \frac{3}{x^2} \to +\infty$ and thus $\log_{10}\left(1 + \frac{3}{x^2}\right) \to +\infty$. Since we only want to consider $x > \frac{2}{3}$ we don't let x got all the way to 0. If we sketch the graph of $f(x)$ for $\frac{2}{3} < x < +\infty$ we see that it looks like what is shown in Figure 5 Thus we see by picking x large enough we can make the values of $f(x)$ as close to zero as needed.

Problem 23

Note that $|x^2 - 4| = |x-2||x+2| < 0.01|x+2|$. Now since $|x-2| < 0.01$ we have $1.99 < x < 2.01$ and thus

$$3.99 < x + 2 < 4.01 \,.$$

Thus the above expression is bounded above by

$$|x^2 - 4| < 0.01(4.01) = 0.0401 \,.$$

Take $N = 0.0401$.

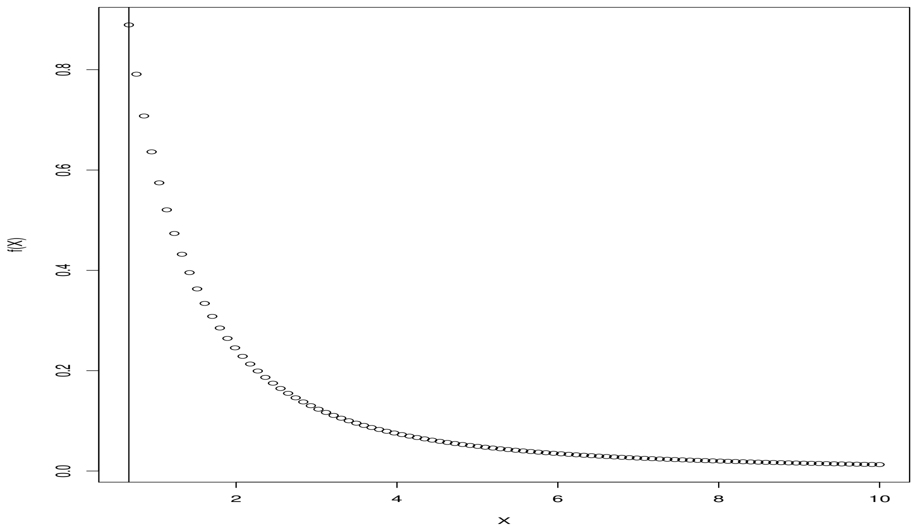

Figure 5: A plot of the function $f(x)$ vs. x for Problem 21 for $x > \frac{2}{3}$.

Problem 24

Since each factor is of the form $10^{i/11}$ the product of n such factors is then

$$\prod_{i=1}^{n}(10^{1/11})^i\,,$$

and we want this product to be larger than 10^5. This can be written as

$$10^{\frac{1}{11}\sum_{i=1}^{n} i} > 10^5\,,$$

or we need to find a n such that

$$\frac{1}{11}\sum_{i=1}^{n} i > 5 \quad \text{or} \quad \sum_{i=1}^{n} > 55\,.$$

Using Equation 22 we get that this means that we need to pick n such that

$$\frac{1}{2}n(n+1) > 55 \quad \text{or} \quad n(n+1) > 110\,.$$

If $n = 10$ then we get $10(11) = 110$, thus we need $n = 11$ to get $11(12) = 132 > 110$.

Problem 26

From the problem we are told that

$$\begin{aligned} m &= \frac{1}{5}(a+b+c+d+e) \\ k &= \frac{1}{2}(a+b) \\ l &= \frac{1}{3}(c+d+e) \\ p &= \frac{1}{2}(k+l)\,. \end{aligned}$$

From these definitions note that $5m = 2k + 3l$ so $m = \frac{2}{5}k + \frac{3}{5}l$ and $p = \frac{1}{2}k + \frac{1}{2}l$. From these we see that

$$\begin{aligned} m - p &= \left(\frac{2}{5} - \frac{1}{2}\right)k + \left(\frac{3}{5} - \frac{1}{2}\right)l \\ &= -\frac{1}{10}k + \frac{1}{10}l\,. \end{aligned}$$

As k and l can be anything (they are not related in any way) the expression for $m - p$ can be made any sign. Thus there is no relationship between m and p.

Problem 27 (what is m?)

We can perform long division by $y-1$ and $y+1$ on the polynomial $y^2 + my + 2$ to get

$$\frac{y^2 + my + 2}{y-1} = y + (m+1) + \frac{m+3}{y-1}\,,$$

and

$$\frac{y^2 + my + 2}{y+1} = y + (m-1) + \frac{-m+3}{y+1}\,.$$

Then we see that $R_1 = m + 3$ and $R_2 = -m + 3$. If these two remainders are equal then we must have that $m = 0$.

Problem 29 (taking classes)

For this problem we can draw the three groups mathematics students, english students, and history students in a Venn diagram and introduce the "set" notation M, E, and H to denote the events that a given student is taking a mathematics, english, or history class respectively. Then using the Venn diagram as a reference in producing the various possible studentship class overlaps for the three classes we end with the following disjoint sets of students

$$
\begin{aligned}
ME^cH^c &= \text{students taking mathematics only} \\
M^cEH^c &= \text{students taking english only} \\
M^cE^cH &= \text{students taking history only} \\
MEH^c &= \text{students taking math and english but not history} \\
ME^cH &= \text{students taking math and history but not english} \\
M^cEH &= \text{students taking english and history but not mathematics} \\
MEH &= \text{students taking mathematics, english, and history}\,.
\end{aligned}
$$

Then the total number of students must be decomposed into the various disjoint sets above. That is if we let $n(\cdot)$ denote the number of students in a given set we first have that

$$28 = n(ME^cH^c) + n(M^cEH^c) + n(M^cE^cH) + n(MEH^c) + n(ME^cH) + n(M^cEH) + n(MEH)\,. \quad (162)$$

From the problem we are told that

$$
\begin{aligned}
n(MEH^c) &= n(ME^cH^c) \\
n(M^cEH^c) &= n(M^cE^cH) = 0 \\
n(ME^cH) &= 6 \\
n(M^cEH) &= 5n(MEH)\,.
\end{aligned}
$$

Then we we put these facts into Equation 162 we get

$$28 = n(MEH^c) + 0 + 0 + n(MEH^c) + n(ME^cH) + 5n(MEH) + n(MEH)\,,$$

or

$$28 = 2n(MEH^c) + 6 + 6n(MEH)\,,$$

or

$$11 = n(MEH^c) + 3n(MEH)\,.$$

We are told that $n(MEH)$ must be even and non-zero. From the above expression $n(MEH) = 2$ since if it is even an larger than this value the above equation will be violated. When $n(MEH) = 2$ we get that $n(MEH^c) = 5$.

Problem 31

Write each expression in terms of the "natural" log. That is we are looking for values x that satisfy

$$\frac{\ln(x)}{\ln(a)}\frac{\ln(x)}{\ln(b)} = \frac{\ln(b)}{\ln(a)}\,.$$

or

$$\ln(x)^2 = \ln(b)^2\,,$$

or taking square roots

$$\ln(x) = \pm|\ln(b)|\,.$$

There are two solution to this one for each sign above given by

$$x = e^{\pm|\ln(b)|}\,.$$

Thus we have two solutions.

Problem 33 (the value of $h + k$)

We write out 15! and then "factor out" as many 12s as we can. For example we have

$$
\begin{aligned}
15! &= 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \\
&= (3 \cdot 5) \cdot (2 \cdot 7) \cdot 13 \cdot (12) \cdot 11 \cdot (2 \cdot 5) \cdot (3 \cdot 3) \cdot (2 \cdot 2 \cdot 2) \cdot 7 \cdot (2 \cdot 3) \cdot 5 \cdot (12) \cdot 2 \\
&= 12^5 (5 \cdot 7 \cdot 13 \cdot 11 \cdot 5 \cdot 7 \cdot 3 \cdot 5 \cdot 2)\,.
\end{aligned}
$$

Thus $k = 5$. We now do the same by factoring out as man 10s as we can. We find

$$
\begin{aligned}
15! &= (3 \cdot 5) \cdot (2 \cdot 7) \cdot 13 \cdot (12) \cdot 11 \cdot (10) \cdot 9 \cdot (2 \cdot 4) \cdot 7 \cdot (2 \cdot 3) \cdot 5 \cdot 4 \cdot (3 \cdot 2) \\
&= 10^3 (3 \cdot 7 \cdot 13 \cdot 12 \cdot 11 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 4 \cdot 3)\,.
\end{aligned}
$$

Thus $h = 3$ and we have that $h + k = 8$.

Problem 34 (the maximum of an expression)

Write the given expression as

$$
\begin{aligned}
\frac{4x^2 + 8x + 13}{6(1+x)} &= \frac{4(x^2 + 2x + 1) + 9}{6(1+x)} \\
&= \frac{4(x+1)^2 + 9}{6(1+x)} = \frac{2(x+1)}{3} + \frac{3}{2(x+1)}\,.
\end{aligned}
$$

If we let $y = \frac{2(x+1)}{3}$ then we have produced a one-to-one mapping of our original domain in x of $0 \le x < +\infty$ into the domain for y of $\frac{2}{3} \le y < \infty$ and are now considering the extream values of the function

$$
y + \frac{1}{y}\,,
$$

over that domain. The derivative of this expression is given by

$$
1 - \frac{1}{y^2}\,.
$$

From this we see that the above is decreasing (has a negative first derivative) when

$$
1 - \frac{1}{y^2} < 0 \quad \text{for} \quad \frac{2}{3} \le y \le 1\,,
$$

and is an increasing function when the derivative is positive or

$$
1 - \frac{1}{y^2} > 0 \quad \text{for} \quad y \ge 1\,.
$$

Thus the point $y = 1$ is the smallest this function can ever be and we have $y + \frac{1}{y} \le 2$.

Problem 38 (getting a job done)

Let v_a, v_b, and v_c the the *rate* at which A, B, and C work on a task independently. This means that $\frac{1}{v_a}$ is the time that it takes for A to complete the job by himself. In the same way $\frac{1}{v_a+v_b}$ is the time it takes A and B to complete the task together etc. Then from what we are told we have

$$\begin{aligned} \frac{1}{v_a} &= \frac{m}{v_b+v_c} \\ \frac{1}{v_b} &= \frac{n}{v_a+v_c} \\ \frac{1}{v_c} &= \frac{x}{v_a+v_b} \,. \end{aligned}$$

If we take the reciprocal of the above expressions we get

$$\begin{aligned} v_b + v_c &= m v_a \\ v_a + v_c &= n v_b \\ v_a + v_b &= x v_c \,. \end{aligned}$$

If we write the above as a system we get

$$\begin{aligned} -m v_a + v_b + v_c &= 0 \\ v_a - n v_b + v_c &= 0 \\ v_a + v_b - x v_c &= 0 \,. \end{aligned}$$

This is three equations and three unknowns. In order for it to not only have the trivial solution (all v's zero) the system as given must be singular, i.e. the determinant of the coefficient matrix must be zero or

$$\left| \begin{array}{ccc} -m & 1 & 1 \\ 1 & -n & 1 \\ 1 & 1 & -x \end{array} \right| = 0 \,.$$

Expanding this determinant we get

$$-m \left| \begin{array}{cc} -n & 1 \\ 1 & -x \end{array} \right| - 1 \left| \begin{array}{cc} 1 & 1 \\ 1 & -x \end{array} \right| + 1 \left| \begin{array}{cc} 1 & -n \\ 1 & 1 \end{array} \right| = 0 \,,$$

or expanding the determinants above we get

$$-m(nx-1) - (-x-1) + (1+n) = 0 \,.$$

Solving for x we get

$$x = \frac{m+n+2}{mn-1} \,.$$

The 1966 Examination

Problem 1 (rationally related)

We are told $\frac{3x-4}{y+15} = C$. When we put in $x = 2$ and $y = 3$ we get that $C = \frac{1}{9}$. Then if $y = 12$ we would have

$$\frac{3x-4}{27} = \frac{1}{9} \quad \text{so} \quad x = \frac{7}{3}\,.$$

Problem 2 (transforming the area)

The original area is $A = \frac{1}{2}bh$. With the transformation of h and b suggested we have that the new values (denoted by primes) must satisfy

$$\begin{aligned} b' &= 1.1b \\ h' &= 0.9h\,. \end{aligned}$$

Then the new area is

$$A' = \frac{1}{2}b'h' = \frac{1}{2}\left(\frac{11}{10}b\right)\left(\frac{9}{10}h\right) = \frac{99}{100}A\,.$$

This is a loss of 1%.

Problem 3 (looking for an equation)

Let r and s be the two unknown numbers. Then we are told that

$$\frac{r+s}{2} = 6 \quad \text{and} \quad \sqrt{rs} = 10\,,$$

or

$$r + s = 16 \quad \text{and} \quad rs = 100\,.$$

If a quadratic polynomial has two roots r and s then we must have

$$(x-r)(x-s) = x^2 - (r+s)x + rs = 0\,.$$

Thus the equation that has the two roots with the given arithmetic and geometric means is then

$$x^2 - 12x + 100 = 0\,.$$

Problem 4 (a ratio of areas)

Let a be the length of the square the two circles refer to. The first circle has a radius given by the length of the hypotenuse of a right triangle with sides $\frac{a}{2}$ and $\frac{a}{2}$. This is the value $\frac{a}{\sqrt{2}}$. Thus the area of the first triangle is then given by

$$A_{C_1} = \pi \left(\frac{a}{\sqrt{2}}\right)^2 = \frac{\pi a^2}{2}\,.$$

The area of the second circle is given by

$$A_{C_2} = \pi \left(\frac{a}{2}\right)^2 = \frac{\pi a^2}{4}\,.$$

The ratio of these two areas is then given by

$$\frac{A_{C_1}}{A_{C_2}} = \frac{1/2}{1/4} = 2\,.$$

Problem 5 (what are the solutions?)

The expression given is equal to (for all possible x values)

$$\frac{2x(x-5)}{x(x-5)} = x - 3\,.$$

As we cannot have a zero in the denominator of the above fraction we must have $x \neq 0$ and $x \neq 5$. In that case the above simplifies to

$$2 = x - 3 \quad \text{or} \quad x = 5\,.$$

As this would result in a zero denominator in the original expression we have no solutions.

Problem 6 (the length of AC)

The chord AC is the longest edge in a triangle with two equal sizes, each of length equal to the radius of the circle or $5/2$ and an angle between the two sides of $180 - 60 = \frac{2\pi}{3}$. Then using the law of cosigns we have that

$$AC^2 = \left(\frac{5}{2}\right)^2 + \left(\frac{5}{2}\right)^2 - 2\left(\frac{5}{2}\right)\left(\frac{5}{2}\right)\cos\left(\frac{2\pi}{3}\right) = \frac{75}{4}\,,$$

when we use the fact that $\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$. This means that the length of AC is $\frac{5\sqrt{3}}{2}$.

Problem 7 (partial fractions)

We recognize that the requested expansion

$$\frac{35x-29}{x^2-3x+2} = \frac{N_1}{x-1} + \frac{N_2}{x-2},$$

is a partial fractions expansion of the left-hand-side. Multiply both sides by x^2-3x+2 to get

$$35x - 29 = N_1(x-2) + N_2(x-1).$$

Let $x = 2$ to get $N_2 = 70 - 29 = 41$ and let $x = 1$ to get $-N_1 = 35 - 29$ or $N_1 = -6$. Then we see that $N_1 N_2 = -6(41) = -246$.

Problem 8 (the distance between centers)

If we draw the two circles and a line connecting the centers of the circles. Let the left-most circle center be denoted C_L and the right-most circle center be denoted as C_R. The common cord intersects the line segment $C_L C_R$ perpendicularly at its midpoint. Thus we have two right triangles that have a height of $\frac{16}{2} = 8$ and two different hypotenuses (corresponding to the radii of the two circles). The two unknown triangle legs (denoted by l_1 and l_2) sum to the desired length of the total distance between centers. Using the Pythagorean theorem in the triangle with a radius of 10 we have

$$l_1^2 = 10^2 - 8^2 = 36 \quad \text{so} \quad l_1 = 6.$$

The same thing for the triangle with radius 17 gives

$$l_2^2 = 17^2 - 8^2 = 225 \quad \text{so} \quad l_2 = 15.$$

Thus the total distance between the two centers is $l_1 + l_2 = 21$.

Problem 9 (evaluating $\log_3(x)$)

Since $\log_2(8) = 3$ the expression for x becomes $x = (\log_8(2))^3$. Now since $\log_8(2) = \frac{1}{3}$ the expression for x becomes $x = \left(\frac{1}{3}\right)^3 = \frac{1}{27}$. Thus

$$\log_3(x) = \log_3\left(\frac{1}{27}\right) = -\log_3(27) = -3.$$

Problem 10 (the sum of their cubes)

Let r and s be the two numbers then $r + s = 1$ and $rs = 1$. Note that we can write $r^3 + s^3$ in factored form

$$r^3 + s^3 = (r+s)(r^2 - rs + s^2).$$

From what we know this can be written as

$$r^3 + s^3 = 1(r^2 - 1 + s^2) = r^2 + s^2 - 1\,.$$

Now note that

$$r^2 + s^2 = (r+s)^2 - 2rs = 1^2 - 2 = -1\,,$$

Thus we finally get $r^3 + s^3 = -1 - 1 = -2$.

Problem 12 (an equation from the exponents)

Note that we can write the given expression as

$$2^{6x+3}2^{6x+12} = 2^{12x+15}\,,$$

or

$$6x + 3 + 6x + 12 = 12x + 15\,.$$

As this is true for all values of x there are more than three values of x that satisfy the given equation (there are infinitely many).

Problem 14 (the area of $\triangle BEF$)

Let the location of the four points A, B, C, and D be located in the Cartesian plane at the locations $(0,0)$, $(0,3)$, $(5,3)$, and $(5,0)$. respectively. By the Pythagorean theorem the length of the segment AC is given by $\sqrt{25+9} = \sqrt{34}$. Thus the length of AE and AF is given by $\frac{1}{3}\sqrt{34}$ and $\frac{2}{3}\sqrt{34}$ respectively. Denote the angle $\angle CAD$ by θ and then from the given rectangle that the line segment AC is in we see that

$$\sin(\theta) = \frac{3}{\sqrt{34}}$$
$$\cos(\theta) = \frac{5}{\sqrt{34}}\,.$$

Using these values (and the lengths of AE and AF) we can explicitly determine the x and y locations of the points E and F. We have

$$E_x = \frac{\sqrt{34}}{3}\cos(\theta) = \frac{\sqrt{34}}{3}\left(\frac{5}{\sqrt{34}}\right) = \frac{5}{3}$$
$$E_y = \frac{\sqrt{34}}{3}\sin(\theta) = \frac{\sqrt{34}}{3}\left(\frac{3}{\sqrt{34}}\right) = 1\,.$$

In the same way, we have $F_x = \frac{10}{3}$ and $F_y = 2$. Now that we have the Cartesian coordinates of the three points of the triangle BEF we can use Equation 12 to compute the area. We find

$$\text{Area } \triangle BEF = \frac{1}{2}\left|\begin{array}{ccc} B_x & B_y & 1 \\ E_x & E_y & 1 \\ F_x & F_y & 1 \end{array}\right| = \frac{1}{2}\left|\begin{array}{ccc} 0 & 3 & 1 \\ 5/3 & 1 & 1 \\ 10/3 & 2 & 1 \end{array}\right| = -\frac{3}{2}\left(\frac{5}{3} - \frac{10}{3}\right) + \frac{1}{2}\left(\frac{10}{3} - \frac{10}{3}\right) = \frac{5}{2}\,.$$

In the above I didn't know beforehand that the determinant above would be positive. If it was not, the area would then the the absolute value of the above expression.

Problem 15 (some inequalities)

From $x - y > x$ we have that $-y > 0$ or $y < 0$. From $x + y < y$ we have that $x < 0$. Thus the two conditions are $x < 0$ and $y < 0$.

Problem 16 (an equation from powers)

We can write the two expressions as

$$2^{2x-(x+y)} = 2^3 \quad \text{and} \quad 3^{2x+2y-5y} = 243\,.$$

Note that $243 = 3^5$ so that the above two equations become

$$x - y = 3 \quad \text{and} \quad 2x - 3y = 5\,.$$

Solving these two equations gives $x = 4$ and $y = 1$ thus $xy = 4$.

Problem 17 (points in common)

We have

$$x^2 + 4y^2 = 1 \quad \text{and} \quad 4x^2 + y^2 = 4\,.$$

Thus the first equation gives $x^2 = 1 - 4y^2$ which when we put this into the second equation we get

$$4(1 - 4y^2) + y^2 = 4 \quad \text{so} \quad y = 0\,,$$

is the only solution. If we put $y = 0$ into the first equation we get $x = \pm 1$.

Problem 18 (the common difference)

For this problem we will use Equations 18 with the numbers given imply

$$155 = \frac{N}{2}(2 + 29) = \frac{N}{2}(2^2 + (N - 1)d)\,.$$

Here we have two equations and two unknowns for N and d. From the first equation we get $31N = 2(155)$ or $N = \frac{310}{31} = 10$. Using the second equation we get $155 = 5(4 + 9d)$ so $d = 3$.

Problem 19 (n when the two sums are equal)

Using the Equations 18 with the numbers given we have

$$\begin{aligned} s_1 &= \frac{N}{2}(2(8) + (N-1)4) = \frac{N}{2}(16 + 4(N-1)) \\ s_2 &= \frac{N}{2}(2(17) + (N-1)2) = \frac{N}{2}(34 + 2(N-1)) \,. \end{aligned}$$

If $s_1 = s_2$ we must have

$$16 + 4(N-1) = 34 + 2(N-1) \quad \text{so} \quad N = 10 \,.$$

Problem 21 (a n-pointed star)

Lets consider the example 5 star given and hope that the result we derive will generalize. Denote the angles of the points of the "star" by θ_i for $i = 1, 2, 3, 4, 5$. We can place θ_i "opposite" the side denoted i for $i = 1, 2, 3, 4, 5$. Consider the triangle that is external to the convex polygon and that has θ_1 as a vertex angle. Since the sum of the angles in a triangle must is 180 we can write θ_1 in terms of the other two internal angles. These two internal angles are themselves supplementary to internal angles of the polygon. Denote the interior angle between sides 3 and 4 as $\angle 34$. This means that we can write θ_1 as

$$\theta_1 = 180 - (180 - \angle 34) - (180 - \angle 23) = \angle 23 + \angle 34 - 180 \,.$$

When we do this for all of the angles in our five pointed star we get

$$\begin{aligned} \theta_2 &= \angle 34 + \angle 45 - 180 \\ \theta_3 &= \angle 15 + \angle 45 - 180 \\ \theta_4 &= \angle 15 + \angle 12 - 180 \\ \theta_5 &= \angle 12 + \angle 23 - 180 \,. \end{aligned}$$

If we now add these expressions together we see that every internal angle in the convex polygon will be represented twice. In the general n sided polygon we would have

$$\sum_{i=1}^{n} \theta_i = 2 \sum_{ij} \angle ij - n180 \,.$$

The sum above represents the sum over all interior angles of a n sided polygon. From Equation 5 we know that this value is $180(n-2)$. When we put that value into the above formula we get that

$$\sum_{i=1}^{n} \theta_i = 180(n-4) \,.$$

Problem 22 (which statements have nonzero solutions)

Part (I): We can find nonzero solutions to $a^2 + b^2 = 0$ by taking $a = 1$ and $b = i$.

Part (II): Squaring this expression we get $a^2+b^2 = a^2b^2$ or when we divide by a^2b^2 (assumed nonzero) on both sides give

$$\frac{1}{a^2} + \frac{1}{b^2} = 0\,.$$

We can then find nonzero solutions if we let $a = 1$ and $b = i$.

Part (III): Squaring this expression we get $a^2 + b^2 = a^2 + 2ab + b^2$ or $2ab = 0$. Thus $a = 0$ or $b = 0$ so a nonzero solution could be $a = 1$ and $b = 0$.

Part (IV): Squaring this expression we get $a^2 + b^2 = a^2 - 2ab + b^2$ or $-2ab = 0$. Thus $a = 0$ or $b = 0$ so a nonzero solution could be $a = 1$ and $b = 0$.

Thus all have nonzero solutions.

Problem 23 (for what values of x are we real)

Write the given expression as

$$(2y + x)(2y + x) - x^2 + x + 6 = 0\,,$$

or

$$(2y + x)^2 - (x^2 - x - 6) = 0\,,$$

or

$$(2y + x)^2 - (x - 3)(x + 2) = 0\,.$$

We want to know the range of x for which y is real. Thus we need $(x - 3)(x + 2)$ to be positive. This happens with $x < -2$ or $x > 3$.

Problem 24 (an equation with logs)

Write the given expression as

$$\frac{\ln(N)}{\ln(M)} = \frac{\ln(M)}{\ln(N)}\,,$$

or

$$\ln(N)^2 = \ln(M)^2\,.$$

This means that $\ln(N) = \pm\ln(M)$ or $\ln(N) = \ln(M^{\pm 1})$ or

$$N = M^{\pm 1}\,.$$

Since we are told that $N \neq M$ we cannot use the $+1$ solution above. Thus we have that $N = M^{-1}$ so that $NM = 1$.

Problem 25

The expression for $F(n+1)$ can be written as $F(n+1) = F(n) + \frac{1}{2}$. If we iterate this expression a few times looking for a pattern we get

$$\begin{aligned}
F(1) &= 2 \\
F(2) &= F(1) + \frac{1}{2} = \frac{5}{2} = \frac{4+1}{2} \\
F(3) &= F(2) + \frac{1}{2} = \frac{6}{2} = \frac{4+2}{2} \\
F(4) &= F(3) + \frac{1}{2} = \frac{7}{2} = \frac{4+3}{2} \\
&\vdots \\
F(n) &= \frac{4+(n-1)}{2} \quad \text{for} \quad n \geq 1\,.
\end{aligned}$$

Using this general formula we find that $F(101) = \frac{4+100}{2} = 52$.

Problem 26

We have

$$\begin{aligned}
13x + 11y = 700 \quad \text{and} \\
y = mx - 1\,.
\end{aligned}$$

Put the second equation into the first and solve for x to get

$$x = \frac{711}{13+11m} = \frac{711}{11(m+1)+2}\,.$$

To find integer solutions for x we ask what are the factors of 711? Note that since it is dividable by 3 (and again divisible by 3) we can write 711 as

$$711 = 3^2(7(11)+2)\,.$$

Thus for x we have

$$x = \frac{9(11(7)+2)}{11(m+1)+2}\,.$$

Thus we need to take $m+1 = 7$ so $m = 6$. We could also try to find an integer m such that

$$11(m+1)+2 = 3(11(7)+2) \quad \text{or} \quad 11(m+1)+2 = 9(11(7)+2)\,,$$

but these don't have integer solutions. Thus $m = 6$ is the only solution.

Problem 27 (rowing with/without a stream)

Let v_s be the velocity of the stream and v_m the velocity of the man. Then when rowing downstream man travels at a rate of $v_s + v_m$ in a time of $\frac{15}{v_s+v_m}$. When rowing upstream he travels at a rate of $v_m - v_s$ in a time of $\frac{15}{v_m-v_s}$. In the problem statement we are told that

$$\frac{15}{v_m + v_s} + 5 = \frac{15}{v_m - v_s} \tag{163}$$

$$\frac{15}{2v_m + v_s} + 1 = \frac{15}{2v_m - v_2} \,. \tag{164}$$

These are two equations and two unknowns from which we want to solve for v_s. Equation 163 is equivalent to

$$15(v_m - v_s) + 5(v_m + v_s)(v_m - v_s) = 15(v_m + v_s) \,,$$

or

$$5(v_m^2 - v_s^2) = 30v_s \quad \text{or} \quad v_m^2 = v_s^2 + 6v_s \,.$$

Equation 164 is

$$15(2v_m - v_s) + (2v_m + v_s)(2v_m - v_s) = 15(2v_m + v_s) \,,$$

or

$$4v_m^2 - v_s^2 = 30v_s \,.$$

From what we know about v_m^2 we thus have

$$4v_s^2 + 24v_s - v_s^2 = 30v_s \quad \text{or} \quad 3v_s^2 = 6v_s \,.$$

Thus $v_s = 0$ or $v_s = 2$.

Problem 28 (the point with a consistent ratio)

Let x be the length from B to P, then from the given problem statement we can conclude that the lengths of the given segments are

$$\begin{aligned}
OA &= a \\
AB &= b - a \\
BC &= c - d \\
CD &= d - c \\
BP &= x \quad \text{so} \\
PC &= c - b - x \\
OP &= b + x \,.
\end{aligned}$$

Then to make $AP : PD = BP : PC$ means that $\frac{AP}{PD} = \frac{BP}{PC}$. This first fraction can be written

$$\frac{AP}{PD} = \frac{AB + BP}{PC + CD} = \frac{b - a + x}{(c - b - x) + (d - c)} \,.$$

The second fraction can be written as

$$\frac{BP}{PC} = \frac{x}{c - b - x}\,.$$

Thus we have the relationship

$$\frac{b - a + x}{d - b - x} = \frac{x}{c - b - x}\,.$$

Since we want to find $OP = b + x$ we introduced the expression $b + x$ into the above to get

$$\frac{(b + x) - a}{d - (b + x)} = \frac{(b + x) - b}{c - (b + x)}\,,$$

or

$$\frac{OP - a}{d - OP} = \frac{OP - b}{c - OP}\,.$$

Clearing denominators gives

$$(OP - a)(c - OP) = (OP - b)(d - OP)\,,$$

or expanding we get

$$OPc - OP^2 - ac + aOP = dOP - OP^2 - bd + bOP\,,$$

or solving for OP we get

$$OP = \frac{ac - bd}{c + a - d - b}\,.$$

Problem 29 (counting numbers divisible by 5 and 7)

Notice that there are $\lfloor \frac{N-1}{5} \rfloor$ numbers in the range $[1, N)$ that are divisible by 5. Thus the number of numbers that are divisible by 5 in the range $[1, 1000)$ is $\lfloor \frac{999}{5} \rfloor = 199$. In the same way there are $\lfloor \frac{N-1}{7} \rfloor$ numbers in the range $[1, N)$ that are divisible by 7. Thus the number of numbers that are divisible by 7 in the range $[1, 1000)$ is $\lfloor \frac{999}{7} \rfloor = 142$. Now there are 999 positive integers in the range $[1, 1000)$ and if we remove the number of multiples of 5 and 7 computed above (i.e. 199 and 142) we have removed any number that is a multiple of $5 \times 7 = 35$ twice. There are $\lfloor \frac{999}{35} \rfloor = 28$ such numbers. The number of numbers less than 1000 and that are not divisible by 5 and 7 is then

$$999 - 199 - 142 + 28 = 686\,.$$

Problem 30 (finding the value of $a + c$)

If we are given the roots of a polynomial then evaluating the polynomial at these values must give zero. This means that we must have

$$\begin{aligned} 1 + a + b + c &= 0 \\ 16 + 4a + 2b + c &= 0 \\ 81 + 9a + 3b + c &= 0\,. \end{aligned}$$

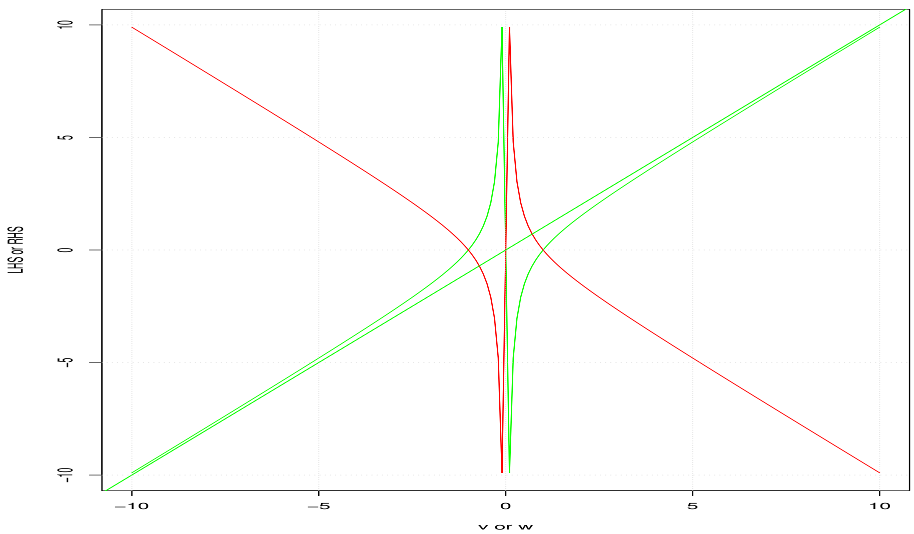

Figure 6: Plots of the function $v - \frac{1}{v}$ and $-w - \frac{1}{w}$.

This gives three equations and three unknowns. If we solve these three equations we get $a = -25$, $b = 60$, and $c = -36$ thus $a + c = -61$.

Problem 33 (counting the number of solutions)

Write the given expression as

$$\frac{x-a}{b} - \frac{b}{x-a} = -\left(\frac{x-b}{a} - \frac{a}{x-b}\right) .$$

Next let $v \equiv \frac{x-a}{b}$ and $w \equiv \frac{x-b}{a}$ then the above is given by

$$v - \frac{1}{v} = -\left(w - \frac{1}{w}\right) .$$

If we plot the left-hand-side of this expression (in green) and the right-hand-side of this expression (in red) we get the plot given in Figure 6. Note that the green and the red curve intersect in *two* places (see below for another discussion on this). Two places where the plots intersect appear to be when the two curves cross the x-axis. This can happen when $v - \frac{1}{v} = 0$ or $v^2 - 1 = 0$ or $v = \pm 1$. If $v = +1$ we have

$$\frac{x-a}{b} = 1 \quad \text{or} \quad x = a + b ,$$

while if $v = -1$ we have

$$\frac{x-a}{b} = -1 \quad \text{or} \quad x = a - b ,$$

In either case we must also have $-\left(w - \frac{1}{w}\right) = 0$ or $w = \pm 1$.

When $v = +1$ then considering the two possible values for w we get that if $w = +1$ we would have

$$\frac{x-b}{a} = 1 \quad \text{or} \quad x = a + b\,,$$

which is consistent. If $w = -1$ we would have

$$\frac{x-b}{a} = -1 \quad \text{or} \quad x = a - b\,,$$

which is not consistent. Thus we have found one solution for $x = a + b$.

If $v = -1$ then in the same logic we would only have the solution corresponding to $w = -1$ which gave $x = a - b$.

Note: the method discussed here gives two solutions but I don't see any way to use it to get the *third* solution (corresponding to $x = 0$). This third solution can be seen to be true by evaluating both sides at $x = 0$. If anyone sees a way to use the above to get the solution $x = 0$ please contact me.

Problem 34 (a rolling wheel)

Let T be the period of the wheel in seconds (the amount of time it takes for the wheel to complete one cycle). Then the wheel lays down an amount of "land" given by the circumference C in time T. Thus $\frac{C}{T} = r$. As we know that $C = 11$ feet we can write this as

$$\frac{11}{T} = r \quad \text{or} \quad 11 = rT\,. \tag{165}$$

Here r is the given velocity of the wheel. If we measure T in seconds then with the 11 in units of feet the r above is in units of feet per second. We are then told that if T shrinks by $\frac{1}{4}$ of a second then r will increase by 5 miles per hour. Lets find the conversion factor from miles per hour to feet per second. We have

$$\frac{1\,\text{mile}}{1\,\text{hour}} = \frac{1\,\text{mile}}{1\,\text{hour}} \times \frac{5280\,\text{feet}}{1\,\text{mile}} \times \frac{1\,\text{hour}}{3600\,\text{seconds}} = \frac{5280\,\text{feet}}{3600\,\text{seconds}} = \frac{22\,\text{feet}}{15\,\text{seconds}}\,. \tag{166}$$

Thus when T shrinks by $\frac{1}{4}$ we have that r will increase by $5\left(\frac{22}{15}\right) = \frac{22}{3}$. We can now write the given relationship between T and r when the values of these variables change as

$$\frac{11}{T - \frac{1}{4}} = r + \frac{22}{3}\,.$$

The above is the same as

$$11 = Tr - \frac{1}{4}r + \frac{22}{3}T - \frac{11}{6}\,.$$

Using Equation 165 we can cancel 11 and Tr to get

$$-\frac{1}{4}r + \frac{22}{3}T - \frac{11}{6} = 0\,.$$

Again using Equation 165 as $T = \frac{11}{r}$ we get

$$-\frac{1}{4}r + \frac{22(11)}{3r} - \frac{11}{6} = 0\,.$$

The above is equivalent to

$$\frac{3}{4}r^2 + \frac{11}{2}r - 2 \cdot 11^2 = 0\,.$$

Solving for r we get

$$r = \frac{-\frac{11}{2} \pm \sqrt{\frac{11^2}{4} - 4\left(\frac{3}{4}\right)(2 \cdot 11^2)}}{2\left(\frac{3}{4}\right)} = \frac{-\frac{11}{2} \pm 5\left(\frac{11}{2}\right)}{\frac{3}{2}}$$

As we expect r to be positive we must take the positive sign above to get $r = \frac{44}{3}$. This is in units of feet per second. To convert to miles per hour we multiply by $\frac{15}{22}$ to get $\frac{44}{3} \cdot \frac{15}{22} = 10$.

Problem 35 (inequalities in a triangle)

Note that for each of the terms in the sum s_1 there is a side of the triangle that is larger than in. Thus we have $s_1 < s_2$. Now considering $s_2 = AB + BC + CA$ not that

$$\begin{aligned} AB &< AO + OB \\ BC &< BO + OC \\ AC &< AO + OC\,. \end{aligned}$$

Adding these three gives

$$s_2 < 2AO + 2OC + 2OB = 2s_1\,.$$

Thus we have $s_1 > \frac{1}{2}s_2$.

Problem 36 (evaluating a sum)

Note that if we let $x = 1$ in the expression given we get

$$(1 + 1 + 1)^n = a_0 + a_1 + a_2 + a_3 + \cdots + a_{2n-1} + a_{2n}\,.$$

If we let $x = -1$ in the expression given we get

$$(1 - 1 + 1)^n = a_0 - a_1 + a_2 - a_3 + \cdots - a_{2n-1} + a_{2n}\,.$$

If we then add these two expressions we get

$$3^n + 1 = 2(a_0 + a_2 + a_4 + \cdots + a_{2n-2} + a_{2n})\,.$$

Solving for the sum requested gives $\frac{1}{2}(1 + 3^n)$.

Problem 37 (doing a job)

Let v_a, v_b, and v_c be the rates at which each member can do the job on their own. Then the statements given mean that

$$\frac{1}{v_a + v_b + v_c} = \frac{1}{v_a} - 6 \tag{167}$$
$$\frac{1}{v_a + v_b + v_c} = \frac{1}{v_b} - 1 \tag{168}$$
$$\frac{1}{v_a + v_b + v_c} = \frac{1}{2v_g}\,. \tag{169}$$

This is three equations and three unknowns. Then let $h = \frac{1}{v_a+v_b}$ be what we want to compute. From Equation 169 by taking the inverse of both sides we have

$$v_a + v_b + v_g = 2v_g\,.$$

Thus $v_g = v_a + v_b$. Since we now know v_g in terms of v_a and v_b we can consider the other two equations i.e. Equations 167 and 168 without the variable v_g. This means we need to solve the following two equations

$$\frac{1}{v_a} - 6 = \frac{1}{v_b} - 1 = \frac{1}{2(v_a + v_b)}\,.$$

If we multiply the first equation by $v_a v_b$ we get

$$v_b - 6v_a v_b = v_a - v_a v_b\,.$$

If we solve this for v_b we get

$$v_b = \frac{v_a}{1 - 5v_a}\,. \tag{170}$$

We then want to put this into $\frac{1}{v_a} - 6 = \frac{1}{2(v_a+v_b)}$, which gives

$$\frac{1}{v_a} - 6 = \frac{1}{2v_a}\left(\frac{1 - 5v_a}{2 - 5v_a}\right)\,.$$

The above is equivalent to

$$60v_a^2 - 29v_a + 3 = 0\,,$$

and has solutions given by

$$v_a = \frac{29 \pm \sqrt{29^2 - 4(60)(3)}}{2(60)} = \frac{29 \pm 11}{120}\,.$$

These two solutions are $v_a = \frac{3}{20}$ and $v_a = \frac{1}{3}$. Given these two value of v_a and Equation 170 we get two values for v_b of $v_b = \frac{3}{5}$ and $v_b = -\frac{1}{2}$. Since we expect $v_b > 0$ we can only consider the first solutions or $(v_a, v_b) = (\frac{3}{20}, \frac{3}{5})$ which gives $v_a + v_b = \frac{3}{4}$. Thus $h = \frac{1}{v_a+v_b} = \frac{4}{3}$.

Problem 39 (repeating decimals in different bases)

For this problem it is useful to think about how one transforms a base 10 repeating decimal into a proper fraction. Given the repeating decimal representation of a number

$$f = 0.37373737\ldots\,,$$

we would multiply by 100 (the base squared) to get

$$100f = 37 + 0.37373737 = 37 + f\,.$$

We can then solve for f to get the fractional representation

$$f = \frac{37}{99}\,.$$

If the number is given in a different base (rather than 10) we must multiply by some power of that base. For example, given a repeating fraction with two repeating digits (in base B) as

$$f = 0.\text{mnmnmnmn}\ldots,$$

we would multiply by B^2 to get

$$B^2 f = mB + n + 0.\text{mnmnmnmn}\ldots = mB + n + f\,.$$

We can now solve for f to get

$$f = \frac{mB+n}{B^2-1}\,. \tag{171}$$

You can check that the above expression works for the base 10 example given earlier where we had $m = 3$, $n = 7$, and $B = 10$. For the fractions given in the problem (with their unknown bases) by using the above we would get

$$F_1 = \frac{3R_1+7}{R_1^2-1} = \frac{2R_2+5}{R_2^2-1}\,, \tag{172}$$

and

$$F_2 = \frac{7R_1+3}{R_1^2-1} = \frac{5R_2+2}{R_2^2-1}\,.$$

Note that these are two equations for the two unknown bases R_1 and R_2. If we take their ratio we get

$$\frac{3R_1+7}{7R_1+3} = \frac{2R_2+5}{5R_2+2}\,.$$

Solving the above for R_2 in terms of R_1 we get

$$R_2 = \frac{1+29R_1}{29+R_1}\,. \tag{173}$$

If we put this expression into Equation 172 (and simplify a bit) we get

$$\frac{3R_1+7}{R_1^2-1} = \frac{(29+R_1)(7+3R_1)}{40(R_1^2-1)}\,.$$

This gives a quadratic equation for R_1 of

$$3R_1^2 - 26R_1 - 77 = 0\,.$$

Solving this for R_1 we get

$$R_1 = \frac{26 \pm \sqrt{26^2 + 4(3)(77)}}{2(3)} = \frac{26 \pm 40}{6}\,.$$

In order that R_1 be positive we must take the positive sign above and get $R_1 = 11$. Then for R_2 using Equation 173 we get $R_2 = 8$. Their sum is then 19.

Problem 40 (a relationship between x and y)

Let F be the point on the line AB when we drop a perpendicular from E and let the $\angle EAF$ be denoted θ. Since $\triangle EAF$ is a right triangle (and from the problem statement) we have

$$AE^2 = x^2 + y^2 = DC^2\,. \tag{174}$$

Now DC is the distance between a point on the circle and the line BC. Introduced a Cartesian coordinate system $(\tilde{x}, \tilde{y})$ with A at the origin $(0, 0)$, O at the point $(a, 0)$, B at the point $(2a, 0)$ etc. In this system the line ADC is given by

$$\tilde{y} = \tan(\theta)\tilde{x}\,.$$

This line will intersect the tangent BC when $\tilde{x} = 2a$ so $\tilde{y} = 2a\tan(\theta)$. Thus the point C is located at $(2a, 2a\tan(\theta))$. Lets now find the coordinate of the point D which is the intersection of the two curves

$$\begin{aligned}(\tilde{x} - a)^2 + \tilde{y}^2 = a^2 \quad &\text{and} \\ \tilde{y} = \tan(\theta)\tilde{x}\,.&\end{aligned}$$

Putting the second equation into the first we get

$$(\tilde{x} - a)^2 + \tan^2(\theta)\tilde{x}^2 = a^2\,.$$

Expanding and canceling we get

$$\tilde{x}^2 - 2a\tilde{x} + \tan^2(\theta)\tilde{x}^2 = 0\,.$$

One solution to this is $\tilde{x} = 0$ and another solution must be

$$\tilde{x} = \frac{2a}{1 + \tan^2(\theta)} = \frac{2a}{\sec^2(\theta)} = 2a\cos^2(\theta)\,.$$

Thus the point D is located at $(2a\cos^2(\theta), 2a\tan(\theta)\cos^2(\theta))$. Now that we know the Cartesian locations of the points C and D we can compute the distance (squared) DC^2 or

$$\begin{aligned}DC^2 &= (D_x - C_x)^2 + (D_y - C_y)^2 \\ &= (2a\cos^2(\theta) - 2a)^2 + (2a\tan(\theta)\cos^2(\theta) - 2a\tan(\theta))^2 \\ &= 4a^2\sin^4(\theta) + 4a^2\tan^2(\theta)\sin^4(\theta) \\ &= 4a^2\sin^4(\theta)(1 + \tan^2(\theta)) \\ &= 4a^2\sin^4(\theta)\sec^2(\theta) = 4a^2\tan^2(\theta)\sin^2(\theta)\,.\end{aligned}$$

Using this in Equation 174 we get

$$x^2 + y^2 = 4a^2 \tan^2(\theta) \sin^2(\theta) = 4a^2 \left(\frac{y^2}{x^2} \right) \frac{y^2}{x^2 + y^2} \,.$$

The above is equivalent to

$$(x^2 + y^2)^2 = 4a^2 \frac{y^4}{x^2} \,.$$

Taking the square root of both sides and using the fact that we know that everything is positive we get

$$x^2 + y^2 = 2a \frac{y^2}{x} \,.$$

When we solve for y^2 in the above we get

$$y^2 = \frac{x^3}{2a - x} \,.$$

The 1967 Examination

Problem 1 (adding two numbers)

When we add the two digits in the tens place we would get $a + 2 = b$ assuming that $a + 2$ was not greater than 10 in which case we would have to carry over a 1 into the hundreds place. Since the two digits in the hundreds place in the summands (2 and 3) do add to the digit in the sum (here 5) there was no carry involved and we know that

$$a + 2 = b \,, \tag{175}$$

where $b \leq 9$. Now if the sum or 5b9 is divisible by 9 that means that the sum of the digits in that number must be divisible by 9 or

$$5 + b + 9 = 0 \,(\mathrm{mod}\, 9) \,.$$

This is the same as $14 + b = 0 \,(\mathrm{mod}\, 9)$ or $5 + b = 0 \,(\mathrm{mod}\, 9)$. For this to happen means that $b = 4$. Using Equation 175 we then have that $a = 2$. Thus $a + b = 6$.

Problem 2 (simplify this)

We have the given expression equal to

$$\frac{x^2y^2 + x^2 + y^2 + 1 + x^2y^2 - x^2 - y^2 + 1}{xy} = \frac{2x^2y^2 + 2}{xy} = 2xy + \frac{2}{xy} \,.$$

Problem 3 (the area of a square)

From the given description, lets compute the distance from any side of the triangle to the center of the inscribed circle. If we draw a segment from a vertex of the triangle to the center of the inscribed circle the angle of this segment to either of its two adjacent sides is $\frac{\pi}{6}$ or $\frac{60}{2} = 30$ degrees. Since this segment is the hypotenuse of a right triangle with a horizontal leg length of $\frac{s}{2}$ we know that the vertical distance from a side of the triangle to the center or the circle is given by

$$y = \frac{s}{2} \tan\left(\frac{\pi}{6}\right) = \frac{s}{2}\left(\frac{\sin(\pi/6)}{\cos(\pi/6)}\right) = \frac{s}{2}\left(\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) = \frac{s}{2\sqrt{3}} \,.$$

This is also the length of $\frac{1}{2}$ of the diagonal of the square. Thus the length of the diagonal of the square is given by $\frac{s}{\sqrt{3}}$. If we denote the length of a side of the internal square as a then a must satisfy

$$a^2 + a^2 = \frac{s^2}{3} \,.$$

Thus $a = \frac{s}{\sqrt{6}}$ and our square has an area given by $\frac{s^2}{6}$.

Problem 4 (simplifying some logs)

From the information given we can compute that $a = x^p$, $b = x^q$, $c = x^r$. Thus the expression we want to compute (in terms of x) looks like

$$\frac{b^2}{ac} = \frac{x^{2q}}{x^p x^r} = x^{2q-p-r}\,.$$

Thus $y = 2q - p - r$.

Problem 5 (a triangle around a circle)

Note that the three sides of the triangle are tangent to the circle at the points where they touch the circle. Now draw three altitudes from the center of the circle to the three edges of the triangle. Note that each of these altitudes has a length of r. Let the sides of the triangle be a, b, and c. Then the total area K of the triangle is given by the sum of the three triangles the altitudes above are drawn in. The bases of these three triangles are a, b, and c. Thus we have

$$K = \frac{1}{2}(ar + br + cr) = \frac{1}{2}r(a + b + c) = \frac{Pr}{2}\,.$$

From this we see that $\frac{P}{K} = \frac{2}{r}$.

Problem 6 (a difference equation)

We have

$$f(x + 1) - f(x) = 44^x - 4^x = 34^x = 3f(x)\,.$$

Problem 8 (an acid solution)

The m ounces of a m percent solution of acid has

$$m\left(\frac{m}{100}\right) = \frac{m^2}{100}\,,$$

amount of pure acid (not water). When we add x ounces of water we get a $(m - 10)$ percent solution. In this new solution the amount of pure acid stays the same. The amount of pure acid in this solution is

$$(m + x)\left(\frac{m - 10}{100}\right)\,.$$

Setting these two expressions equal to each other gives

$$\frac{m^2}{100} = (m + x)\left(\frac{m - 10}{100}\right)\,.$$

Solving this for x gives

$$x = \frac{10m}{m - 10}\,.$$

Problem 9 (values for the area K)

If we let s, a, and l be the short side, the altitude, and the long side of the trapezoid respectively then since we know these three numbers are three terms in an arithmetic sequence we have from Equation 17 that we can write s, a, and l in terms of (the unknown a_1 and d) as

$$\begin{aligned} s &= a_1 \\ a &= a_1 + d \\ l &= a_1 + 2d \,. \end{aligned}$$

Then the area K of this trapezoid is given by

$$K = \frac{1}{2}a(s+l) = \frac{1}{2}(a_1+d)(2a_1+2d) = (s_0+h)^2 \,.$$

As this expression can be an integer, rational, or irrational depending on the values of s_0 and h the value of K can be any value also.

Problem 10 (partial fractions in disguise)

Write the given expression with $y = 10^x$ so that we want to write

$$\frac{2y+3}{(y-1)(y+2)} = \frac{a}{y-1} + \frac{b}{y+2} \,.$$

This is equivalent to

$$2y+3 = a(y+2) + b(y-1) \,.$$

If we let $y = 1$ we get $a = \frac{5}{3}$. If we let $y = -2$ we get $b = \frac{1}{3}$. Thus $a - b = \frac{4}{3}$.

Problem 11 (dimensions of a rectangle)

Let h and w be the height and width of the given rectangle. The perimeter P is then given by $P = 2h + 2w$. The diagonal d of this rectangle must satisfy $d^2 = h^2 + w^2$. We want to minimize d^2 given that P is a constant. Solving for h we have $h = \frac{P}{2} - w$. Using this expression for h we get that d^2 as a function of w is given by

$$d^2(w) = \left(\frac{P}{2} - w\right)^2 + w^2 = \frac{P^2}{4} - Pw + 2w^2 \,.$$

Taking the w derivative of the above expression and setting it equal to zero gives

$$-P + 4w = 0 \quad \text{or} \quad w = \frac{P}{4} \,.$$

The second derivative of the expression above is $\frac{1}{2} > 0$ and so the value of w above is a minimum. Since $P = 20$ we have $w = 5$ and $h = 10 - 5 = 5$. Thus

$$d^2 = 5^2 + 5^2 = 50 \quad \text{so} \quad d = \sqrt{50} \,.$$

Problem 12 (a convex region)

If we assume $m < 0$ (all of the suggested answers have $m < 0$) then drawing the given region in the x-y plane shows that the convex region is a trapezoid. The height of this trapezoid can be measured along the x-axis and is $h = 4 - 1 = 3$. The longer base has a length equal to the height of the point on the the line above $x = 1$ or $y = 4m + 4$. The shorter base has a length equal to the height of the point on the the line above $x = 4$ or $y = m + 4$. The area of the trapezoid is then given by

$$A = \frac{1}{2}h(b_1 + b_2) = \frac{3}{2}(m + 4 + 4m + 4)\,.$$

Setting this equal to 7 gives $m = -\frac{2}{3}$.

Problem 13 (the number of triangles)

Since we are told the length BC (with a value of a) and the angle $\angle ABC$ together these two pieces of information allow us to construct the hypotenuse of a right triangle with an interior angle of $\angle ABC$. Dropping the perpendicular from the end point C to the line segment AB would complete the right triangle. If this dropped perpendicular is exactly the length of h_c then there are a multiple (an infinite number) of points where we can place the point A. If this dropped perpendicular is not equal to h_c then there are no such triangles.

Problem 14 (inverting a function)

In the expression $y = f(x) = \frac{x}{1-x}$ when we solve for x we get

$$x = \frac{y}{1 + y}\,.$$

This is equal to $-f(-y)$.

Problem 16 (multiplication in different bases)

The given multiplication is

$$(12)(15)(16) = (3146)\,.$$

By multiplying out we can write the left-hand-side as

$$(b + 2)(b + 5)(b + 6) = (b + 2)(b^2 + 11b + 30) = b^3 + 13b^2 + 52b + 60\,.$$

While the right-hand-side is given by

$$3b^3 + b^2 + 4b + 6\,.$$

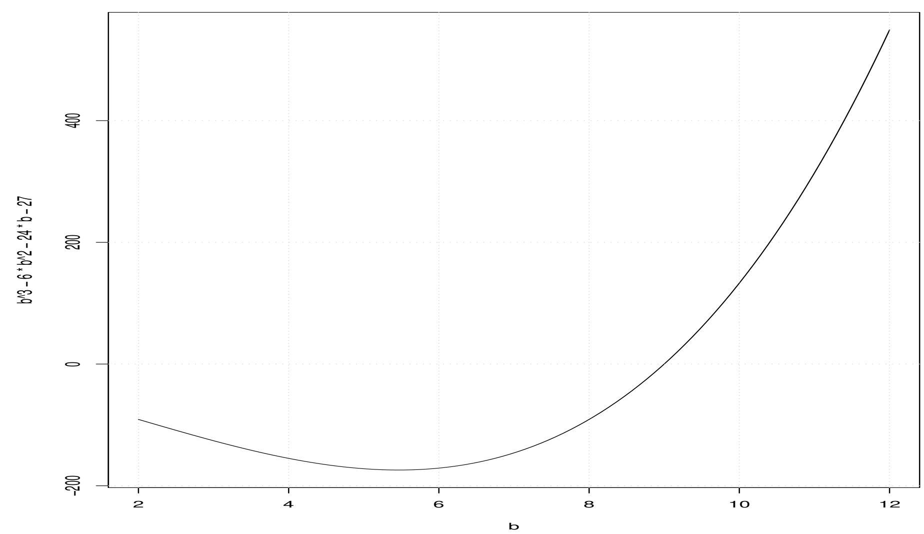

Figure 7: A plot of the function $b^3 - 6b^2 - 24b - 27$ vs. b.

If we set these two expressions equal to each other and then simplify we get the following polynomial equation

$$b^3 - 6b^2 - 24b - 27 = 0\,.$$

In the R function `prob_16_1967.R` when we plot this equation as a function of b we get the plot shown in Figure 7. From that we see that for $b \approx 9$ this polynomial is near zero. If we evaluate this polynomial at $b = 9$ we see that it is *exactly* zero. Thus Given this value of b we can then compute s in base 10 where we find $s = 40$. Since this can be written as

$$40 = 4(9) + 4\,,$$

or in base b this number is 44.

Problem 17

The two roots of this quadratic is given by

$$r_{1,2} = \frac{-p \pm \sqrt{p^2 - 4(8)}}{2} = \frac{-p \pm \sqrt{p^2 - 32}}{2}\,.$$

Since we know that $r_{1,2}$ are both real we must have that $p^2 - 32 > 0$ or $|p| > 4\sqrt{2}$. Given this fact by adding the two solutions above we see that

$$r_1 + r_2 = -p\,.$$

Thus we have that

$$|r_1 + r_2| = |p| > 4\sqrt{2}\,.$$

Problem 18 (bounding an expression)

Note that
$$x^2 - 5x + 6 < 0 \quad \text{so} \quad (x-2)(x-3) < 0 \,,$$
which means that $2 < x < 3$. Since $x^2 + 5x + 6 = (x+2)(x+3)$ and the bounds on x we have that
$$\begin{aligned} 4 &< x + 2 < 5 \\ 5 &< x + 3 < 6 \,, \end{aligned}$$
thus the product of these two factors is bounded as $20 < (x+2)(x+3) < 30$.

Problem 19 (rectangle areas)

Let l and w be the length and width of the original rectangle. Then from the problem we are told that
$$\left(l + \frac{5}{2}\right)\left(w - \frac{2}{3}\right) = lw = \left(l - \frac{5}{2}\right)\left(w + \frac{4}{3}\right) \,.$$
Expanding everything we get
$$lw - \frac{2}{3}l + \frac{5}{2}w - \frac{5}{3} = lw = lw + \frac{4}{3}l - \frac{5}{2}w - \frac{10}{3} \,.$$
The first of these two equations gives
$$-4l + 15w - 10 = 0 \,,$$
while the second of these two equations gives
$$8l - 15w - 20 = 0 \,.$$
These are two equations and two unknowns. As a system this is
$$\begin{bmatrix} -4 & 15 \\ 8 & 15 \end{bmatrix} \begin{bmatrix} l \\ w \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \end{bmatrix} \,.$$
Using Cramer's rule we find the two solutions for l and w given by
$$l = \frac{\begin{vmatrix} 10 & 15 \\ 20 & -15 \end{vmatrix}}{\begin{vmatrix} -4 & 15 \\ 8 & -15 \end{vmatrix}} = \frac{15(10)\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix}}{15(4)\begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix}} = \frac{5(-1-2)}{2(+1-2)} = \frac{15}{2}$$
$$w = \frac{\begin{vmatrix} -4 & 10 \\ 8 & 20 \end{vmatrix}}{\begin{vmatrix} -4 & 15 \\ 8 & -15 \end{vmatrix}} = \frac{4(10)\begin{vmatrix} -1 & 1 \\ 2 & 2 \end{vmatrix}}{15(4)\begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix}} = \frac{10(-2-2)}{15(+1-2)} = \frac{8}{3} \,.$$
Now that we know l and w the original area is given by
$$lw = \left(\frac{15}{2}\right)\frac{8}{3} = 20 \,.$$

Problem 20 (nesting squares)

Let the ith square have a side length of S_i. The the radius of the inscribed circle is $\frac{S_i}{2}$. The square that is inscribed inside that circle will have a diagonal equal to $2\left(\frac{S_i}{2}\right) = S_i$. Thus its side S_{i+1} must be given by

$$S_{i+1}^2 + S_{i+1}^2 = S_i{}^2 \quad \text{so} \quad S_{i+1} = \frac{S_i}{\sqrt{2}}\,.$$

Thus it looks like the sides of the squares satisfy the above recursion relationship. If we start this relationship with $S_0 = m$ we would have that

$$S_i = \frac{m}{2^{\frac{i}{2}}} \quad \text{for} \quad i \geq 0\,.$$

The radius of the circle inscribed inside this square has a radius

$$R_i = \frac{S_i}{2} = \frac{m}{2^{\frac{i}{2}+1}}\,.$$

We want to evaluate

$$\sum_{i=0}^{\infty} \pi R_i^2 = \frac{\pi m^2}{4} \sum_{i=0}^{\infty} \frac{1}{2^i} = \frac{\pi m^2}{4}\left(\frac{1}{1-\frac{1}{2}}\right) = \frac{\pi m^2}{2}\,.$$

Problem 21 (some triangles)

If we recall the angle bisector theorem (Page 6) the angle bisector divides the opposite segment into sides that are proportional to the adjacent sides. Thus using the notation in the problem we would have

$$\frac{A_1B}{A_1C} = \frac{5}{3}\,.$$

Since the right triangle is a "345" triangle we know that $BC = 4$ and the above becomes

$$\frac{A_1B}{4 - A_1B} = \frac{5}{3}\,.$$

If we solve the above for A_1B we find $A_1B = \frac{5}{2}$. With this we know that

$$A_1C = 4 - A_1B = 4 - \frac{5}{2} = \frac{3}{2}\,.$$

We now have the lengths of two of the three sides in the triangle PRQ. Using the Pythagorean theorem we would have that $PQ^2 = PR^2 + RQ^2$ or

$$A_1B^2 = A_1C^2 + RQ^2\,.$$

Putting in what we know we get

$$\frac{25}{4} = \frac{9}{4} + RQ^2 \quad \text{so} \quad RQ = 2\,.$$

Again using the fact that in a triangle the angle bisector divides the impending side into pieces that are proportional to the adjacent sides we have

$$\frac{P_1R}{QP_1} = \frac{A_1C}{A_1B} = \frac{3/2}{5/2} = \frac{3}{5}.$$

Since $QP_1 = QR - P_1R = 2 - P_1R$ the above is equivalent to

$$\frac{P_1R}{2 - P_1R} = \frac{3}{5}.$$

We can solve for P_1R in the above and find $P_1R = \frac{3}{4}$. Using this value we can finally compute the length of PP_1 since it is a hypotenuse to the triangle PRP_1. We have

$$PP_1{}^2 = PR^2 + P_1R^2 = A_1C^2 + P_1R^2 = \frac{9}{4} + \frac{9}{16} = \frac{45}{16},$$

thus $PP_1 = \frac{\sqrt{45}}{4} = \frac{3\sqrt{5}}{4}$.

Problem 22 (the remainder when dividing by DD')

From what we are told we have that

$$\begin{aligned} P &= QD + R \quad \text{with} \quad R \leq D - 1 \\ Q &= Q'D' + R' \quad \text{with} \quad R' \leq D' - 1. \end{aligned}$$

From the expression for Q we get

$$P = D(Q'D' + R') + R = Q'DD' + DR' + R.$$

Thus the remainder when we divide by DD' is $DR' + R$. This is assuming that we can show that $DR' + R$ is smaller than or equal to DD'. From the two conditionals on the division above we see that the proposed remainder is bounded above by

$$R + R'D \leq D - 1 + D'D - D = DD' - 1,$$

showing the required inequality.

Problem 23 (a limit)

Write the given expression as

$$\log_3\left(\frac{6x-5}{2x+1}\right).$$

Now as $x \to \infty$ note that $\frac{6x-5}{2x+1} \to 3$. Thus we have

$$\log_3\left(\frac{6x-5}{2x+1}\right) \to \log_3(3) = 1.$$

Problem 24 (counting solutions)

We want to count the number of positive integer solutions to the linear equation $3x + 5y = 501$. Note that if $x = 7$ then we must find a value of y such that $5y = 480$ or $y = 96$. Based on this one solution lets look for more solutions for x and y that we can write as

$$x = 7 + \xi$$
$$y = 96 + \eta ,$$

for some values of ξ and η. For the above representation we are looking for values of ξ and η that satisfy

$$3x + 5y = 21 + 3\xi + 5(96) + 5\eta = 501 ,$$

or

$$3\xi + 5\eta = 0 . \quad (176)$$

Thus we need to find integer values of ξ and η such that the above holds and $x \geq 1$ and $y \geq 1$. In terms of ξ and η these are

$$7 + \xi \geq 1 \quad \text{or} \quad \xi \geq -6 \quad (177)$$
$$96 + \eta \geq 1 \quad \text{or} \quad \eta \geq -95 . \quad (178)$$

From Equation 176 we have that $\eta = -\frac{3}{5}\xi$. When we put this into Equation 178 we get

$$-\frac{3}{5}\xi \geq -95 \quad \text{or} \quad \xi \leq \frac{475}{3} = 158\,\frac{1}{3} .$$

Thus we need ξ and η to be integers with ξ in the range

$$-6 \leq \xi \leq 158\,\frac{1}{3} ,$$

and ξ a multiple of 5 so that when we compute $\eta = -\frac{3}{5}\xi$ we get an integer. There are

$$158 - (-6) + 1 = 165 ,$$

integers (not necessarily divisible by five) that we could consider. The number of these that are divisible by five are

$$\left\lfloor \frac{165}{5} \right\rfloor = 30 + \left\lfloor \frac{15}{5} \right\rfloor = 33 .$$

Thus this is the number of solutions to the given initial equation.

Problem 25 (can we divide?)

Write the unknown p as $p = 2k + 1$ for $k \geq 1$. Then we can evaluate the expression

$$(p - 1)^{\frac{1}{2}(p-1)} = (2k)^{\frac{1}{2}(2k)} = (2k)^k = 2^k k^k .$$

Note that this is an *even* number. Given this result we can eliminate some of the possible answers. Since the above is even and p is odd, the answer (C) cannot be true. Now as $(p-1)^{\frac{1}{2}(p-1)}$ is even we have that $(p-1)^{\frac{1}{2}(p-1)} - 1$ is given by

$$2^k k^k - 1\,,$$

and is odd. As $p-1$ is even (E) cannot be true. Now $p-2$ is $2k-1$. Lets see if $p-2$ divides $(p-1)^{\frac{1}{2}(p-1)} - 1$. That means we need to consider the fraction

$$\frac{(2k)^k - 1}{2k-1}\,.$$

By recalling the expression for a geometric sum given by Equation 20 we see that the above fraction is given by

$$\sum_{n=0}^{k-1} (2k)^n\,.$$

Thus this division is possible and (A) is correct.

Problem 26 (bounding $\log_2(10)$)

Note that from the information given we have

$$10^3 = 1000 < 1024 = 2^{10}\,.$$

If we take the $\log_{10}$ of both sides we get

$$3 < 10\log_{10}(2) \quad \text{so} \quad \log_{10}(2) > \frac{3}{10}\,.$$

We also have

$$10^4 = 10000 > 8192 = 2^{13}\,.$$

If we take the $\log_{10}$ of both sides we get

$$4 > 13\log_{10}(2) \quad \text{so} \quad \log_{10}(2) < \frac{4}{13}\,.$$

Combining these two we have

$$\frac{3}{10} < \log_{10}(2) < \frac{4}{13}\,.$$

Problem 27 (burning candles)

Let l_1 be the length of the first candle as a function of time as it burns. Assume its initial length is l_0, and after 3 hours its length is 0. Then assuming $l_1(t) = A + Bt$ we must have $l_1(0) = l_0$ and $l_1(3) = 0$. This means that

$$B = l_0 \quad \text{and} \quad 3A + l_0 = 0 \quad \text{so} \quad A = -\frac{l_0}{3}\,.$$

Thus we have that $l_1(t)$ is given by

$$l_1(t) = l_0 \left(1 - \frac{t}{3}\right) .$$

In the same way the second candle has its length as a function of time given by

$$l_1(t) = l_0 \left(1 - \frac{t}{4}\right) .$$

Note that the first candle burns faster than the second since it burns to zero in less time (three hours vs. four hours). If we light them at the same time, the time t for the first candle become one half the length of the second candle will happen when

$$l_1(t) = \frac{1}{2} l_2(t) .$$

When we solve this for t we get $t = \frac{12}{5}$ in units of hours. This is two hours and 12 minutes. Since we want this to happen at 4 PM we should subtract this amount of time from then. This gives $\frac{20}{5} - \frac{12}{8} = \frac{8}{5}$ of an hour from 12 or 1:36 PM.

Problem 30 (selling radios)

To buy n radios at d dollars means that we are paying $\frac{d}{n}$ dollars per radio. Giving two radios to the bazaar at one-half their cost means that we suffer a loss of $\frac{1}{2}\left(\frac{d}{n}\right)$ on each radio giving a total loss of $\frac{d}{n}$ on both radios. The rest $n-2$ radios are sold at a profit of \$8 each for a total profit of $8(n-2)$. The overall profit is then

$$-\frac{d}{n} + 8(n-2) .$$

which we are told equals 72. If we set the above equal to 72 and solve for d we find

$$d = 8n^2 - 88n = 8n(n-11) .$$

Now if $n = 11$ then $d = 0$ which is not possible. For $n \geq 12$ we have d a positive integer. Thus the smallest value for n is 12.

Problem 31

Given the value of a then we would have $b = a + 1$ and $c = ab = a(a+1)$. Using these lets form an expression for D. We have

$$\begin{aligned} D &= a^2 + (a+1)^2 + a^2(a+1)^2 = a^2 + a^2 + 2a + 1 + a^2(a^2 + 2a + 1) \\ &= 2a^2 + 2a + 1 + a^4 + 2a^3 + a^2 = a^4 + 2a^3 + 3a^2 + 2a + 1 . \end{aligned}$$

Since we will be taking the square root of this expression lets look and see if it factors into a square. Consider the most general square of a quadratic that would have a leading coefficient of a^4. We have

$$\begin{aligned}(a^2 + Aa + B)^2 &= (a^2 + Aa + B)(a^2 + Aa + B) \\ &= a^4 + Aa^3 + Ba^2 + Aa^3 + A^2a^2 + ABa + Ba^2 + ABa + B^2 \\ &= a^4 + 2Aa^3 + (2B + A^2)a^2 + 2ABa + B^2\,.\end{aligned}$$

To make this match D we will need to take $2A = 2$ so that $A = 1$ and

$$2B + A^2 = 3 \quad \text{so} \quad 2B = 2 \quad \text{so} \quad B = 1\,.$$

Thus using these values we have shown that

$$D = (a^2 + a + 1)^2\,.$$

Note that if we can show that $a^2 + a + 1 > 0$ for all a then $\sqrt{D} = a^2 + a + 1$. We can show that $a^2 + a + 1 > 0$ by finding the minimum of this expression and showing that it its value is larger than 0. To find the minimum we take the first derivative of $a^2 + a + 1$ with respect to a to get

$$2a + 1 = 0 \quad \text{so} \quad a = -\frac{1}{2}\,.$$

Since the second derivative of $a^2 + a + 1$ is the number $2 > 0$ the value of $a = \frac{1}{2}$ gives a minimum of $a^2 + a + 1$. The value of $a^2 + a + 1$ at this minimum is $\frac{3}{4} > 0$. Thus we have shown that $\sqrt{D} = a^2 + a + 1$. From this we note that a is an integer then $\sqrt{D}$ will not be a rational or irrational number. This observation eliminates some choices. Lets try a few values of a and see if we can observe a pattern. We have

$$\begin{aligned} a = 0 &\quad \text{gives} \quad \sqrt{D} = 1 \\ a = 1 &\quad \text{gives} \quad \sqrt{D} = 3 \\ a = 2 &\quad \text{gives} \quad \sqrt{D} = 7\,. \end{aligned}$$

Thus it looks like this is always an odd integer so we will try to show that. Notice when a is an integer then $a(a + 1)$ is the product of an odd and an even integer and thus will be an even number. Since $a(a + 1) = a^2 + a$ is always even we must have that $a^2 + a + 1$ is always an odd number.

Problem 32

In the given problem we are told the lengths of the distance from each vertex to the point O which is the intersection of the diagonals of the quadrilateral $ABCD$ *and* the length of one side namely AB. Since we have so many lengths we will use the law of cosigns to determine the angles of the segments that meet at the point O. For example, since we know the length of the segment AB we will use the law of cosigns on the triangle $\triangle ABO$ to get

$$AB^2 = AO^2 + BO^2 - 2AOBO\cos(\angle AOB)\,.$$

When we put in the numbers for this problem this gives

$$36 = 64 + 16 - 2(8)(4)\cos(\angle AOB)\,.$$

Solving we find

$$\cos(\angle AOB) = \frac{11}{16}\,.$$

Since we want to determine the length AD we can write the law of cosigns for the triangle $\triangle AOD$ where we have

$$AD^2 = AO^2 + DO^2 - 2AODO\cos(\angle AOD)\,.$$

When we put in the numbers given in the problem we get

$$AD^2 = 64 + 36 - 2(8)(6)\cos(\angle AOD)\,. \tag{179}$$

To evaluate this we need to know $\cos(\angle AOD)$. Now we have that

$$\angle AOD + \angle AOB = \pi\,,$$

and thus

$$\cos(\angle AOD) = \cos(\pi - \angle AOB) = -\cos(\angle AOB) = -\frac{11}{16}\,.$$

Using this in Equation 179 we get

$$AD^2 = 100 + 96\left(\frac{11}{16}\right) = 166 \quad \text{thus} \quad AD = \sqrt{166}\,.$$

Problem 33

Let the segment AB lie along an x-axis of a coordinate system so that the point A is located at $(0,0)$ and the point B is on the x-axis at the location $(b,0)$. Then point O is located at $\left(\frac{b}{2},0\right)$ and is the center of the circle of radius $\frac{b}{2}$. Let the point C be located at a distance x (an unknown) to the right of the point O.

We first note (to be used later) that with these definitions the segment CD is a leg of the right triangle OCD which has a hypotenuse of length $\frac{b}{2}$. Using the Pythagorean theorem we thus have that

$$CD = \sqrt{\left(\frac{b}{2}\right)^2 - x^2}\,. \tag{180}$$

Next the center of the left-most circle is located at the midpoint between the points A and C or at the location

$$\frac{1}{2}\left(0 + \frac{b}{2} + x\right) = \frac{1}{2}\left(\frac{b}{2} + x\right)\,.$$

This is also the length of the radius of this left-most circle. The center of the right most circle is located at the midpoint between the points C and D or at

$$\frac{1}{2}\left(b + \frac{b}{2} + x\right) = \frac{1}{2}\left(\frac{3b}{2} + x\right)\,.$$

Using this expression, the radius of the right-most circle is given by

$$b - \frac{1}{2}\left(\frac{3b}{2} + x\right) = \frac{b}{4} - \frac{x}{2} = \frac{1}{2}\left(\frac{b}{2} - x\right).$$

With all of these variables introduced we can now compute the area of the shaded section. We have

$$\begin{aligned}
2A_{\text{shaded}} &= \text{area of large circle} - \text{area of left-most circle} - \text{area of right-most circle} \\
2A_{\text{shaded}} &= \pi\left(\frac{b}{2}\right)^2 - \pi\left(\frac{1}{2}\left(\frac{b}{2} + x\right)\right)^2 - \pi\left(\frac{1}{2}\left(\frac{b}{2} - x\right)\right)^2 \\
&= \frac{\pi}{4}\left(\frac{b^2}{2} - 2x^2\right),
\end{aligned}$$

when we simplify. To compute the ratio desired in the problem we have to divide this area by the area of the circle with radius CD. This later circle has a radius CD given by Equation 180 and has an area given by

$$\pi\left(\frac{b^2}{4} - x^2\right).$$

Thus the desired ratio is given by

$$\frac{\frac{\pi}{8}\left(\frac{b^2}{2} - 2x^2\right)}{\pi\left(\frac{b^2}{4} - x^2\right)} = \frac{1}{4}.$$

Problem 34

One thing to notice is from the problem statement if $n \to \infty$ we would have that the triangle DEF lays directly "on top" of the triangle ABC since $D \to A$, $E \to B$, and $F \to C$ in that limit. Thus the ratio of the two areas must be one. Thus taking the limit of each of the choices we have that only solutions (A) and (D) have this property. In the same vein taking the limit of $n \to 0$ we would in this case have $D \to B$, $E \to C$, and $F \to A$ and the ratio is again one. Taking the limits of solutions (A) and (D) as $n \to 0$ we see that only solution (A) has this property and thus must be the solution.

Problem 35

Start by dividing the given polynomial by its leading coefficient to get

$$x^3 - \frac{144}{64}x^2 + \frac{92}{64}x - \frac{15}{64} = x^3 - \frac{9}{4}x^2 + \frac{23}{16}x - \frac{15}{64} = 0. \tag{181}$$

Now the fundamental theorem of algebra states that we can factor the above polynomial as

$$(x - r_1)(x - r_2)(x - r_3) = 0,$$

here r_i for $i = 1, 2, 3$ are the three roots. If we multiply these factors together we get

$$\begin{aligned}(x - r_1)(x - r_2)(x - r_3) &= (x - r_1)(x^2 - (r_2 + r_3)x + r_2 r_3) \\ &= x^3 - (r_2 + r_3)x^2 + r_2 r_3 x - r_1 x^2 + r_1(r_2 + r_3)x - r_1 r_2 r_3 \\ &= x^3 - (r_1 + r_2 + r_3)x^2 + (r_1 r_2 + r_1 r_3 + r_2 r_3)x - r_1 r_2 r_3 \,.\end{aligned}$$

Thus the constant term in the expanded expression $-r_1 r_2 r_3$ is the negative of the product of the roots and the coefficient of x^2 is $-(r_1 + r_2 + r_3)$ or the negative of the sum of the roots. If we take the three roots to be $a - d$, a, and $a + d$ then from the coefficient of x^2 we have

$$r_1 + r_2 + r_3 = 3a = \frac{9}{4} \quad \text{so} \quad a = \frac{3}{4} \,.$$

The negative product of the roots is the constant term so we have

$$(a - d)a(a + d) = \frac{15}{64} \,,$$

or using what we know for a we have

$$\frac{3}{4}\left(\frac{3}{4} - d\right)\left(\frac{3}{4} + d\right) = \frac{15}{64} \,.$$

Solving for d we get $d = \pm\frac{1}{2}$. Thus the three roots are

$$\frac{1}{4}, \frac{3}{4}, \frac{5}{4} \,.$$

Given these values the difference between the largest and smallest root is $\frac{5}{4} - \frac{1}{4} = 1$.

Problem 39

Let l_n be the smallest element in the nth set. For the sets given as examples we have

$$\begin{aligned}l_1 &= 1 \\ l_2 &= 2 \\ l_3 &= 4 \\ l_4 &= 7 \,.\end{aligned}$$

From this we can see that l_n is given by (starting with $l_1 = 1$)

$$l_n = l_{n-1} + (n - 1) \,.$$

Since the nth set has n elements in it. We can check that this formula is correct

$$\begin{aligned}l_2 &= l_1 + (2 - 1) = 2 \\ l_3 &= l_2 + (3 - 1) = 2 + 2 = 4 \\ l_4 &= l_3 + 3 = 4 + 3 = 7 \,.\end{aligned}$$

Using this recursion relationship we have $l_n - l_{n-1} = n - 1$. To evaluate l_n explicitly in terms of n we can sum both sides from $n = 2$ to $n = N$ where we have

$$\sum_{n=2}^{N}(l_n - l_{n-1}) = \sum_{n=2}^{N}(n-1)\,.$$

For the left-hand-side we get

$$(l_N - l_{N-1}) + (l_{N-1} - l_{N-2}) + \cdots + (l_3 - l_2) + (l_2 - l_1) = l_N - l_1\,.$$

For the right-hand-side we get

$$\sum_{n=2}^{N}(n-1) = \sum_{n=1}^{N-1} n = \frac{1}{2}N(N-1)\,.$$

This means that

$$l_n = l_1 + \frac{1}{2}N(N-1) = 1 + \frac{1}{2}N(N-1)\,.$$

Now that we know the number of the first element in our nth set we can evaluate the sum requested. We have

$$\begin{aligned} S_n &= \sum_{k=l_n}^{l_n+n-1} k = \sum_{k=1}^{l_n+n-1} k - \sum_{k=1}^{l_n-1} k \\ &= \frac{1}{2}(l_n + n - 1)(l_n + n) - \frac{1}{2}(l_n - 1)l_n \\ &= \frac{1}{2}\left(2nl_n + n^2 - n\right)\,. \end{aligned}$$

Using this we can evaluate what we are asked and find

$$S_{21} = \frac{1}{2}\left(2(21)l_{21} + 21^2 - 21\right)\,.$$

So we need to evaluate l_{21} where we find from what we did before that

$$l_{21} = 1 + \frac{21(20)}{2} = 1 + 10(21) = 211\,.$$

Given this we find

$$S_{21} = \frac{1}{2}\left(42(211) + 21^2 - 21\right) = 4641\,.$$

Problem 40

Let the equilateral triangle have sides with length l. We position the triangle with its lower line segment AC along the x-axis so that in Cartesian coordinates we have that $A = (0,0)$ and $C = (l,0)$. Since the internal angles in a equilateral triangle are $\frac{\pi}{3}$ we know the location of the point B in terms of l. Namely it is located at

$$\left(\frac{l}{2}, l\sin\left(\frac{\pi}{3}\right)\right) = \left(\frac{l}{2}, l\frac{\sqrt{3}}{2}\right)\,.$$

Now lets place the point P inside the triangle at a location $P = (x, y)$. Then since we know the distance between P and each of the vertices of the triangle we can write three equations as

$$|AP| = 6 \Rightarrow x^2 + y^2 = 36 \tag{182}$$
$$|BP| = 8 \Rightarrow \left(\frac{l}{2} - x\right)^2 + \left(\frac{\sqrt{3}}{2}l - y\right)^2 = 64$$
$$|CP| = 10 \Rightarrow (x - l)^2 + y^2 = 100\,,$$

Note that these are three equations for the three unknowns x, y, and l. We can solve these as follows. To begin we subtract the first one from the second and third ones to get

$$-2xl + l^2 = 64 \tag{183}$$
$$l^2 - lx - \sqrt{3}ly = 28\,. \tag{184}$$

Note that the first equation is linear in x so we can solve for it to get

$$x = \frac{l^2 - 64}{2l}\,.$$

We can put this into Equation 184 to get

$$\frac{l^2}{2} - \sqrt{3}ly = -4\,.$$

This is a linear equation for y which we can solve to get

$$y = \frac{4 + \frac{l^2}{2}}{\sqrt{3}l}\,.$$

Putting the expressions for x and y in terms of l into Equation 182 we get

$$\left(\frac{l^2 - 64}{2l}\right)^2 + \left(\frac{4 + \frac{l^2}{2}}{\sqrt{3}l}\right)^2 = 36\,.$$

We can simplify this to get the following equation for l

$$4l^4 - 368l^2 + 11920 = 0\,.$$

We can use the quadratic equation to solve for l^2 where we find

$$l^2 \in \{16.86156, 183.13844\}\,.$$

We know that l^2 can be the first value since $l^2 > 10^2 = 100$. Since the area of an equilateral triangle (given the side length l is) $A = \frac{\sqrt{3}}{4}l^2$ we can compute the area under the valid choice to be

$$A = 79.30127\,.$$

From the choices given in the problem the closest area is 79.

The 1968 Examination

Problem 1 (increasing the circumference)

The circumference of the original circle (in terms of the diameter d) is given by $C = \pi d$. If d goes to $d + \pi$ then the circumference goes from C to $C + \Delta C$ with $\Delta C \geq 0$. Using the expression for the circumference in terms of d we have that

$$C + \Delta C = \pi(d + \pi) \quad \Rightarrow \quad \Delta C = \pi^2 .$$

Problem 2 (some ratios)

Write the expression involving division suggested as

$$\frac{64^{x-1}}{4^{x-1}} = \frac{4^{3x-3}}{4^{x-1}} = 4^{2x-2} = 2^{4x-4} .$$

Write the expression in terms of 256 as

$$256^{2x} = 2^{8(2x)} = 2^{16x} .$$

Equating exponents of these two we get $4x - 4 = 16x$ or $x = -\frac{1}{3}$.

Problem 3

The line $x - 3y - 7 = 0$ has a slope of $\frac{1}{3}$, so the line perpendicular to it must have a slope of -3 and have the representation

$$y - y_0 = -3(x - x_0) .$$

To go through the point $(0, 4)$ means the above equation is $y = -3x + 4$.

Problem 4 (an expression)

From the given operator definition we have

$$4 \star 4 = \frac{16}{8} = 2 .$$

Then we have

$$4 \star (4 \star 4) = 4 \star 2 = \frac{8}{6} = \frac{4}{3} .$$

Problem 5 (evaluating $f(r) - f(r-1)$)

From the given expression for $f(n)$ we get

$$\begin{aligned} f(r) - f(r-1) &= \frac{1}{3}r(r+1)(r+2) - \frac{1}{3}(r-1)r(r+1) \\ &= \frac{1}{3}r(r+1)(r+2-(r-1)) = r(r+1) \,. \end{aligned}$$

Problem 6

Note if we draw the segment DC *parallel* to AB then we have

$$\begin{aligned} \angle CDE &= \angle BAD \\ \angle DCE &= \angle ABC \,, \end{aligned}$$

Thus

$$S = \angle CDE + \angle DCE = \angle BAD + \angle ABC = S' \,,$$

so that $\frac{S}{S'} = 1$ at least in this case. In the general case we recall two facts

- all of the angles in the quadrilateral must add to 360 degrees and
- the angles $\angle CDE$ and $\angle CDA$ are supplementary and the angles $\angle DCE$ and $\angle DCB$ are supplementary

Using these we get that

$$360 = \angle BAD + \angle ABC + (180 - \angle CDE) + (180 - \angle DCE) \,,$$

or when we simplify we get

$$0 = \angle BAD + \angle ABC - (\angle CDE + \angle DCE) \quad \text{or} \quad 0 = S' - S \,,$$

and we see that $\frac{S}{S'} = 1$ in all cases.

Problem 7

Using Equation 14 we find that the radius requested is given by 6.25.

Problem 8

The error in percent (relative to the truth) is the expression $\frac{\text{truth}-\text{approximation}}{\text{truth}}$. In this case this equals

$$\frac{6x - \frac{x}{6}}{6x} = \frac{6 - \frac{1}{6}}{6} = \frac{35}{36} = 1 - \frac{1}{36} = 0.9722222 \,,$$

or about 97%.

Problem 9

The given expression

$$|x+2| = 2|x-2|\,,$$

is equivalent to

$$x+2 = \pm 2(x-2)\,.$$

If we consider the plus sign we get $x+2 = 2(x-2)$ which has $x = 6$ as a solution. For the negative sign we have $x+2 = -2(x-2)$ which has the solution $x = \frac{2}{3}$. The sum of these two solutions is $6 + \frac{2}{3} = \frac{20}{3} = 6\,\frac{2}{3}$.

Problem 10

Some students are not fraternity members for if they all were then they are all honest (which we know is not true from assumption I).

Problem 11

The arc length is given by $r\theta$ where θ is the angle measured in radians. Thus for the two circles we have

$$\begin{aligned} \text{Arc Length}(60\,^\circ) &= r_I\left(\frac{\pi}{3}\right) \\ \text{Arc Length}(45\,^\circ) &= r_{II}\left(\frac{\pi}{4}\right)\,. \end{aligned}$$

If these are equal then we have

$$r_I = \frac{3}{4} r_{II}\,.$$

Because of this relationship the areas are related as

$$A_I = \pi r_I^2 = \pi\left(\frac{9}{16} r_{II}^2\right) = \frac{9}{16} A_{II}\,.$$

Problem 12 (the radius of the circumcircle)

This problem is asking for the circumcircle of the triangle. Reviewing our notes on triangles on Page 6 we start by considering if $a^2 + b^2 = c^2$ to see if the triangle given is in fact a right triangle. It turns out that we do and because of this the hypotenuse is a diameter of the circumcircle, and its center is exactly at the midpoint of the hypotenuse. Thus the radius of the circumcircle is $\frac{1}{2}\left(12\frac{1}{2}\right) = \frac{25}{4}$.

Problem 13

If the polynomial $x^2+mx+n=0$ has roots m and n then it can be factored as $(x-m)(x-n)$. Expanding this product we get $x^2-(m+n)x+mn$. Setting this equal to the original quadratic polynomial we get that

$$mn = n \quad \text{and} \quad -(m+n) = m\,.$$

The first equation can have a solution $n = 0$ and m can be arbitrary or $n \neq 0$ and then $m = 1$. If we consider the first case and assume $n = 0$ then the second equation implies that $m = 0$ also. This cannot be the solution values for m and n we are looking for since in the problem we are told that $n \neq 0$ and $m \neq 0$. If we consider the second case where $n \neq 0$ and $m = 1$ then the second equation requires that $n = -2$ and we have found that $(n, m) = (-2, 1)$. The sum of these two roots is -1.

Problem 14

From the two equations given we can write

$$y = 1 + \frac{1}{x} = 1 + \frac{1}{1+\frac{1}{y}} = 1 + \frac{y}{y+1} = \frac{2y+1}{y+1}\,.$$

Multiplying by $y+1$ on both sides gives

$$y^2 + y = 2y + 1 \quad \text{or} \quad y^2 - y - 1 = 0\,.$$

This is a quadratic equation and has roots given by

$$y = \frac{1 \pm \sqrt{1-4(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}\,.$$

Notice that the single equation for x is

$$x = 1 + \frac{1}{y} = 1 + \frac{1}{1+\frac{1}{x}}\,.$$

Since this is the same equation as for y we have that $y = x$.

Problem 15 (the product of consecutive positive odd integers)

The product P we are discussing would be of the form $P = (2k-1)(2k+1)(2k+3)$ for $k \geq 1$. If we start by trying to get a feel for what type of numbers P can be we find the first ten value of P are given by

```
 15   105   315   693  1287  2145  3315  4845  6783  9177
```

Note that when $k = 1$ the value of P is divisible by 1, 3, 5, and 15. When $k = 2$ second value of $P = 3(5)(7) = 105$ is divisible by three and five. When $k = 3$ the third value of $P = 5(7)(9) = 315$ is divisible by three and five. Finally, when $k = 4$ we note that $P = 7(9)(11) = 693$ which is not divisible by five. Thus we will guess that every value of P is divisible by three. To prove this we know that whatever the value of k we must have one of the following true

$$\begin{aligned} k \bmod 3 &= 0 \\ k \bmod 3 &= 1 \\ k \bmod 3 &= 2 \,. \end{aligned}$$

If the first condition on k is true ($k \bmod 3 = 0$) then $k = 3m$ for some $m \geq 1$ and the product for P above looks like

$$(6m - 1)(6m + 1)(6m + 3) \,.$$

Notice that the last factor in the product is divisible by three. If the second condition on k is true ($k \bmod 3 = 1$) then $k = 3m + 1$ for some $m \geq 0$ and the product P above looks like

$$(6m + 2 - 1)(6m + 2 + 1)(6m + 2 + 3) = (6m + 1)(6m + 3)(6m + 5) \,.$$

Notice that the second factor in the above product is divisible by three. If the third condition on k is true ($k \bmod 3 = 2$) then $k = 3m + 2$ for some $m \geq 0$ and the product P looks like

$$(6m + 4 - 1)(6m + 4 + 1)(6m + 4 + 3) = (6m + 3)(6m + 5)(6m + 7) \,.$$

Notice that the first factor in the above product is divisible by three and thus every P computed in this manner is also.

Problem 16

Lets first assume that $x > 0$ then under this assumption the inequality $\frac{1}{x} < 2$ means that

$$1 < 2x \quad \text{or} \quad x > \frac{1}{2} \,.$$

While the inequality $\frac{1}{x} > -3$ means that

$$1 > -3x \quad \text{or} \quad x > -\frac{1}{3} \,.$$

The intersection of all three inequality regions mean that $x > \frac{1}{2}$. Next consider the case where $x < 0$. Under this assumption the inequality $\frac{1}{x} < 2$ means that

$$1 > 2x \quad \text{or} \quad x < \frac{1}{2} \,.$$

While the inequality $\frac{1}{x} > -3$ means that

$$1 < -3x \quad \text{or} \quad x < -\frac{1}{3} \,.$$

The intersection of all three inequality regions mean that $x < -\frac{1}{3}$. Combining these two regions means that we get

$$x > \frac{1}{2} \quad \text{or} \quad x < -\frac{1}{3} \,.$$

Problem 17

From the definition of $x_k = (-1)^k$ and $f(n)$ we have

$$\begin{aligned}
f(1) &= \frac{x_1}{1} = -1 \\
f(2) &= \frac{-1+1}{2} = 0 \\
f(3) &= -\frac{1}{3} \\
f(4) &= 0 \\
f(5) &= -\frac{1}{5}\,,
\end{aligned}$$

and so on. These values look like $\left\{0, -\frac{1}{n}\right\}$.

Problem 18

Drawing a picture of the given construction we can form the following conclusions.

- We have $\angle FEG = \angle BAE$ by the parallel lines DEF and AB.
- We are told that $\angle FEG = \angle GEC$ in the problem statement.
- We have $\angle GEC = \angle BEA$ by opposite angles.

Thus the triangle BEA has two equal angles namely $\angle BAE$ and $\angle BEA$ and so is an isosceles triangle. Since $AB = 8$ we have that $BE = AB = 8$. Next use the fact that $\triangle ABC$ is similar to $\triangle ECD$ to write

$$\frac{5}{8} = \frac{CE}{8 + CE}\,.$$

Solving the above for CE we have $CE = \frac{40}{3}$.

Problem 19 (making change)

Now ten dollars is $10 \cdot 100 = 1000$ cents and we want to count the number of ways to write this in the form

$$1000 = 10n + 25m\,,$$

for some $n \geq 1$ and $m \geq 1$. Since the left-hand-side is a multiple of ten m needs to be even so we take $m = 2m'$ to enforce this. Thus we want to count the number of solutions to

$$1000 = 10n + 50m' = 10(n + 5m') \quad \text{or} \quad 100 = n + 5m'\,.$$

In this expression we see that n needs to be a multiple of five so we let $n = 5n'$ to enforce this and we want to count the number of solutions to

$$20 = n' + m' \,,$$

where $n' \geq 1$ and $m' \geq 1$. To solve the above we can have $n' \in \{1, 2, \cdots, 18, 19\}$ and then m' is determined via $m' = 20 - n'$. Thus there are 19 ways to make the desired change.

Problem 20

Let α_k be the kth interior angle (in degrees) then from the problem statement we have

$$\alpha_k = \alpha_0 + 5k \,,$$

for some α_0. Now we know that $\alpha_n = \alpha_0 + 5n = 160$ thus $\alpha_0 = 160 - 5n$. Recalling via Equation 5 that the sum of the internal angles in a concave n-polygon is $180(n-2)$, we then must have

$$\begin{aligned}
\sum_{k=1}^{n}(\alpha_0 + 5k) = \alpha_0 n + 5\sum_{k=1}^{n} k &= \alpha_0 n + \frac{5}{2}n(n+1) \\
&= (160 - 5n)n + \frac{5}{2}n(n+1) \,.
\end{aligned}$$

Setting this equal to $180(n-2)$ we get the quadratic for n of

$$n^2 + 7n - 144 = 0 \,.$$

Solving this with using the quadratic equation gives

$$n = \frac{-7 \pm \sqrt{49 - 4(-144)}}{2} = \frac{1}{2}\{18, -32\} \,.$$

Taking the positive root (since we know $n > 0$) we have $n = 9$.

Problem 21

For this sum notice that for all terms larger than 10! each term in the sum will have 10 as a factor and thus must end in a zero. Thus all terms larger than 10! don't affect the value of the ones digit. In fact if we compute several of the values in the sum we get

$$S = 1 + 2 + 6 + 24 + 120 + \cdots \,.$$

Since the term $5! = 120$ ends with a zero digit all terms in the sum S that follow this term will also end in a zero digit and cannot affect the value of the ones digit. Thus the value of the ones digit for S will be the same as the value of the ones digit for the sum

$$1 + 2 + 6 + 24 = 33 \,,$$

or the value of three.

Problem 22

To solve this problem recall that a quadrilateral can only exist if each of its sides is less than the sum of the other three sides in length. Let the four sides be denoted s_i for $i = 1, 2, 3, 4$. Then using the above constraint on the side s_1 we would have

$$s_1 < s_2 + s_3 + s_4 \,, \tag{185}$$

but since $s_1 + s_2 + s_3 + s_4 = 1$ we can solve for $s_2 + s_3 + s_4 = 1 - s_1$ and put this into the right-hand-side of the above inequality to get

$$s_1 < 1 - s_1 \quad \text{or} \quad s_1 < \frac{1}{2} \,.$$

Since the first side is not special we have that $s_i < \frac{1}{2}$ for all i. If we replace s_1 with $1 - s_2 - s_3 - s_4$ in Equation 185 we get

$$1 - s_2 - s_3 - s_4 < s_2 + s_3 + s_4 \quad \text{or} \quad s_2 + s_3 + s_4 > \frac{1}{2} \,.$$

Thus there must be at least one of s_2, s_3, or s_4 that is larger than $\frac{1}{2(3)} = \frac{1}{6} = 0.1666667 > 0.125 = \frac{1}{8}$. Again since the first side is not special the above logic must hold for all four sides. Since choices (B), (C), and (D) allow sides that are less than $\frac{1}{6}$ they cannot be correct and the answer must be (E).

Problem 23

To have the left-hand-side be real we need the argument of each logarithm to be positive which will happen if $x > -3$ and $x > 1$. The right-hand-side of the given expression factors as

$$\log((x - 3)(x + 1)) \,.$$

To have the argument of this logarithm positive we need $x < -1$ or $x > 3$. Since we are told that all logarithms are real numbers any possible solution x must satisfy all of these requirements and so $x > 3$. Now that we know the required domain for x we can subtract the left-hand-side from the right-hand-side to get

$$\log\left(\frac{(x - 3)(x + 1)}{(x + 3)(x - 1)}\right) = 0 \,.$$

Thus a solution x must satisfy

$$\frac{(x - 3)(x + 1)}{(x + 3)(x - 1)} = 1 \,.$$

or

$$x^2 - 2x - 3 = x^2 + 2x - 3 \,,$$

which has only the solution $x = 0$. Since this does not satisfy $x > 3$ there are no solutions to this equation under the conditions required.

Problem 24 (a wooden frame)

The wording of this problem was difficult for me and I had to look at the diagram given in the book's solutions to understand what it meant. After consulting that diagram I was able to do the problem. We put a frame around the given picture in such a way that the widths of the vertical sides are w and the heights of the horizontal sides are $2w$. Now the area of the picture is $18(24) = 432$ inches squared. With the previous labels the area of the frame is given by

$$A_{\text{frame}} = 2(18 + 2w)(2w) + 2(24w)\,,$$

where the first term is the area of the top and bottom pieces of the frame and the second terms is the area of the side pieces of the frame (excluding the top horizontal parts that were included in the first term). Equating these two expressions gives a quadratic for w that has its only positive solution $w = 3$. Then the picture frame has its height given by $24+4w = 36$ and its width given by $18 + 2w = 24$. The small to large dimensions are then $24 : 36 = 2 : 3$.

Problem 25 (Flash vs. Ace)

Let $f(t)$ be the position of Flash at time t and $a(t)$ the position of Ace at time t. Then we are told that

$$\begin{aligned} a(t) &= y + v_{\text{ace}}t \\ f(t) &= xv_{\text{ace}}t\,, \end{aligned}$$

since Ace gets a head start of y. Here v_{ace} is the running velocity of Ace. Flash will overtake Ace at a time t^* where

$$xv_{\text{ace}}t^* = y + v_{\text{ace}}t^* \quad \text{or} \quad t^* = \frac{y}{v_{\text{ace}}(x-1)}\,.$$

The distance that Flash must run to catch Ace is then $f(t^*)$ or

$$\frac{xy}{x-1}\,.$$

Problem 26

We have S given by

$$S = 2 + 4 + 6 + \cdots + 2N = 2(1 + 2 + 3 + \cdots N) = 2\left(\frac{N(N+1)}{2}\right) = N(N+1)\,.$$

If we want $S > 10^6$ then we need $N(N + 1) > 10^6$. The solution for the first N to satisfy the previous inequality will be close to the solution to $N^2 > 10^6$ which is $N > 10^3$. We can check if this value of N actually works in our original inequality i.e. consider

$$N(N+1) = 1000(1001) = 1001000 > 10^6\,.$$

To see if a smaller value of N works as well (remember we found an approximate solution to the original problem) we can try the value $N = 999$. In that case we have

$$N(N+1) = 999(1000) = 999000\,,$$

which is not larger than 10^6 and thus this value of N does not satisfy $N(N+1) > 10^6$ so $N = 1000$ is the first integer where $N(N+1)$ is larger than 10^6. The sum of the digits in N is one.

Problem 27

We are given S_n defined as

$$S_n = 1 - 2 + 3 - 4 + \cdots + (-1)^{n-1} n\,.$$

for $n = 1, 2, 3, \ldots$. To start let n be even. Then in that case S_n has the last two terms in its sum given by $+(n-1) - n$. Thus we can write S_n as

$$\begin{aligned}
S_n &= [1 + 3 + 5 + \cdots + (n-3) + (n-1)] - [2 + 4 + \cdots + (n-2) + n] \\
&= [1 + 2 + 3 + 4 + \cdots + (n-3) + (n-2) + (n-1) + n] \\
&- 2[2 + 4 + \cdots + (n-2) + n] \\
&= \sum_{k=1}^{n} k - 2(2) \sum_{k=1}^{n/2} k \\
&= \frac{n(n+1)}{2} - 4\left(\frac{(n/2)(n/2+1)}{2}\right) = -\frac{n}{2}\,,
\end{aligned}$$

when we simplify. For a few values of n lets check that this expression gives the same value for S_n as the definition. Using it we have

$$\begin{aligned}
S_2 &= -1 \\
S_4 &= -2\,,
\end{aligned}$$

which are both correct. If n is odd then we can write it as $n = 2n' + 1$ for some n' and get

$$S_n = S_{2n'} + (2n' + 1) = -\frac{(n-1)}{2} + n = \frac{n+1}{2}\,.$$

We again can check that this gives the correct expression for a few values of n

$$\begin{aligned}
S_1 &= 1 \\
S_3 &= 2\,,
\end{aligned}$$

both of which are correct. Then with these two expressions for S_n we can compute the desired sum since

$$\begin{aligned}
S_{17} &= \frac{18}{2} = 9 \\
S_{33} &= \frac{34}{2} = 17 \\
S_{50} &= -\frac{50}{2} = -25\,.
\end{aligned}$$

Thus adding these numbers we find

$$S_{17} + S_{33} + S_{50} = 1\,.$$

Problem 28

From the problem statement we are told that

$$\frac{1}{2}(a+b) = \sqrt{ab}\,,$$

or

$$a + b = 4\sqrt{ab}\,.$$

If we divide by b we get

$$\frac{a}{b} + 1 = 4\sqrt{\frac{a}{b}}\,.$$

Let $r = \sqrt{\frac{a}{b}}$ then the above expression is equivalent to

$$r^2 - 4r + 1 = 0\,.$$

We can solve for r using the quadratic equation to get

$$r = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}\,.$$

Thus

$$r^2 = \frac{a}{b} = (2+\sqrt{3})^2 = 4 + 4 + 4\sqrt{3} + 3 = 7 + 4\sqrt{3} \approx 7 + 4(1.7) = 13.8\,,$$

which is close to 14.

Problem 29

When $x < 1$ we have that $\log(x) < 0$. If we take the inequality $x < 1$ and multiplying both sides by $\log(x)$ (and remember that $\log(x)$ is negative) we get $x\log(x) > \log(x)$. Taking the exponential of both sides of this expression gives

$$x^x > x\,, \tag{186}$$

and shows that $y > x$.

Lets now try to compare x^{x^x} and x^x. When we take their ratio we get

$$\frac{x^{x^x}}{x^x} = x^{x^x - x}\,.$$

We claim the above is less than one. To show that statement we will assume it is true and derive a true statement that is equivalent to the one we want to prove. To do this we take the logarithm of $x^{x^x - x)} < 1$ to get

$$(x^x - x)\log(x) < 0\,.$$

This is true since we know that $x^x - x > 0$ by Equation 186 and that $\log(x) < 0$. Thus we have shown that $z < y$.

Lets now try to compare x^{x^x} and x. To do that consider their ratio to get

$$\frac{x^{x^x}}{x} = x^{x^x - 1}\,.$$

I claim that this ratio is larger than one. To show that, we take the logarithm of $x^{x^x-1} > 1$ to get

$$(x^x - 1)\log(x) > 0\,.$$

Then to show that this is true we need to show that $x^x < 1$ which taking logarithms of that expression gives $x\log(x) < 0$ which is true. Thus we have shown that $z > x$.

In summary, we have shown that $y > x$, $z < y$ and $z > x$ so their ordering from smallest to largest must be

$$x < z < y\,.$$

Problem 30

To get some understanding of this problem we will consider some simple cases first. If we let P_1 be a triangle (so that $n_1 = 3$) and P_2 be a square (so that $n_2 = 4$) then we have maximal intersection with the segments of the square when each of the corners of the triangle "punctures" one side of the square. This will give rise to $2(3) = 2n_1 = 6$ intersections.

As another example, if P_1 is a polygon with four sides (so that $n_1 = 4$) then using each of the corners of this polygon we can puncture one of the sides of P_2. This gives $2(4) = 2n_1 = 8$ intersections. Continuing this argument for polygons of all sizes, by putting a corner of P_1 through each of the sides of P_2 we will have a total of $2n_1$ intersections.

Problem 31

Recall that the area of an equilateral triangle with a side length of a is given by $\frac{\sqrt{3}}{4}a^2$. Using this we have that triangle I must have a side length given by

$$\frac{\sqrt{3}}{4}a_I^2 = 32\sqrt{3} \quad \text{or} \quad a_I = 8\sqrt{2}\,.$$

We can also conclude that triangle III has a side length that is given by

$$\frac{\sqrt{3}}{4}a_{III}^2 = 8\sqrt{3} \quad \text{or} \quad a_I = 4\sqrt{2}\,.$$

Finally the side length of the square II must satisfy

$$a_{II}^2 = 32 \quad \text{or} \quad a_{II} = 4\sqrt{2}\,.$$

Given that we know the three lengths above we know the initial length of AD and find it to be

$$a_I + a_{II} + a_{III} = 8\sqrt{2} + 4\sqrt{2} + 4\sqrt{2} = 16\sqrt{2}\,.$$

Next we are told that the length of AD is reduced by 12.5% of itself while the lengths of $a_I = AB$ and $a_{III} = CD$ don't change. The new size of AD is given by

$$(1 - 0.125)AD = 0.875AD\,.$$

Given the constraints on AB and CD the new length of $a'_{II} = BC'$ is given by

$$0.875(16\sqrt{2}) = a_I + a'_{II} + a_{III} = 8\sqrt{2} + a'_{II} + 4\sqrt{2} = 12\sqrt{2} + a'_{II}\,.$$

Solving for a''_{II} we get $a'_{II} = 2\sqrt{2}$. The new area of the square is then ${a'_{II}}^2 = 8$. The percentage change in the area of II is then

$$\frac{8-32}{32} = -\frac{3}{4}\,,$$

or a decrease of 75%.

Problem 32

Let A be moving from left to right along the x-axis and B moving from "bottom to top" along the y-axis. Take $t = 0$ to be the time when A is at the origin $(x, y) = (0, 0)$. Since they move at a uniform speed their positions are given by

$$\begin{aligned} x_A(t) &= v_A t \\ y(B(t) &= v_B t - 500\,, \end{aligned}$$

with v_A and v_B both positive. Then in two minutes we have them equidistant from the origin O or

$$x_A(2) = |y_B(2)|\,,$$

or

$$2v_A = |2v_B - 500|\,.$$

Dividing this by two we get $v_A = |v_B - 250|$. In eight minutes *more* they are again equidistant from O again or

$$x_A(10) = |y_B(10)| \quad \text{or} \quad 10v_A = |10v_B - 500|\,.$$

Dividing this by ten we get $v_A = |v_B - 50|$. Equating these two expressions for v_A gives

$$|v_B - 250| = |v_B - 50| \,.$$

We are looking for the value of v_B that will satisfy this. Notice that the absolute values in the above expression are needed (or not) depending on where v_B is relative to the value of 50 and 250. If for example $0 < v_B < 50$ then the above expression would become

$$-(v_B - 250) = -(v_B - 50) \,,$$

which is an expression that simplifies to a contradiction. Thus there is no solution to the above with v_B in this range. The same conclusion is reached if we assume that $v_B > 250$. In the case where $50 < v_B < 250$ then the above becomes

$$v_B - 250 = -(v_B - 50) \quad \text{so} \quad v_B = 150 \,.$$

We can then compute v_A using either of the two formulas above. For example we have

$$\begin{aligned} v_A &= |v_B - 250| = |-100| = 100 \quad \text{or} \\ v_A &= |v_B - 50| = |150 - 50| = 100 \,. \end{aligned}$$

Using these value for v_A and v_B we compute

$$\frac{v_A}{v_B} = \frac{100}{150} = \frac{2}{3} \,.$$

Problem 33 (a number in base 7 and 9)

Let the digits when N is written in base seven be d_1, d_2, and d_3. Then from the problem statement we have that

$$\begin{aligned} N &= (d_1 d_2 d_3)_7 = d_1 7^2 + d_2 7 + d_3 \\ N &= (d_3 d_2 d_1)_9 = d_3 9^2 + d_2 9 + d_1 \,. \end{aligned}$$

As these two are equal we can equate to get the constraint

$$24 d_1 - d_2 - 40 d_3 = 0 \,.$$

We now must have three integers $0 \le d_i < 7$ that satisfy the above equation. Solving the above for d_2 we get

$$d_2 = 24 d_1 - 40 d_3 = 8(3 d_1 - 5 d_3) \,.$$

Only one solution for d_1, d_2, and d_3 will satisfy this equation and have the correct bounds. To find this solution we will specify possible values for d_1, d_2, and d_3 and show that the equation above gives an inconsistent value for d_2 in all but one case.

- If we take $d_1 = 0$ then we don't have a three digit number and so we don't need to consider this case.

- If we take $d_1 = 1$ then from the above $d_2 = 8(3 - 5d_3)$. If $d_3 = 0$ the value for d_2 is larger than 7. If $d_3 \geq 1$ then d_2 is negative. In each of these cases we have a violation of the assumptions on the allowed values for d_2.
- If we take $d_1 = 2$ then from the above $d_2 = 8(6 - 5d_3)$. No valid values of d_3 give valid values for d_2.
- If we take $d_1 \in \{3, 4, 5, 6\}$ then again no valid values of d_3 give valid values for d_2.
- If we take $d_1 = 5$ then from the above $d_2 = 8(15 - 5d_3)$. If $d_3 \in \{0, 1, 2\}$ the computed value of d_2 is too large. If $d_3 \in \{4, 5, 6\}$ the computed value of d_2 is negative. If $d_3 = 3$ then $d_2 = 0$.

From the above our digits are $d_1 = 5$, $d_2 = 0$ and $d_3 = 3$. We can check that these values indeed satisfy the required conditions of the problem as

$$(503)_7 = 5(49) + 3 = 248$$
$$(305)_9 = 3(81) + 5 = 248\,,$$

which are equal as they should be.

Problem 34

Let n and a be the number of nays and ayes in the first vote and n' and a' be the number of nays and ayes in the second vote. Then we know that

$$n + a = 400 = n' + a'\,,$$

since that is the total number of voters. In the first vote the bill was defeated by a margin m so that

$$n - a = m > 0\,.$$

On the re-vote the bill passed by a margin twice of that when it was first defeated or

$$a' - n' = 2m = 2(n - a)\,.$$

The number of ayes on the second vote in terms of nays on the first vote is given by

$$a' = \frac{12}{11}n\,.$$

There are four equations here and four unknowns n, a, n', and a' so we can determine each variable in the above. These four equations are

$$\begin{aligned} n + a &= 400 \\ n' + a' &= 400 \\ 2n - 2a + n' - a' &= 0 \\ \frac{12}{11}n - a' &= 0\,. \end{aligned}$$

Solving this system we get $n = 220$, $a = 180$, $n' = 160$, and $a' = 240$. Then the problem asks us to compute $a' - a = 240 - 180 = 60$.

Problem 35

Let the center of the circle be located at the origin of a Cartesian coordinate system. A circle with radius a centered at the origin has an equation

$$x^2 + y^2 = a^2 .$$

Let the distance OG be denoted as d. Then since $GH = HJ$ the distance GH must satisfy

$$d + 2GH = a \quad \text{so} \quad GH = \frac{a-d}{2} .$$

Note that the y coordinate of the point D is d and the y coordinate of the point F is the same as $OH = d + GH$ or $\frac{a+d}{2}$. As these two points are on the circle we can compute their x coordinate in the first quadrant and find

$$\begin{aligned} x_D &= \sqrt{a^2 - d^2} \quad \text{for the point } D \\ x_F &= \sqrt{a^2 - \left(\frac{a+d}{2}\right)^2} = \frac{1}{2}\sqrt{3a^2 - 2ad - d^2} \quad \text{for the point } F . \end{aligned}$$

Now that we know these points we can compute the required areas. For the area of the trapezoid K we have

$$\begin{aligned} K &= \frac{1}{2}\text{height} \times (\text{base}_1 + \text{base}_2) \\ &= \frac{1}{2}GH\,(2x_D + 2x_F) = GH\left(\sqrt{a^2 - d^2} + \frac{1}{2}\sqrt{3a^2 - 2ad - d^2}\right) . \end{aligned}$$

For the area of the rectangle R we have

$$\begin{aligned} R &= \text{base} \times \text{height} = 2x_F \times GH \\ &= \sqrt{3a^2 - 2ad - d^2} . \end{aligned}$$

The ratio $K : R$ is then given by

$$\frac{K}{R} = \frac{1}{2} + \frac{\sqrt{a^2 - d^2}}{\sqrt{3a^2 - 2ad - d^2}} = \frac{1}{2} + \sqrt{\frac{a^2 - d^2}{3a^2 - 2ad - d^2}} .$$

If we let d approach the value of a we get that the limit inside the square root is of the type $0/0$ and must be evaluated using L'Hospital's rule. This then gives

$$\begin{aligned} \lim_{d \to a} \frac{K}{R} &= \frac{1}{2} + \sqrt{\lim_{d \to a}\left(\frac{a^2 - d^2}{3a^2 - 2ad - d^2}\right)} \\ &= \frac{1}{2} + \sqrt{\lim_{d \to a}\left(\frac{-2d}{-2a - 2d}\right)} = \frac{1}{2} + \sqrt{\frac{1}{2}} . \end{aligned}$$

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Problem 1

From the problem statement we have

$$\frac{a+x}{b+x} = \frac{c}{d},$$

or

$$d(a+x) = c(b+x) \quad \text{so} \quad x = \frac{ad-bc}{c-d}.$$

Problem 2

Let c be the cost then we are told that $x = 0.85c$ and $y = 1.15c$. Thus

$$\frac{y}{x} = \frac{1.15}{0.85} = \frac{115}{85} = \frac{23}{17}.$$

Problem 3

This is just like subtraction in base 10 where we have to borrow from higher order digits and the answer is (E).

Problem 4

From the given definition we have

$$(3,2) \star (0,0) = (3-0, 2+0) = (3,2),$$

and

$$(x,y) \star (3,2) = (x-3, y+2).$$

If these two are equal then we must have $x = 6$ and $y = 0$.

Problem 5

From the problem we are told that

$$N - 4\left(\frac{1}{N}\right) = R.$$

As an equation for N this is equal to

$$N^2 - RN - 4 = 0\,.$$

Solving this with the quadratic equation we get

$$N = \frac{R \pm \sqrt{R^2 - 4(-4)}}{2} = \frac{R \pm \sqrt{R^2 + 16}}{2}\,.$$

Thus adding the positive and negative roots we get the value of R.

Problem 6 (nested circles)

Let r_2 and r_1 be the radii of the larger and the smaller circle respectively. Draw a segment (of length r_2) connecting the common center to one of the locations where the cord touches the larger circle. Draw a second segment (of length r_1) from the common center to the point of tangency. Then since this second segment is perpendicular to the chord we have that r_2 is the hypotenuse of a right triangle with one leg of length r_1 and the other legs must be

$$\sqrt{r_2^2 - r_1^2}\,.$$

We are told that

$$\pi r_2^2 - \pi r_1^2 = 12.5\pi\,,$$

and thus $r_2^2 - r_1^2 = 12.5$ thus the total cord length (the above is 1/2 the total chord length) is given by

$$2\sqrt{r_2^2 - r_1^2} = 2\sqrt{\frac{25}{2}} = 5\sqrt{2}\,.$$

Problem 7

From the given expression for y we have that

$$y_1 = a + b + c \tag{187}$$
$$y_2 = a - b + c\,. \tag{188}$$

If we subtract Equation 188 from Equation 187 we get

$$y_1 - y_2 = 2b = -6 \quad \text{so} \quad b = -3\,.$$

Problem 8

From the given minor arc measures the angle measure of the vertex opposite the arc is given by one-half the arc measures. Thus

- the angle measure of $\angle C$ is $\frac{1}{2}(x+75)$
- the angle measure of $\angle A$ is $\frac{1}{2}(2x+25)$
- the angle measure of $\angle B$ is $\frac{1}{2}(3x-22)$

Since the angles in a triangle must sum to 180 degrees we know that

$$x + 75 + 2x + 25 + 3x - 22 = 360\,.$$

Solving this for x we get $x = 47$ degrees. With this value of x the three interior angles take the values

$$61\,,\,59.5\,,\,59.5\,.$$

Problem 9

For this we need to compute

$$\begin{aligned}\frac{1}{52}\sum_{k=2}^{53} k &= \frac{1}{52}\left(\frac{1}{2}(53)(53+1)-1\right)\\ &= \frac{1}{52}\left(27(53)-1\right) = \frac{1430}{52} = 27.5\,.\end{aligned}$$

Problem 10

The set of points that are equidistant from the two parallel tangents will be on a line that bisects them and passes through the center of the circle. If r is the radius of the circle, then each point on this bisecting line is of a distance r from either of the two parallel tangent. Thus we need to find any points on this bisecting line that are a distance r from the circle. There will be two points on this bisecting line and at a distance of $2r$ from the center and the center of the circle itself. This gives three points total.

Problem 11

The distance $PR + RQ$ will be smallest when the point R is on the line connecting the two points. That line is

$$y - (-2) = \left(\frac{-2-2}{-1-4}\right)(x+1)\,,$$

or

$$y = \frac{4}{5}(x+1) - 2\,.$$

If we let $x = 1$ to find the value of m we get

$$y = \frac{4}{5}(2) - 2 = -\frac{2}{5}\,.$$

Problem 12

Write F as

$$F = \frac{6x^2 + 16x + 3m}{6} = x^2 + \frac{8}{3}x + \frac{m}{2} = \left(x + \frac{4}{3}\right)^2 - \frac{16}{9} + \frac{m}{2}\,.$$

To have this be only the square of a linear expression in x means that

$$\frac{m}{2} - \frac{16}{2} = 0\,,$$

or $m = \frac{2(16)}{9} = \frac{32}{9}$. Note that

$$3 = \frac{27}{9} < m < \frac{36}{9} = 4\,.$$

Problem 13

From the statement of the problem we have

$$\pi R^2 = \frac{a}{b}(\pi R^2 - \pi r^2)\,.$$

As the larger circle has more area (i.e. the left-hand-side of the above) than the amount of area "inside the larger but outside the smaller" (i.e. the expression $\pi R^2 - \pi r^2$) for the above expression to make sense we must have $\frac{a}{b} > 1$ or $a > b$. We can solve the above for R^2 in terms of r^2 to get

$$R^2 = \frac{a}{a-b} r^2\,,$$

from which we see that

$$R : r = \sqrt{a} : \sqrt{a-b}\,.$$

The argument given earlier show that the square root is defined in the real numbers.

Problem 14

From the expression

$$\frac{x^2 - 4}{x^2 - 1} > 0\,, \tag{189}$$

if $x^2 - 1 > 0$ then the above is equivalent to $x^2 - 4 > 0$. This later inequality has the solution $|x| > 2$. For $x^2 - 1 > 0$ we must have $|x| > 1$ and so the solution combining both of these conditions is $|x| > 2$.

Now if $x^2 - 1 < 0$ then Equation 189 is equivalent to $x^2 - 4 < 0$ which has the solution $|x| < 2$. For $x^2 - 1 < 0$ then we must have $|x| < 1$ and so the solution combining both of these conditions is $|x| < 1$.

Combining both of the possible solutions above we have that for Equation 189 to be true that x must satisfy $|x| > 2$ or $|x| < 1$.

Problem 15

We start by drawing the given picture. We are told that the length of AO is r. Since the radius from O to M is perpendicular to AB we have that it bisects the segment AB and thus the length of AM is $r/2$. From these two facts and the fact that $\triangle AOM$ is a right triangle we have that

$$OM^2 + AM^2 = AO^2 .$$

From this we can solve for OM and find $OM = \frac{\sqrt{3}}{2} r$.

Since MD meets AO at a perpendicular we have the following similar triangles

$$\triangle AOM \sim \triangle AMD \sim \triangle MOD .$$

Using $\triangle AOM \sim \triangle AMD$ we have

$$\frac{AD}{DM} = \frac{\frac{r}{2}}{\frac{\sqrt{3}}{2} r} = \frac{1}{\sqrt{3}} . \tag{190}$$

Using $\triangle AOM \sim \triangle MOD$ we have

$$\frac{DO}{DM} = \frac{\frac{\sqrt{3}}{2} r}{\frac{r}{2}} = \sqrt{3} . \tag{191}$$

Since $AD + DO = r$ we have that $DO = r - AD$. If we put this into Equation 191 we get

$$\frac{r - AD}{DM} = \sqrt{3} .$$

Replacing $\frac{AD}{DM}$ in the above with Equation 190 we get

$$\frac{r}{DM} - \frac{1}{\sqrt{3}} = \sqrt{3} \quad \text{so} \quad DM = \frac{\sqrt{3}}{4} r .$$

Now that we know DM we have

$$AD = \frac{1}{\sqrt{3}} DM = \frac{r}{4} .$$

Thus the area we want is given by

$$\frac{1}{2} AD \cdot MD = \frac{1}{2} \left(\frac{r}{4}\right) \left(\frac{\sqrt{3}}{4} r\right) = \frac{\sqrt{3}}{32} r^2 .$$

Problem 16

The first few terms of the binomial expansion are

$$(a - b)^n = a^n - \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \cdots .$$

Using the values of the binomial coefficients we are told that

$$-na^{n-1}b - \frac{n(n-1)}{2}a^{n-2}b^2 = 0\,,$$

when $a = kb$. In that case the above is

$$-n(kb)^{n-1}b - \frac{n(n-1)}{2}(kb)^{n-2}b^2 = 0\,.$$

If we divide by $nb^n k^{n-2}$ this is

$$-k + \frac{n-1}{2} = 0 \quad \text{or} \quad n = 1 + 2k\,.$$

Problem 17

If we let $v = 2^x$ then our equation becomes

$$v^2 - 8v + 12 = 0\,,$$

which factors to give

$$(v-6)(v-2) = 0\,.$$

The two solutions to this are

$$\begin{aligned} 2^x &= 6 \quad \text{or} \quad x = \frac{\log(6)}{\log(2)} \quad \text{and} \\ 2^x &= 2 \quad \text{or} \quad x = 1\,. \end{aligned}$$

Note that we can write the first solution above as

$$x = \frac{\log(2) + \log(3)}{\log(2)} = 1 + \frac{\log(3)}{\log(2)}\,.$$

Problem 18

We will have solutions to the two equations if one of the factors in the product in each expression is zero. Thus we should find the zeros to the first factor in each product

$$\begin{aligned} x - y + 2 &= 0 \\ x + y - 2 &= 0 \quad \text{or} \quad (x, y) = (0, 2)\,, \end{aligned}$$

zeros for the first factor in the first product and the second factor in the second product

$$\begin{aligned} x - y + 2 &= 0 \\ 2x - 5y + 7 &= 0 \quad \text{or} \quad (x, y) = (-1, 1)\,, \end{aligned}$$

zeros for the second factor in the first product and the first factor in the second product

$$\begin{aligned} 3x + y - 4 &= 0 \\ x + y - 2 &= 0 \quad \text{or} \quad (x, y) = (1, 1)\,, \end{aligned}$$

zeros for the second factor in each product

$$\begin{aligned} 3x + y - 4 &= 0 \\ 2x - 5y + 7 &= 0 \quad \text{or} \quad (x, y) = (0.7647059, 1.7058824)\,. \end{aligned}$$

Thus we have four solutions.

Problem 19

We can factor the given expression as

$$(x^2y^2 - 1)(x^2y^2 - 9) = 0\,.$$

This has solutions when

$$\begin{aligned} x^2y^2 &= 1 \\ x^2y^2 &= 9\,, \end{aligned}$$

or taking square roots of both sides when

$$\begin{aligned} xy &= \pm 1 \\ xy &= \pm 3\,. \end{aligned}$$

Now as we are told that both x and y must be positive we don't need to consider the negative sign in the above expressions. The number of integer solutions where $xy = 1$ is one when both $x = 1$ and $y = 1$. The number of integer solutions where $xy = 3$ is two when $(x, y) = (1, 3)$ and $(x, y) = (3, 1)$. Thus we have three total solutions of the type requested.

Problem 20

Name the two factors in the definition of P as p_1 and p_2 so that $P = p_1p_2$. From the numbers given we know that

$$\begin{aligned} 3.6\,10^{18} < \; & p_1 \; < 3.7\,10^{18} \\ 3.4\,10^{14} < \; & p_2 \; < 3.5\,10^{14}\,. \end{aligned}$$

Thus we then have that

$$3.6(3.4)\,10^{32} < p_1p_2 < 3.7(3.5)\,10^{32}\,.$$

or

$$12.24\,10^{32} < P < 12.95\,10^{32}\,,$$

or

$$1.224\,10^{33} < P < 1.295\,10^{33}\,,$$

Thus P must have $33 + 1 = 34$ digits in it.

Problem 21

In the x-y plane the curve $x^2 + y^2 = m$ is a circle with radius $\sqrt{m}$ and the line $x + y = \sqrt{2m}$ is a line with a negative slope. When we sketch each of these curves we see that the point of tangency must be in the first quadrant. Let the point of tangency be denoted as P. At the point of tangency the segment from the origin $(0, 0)$ (denoted by the letter O) to the point P and the line $x + y = \sqrt{2m}$ form a right angle. The intersection of the line $x + y = \sqrt{2m}$ and the x-axis is the point $(\sqrt{2m}, 0)$ (denoted by the point Q). Thus one leg of this right triangle OPQ has a length of $\sqrt{m}$ (the radius of the circle) and the hypotenuse of this right triangle has a length $\sqrt{2m}$. The unknown leg PQ of then has a length given by the Pythagorean theorem of

$$PQ^2 = (\sqrt{2m})^2 - (\sqrt{m})^2 = 2m - m = m\,,$$

so $l = \sqrt{m}$. Since triangle OPQ has two equal sides of length $OP = PQ = \sqrt{m}$ its acute angles must equal $\frac{180-90}{2} = 45$ degrees (or $\frac{\pi}{4}$ radians). This point on the circle will be located at

$$(x, y) = \left(\sqrt{m}\cos\left(\frac{\pi}{4}\right), \sqrt{m}\sin\left(\frac{\pi}{4}\right)\right) = \left(\frac{\sqrt{m}}{\sqrt{2}}, \frac{\sqrt{m}}{\sqrt{2}}\right).$$

Note that if we compute $x + y$ we find

$$\frac{2\sqrt{m}}{\sqrt{2}} = \sqrt{2m}\,,$$

i.e. that point is on the line $x + y = \sqrt{2m}$.. Sine every value of m that is nonnegative will have this property we have that m can be any non-negative real number.

Problem 22

If we draw the given region K in the x-y plane we can break the area up into three parts: the triangle with vertices given by

$$(0, 0),\ (5, 0),\ (5, 5)\,,$$

with an area of $\frac{1}{2}(5)(5) = \frac{25}{2}$. The rectangle with vertices

$$(5, 0),\ (8, 0),\ (8, 5),\ (5, 5)\,,$$

with an area of $3(5) = 15$. The triangle with vertices given by

$$(5, 5),\ (8, 5),\ (8, 11)\,,$$

with an area of $\frac{1}{2}(3)(11 - 5) = 9$. The total area is then the sum of these three pieces or 36.5.

Problem 23

If we let $n = 2$ we have that $n! + 1 = 3$ and $n! + n = 4$ from which we see there are no primes p that satisfy

$$3 < p < 4\,.$$

If we let $n = 3$ we have that $n! + 1 = 6 + 1 = 7$ and $n! + n = 6 + 3 = 9$ from which we see that there are also no primes p that satisfy

$$7 < p < 9\,.$$

If we let $n = 4$ we have that $n! + 1 = 24 + 1 = 25$ and $n! + n = 24 + 4 = 28$ from which we see that there are again no primes p that satisfy

$$25 < p < 28\,.$$

The only answer that satisfies these conditions is A.

Problem 24

The information about the remainders of P and P‘ mean that

$$\begin{aligned} P &= DQ + R \\ P' &= DQ' + R\,, \end{aligned}$$

for some natural numbers Q and Q'. From the above expression we have that

$$\begin{aligned} PP' &= (DQ + R)(DQ' + R) \\ &= D^2QQ' + DQR' + DRQ' + RR' \\ &= D(DQQ' + QR' + RQ') + RR'\,. \end{aligned}$$

Since we know the remainder of RR' when divided by D is r' we can write

$$RR' = Dq' + r'\,,$$

for some natural number q'. Putting this into the last expression for PP' we get

$$PP‘ = D(DQQ' + QR' + RQ') + Dq' + r' = D(DQQ' + QR' + RQ' + q') + r'\,.$$

Thus the remainder of dividing PP' by D is r'. As we are also told that it is r we have that $r = r'$.

Problem 25

The given expression is equivalent to

$$\log_2(ab) \geq 6 \quad \text{or} \quad ab \geq 2^6\,.$$

Thus we have a lower bound on the product of a and b. Since we want to consider the sum of a and b we will recall the arithmetic and geometric mean inequality

$$\frac{x+y}{2} \geq \sqrt{xy}\,. \tag{192}$$

To use this we take a square-root of the lower bound for ab to get

$$\sqrt{ab} \geq 2^3\,.$$

Thus using Equation 192 we have that

$$a+b \geq 2\sqrt{ab} \geq 2^4 = 16\,.$$

Problem 26

To solve this problem introduce a Cartesian coordinate system and put the point A at $(0,0)$ and the point B at $(40,0)$. Then the point M is at $(20,0)$ and the point C is located at $(20,16)$. Next let the functional form form for our quadratic be given by $y(x) = -ax^2+bx+c$. We need to determine the values of a, b, and c. To have $y(0) = 0$ we must have $c = 0$. To have $y(40) = 0$ we must have

$$-40^2 a + 40b = 0 \quad \text{so} \quad b = 40a\,.$$

Thus at this point we know

$$y(x) = -ax^2 + 40ax\,.$$

To have $y(20) = 16$ we must have

$$-a(400) + 40a(20) = 16 \quad \text{so} \quad a = \frac{1}{25}\,.$$

Thus using this value of a we now have

$$y(x) = -\frac{1}{25}x^2 + \frac{40}{25}x = -\frac{1}{25}x^2 + \frac{8}{5}x\,.$$

For the problem we want to evaluate

$$y(20+5) = y(25) = -\frac{1}{25}(25)^2 + \frac{8}{5}(25) = 15\,.$$

Problem 27 (a moving particle)

Assume the particle starts at the origin and travels one mile. Then the speed during the second mile is $\frac{C}{1} = C$ since at that point the particle has traveled one integer miles. When the particle is between two and three miles its speed is $\frac{C}{2}$. When the particle is between three and four miles its speed is $\frac{C}{3}$. In the same way when the particle is between miles n

and $n+1$ its speed is $\frac{C}{n}$. Since we are told that in the second mile (when it has a speed of $\frac{C}{1} = C$) the particle takes two hours to travel this mile we have

$$2C = 1 \quad \text{so} \quad C = \frac{1}{2}\,.$$

Thus the speed when the particles location x is between miles n and $n+1$ (or the $n+1$st mile) is $\frac{1}{2n}$. The speed to travel the nth mile is then $\frac{1}{2(n-1)}$ and so the *time* to travel the nth mile is $2(n-1)$.

We can use the given statement that the second mile $n = 2$ took two hours to travel to eliminate all choices but A and E. As the velocity of the particle decreases as the number of miles traveled increases the time to travel each subsequent mile must increase. Only solution E has this property and is the correct answer.

Problem 28

The point P must be inside the unit circle and thus must satisfy

$$x^2 + y^2 < 1\,.$$

Now without loss of generality let the diameter of the circle be the x-axis so the two endpoints of the diameter are located at $(1,0)$ and $(-1,0)$ then we want the point $P = (x,y)$ to satisfy

$$(x-(-1))^2 + y^2 + (x-1)^2 + y^2 = 3\,.$$

Expanding the left-hand-side we get the above is equivalent to

$$x^2 + y^2 = \frac{1}{2}\,,$$

which is an equation of a circle with a radius $\frac{1}{\sqrt{2}}$. As there are an infinite number of points on this circle we have an infinite number of solutions.

Problem 29

We are told that $x = t^{\frac{1}{t-1}}$ and $y = t^{\frac{t}{t-1}}$. First write the expression for y as

$$y = \left(t^{\frac{1}{t-1}}\right)^t = x^t\,.$$

Next if we consider the expression for y we have

$$y = t^{\frac{t-1+1}{t-1}} = t^{1+\frac{1}{t-1}} = tx\,,$$

and we have $t = \frac{y}{x}$. Using this in the relationship $y = x^t$ we get

$$y = x^{\frac{y}{x}}\,.$$

Take the x power of both sides of the above gives

$$y^x = x^y\,.$$

Problem 30

Let the point C be located in a Cartesian coordinate system at the point $(0, 0)$, let the side AC and BC be of unit length length so that we can have A located at the point $(-1, 0)$ and B located at the point $(0, 1)$. Then a point $P = (x, y)$ will be located on the hypotenuse of the right triangle ABC if it is on the line

$$y - 0 = \left(\frac{1-0}{0+1}\right)(x+1) \quad \text{so} \quad y = x + 1\,.$$

For a general point $P = (x, y)$ we have that

$$\begin{aligned} AP^2 &= (-1-x)^2 + y^2 = (1+x)^2 + y^2 \\ PB^2 &= (0-x)^2 + (1-y)^2 = x^2 + (1-y)^2\,. \end{aligned}$$

Thus

$$s = AP^2 + PB^2 = 1 + 2x + x^2 + y^2 + x^2 + 1 - 2y + y^2 = 2 + 2x + 2x^2 - 2y + 2y^2\,.$$

For P to be on the hypotenuse we need to have $y = x + 1$ so that s is given by

$$\begin{aligned} s &= 2 + 2x + 2x^2 - 2(x+1) + 2(x+1)^2 \\ &= 2 + 2x + 2x^2 - 2x - 2 + 2x^2 + 4x + 2 \\ &= 2 + 4x + 4x^2\,. \end{aligned}$$

The distance to the origin squared is CP^2 or

$$x^2 + y^2 = x^2 + (x+1)^2 = 2x^2 + 2x + 1\,.$$

From this we see that $s - 2CP^2 = 0$ for all values of x.

Problem 31

From the given mapping from (x, y) into (u, v) we see that the four corners of the initial square get mapped to

$$\begin{aligned} (0,0) &\to (0,0) \\ (1,0) &\to (1,0) \\ (1,1) &\to (0,2) \\ (0,1) &\to (-1,0)\,. \end{aligned}$$

Since answers A, B, and E don't pass through the point $(0, 0)$ they cannot be the answer. Points on the right-hand-side of the initial square (where $(x, y) = (1, y)$ get mapped to

$$(1 - y^2, 2y)\,.$$

Notice that this is a quadratic in the first component thus answer C cannot be correct since each of the boundaries in (u, v) are linear. Thus the answer must be D.

Problem 32

Introduce the notation $\Delta u_n \equiv u_n - u_{n-1}$ then in terms of Δ we have

$$\Delta u_n = 3 + 4(n-2) = 4n - 5\,.$$

Then following the techniques in [1] by summing both sides we have

$$\begin{aligned}\sum_{k=2}^{n} \Delta u_k = u_n - u_1 &= 4\sum_{k=2}^{n} k - 5\sum_{k=2}^{n} 1 \\ &= 4\left(\frac{n(n+1)}{2} - 1\right) - 5(n-1) = 2n^2 - 3n + 1\,.\end{aligned}$$

If we solve for u_n given that $u_1 = 5$ we get

$$u_n = 2n^2 - 3n + 6\,.$$

The sum of these coefficients is $2 - 3 + 6 = 5$.

Problem 33

Using the expression given by Equation 18 for S_n and T_n we have

$$\begin{aligned}S_n &= \sum_{k=1}^{n} a_k = \frac{n}{2}(2a_1 + (n-1)d_a) \\ T_n &= \sum_{k=1}^{n} b_k = \frac{n}{2}(2b_1 + (n-1)d_b)\,,\end{aligned}$$

for some values of a_1, d_a, b_1, and d_b. Using the above expressions and what we are told in the problem statement we have that we can write the ratio of S_n to T_n as

$$\frac{S_n}{T_n} = \frac{7n+1}{4n+27} = \frac{2a_1 + (n-1)d_a}{2b_1 + (n-1)d_b}\,. \tag{193}$$

We want to know the value of $\frac{a_{11}}{b_{11}} = \frac{a_1+10d_a}{b_1+10d_b}$. We can get this ratio from Equation 193 if we take $n = 21$. Then we have

$$\frac{a_{11}}{b_{11}} = \frac{2(a_1+10d_a)}{2(b_1+10d_b)} = \frac{7(21)+1}{4(21)+27} - \frac{148}{111} = 1.333333 = \frac{4}{3}\,.$$

Problem 34

The information about the quotient means that

$$\frac{x^{100}}{x^2 - 3x + 2} = Q(x) + \frac{R(x)}{x^2 - 3x + 2}\,,$$

where $Q(x)$ and $R(x)$ are the quotient and remainder polynomials respectively. Solving for $R(x)$ we get

$$R(x) = x^{100} - Q(x)(x^2 - 3x + 2) = x^{100} - Q(x)(x-1)(x-2)\,. \tag{194}$$

Since we are told that $R(x)$ is a first degree polynomial it has only two terms so we can write it as $R(x) = r_1 x + r_0$. We can find the values for r_0 and r_1 by evaluating $R(x)$ at $x = 1$ and $x = 2$. We get

$$\begin{aligned} R(1) &= r_1 + r_0 = 1 \\ R(2) &= 2r_1 + r_0 = 2^{100}\,. \end{aligned}$$

Solving the above we for r_0 and r_1 we find $r_0 = 2 - 2^{100}$ and $r_1 = 2^{100} - 1$ so that $R(x)$ looks like

$$R(x) = 2 - 2^{100} + \left(2^{100} - 1\right) x = 2^{100}(x-1) - x + 2\,,$$

which is one of the choices.

Problem 35

The function $L(m)$ is the left-most solution x to the equation $x^2 - 6 = m$ or $x = -\sqrt{6+m}$. Then we want to compute

$$\frac{L(-m) - L(m)}{m} = \frac{-\sqrt{6-m} + \sqrt{6+m}}{m}\,,$$

as m shrinks to zero. The easiest way to do this is to write the above as

$$2\left(\frac{\sqrt{6+m} - \sqrt{6-m}}{2m}\right)\,,$$

and then to recognize that as $m \to 0$ the term in parenthesis will tend to the derivative of the function $\sqrt{6+x}$ evaluated at $x = 0$. This derivative is

$$\frac{d}{dx}\sqrt{6+x}\bigg|_{x=0} = \frac{1}{2\sqrt{6+x}}\bigg|_{x=0} = \frac{1}{2\sqrt{6}}\,.$$

Thus the limit we seek is then twice this value or $\frac{1}{\sqrt{6}}$.

Another way to solve this problem is to recall that for small m we have

$$\begin{aligned} \sqrt{6-m} &= \sqrt{6}\sqrt{1 - \frac{m}{6}} \approx \sqrt{6}\left(1 - \frac{m}{12}\right) \\ \sqrt{6+m} &= \sqrt{6}\sqrt{1 + \frac{m}{6}} \approx \sqrt{6}\left(1 + \frac{m}{l2}\right)\,. \end{aligned}$$

Thus the fraction above is close to

$$\frac{\sqrt{6}\left(-1 + \frac{m}{12}\right) + \sqrt{6}\left(1 + \frac{m}{12}\right)}{m}\frac{2\sqrt{6}}{12} = \frac{1}{\sqrt{6}}\,,$$

the same result as before.

The 1970 Examination

Problem 1

The given expression is equal to

$$\sqrt{1+\sqrt{2}}\,.$$

Taking the fourth power of this we get

$$(1+\sqrt{2})^2 = 1 + 2\sqrt{2} + 2 = 3 + 2\sqrt{2}\,.$$

Problem 2

We are told that

$$P = 4l = 2\pi r\,.$$

where P is the perimeter, l is the length of a side of the square, and r is the circles radius. Then we want to compute

$$\frac{A_{\text{circle}}}{A_{\text{square}}} = \frac{\pi r^2}{l^2} = \frac{\pi r^2}{\left(\frac{\pi r}{2}\right)^2} = \frac{4}{\pi}\,.$$

Problem 3

From the equation for x we have $2^p = x - 1$. Then from the equation for y we have

$$y = 1 + \frac{1}{x-1} = \frac{x}{x-1}\,.$$

Problem 4

Let the three integers be denoted by $i-1$, i, and $i+1$ (it is usually advantageous to make things as symmetric as possible). Then the sum of the squares of these three integers (denoted by s) is given by

$$s = (i-1)^2 + i^2 + (i+1)^2 = i^2 - 2i + 1 + i^2 + i^2 + 2i + 1 = 3i^2 + 2\,.$$

If $i = 0$ then $s = 2$ and an element of S is dividable by two. If $i = \pm 1$ then $s = 5$ and a member of S is divisible by five. If $i = \pm 2$ then $s = 14$ and a member if S is divisible by seven. The given expression for s (i.e. $s = 3i^2 + 2$) means that when we divide by three we have a remainder of two. Thus no member of the set S is divisible by three. We look to see if a member of S is divisible by 11. If we take $i = 5$ we get $s = 77$ which is divisible by 11 and thus (B) is correct.

Problem 5

When $x = i$ we have that $x^4 = 1$ and $x^2 = -1$ so that $f(i) = \frac{1-1}{i+1} = 0$.

Problem 6

We write the given expression as

$$x^2 + 8x = x^2 + 8x + 16 - 16 = (x+4)^2 - 16\,.$$

Thus we see that this expression is always larger than the value of -16.

Problem 7

Let the square have vertices in an x-y coordinate system at the locations

- $(0, 0)$ for the point A
- $(s, 0)$ for the point B
- (s, s) for the point C and
- $(0, s)$ for the point D.

Then the circle through A with a radius s is given by

$$x^2 + y^2 = s^2\,,$$

The circle though the point B with a radius s is given by

$$(x-s)^2 + y^2 = s^2\,.$$

Now the distance from the point of intersection of these two circle to the segment CD is the value of $s - y$. If we subtract the first equation from the second equation we get

$$-2sx + s^2 = 0 \quad \text{or} \quad x = \frac{s}{2}\,.$$

Put this into the first equation and we get $y^2 = s^2 - \frac{s^2}{4} = \frac{3s^2}{4}$. Thus $y = \pm\frac{\sqrt{3}}{2}s$. Given this the distance from the side CD is

$$s - \left(\frac{\sqrt{3}s}{2}\right) = \frac{s}{2}(2 - \sqrt{3})\,.$$

Problem 8

Write the expression for a as

$$a = \log_8(225) = \frac{\log(225)}{\log(8)} = \frac{\log(15^2)}{\log(2^3)} = \frac{2\log(15)}{3\log(2)}$$

Next write the expression for b as

$$b = \log_2(15) = \frac{\log(15)}{\log(2)}\,.$$

From these two we see that $a = \frac{2}{3}b$.

Problem 9

From the problem we are told that

$$\frac{AP}{PB} = \frac{2}{3}\,, \quad \frac{AQ}{QB} = \frac{3}{4} \text{ and } PQ = 2\,.$$

Here the points of the segment are (in order): A, P, Q, and then B. For ease of notation let $AP = x$ and $AB = l$ then we want to evaluate l. From the first ratio we have

$$\frac{x}{l-x} = \frac{2}{3}\,,$$

while from the second ratio we have

$$\frac{x+2}{l-x-2} = \frac{3}{4}\,.$$

These are two equations for two unknowns x and l. Using the first expression we have that $x = \frac{2}{5}l$ which when we put into the second expression gives $l = 70$.

Problem 10

Multiply the expression for F by 10 to get

$$10F = 4.8181\cdots\,.$$

Next multiply that expression by 100 to get

$$1000F = 481.818181\cdots\,.$$

Now subtract these two expressions to get

$$1000F - 10F = 990F = 481 - 4 = 477\,.$$

Thus we have that $F = \frac{477}{990}$. This can be reduced (by dividing by three twice) to get

$$F = \frac{159}{330} = \frac{53}{110}\,.$$

Thus $110 - 53 = 57$.

Problem 11

Since the given expression is a third order polynomial with a leading coefficient of two we can write it in terms of the two given factors as a product like $(x+2)(x-1)(2x+r)$. When we expand this we get

$$\begin{aligned}(x+2)(x-1)(2x+r) &= (x^2+x-2)(2x+r) = 2x^3+2x^2-4x+rx^2+rx-2r \\ &= 2x^3+(2+r)x^2+(r-4)x-2r\,.\end{aligned}$$

If this is to equal $2x^3-hx+k$ then we must have $r=-2$. When $r=-2$ we then have $h=6$ and $k=4$. The expression we want to evaluate is then given by

$$|2h-3k| = |12-12| = 0\,.$$

Problem 12

Draw the circle in the rectangle as specified. Then since the circle is tangent to the sides AB and CD a line connecting these two tangent points must go through the center of the circle (which we denote Q). Thus this dimension of the rectangle (call it this height) has a length of $2r$. Since the circle is also tangent to the segment AD we have that a line from the center of the circle to its point of tangency (call that the point R) must be parallel to the segments AB and CD and bisects the segment AD. Because of this bisecting property if we extend the segment RQ it must also bisect the diagonal of the rectangle or AC (call this the point M). Since we are told that M is also on the circle, if we drop a perpendicular from M to the line AB we have that $\frac{1}{2}AC$ is the length of a right triangle with legs of lengths $2r$ and r. Thus using that the length AC can be determined from the Pythagorean theorem or

$$\left(\frac{1}{2}AC\right)^2 = (2r)^2+r^2\,.$$

This gives $AC=2\sqrt{5}r$. Using this value for the length of the diagonal of the rectangle $ABCD$ and the fact that the height of the rectangle is of length $2r$ we have that the width of the rectangle (denoted by w) is given by

$$AC^2 = 4r^2+w^2 \quad \text{so} \quad w = 4r\,,$$

when we put in the expression for AC in terms of r. Thus the area of the rectangle is given by $(2r)(4r)=8r^2$.

Problem 13

We will consider each of the given choices and show if they are correct or incorrect.

- $a*b=a^b$ while $b*a=b^a$ which are not equal.

- $a*(b*c) = a^{b^c}$ while $(a*b)*c = (a^b)^c = a^{bc}$ which are not equal.
- $(a*b^n) = a^{b^n}$ while $(a*n)*b = (a^n)*b = (a^n)^b = a^{nb}$ which are not equal.
- $(a*b)^n = (a^b)^n = a^{nb}$ while $a*(bn) = a^{bn}$ which are the same.

Problem 14

The two roots of the given quadratic are given by

$$\frac{-p \pm \sqrt{p^2-4q}}{2}.$$

If we take the difference of these two roots we get

$$\left(\frac{-p+\sqrt{p^2-4q}}{2}\right) - \left(\frac{-p-\sqrt{p^2-4q}}{2}\right) = \sqrt{p^2-4q}.$$

Since we are told that this equals one we have when we solve for p

$$p = \pm\sqrt{1+4q}.$$

Since we are told that p is positive we must take the positive sign in the above expression. Note if we took the difference of the roots in the opposite order we would get the same answer.

Problem 15

All lines go though the point $(3,4)$ so we can't eliminate any of them using the fact that the resulting line must go through that point. We next find the trisection points of the line segment which are given by

$$\frac{1}{3}(-4,5) + \frac{2}{3}(5,-1) = (-2,1),$$

and

$$\frac{2}{3}(-4,5) + \frac{1}{3}(5,-1) = (-1,3).$$

None of the lines given go through the first point, but the line $x - 4y + 13 = 0$ goes through the second point and must be the answer.

Problem 16

From the given recurrence relationship we compute

$$\begin{aligned} F(4) &= 2 \\ F(5) &= \frac{2+1}{1} = 3 \\ F(6) &= \frac{3(2)+1}{1} = 7\,. \end{aligned}$$

Problem 17

Since $r > 0$ we can divide by it to get $p > q$. If we negate both sides of that expression we get $-p < -q$ so the first choice (a) cannot be true. We can try to eliminate some remaining choices by selecting values of p and q such that $p > g$ and seeing if they satisfy the remaining inequalities. To this end let $p = 2$ and $q = 1$ then

- Part (B) or $-p > q$ is the statement $-2 > 1$ which is not true
- Part (C) or $1 > -q/p$ is the statement $1 > -1/2$ which is true
- Part (D) or $1 < q/p$ is the statement $1 < 1/2$ which is not true

Thus if we can prove that $1 > -q/p$ given $p > q$ we are done. Rather than attempt to do this directly, lets consider two different values of p and q such that $p > q$ and see if the inequality in Part (c) still holds. Towards that end let $p = \frac{1}{2}$ and $q = -1$ then $1 > -q/p$ is the statement $1 > 2$ which is not true. Thus none of the given inequalities are true.

Problem 18

Denote this expression as E. If we square this expression we get

$$\begin{aligned} E^2 &= 3 + 2\sqrt{2} - 2\left(\sqrt{3+2\sqrt{2}}\right)\left(\sqrt{3-2\sqrt{2}}\right) + 3 - 2\sqrt{2} \\ &= 6 - 2\sqrt{9-4(2)} = 6 - 2\sqrt{1} = 4\,. \end{aligned}$$

Thus $E = \pm 2$. Since we know that $\sqrt{3+2\sqrt{2}} > \sqrt{3-2\sqrt{2}}$ the value of E must be positive and thus we have $E = 2$.

Problem 19

Using Equation 20 (with $N \to \infty$) we are told that

$$\frac{a_1}{1-d} = 15$$
$$\frac{a_1^2}{1-d^2} = 45\,.$$

Solving these two equation we find $a_1 = 5$ and $d = \frac{2}{3}$.

Problem 20

If BH and CK are both perpendicular to HK then we can draw HK as a "y-axis", take CK as a "x-axis", and finally draw BH as a horizontal line parallel to CK. Then if we draw a horizontal line through the midpoint of BC by using a property of parallel lines this horizontal line must pass thought the midpoint of HK. Denote the midpoint of HK by the letter M'. Then as the right triangles $HM'M$ and $KM'M$ have equal legs ($KM' = HM'$ and $M'M$ is a common leg) they must be equivalent triangles. This means that $MK = MH$.

Problem 21 (use the snow tires?)

When the cars instrument panel is calibrated with the original tires the number of rotations N, times the circumference of the original tire $2\pi r$, must equal the distance traveled D or

$$2\pi r N = D\,. \tag{195}$$

On the first trip we know

$$2\pi r N = 450\,.$$

On the way back with the snow tires on, the radius of the tires r' is different, the number of rotations N' is different but the total distance traveled is the same or

$$2\pi r' N' = 450\,. \tag{196}$$

We are told that the instrument panel (thinking the old tires are still on) felt that the car traveled 440 miles and thus N' must satisfy

$$2\pi r N' = 440\,. \tag{197}$$

Solving this for N' gives $N' = \frac{440}{2\pi r}$. Putting this into Equation 196 gives

$$\frac{r'}{r} = \frac{450}{440} = 1 + \frac{1}{44}\,.$$

Thus

$$r' = r + \frac{r}{44} = r + \frac{15}{44} = r + 0.3409091\,.$$

Problem 22

Using Equation 22 we can write the expression we are told in the problem as

$$\sum_{k=1}^{3n} k - \sum_{k=1}^{n} k = 150\,,$$

as

$$\frac{1}{2}(3n)(3n+1) - \frac{1}{2}n(n+1) = 150\,.$$

Simplifying this we get

$$\frac{n}{2}(8n+2) = 150\,,$$

or

$$4n^2 + n - 150 = 0\,.$$

This is a quadratic equation for n which has solutions

$$n = \frac{-1 \pm \sqrt{1^2 - 4(4)(-150)}}{2(4)} = \frac{-1 \pm 49}{8}\,,$$

which gives the two values $-\frac{25}{4}$ and 6. Since n must be an integer we must have $n = 6$ then we want to evaluate

$$\sum_{k=1}^{24} k = \frac{1}{2}(24)(25) = 300\,.$$

Problem 23

Note that 10! can be written in its prime factors as

$$\begin{aligned}
10! &= 10(9)(8)(7)(6)(5)(4)(3)(2)1 \\
&= (2\cdot 5)(3^2)(2^3)(7)(2\cdot 3)(5)(2^2)(3)(2) \\
&= 2^8(3^4)(5^2)7 = (2^2)^4 \cdot 3^4 \cdot 5^2 \cdot 7 = (4\cdot 3)^4 \cdot 5^2 \cdot 7 \\
&= 12^4 \cdot 5^2 \cdot 7\,.
\end{aligned}$$

Now $5^2 \cdot 7 = 175$ and this can be written in terms of 12 as

$$175 = 12^2 + 2\cdot 12 + 7\,.$$

Thus we have

$$10! = 12^4(12^2 + 2\cdot 12 + 7) = 12^6 + 2\cdot 12^5 + 7\cdot 12^4\,.$$

In terms of base 12 we have just shown that

$$10! = (1,2,7,0,0,0,0)_{12}\,,$$

and thus ends with $k = 4$ zeros.

Problem 24

Let the length of the side of the equilateral triangle be denoted s and the length of a side of the regular hexagon be l. Then in the problem statement we are told that $3s = 6l$ so $l = \frac{s}{2}$. The area of an equilateral triangle (given the length of its side) is given by

$$A_{\text{equilateral triangle}} = \frac{\sqrt{3}}{4}s^2 \,. \tag{198}$$

We are also told that the area of the triangle is two so using the above equation and solving for s we find $s = 2\left(\frac{2}{3}\right)^{1/2}$ and thus that $l = \left(\frac{2}{3}\right)^{1/2}$. Now the area of a regular hexagon with a side length l is given by

$$A_{\text{regular hexagon}} = \frac{3\sqrt{3}}{2}l^2 \,, \tag{199}$$

when we put in what we know for l we get that the hexagon has an area of three.

Problem 25 (the cost of postage)

The best way to solve this problem is to describe the cost function using inequalities and then find which of the given functions equals this description. From the problem the cost as a function of weight is given by

$$\text{Cost}(W) = \begin{cases} 6 & 0 < W \leq 1 \\ 12 & 1 < W \leq 2 \\ 18 & 2 < W \leq 3 \\ \cdots & \end{cases}$$

If we take $W = 1$ our cost is six cents since $[1] = 1$ we can drop from consideration the solution (C) (which would give a cost of 0) and (D) (which would give a cost of 12 cents). If we take $W = 1/2$ we can drop from consideration (A) (which would give a cost of 3 cents) and (B) (which would give a cost of 0). Since we have eliminated all of the others the answer must be (E). Note that for $W = 1/2$ (E) gives a cost of six cents as it must.

Problem 26

To satisfy the first graph we must have one of the two products be zero or

$$x + y - 5 = 0 \quad \text{or} \quad 2x - 3y + 5 = 0 \,.$$

The same type of condition hold to satisfy the second graph. To satisfy both graphs at the same time we then must find (x, y) pairs that make one of the factors in each graph vanish or

$$\begin{aligned} x + y - 5 = 0 &\quad \text{and} \quad x - y + 1 = 0 \quad \text{or} \quad (x, y) = (2, 3) \\ x + y - 5 = 0 &\quad \text{and} \quad 3x + 2y - 12 = 0 \quad \text{or} \quad (x, y) = (2, 3) \\ 2x - 3y + 5 = 0 &\quad \text{and} \quad x - y + 1 = 0 \quad \text{or} \quad (x, y) = (2, 3) \\ 2x - 3y + 5 = 0 &\quad \text{and} \quad 3x + 2y - 12 = 0 \quad \text{or} \quad (x, y) = (2, 3) \,. \end{aligned}$$

As each of these pairs is satisfied by only one point.

Problem 27

Using Equation 16 we have that

$$r = \frac{K}{\frac{1}{2}K} = 2\,.$$

Problem 28

We first recall the facts that the medians of a triangle meet at a point (called the centroid) and that this point divides each median into two segments in the ratio of 1:2. If we draw the triangle as specified and let the centroid be denoted by O, the median of the side BC be denoted by M, and the median of the side AC be denoted by N. Then to use the fact that the medians are divided at the centroid into two segments in the ratio of 1:2 let the length OM be denoted by u (so that OA is of length $2u$) and the length ON be denoted by v (so that OB is of length $2v$). Then since the medians intersect at right angles by the Pythagorean theorem we have that for the right triangle BOM that

$$BM^2 = OM^2 + OB^2\,,$$

or

$$\left(\frac{7}{2}\right)^2 = u^2 + 4v^2\,.$$

For the right triangle AON we have that

$$AN^2 = AO^2 + ON^2\,,$$

or

$$3^2 = 4u^2 + v^2\,.$$

These give two equations for the two unknowns u and v. Solving them we find $u^2 = \frac{19}{12}$ and $v^2 = \frac{8}{3}$. To determine the length of AB use the right triangle AOB and the Pythagorean theorem as

$$AB^2 = BO^2 + OA^2 \quad \text{or} \quad AB^2 = 4v^2 + 4u^2\,.$$

Using what we know about u and v we get that $AB^2 = 17$.

Problem 29

If we measure angles clockwise starting at 12:00 (directly north looking at the fact of the clock) when we are just at 10:00 o'clock the minute hand is at the angular position of O and the hour hand is at the angular position

$$10\left(\frac{2\pi}{12}\right) = \frac{5\pi}{3}\,.$$

Let the current time be t minutes past 10:00 o'clock. Then for $0 \leq t \leq 60$ the angular position of the minute and the hour hand is given by

$$\theta_{\text{min}}(t) = \frac{2\pi}{60} t$$
$$\theta_{\text{hr}}(t) = \frac{5\pi}{3} + \frac{2\pi}{12(60)} t \,.$$

We are told that

$$\theta_{\text{min}}(t+6) + \pi = \theta_{\text{hr}}(t-3)\,.$$

Using the expressions above this becomes

$$\frac{2\pi}{60}(t+6) + \pi = \frac{5\pi}{3} + \frac{2\pi}{12(60)}(t-3)\,.$$

if we solve for t we find $t = 15$. Thus the time is 10:15.

Problem 30

Drop a perpendicular from the point D to the segment AB and call the point where this perpendicular intersects AB the point D'. Drop a perpendicular from the point C to the segment AB and call the point where this perpendicular intersects AB the point C'. Then since AB and CD are parallel we have that $DD' = CC'$. Lets call that distance h. Define the angle at B to be θ. Then again, as AB is parallel to CD we have that $\angle C = \pi - \theta$. From the problem we are told that $\angle D = 2\theta$ and again using parallel lines we have that $\angle A = \pi - \angle D = \pi - 2\theta$. Thus we have all angles of the quadrilateral in terms of θ. We can write the length AB as

$$AB = AD' + D'C' + C'B\,.$$

Using trigonometry we can write the above as

$$\begin{aligned}
AB &= a\cos(\pi - 2\theta) + b + \frac{h}{\tan(\theta)} \\
&= a(\cos(\pi)\cos(2\theta) + \sin(\pi)\sin(2\theta)) + b + \frac{1}{\tan(\theta)}\,(a\sin(\pi - 2\theta)) \\
&= -a\cos(2\theta) + b + \frac{a}{\tan(\theta)}\,(\sin(\pi)\cos(2\theta) - \cos(\pi)\sin(2\theta)) \\
&= -a\cos(2\theta) + b + a\frac{\sin(2\theta)}{\tan(\theta)} = -a(\cos^2(\theta) - \sin^2(\theta)) + b + \frac{2a\sin(\theta)\cos(\theta)}{\tan(\theta)} \\
&= -a(\cos^2(\theta) - \sin^2(\theta)) + b + 2a\cos^2(\theta) = a\sin^2(\theta) + b + a\cos^2(\theta) = a + b\,.
\end{aligned}$$

Problem 31

The largest digit sum we can get from a five digit number will happen when each digit is a nine where we would get the digit sum of $5 \times 9 = 45$. Because we are told that the digit sum

Number	Alternating Digit Sum
79999	7-9+9-9+9=7
97999	9-7+9-9+9=11
99799	9-9+7-9+9=7
99979	9-9+9-7+9=11
99997	9-9+9-9+7=7
88999	8-8+9-9+9=9
89899	8-9+8-9+9=7
89989	8-9+9-8+9=9
89998	8-9+9-9+8=7
98899	9-8+8-9+9=9
98989	9-8+9-8+9=11
98998	9-8+9-9+8=9
99889	9-9+8-8+9=9
99898	9-9+8-9+8=7
99988	9-9+9-8+8=9

Table 6: The five digits numbers that have their digits sum to 43.

of our five digit number is 43 we know that some of the digits must be less than nine. We can get a sum of 43 by replacing one of the of the nines in an all nine digit number with a seven *or* by replacing two nines with two eights. For each of these numbers we will want to determine if it is divisible by eleven or not. To determine if a number is divisible by eleven we will use the rule on Page 8. Thus we have the numbers given in Table 6. From that table we see that from 15 numbers only the numbers 97999, 99979, and 98989 are divisible by eleven. This gives a probability of $\frac{3}{15} = \frac{1}{5}$.

Problem 32

Draw a circle of radius r in an x-y plane and assume that A starts going counterclockwise from the point $(r, 0)$ and B starts going clockwise from the point $(-r, 0)$. Let v_A and v_B be the velocities of A and B respectively. Then if at time t_1 they first meet after B has gone 100 yards this means that

$$v_A t_1 = \frac{1}{2}C - 100$$
$$v_B t_1 = 100\,,$$

where C is the circumference of the circle. If they meet again at time t_2 or 60 yards before A has completed one lap we have

$$v_A t_2 = C - 60$$
$$v_B t_2 = \frac{1}{2}C + 60\,.$$

If we take the ratios of the first two equations and the second two equations to get expressions for $\frac{v_A}{v_B}$ we get

$$\frac{v_A}{v_B} = \frac{\frac{1}{2}C - 100}{100} = \frac{C - 60}{\frac{1}{2}C + 60}.$$

If we solve this last equation for C we find $C = 480$.

Problem 33

We will solve this problem by finding an expression for the sum S_k of the digits $1, 2, \ldots, 10^k - 2, 10^k - 1$ by induction. Then the value of the expression we derive for $k = 4$ is one less than the number we seek. $k = 1$ then we want to sum the digits

$$1, 2, 3, 4, 5, 6, 7, 8, 9.$$

This is given by

$$S_1 = \frac{1}{2}(9)(10) = 45.$$

If $k = 2$ then we want to sum the digits of the numbers in the range

$$1 - 9, 10 - 19, 20 - 29, \ldots, 80 - 89, 90 - 99.$$

Note that there are *ten* groups that will have their ones digits sum to the same $S_1 = 45$ we computed above and nine groups that have their tens digits constant in each of the ten ranges above. In each group the tens digit is repeated ten times. This gives

$$\begin{aligned} S_2 &= 10S_1 + 10(1) + 10(2) + \cdots + 10(8) + 10(9) \\ &= 10S_1 + 10(1 + 2 + \cdots + 8 + 9) = 10S_1 + 10S_1 = 20S_1 = 900. \end{aligned}$$

If $k = 3$ then we want to sum the digits of the numbers in the range

$$1 - 99, 100 - 199, 200 - 299, \ldots, 800 - 899, 900 - 999.$$

Again there are *ten* groups that will have their tens and ones digits sum to the same S_2 we computed above and nine groups that have their hundreds digits constant for all 100 numbers in each of the ten ranges above. This gives

$$S_3 = 10S_2 + 10^2 S_1 = 10(900) + 100(45) = 13500.$$

The pattern for general k seems now clear

$$S_k = 10S_{k-1} + 10^{k-1} S_1 = 10S_{k-1} + 45 \cdot 10^{k-1}.$$

Thus we want to evaluate

$$S_4 = 10S_3 + 45(1000) = 180000.$$

The true sum we want is one larger than this or 180001.

Problem 34

Let n be the integer we seek. Then the conditions of the problem imply that

$$\begin{aligned}13511 &= an + b\\ 13903 &= cn + b\\ 14589 &= dn + b\,,\end{aligned}$$

for some b such that $0 \leq b \leq n-1$. Subtracting pairs of these equations we get

$$13903 - 13511 = 392 = (c-a)n \tag{200}$$
$$14589 - 13905 = 686 = (d-a)n \tag{201}$$
$$14589 - 13511 = 1078 = (d-a)n\,. \tag{202}$$

Note that if from Equation 202 we subtract Equation 200 we get

$$1078 - 392 = 686 = [(d-a) - (c-a)]n = (d-c)n\,,$$

which is the same as Equation 201. Thus there are only two unique equations in the above set of three. We next factor the numbers on the left-hand-side of Equation 200 and Equation 201 as

$$\begin{aligned}392 &= 2(196) = 2^2(98) = 2^3(49) = 2^3 7^2\\ 686 &= 2(343) = 2(7)(49) = 2\cdot 7^3\,.\end{aligned}$$

to make n as large as possible we should take n the greatest common denominator of these two numbers or $n = 2\cdot 7^2 = 98$. Then we have

$$\begin{aligned}393 &= 2^2(98) = 4(98)\\ 686 &= 7(98)\,.\end{aligned}$$

Thus $c - a = 4$ and $d - c = 7$ but these relationships are not needed in this problem.

Problem 35

From the problem statement we must have the pension P given by $P = C\sqrt{T}$ for some constant C and T the variable representing the years of service (i.e. time). The given two conditions on the pension imply that

$$C\sqrt{T+a} = C\sqrt{T} + p \tag{203}$$
$$C\sqrt{T+b} = C\sqrt{T} + q\,. \tag{204}$$

We want to derive an expression for $C\sqrt{T}$. The above is a system of two equations in the two unknowns C and T and thus should have a solution we can find. To find it we first write the two equations as

$$\begin{aligned}\sqrt{T+a} &= \sqrt{T} + \frac{p}{C}\\ \sqrt{T+b} &= \sqrt{T} + \frac{q}{C}\,.\end{aligned}$$

If we square each of these we get

$$T + a = T + \frac{2p}{C}\sqrt{T} + \frac{p^2}{C^2} \quad \text{or} \quad a = \frac{2p}{C}\sqrt{T} + \frac{p^2}{C^2} \tag{205}$$

$$T + b = T + \frac{2q}{C}\sqrt{T} + \frac{q^2}{C^2} \quad \text{or} \quad b = \frac{2q}{C}\sqrt{T} + \frac{q^2}{C^2}\,. \tag{206}$$

We solve for $\sqrt{T}$ in Equation 205 to get

$$\sqrt{T} = \frac{Ca}{2p} - \frac{p}{2C}\,. \tag{207}$$

Put this in Equation 206 to get

$$b = \frac{2q}{C}\left(\frac{Ca}{2p} - \frac{p}{2C}\right) + \frac{q^2}{C^2} = \frac{qa}{p} + \frac{q(q-p)}{C^2}\,.$$

Solving for C^2 in the above we get

$$C^2 = \frac{qp(q-p)}{bp - aq}\,.$$

Thus taking the positive square root we have

$$C = \frac{\sqrt{qp(q-p)}}{\sqrt{bp - aq}}\,.$$

Next we use Equation 206 to get and expression for $\sqrt{T}$

$$\sqrt{T} = \frac{a}{2p}\frac{\sqrt{qp(q-p)}}{\sqrt{bp-aq}} - \frac{p}{2}\frac{\sqrt{bp-aq}}{\sqrt{pq(q-p)}}\,.$$

The pension (which is $C\sqrt{T}$) is then given by

$$C\sqrt{T} = \frac{c^2 a}{2p} - \frac{p}{2} = \frac{aq(q-p)}{2(bp-aq)} - \frac{p(bp-aq)}{2(bp-aq)} = \frac{aq^2 - bp^2}{2(bp-aq)}\,.$$

As another solution method it might have been faster to take each of the five given expressions for the pension and verify that they satisfy both of Equations 203 and 204. That approach might have been faster than the algebraic one given here.

The 1971 Examination

Problem 1

Write the given product as

$$2^{12}5^8 = 2^8 5^8 2^4 = (2 \cdot 5)^8 2^4 = 10^8 \cdot 16\,,$$

this is the number 16 followed by 8 zeros giving a total number of digits of 10.

Problem 2

From the given problem we have that the fraction $\frac{f}{bc}$ is the number of bricks laid per day per man (or the rate of brick laying). Then to lay a total of b bricks with c men will take x days where x must satisfy

$$xc\left(\frac{f}{bc}\right) = b \quad \text{so} \quad x = \frac{b^2}{f}\,,$$

when we solve for x.

Problem 3

The slope of the line between the two points is

$$m = \frac{8-0}{0+4} = 2\,.$$

Thus the line is $y = 2x + 8$. If we set $y = -4$ and solve for x we get $x = -\frac{12}{2} = -6$.

Problem 4

Let x be the amount of money the boy scouts started with. Then from what we are told

$$x\left(1 + \frac{0.05}{12}\right)^2 = 255.31\,.$$

Solving for x gives $x = 253.1956$. Thus the interest credited is then $255.31 - x = 2.114359$ which gives 11 cents.

Problem 6

Note that (A) and (B) must be true since "regular" multiplication has these properties. Note that for (C) we have $\frac{1}{2} \star a = a$ and so $\frac{1}{2}$ is the multiplicative inverse. For (D) to be true we must be able to find an inverse for every element a in S. If we denote a^{-1} by this inverse we must have $a^{-1} \star a = \frac{1}{2}$ and thus

$$2(a^{-1})a = \frac{1}{2} \quad \text{so} \quad a^{-1} = \frac{1}{4a}\,.$$

Thus the inverse of a under the operation $\star$ exists and is given by $\frac{1}{4a}$ for all $a > 0$. Note that in (E) we have

$$a \star \left(\frac{1}{2a}\right) = 2a\left(\frac{1}{2a}\right) = 1\,.$$

But 1 is not the multiplicative inverse for this operation, since $a \star 1 = 2a \neq a$. Thus the element $\frac{1}{2a}$ is not what gives us the identity (which is $\frac{1}{2}$ for this product) but $\frac{1}{4a}$ is.

Problem 7

We can write the given expression as

$$2^{-(2k+1)} - 2^{-(2k-1)} + 2^{-2k} = 2^{-2k}(2^{-1} - 2 + 1) = 2^{-2k}(2^{-1} - 1) = 2^{-2k}(-2^{-1}) = -2^{-(2k+1)}\,.$$

Problem 8

Write the inequality as

$$6x^2 + 5x - 4 < 0\,.$$

The roots of the left-hand-side are given by the quadratic equation where we find

$$x = \frac{-5 \pm \sqrt{25 - 4(6)(-4)}}{2(6)}\,,$$

which simplify to $-\frac{4}{3}$ and $\frac{1}{2}$. Now if $x < -\frac{4}{3}$ or $x > \frac{1}{2}$ the left-hand-side of the original inequality is positive. If x is between these two limits (i.e. $-\frac{4}{3} < x < \frac{1}{2}$) then the left-hand-side is negative and the desired inequality is satisfied.

Problem 10

In this universe a person can only have blond or brunette hair and blue or brown eyes. Thus all 50 people must be a member of one and only one of the following sets

$$\text{Blond} \cap \text{Blue}\,, \text{Blond} \cap \text{Brown}\,, \text{Brunette} \cap \text{Blue}\,, \text{Brunette} \cap \text{Brown}\,.$$

Since there are 31 brunette people there must be $50 - 31 = 19$ blond people. Since there are 18 brown eyed people there must be $50 - 18 = 32$ blue eyed people. Since there are 14 blond haired and blue eyed people there must be $19 - 14 = 5$ blond haired and brown eyed people. Also since there are 14 blond haired and blue eyed people there must be $32-14 = 18$ brunette and blue eyed people. Since we have the number of members in each of the above sets but the one that we want the number of brunette and brown eyed people must be

$$50 - 14 - 5 - 18 = 13\,.$$

Problem 11

Note that both a and b must be larger than the largest digit in either of 47 or 74 which is seven. The number 47 in base a means that

$$47_a = 4a + 7\,,$$

and the number 74 in base b means that

$$74_b = 7b + 4\,.$$

As these numbers are the same we can subtract them to get zero or the relationship

$$4a - 7b + 3 = 0\,,$$

or solving for a we get

$$a = \frac{1}{4}(7b - 3)\,.$$

We now need to find the two smallest integers a and b such that the above equation hold true. Starting with $b \in \{8, 9, 10, 11, 12, 13, 14, 15\}$ using the above equation to solve for a we would find that the smallest value of b that gives a as an integer is $b = 9$ where we get $a = 15$. We can check that we can't get two smaller values by solving for b to get

$$b = \frac{1}{7}(4a + 3)\,.$$

When $a \in \{8, 9, 10, 11, 12, 13, 14, 15\}$ the smallest value of a such that $b > 7$ is again given by $a = 15$. Thus $a + b = 24$ which is XXIV as a Roman numeral.

Problem 12

We first seek to find the number N that when we divide 69, 90, and 125 by that number we get the same remainder. As 90 is divisible by two and three while at least one other number in our list is not, we know that $N \neq 2$ and $N \neq 3$. As

$$69 \bmod 4 = 1\,,$$

while

$$90 \bmod 4 = 2\,,$$

so that we know $N \neq 4$. As 125 is divisible by five and 69 is not we know that $N \neq 5$. As

$$
\begin{aligned}
69 \bmod 6 &= 3 \\
90 \bmod 6 &= 4\,,
\end{aligned}
$$

we know $N \neq 6$. For $N = 7$ note that

$$
\begin{aligned}
69 \bmod 7 &= 6 \\
90 \bmod 7 &= 6 \\
125 \bmod 7 &= 6\,,
\end{aligned}
$$

and for this value of N we have $81 \bmod 7 = 4$. The value of four is such that $4 \bmod 7 = 4$.

Problem 13

Write the expression we want to evaluate as

$$
(1 + 0.0025)^{10} = (1 + 2.510^{-3})^{10}\,.
$$

Then using the binomial theorem

$$
(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}\,,
$$

we can write the expression we are attempting to evaluate as

$$
\begin{aligned}
(1 + 2.5\,10^{-3})^{10} = (2.5\,10^{-3} + 1)^{10} &= \sum_{k=0}^{10} \binom{10}{k} (2.5\,10^{-3})^k \\
&= \sum_{k=0}^{10} \binom{10}{k} \left(\frac{5}{2}\right)^k 10^{-3k} \\
&= 1 + 10\left(\frac{5}{2}\right) 10^{-3} + \binom{10}{2}\left(\frac{5}{2}\right)^2 10^{-6} + \binom{10}{3}\left(\frac{5}{2}\right)^3 10^{-9} + \dots \\
&\approx 1 + 25\,10^{-3} + 45\left(\frac{25}{4}\right) 10^{-6} + 120\left(\frac{125}{8}\right) 10^{-9} \\
&= 1 + 0.025 + c_2 10^{-6} + c_3 10^{-9}\,.
\end{aligned}
$$

Because of their values the first and second terms won't contribute to the digit in the fifth decimal place. We next evaluate the two "coefficients" c_2 and c_3. For c_2 we have

$$
c_2 = 45\left(\frac{24 + 1}{4}\right) = 45(6.25) = 281.25\,,
$$

while for c_3 we have

$$
c_3 = 30\left(\frac{125}{2}\right) = 15(125) = 1875\,.
$$

From these numbers the contribution of the third and fourth terms in the above sum are then given by

$$
0.00028125 \quad \text{and} \quad 0.000001875\,.
$$

The only nonzero digit in the fifth decimal place comes from the first number and has the value eight.

Problem 14

Factor the given expression as follows

$$2^{48} - 1 = (2^{24} - 1)(2^{24} + 1) = (2^{12} - 1)(2^{12} + 1)(2^{24} + 1) = (2^6 - 1)(2^6 + 1)(2^{12} + 1)(2^{24} + 1)\,.$$

The values of two of the products in the above expansion namely $2^6 - 1 = 63$ and $2^6 + 1 = 65$ are between 60 and 70 we have found the requested numbers.

Problem 15 (tipping the aquarium)

We are told that the height of the aquarium is 8 inches and the width of the aquarium is 10 inches. Let the depth of the aquarium be denoted as d and the unknown height of the water (when the aquarium is flat) as h. Then the volume of the water when the aquarium is flat is

$$10hd\,.$$

Next we tip the aquarium forward slowly and then stop when the water just touches the top and would spill out if we tipped forward any more. When viewed from the side the shape of the water in that configuration has a triangular base with a height the full 10 inches of the width. One length of the triangular base is the 8 inch height and the water goes $\frac{3}{4}$ up the bottom the volume of the water in this configuration is

$$\text{height} \times \text{area of triangular base} = 10\left(\frac{1}{2}\left(\frac{3}{4}d\right)(8)\right) = 30d\,.$$

Setting this equal to the expression above we can cancel d and get $h = 3$.

Problem 16

The correct average of the thirty-five numbers would be given by $\bar{x} = \frac{1}{35}\sum_{i=1}^{35} x_i$. Then from what we are told the student did the average they computed would be given by

$$\bar{x}' = \frac{1}{36}\left[\sum_{i=1}^{35} x_i + \bar{x}\right] = \frac{1}{36}\left[35\bar{x} + \bar{x}\right] = \bar{x}\,.$$

That is the student actually got the correct value for the average and the ratio is one.

Problem 17

Note that all answers are linear in n. This means that if we can determine the value of this function (i.e. the maximum number of non-overlapping areas) for two values of n we can find which of the given expressions is correct by observing if they give the known values on these two cases. For example when $n = 1$ the disk is split with 2 equally spaced radii (i.e. by a diameter) the chord that will given the maximum number of non-overlapping areas is one that does not go through the center and gives four areas. When $n = 2$ the disk is split into four quadrants and (by observation) the maximal number of non-overlapping areas is when the chord crosses through one point on an "up-down" radius and one point on a "right-left" radius. Doing this will give seven areas. Evaluating each of the choices at $n = 1$ and $n = 2$ only the expression $3n + 1$ satisfies these two "base cases" and must be the right answer.

Problem 18 (a boat in a river)

Let v_r and v_b be the velocity of the river and boat respectively. We are told that $v_r = 3$ and that the length of each leg of the journey was four miles. Let t_d and t_u be the time it takes to go downstream/upstream. As equations the downstream and upstream journeys must then satisfy

$$(v_r + v_b)t_d = 4 \quad (208)$$
$$(v_b - v_r)t_u = 4\,. \quad (209)$$

The total time takes one hour means that $t_d + t_u = 1$. Using the fact that $v_r = 3$ we get the three equations

$$\begin{aligned}(3 + v_b)t_d &= 4 \\ (v_b - 3)t_u &= 4 \\ t_d + t_u &= 1\,.\end{aligned}$$

in the three unknowns t_d, t_u, and v_b. Using the last equation in the other two we get

$$\begin{aligned}(3 + v_b)(1 - t_u) &= 4 \\ (v_b - 3)t_u &= 4\,.\end{aligned}$$

Thus from the last equation we get

$$t_u = \frac{4}{v_b - 3}\,.$$

Putting this into the first gives

$$(3 + v_b)\left(1 - \frac{4}{v_b - 3}\right) = 4\,.$$

Simplifying this we get the quadratic

$$v_b^2 - 8v_b + 9 = 0\,,$$

which can be factored as

$$(v_b - 9)(v_b + 1) = 0 \quad \text{so} \quad v_b = -1 \quad \text{or} \quad v_b = 9\,.$$

The velocity of the boat must be positive so the ratio requested is

$$\frac{v_r + v_b}{v_b - v_r} = \frac{3 + 9}{9 - 3} = 2\,.$$

Problem 19

The line $y = mx + 1$ goes through the point $(0, 1)$ and points either upwards or downwards depending on the sign of m. The ellipse $x^2 + 4y^2 = 1$ can be written as

$$x^2 + \frac{y^2}{\left(\frac{1}{2}\right)^2} = 1\,,$$

which shows that the ellipse goes through the points $(1, 0)$, $\left(0, \frac{1}{2}\right)$, $(-1, 0)$ and $\left(0, -\frac{1}{2}\right)$. Drawing this ellipse and drawing the line $y = mx + 1$ we see that if $m < 0$ the line could intersect the ellipse at points in the first quadrant while if $m > 0$ the line could intersect the ellipse at points in the second quadrant.

To find the locations x where these two curves intersect, put the given expression for y into the equation for the ellipse to get

$$x^2 + 4(mx + 1)^2 = 1\,,$$

or

$$(1 + 4m^2)x^2 + 8mx + 3 = 0\,.$$

Solving this with the quadratic equation we get

$$x = \frac{-8m \pm \sqrt{64m^2 - 4(1 + 4m^2)(3)}}{2(1 + 4m^2)} = \frac{-4m \pm \sqrt{4m^2 - 3}}{1 + 4m^2}\,.$$

From this expression we will have zero, one, or two solutions depending on the value of $4m^2 - 3$. We will have only one intersection if $4m^2 - 3 = 0$ or $m^2 = \frac{3}{4}$.

Problem 20 (the absolute value of h)

Method I: Solving the quadratic $x^2 + 2hx - 3 = 0$ we get

$$x = \frac{-2h \pm \sqrt{4h^2 - 4(-3)}}{2} = -h \pm \sqrt{h^2 + 3}\,.$$

Thus the sum of the two roots squared is given by

$$\begin{aligned}
x_-^2 + x_+^2 &= (-h - \sqrt{h^2 + 3})^2 + (-h + \sqrt{h^2 + 3})^2 \\
&= (h + \sqrt{h^2 + 3})^2 + (h - \sqrt{h^2 + 3})^2 \\
&= h^2 + 2h\sqrt{h^2 + 3} + h^2 + 3 + h^2 - 2h\sqrt{h^2 + 3} + h^2 + 3 \\
&= 4h^2 + 6\,.
\end{aligned}$$

We are told this equals 10 which if we solve for h^2 gives $h^2 = 1$ or $h = \pm 1$. Thus $|h| = 1$.

Method II: From the form of the quadratic given in the problem the sum of the two roots must take the form

$$x_-^2 + x_+^2 = (3 - 2hx_+) + (3 - 2hx_-) = 6 - 2h(x_+ + x_-)\,.$$

To evaluate this we recall that the two solutions to a quadratic

$$x^2 + bx + c = 0 \,,$$

have roots that have their sum to $-b$ and have their product of c. Thus the sum of $x_+ + x_-$ in this case must equal $-2h$ and the above becomes

$$x_-^2 + x_+^2 = 6 + 4h^2 \,,$$

the same result we had earlier. The rest of the solution is as above.

Problem 21

We start by solving each of the given expressions for x, y, and z respectively. For x we have

$$\begin{aligned}
\log_3(\log_4(x)) &= 1 \\
\log_4(x) &= 3 \\
x &= 4^3 = 64 \,.
\end{aligned}$$

For y we have

$$\begin{aligned}
\log_4(\log_2(y)) &= 1 \\
\log_2(y) &= 4 \\
y &= 2^4 = 16 \,.
\end{aligned}$$

For z we have

$$\begin{aligned}
\log_2(\log_3(z)) &= 1 \\
\log_3(z) &= 2 \\
z &= 3^2 = 9 \,.
\end{aligned}$$

Thus using these values we find $x + y + z = 89$.

Problem 22

We start by expanding the product $(1 - w + w^2)(1 + w - w^2)$ as

$$\begin{aligned}
(1 - (w - w^2))(1 + (w - w^2)) &= 1 - (w - w^2)^2 = 1 - (w^2 - 2w^3 + w^4) \\
&= 1 - (w^2 - 2 + w) = 3 - w^2 - w \,. \qquad (210)
\end{aligned}$$

In the above we have used the fact that $w^3 = 1$. The solutions of the equation $w^3 = 1$ are the numbers

$$w = e^{\frac{2\pi i}{3} n} \quad \text{for} \quad n = 0, 1, 2 \,.$$

Since we are told that w is imaginary we know that $n \neq 0$ since that would give the root $w = 1$ (which is not imaginary). Note that

$$w^2 = e^{\frac{4\pi i}{3} n} = e^{\frac{6\pi i - 2\pi i}{3} n} = e^{\frac{6\pi i}{3} n} e^{-\frac{2\pi i}{3} n} = \bar{w} \,,$$

the complex conjugate of w. Thus the expression we want to simplify in Equation 210 above is given by

$$3 - \bar{w} - w = 3 - (w + \bar{w}) \,.$$

If $n = 1$ then we have $w = w_1 = e^{\frac{2\pi i}{3}}$ and $\bar{w} = e^{-\frac{2\pi i}{3}}$ so that in this case we get

$$3 - 2\cos\left(\frac{2\pi}{3}\right) = 3 - 2\left(-\frac{1}{2}\right) = 4 \,.$$

If $n = 2$ then we have $w = w_2 = e^{\frac{4\pi i}{3}}$ and $\bar{w} = e^{-\frac{4\pi i}{3}}$ so that in this case we get

$$3 - 2\cos\left(\frac{4\pi}{3}\right) = 3 - 2\left(-\frac{1}{2}\right) = 4 \,,$$

the same value.

Problem 23 (two teams)

Let $p_{n,m}$ be the probability that A wins the series of games given that A needs n wins and B needs m wins to win the series. Let p be the probability that A wins a single game when playing against B (here we are told that $p = \frac{1}{2}$). Conditioning on the outcome of the next game gives

$$p_{n,m} = p p_{n-1,m} + (1 - p) p_{n,m-1} \,.$$

Using the above we desire to compute $p_{2,3}$. Some simple boundary cases are

$$\begin{aligned} p_{n,0} &= 0 \\ p_{0,m} &= 1 \\ p_{1,1} &= p = \frac{1}{2} \,. \end{aligned}$$

Using these we can compute

$$\begin{aligned} p_{1,2} &= p p_{0,2} + (1 - p) p_{1,1} = \frac{1}{2}(1) + \frac{1}{2}\left(\frac{1}{2}\right) = \frac{3}{4} \\ p_{2,1} &= p p_{1,1} + (1 - p) p_{2,0} = \frac{1}{2}\left(\frac{1}{2}\right) + \frac{1}{2}(0) = \frac{1}{4} \\ p_{2,2} &= p p_{1,2} + (1 - p) p_{2,1} = \frac{1}{2}\left(\frac{3}{4}\right) + \frac{1}{2}\left(\frac{1}{4}\right) = \frac{1}{2} \,. \end{aligned}$$

With the above values we now compute the probability of interest

$$\begin{aligned} p_{2,3} = p p_{1,3} + (1 - p) p_{2,2} &= \frac{1}{2}\left(\frac{1}{2} p_{0,3} + \frac{1}{2} p_{1,2}\right) + \frac{1}{2}\left(\frac{1}{2}\right) \\ &= \frac{1}{2}\left(\frac{1}{2}(1) + \frac{1}{2}\left(\frac{3}{4}\right)\right) + \frac{1}{4} = \frac{1}{2} + \frac{3}{16} = \frac{11}{16} \,. \end{aligned}$$

Thus the odds for A to win are 11 to 5.

Problem 24 (counting Pascal's triangle)

Row n has n elements with two (the first and the last) that have the value one and so $n-2$ non-ones. The number of non-ones up to and including row n is then

$$\sum_{k=2}^{n}(k-2) = \sum_{k=3}^{n}(k-2) = \sum_{k=1}^{n-2} = \frac{1}{2}(n-1)(n-2)\,.$$

The number of ones up to and including row n is then

$$1 + 2 + 2 + \cdots + 2 = 1 + 2(n-1) = 2n - 1\,.$$

Thus the quotient of the number of non-ones to the number of ones is given by

$$\frac{(n-1)(n-2)}{2(2n-1)} = \frac{n^2 - 3n + 2}{4n - 2}\,.$$

Problem 25

Let f and b be the fathers and boys age respectfully. Then the problem states that

$$100f + b - (f - b) = 4289\,,$$

or

$$99f + 2b = 4289\,.$$

As $4289 = 43(99) + 32$ lets write $f = \hat{f} + 43$ to get

$$99\hat{f} + 2b = 32\,.$$

One solution to the above is $\hat{f} = 0$ and $b = 16$. Then one solution to the original equation is $f = 43$ and $b = 16$. Using this information we have that *all* solutions to the original equation are given by

$$\begin{aligned} f &= 43 - 2t \\ b &= 16 + 99t\,, \end{aligned}$$

for t an integer. We can verify that these expressions are solutions as

$$99(43 - 2t) + 2(16 + 99t) = 4289\,,$$

as required. From the expression for b given above we see that if $t \geq 1$ the boy will not be a teenage (if $t < 0$ his age is negative). The only value of t that does not give a contradiction is when $t = 0$. In that case we find $f + b = 59$.

Problem 26

Draw a line through F and parallel to AE. Let the point of intersection of this line with the segment BC be denoted F'. Then triangles $BF'F$ is similar to BEG and thus we have

$$\frac{BG}{BE} = \frac{BF}{BF'} = \frac{2BG}{BF'}\,.$$

Thus canceling BG from both sides we have

$$\frac{1}{BE} = \frac{2}{BF'}\,,$$

or $BF' = 2BE$. Now as $BF' = BE + EF'$ we get that $EF' = BE$.

Now triangle AEC is similar to $FF'C$ and thus

$$\frac{AC}{EC} = \frac{FC}{F'C} \quad \text{or} \quad \frac{3AF}{F'C + EF'} = \frac{2AF}{F'C}\,.$$

In this last equation we cancel AF from both sides we get

$$3F'C = 2(F'C + EF') \quad \text{or} \quad F'C = 2EF'\,.$$

Using this we can compute EC as

$$EC = EF' + F'C = 3EF'\,,$$

and since $BE = EF'$ so the point E divides the side in the ratio of 1:3.

Problem 27

If we let r, b, and w be the number of red, blue, and white chips respectively then we are told that

$$b \geq \frac{1}{2}w \tag{211}$$

$$b \leq \frac{1}{3}r \tag{212}$$

$$w + b \geq 55\,. \tag{213}$$

If we consider the w-b plane then Equation 211 and Equation 213 place constraints on how small b can be for a given value of w. Thus in the b-w space we draw the lines $b = \frac{1}{2}w$ and $b = 55 - w$. These two lines form a "V" in the b-w space such that the value of b must be above this 'V" for any given value of w. These two lines have their vertex (i.e. intersect) at the point

$$(w, b) = \left(\frac{110}{3}, \frac{55}{3}\right) = (36.66, 18.33)\,.$$

This is the location of the smallest value that b could be. As r, b, and w must be integers we can have $w \in \{36, 37\}$. If $w = 36$ then $b \geq 55 - w = 19$. If $w = 37$ then $b \geq \frac{1}{2}w = 18.5$. Thus the smallest b can be in each case is 19. Then using Equation 212 the smallest r can be is $3b = 57$.

Problem 28

When we divide the triangle into n pieces with $n-1$ parallel lines we take the original triangles height of h and cut it into segments of length $\frac{h}{n}$. If we envision the base of the triangle "at the bottom" and of length b then each parallel line in the triangle is smaller than the one before it by the fraction $\frac{1}{n}b$. That is the base has a length b, the parallel line above this one is of length $\left(\frac{n-1}{n}\right)b$, the parallel line above that one is of length $\left(\frac{n-2}{n}\right)b$, etc. Since we are told that we have $n-1=9$ parallel lines we have $n=10$ and the first parallel line above the base has a length $\frac{9}{10}b$ and a height of $\frac{h}{10}$. The area bounded by the base at the bottom and the first parallel line is a trapezoid and thus has an area of "one half times the height times the sum of the bases" or

$$\frac{1}{2}\left(\frac{h}{10}\right)\left(b+\frac{9}{10}b\right).$$

As we are told this equals 38 we can solve for $\frac{1}{2}bh$ (the area of the original triangle) to get

$$\frac{1}{2}bh = 200\,.$$

Problem 29

We have

$$\prod_{k=1}^{n} a_k = \prod_{k=1}^{n} 10^{k/11} = 10^{\frac{1}{11}\sum_{i=1}^{n} k} = 10^{\frac{1}{11}\left(\frac{n(n+1)}{2}\right)}.$$

If we want this larger than $100000 = 10^5$ we must have

$$\frac{n(n+1)}{22} > 5\,,$$

or

$$n^2+n-110>0\,.$$

The roots of the quadratic on the left-hand-side are

$$n = \frac{-1 \pm \sqrt{1^2+4(110)}}{2} = \frac{-1\pm\sqrt{441}}{2} = \frac{-1\pm 21}{2}\,.$$

The only positive value root is $n=10$. Thus we need to take $n>10$ or $n\geq 11$.

Problem 30

Now as $f_{35}=f_5$ we must have $f_{34}=f_4$. To show this is true we assume that it is not i.e. that is $f_{34}\neq f_4$ then $f_{35}=f_1(f_{34})\neq f_1(f_4)=f_5$ which is a contradiction on what we are told. In the same way we can conclude that

$$
\begin{aligned}
f_{33} &= f_3 \\
f_{32} &= f_2 \\
f_{31} &= f_1 \\
f_{30} &= x \,.
\end{aligned}
$$

Thus one way to solve this problem it to consider the given expressions for f_{28} and from them compute f_{30} and see for which of the choices we get x for f_{30}.

For answer A if $f_{28} = x$ then $f_{29} = f_1(x)$ and then $f_{30} = f_1(f_1(x))$ which one can compute is not equal to x. Thus A cannot be true.

For answer B if $f_{28} = \frac{1}{x}$ then

$$
f_{29} = \frac{2\left(\frac{1}{x}\right) - 1}{\left(\frac{1}{x}\right) + 1} = \frac{2 - x}{1 + x} \,,
$$

and

$$
f_{30} = \frac{\frac{4-2x}{1+x} - 1}{\frac{2-x}{1+x} + 1} = 1 - x \neq x \,.
$$

Thus B cannot be true.

For answer C if $f_{28} = \frac{x-1}{x}$ then

$$
f_{29} = \frac{\frac{2x-2}{x} - 1}{\frac{x-1}{x} + 1} = \frac{x - 2}{2x - 1} \,,
$$

and

$$
f_{30} = \frac{\frac{2x-4}{2x-1} - 1}{\frac{x-2}{2x-1} + 1} = \frac{1}{1 - x} \neq x \,.
$$

Thus C cannot be true.

For answer D if $f_{28} = \frac{1}{1-x}$ then

$$
f_{29} = \frac{\frac{2}{1-x} - 1}{\frac{1}{1-x} + 1} = \frac{1 + x}{2 - x} \,,
$$

and

$$
f_{30} = \frac{\frac{2+2x}{2-x} - 1}{\frac{1+x}{2-x} + 1} = x \,,
$$

when we simplify. Thus answer D is the correct one.

Problem 31

Let the center of the circle be denoted the point O. Then the length CD is the length of one side of the triangle $\triangle COD$. If we knew the value of the angle $\angle COD$ (or its cosign) then we could use the law of cosigns to compute the length of CD since we know the value of the lengths of the adjacent two sides $CO = DO = 2$ (as the radius of the given circle is two). Given this we will work to compute $\cos(\angle COD)$.

Now as the radius of this circle is of length two we have that the two triangles $\triangle ABO$ and $\triangle BCO$ both have two sides of length two and one side of length one. Since we know all three sides of these triangles we can use the law of cosigns to compute the cosign of the angles $\angle AOB = \angle BOC$. We have

$$\cos(\angle AOB) = \frac{1^2 - 2^2 - 2^2}{-2(2)(2)} = \frac{7}{8}\,.$$

Next note that we have

$$\pi = 2\angle AOB + \angle COD \quad \text{so} \quad \angle COD = \pi - 2\angle AOB\,.$$

The cosign that we want to evaluate is then given by

$$\cos(\angle COD) = \cos(\pi - 2\angle AOB) = -\cos(2\angle AOB)\,.$$

Using the double angle formula for cosign

$$\cos(2\theta) = \cos(\theta)^2 - \sin(\theta)^2 = 2\cos^2(\theta) - 1\,, \tag{214}$$

we get that

$$\cos(\angle COD) = -(2\cos^2(\angle AOB) - 1) = -\frac{17}{32}\,,$$

when we use the known value for $\cos(\angle AOB)$. Using this with the law of cosigns we can compute the length CD as

$$CD^2 = 2^2 + 2^2 - 2(2)(2)\cos(\angle COD) = \frac{49}{4}\,.$$

Thus $CD = \frac{7}{2}$.

Problem 32

Let $r = 2^{-1/32}$ and then the product we want to evaluate is

$$(1+r)(1+r^2)(1+r^4)(1+r^8)(1+r^{16})\,.$$

Consider just multiplying two factors in the above product

$$(1+r)(1+r^2) = 1 + r^2 + r + r^3 = \sum_{k=0}^{3} r^k\,.$$

What about multiplying three factors in the above product

$$(1+r)(1+r^2)(1+r^4) = \left(\sum_{k=0}^{3} r^k\right)(1+r^4) = \sum_{k=0}^{3} r^k + \sum_{k=0}^{3} r^{k+4} = \sum_{k=0}^{7} r^k\,.$$

In the same way we compute the product of the first four and five factors to be

$$\sum_{k=0}^{15} r^k \quad \text{and} \quad \sum_{k=0}^{31} r^k\,.$$

The above can be evaluated using the geometric series to get

$$\frac{1-r^{31+1}}{1-r} = \frac{1-r^{32}}{1-r} = \frac{1}{2(1-2^{-1/32})}\,.$$

Problem 33

A geometric progression takes the following form

$$a_0\,, a_0r\,, a_0r^2\,, a_0r^3\,, \dots\,,$$

so the product of n of its terms is

$$P = \prod_{k=0}^{n-1} a_o r^k = a_0^n \prod_{k=0}^{n-1} r^k = a_0^n r^{\sum_{k=0}^{n-1}} = a_0^n r^{\frac{n(n-1)}{2}}\,.$$

The sum of n of its terms is

$$S = \sum_{k=0}^{n-1} a_0 r^k = a_0 \frac{1-r^n}{1-r}\,,$$

and the sum of n reciprocals of its terms is

$$S' = \sum_{k=0}^{n-1} \frac{1}{a_0} r^{-k} = \frac{1}{a_0}\left(\frac{1-r^{-n}}{1-r^{-1}}\right) = \frac{1}{a_0}\left(\frac{r^n-1}{r^n-r^{n-1}}\right) == \frac{1}{a_0}\left(\frac{r^n-1}{r-1}\right)\frac{1}{r^{n-1}}\,.$$

Note that we can write the above as

$$S' = \frac{1}{a_0}\left(\frac{S}{a_0}\right)\frac{1}{r^{n-1}} \quad \text{so} \quad \frac{S'}{S} = \frac{1}{a_0^2}\frac{1}{r^{n-1}} \tag{215}$$

Also note that P can be written

$$P = \left[a_0 r^{\frac{1}{2}(n-1)}\right]^n = \left[a_0^2 r^{n-1}\right]^{\frac{n}{2}}\,. \tag{216}$$

Solving for $a_0^2 r^{n-1}$ in Equation 215 we get

$$a_0^2 r^{n-1} = \frac{S}{S'}\,.$$

If we put this into Equation 216 we get

$$P = \left(\frac{S}{S'}\right)^{n/2}\,.$$

Problem 34

Note: I'm not sure why this is wrong. If anyone sees what is wrong with this approach please contact me.

The minute hand has a position given by $\theta_{\text{minute hand}} = 2\pi\left(\frac{t}{69}\right)$ with the angle "zero" be when the minute hand points north (to the 12 on the clock). The worker work eight hours according to this slow clock. This means the minute hand cycles eight times and each cycle is 69 minutes so $8(69) = 552$ minutes must pass This is

$$\frac{552}{60} = 9.2\,,$$

hours. Thus an extra 1.2 hour of work. At time-and-one half we would pay six dollars per hour for a total payment of $1.2(6) = 7.2$ dollars. The "extra" pay would then be

$$7.2 - 1.2(4) = 2.4\,,$$

dollars.

Problem 35

It helps with this problem to draw a right angle with two circles (one large and one small) that are tangent to the rays emanating from the right angle. The smaller circle is "closer to the corner" than the larger one. In my picture I had the right angle in the top-left of my drawing and in the larger circle I drew two radii: one pointing upwards (and touching the tangent point of the right angle) and one pointing to the center of the small circle (and touching the tangent point between the two circles). In the smaller circle I drew one radius pointing upwards (and touching the tangent point of the right angle).

We will now derive the radius of the larger circle in terms of the radius of the smaller circle. Let the radius of the larger circle be denoted as R_0 and the radius of the smaller circle be denoted as R_1. Notice that as measured along the radius that points upwards (North in the picture) R_0 is equal to the projection of the line segment that points from the center of the larger circle to the center of the little circle plus the radius R_1 of the smaller circle. Since the angle between the two radii of the larger circle is $\frac{\pi}{4}$ we have that

$$R_0 = (R_0 + R_1)\cos\left(\frac{\pi}{4}\right) + R_1 = \frac{1}{\sqrt{2}}(R_0 + R_1) + R_1\,.$$

If we multiply by $\sqrt{2}$ and then solve for R_1 we get

$$\sqrt{2}R_0 = R_0 + R_1 + \sqrt{2}R_1 \quad \text{so} \quad R_1 = \frac{\sqrt{2}-1}{\sqrt{2}+1}R_0\,.$$

We can simply the above expression by removing all square roots from the denominator as

$$R_1 = \frac{\sqrt{2}-1}{\sqrt{2}+1}\left(\frac{\sqrt{2}-1}{\sqrt{2}-1}\right)R_0 = \frac{(\sqrt{2}-1)^2}{2-1}R_0 = (\sqrt{2}-1)^2 R_0\,.$$

I doing this we have shown the identity (which we will use later)

$$\frac{1}{\sqrt{2}+1} = \sqrt{2} - 1\,.$$

As the above holds for just two circles we can repeat this process an infinite number of times. When we do that we get that the kth circle has a radius given by $R_k = (\sqrt{2}-1)^{2k} R_0$ for $k = 0, 1, 2, \cdots$. The ratio we are asked to compute for this problem πR_0^2 divided by the sum

$$\sum_{k=1}^{\infty} (\sqrt{2}-1)^{4k} \pi R_0^2 = R_0^2 \pi \left(\frac{(\sqrt{2}-1)^4}{1-(\sqrt{2}-1)^4} \right)\,.$$

Now as

$$(\sqrt{2}-1)(\sqrt{2}+1) = 1\,,$$

if we multiply the top and bottom of this expression by $(\sqrt{2}+1)^4$ we get

$$\begin{aligned} \pi R_0^2 \left(\frac{1}{(\sqrt{2}+1)^4 - 1} \right) &= \pi R_0^2 \left(\frac{1}{(2+2\sqrt{2}+1)^2 - 1} \right) = \pi R_0^2 \left(\frac{1}{(3+2\sqrt{2})^2 - 1} \right) \\ &= \pi R_0^2 \left(\frac{1}{(9+12\sqrt{2}+8)^2 - 1} \right) = \pi R_0^2 \left(\frac{1}{16+12\sqrt{2}} \right)\,. \end{aligned}$$

Which gives the answer C.

The 1972 Examination

Problem 1

Lets check that for the given three sides that we have $a^2 + b^2 = c^2$ as they must when they are the sides of a right triangle. We have

- I: $3^2 + 4^2 = 5^2$ is true. This is the most "common" right triangle.
- II: We have that $4^2 + 7.5^2 = 8.5^2$ which is true.
- III: We have $7^2 + 24^2 = 49 + 576 = 625 = 25^2$ and so we have another right triangle.
- IV: We have $3.5^2 + 4.5^2 = 32.5 \neq 5.5^2 = 30.25$ and thus this is not a right triangle.

Problem 2

Lets assume that we purchase the goods at a price g and sell the goods are a price s for a profit of

$$\frac{s-g}{g} = x \quad \text{so} \quad s = (1+x)g\,. \tag{217}$$

If the price of goods decreases and goes to $0.92g$ while the profit increase to $x + 0.1$ then we must have

$$\frac{s - 0.92g}{0.92g} = x + 0.1\,,$$

or

$$s = 0.92g + 0.92g(x + 0.1)\,. \tag{218}$$

If we put Equation 217 into the left-hand-side of the above to get

$$(1+x)g = 0.92g + 0.92g(x+0.1)\,,$$

which gives $x = 0.15$ when we cancel g and solve for x.

Problem 3

We write the given expression as

$$\frac{1}{x^2 - x} = \frac{1}{x(x-1)}\,.$$

For the given value of x we have that

$$x - 1 = \frac{1 - i\sqrt{3}}{2} - \frac{2}{2} = \frac{-1 - i\sqrt{3}}{2}\,.$$

Thus the denominator is given by

$$x(x-1) = \left(\frac{1-i\sqrt{3}}{2}\right)\left(\frac{-1-i\sqrt{3}}{2}\right) = \frac{-1-i\sqrt{3}+i\sqrt{3}-3}{4} = -1\,.$$

Thus $\frac{1}{x(x-1)} = -1$.

Problem 4 (an equation on sets)

From the description the set X could be any of

$$\{1,2\}\,,\{1,2,3\}\,,\{1,2,4\}\,,\{1,2,5\}\,,\{1,2,3,4\}\,,\{1,2,3,5\}\,,\{1,2,4,5\}\,,\{1,2,3,4,5\}\,,$$

which gives eight solutions.

Problem 5

Consider the function $y = x^{1/x}$ then $\ln(y) = \frac{1}{x}\ln(x)$. Taking the derivative of this we get

$$\frac{1}{y}\frac{dy}{dx} = -\frac{1}{x^2}\ln(x) + \frac{1}{x^2} = \frac{1}{x^2}\left[1-\ln(x)\right]\,.$$

Now $\ln(x) > 1$ if $x > e = 2.718282$ so $\frac{dy}{dx} < 0$ i.e. y decreases as x increases and is larger than the number e. Thus we have shown that

$$3^{1/3} > 8^{1/8} > 9^{1/9}\,.$$

We now need to compare $2^{1/2}$ and $3^{1/3}$. Comparing each of these is the same as comparing each number when raised to the sixth power or

$$2^3 = 8 \quad \text{vs.} \quad 3^2 = 9\,.$$

As nine is larger than eight we have that $3^{1/3} > 2^{1/2}$. We now need to compare $2^{1/2}$ to $8^{1/8}$. Raising both to the eighth power we get

$$2^4 = 16 \quad \text{vs.} \quad 8\,,$$

showing that $2^{1/2} > 8^{1/8}$. Thus the largest and next largest numbers in this list are

$$3^{1/3}\,, 2^{1/2}\,.$$

Problem 6

Write the given expression as

$$(3^x)^2 - 10(3^x) + 9 = 0\,,$$

which is a quadratic equation in 3^x. Solving this quadratic we get

$$3^x = \frac{10 \pm \sqrt{100 - 4(9)}}{2} = \frac{10 \pm 8}{2},$$

which has the two roots 1 and 9. If $3^x = 1$ then $x = 0$. If $3^x = 9$ then $x = 2$ thus there are two values for $x^2 + 1$ which are one and five.

Problem 7

In the problem statement we are told that $yz : zx : xy = 1 : 2 : 3$ so we have

$$\frac{yz}{zx} = \frac{1}{2} \quad \text{or} \quad \frac{y}{x} = \frac{1}{2},$$

and

$$\frac{zx}{xy} = \frac{2}{3} \quad \text{or} \quad \frac{z}{y} = \frac{2}{3}.$$

From these we compute that $\frac{x}{yz} : \frac{y}{zx}$ is

$$\frac{\frac{x}{yz}}{\frac{y}{zx}} = \frac{x^2}{y^2} = 4.$$

Problem 8

In the expression

$$|x - \log(y)| = x + \log(y),$$

if we assume that $x > \log(y)$ we get

$$x - \log(y) = x + \log(y) \quad \text{or} \quad \log(y) = 0,$$

Thus $y = 1$. If we assume that $x < \log(y)$ we get

$$-x + \log(y) = x + \log(y) \quad \text{or} \quad x = 0.$$

Our two solutions are $x = 0$ and $y = 1$ which can be written as

$$x(y - 1).$$

Problem 9

Let P be the number of papers in each box initially and let E be the number of envelopes in the box initially. Then for Ann we have

$$E = P - 50,$$

since she uses all the envelopes with one sheet letters and in doing so uses all but 50 pieces of paper. For Sue we have

$$\frac{P}{3} = E - 50\,,$$

since she used all of the paper (into letters with three pages each) and in doing so used all but 50 envelopes. These are two equations in the two unknowns E and P. Solving them we get $P = 150$ and $E = 100$.

Problem 10

Recall that the function $|x-2|$ is the function $|x|$ shifted to the right by two units. Thus we can plot the function $|x-2|$ and then draw horizontal lines at $y = 1$ and $y = 7$. The region of xs where the function $|x-2|$ is greater than $y = 1$ and less than $y = 7$ is the region we seek. The region of x where $1 \le |x-2|$ is given by

$$(-\infty, 1] \quad \text{and} \quad [3, +\infty)\,.$$

The region of x where $|x-2| \le 7$ is given by

$$[-5, +9]\,.$$

The intersection of these two regions is

$$[-5, 1] \quad \text{or} \quad [3, 9]\,.$$

Problem 11

Solve for x^2 in the second equation to get

$$x^2 = 3y - 12\,.$$

Put that into the first equation to get

$$3y - 12 + y^2 - 16 = 0\,.$$

This is a quadratic equation with the solutions $y = -7$ and $y = 4$. If we put the solution $y = -7$ back into the expression for x^2 above we find that $x^2 = 3(-7) - 12 = -33$ and thus for x so solve this x would have to be imaginary. Thus the only solution we have is $y = 4$.

Problem 12

Let E be the edge length in feet. Then we are told that

$$E^3 = 6(12E)^2 \quad \text{or} \quad E = 6(12^2) = 864\,,$$

feet.

Problem 13

Draw a line through the point M parallel to AB (and CD) with the point of intersection of AD denoted by R and the point of intersection of BC denoted by S. Then because M bisects AE using the similar triangles AMR and AED we have that

$$RM = \frac{1}{2}DE = \frac{5}{2}.$$

Thus the length of MS is

$$MS = 12 - \frac{5}{2} = \frac{19}{2}.$$

Now triangle MRP is similar to MSQ so we have

$$\frac{RM}{MS} = \frac{PM}{MQ} \quad \text{so} \quad \frac{PM}{MQ} = \frac{5/2}{19/2} = \frac{5}{19}.$$

Problem 14

Using the "law of sines" if L is the side opposite the angle $\frac{\pi}{6}$ then we have

$$\frac{L}{\sin\left(\frac{\pi}{6}\right)} = \frac{8}{\sin\left(\frac{\pi}{4}\right)},$$

or

$$2L = 8\sqrt{2} \quad \text{so} \quad L = 4\sqrt{2}.$$

Problem 15

Let B be the number of bricks needed to finish the full job. If b_1 is the number of bricks per hour that the first bricklayer can lay and b_2 is the number of bricks per hour that the second bricklayer can lay then we are told that

$$9b_1 = B$$
$$10b_2 = B.$$

We are told that when the two bricklayers work together their combined rate is actually ten bricks per hour *less* than what it would be otherwise. This means that combined their rate of bricklaying is

$$b_1 + b_2 - 10.$$

When they work together they finish the job in five hours or

$$5(b_1 + b_2 - 10) = B.$$

This gives three equations and three unknowns for us to solve for. Using the first two equations we can write b_i in terms of B and put them into the last expression to get

$$5\left(\frac{B}{9} + \frac{B}{10} - 10\right) = B.$$

Solving this for B gives $B = 900$.

Problem 16

Let the two numbers between three and nine be given by x and y. So the sequence is

$$3\,,x\,,y\,,9\,.$$

Since the first three numbers are in a geometric progression we can write them as

$$3\,,3r\,,3r^2\,.$$

Since the last three numbers are in an arithmetic progression we can write them as

$$9-2h\,,9-h\,,9\,.$$

Equating the expressions for x and y give

$$\begin{aligned} x &= 3r = 9-2h \\ y &= 3r^2 = 9-h\,. \end{aligned}$$

These are two equations for the two unknowns r and h. Solving the second equation for h we get $h = 9 - 3r^2$. Putting this into the first equation gives

$$6r^2 - 3r - 9 = 0\,.$$

Solving this for r we get the two roots

$$\{-1, 1.5\}\,.$$

We can't have $r = -1$ since in that case $3r = -3$ is not between three and nine. Thus $r = 1.5$ and the sequence is

$$3\,,3(1.5)\,,3(1.5^1)\,,9\,,$$

or

$$3\,,4.5\,,6.75\,,9\,.$$

The sum requested is $4.5 + 6.75 = 11.25$.

Problem 17

Consider a unit string cut at the point p where $0 \le p \le 1$. We want to know the probability that

$$\max(p, 1-p) \ge x \min(p, 1-p) \quad \text{or} \quad \frac{\max(p, 1-p)}{\min(p, 1-p)} \ge x\,.$$

The left-hand-side of this last expression is a function of p that we can evaluate for $0 \le p \le 1$ where we find

$$\frac{\max(p, 1-p)}{\min(p, 1-p)} = \begin{cases} \frac{1-p}{p} & 0 \le p \le \frac{1}{2} \\ \frac{p}{1-p} & \frac{1}{2} \le p \le 1 \end{cases}\,.$$

This will be greater than x when

$$\frac{1-p}{p} \geq x \quad \text{or} \quad p \geq \frac{x}{x+1} \,.$$

Thus the given probability that this happens is given by

$$1\left(\frac{1}{x+1}\right) + \left(1 - \frac{x}{x+1}\right) 1 = \frac{2}{x+1} \,,$$

when we simplify.

Problem 18

One way to solve this is the following. In the drawing I did for this problem I have the trapezoids bottom given by AB (of length $2b$) and the trapezoids top given by CD (of length b), such that starting at A and walking counterclockwise around the trapezoid we visit the vertices in the order $ABCD$. Next extend the nonparallel segments AD and BC until the meet at a vertex (denoted O) forming a triangle that is cut into two pieces with the two parallel lines AB and CD. Then because $AB = 2b = 2DC$ we also have $CB = CO$ and $AD = DO$. This is just the theorem that "in a triangle parallel lines grow in proportion to the distance they are from their common vertex" and can be proved with similar triangles. Note that because of this the diagonals AC and BD both go from a vertex of the triangle ABO to the midpoint of their opposite side and are thus "medians" of the triangle ABO. Because they are medians a theorem about medians of a triangle state that their common point of intersection divides them in the ratio of $2:1$. This means that

$$EC = \frac{1}{3} AC = \frac{11}{3} \,.$$

Another way to solve this problem is to recognize that triangle ABE is similar to triangle CDE. Thus

$$\frac{AB}{CD} = \frac{AE}{CE} \,,$$

or

$$\frac{2b}{b} = \frac{AC - CE}{CE} \,,$$

or

$$2 = \frac{11 - CE}{CE} \,.$$

This later expression has the solution $CE = \frac{11}{3}$ the same as we had before.

Problem 19

Note that when $n = 1$ the sum of n terms is one. The only expressions where this is true are B and D. When $n = 2$ the sum of n terms is four. From B and D the only option that gives four is D. We can prove that the given expression in D is true in the following way. Let s_k be defined as the sum

$$s_k = 1 + 2 + \cdots + 2^{k-1} = \sum_{l=0}^{k-1} 2^l = \frac{1-2^k}{1-2} = 2^k - 1\,.$$

Then the true sum we want to evaluate is given by

$$\begin{aligned} s_n &= \sum_{k=1}^{n} s_k = \sum_{k=1}^{n} (2^k - 1) = \sum_{k=1}^{n} 2^k - n \\ &= \sum_{k=0}^{n} 2^k - 1 - n = \frac{1-2^{n+1}}{1-2} - 1 - n = 2^{n+1} - 2 - n\,. \end{aligned}$$

Problem 20

For the tangent of x to be as given means that the length of the side opposite the angle x must be proportional to $2ab$ while the side adjacent to the angle x must be proportional to $a^2 - b^2$. Thus the hypotenuse h of this triangle is given by

$$h^2 = (a^2-b^2)^2 + 4a^2b^2 = a^4 - 2a^2b^2 + b^4 + 4a^2b^2 = a^4 + 2a^2b^2 + b^4 = (a^2+b^2)^2\,.$$

Thus $h = a^2 + b^2$. Using this we have that

$$\sin(x) = \frac{2ab}{h} = \frac{2ab}{a^2+b^2}\,.$$

Problem 21

Introduce a point P where the segment F intersects AD and a point Q where the segment EC intersects AD. For this problem we want to evaluate the sum of the angles

$$A + B + C + D + E + F\,.$$

To do this, we note that the sum $B + D$ are two of the angles in the triangle BPD and so

$$B + D = 180 - \angle BPD\,.$$

Also the sum $A + C$ are two of the angles in the triangle CQA and so

$$A + C = 180 - \angle CQA\,.$$

Next the sum $E + F$ are two of the angles in the quadrilateral $EFPQ$ and so

$$E + F = 360 - \angle FPQ - \angle EQP\,.$$

Putting these three expressions together we get

$$\begin{aligned} B + D + A + C + E + F &= (180 - \angle BPD) + (180 - \angle CQA) + (360 - \angle FPQ - \angle EQP) \\ &= 720 - (\angle BPD + \angle FPQ) - (\angle CQA - \angle EQP) \\ &= 720 - 180 - 180 = 360\,, \end{aligned}$$

using supplementary angles.

Problem 22

If we let the roots be denoted x_1 and x_2 then putting them into the given expression means that

$$\begin{aligned} x_1 q + r &= -x_1^3 \\ x_2 q + r &= -x_2^3 \,. \end{aligned}$$

This gives two equations for the two unknowns q and r. Solving the first equation for r we have $r = -x_1^3 - x_1 q$. Putting this into the second equation gives

$$(x_2 - x_1)q = -x_2^3 + x_1^3 \quad \text{so} \quad q = -\frac{x_2^3 - x_1^3}{x_2 - x_1} \,.$$

Note that since

$$x_2^3 - x_1^3 = (x_2 - x_1)(x_2^2 + x_2 x_1 + x_1^2) \,,$$

we can write the above as

$$q = -(x_2^2 + x_1 x_2 + x_1^2) \,.$$

If we let $x_2 = a + ib$ and $x_1 = a - ib$ then we find the above is equal to

$$\begin{aligned} q &= -((a+ib)^2 + (a+ib)(a-ib) + (a-ib)^2) \\ &= -(a^2 + 2iab - b^2 + a^2 + b^2 + a^2 - 2iab - b^2) \\ &= -(3a^2 - b^2) = b^2 - 3a^2 \,. \end{aligned}$$

Problem 23

If we let O denote the center of a $x-y$ coordinate system with the x axis lying along the base of the figure and the y axis lying perpendicular to the x axis though O. Then by symmetry, the x-coordinate of the center of the enclosing circle must lie on the y axis and thus has coordinate $x_c = 0$. Let the enclosing circle have a center of $(0, y_c)$ and a radius r. From the drawing it looks like the point $(0, 1)$ *could* be the circles center. Visually it looks like the farthest two points from a center on the y axis would be the point $(1, 0)$ and the point $\left(\frac{1}{2}, 2\right)$. The distance from the point $(1, 0)$ to the point $\left(\frac{1}{2}, 2\right)$ is $\frac{\sqrt{5}}{2} = 1.118$, while the distance from $(0, 1)$ to the point $(1, 0)$ is $\sqrt{2} = 1.4142 > \frac{\sqrt{5}}{2}$. This means that we could find a circle with a *smaller* radius by selecting a value for y_c that is less than one and going exactly though the two points $(1, 0)$ and $\left(\frac{1}{2}, 2\right)$. This gives two equations

$$\begin{aligned} (0-1)^2 + (y-0)^2 &= r^2 \\ \left(0 - \frac{1}{2}\right)^2 + (y-2)^2 &= r^2 \,. \end{aligned}$$

If we solve these we find that $y = \frac{13}{16} < 1$ and $r^2 = \frac{425}{16}$ so $r = \frac{5\sqrt{17}}{16}$.

Problem 24

Let the distance, rate, and time be denoted by d, r, and t. Then the problem statement

$$d = rt \tag{219}$$
$$d = \left(r + \frac{1}{2}\right)\left(\frac{4}{5}t\right) \tag{220}$$
$$d = \left(r - \frac{1}{2}\right)\left(t + \frac{5}{2}\right). \tag{221}$$

These are three equations and three unknowns so we should be able to solve for each variable. Now Equation 220 is

$$\frac{5}{4}d = \left(r + \frac{1}{2}\right)t \quad \text{or} \quad \frac{5}{4}d = rt + \frac{t}{2}.$$

Replacing rt with Equation 219 (i.e. d) we get

$$\frac{5}{4}d = d + \frac{t}{2} \quad \text{so} \quad t = \frac{d}{2}.$$

Equation 221 is given by

$$d = rt + \frac{5}{2}r - \frac{1}{2}t - \frac{5}{4}.$$

Since $rt = d$ we get

$$0 = 5r + t - \frac{5}{4} \quad \text{so} \quad r = \frac{(5/4) - t}{5}.$$

But we can write t in terms of d to get

$$r = \frac{(5/4) - (d/2)}{5}.$$

Now that we have r and t in terms of d we can write Equation 219 all in terms of d as

$$d = \left(\frac{(5/4) - (d/2)}{5}\right)\left(\frac{d}{2}\right).$$

We can solve this for d to find $d = 15$ miles.

Problem 25

When placing the given quadrilateral into the circle we are told that the four corners of the quadrilateral are on the circle. Breaking the quadrilateral up into four triangles by the using the diagonals of the quadrilateral we get four triangles that have the same circle as their circumcircle. Lets determine the length of the diagonal BD using the law of cosign. From $\triangle BCD$ we have

$$BD^2 = 39^2 + 52^2 - 2(39)(52)\cos(C) = 4225 - 4056\cos(C).$$

Using $\triangle BAD$ we have

$$BD^2 = 25^2 + 60^2 - 2(25)(60)\cos(A) = 4225 + 3000\cos(A)\,.$$

Since $\angle A$ and $\angle C$ are opposite angles of the inscribed quadrilateral we have that $\angle A + \angle C = \pi$ and thus $\cos(A) = -\cos(C)$. Using this we can use the above two equations to solve for BD and $\cos(C)$. We find $\cos(C) = 0$ so $\angle C = \frac{\pi}{2}$ and $BD = 65$. The diameter of the circumcircle of any triangle is given by any one of the triangles sides divided by the sin of the angle opposite that side. Thus the diameter of the circumcircle for this problem is given by

$$\frac{BD}{\sin(C)} = BD = 65\,.$$

Problem 27

Let the base of the triangle be the segment AB and the height is then the length of the segment from the base AB to the point C. We are told that

$$\frac{1}{2}AB \cdot h = 64\,. \tag{222}$$

The information about the geometric mean is that

$$\sqrt{AB \cdot AC} = 12\,. \tag{223}$$

Let the height intersect the line the base is on at the point D. Then since the height is a leg of the right triangle $\triangle CAD$ we have that $h = AC\sin(A)$. If we put this expression for h into Equation 222 we get

$$ABAC\sin(A) = 128\,.$$

Using Equation 223 we have that

$$\sin(A) = \frac{128}{144} = \frac{8}{9}\,.$$

Problem 28

If we imaging placing the origin of a x-y Cartesian coordinate system on the center of the checkerboard then the corners of the checkerboard are located at the locations (counterclockwise from North-East to South-West)

$$(4,4)\,,(-4,4)\,,(-4,-4)\,,(4,-4)\,.$$

On this x-y system we also place a circle with a radius $r = 4$. This means that the points (x,y) that are inside (or on) the circle satisfy

$$x^2 + y^2 \leq 16\,.$$

We want to then count the number of unit squares from the checkerboard that are fully inside this circle. By symmetry the number of complete unit squares that are in the entire circle will be four times the number of complete unit squares inside the first quadrant. There are $4 \times 4 = 16$ squares in the upper right corner of the checkerboard. Thus one technique to solve this problem would be to determine which of these 16 squares have all four of their corners inside the circle. We can simply the amount of calculations we have to do by working row by row and then stopping when we find the first square that has one of its corners outside the circle.

For the first row of squares the corner point that is farthest from the origin are

$$(1,1)\,,(2,1)\,,(3,1)\,,(4,1)\,.$$

Only the first three are inside the circle.

For the second row of squares the corner point that is farthest from the origin are

$$(1,2)\,,(2,2)\,,(3,2)\,,(4,2)\,.$$

Again the first three are inside the circle.

For the third row of squares the corner point that is farthest from the origin are

$$(1,3)\,,(2,3)\,,(3,3)\,,(4,3)\,.$$

Only the two are inside the circle.

For the fourth row of squares the corner point that is farthest from the origin are

$$(1,4)\,,(2,4)\,,(3,4)\,,(4,4)\,.$$

None of these points are inside the circle. Thus we have

$$3+3+2=8\,,$$

complete squares in the first quadrant and thus $4 \times 8 = 32$ total squares covered by the circular disk.

Problem 29

From what we are given lets compute

$$\begin{aligned}
1+\frac{3x+x^3}{1+3x^2} &= \frac{1+3x^2+3x+x^3}{1+3x^2} = \frac{(x+1)^3}{1+3x^2} \\
1-\frac{3x+x^3}{1+3x^2} &= \frac{1+3x^2-3x-x^3}{1+3x^2} = \frac{(1-x)^3}{1+3x^2}\,.
\end{aligned}$$

Then we have

$$\begin{aligned}
f\left(\frac{3x+x^3}{1+3x^2}\right) &= \log\left[\frac{(x+1)^3}{1+3x^2}\left(\frac{1+3x^2}{(1-x)^3}\right)\right] \\
&= \log\left[\left(\frac{1+x}{1-x}\right)^3\right] = 3\log\left[\frac{1+x}{1-x}\right] = 3f(x)\,.
\end{aligned}$$

Problem 30

Let the point of fold on the line AB be denoted E and the point where the corner B (when folded) intersects the AD segment be denoted by F. Then the triangle EBC and EFC are the same i.e. they have equal side lengths and equal angles. Because of this $\angle BEC = \angle FEC = \frac{\pi}{2} - \theta$, $EB = EF$, and

$$\angle AEF = \pi - \left(\frac{\pi}{2} - \theta\right) - \left(\frac{\pi}{2} - \theta\right) = 2\theta\,.$$

Now from the angle θ in triangle ECB we have that

$$EB = L\sin(\theta)\,. \tag{224}$$

From $\angle AEF$ in triangle AEF we have that

$$6 - EB = EF\cos(2\theta) = EB\cos(2\theta)\,.$$

This last expression means that

$$\begin{aligned} EB &= \frac{6}{1+\cos(2\theta)} = \frac{6}{1+\cos^2(\theta)-\sin^2(\theta)} = \frac{6}{1+\cos^2(\theta)-(1-\cos^2(\theta))} \\ &= \frac{6}{2\cos^2(\theta)} = \frac{3}{\cos^2(\theta)}\,. \end{aligned}$$

If we put this expression for EB into Equation 224 we get

$$\frac{3}{\cos^2(\theta)} = L\sin(\theta) \quad \text{or} \quad L = 3\sec^2(\theta)\csc(\theta)\,.$$

Problem 31

Lets see if we can determine a pattern for the remainder of 2^n when we divide by 13 and then apply this pattern when $n = 1000$. Notice that

$$\begin{aligned}
2^0 \bmod 13 &= 1 \bmod 13 = 1 \\
2^1 \bmod 13 &= 2 \bmod 13 = 2 \\
2^2 \bmod 13 &= 4 \bmod 13 = 4 \\
2^3 \bmod 13 &= 8 \bmod 13 = 8 \\
2^4 \bmod 13 &= 16 \bmod 13 = 3 \\
2^5 \bmod 13 &= 32 \bmod 13 = 6 \\
2^6 \bmod 13 &= 64 \bmod 13 = 12 \\
2^7 \bmod 13 &= 128 \bmod 13 = 11 \\
2^8 \bmod 13 &= 256 \bmod 13 = 9 \\
2^9 \bmod 13 &= 512 \bmod 13 = 5 \\
2^{10} \bmod 13 &= 1024 \bmod 13 = 10 \\
2^{11} \bmod 13 &= 7 \\
2^{12} \bmod 13 &= 1 \\
2^{13} \bmod 13 &= 2 \\
2^{14} \bmod 13 &= 4 \\
2^{15} \bmod 13 &= 8 \\
2^{16} \bmod 13 &= 3 \\
2^{17} \bmod 13 &= 6 \,.
\end{aligned}$$

Thus it looks like the cycle of remainders above repeats (from one to seven) as the powers of two change from 0 to 11. Notice that every 12 powers of two bring us back to a remainder of one. If we next break 1000 into groups where the remainder cycles as above we have

$$1000 = 12(83) + 4 \,.$$

Thus the first 12(83) powers of two will have remainders that cycle through the above numbers and end back with the value of one. We then need to count from the pattern of remainders starting at one *four* more values. This gets us to the remainder of three. Thus

$$2^{1000} \bmod 13 = 3 \,.$$

Problem 32

If we take E as the origin of a (x, y) coordinate system and let the center of the circle be located at (x_0, y_0) then the three points A, B, and D must satisfy the equation of a circle or

$$(x - x_0)^2 + (y - y_0)^2 = r^2 \,,$$

for some radius r. Putting in the known (x, y) values for the three points we get the equations

$$(6 - x_0)^2 + y_0^2 = r^2 \quad (225)$$
$$(-2 - x_0)^2 + y_0^2 = r^2 \quad (226)$$
$$x_0^2 + (-3 - y_0)^2 = r^2 \,. \quad (227)$$

Taking the difference of Equations 225 and 226 we get

$$(6 - x_0)^2 - (2 + x_0)^2 = 0 \,.$$

This gives a single equation for x_0 which has the solution $x_0 = 2$. Putting this into Equations 225 and 227 we get

$$16 + y_0^2 = r^2 \quad (228)$$
$$4 + (3 + y_0)^2 = r^2 \,. \quad (229)$$

Setting these two equations equal to each other gives

$$16 + y_0^2 = 4 + 9 + 6y_0 + y_0^2 \quad \text{or} \quad y_0 = \frac{1}{2} \,.$$

Now that we know (x_0, y_0) we can put these into Equation 225 to get $r^2 = \frac{65}{4}$. Thus the radius is the square root of this and the diameter is then $\sqrt{65}$.

Problem 33

Let our three digit number be abc with value $100a + 10b + c$ and each digit is in the range $[0, 9]$. Then

$$f = \frac{100a + 10b + c}{a + b + c} \,.$$

We want to minimize f. Lets first write f as

$$f = 1 + \frac{9(11a + b)}{a + b + c} \,.$$

To minimize f we must then minimize the fraction in the above expression for f. As c only appears in the denominator of the above fraction we can make f as small as possible by taking c as large as possible. Thus we take $c = 9$ and we see to minimize

$$f = 1 + \frac{9(11a + b)}{a + b + 9} \,.$$

We now write f as

$$f = 1 + \frac{9(a + b + 9 + 10a - 9)}{a + b + 9} = 1 + 9\left(1 + \frac{10a - 9}{a + b + 9}\right) = 10 + \frac{9(10a - 9)}{a + b + 9} \,.$$

Now $a \neq 0$ or otherwise we don't have a true three digit number thus $a \geq 1$ and $10a - 9 \geq 1$. To minimize f we thus need to minimize the fraction on the right-hand-side of the above.

Since b only appears in the denominator of the above we can minimize the fraction by taking b as large as possible. We can't take $b = 9$ since we know that $c = 9$ thus we take $b = 8$ and get

$$f = 10 + \frac{9(10a - 9)}{a + 17}.$$

This expression we will write as follows

$$\begin{aligned} f &= 10 + \frac{9(10(a + 17 - 17) - 9)}{a + 17} \\ &= 10 + \frac{9(10(a + 17) - 179)}{a + 17} \\ &= 100 - \frac{9(179)}{a + 17}. \end{aligned}$$

To make f as small as possible we want to make the fraction above as large as possible thus we take $a = 1$ Thus our three digit number is 189 and the smallest quotient has the value $\frac{189}{1+8+9} = 10.5$.

Problem 34

From the problem statement we are told (using the obvious notation) that

$$\begin{aligned} 3d + t &= 2h \quad \text{and} \\ 2h^3 &= 3d^3 + t^3. \end{aligned}$$

To start the solution we note that we can write the first equation as

$$2(h - d) = d + t. \tag{230}$$

Next note that we can write the second equation as

$$2(h^3 - d^3) = d^3 + t^3, \tag{231}$$

which if we factor both sides gives

$$2(h - d)(h^2 + hd + d^2) = (d + t)(d^2 - dt + t^2).$$

Using Equation 230 we can cancel the expression $2(h - d)$ on the left with the expression $(d + t)$ on the right to get

$$h^2 + hd + d^2 = d^2 - dt + t^2,$$

or canceling d^2 on both sides gives

$$h^2 + hd = -dt + t^2.$$

This we can write as

$$h^2 - t^2 = -d(t + h),$$

or factoring we have

$$(h - t)(h + t) = -d(t + h).$$

When we cancel $(t+h)$ on both sides we get

$$h - t = -d \quad \text{or} \quad t = d + h\,.$$

Use this expression for t in Equation 230 to get $4d = h$. As we are told that h and d are relatively prime the only way this equation can hold is if $d = 1$ so that $h = 4$. With these values we get that $t = d + h = 1 + 4 = 5$. and the expression we seek to evaluate

$$d^2 + h^2 + t^2 = 1^2 + 4^2 + 5^2 = 42\,.$$

Problem 35

To start this problem it can be helpful to draw the triangle rotating around inside the square for several steps of its path. One thing to notice is that since the side of inner triangle has a length of two the distance each point moves on a single rotation will be some fraction of the circumference of a circle with this radius i.e. a fraction of $2\pi(2) = 4\pi$. Solving this problem is then really just a careful accounting of the number of times and the fractional amount that each corner moves under as the triangle translates around. As the triangle is an equilateral triangle the angular amount that an point can move under one rotation is either $180 - 60 = 120$ degrees or $90 - 60 = 30$ degrees.

We document the steps (and angular distance traveled by each corner of the triangle) for one movement of the triangle and then compute the solution to the requested problem by summing the total distance each point travels under all rotations. To begin notice that the triangle starts at the "position" ABP where A is in the left corner, B is along the bottom side of the square, and P is the point in the interior of the square.

Then

- In step 1: P moves clockwise 120 degrees, A moves clockwise 120 degrees, and B does not move.
- In step 2: P does not move, A moves 30 degrees, and B moves 30 degrees.
- In step 3: P moves 120 degrees, A does not move, and B moves 120 degrees.
- In step 4: P moves 30 degrees, A moves 30 degrees, and B does not move.
- In step 5: P does not move, A moves 120 degrees, and B moves 120 degrees.
- In step 6: P moves 30 degrees, A does not move, and B moves 30 degrees.
- In step 7: P moves 120 degrees, A moves 120 degrees, and B does not move.
- In step 8: P does not move, A moves 30 degrees, and B moves 30 degrees.

During this sequence of rotations the point A, B, and P moved a total of

- P moved $120 + 0 + 120 + 30 + 0 + 30 + 120 + 0 = 420$ degrees.
- A moved $120 + 30 + 0 + 30 + 120 + 0 + 120 + 30 = 450$ degrees.
- B moved $0 + 30 + 120 + 0 + 120 + 30 + 0 + 30 = 330$ degrees.

At this point the triangle is in the position PAB namely P is in the left corner, A along the bottom side of the square, and B in the interior of the square. If we repeat the same sequence of rotations we will end with the triangle in position BPA and repeating the sequence of rotations again finally in position ABP which is the position we started with. The total angular amount that the point P traveled to get back to its starting locations during all of these rotations is then the sum of

- 420 degrees for the triangle to go from positions ABP to PAB.
- 450 degrees for the triangle to go from positions PAB to BPA since now P is at the location of A in the initial triangle whose movement we documented in detail.
- 330 degrees for the triangle to go from positions BPA to ABP since now P is at the location of B in the initial triangle whose movement we documented in detail.

Summing these we get 1200 degrees of travel. This is $\frac{1200}{360} = \frac{10}{3}$ of a full circle giving a length traveled of

$$\frac{10}{3}(4\pi) = \frac{40}{3}\pi \,.$$

The 1973 Examination

Problem 1

Drawing a picture of this situation we have two radii from the center to the circle to the intersection of the chord and the circle and a third radius that is bisected by this chord. This gives two right triangles each with a hypotenuse of length twelve and one leg of length six. The only unknown in each right triangle is one of the legs which is also 1/2 the total length of the chord. Thus the length of the chord is

$$2\sqrt{12^2 - 6^2} = 2\sqrt{108} = 12\sqrt{3}\,.$$

Problem 2

Let the smaller unit cubes be called cubits. Of the 1000 cubits, the ones that have at least one face painted are the "outer" cubits. Notice that these outer cubits surround an inner cube of entirely unpainted cubits. Thus if we "shave" off one cubit from each of the six faces of the large cube we end up with an inner cube of all unpainted cubits. In this inner cube there are 8^3 of these cubits. Thus the number of painted cubits are

$$1000 - 8^3 = 488\,.$$

Problem 3

If the stronger Goldbach conjecture is true then we should be able to write 126 as

$$126 = p_1 + p_2\,,$$

where p_1 and p_2 are both prime and $p_1 \neq p_2$. Without loss of generality lets assume that $p_1 < p_2$. Then to have the largest value of $p_2 - p_1$ we would like to take p_1 "as small as possible". We can consider the first few primes for p_1 and determine the first time if they determine a value for p_2 that is also prime. For example

$$\begin{aligned}
p_1 &= 2 \quad \text{so } p_2 = 124 \text{ which is not prime since it is divisible by } 2\\
p_1 &= 3 \quad \text{so } p_2 = 123 \text{ which is not prime since it is divisible by } 3\\
p_1 &= 5 \quad \text{so } p_2 = 121 \text{ which is not prime since it is divisible by } 11\\
p_1 &= 7 \quad \text{so } p_2 = 119 \text{ which is not prime since it is divisible by } 7\\
p_1 &= 11 \quad \text{so } p_2 = 115 \text{ which is not prime since it is divisible by } 5\\
p_1 &= 13 \quad \text{so } p_2 = 113 \text{ which is prime}\,.
\end{aligned}$$

Thus we should take $p_1 = 13$ so that $p_2 = 113$ and $p_2 - p_1 = 100$ for the largest difference.

Problem 4

If we lay the hypotenuse along the x-axis then the "inner" triangle we want to compute the area of is an isosceles triangle with a base length of twelve and two equal angles of 30 degrees. By symmetry the height of this triangle is located 1/2 way along the hypotenuse. Thus because the two equal angles are 30 degrees this height has a value of

$$6\tan(30^\circ) = \frac{6}{\sqrt{3}}\,.$$

Thus the area of the triangle in question is $\frac{1}{2}bh = \frac{1}{2}(12)\left(\frac{6}{\sqrt{3}}\right) = 12\sqrt{3}$.

Problem 5

Define the operator $*$ as

$$a * b = \frac{1}{2}(a+b)\,.$$

With this we will see which of the four suggested properties are true by computing both sides and seeing whether or not they are equal.

For the first property I the left-hand-side of $(a*b)*c = a*(b*c)$ is

$$\begin{aligned}(a*b)*c &= \left(\frac{1}{2}(a+b)\right)*c \\ &= \frac{1}{2}\left(\frac{1}{2}(a+b)+c\right)\,,\end{aligned}$$

while the right-hand-side is

$$\begin{aligned}a*(b*c) &= a*\left(\frac{1}{2}(b+c)\right) \\ &= \frac{1}{2}\left(a+\frac{1}{2}(b+c)\right)\,,\end{aligned}$$

which is not equal to the left-hand-side.

The second property II is the statement that $a*b = b*a$ which when we simplify both sides can be seen to be true.

The third property III is the statement that $a*(b+c) = (a*b)+(a*c)$. This has a left-hand-side given by

$$a*(b+c) = \frac{1}{2}(a+b+c)\,,$$

while the right-hand-side is

$$(a*b)+(a*c) = \frac{1}{2}(a+b)+\frac{1}{2}(a+c)\,,$$

which is not equal to the left-hand-side.

For the fourth IV property $a + (b * c) = (a + b) * (a + c)$ the left-hand-side is equal to

$$a + (b * c) = a + \frac{1}{2}(b + c)\,,$$

while the right-hand-side is equal to

$$(a + b) * (a + c) = \frac{1}{2}(a + b + a + c) = a + \frac{1}{2}(b + c)\,.$$

which is equal to the left-hand-side.

For the fifth V property to have averaging have an identity element e we would need to have e satisfy $a * e = a$ or

$$\frac{1}{2}(a + e) = a\,,$$

or $e = a$. Since this changes for each a there is no identity.

Problem 6

The number 24 in base b is $2b + 4$ in decimal. Squaring that number we get

$$(2b + 4)^2 = 4b^2 + 16b + 16\,.$$

We are told that is the number 554 in base b or $5b^2 + 5b + 4$ in decimal. Equating these two decimal numbers gives

$$4b^2 + 16b + 16 = 5b^2 + 5b + 4\,,$$

which can be solved for b. We get $(b - 12)(b + 1) = 0$ so $b = 12$ or $b = -1$. Since b must be positive we have that $b = 12$.

Problem 7

We want to evaluate

$$\begin{aligned}
\sum_{k=5,6,7,\cdots}^{34} (10k + 1) &= 10\sum_{k=5}^{34} k + \sum_{k=5}^{34} 1 \\
&= 10\left(\sum_{k=1}^{34} k - \sum_{k=1}^{4} k\right) + (34 - 5 + 1) \\
&= 10\left(\frac{34(35)}{2} - \frac{4(5)}{2}\right) + 30 = 5880\,.
\end{aligned}$$

Problem 8

We would have the amount of paint needed P given by an expression of the form $P = knh^2$ where k is a constant, n is the number of statues to paint, and h is the statue height. We are told that $P = 1$ when $n = 1$ and $h = 6$ so that using our formula for P we have

$$1 = k(1)(36) \quad \text{so} \quad k = \frac{1}{36}\,.$$

Thus $P = \frac{1}{36}nh^2$. If we then have $n = 540$ and $h = 1$ we find

$$P = \frac{1}{36}540(1^2) = 15\,.$$

Problem 9

For this problem we are told that the area of $\triangle CHM$ is K. This means that

$$\frac{1}{2}MH \times CH = K\,.$$

Since the segments CM and CH trisect the 90 degree angle $\angle ACB$ we have that $\angle ACM$, $\angle MCH$, and $\angle HCB$ are all 30 degrees. This in tern means that

$$MH = CH\tan(30^\circ) = \frac{CH}{\sqrt{3}}\,.$$

Using this in the formula for the area of $\triangle CHM$ we get

$$\frac{1}{2}\left(\frac{1}{\sqrt{3}}\right)CH^2 = K \quad \text{or} \quad CH^2 = 2\sqrt{3}K\,.$$

Next note that

$$\tan(\angle HCB) = \frac{HB}{CH} \quad \text{so} \quad HB = \frac{CH}{\sqrt{3}}\,,$$

as $\angle HCB = 30^\circ$. Since the area of $\triangle ABC$ is $\frac{1}{2}AB \times CH$ and we know what CH is in terms of K we need to get AB terms of K. To do that note that

$$\frac{1}{2}AB = MB = MH + HB = \frac{CH}{\sqrt{3}} + \frac{CH}{\sqrt{3}} = \frac{2CH}{\sqrt{3}}\,,$$

so $AB = \frac{4CH}{\sqrt{3}}$ which can be written in terms of K. Using everything so far we find the area of $\triangle ABC$ given by

$$\frac{1}{2}AB \times CH = \frac{1}{2}\left(\frac{4CH}{\sqrt{3}}\right)CH = \frac{2}{\sqrt{3}}CH^2 = \frac{2}{\sqrt{3}}(2\sqrt{3}K) = 4K\,.$$

Problem 10

There is no solution to this set of equations if the determinant of the matrix A in the linear system $Ax = b$ is zero or in this case that is

$$\begin{vmatrix} n & 1 & 0 \\ 0 & n & 1 \\ 1 & 0 & n \end{vmatrix} = 0\,,$$

or expanding about the bottom row this is

$$\begin{vmatrix} 1 & 0 \\ n & 1 \end{vmatrix} + n \begin{vmatrix} 0 & 1 \\ 1 & n \end{vmatrix} = 0\,,$$

or

$$1 + n^3 = 0\,.$$

This has the real solution of $n = -1$.

Problem 11

Let the value of a (described in the footnote) take the value $a = 2$. Then inequality $|x|+|y| \leq 2$ is a "diamond" shape with its "points" at the North-East-West-South coordinate locations i.e. the locations $(0, 2)$, $(2, 0)$, $(0, -2)$, and $(-2, 0)$. The inequality $\sqrt{2(x^2 + y^2)} \leq 2$ is equal to

$$x^2 + y^2 \leq 2 = \sqrt{2}^2 = 1.414214^2\,,$$

which is a circle with a radius $1.414214 < 2$ and thus the points that satisfy this inequality lie *inside* the previously inequality $|x| + |y| \leq 2$. Finally the inequality $2 \max(|x|, |y|) \leq 2$ becomes

$$\max(|x|, |y|) \leq 1\,,$$

which is a square with corners at the locations $(1, 1)$, $(-1, 1)$, $(-1, -1)$, and $(1, -1)$. The points that satisfy this inequality lie *inside* the previous circular disk region. Thus we have that choice II (or B) is the correct one.

Problem 12

From the problem statement we are told that

$$35 = \frac{1}{D} \sum_{i=1}^{D} d_i \quad (232)$$

$$50 = \frac{1}{L} \sum_{j=1}^{L} l_j\,. \quad (233)$$

Here D is the number of doctors with d_i their individual ages while L is the number of lawyers with l_i their individual ages. We are also told that

$$\frac{1}{D+L}\left(\sum_{i=1}^{D} d_i + \sum_{j=1}^{L} l_j\right) = 40\,.$$

If we solve for the sums in Equation 232 and 233 and put them into the above expression we get

$$35D + 50L = 40(D+L) \quad \text{or} \quad 10L = 5D\,.$$

Thus we have that $\frac{D}{L} = 2$.

Problem 13

Let the given expression be denoted x and then square its value to get

$$\begin{aligned} x^2 &= \frac{4}{9}\left(\frac{2+2\sqrt{12}+6}{2+\sqrt{3}}\right) = \frac{4}{9}\left(\frac{8+2\sqrt{12}}{2+\sqrt{3}}\right) \\ &= \frac{8}{9}\left(\frac{4+\sqrt{12}}{2+\sqrt{3}}\right) = \frac{16}{9}\,. \end{aligned}$$

Taking the square root (and taking the positive root) gives $x = \frac{4}{3}$.

Another (more complicated) way to solve this problem is to write it as

$$\begin{aligned} \frac{2(\sqrt{2}+\sqrt{6})}{3\sqrt{2+\sqrt{3}}} &= \frac{2\sqrt{2}(1+\sqrt{3})}{3\sqrt{2}\sqrt{1+\frac{\sqrt{3}}{2}}} = \frac{2}{3}\left(\frac{1+\sqrt{3}}{\sqrt{1+\frac{\sqrt{3}}{2}}}\right) \\ &= \frac{2}{3}\left(\frac{1+\sqrt{3}}{\sqrt{1+\frac{\sqrt{3}}{2}}}\right)\left(\frac{\sqrt{1-\frac{\sqrt{3}}{2}}}{\sqrt{1-\frac{\sqrt{3}}{2}}}\right) = \frac{2}{3}\left(\frac{\sqrt{(1+\sqrt{3})^2\left(1-\frac{\sqrt{3}}{2}\right)}}{\sqrt{1-\frac{3}{4}}}\right) \\ &= \frac{2}{3}\left(\frac{\sqrt{(1+2\sqrt{3}+3)\left(1-\frac{\sqrt{3}}{2}\right)}}{\frac{1}{2}}\right) = \frac{4}{3}\sqrt{(4+2\sqrt{3})\left(1-\frac{\sqrt{3}}{2}\right)} \\ &= \frac{4}{3}\sqrt{4+2\sqrt{3}-2\sqrt{3}-3} = \frac{4}{3}\sqrt{1} = \frac{4}{3}\,, \end{aligned}$$

the same value as before.

Problem 14

Let A, B, C be the rates at which water from the corresponding valves flow into the tank. Let V be the total amount of water that the tank can hold. Then we are told that

$$\begin{aligned}(A+B+C)1 &= V \\ (A+C)1.5 &= V \\ (B+C)2 &= V\,.\end{aligned}$$

We want to know the value of n such that $(A+B)n = V$. The above is three equations and three unknowns which we can solve. Doing this we find

$$\begin{aligned}A &= \frac{V}{2} \\ B &= \frac{V}{3} \\ C &= \frac{V}{6}\,.\end{aligned}$$

From these we have that $A+B = \frac{5}{6}V$. Thus $n = \frac{6}{5} = 1.2$ hours to fill the tank with A and B.

Problem 15

The circle that circumscribes the sector will be the same one that circumscribes the triangle we obtain from the two radial edges and the cord that their intersection with the circle makes (drawing a few pictures of this case should make this argument seem reasonable). In that case we will use Equation 15 to find the radius of this circumscribing circle. To use that formula we need to know the length of the side of the triangle that is opposite to the angle θ. If we denote this length by l it can be computed using the law of cosigns or

$$l^2 = r^2 + r^2 - 2r^2\cos(\theta) = 2r^2(1-\cos(\theta))\,.$$

Using the the fact that $\sin^2(\theta) = \frac{1-\cos(2\theta)}{2}$ we have that

$$1 - \cos(\theta) = 2\sin^2\left(\frac{\theta}{2}\right)\,.$$

Using this we can write the above as

$$l^2 = 4r^2\sin^2\left(\frac{\theta}{2}\right) \quad \text{so} \quad l = 2r\sin\left(\frac{\theta}{2}\right)\,.$$

With this the radius of the circumscribing circle is then given by Equation 15 where we find

$$\frac{2r\sin\left(\frac{\theta}{2}\right)}{2\sin(\theta)} = \frac{r\sin\left(\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)} = \frac{r}{2\cos\left(\frac{\theta}{2}\right)}\,.$$

If we take $r = 6$ the above becomes $3\sec\left(\frac{\theta}{2}\right)$.

Problem 16

Recall that the sum of the interior angles of a n sided polygon is given by Equation 5. If we let θ^* be the angle that we did not include in our sum we must have that

$$180(n-2) = 2190 + \theta^* .$$

If we consider the given values for n and subtract the "sum minus the angle θ^*" (or the value 2190) from the products of $180(n-2)$ we find that for the suggested values of n the possible values for θ^* would then be

```
-210   150   510   870  1230
```

Since any convex polygon must have its internal angle less than 180 degrees the only value of the second one or $n = 15$.

Problem 17

We are told that

$$\sin\left(\frac{1}{2}\theta\right) = \sqrt{\frac{x-1}{2x}} ,$$

and θ is a acute angle. From this value we have

$$\cos\left(\frac{1}{2}\theta\right) = \sqrt{1 - \left(\frac{x-1}{2x}\right)} = \sqrt{\frac{2x-x+1}{2x}} = \sqrt{\frac{x+1}{2x}} .$$

Now using $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ we have

$$\begin{aligned}\sin(\theta) &= 2\sin\left(\frac{1}{2}\theta\right)\cos\left(\frac{1}{2}\theta\right) = 2\sqrt{\left(\frac{x-1}{2x}\right)\left(\frac{x+1}{2x}\right)} \\ &= 2\sqrt{\frac{x^2-1}{4x^2}} = \frac{\sqrt{x^2-1}}{x} .\end{aligned}$$

Now for $\cos(2\theta)$ we have the identities

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta) .$$

Using this we have

$$\begin{aligned}\cos(\theta) &= 1 - 2\sin^2\left(\frac{1}{2}\theta\right) = 1 - 2\left(\frac{x-1}{2x}\right) \\ &= \frac{x-(x-1)}{x} = \frac{1}{x} .\end{aligned}$$

Thus with expressions for $\sin(\theta)$ and $\cos(\theta)$ we can now compute

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \sqrt{x^2-1} .$$

Problem 18

For p a prime number for this question we want to know when

$$\frac{p^2-1}{24} = \frac{(p-1)(p+1)}{24},$$

has no remainder.

By trying several small prime values for p we notice that when $p=7$ we have that $p^2-1=48$ which 24 will divide. Thus we have at least one value of p (other than $p=5$) that works.

Now as p is prime it must be odd so $p-1$ and $p+1$ are both even. As they are *consecutive* even integers they are both divisible by two and one of them is divisible by four. Thus their product is divisible by $2\times 4=8$. From the three consecutive integers $p-1$, p, and $p+1$ we must have that one of them is divisible by three. It cannot be the prime number p and thus one of $p-1$ or $p+1$ is divisible by three. Thus the product $(p-1)(p+1)$ is divisible by $8\times 3=24$ and thus the given statement is true for all prime $p\geq 5$.

Problem 19

From the given definition we have

$$\frac{72_8!}{18_2!} = \frac{72(72-8)(72-16)(72-24)(72-32)(72-40)(72-48)(72-56)(72-64)}{18(18-2)(18-4)(18-6)(18-8)(18-10)(18-12)(18-14)(18-16)}.$$

As $72=8(9)$ and $18=2(9)$ the above is given by

$$\begin{aligned}\frac{72_8!}{18_2!} &= \frac{9(8)\times 8(8)\times 7(8)\times 6(8)\times 5(8)\times 4(8)\times 3(8)\times 2(8)\times 1(8)}{9(2)\times 8(2)\times 7(2)\times 6(2)\times 5(2)\times 4(2)\times 3(2)\times 2(2)\times 1(2)}\\ &= \frac{8^9}{2^9} = \left(\frac{8}{2}\right)^9 = 4^9.\end{aligned}$$

Problem 20

Introduce a Cartesian coordinate system where the stream runs East-West along the x-axis, the cowboy is located at the point $(0,-4)$, and the cabin is located at $(8,-4-7)=(8,-11)$. Let the location where the cowboy waters his horse be located on the stream at $(w,0)$ where $0\leq w\leq 8$. Then the total distance the cowboy travels to get to the stream and then to his cabin is

$$D = \sqrt{w^2+16} + \sqrt{(w-8)^2+121}.$$

We want to find the minimum of D as a function of w. Taking the derivative of the above expression with respect to w and then setting the result equal to zero gives

$$\frac{dD}{dw} = \frac{2w}{2\sqrt{w^2+16}} + \frac{2(w-8)}{2\sqrt{(w-8)^2+121}} = 0.$$

This is equivalent to the equation

$$w\sqrt{(w-8)^2+121} + (w-8)\sqrt{w^2+16} = 0\,,$$

or

$$-w\sqrt{(w-8)^2+121} = (w-8)\sqrt{w^2+16}\,.$$

If we square both sides of this expression we get

$$w^2((w-8)^2+121) = (w-8)^2(w^2+16)\,.$$

Expanding everything and simplifying we end up with

$$105w^2 + 256w - 1024 = 0\,,$$

which has solutions given by

$$w = \frac{-256 \pm \sqrt{256^2 + 4(105)(1024)}}{2(105)} = \frac{-256 \pm 704}{2(105)} = \left\{ -\frac{2^5}{7}, \frac{2^5}{15} \right\}.$$

Only the positive value of w is valid. With this value of w to compute the total distance traveled we need to also compute

$$\begin{aligned} w^2 + 16 &= 20.55111 \\ (w-8)^2 + 121 &= 155.4178 \\ D &= 17\,. \end{aligned}$$

Problem 21

Let the first number in the set be denoted by $1 \le x < 100$ then if the set has $t \ge 2$ terms it will consist of the numbers

$$\{x, x+1, x+2, \cdots, x+t-2, x+t-1\}\,.$$

The sum of these terms is given by

$$tx + \sum_{k=0}^{t-1} k = tx + \frac{t(t-1)}{2}\,.$$

If this is to equal to 100 then we must have

$$tx + \frac{t(t-1)}{2} = 100 \quad \text{or} \quad t(2x + (t-1)) = 200\,. \tag{234}$$

Thus we need to now consider the factors of 200. Since $200 = 2^3 \cdot 5^2$ we have that possible values for t could be any number we could make from the product $2^3 \cdot 5^2$ and satisfying the constraints that $t \ge 2$. Thus for t we could have numbers of the form $2^i 5^j$ for $0 \le i \le 3$ and $0 \le j \le 2$. These give

$$t \in \{2, 4, 8, 5, 10, 20, 40, 25, 50\}\,.$$

For each of these values of t we can put it into Equation 234 and solve for x. If we have a valid value of x then we have found one solution to our problem. For the given possible value for t we find x given by

```
[1] 49.5 23.5 9.0 18.0 5.5 -4.5 -17.0 -8.0 -22.5
```

There are only two numbers above that are valid values for x. Thus we have only two solutions.

Problem 22

We want to find all solutions to

$$|x-1|+|x+2|<5\,.$$

To solve this problem we recall the definition of the absolute value function

$$|x| = \left\{ \begin{array}{cc} x & x \geq 0 \\ -x & x < 0 \end{array} \right. .$$

In what follows for simplicity we will ignore the "equality at $x=0$" in the greater than or equal to part in the above definition. Then using the definition of the absolute value the first term in the above can be written

$$\begin{aligned} |x-1| &= \left\{ \begin{array}{cc} x-1 & x-1>0 \\ -(x-1) & x-1<0 \end{array} \right. \\ &= \left\{ \begin{array}{cc} x-1 & x>1 \\ -x+1 & x<1 \end{array} \right. . \end{aligned}$$

In the same way the second term can be written as

$$\begin{aligned} |x+2| &= \left\{ \begin{array}{cc} x+2 & x+2>0 \\ -x-2 & x+2<0 \end{array} \right. \\ &= \left\{ \begin{array}{cc} x+2 & x>-2 \\ -x-1 & x<-2 \end{array} \right. . \end{aligned}$$

Therefore the sum of these two expressions can be written as

$$\begin{aligned} |x-1|+|x+2| &= \left\{ \begin{array}{cc} -x+1-x-2 & x<-2 \\ -x+1+x+2 & -2<x<1 \\ x-1+x+2 & x>1 \end{array} \right. \\ &= \left\{ \begin{array}{cc} -2x-1 & x<-2 \\ 3 & -2<x<1 \\ 2x+1 & x>1 \end{array} \right. . \end{aligned}$$

Now with this expression we see that $|x-1|+|x+2|<5$ will certainly be true when $-2<x<1$ (since it evaluates to the value of three (which is less than five). For other regions we can expand our valid region of x "to the left" until

$$-2x-1=5 \quad \text{or} \quad x=-3\,.$$

We can expand our valid region "to the right" until

$$2x+1=5 \quad \text{or} \quad x=2\,.$$

Thus combining these two results the region of interest is $-3<x<+2$.

Problem 23

Let R be the event that the card is red on both sides. Let B be event that the card is red on one side and blue on the other. Finally let F be the event that the face up is red. Then we want to evaluate $P(R|F)$. Using Bayes' rule we have

$$P(R|F) = \frac{P(FIR)P(R)}{P(F|R)P(R) + P(R|B)P(B)} = \frac{1(1/2)}{1(1/2) + (1/2)(1/2)} = \frac{2}{3} .$$

Problem 24

Let s, c, and p be the price of one sandwich, one cup of coffee, and one piece of pie respectively. From the problem statement we are told that

$$\begin{aligned} 3s + 7c + p &= 3.15 \\ 4s + 10c + p &= 4.20 . \end{aligned}$$

We can solve for s and c in terms of p. When we do that we get

$$\begin{aligned} s &= -\frac{3}{2}p + 1.05 \\ c &= \frac{1}{2}p . \end{aligned}$$

Thus we find that the sum desired is given by

$$s + c + p = -p + 1.05 + p = 1.05 .$$

Problem 25

The area of the full grass plot is $\pi 6^2 = 36\pi$. If we draw a circle in the x-y plane then the y value of the points on the circle is given by

$$y = \pm\sqrt{6^2 - x^2} .$$

Thus the area of the gravel path is given by

$$A = 2\int_0^3 y(x)dx = 2\int_0^3 \sqrt{36 - x^2}dx .$$

To evaluate this let $x = 6\sin(\theta)$ then $dx = 6\cos(\theta)d\theta$ and we have

$$\begin{aligned} A &= 2(36)\int_0^{\pi/6} \sqrt{1 - \sin^2(\theta)}\cos(\theta)d\theta \\ &= 72\int_0^{\pi/6} \cos^2(\theta)d\theta = 36\int_0^{\pi/6} (1 + \cos(2\theta))d\theta \\ &= 36\left(\frac{\pi}{6} - \left(\frac{\sin(2\theta)}{2}\Big|_0^{\pi/6}\right)\right) = 36\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right) . \end{aligned}$$

Thus the area of the circular grass with the gravel path removed is

$$36\pi - 36\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right) = 30\pi - 9\sqrt{3}\,.$$

Problem 26

We are told that (with $2n$ the index of the last term)

$$\begin{aligned} a_1 + a_3 + \cdots + a_{2n-1} &= 24 \\ a_2 + a_4 + \cdots + a_{2n} &= 30\,. \end{aligned}$$

If a_k is an arithmetic sequence with a common difference d then the odd and even terms are another arithmetic sequence with common difference $2d$. Thus using Equation 18 we can evaluate the given sums on the left-hand-side to get

$$\begin{aligned} \sum_{k=1}^{n} a_{2k-1} &= \frac{n}{2}(2a_1 + 2(n-1)d) = 24 \\ \sum_{k=1}^{n} a_{2k} &= \frac{n}{2}(2a_2 + 2(n-1)d) = \frac{n}{2}(2(a_1 + d) + 2(n-1)d) = 30\,. \end{aligned}$$

The first of these two equations means that

$$n(a_1 + (n-1)d) = 24\,,$$

while the second means that

$$n(a_1 + (n-1)d + d) = 30\,.$$

If we use this first expression in the second expression we get

$$nd = 30 - 24 = 6\,.$$

We are also told that $a_{2n} - a_1 = 10.5$. Now using Equation 17 this means that

$$a_1 + (2n-1)d - a_1 = 10.5 \quad \text{or} \quad (2n-1)d = 10.5\,.$$

As we know that $nd = 6$ when we use that in the above we get $d = 1.5$. Solving for n then gives $n = \frac{6}{d} = 4$. As this is one-half of the total number of terms we must have *eight* terms in the original sum.

Problem 27

To solve this problem we will need to recall the definition of the average velocity (when the velocity $v(t)$ can be a function of time)

$$\bar{v} = \frac{1}{T}\int_0^T v(t)dt\,, \tag{235}$$

we will use this relationship to compute values for x and y. Next, let the distance both cars travel be denoted as D. For car A, let t_u and t_v be the times that the car travels with speeds u and v respectively. As each of these times correspond to one half of the distance D we have

$$
\begin{aligned}
t_u &= \frac{D}{2u} \\
t_v &= \frac{D}{2v} \,.
\end{aligned}
$$

Then as the total distance traveled is D we know that

$$
ut_u + vt_v = D \,.
$$

Let the total time traveled by car A be denoted T_A then

$$
T_A = t_u + t_v = \frac{D}{2}\left(\frac{1}{u} + \frac{1}{v}\right) \,.
$$

Using Equation 235 we can compute the average velocity of car A

$$
\begin{aligned}
x &= \frac{1}{T_A}\int_0^{T_A} \text{velocity}(t)dt \\
&= \frac{1}{T_A}\left[\int_0^{t_u} udt + \int_{t_u}^{t_u+t_v} vdt\right] = \frac{1}{T_A}[ut_u + vt_v] = \frac{D}{T_A} \\
&= \frac{D}{\frac{D}{2}\left(\frac{1}{u}+\frac{1}{v}\right)} = \frac{2}{\frac{1}{u}+\frac{1}{v}} \,.
\end{aligned}
$$

For car B, let T_B be the amount of time it takes this car to drive the distance D. As we drive with the velocity u for one-half of the time (and with velocity v for the other one-half of the time) we have

$$
\left(\frac{T_B}{2}\right)u + \left(\frac{T_B}{2}\right)v = D \,.
$$

Again using Equation 235 we can compute the average velocity of car B

$$
\begin{aligned}
y &= \frac{1}{T_B}\int_0^{T_B} \text{velocity}(t)dt \\
&= \frac{1}{T_B}\left[\int_0^{T_B/2} udt + \int_{T_B/2}^{T_B} vdt\right] = \frac{1}{T_B}\left[u\left(\frac{T_B}{2}\right) + v\left(\frac{T_B}{2}\right)\right] \\
&= \frac{u+v}{2} \,.
\end{aligned}
$$

From the above two expressions we want to compare the values of

$$
x = \frac{2}{\frac{1}{u}+\frac{1}{v}} \quad \text{vs.} \quad y = \frac{1}{2}(u+v) \,.
$$

We can get a hint of what we should try to prove if we take $u = 1$ and $v = \frac{1}{3}$ we get

$$
x = \frac{2}{1+3} = \frac{1}{2} \,,
$$

and

$$y = \frac{1}{2}\left(1 + \frac{1}{3}\right) = \frac{2}{3}\,.$$

Thus we see that for this example $x < y$. We can get equality between x and y if we let $u = v = 1$. Based on these examples we should try to prove

$$\frac{2}{\frac{1}{u} + \frac{1}{v}} \le \frac{1}{2}(u + v)\,.$$

We will assume that this statement is true and see if we can derive a true statement from it. If we can (and we can reverse all steps taken) then we have proved the desired inequality. The above is equivalent to

$$(u + v)\left(\frac{1}{u} + \frac{1}{v}\right) \ge 4\,,$$

which is equivalent to

$$1 + \frac{u}{v} + \frac{v}{u} + 1 \ge 4 \quad \text{or} \quad \frac{u}{v} + \frac{v}{u} \ge 2\,.$$

If we let $\xi = \frac{u}{v}$ the above is

$$\xi + \frac{1}{\xi} \ge 2 \quad \text{or} \quad \xi^2 + 1 \ge 2\xi \quad \text{or} \quad \xi^2 - 2\xi + 1 \ge 0 \quad \text{or} \quad (\xi - 1)^2 \ge 0\,.$$

Since this true we have the original statement $x \le y$ is true.

Problem 28

Here we are told that the terms a, b, and c are in a geometric progression which means that $\frac{b}{a} = r$ and $\frac{c}{b} = r$ for some r. From that we can conclude that $b = ar$ and $c = ar^2$. As we know that $1 < a < b < c$ we can conclude that $r > 1$. Now we have

$$\begin{aligned}
\log_a(n) &= \frac{\log(n)}{\log(a)} \\
\log_b(n) &= \frac{\log(n)}{\log(b)} = \frac{\log(n)}{\log(a) + \log(r)} \\
\log_c(n) &= \frac{\log(n)}{\log(c)} = \frac{\log(n)}{\log(a) + 2\log(r)}\,.
\end{aligned}$$

Thus the reciprocals of these numbers look like

$$\begin{aligned}
\frac{1}{\log_a(n)} &= \frac{\log(a)}{\log(n)} \\
\frac{1}{\log_b(n)} &= \frac{\log(a) + \log(r)}{\log(n)} \\
\frac{1}{\log_c(n)} &= \frac{\log(a) + 2\log(r)}{\log(n)}\,.
\end{aligned}$$

These numbers are in a arithmetic progression with a starting value of $\frac{\log(a)}{\log(n)}$ and a common difference of $\frac{\log(r)}{\log(n)}$.

Problem 29

Let the two boys be denoted by 1 and 2 and the radius of the track be R. Then the time it takes each boy to go around the track will be

$$T_1 = \frac{2\pi R}{5}$$
$$T_2 = \frac{2\pi R}{9}.$$

Thus each boy will be back at the start in T_i time $i = 1, 2$. They will stop running when the both meet at the starting point again which means that the first time we have

$$n_1 T_1 = n_2 T_2 ,$$

where n_1 and n_2 are integers that are as small as possible. This means that

$$\frac{n_1}{n_2} = \frac{T_2}{T_1} = \frac{5}{9}.$$

The solution with the smallest values of n_1 and n_2 is $n_1 = 5$ and $n_2 = 9$.

With these, the total amount of time each boy is running is then given by

$$n_1 T_1 = n_2 T_2 = 5\left(\frac{2\pi R}{5}\right) = 2\pi R.$$

Assume that the first boy starts running counter-clockwise while the second by runs clockwise and measure angles clockwise positive from the x-axis. The angular frequency of each boy is given by

$$\omega_1 = \frac{2\pi}{T_1} = \frac{5}{R}$$
$$\omega_2 = -\frac{2\pi}{T_2} = -\frac{9}{R}.$$

With these values, the *angular* position of each boy as a function of time is given by

$$\theta_1(t) = \omega_1 t = \left(\frac{5}{R}\right) t$$
$$\theta_2(t) = \omega_2 t = -\left(\frac{9}{R}\right) t.$$

Now we want to know for how many times the boys meet before the each meet at where they started. This will happen when

$$\theta_1(t) - \theta_2(t) = 2\pi m ,$$

for integer values of m. Using what we know for $\theta_1(t)$ and $\theta_2(t)$ from the above we have this is equal to

$$\left(\frac{5}{R}\right) t + \left(\frac{9}{R}\right) t = 2\pi m .$$

Solving the above for t we get

$$t = \frac{2\pi m R}{14}.$$

Now as we know that $0 \le t \le 2\pi R$ we can put the expression for t into that inequality to get

$$0 \le \frac{2\pi m R}{14} \le 2\pi R,$$

or

$$0 \le n \le 14.$$

The value of $n = 0$ corresponds to $t = 0$ which is when the two boys start running. The value of $n = 14$ corresponds to $t = 2\pi R$ and is when the boys stop running thus there are $15 - 2 = 13$ crossings.

Problem 30

Note that the set of points in S are the ones inside a circle with a center at the point $(T, 0)$ and a radius of T. Thus the center of the circle is not on the line $y = x$ (since it is on the x-axis only). This also means that points in S are in the fourth quadrant.

Next we will try to understand the function $[t]$ and $t - [t]$. For the first function using the definition given we have

$$[t] = \begin{cases} 0 & 0 \le t < 1 \\ 1 & 1 \le t < 2 \\ 2 & 2 \le t < 3 \\ 3 & 3 \le t < 4 \\ 4 & 4 \le t < 5 \\ \vdots & \vdots \end{cases}.$$

Using this we find that

$$t - [t] = \begin{cases} t - 0 & 0 \le t < 1 \\ t - 1 & 1 \le t < 2 \\ t - 2 & 2 \le t < 3 \\ t - 3 & 3 \le t < 4 \\ t - 4 & 4 \le t < 5 \\ \vdots & \vdots \end{cases}.$$

If we plot this function we find that it is a "sawtooth" or linear segments that start on the x-axis at the values of $t = 0, 1, 2, \cdots$ with a slope of one and stop when they get to the next integer (i.e. $t + 1$) and don't include the right most endpoint. Thus $|t - [t]| < 1$ for all t. This means that the area of S must be less than that of a circle with a radius of one or less than $\pi 1^2 = \pi$.

Next consider the value of $t = 0.5$ then we have $t - [t] = 0.5 - 0 = 0.5$ so S is the set of points that satisfy

$$(x - 0.5)^2 + y^2 \le 0.5^2.$$

Note that $(0, 0)$ is in the above set.

Problem 31

We are told that

$$YE \cdot ME = TTT\,.$$

Notice that for any digit of T the right-hand-side TTT can be factored as $T(111) = T \cdot 3 \cdot 37$. Thus the left-hand-side must have the same factors 3, T, and 37. Because of this one of YE or ME must be a two digit number that has a factor of 37. The two choices thus are

$$\begin{aligned} 1 \times 37 &= 37 \quad \text{or} \\ 2 \times 37 &= 74\,. \end{aligned}$$

If we assume that YE is 37 then we have $E = 7$ and we must find digits M and T such that

$$37 \cdot M7 = T \cdot 3 \cdot 37\,,$$

or

$$M7 = 3 \cdot T$$

For the values of $T \in \{1, 2, \cdots, 8, 9\}$ the only product $3 \cdot T$ that ends in a seven is when $T = 9$ and this gives $M = 2$. With this case we have

$$E + M + T + Y = 7 + 2 + 9 + 3 = 21\,.$$

Trying to let YE equal 74 will result in a contradiction downstream. Thus we have found the only solution above.

Problem 32

The volume of a pyramid is $V = \frac{1}{3}Ah$ where A is the area of the base and h is the height of the pyramid. Since the base is an equilateral triangle with a side length $s = 6$ we have that its area is (using the formula for the area of an equilateral triangle)

$$A = \frac{\sqrt{3}}{4}s^2 = 9\sqrt{3}\,.$$

We now need to determine the height h of the pyramid. To do this note that the vertex of the pyramid is right over the centroid of the equilateral triangle that is its base. This centroid is located at a point that is a distance r from any vertex where r satisfies

$$r\cos(30^\circ) = \frac{6}{2} = 3 \quad \text{so} \quad r = 2\sqrt{3}\,.$$

The height of the pyramid is then the length of a leg of a right triangle with a hypotenuse of $\sqrt{15}$ and an other leg of length $r = 2\sqrt{3}$ thus h must satisfy

$$15 - 4(3) = h^2 \quad \text{so} \quad h = \sqrt{3}\,.$$

Using all of these parts we find $V = 9$.

Problem 33

Let p be the percentage of acid in the original mixture. Let V be the volume of the original mixture in ounces. Then pV is the amount of acid and $(1-p)V$ is the amount of water in ounces. When we add one ounce of water we have pV ounces of acid and $(1-p)V+1$ ounces of water. This has a percentage of acid given by $\frac{1}{5}$ or

$$\frac{pV}{pV+(1-p)V+1} = \frac{pV}{V+1} = \frac{1}{5}\,.$$

Next we add one ounce of acid to this mixture we get $pV+1$ ounces of acid and $(1-p)V+1$ ounces of water which gives a percentage of acid given by

$$\frac{pV+1}{pV+1+(1-p)V+1} = \frac{pV+1}{V+2} = \frac{1}{3}\,.$$

Solving for pV in the first equation and putting this into the second equation gives

$$\frac{\frac{1}{5}(V+1)+1}{V+2} = \frac{1}{3}\,.$$

Solving this for V gives $V=4$. Putting this into the first equation we can solve for p. Doing this we get $p=\frac{1}{4}=0.25$.

Problem 34

Let v be the velocity of the plane in still air and v_w the velocity of the wind. Then if D is the distance between the two towns then on the trip that was against the wind we have

$$84(v-v_w) = D\,.$$

On the return trip the plane took nine minutes less than T or

$$(T-9)(v+v_w) = D\,,$$

where T is the time to fly in a still wind and so must satisfy

$$Tv = D \quad \text{or} \quad T = \frac{D}{v}\,.$$

If we divide each of these two equations by v (and use the above) we get

$$84\left(1-\frac{v_w}{v}\right) = T$$
$$(T-9)\left(1+\frac{v_w}{v}\right) = T\,.$$

This is two equations and two unknowns T and $\frac{v_w}{v}$ that we can solve. Using the first we get $\frac{v_w}{v} = 1-\frac{T}{84}$. If we put that into the second we get

$$(T-9)\left(2-\frac{T}{84}\right) = T\,,$$

or

$$T^2 - 93T + 1512 = 0\,.$$

Solving this we get $T=21$ or $T=72$. The number of minutes for the return trip is $T-9$ which could be 12 or 63 minutes.

Problem 35 (d vs. s)

From the diagram we know that the angles $\angle NOR$, $\angle POQ$, and $\angle MOP$ must all be the same. Since MN, PQ, and OR are parallel then $QN = PM = NR = s$. Thus the angles $\angle QON$ and $\angle MOR'$ are all equal. This means that each "wedge" of the circle in the upper 1/2 plane has an angle of $\frac{180}{5} = 36$ degrees. We can use the law of cosigns to compute the lengths s and d. We have

$$
\begin{aligned}
s^2 &= 1^2 + 1^2 - 2(1)(1)\cos(36^\circ) = 2 - 2\cos(36^\circ) \\
d^2 &= 1^2 + 1^2 - 2(1)(1)\cos(3(36^\circ)) = 2 - 2\cos(108^\circ)\,.
\end{aligned}
$$

Using the identity that

$$
1 - \cos(\theta) = 2\sin^2\left(\frac{\theta}{2}\right)\,,
$$

we can write the above as

$$
\begin{aligned}
s^2 &= 4\sin^2(18^\circ) \quad \text{so} \quad s = 2\sin(18^\circ) \\
d^2 &= 4\sin^2(54^\circ) \quad \text{so} \quad d = 2\sin(54^\circ)\,.
\end{aligned}
$$

Now that we have expressions for s and d we can see which of the given expressions hold true. Using the following trigonometric identities

$$
\begin{aligned}
\sin(\theta) &= \cos(90 - \theta) \\
\cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta) \\
&= 1 - 2\sin^2(\theta) \\
&= 2\cos^2(\theta) - 1\,,
\end{aligned}
$$

we can write s (dropping the degree notation for now) as

$$
\begin{aligned}
s &= 2\sin(18) = 2\cos(72) = 2\cos(2(36)) \\
&= 2(2\cos^2(36) - 1) = 2(2\sin^2(54) - 1) \\
&= 2\left(\frac{d^2}{2} - 1\right) = d^2 - 2\,.
\end{aligned} \tag{236}
$$

We can also write d as

$$
\begin{aligned}
d &= 2\sin(54) = 2\cos(36) = 2\cos(2(18)) \\
&= 2(1 - 2\sin^2(18)) \\
&= 2\left(1 - \frac{s^2}{2}\right) = 2 - s^2\,.
\end{aligned} \tag{237}
$$

If we add these two equations together we get

$$
s + d = d^2 - 2 + (2 - s^2) = d^2 - s^2 = (d - s)(d + s) \quad \text{so} \quad d - s = 1\,.
$$

Since this means that $d = 1 + s$ if we put that into Equation 236 we get

$$
s = (1 + s)^2 - 2\,.
$$

Which has solutions $s = \frac{-1\pm\sqrt{5}}{2}$. We have to take the positive sign to make sure that $s > 0$. This means that

$$d = 1 + s = \frac{1+\sqrt{5}}{2}\,.$$

This means that

$$ds = \left(\frac{-1+\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{2}\right) = \frac{5-1}{4} = 1\,.$$

If we then use Equations 236 and 237 we compute

$$d^2 - s^2 = (s+2) - (2-d) = s + d = \frac{-1+\sqrt{5}}{2} + \frac{1+\sqrt{5}}{2} = \sqrt{5}\,.$$

The 1974 Examination

Problem 1

Solving for y we get

$$y = \frac{6x}{x-4}\,,$$

which is not listed. Solving for x we get

$$x = \frac{4y}{y-6}\,,$$

which is one of the choices.

Problem 2

The two equations that must be true are

$$\begin{aligned} 3x_1^2 - hx_1 &= b \\ 3x_2^2 - hx_2 &= b\,. \end{aligned}$$

If we subtract these two we get

$$3(x_1^2 - x_2^2) - h(x_1 - x_2) = 0\,,$$

or factoring the first term we get

$$3(x_1 - x_2)(x_1 + x_2) - h(x_1 - x_2) = 0\,.$$

Since we know that $x_1 \neq x_2$ we have that $x_1 - x_2 \neq 0$ and we can divide by this to get

$$3(x_1 + x_2) - h = 0\,.$$

Which we can then solve for the sum $x_1 + x_2$.

Problem 3

We have

$$(1 + x(2-x))^4 = \sum_{k=0}^{4} \binom{4}{k} (x(2-x))^k = \sum_{k=0}^{4} \binom{4}{k} x^k (2-x)^k\,.$$

This last expression expands as

$$1 + 4x(2-x) + \binom{4}{2} x^2 (2-x)^2 + \binom{4}{3} x^3 (2-x)^3 + x^4 (2-x)^4\,.$$

Only the last term of $x^4(2-x)^4$ will have any terms of the form x^7. This means that we need to find the coefficient of x^7 in

$$x^4(2-x)^4 = x^4(x-2)^4 = x^4 \sum_{k=0}^{4} \binom{4}{k} x^k (-2)^{4-k}\,.$$

Thus we need to consider $k=3$ which gives the coefficient

$$\binom{4}{3}(-2)^1 = 4(-2) = -8\,.$$

when we recall that $\binom{4}{3} = 4$.

Problem 4

We will write our division problem as

$$\frac{x^{51}+51}{x+1} = Q(x) + \frac{R}{x+1}\,.$$

Here $Q(x)$ is a polynomial of degree 50 and R is a scalar. Multiplying this by $x+1$ we get

$$x^{51} + 51 = Q(x)(x+1) + R\,,$$

and solving for R we get

$$R = x^{51} + 51 - Q(x)(x+1)\,.$$

If we take $x=-1$ we get

$$R = -1 + 51 - Q(-1)(0) = 50\,.$$

Thus the remainder of dividing $x^{51}+51$ by $x+1$ is just 50.

Problem 6

From the definition we can see that the operator is commutative. To show it is associative we need to show that $(x \star y) \star z = x \star (y \star z)$. The left-hand-side of this is given by

$$\begin{aligned}(x \star y) \star z &= \left(\frac{xy}{x+y}\right) \star z \\ &= \frac{\frac{xyz}{x+y}}{\frac{xy}{x+y} + z} = \frac{xyz}{xy+xz+yz}\,.\end{aligned}$$

The right-hand-side of this is given by

$$\begin{aligned}x \star (y \star z) &= x \star \left(\frac{yz}{y+z}\right) \\ &= \frac{\frac{xyz}{y+z}}{x + \frac{yz}{y+z}} = \frac{xyz}{xy+xz+yz}\,.\end{aligned}$$

Notice that these two are equal and our operator is associative.

Problem 7

If we let P be the initial population, then from then problem statement we have that

$$(P + 1200)(1 - 0.11) = P - 32\,.$$

If we solve for P we get $P = 10000$.

Problem 8

Note that 3^{11} is a product that starts with the number three (an odd number) and each additional multiplication by three gives another odd number. The product 5^{13} starts with five (an odd number) and each additional multiplication gives another odd number. Thus the sum of these two numbers is even and thus divisible by two.

Problem 9

From the description when we have an odd row say $1, 3, 5, \ldots$, if we index these odd rows as $2k + 1$ where we would need to take $k = 0, 1, 2, \ldots$, then the first element in each row is $2 + 8k$. That odd row then will hold the four numbers

$$2 + 8k\,, 3 + 8k\,, 4 + 8k\,, 5 + 8k\,,$$

in the columns two, three, four, and five. The number 1000 will be "close" to the row where $2 + 8k = 1000$ which would have $k = 124.75$. As k must be an integer that value is not possible. If we take $k = 124$ then the four elements in that row are the numbers 994, 995, 996, 997. As the even rows count backwards from column four to column one, the number 998 would be in column four, the number 999 in column three, and finally the number 1000 would be in column two.

Problem 10

Expanding the given expression to produce a quadratic equation in "standard form" we get

$$(2k - 1)x^2 - 8x + 6 = 0\,.$$

The discriminant of this quadratic is given by

$$b^2 - 4ac = 64 - 4(2k - 1)6 = 8(11 - 6k)\,.$$

To have no real roots this must be negative which will happen when $k > \frac{11}{6} = 1\,\frac{5}{6}$. Thus the smallest integer value where this happens is $k = 2$.

Problem 11

The slope is given by $m = \frac{d-b}{c-a}$ and the distance (squared) between these two points is given by

$$d^2 = (a-c)^2 + (b-d)^2 = (a-c)^2 \left(1 + \frac{(b-d)^2}{(a-c)^2}\right) = (a-c)^2(1+m^2)\,.$$

Thus

$$d = |a-c|\sqrt{1+m^2}\,.$$

Problem 12

To evaluate $f(1/2)$ lets find the value of x such that $g(x) = 1/2$. This means that

$$1 - x^2 = \frac{1}{2} \quad \text{so} \quad x = \pm\frac{1}{\sqrt{2}}\,.$$

Then

$$f\left(g\left(\pm\frac{1}{\sqrt{2}}\right)\right) = f\left(\frac{1}{2}\right) = \frac{1-1/2}{1/2} = 1\,.$$

Problem 13

D is equivalent to the given statement.

Problem 14

- A is true since if $x < 0$ then $x^2 > 0$ so $x^2 > x$.
- B is not true if $x = -1$ since $x^2 > 0$ but x is not greater than zero.
- C is not true when $x = -2$.
- D is not true when $x = 2$.
- E is not true when $x = -2$.

Problem 15 (some absolute values)

We have that

$$|1+x| = \begin{cases} 1+x & \text{if} \quad x > -1 \\ -1-x & \text{if} \quad x < -1 \end{cases}\,.$$

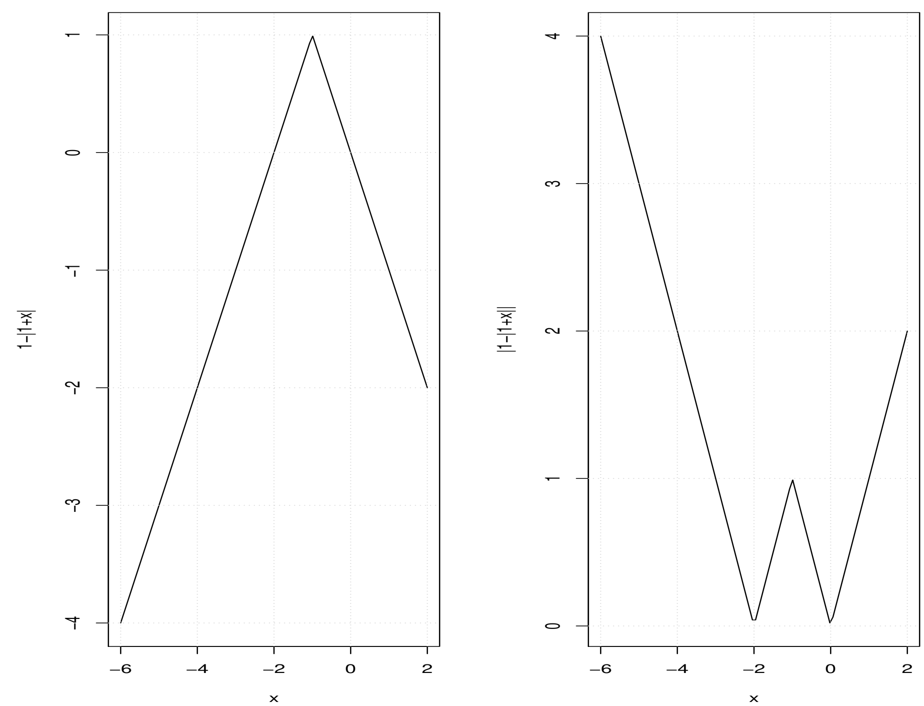

Figure 8: **Left:** A plot of the function $1-|1+x|$. **Right:** A plot of the function $|1-|1+x||$.

Thus

$$1-|1+x| = \begin{cases} 1-1-x & \text{if} \quad x > -1 \\ 1-(-1-x) & \text{if} \quad x < -1 \end{cases} = \begin{cases} -x & \text{if} \quad x > -1 \\ 2+x & \text{if} \quad x < -1 \end{cases}$$

Then taking the absolute value of the above we get

$$|1-|1+x|| = \begin{cases} |x| & \text{if} \quad x > -1 \\ |2+x| & \text{if} \quad x < -1 \end{cases}$$

Now as $x < -2$ we have $2+x < 0$ so $|2+x| = -2-x$ which is B.

Another way to solve this problem is to construct the function $|1-|1+x||$ graphically. The function $|1+x|$ is the absolute value function (a "V" shape pointing upwards out of the origin) but shifted to start and $x = -1$. Then $-|1+x|$ is this shape reflected about the x-axis. To get the curve $1-|1+x|$ we shift this curve upwards by one unit. This gives the plot in Figure 8 (left). Taking the absolute value of this curve gives a plot like that in Figure 8 (right). From that second plot we see that the curve when $x < -2$ is the line $y = -2-x$ which is again B.

Problem 16

We first draw the triangle with the inscribed and circumscribed circles. Because the triangle is a right triangle the hypotenuse of the triangle is the *diameter* of the circumscribed circle.

Because the triangle is isosceles the top vertex (the right angle) is located above the center of the diameter of the circumscribed circle. This configuration also puts the inscribed circle such that its center is also above the center of the diameter.

From the center of the inscribed circle we drop a perpendicular to center of the diameter of the circumscribed circle. We also draw a line segment from the center of the inscribed circle to one of the acute angles of the isosceles right triangle. Then the perpendicular we dropped (of length r) from the center of the inscribed circle to the center of the diagonal of the circumscribed circle is one leg of a right triangle. Another leg is the radius of the circumscribed circle (or length R). The smallest acute angle of this right triangle has a size

$$\frac{1}{2}\left(\frac{\pi}{4}\right) = \frac{\pi}{8}\,.$$

Using the definition of the tangent of this angle we thus have

$$\tan\left(\frac{\pi}{8}\right) = \frac{r}{R}\,.$$

Using this we have

$$\frac{r}{R} = \tan\left(\frac{\pi}{8}\right) = \tan\left(\frac{1}{2}\left(\frac{\pi}{4}\right)\right)\,.$$

To evaluate this we will use the half-angle formula for $\tan\left(\frac{1}{2}\theta\right)$ or

$$\tan\left(\pm\frac{\theta}{2}\right) = \frac{\pm\sin(\theta)}{1+\cos(\theta)}\,. \tag{238}$$

When we take $\theta = \frac{\pi}{4}$ we have

$$\tan\left(\frac{\pi}{8}\right) = \frac{\sin\left(\frac{\pi}{4}\right)}{1+\cos\left(\frac{\pi}{4}\right)} = \frac{1/\sqrt{2}}{1+1/\sqrt{2}} = \frac{1}{\sqrt{2}+1}\,.$$

Therefore

$$\frac{R}{r} = \sqrt{2}+1\,.$$

Problem 17

From the binomial expansion we have

$$\begin{aligned}(1+i)^{20} - (1-i)^{20} &= (i+1)^{20} - (i-1)^{20} \\ &= \sum_{k=0}^{20}\binom{20}{k} i^k - \sum_{k=0}^{20}\binom{20}{k} i^k (-1)^{20-k} \\ &= \sum_{k=0}^{20}\binom{20}{k} i^k (1-(-1)^{20-k})\,.\end{aligned}$$

If $20-k$ is even then $1-(-1)^{20-k}=0$. This will happen when $k=0,2,4,\cdots,16,18,20$. Thus the above sum becomes

$$\begin{aligned}(1+i)^{20}-(1-i)^{20} &= \sum_{k=1,3,\cdots,17,19}\binom{20}{k}i^k(1-(-1)^{20-k}) \\ &= 2\sum_{k=1,3,\cdots,17,19}\binom{20}{k}i^k \\ &= 2\sum_{l=0}^{9}\binom{20}{2l+1}i^{2l+1}\,.\end{aligned}$$

In the above sum the terms can be paired since

$$\begin{aligned}\binom{20}{19} &= \binom{20}{1} \\ \binom{20}{17} &= \binom{20}{3} \\ &\text{etc.}\end{aligned}$$

When we do that we get

$$\begin{aligned}(1+i)^{20}-(1-i)^{20} &= 2\sum_{l\in\{0,9\}}\binom{20}{2l+1}i^{2l+1}+2\sum_{l\in\{1,8\}}\binom{20}{2l+1}i^{2l+1} \\ &+2\sum_{l\in\{2,7\}}\binom{20}{2l+1}i^{2l+1}+2\sum_{l\in\{3,6\}}\binom{20}{2l+1}i^{2l+1} \\ &+2\sum_{l\in\{4,5\}}\binom{20}{2l+1}i^{2l+1}\,.\end{aligned}$$

Notice that each of above sums evaluates to

$$\begin{aligned}(1+i)^{20}-(1-i)^{20} &= 2\binom{20}{1}\sum_{l\in\{0,9\}}i^{2l+1}+2\binom{20}{3}\sum_{l\in\{1,8\}}i^{2l+1} \\ &+2\binom{20}{5}\sum_{l\in\{2,7\}}i^{2l+1}+2\binom{20}{7}\sum_{l\in\{3,6\}}i^{2l+1} \\ &+2\binom{20}{9}\sum_{l\in\{4,5\}}i^{2l+1}\,.\end{aligned}$$

and that each of these sums evaluates to zero. Thus the total sum is zero.

Problem 18

We will convert all logarithms into logarithms with respect to a common base of e as

$$\begin{aligned}p &= \frac{\ln(3)}{\ln(8)}=\frac{\ln(3)}{3\ln(2)} \\ q &= \frac{\ln(5)}{\ln(3)}\,.\end{aligned}$$

Thus the expression we seek to evaluate can be written as

$$\log_{10}(5) = \frac{\ln(5)}{\ln(10)} = \frac{\ln(5)}{\ln(5)+\ln(2)} = \frac{1}{1+\frac{\ln(2)}{\ln(5)}}\,.$$

From the expressions for p and q above we can get the ratio

$$\frac{\ln(2)}{\ln(5)} = \frac{\frac{\ln(3)}{3p}}{q\ln(3)} = \frac{1}{3pq}\,.$$

Thus we have

$$\log_{10}(5) = \frac{1}{1+\frac{1}{3pq}} = \frac{3pq}{1+3pq}\,.$$

Problem 19

Each side of the square $ABCD$ has a length of one. Let the distance BN be denoted x (so that $AN = 1-x$). Then by the Pythagorean theorem we have that the length of CN is given by

$$CN = \sqrt{1+x^2}\,.$$

As the triangle CMN is equilateral we know $CN = MN = CM = \sqrt{1+x^2}$. Again using the Pythagorean theorem we have that

$$AM^2 + AN^2 = MN^2 \quad \text{or} \quad AM^2 + (1-x)^2 = 1 + x^2\,.$$

This give $AM = \sqrt{2x}$. As triangle CDM has $CD = 1$ and $CM = \sqrt{1+x^2}$ again using the Pythagorean theorem we can compute that $DM = x$. Then

$$DM + AM = 1 \quad \text{we have} \quad x + \sqrt{2x} = 1\,.$$

Solving this for $\sqrt{x}$ gives

$$\sqrt{x} = \frac{-\sqrt{2}\pm\sqrt{6}}{2}\,.$$

We must take the positive sign or else $x > 1$. Then we have

$$x = 2 - \sqrt{3}\,.$$

With this value of x, the length of a side of the equilateral triangle CMN is

$$\sqrt{1+x^2} = \sqrt{8-4\sqrt{3}}\,.$$

Using the formula for the area of an equilateral triangle with a side length s given by $A = \frac{\sqrt{3}s^2}{4}$ we have that the area of triangle CMN is

$$\frac{\sqrt{3}}{4}\left(8-4\sqrt{3}\right) = 2\sqrt{3} - 3\,.$$

Problem 20

Consider rationalizing the denominator of the fraction

$$\begin{aligned}\frac{1}{\sqrt{n+1}-\sqrt{n}} &= \frac{1}{\sqrt{n+1}-\sqrt{n}}\left(\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}\right)\\ &= \frac{\sqrt{n+1}+\sqrt{n}}{(n+1)-n} = \sqrt{n+1}+\sqrt{n}\,.\end{aligned}$$

Doing this to each term the sum T can be written

$$\begin{aligned}T &= (\sqrt{9}+\sqrt{8})-(\sqrt{8}+\sqrt{7})+(\sqrt{7}+\sqrt{6})-(\sqrt{6}+\sqrt{5})+(\sqrt{5}+\sqrt{4})\\ &= \sqrt{9}+\sqrt{4} = 5\,.\end{aligned}$$

Problem 21

Recalling Equation 19 from the problem statement we are told that

$$a_5 - a_4 = a_1 r^4 - a_1 r^3 = 576 \quad (239)$$

$$a_2 - a_1 = a_1 r - a_1 = 9\,. \quad (240)$$

From the first of these equations we have

$$a_1 r^3 (r-1) = 576\,.$$

Putting Equation 240 into the previous one gives

$$9r^3 = 576 \quad \text{so} \quad r = 4\,.$$

Then again using Equation 240 we find $a_1 = 3$. The sum of the first five terms is then computed with Equation 20 where we find $S_5 = 1023$.

Problem 22

The given function has as its first derivative

$$f'(A) = \frac{1}{2}\cos\left(\frac{A}{2}\right) + \frac{\sqrt{3}}{2}\sin\left(\frac{A}{2}\right)\,.$$

If we set this equal to zero we get that

$$\tan\left(\frac{A}{2}\right) = -\frac{1}{\sqrt{3}}\,.$$

Now since $\tan\left(\frac{5\pi}{6}\right) = -\frac{1}{\sqrt{3}}$ we have that

$$\frac{A}{2} = \frac{5\pi}{6} \quad \text{so} \quad A = \frac{5\pi}{3}\,.$$

This is 300 degrees and is not one of the listed options.

Problem 23

Let the center of the circle be located at the origin of a coordinate system. Then the point T' is located at the location $(0, -r)$, the point T is located at the location $(0, r)$, the point P is located at $(4, r)$, the point Q is located at $(9, -r)$.

The line PQ is then the equation

$$y - r = \left(\frac{r - (-r)}{4 - 9} \right) (x - 4) ,$$

or

$$y = r - \frac{2r}{5}(x - 4) . \tag{241}$$

The line OT'' must be perpendicular to the the segment PQ and thus has a slope given by

$$-\frac{1}{-\frac{2r}{5}} = \frac{5}{2r} .$$

Thus the line OT'' is given by

$$y = \frac{5}{2r} x . \tag{242}$$

If we solve Equations 241 and 242 for x and y (the coordinates of the point T'' in terms of r) we find

$$x = \frac{26r^2}{25 + 4r^2}$$
$$y = \frac{65r}{25 + 4r^2} .$$

As this point (x, y) must also be on the circle $x^2 + y^2 = r^2$ we have that

$$\frac{26^2 r^4}{(25 + 4r^2)^2} + \frac{65^2 r}{(25 + 4r^2)^2} = r^2 .$$

If we simplify this we get

$$4r^4 - 119r^2 - 900 = 0 .$$

This has roots $r^2 = -6.25$ and $r^2 = 36$. Taking the positive root we get $r = 6$.

Problem 24

The statement "at least a five" means on a single roll we can roll a five or a six. We can roll at least a five with a probability of $p = \frac{2}{6} = \frac{1}{3}$. The number of times this happens in n (here $n = 6$) rolls is a binomial random variable with $n = 6$ and $p = \frac{1}{3}$. Thus the probability we seek is

$$\binom{6}{5} \left(\frac{1}{3}\right)^5 \left(\frac{2}{3}\right) + \binom{6}{6} \left(\frac{1}{3}\right)^6 \left(\frac{2}{3}\right)^0 .$$

This simplifies to give $\frac{13}{3^6} = \frac{13}{729}$.

Problem 25

From the diagram in terms of area we have

$$\triangle QAM = \triangle CDM$$
$$\triangle PBM = \triangle DCN\,.$$

Then building the area we want out of the areas we have we have

$$\begin{aligned}\triangle QPO &= \square OMABN + \triangle QAM + \triangle PBN \\ &= \square OMABN + \triangle CDM + \triangle DCN \\ &= \square OMABN + \triangle CON + \triangle CDO + \triangle DOM + \triangle CDO \\ &= \square ABCD + \triangle CDO\,,\end{aligned}$$

since we have constructed the area of $\square ABCD$ from its four internal regions. Thus we need to find the area of $\triangle CDO$ in terms of the area of $\square ABCD$. Next notice that the height of $\triangle OCD$ "looks to be about" $\frac{1}{4}$ that of the original parallelogram $ABCD$. If we assume this idea then we have that the area of $\triangle CDO$ is given by $\frac{1}{2}bh$ or

$$\frac{1}{2} \cdot CD \cdot \left(\frac{1}{4}\text{height of parallelogram } ABCD\right) = \frac{1}{8}k\,.$$

Then the area of $\triangle QPO$ is given by

$$k + \frac{1}{8}k = \frac{9k}{8}\,.$$

Problem 26

The number given can be written

$$(30)^4 = (2 \cdot 3 \cdot 5)^4 = 2^4 \cdot 3^4 \cdot 5^4\,.$$

Then any number of the form $2^i \cdot 3^j \cdot 5^k$ for i, j, and k integers in the set $\{0, 1, 2, 3, 4\}$ will divide this number. As there are five choices for each of i, j, and k there are

$$5^3 = 125\,,$$

possible integer divisors. Two of these are the values one (when $i = j = k = 0$) and 30^4 (when $i = j = k = 4$). Thus the number of divisors excluding these two numbers is $125 - 2 = 123$.

Problem 27

Start with the statement that

$$|f(x) + 4| < a \quad \text{when} \quad |x + 2| < b\,.$$

From the expression for $f(x)$ is this is the statement that

$$|3x + 2 + 4| < a \quad \text{when} \quad |x + 2| < b \,,$$

or the statement

$$|x + 2| < \frac{a}{3} \quad \text{when} \quad |x + 2| < b \,.$$

Using this we can work backwards. If $b \leq \frac{a}{3}$ then we have

$$|x + 2| < b \leq \frac{a}{3} \,.$$

Reversing logic above show that this implies $|f(x) + 4| < 3b \leq a$ or $|f(x) + 4| < a$.

Problem 28

Note that if $a_i = 0$ for all i then $x = 0$ and thus the value of zero must be in the range of valid x values. Next if $a_i = 2$ for all i then

$$\begin{aligned} x &= 2\sum_{i=1}^{25} \frac{1}{3^i} = 2\sum_{i=0}^{24} \frac{1}{3^{i+1}} = \frac{2}{3}\sum_{i=0}^{24} \frac{1}{3^i} \\ &= \frac{2}{3}\left(\frac{1 - \left(\frac{1}{3}\right)^{25}}{1 - \frac{1}{3}}\right) \leq \frac{2}{3}\left(\frac{1}{\frac{2}{3}}\right) = 1 \,. \end{aligned}$$

Thus $x < 1$. Finally note that if $a_1 = 2$ while $a_i = 0$ for all other i then $x = \frac{2}{3}$ thus the value of $\frac{2}{3}$ must be in the range of valid x values. There is only one interval listed that will cover all of these cases.

Problem 29

Using Equation 18 we have that

$$S_p = \frac{40}{2}(2p + 39(2p - 1)) = 20(80p - 39) \,.$$

Thus we want to evaluate

$$\begin{aligned} \sum_{p=1}^{10} S_p &= 20\sum_{p=1}^{10}(80p - 39) = 1600\sum_{p=1}^{10} p - 780\sum_{p=1}^{10} 1 \\ &= \frac{1600}{2}(10)(11) - 780(10) = 80200 \,. \end{aligned}$$

As another way to solve this problem, from the problem statement we have that

$$p = \sum_{k=1}^{40}(p + (2p - 1)(k - 1)) \,.$$

We can evaluate this as

$$S_p = 40p + (2p-1)\sum_{k=1}^{40}(k-1) = 40p + (2p-1)\left(\frac{1}{2}(39)(40)\right)$$
$$= 40p + 20(39)(2p-1) = 1600p - 780\,.$$

Then we have

$$\sum_{k=1}^{10} S_k = 1600\sum_{k=1}^{10} k - 780(10)$$
$$= 1600\left(\frac{1}{2}(10)(11)\right) - 7800 = 80200\,.$$

Problem 30

Let the segment be of length L and let x such that $0 \le x \le \frac{1}{2}$ be the fraction of L that is the smaller one i.e. $xL < (1-x)L$. Then the statement about x is that

$$\frac{xL}{L - xL} = \frac{(1-x)L}{L} \quad \text{or} \quad \frac{x}{1-x} = 1 - x\,. \tag{243}$$

Thus we see that x satisfies

$$x = (1-x)^2 \quad \text{or} \quad x = 1 - 2x + x^2 \quad \text{or} \quad x^2 - 3x + 1 = 0\,. \tag{244}$$

Using this last expression, we see that in future expressions we can replace x^2 with a linear expression since

$$x^2 = 3x - 1\,. \tag{245}$$

Then R is given by

$$R = \frac{xL}{(1-x)L} = \frac{x}{1-x}\,.$$

Using Equation 243 we see that

$$R^2 = (1-x)^2\,,$$

and

$$R^{-1} = 1 - x\,.$$

Using these two we have

$$R^2 + \frac{1}{R} = (1-x)^2 + \frac{1}{1-x}\,.$$

We can simplify this as

$$R^2 + \frac{1}{R} = \frac{(1-x)^3 + 1}{1-x} = \frac{1 - 3x + 3x^2 - x^3 + 1}{1-x}$$
$$= \frac{2 - 3x + 3x^2 - x^3}{1-x}\,.$$

Using Equation 245 we see that this is equal to

$$
\begin{aligned}
R^2 + \frac{1}{R} &= \frac{2 - 3x + 3(3x-1) - x(3x-1)}{1-x} \\
&= \frac{-1 + 7x - 3x^2}{1-x} = \frac{-1 + 7x - 3(3x-1)}{1-x} \\
&= \frac{2-2x}{1-x} = 2\,.
\end{aligned}
$$

Thus when we track this through the expression the expression we are given we end with the value of two.

The 1975 Examination

Problem 1

For this problem we just simplify from the outside inwards. We have that

$$2 - \frac{1}{2} = \frac{3}{2}\,,$$

and that

$$2 - \frac{2}{3} = \frac{4}{3}\,,$$

and that

$$2 - \frac{3}{4} = \frac{5}{4}\,,$$

so the expression evaluates to $\frac{4}{5}$.

Problem 2

Since the two lines have a different y intercepts they will have a point of intersection if they have different slopes. This means that

$$m \neq (2m-1) \quad \text{or} \quad m \neq 1\,.$$

Problem 3

To start, recall that we are told that $x < a$, $y < b$, and $z < c$.

The relationship $xy + yz + zx < ab + bc + ca$ is false. To show that let $x = -10$, $a = 1$, $y = -10$, $b = 1$, $z = -10$, and $c = 1$. Then $xy + yz + zx = 300$ and $ab + bc + ca = 3$ so $xy + yz + zx > ab + bc + ca$ in that case.

The relationship $x^2 + y^2 + z^2 < a^2 + b^2 + c^2$ is also false. The previous example would get $x^2 + y^2 + z^2 = 300$ and $a^2 + b^2 + c^2 = 3$ so $x^2 + y^2 + z^2 > a^2 + b^2 + c^2$ in this case.

The relationship $xyz < abc$ is false. To show that let $x = -1$, $a = 1$, $y = -2$, $b = 1$, $z = \frac{2}{3}$, and $c = 1$. Then $xyz = \frac{4}{3}$ and $abc = 1$ so $xyz > abc$.

Problem 4

Let the side of the first square be d and the side of the second square be s. Then as d is the diagonal of the second square we have that

$$2s^2 = d^2 \quad \text{so} \quad s^2 = \frac{d^2}{2}\,.$$

The ratio we seek is $\frac{d^2}{s^2} = 2$.

Problem 5

Expanding the product as desired we would have

$$(x+y)^9 = \sum_{k=0}^{9} \binom{9}{k} y^k x^{9-k}\,.$$

Then $k = 1$ is the second term and $k = 2$ is the third term. These terms are

$$\binom{9}{1} y^1 x^8 = 9(1-p)p^8$$
$$\binom{9}{2} y^2 x^7 = 36(1-p)^2 p^7\,.$$

If we set these two expressions equal we get

$$p = 4(1-p) \quad \text{or} \quad p = \frac{4}{5}\,.$$

Problem 6

We want to evaluate

$$\sum_{k=1}^{80} 2k - \sum_{k=1}^{80} (2k-1) = 80\,.$$

Problem 7

As the absolute value function is always positive the given fraction will be positive is $x > 0$. When $x > 0$ the numerator is $|x - |x|| = 0$ thus the fraction is never positive (it is always zero).

Problem 8

This would be II and IV.

Problem 9

From Equation 17 we have

$$\begin{aligned} a_n &= 25 + d(n-1) \\ b_n &= 75 + h(n-1) \,. \end{aligned}$$

From this we have

$$a_{100} + b_{100} = 100 + 99(d+h) = 100 \,.$$

This means that $d + h = 0$. If we then want to evaluate

$$\sum_{n=1}^{100}(a_n + b_n) = \sum_{n=1}^{100}(100 + (n-1)(d+h)) = \sum_{n=1}^{100} 100 = 10000 \,.$$

Problem 10

Consider $(10^p + 1)^2$ from which we find

$$(10^p + 1)^2 = 10^{2p} + 2\,10^p + 1 \,.$$

The number 10^{2p} has a single digit 1 with $2p$ zeros. The number $2\,10^p$ has a single digit 2 with p zeros. The number 1 has only the single digit one. Adding these parts together gives a number that has a total of $1 + 2 + 1 = 4$ ones. This result is independent of the power p.

Problem 11

One chord (the chord drawn through the center of the circle and P) will have its midpoint at the center of the circle O. Another chord (the one perpendicular to this last one drawn) will have its midpoint at P. As we draw chords around the point P the midpoint of the remaining chords trace a circle with a diameter of PO.

Problem 12

Factor the left-hand-side to get

$$(a-b)(a^2 + ab + b^2) = 19(a-b)^3 \,.$$

Since $a \neq b$ we know that $a - b \neq 0$ and the above equals

$$a^2 + ab + b^2 = 19(a-b)^2 \,.$$

Expanding the right-hand-side of this expression gives

$$a^2 + ab + b^2 = 19(a^2 - 2ab + b^2) \,.$$

When we simplify this we get

$$6a^2 - 13ab + 6b^2 = 0 \,.$$

This expression factors as

$$(2a - 3b)(3a - 2b) = 0 \,.$$

Thus $b = \frac{2}{3}a$ or $b = \frac{3}{2}a$. Since $x = a - b$ for each of these solutions we have

$$\begin{aligned} x &= a - \frac{2}{3}a = \frac{1}{3}a \quad \text{for the first solution} \\ x &= a - \frac{3}{2}a = -\frac{1}{2}a \quad \text{for the second solution} \,. \end{aligned}$$

Problem 13

If we write the given expression as

$$x^6 + 8 = x(3x^4 + 6x^2 + 1) \,,$$

we see that there can be no negative roots since when $x < 0$ the left-hand-side is always positive while the right-hand-side is always negative. That we have at least one positive root can be seen by evaluating the polynomial at $x = 0$ where we find

$$x^6 - 3x^5 - 6x^3 - x + 8\big|_{x=0} = 8 > 0 \,.$$

Evaluating this same polynomial at $x = 1$ we find

$$x^6 - 3x^5 - 6x^3 - x + 8\big|_{x=1} = 1 - 3 - 6 - 1 + 8 = -1 < 0 \,.$$

Thus there must be at least one real zero in the range $0 < x < 1$.

Problem 14

Here the expression "so and so" means two values of "so". To simplify notation we let

$$\begin{aligned} W &= \text{whatsis} \\ H &= \text{whosis} \\ I &= \text{is} \\ S &= \text{so} \,. \end{aligned}$$

The first statement we are told is that if $H = I$ and $2S = IS$ then we can conclude that $W = S$. Then we are asked value of HW when $H = S$, $2S = S^2$, and $I = 2$. The second of these statements or $2S = S^2$ when S is not zero means that $S = 2$. Thus taking the two statements $2S = S^2$ and $I = 2$ together we have that $(I, S) = (2, 2)$. Including the first statement (or $H = S$) means that $H = 2$ also. Thus we want the value of HW in the situation where

$$(H, I, S) = (2, 2, 2) .$$

We now use the first statement (since both $H = I$ and $2S = IS$ are true) to conclude that $W = S = 2$. Thus the value of $HW = 4$ this is the same as "so and so" or $S^2 = 4$.

Problem 15

We are told that $a_1 = 1$ and $a_2 = 3$ and that the rest of the elements in the sequence can be obtained by

$$a_n = a_{n-1} - a_{n-2} \quad \text{for} \quad n \geq 3 . \tag{246}$$

Then we could to compute the sum the first one-hundred elements as

$$\begin{aligned}
S = \sum_{n=1}^{100} a_n &= a_1 + a_2 + \sum_{n=3}^{100} a_n = a_1 + a_2 + \sum_{n=3}^{100} (a_{n-1} - a_{n-2}) \\
&= a_1 + a_2 + \sum_{n=3}^{100} a_{n-1} - \sum_{n=3}^{100} a_{n-2} = a_1 + a_2 + \sum_{n=2}^{99} a_n - \sum_{n=1}^{98} a_n \\
& a_1 + a_2 + \left(\sum_{n=2}^{98} a_n + a_{99} \right) - \left(a_1 + \sum_{n=2}^{98} a_n \right) \\
&= a_2 + a_{99} .
\end{aligned}$$

Using the given recurrence Equation 246 to compute additional terms in this sequence we find

$$1 , 3 , 2 , -1 , -3 , -2 , 1 , 3 , 2 , -1 , \cdots$$

Thus we see a pattern of values that repeats. From the above we can conclude that the value of a_n for all n is given by

$$\begin{aligned}
a_{1+3k} &= (-1)^k \\
a_{2+3k} &= 3(-1)^k \\
a_{3+3k} &= 2(-1)^k ,
\end{aligned}$$

for $k \geq 0$. Using this, we see that $a_{99} = a_{3+3(32)} = 2(-1)^{32} = 2$. Thus $S = 3 + 2 = 5$.

Problem 16

Let the terms in the geometric sequence be given by $a_0 r^k$ for $k \geq 0$. Then we are told that

$$\sum_{k \geq 0} a_0 r^k = 3 ,$$

or

$$\frac{a_0}{1-r} = 3 \quad \text{so} \quad a_0 = 3(1-r)\,.$$

Now as $r = \frac{1}{n}$ for some positive integer n we can write a_0 as

$$a_0 = 3\left(1 - \frac{1}{n}\right) = 3\left(\frac{n-1}{n}\right)\,.$$

Since a_0 is a positive integer we must have $n = 3$ so that $a_0 = 2$. From that we know that $r = \frac{1}{n} = \frac{1}{3}$ and we have the sum of the first two terms given by

$$a_0 + a_0 r = 2 + \frac{2}{3} = \frac{8}{3}\,.$$

Problem 17

Let the symbols T and B stand for train and bus respectively. Then without any conditions we might expect that the following would be possible values for his morning and evening commutes respectively

$$(T,B)\,,(T,T)\,,(B,T)\,,(B,B)\,.$$

From the problem statement the pair (T,B) is possible since if he takes the train in the morning he then will take the bus in the evening. The pair (B,T) is also possible since if he takes the train in the evening he must have taken the bus in the morning. The pair (T,T) is not possible for the same reason (he couldn't have taken the train in the morning if we see him taking the train in the evening). The pair (B,B) is also possible. The the variables x_{TB}, x_{BT}, and x_{BB} be the number of times our commuter commutes as (T,B), (B,T), and (B,B) respectively. Then we have

$$\begin{aligned} 8 &= x_{BT} + x_{BB} \\ 15 &= x_{TB} + x_{BB} \\ 9 &= x_{TB} + x_{BT}\,. \end{aligned}$$

This is a system of three equations and three unknowns. Solving it we get

$$\begin{aligned} x_{TB} &= 8 \\ x_{BT} &= 1 \\ x_{BB} &= 7\,. \end{aligned}$$

Then x the number of working days is the sum of all the different possible day types

$$x = x_{TB} + x_{BT} + x_{BB} = 16\,.$$

Problem 18

There are $9 \times 10 \times 10 = 900$ non-reducible three digit numbers. In order for $\log_2(N)$ to be a integer it must be one of

$$\begin{aligned} 2^7 &= 128 \\ 2^8 &= 256 \quad \text{or} \\ 2^9 &= 512 \end{aligned}$$

This gives a probability of $\frac{3}{900} = \frac{1}{300}$.

Problem 19

The given expression can be written

$$\left(\frac{\log(x)}{\log(3)}\right)\left(\frac{\log(5)}{\log(x)}\right) = \frac{\log(5)}{\log(3)},$$

or

$$\frac{\log(x)}{\log(x)} = 1 .$$

This is true for any x for which we can perform the above division or all positive $x \neq 1$.

Problem 20

Introduce a coordinate system with BC along the x-axis and the point B be located at the origin $(0, 0)$. Let the point A be located at (x, y), the point M be at $(r, 0)$, and the point C at $(2r, 0)$. From the given distances we know that

$$\begin{aligned} x^2 + y^2 &= 16 \\ (x - r)^2 + y^2 &= 9 \\ (x - 2r)^2 + y^2 &= 64 . \end{aligned}$$

Expanding the second and the third equations and using the first equation to replace any $x^2 + y^2$ with 16 we get

$$\begin{aligned} -2rx + r^2 &= -7 \\ -rx + r^2 &= 12 . \end{aligned}$$

These give two equations and two unknowns x and r. If we subtract the first equation from the second equation we get $rx = 19$. If we put this into the second equation above we get

$$-19 + r^2 = 12 \quad \text{or} \quad r = \sqrt{31} .$$

Thus the length of BC is $2r = 2\sqrt{31}$.

Problem 21

If we let $a = b = 0$ the given expression gives $f(0)^2 = f(0)$ and since $f(0) > 0$ we have $f(0) = 1$. Next if we take $b = -a$ we get

$$f(a)f(-a) = f(0) = 1 \quad \text{or} \quad f(-a) = \frac{1}{f(a)}\,.$$

Next if we consider $f(3a)$ we get

$$f(3a) = f(a + a + a) = f(a)f(a + a) = f(a)^3 \quad \text{so} \quad f(a) = \sqrt[3]{f(3a)}\,.$$

We then need to check if $f(b) > f(a)$ if $b > a$. That this does not have to be true can be seen by considering the function $f(a) = 0.5^a$ which satisfies all of the previous statements but has

$$f(a + 1) = f(1)f(a) = 0.5f(a) < f(a)\,.$$

Problem 22

Factor the given polynomial into its two roots as

$$(x - r_1)(x - r_2) = x^2 - (r_1 + r_2)x + r_1r_2\,.$$

Then $r_1r_2 = q$. Since q is prime one of r_1 and r_2 must take the value of one. Assume that $r_1 = 1$ then $r_2 = q$ and we have

$$(x - 1)(x - q) = x^2 - (q + 1)x + q\,.$$

This means that $p = q + 1$. The only two consecutive primes are two and three and we have that $q = 2$ and $p = 3$. Using these numbers we see that all of the given expressions are true.

Problem 23

Let the square have a side of length l. Then the area of $AOCD$ is the area of the square $ABCD$ minus the area of the triangles MBC and the triangle AMO. The area of $ABCD$ is l^2. The area of MBC is $\frac{1}{2}\left(\frac{l}{2}\right)l = \frac{l^2}{4}$. The triangle AMO has a base length of $\frac{l}{2}$ and we need to determine the height of this triangle. The height of this triangle is the y location of the intersection of the segments AN and MC. If we put this square in a coordinate system with the point A at the origin then the line segment AN is the line

$$y = \frac{l/2}{l}x = \frac{x}{2}\,.$$

The line segment MC is the line

$$y - 0 = \left(\frac{l - 0}{l - l/2}\right)\left(x - \frac{l}{2}\right) = 2x - l\,.$$

These two lines intersect at the point

$$x = \frac{2}{3}l$$
$$y = \frac{1}{3}l\,.$$

Thus the triangle AMO has an area of

$$\frac{1}{2}\left(\frac{l}{2}\right)\left(\frac{l}{3}\right) = \frac{l^2}{12}\,.$$

With this we compute the area of $AOCD$ is

$$l^2 - \frac{l^2}{4} - \frac{l^2}{12} = \frac{2}{3}l^2\,.$$

This has a ratio with l^2 of $\frac{2}{3}$.

Problem 27

If we write the polynomial in factored form as

$$(x-q)(x-q)(x-r) = 0\,,$$

we can expand the left-hand-side of this expression to get

$$x^3 - (p+q+r)x^2 + (rq+pq+pr)x - pqr = 0\,.$$

Comparing this to the original polynomial we see that

$$p+q+r = 1 \tag{247}$$
$$rq+pq+pr = 1 \tag{248}$$
$$pqr = 2\,. \tag{249}$$

We could try to solve this system of nonlinear equations for p, q, and r. Since we know the value of $p+q+r$ we might try *cubing* this expression since we know that will give us

$$p^3 + q^3 + r^3 + \text{other terms}\,,$$

and maybe we can evaluate the values of the "other terms" using the algebraic relationships above. To start down this path we note that

$$\begin{aligned}1 = 1^3 = (p+q+r)^3 &= (p+q+r)(p+q+r)^2 \\ &= (p+q+r)(p^2+pq+pr+pq+q^2+qr+rp+rq+r^2) \\ &= (p+q+r)(p^2+q^2+r^2+2pq+2pr+2qr) \\ &= (p+q+r)(p^2+q^2+r^2+2)\,,\end{aligned}$$

where we have used Equation 248 to evaluate and inner product like sum in the second to last equation. Continuing to expand the right-hand-side of the above we have

$$
\begin{aligned}
1 &= p^3 + pq^2 + pr^2 + 2p + qp^2 + q^3 + qr^2 + 2q + rp^2 + rq^2 + r^3 + 2r \\
&= p^3 + q^3 + r^3 + pq^2 + pr^2 + qp^2 + qr^2 + rp^2 + rq^2 + 2(p + q + r) \\
&= p^3 + q^3 + r^3 + q(pq + rq) + r(pr + qr) + p(pq + rp) + 2 \\
&= p^3 + q^3 + r^3 + q(1 - pr) + r(1 - qp) + p(1 - rq) + 2\,.
\end{aligned}
$$

Here we have used Equations 247 and 248. Continuing to expand the above we have

$$
\begin{aligned}
1 &= p^3 + q^3 + r^3 + (q + r + p) - 3pqr + 2 \\
&= p^3 + q^3 + r^3 + 1 - 3(2) + 2 = p^3 + q^3 + r^3 - 3\,.
\end{aligned}
$$

In deriving the last line we have used Equation 247 and 249. Solving for the expression desired we find $p^3 + q^3 + r^3 = 4$.

Problem 29

Warning: I tried to work this problem using a simple approximation $(1 + x)^{1/2} \approx 1 + \frac{x}{2}$. I quickly found that this was not accurate enough and then tried to use higher order approximations of the above expression. While this must work in principle all of the steps I did by "by-hand" gave approximations that were not accurate enough. Rather than push this through I ended up stopping. Below is what I initially tried in case anyone finds it useful.

To undo square roots we need to "square" things. We can do that by writing the expression as

$$
E \equiv (\sqrt{3} + \sqrt{2})^6 = ((\sqrt{3} + \sqrt{2})^2)^3 = (3 + 2\sqrt{6} + 2)^3 = (5 + 2\sqrt{6})^3\,.
$$

From this expression we first evaluate

$$
(5 + 2\sqrt{6})^2 = 25 + 20\sqrt{6} + 4(6) = 49 + 20\sqrt{6}\,.
$$

Multiplying this expression by another $5 + 2\sqrt{6}$ we get

$$
E = 245 + 100\sqrt{6} + 98\sqrt{6} + 240 = 485 + 198\sqrt{6}\,. \tag{250}
$$

We now need to estimate the value of $\sqrt{6}$. We can first try to do this by "taking out the big part" using methods from [2]. We have

$$
\sqrt{6} = \sqrt{9 - 3} = \sqrt{9}\sqrt{1 - \frac{1}{3}} = 3\sqrt{1 - \frac{1}{3}} \leq 3\left(1 - \frac{1}{6}\right) = \frac{5}{2}\,.
$$

This has an error of order $\left(\frac{1}{3}\right)^2 = \frac{1}{9} \approx 0.1$. Using this we find

$$
E \leq 485 + 198\left(\frac{5}{2}\right) = 980\,.
$$

As this is not one of the choices our approximation used to compute $\sqrt{6}$ above is not accurate enough. In fact we could have known that before we computed the full value of E since the approximation error is of order 0.1 and when we multiply by 3(198) as needed to evaluate Equation 250 the error in E will be of order $198(3)(0.1) = 59.4 \approx 60$ which is way too large if we want to estimate E to the nearest integer. One way to get higher accuracy is to use higher order Taylor expansions of the square root function. We know the number of terms we need to use since we must have the total error in our approximation of E to be less than one or

$$198 \times \text{error in Taylor series approximation of } \sqrt{6} < 1\,.$$

This will be true if we require

$$\text{error in Taylor series approximation of } \sqrt{6} < \frac{1}{200}\,.$$

Since if we enforce that expression the error will certainly be less than $\frac{1}{198}$ (as $\frac{1}{200} < \frac{1}{198}$). Or since we really are approximating $\sqrt{6}$ as $3\sqrt{1-\frac{1}{3}}$ that

$$\text{error in Taylor series approximation of } \sqrt{1-\frac{1}{3}} < \frac{1}{600}\,.$$

Since the previous approximation is not accurate enough we can use a larger number of terms in the Taylor series of $(1+x)^{1/2}$ to increase the accuracy. The Taylor series of $(1+x)^{1/2}$ is given by[4]

$$(1+x)^{1/2} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \cdots\,.$$

Then with $x = -\frac{1}{3}$ we find that some of the various terms are given by

$$\begin{aligned}
\frac{1}{2}x &= -\frac{1}{6} \\
-\frac{1}{8}x^2 &= -\frac{1}{8}\left(\frac{1}{9}\right) = -\frac{1}{72} \\
\frac{1}{16}x^3 &= \frac{1}{16}\left(-\frac{1}{27}\right) = -\frac{1}{432} \\
-\frac{5}{128}x^4 &= -\frac{5}{128}\left(\frac{1}{81}\right) = -\frac{5}{10368} \\
\frac{7}{256}x^5 &= -\frac{7}{256}\left(\frac{1}{243}\right) = -\frac{7}{62208}\,.
\end{aligned}$$

To have the error small enough we need to use a Taylor series that includes the first *four* terms. In that case we have

$$\sqrt{1-\frac{1}{3}} \approx 1 - \frac{1}{6} - \frac{1}{72} - \frac{1}{432} - \frac{5}{10368} - O\left(\frac{7}{62208}\right) = \frac{8467}{10368} - O\left(\frac{7}{62208}\right)\,.$$

Using this approximation we then get that E is given by

$$\begin{aligned}
E &\approx 485 + 198\left(3\left(\frac{8467}{10368}\right)\right) - O\left(198(3)\left(\frac{7}{62208}\right)\right) \\
&= 970.0885 - O(0.06684028)\,.
\end{aligned}$$

[4] https://en.wikipedia.org/wiki/Taylor_series

This would make one conclude that the result is 971 which is incorrect. In fact even high order approximations would need to be taken to solve the problem using this method and I stopped working. It should be said that I wouldn't expect anyone to be able to work this problem in this way on the actual test and in fact while the manipulations above "could" be done "by-hand" at some point it would take too much time to work the problem in this manner on the actual test.

Problem 30

Let $w = \cos(36)$ and $y = \cos(72)$ if we take $\theta = 36$ in

$$\cos(2\theta) = 2\cos^2(\theta) - 1\,,$$

we get

$$y = 2w^2 - 1\,.$$

Then if we take $\theta = 18$ in

$$\cos(2\theta) = 1 - 2\sin^2(\theta) = 1 - 2\cos^2(90 - \theta)\,,$$

we get

$$w = 1 - 2\cos^2(90 - 18) = 1 - 2\cos^2(72) = 1 - 2y^2\,.$$

Adding these two equations for w and y gives

$$y + w = 2(w^2 - y^2) = 2(w - y)(w + y) \quad \text{or} \quad 2(w - y) = 1\,.$$

Solving for $x = w - y$ we get $x = \frac{1}{2}$.

The 1976 Examination

Problem 1

The given statement means that

$$1 - \frac{1}{1-x} = \frac{1}{1-x}\,.$$

Solving for x we get $x = -1$.

Problem 2

The expression under the square root will be negative unless it is zero which will only happen if $x = -1$. Thus there is only one real number where the given expression is a real number.

Problem 3

If we put the square in a coordinate system with the corners at $(0,0)$, $(2,0)$, $(2,2)$, and $(0,2)$ then the distances from $(0,0)$ for midpoints of the sides is

$$1\,, \sqrt{2^2+1^2}\,, \sqrt{2^2+1^2}\,, 1\,.$$

Summing these we get $2 + 2\sqrt{5}$.

Problem 4

We are told that

$$\sum_{k=1}^{n} r^{k-1} = \sum_{k=0}^{n-1} r^k = \frac{1-r^n}{1-r} = s\,,$$

which is to be compared to

$$\sum_{k=0}^{n-1} r^{-k} = \frac{1-r^{-n}}{1-r^{-1}} = \frac{r^n - 1}{r^n(1-r^{-1})}$$
$$= \frac{1}{r^{n-1}}\left(\frac{1-r^n}{1-r}\right) = \frac{s}{r^{n-1}}\,.$$

Problem 5

Since our integer n is such that $10 < n < 100$ we can write it as the two digit number $n_1 n_2$ where

$$\begin{aligned}1 \leq n_1 \leq 9 \\ 0 \leq n_2 \leq 9 \,.\end{aligned}$$

Then the problem is to count how many solutions to the following two digit subtraction problem there are

$$n_2 n_1 - n_1 n_2 = 9 \,.$$

The above expression is equal to

$$(10 n_2 + n_1) - (10 n_1 + n_2) = 9 \,,$$

or

$$10(n_2 - n_1) + n_1 - n_2 = 9 \,.$$

Thus we need to count the number of solutions to $-10k + k = 9$ for $k \equiv n_1 - n_2$. That last equation has the unique solution $k = -1$. We are then lead to consider for how many values of n_1 and n_2 do we have $n_1 - n_2 = -1$. We can have this later equation satisfied if

- If we have $n_1 = 1$ then we need to have $n_2 = 2$.
- If we have $n_1 = 2$ then we need to have $n_2 = 3$.
- If we have $n_1 = 3$ then we need to have $n_2 = 4$.
- (pattern continues)
- If we have $n_1 = 8$ then we need to have $n_2 = 9$.

This gives eight possible solutions.

Problem 6

The solutions to

$$x^2 - 3x + c = 0 \,, \tag{251}$$

are

$$x = \frac{3 \pm \sqrt{9 - 4c}}{2} \,. \tag{252}$$

Next the solutions to

$$x^2 + 3x - c = 0 \,, \tag{253}$$

are

$$x = \frac{-3 \pm \sqrt{9 + 4c}}{2} \,. \tag{254}$$

If we consider one solution to Equation 251 say the positive solution from 252 and take the negative of that expression we get

$$\frac{-3-\sqrt{9-4c}}{2}.$$

For this to be a solution to Equation 253 means that

$$\frac{-3-\sqrt{9-4c}}{2}=\frac{-3\pm\sqrt{9-4c}}{2},$$

or

$$-\sqrt{9-4c}=\pm\sqrt{9+4c},$$

or squaring both sides we have $-4c=+4c$ so that $c=0$. Using the other root we also conclude that $c=0$. Thus Equation 251 is really

$$x^2-3x=0,$$

so $x=0$ or $x=3$.

Problem 7

The given expression will be positive if both factors are positive or both factors are negative. If both factors are positive we need to have

$$\begin{aligned} 1-|x|>0 \quad &\text{or} \quad |x|<1 \quad \text{or} \quad -1<x<+1 \\ 1+x>0 \quad &\text{or} \quad x>-1\,. \end{aligned}$$

Thus for both these conditions to hold we need to have $-1<x<+1$. If both factors are negative we need to have

$$\begin{aligned} 1-|x|<0 \quad &\text{or} \quad |x|>1 \quad \text{or} \quad x<-1 \quad \text{or} \quad x>+1 \\ 1+x<0 \quad &\text{or} \quad x<-1\,. \end{aligned}$$

Thus for both these conditions to hold we need to have $x<-1$. These give two conditions under which the given expression is positive

$$-1<x<+1 \quad \text{or} \quad x<-1\,.$$

These cannot be combined to be $x<+1$ since if $x=-1$ the product is zero and not positive.

Problem 8

The region specified is a square with vertices $(+4,-4)$, $(+4,+4)$, $(-4,+4)$, and $(-4,-4)$. There are nine integer points per row and nine rows giving a total if 81 points that satisfy the conditions stated.

We now need to count how many integer points (i, j) satisfy

$$\sqrt{i^2 + j^2} \leq 2 \quad \text{or} \quad i^2 + j^2 \leq 2^2 \,.$$

This is a circle of radius 2 inside the square drawn earlier. Drawing it in the square we can count the number of points that fall in side of it. We find thirteen points do. Thus the desired probability is $\frac{13}{81}$.

Problem 9

Let the base of triangle ABC be AB. As D bisects AB and E bisects DB if we let the length of EB be x we have

$$\begin{aligned} EB &= x \\ DE &= x \\ AD &= 2x \\ AB &= 4x \,. \end{aligned}$$

Let the height of triangle ABC be denoted h. Then since we know triangle ABC's area we have

$$96 = \frac{1}{2} ABh = \frac{1}{2}(4x)h \,.$$

Thus $xh = 48$. Now for triangle AEF as F is the midpoint of leg BC this triangle has a height of one-half that of triangle ABC. Thus its area is given by

$$\frac{1}{2} AE \left(\frac{1}{2} h\right) = \frac{1}{2}(3x)\left(\frac{1}{2} h\right) = \frac{3}{4} xh \,.$$

Using what we know about xh we find this to be equal to 36.

Problem 10

The expression $f(g(x))$ is given by

$$f(g(x)) = m(px + q) + n = mpx + mq + n \,,$$

while the expression $g(f(x))$ is given by

$$g(f(x)) = p(mx + n) = pmx + pn + q \,.$$

If these are two be equal we must have

$$mq + n = pn + q \,.$$

We can write the above as

$$n(1 - p) - q(1 - m) = 0 \,.$$

Problem 11

Statements I and II directly contradict the given statement. Thus statement III and IV are true.

Problem 12 (the largest number of crates)

To "make sure" we have crates with duplicate number of apples we could give ourselves the worst case situation by putting as many different apples in each crate as possible. If we let A be the number of apples in a crate then from the problem statement we are told that $120 \leq A < 144$. Then to spread the number of apples out as much as possible we can imagine a scenario where we start with

- we could have the first crate with 120 apples
- we could have the second crate with 121 apples
- we could have the third crate with 122 apples
- etc.

The crate number c to apple number a are related by $c = a - 119$. Thus the crate number when we get the largest possible number of apples is $c = 143 - 119 = 24$. Thus every 24 creates we have spread the apples out as thinly as they will go. We can fill five blocks of creates like this with eight left over as

$$128 = 5(24) + 8\,.$$

Because we have this left over the largest value of n will be $n = 5 + 1 = 6$.

Problem 13

From the problem statement we have that x cows will produce $\frac{x+1}{x+2}$ cans of milk per-day and thus the per-cow-per-day rate of milk production is given by dividing this expression by x or

$$\frac{x+1}{x(x+2)}\,.$$

With d days and $x + 3$ cows we would produce an amount of milk given by

$$d(x+3)\frac{x+1}{x(x+2)}\,.$$

To have this be equal to the value of $x + 5$ we must have d such that

$$d(x+3)\frac{x+1}{x(x+2)} = x+5 \quad \text{so} \quad d = \frac{x(x+2)(x+5)}{(x+1)(x+3)}\,.$$

Problem 14

The sum of all the interior angles in a regular n-gon is given by Equation 5. Then using Equation 18 as we know the values for the first and last element in the sequence we can equate this to $\frac{n}{2}(100 + 140)$ and then solve for n. We find $n = 6$.

Problem 15

The given information means that when we divide each of the numbers by d we have

$$\begin{aligned}1059 &= n_1 d + r \\ 1417 &= n_2 d + r \\ 2312 &= n_3 d + r \,.\end{aligned}$$

For some positive numbers n_1, n_2, and n_3. If we subtract pairs of these equations we get

$$\begin{aligned}358 &= (n_2 - n_1)d \\ 1253 &= (n_3 - n_1)d \\ 895 &= (n_3 - n_2)d \,.\end{aligned}$$

Thus we see that d must be common divisor of the three numbers listed. If we look for the largest common divisor we will have the smallest value for r. Using `python` we can find the largest common divisor of the three numbers on the left-hand-side as

```
from fractions import gcd
gcd(358,gcd(1253,895)) # gives 179
```

Note that since 179 is prime there can be no smaller value for d (other than the value of one, which is not allowed by the problem). Now that we know the value of d we go back to the three equations we wrote down at the beginning and write

$$\begin{aligned}1059 &= 5(179) + 164 \\ 1417 &= 7(179) + 164 \\ 2312 &= 12(179) + 164 \,.\end{aligned}$$

Thus $r = 164$. Using what we know we have $d - r = 179 - 164 = 15$.

Problem 16

We first draw both triangles. Then drop a perpendicular from vertex C to the segment AB and denote that point G. This will bisect the segment AB which means that it divides the triangle ABC into two triangles of equal area. The sides of the triangle AGC have lengths of h, GC and AC and angles of α, $\frac{\pi}{2}$, and β. The altitude in triangle DEF (from F to the segment DE intersecting at H) also divides this triangle into two other triangles one of which is DHF. The lengths of this triangle are DH, h and DF. As we are told that $AC = DF$ we have that the triangles AGC and DHF are congruent and $DH = GC$. This means that

$$\angle ACB + \angle DFE = 2\beta + 2\alpha = 2(\beta + \alpha) = 2\left(\frac{\pi}{2}\right) = \pi\,,$$

and these two angles are supplementary. As there are two congruent triangles per big triangle the areas of ABC and DEF are equal also.

Problem 17

Consider squaring the expression given. We have

$$\begin{aligned}(\sin(\theta) + \cos(\theta))^2 &= \sin^2(\theta) + \cos^2(\theta) + 2\sin(\theta)\cos(\theta)\\ &= 1 + \sin(2\theta) = 1 + a\,.\end{aligned}$$

Taking the square root of this gives $\sin(\theta) + \cos(\theta) = \sqrt{1 + a}$. We take the positive square root since we are told that θ is an acute angle.

Problem 18

We extend the segment DB until it intersects the circle at a point E. We have $BCBE = AB^2$ or

$$3(DE + 6) = 36 \quad \text{so} \quad DE = 6\,.$$

Now draw segments OE and OC each of length r the radius of the circle. Then if we use the law of cosigns for triangle ODC we get

$$r^2 = 2^2 + 3^2 - 2(2)(3)\cos(\theta) = 13 - 12\cos(\theta)\,,$$

where θ is the angle $\angle ODC$. Again using the law of cosigns this time for the triangle ODE we get

$$r^2 = 6^2 + 2^2 - 2(6)(2)\cos(\pi - \theta) = 40 + 24\cos(\theta)\,.$$

If we equate these two expressions and solve for θ we get $\cos(\theta) = -\frac{3}{4}$. Putting this value into either of the above two expressions gives $r^2 = 22$ so $r = \sqrt{22}$.

Problem 19

The degree of the remainder when we divide $p(x)$ by $(x - 1)(x - 3)$ must be one less than $(x - 1)(x - 3)$ and so must be of the form $ax + b$ for constants a and b. Thus we have

$$p(x) = q(x)(x - 1)(x - 3) + ax + b\,. \tag{255}$$

From what we are told we know that

$$
\begin{aligned}
p(x) &= q_1(x)(x-1)+3 \\
p(x) &= q_2(x)(x-3)+5\,.
\end{aligned}
$$

Thus we know that $p(1)=3$ and $p(3)=5$. Using these two facts in Equation 255 we get

$$
\begin{aligned}
3 &= a+b \\
5 &= 3a+b\,.
\end{aligned}
$$

If we solve for a and b we get $a=1$ and $b=2$. Thus the remainder is $x+2$.

Problem 20

Our given expression is

$$4(\log_a(x))^2 + 3(\log_b(x))^2 = 8(\log_a(x))(\log_b(x))\,.$$

Converting everything into natural logarithms we have

$$4\left(\frac{\log(x)}{\log(a)}\right)^2 + 3\left(\frac{\log(x)}{\log(b)}\right)^2 = 8\left(\frac{\log(x)}{\log(a)}\right)\left(\frac{\log(x)}{\log(b)}\right)\,.$$

If we divide by $\log(x)^2$ (since x is never one) we get

$$\frac{4}{\log(a)^2} + \frac{3}{\log(b)^2} = \frac{8}{\log(a)\log(b)}\,,$$

or

$$4(\log(b))^2 + 3(\log(a))^2 = 8\log(a)\log(b)\,,$$

or

$$4(\log(b))^2 - 8\log(a)\log(b) + 3(\log(a))^2 = 0\,.$$

If we let $x=\log(b)$ and $y=\log(a)$ then this is

$$4x^2 - 8xy + 3y^2 = 0\,,$$

which factors as

$$(2x-3y)(2x-y) = 0\,.$$

Thus the two solutions to the above are

$$
\begin{aligned}
2\log(b) &= 3\log(a) \quad \text{or} \\
2\log(b) &= \log(a)\,,
\end{aligned}
$$

or simplifying a bit

$$
\begin{aligned}
b^2 &= a^3 \quad \text{or} \\
b^2 &= a\,.
\end{aligned}
$$

From which we see that none of the suggested solutions are correct.

Problem 21

The expression given has an exponent that equals

$$\begin{aligned}\frac{1}{7}\sum_{k=0}^{n}(2k+1) &= \frac{1}{7}\left(2\sum_{k=0}^{n}k+(n+1)\right)\\ &= \frac{1}{7}\left(2\sum_{k=1}^{n}k+n+1\right)\\ &= \frac{1}{7}\left(2\left(\frac{n(n+1)}{2}\right)+n+1\right)\\ &= \frac{1}{7}(n+1)^2\,.\end{aligned}$$

Now we want to have

$$2^{\frac{(n+1)^2}{7}} > 1000 = 10^3 = 2^3\cdot 5^3\,.$$

We can find an approximate value for n by approximating the right-hand-side as

$$2^3\cdot 5^3 > 2^3\cdot 4^3 = 2^3\cdot 2^6 = 2^9\,.$$

This means we want

$$\frac{(n+1)^2}{7} > 9\,,$$

or

$$n+1 > 8\sqrt{1-\frac{1}{64}} \approx 8\left(1-\frac{1}{128}\right)\,.$$

Thus

$$n > 7-\frac{8}{128} = 7-\frac{1}{16}\,.$$

Thus we might try $n=7$ and see if this works. In that case we have

$$\frac{(n+1)^2}{7} = \frac{64}{7} = \frac{63+1}{7} = 9+\frac{1}{7}\,.$$

Then with this we have

$$2^{\frac{(n+1)^2}{7}} = 2^9\cdot 2^{1/7} < 512\cdot 2^{1/2} \approx 512(1.41) = 716.8 < 1000\,.$$

Lets then try $n=8$ and see if this works. We have

$$\frac{(n+1)^2}{7} = \frac{81}{7} = \frac{70+11}{7} = 10+\frac{11}{7}\,.$$

Then with this we have

$$2^{\frac{(n+1)^2}{7}} = 2^{10}\cdot 2^{11/7} = 1024\cdot 2^{11/7} > 1000\,.$$

Thus we have $n=8$. Note the solution to this problem is wrong. We can use a computer to calculate

$$\begin{aligned} n &= 7 \quad \text{where} \quad 2^{\frac{(n+1)^2}{7}} = 567\\ n &= 8 \quad \text{where} \quad 2^{\frac{(n+1)^2}{7}} = 3043.3 > 1000\,.\end{aligned}$$

Problem 22

Lets put the vertices of the triangle at the locations $(0,0)$, $(s,0)$, and

$$\left(s\cos\left(\frac{\pi}{3}\right), s\sin\left(\frac{\pi}{3}\right)\right) = \left(\frac{1}{2}s, \frac{\sqrt{3}}{2}s\right).$$

Then the sum of the distance squared (lets call it D) to each of the vertices is the expression

$$\begin{aligned} D &= x^2 + y^2 + (s-x)^2 + y^2 + \left(\frac{s}{2} - x\right)^2 + \left(\frac{\sqrt{3}}{2}s - y\right)^2 \\ &= 3(x^2+y^2) - 3sx - \sqrt{3}sy + 2s^2\,, \end{aligned}$$

when we expand and simplify. If we set this equal to a we get

$$x^2 - sx + y^2 - \frac{1}{\sqrt{3}}sy = \frac{1}{3}(a - 2s^2)\,.$$

If we "complete the square" of the quadratic expressions for x and y we get

$$\left(x - \frac{s}{2}\right)^2 + \left(y - \frac{s}{2\sqrt{3}}\right)^2 = \frac{1}{3}(a - s^2)\,.$$

This is a circle as long as $a > s^2$.

Problem 23

The given expression is equal to

$$\begin{aligned} \left(\frac{n-k-k-1}{k+1}\right)\binom{n}{k} &= \left(\frac{n-k-(k+1)}{k+1}\right)\binom{n}{k} \\ &= \left(\frac{n-k}{k+1} - 1\right)\binom{n}{k} \\ &= \left(\frac{n-k}{k+1}\right)\binom{n}{k} - \binom{n}{k}. \end{aligned}$$

As we know that $\binom{n}{k}$ is always an integer our given expression will be an integer if the first part of the above is. To study that part we write it as

$$\begin{aligned} \left(\frac{n-k}{k+1}\right)\binom{n}{k} &= \frac{n-k}{k+1}\left(\frac{n!}{k!(n-k)!}\right) \\ &= \frac{n!}{(k+1)!(n-k-1)!} = \frac{n!}{(k+1)!(n-(k+1))!} \\ &= \binom{n}{k+1}, \end{aligned}$$

which is an integer for all n and k.

Problem 25

Using the definition of Δ we can compute

$$\begin{aligned}
\Delta 1 &= 1 - 1 = 0 \\
\Delta n &= (n+1) - n = 1 \\
\Delta n^2 &= (n+1)^2 - n^2 = n^2 + 2n + 1 - n^2 = 2n + 1 \\
\Delta n^3 &= (n+1)^3 - n^3 = n^3 + 3n^2 + 3n + 1 - n^3 = 3n^2 + 3n + 1\,.
\end{aligned}$$

Using these we have

$$\begin{aligned}
\Delta^2 n^3 &= \Delta(3n^2 + 3n + 1) = 3\Delta n^2 + 3\Delta n + 0 \\
&= 3(2n+1) + 3(1) = 6n + 6\,,
\end{aligned}$$

and

$$\Delta^3 n^3 = 6\Delta n + 0 = 6\,.$$

Thus we see that $\Delta^4 n^3 = 0$.

Problem 27

Lets denote the total expression as N. Then to start this problem we move the expression $\sqrt{3 - 2\sqrt{2}}$ to the left-hand-side and square. When we do this we get

$$(N - \sqrt{3 - 2\sqrt{2}})^2 = \left(\frac{\sqrt{\sqrt{5} + 2} + \sqrt{\sqrt{5} - 2}}{\sqrt{\sqrt{5} + 1}} \right)^2\,.$$

The right-hand-side can be simplified as

$$\frac{\sqrt{5} + 2 + 2\sqrt{5 - 4} + \sqrt{5} - 2}{\sqrt{5} + 1} = \frac{2\sqrt{5} + 2}{\sqrt{5} + 1} = 2\,.$$

If we then use this and then solve for N we get

$$N = \pm\sqrt{2} - \sqrt{3 - 2\sqrt{2}}\,.$$

From the expression originally given in the problem statement for N we know that $N > 0$ and thus we need to take the positive square root. Now note that we can write

$$3 - 2\sqrt{2} = 2 - 2\sqrt{2} + 1 = (\sqrt{2} - 1)^2\,,$$

thus when we take the square root of this required by the expression above for N we get

$$N = \sqrt{2} - (\sqrt{2} - 1) = 1\,.$$

Problem 28

In this problem we want to count *distinct* intersections. As there are 100 total lines there would be $\left(\begin{array}{c} 100 \\ 2 \end{array}\right) = 4950$ intersections if the lines were assumed to have no structure. We know that 25 of these lines are parallel and thus cannot have an intersection. Thus we need to remove $\left(\begin{array}{c} 25 \\ 2 \end{array}\right) = 300$ from the previous number. In addition, another 25 (the lines $L_1, L_5, \cdots L_{97}$) intersect at a single point and thus from all $\left(\begin{array}{c} 25 \\ 2 \end{array}\right) = 300$ pairs of these single point intersection lines there is only one distinct point added to the count (and not $\left(\begin{array}{c} 25 \\ 2 \end{array}\right)$). Thus the number of distinct intersections is then

$$\left(\begin{array}{c} 100 \\ 2 \end{array}\right) - \left(\begin{array}{c} 25 \\ 2 \end{array}\right) - \left(\left(\begin{array}{c} 25 \\ 2 \end{array}\right) - 1\right) = 4950 \,.$$

Problem 30

The first time I tried to solve this problem I was unable to do so. When I revisited this problem when working the "review" problems for the test in 2000 I was able to make more progress and with a "peak" at the solutions was able to derive a solution.

We can make these equations more "symmetric" if we let

$$\begin{aligned} u &= x \\ v &= 2y \\ w &= 4z \,. \end{aligned}$$

These are almost "trivial" transformations and given (u, v, w) we can immediately determine (x, y, z). Using the above transformations the given equations become

$$\begin{aligned} u + v + w &= 12 \\ uv + vw + uw &= 44 \\ uvw &= 48 \,. \end{aligned}$$

Its hoped that the above are "recognized" as the coefficients of t in the expansion of

$$(t - u)(t - v)(t - w) \,.$$

What I mean is that expanding the above gives

$$t^3 - (u + v + w)t^2 + (uv + uw + vw)t - uvw \,,$$

and from the equations given we can evaluate the coefficients of t^i for $i \in \{0, 1, 2\}$ to find

$$(t - u)(t - v)(t - w) = t^3 - 12t^2 + 44t - 48 \,.$$

Now the right-hand-side of the above is a third degree polynomial in t and as such has three roots. From the left-hand-side of the above this means that each root could equal one of u, v, or w. Factoring the right-hand-side of the above gives the roots $t \in \{2, 4, 6\}$. Thus (u, v, w) for example can equal $(2, 4, 6)$ or any permutation of these three numbers. There are $3! = 6$ such permutations and thus six solutions to the given original system.

The 1977 Examination

Problem 1

For this we have

$$x + y + z = x + 2x + 2(2x) = 7x\,.$$

Problem 2

We could have equilateral triangles that have different edge lengths and are thus not congruent to each other.

Problem 3

We have

$$50n + 25n + 10n + 5n + n = 273\,.$$

Solving for n gives $n = 3$. The total number of coins he has is then $5n = 15$.

Problem 4

We start this problem by defining a few angles. First let $x = \angle ECD$ and $y = \angle FBD$. Then since $\triangle ECD$ is isosceles we have $\theta = \angle CDE = \angle DEC$. Since $\triangle FBD$ is isosceles we have $\phi = \angle FDB = \angle DFB$. As we know that $\angle CAB = 80°$ we have that

$$x + y = 180 - 80 = 100\,.$$

The same theorem for the sum of the angles in the triangles ECD and DFB gives

$$x + 2\theta = 180 \quad (256)$$

$$y + 2\phi = 180\,. \quad (257)$$

If we add all of the angles in the quadrilateral $AEDF$ up we get

$$80 + (180 - \theta) + (180 - \phi) + \angle EDF = 360\,,$$

or

$$80 - \theta - \phi + \angle EDF = 0\,. \quad (258)$$

If we add Equations 256 and 257 together we get

$$x + y + 2(\theta + \phi) = 360\,.$$

But we know that $x + y = 100$ and so the above becomes $\theta + \phi = 130$. Using this in Equation 258 we get

$$80 - 130 + \angle EDF = 0 \quad \text{so} \quad \angle EDF = 50\,.$$

Problem 5

Points on the line segment between A and B will certainly have their undirected distance sum to the length between AB. Note that an ellipse has the sum of the distances equal to a fixed constant *larger* than the distance between A and B.

Problem 6

Let $a = 2x$ and $b = \frac{y}{2}$ then the expression we are given is equivalent to

$$(a+b)^{-1}\left(\frac{1}{a}+\frac{1}{b}\right) = (a+b)^{-1}\left(\frac{a+b}{ab}\right) = \frac{1}{ab}\,.$$

From what we know about a and b this is equal to

$$\frac{1}{2x\left(\frac{y}{2}\right)} = \frac{1}{xy}\,.$$

Problem 7

Write this expression for t as

$$\begin{aligned}\frac{1}{1-2^{1/4}} &= \frac{1+2^{1/4}}{(1-2^{1/4})(1+2^{1/4})} = \frac{1+2^{1/4}}{1-2^{1/2}}\\ &= \frac{(1+2^{1/4})(1+2^{1/2})}{(1-2^{1/2})(1+2^{1/2})} = \frac{(1+2^{1/4})(1+2^{1/2})}{(1-2)}\\ &= -(1+2^{1/4})(1+2^{1/2})\,.\end{aligned}$$

Problem 8

Notice that if $x < 0$ then $\frac{x}{|x|} = -1$ and if $x > 0$ then $\frac{x}{|x|} = +1$. Now the expression abc can be positive or negative depending on the signs of a, b, and c. In Table 7 we enumerate the possible signs of a, b, the sign of the product abc, and the value of the requested sum. Looking in that table we see that there are only three possible values for the the sum $\{-4, 0, +4\}$.

Problem 9

Let the arc AB be x and AD be y. Then from the problem statement we have that

$$\begin{aligned}3x + y &= 360\\ \frac{1}{2}(x-y) &= 40\,.\end{aligned}$$

sign of a	sign of b	sign of c	sign of abc	value
$+$	$+$	$+$	$+$	4
$+$	$+$	$-$	$-$	0
$+$	$-$	$+$	$-$	0
$-$	$+$	$+$	$-$	0
$-$	$-$	$+$	$+$	0
$-$	$+$	$-$	$+$	0
$+$	$-$	$-$	$+$	0
$-$	$-$	$-$	$-$	-4

Table 7: The possible values for the desired expression.

Solving these for x and y we find $x = 110$ and $y = 30$. We want $\angle ACD$ which is $\frac{1}{2}(y - 0) = \frac{30}{2} = 15$.

Problem 10

We want the value of the sum $a_7 + a_6 + \cdots + a_1 + a_0$ which we will get if we set $x = 1$ in the function given. This gives the desired sum as the value

$$(3 - 1)^7 = 2^7 = 128\,.$$

Problem 11

The value of $[x]$ is the unique value of n such that $n \leq x < n + 1$. If we add one to this we get

$$n + 1 \leq x + 1 < (n + 1) + 1\,.$$

Thus we see that $[x + 1] = n + 1 = [x] + 1$.

We can show that $[x + y] \neq [x] + [y]$ by considering the following case. Let $x = y = 0.5$ so that $x + y = 1$. From these values we get $[x] = [y] = 0$ and $[x + y] = [1] = 1$. Thus $[x + y] = 1 \neq [x] + [y] = 0$.

Next we can show that $[xy] \neq [x][y]$ by considering the following example. Let $x = 0.5$ and $y = 2$. Then we have $[x] = 0$ and $[y] = 2$. The product however is $[xy] = [1] = 1 \neq [x][y] = 0$.

Problem 12

From the given problem statement (using obvious notation) we have

$$a = b + c + 16 \tag{259}$$
$$a^2 = (b + c)^2 + 1632\,. \tag{260}$$

Using Equation 259 in Equation 260 we have

$$a^2 = (a - 16)^2 + 1632\,,$$

which simplifies to $a = 59$. Using this in Equation 260 we then have

$$59^2 = (b + c)^2 = 1632\,,$$

which simplifies to $b + c = \pm 43$. As a, b, and c are ages they are all positive and we have $b + c = 43$. From all of this we have that

$$a + b + c = 102\,.$$

Problem 13

To be a geometric progression we need to have $a_{n+1} = r a_n$ for some constant r. If we iterate the given expression (assuming we are given values for a_0 and a_1) we have

$$\begin{aligned}
a_3 &= a_1 a_2 \\
a_4 &= a_2 a_3 = a_1 a_2^2 = r(a_1 a_2) \quad \text{so} \quad r = a_2 \\
a_5 &= a_4 a_3 = (a_1 a_2^2)(a_1 a_2) = a_1^2 a_2^3 = r(a_1 a_2^2) \quad \text{so} \quad r = a_1 a_2 \\
a_6 &= a_5 a_4 = (a_1^2 a_2^3)(a_1 a_2^2) = a_1^3 a_2^5 = r(a_1^2 a_2^3) \quad \text{so} \quad r = a_1 a_2^2 \\
a_7 &= a_6 a_5 = (a_1^3 a_2^5)(a_1^2 a_2^3) = a_1^5 a_2^8 = r(a_1^3 a_2^5) \quad \text{so} \quad r = a_1^2 a_2^3\,.
\end{aligned}$$

In order to make this true for all n we would need a value of r that satisfied the above expressions. This can only happen if $a_1 = a_2 = 1$.

Problem 14

If $m = 0$ then we must have $n = 0$ and if $n = 0$ then we must have $m = 0$. Thus one solution is $(m, n) = (0, 0)$. Solving for n gives

$$n = \frac{m}{m - 1}\,.$$

Note we can't have $m = 1$ to have a valid solution for n. If $m = 1$ then we would need to have $1 + n = n$ which cannot be true for any integer value of n. Thus $m \neq 1$. The above solution for n will not be a integer unless $m - 1$ divides m. This can only happen if $m = 2$. Then $n = 2$ and we have our only other solution.

Problem 15

We need to determine the length of one side of this triangle to answer this problem. We can do this if from the centers of the bottom two circles we drop perpendiculars to the x-axis and draw segments into the two corners on the base of the triangle. Consider the left-most bottom circle. Note that the segment drawn from its center to the left-most corner of the triangle is the hypotenuse of a right triangle with a leg equal to the radius of the circle of three. Thus the other leg has a length h where h is given by

$$\tan(30^\circ) = \frac{3}{h} \quad \text{so} \quad h = 3\sqrt{3}\,.$$

Using this the length of one side of the triangle is given by

$$3\sqrt{3} + 2(3) + 3\sqrt{3} = 6(1+\sqrt{3})\,.$$

The perimeter of the triangle is then given by three times this or $18(1+\sqrt{3})$.

Problem 16

Note that we can write the expression we are summing as

$$\begin{aligned}\cos(45+90n) &= \cos(45)\cos(90n) - \sin(45)\sin(90n)\\ &= \frac{1}{\sqrt{2}}(\cos(90n) - \sin(90n))\,.\end{aligned}$$

Now for $n = 1$ the expression without the $\sqrt{2}$ fraction evaluates

$$0 - 1 = -1$$

For $n = 2$ the expression without the $\sqrt{2}$ fraction evaluates to

$$-1 - 0 = -1$$

For $n = 3$ the expression without the $\sqrt{2}$ fraction evaluates to

$$0 - (-1) = 1\,.$$

For $n = 4$ the expression without the $\sqrt{2}$ fraction evaluates to

$$1 - 0 = 1\,.$$

For larger values of n these values repeat periodically with a period of four. Using this information we can write the sum we want to evaluate as

$$\begin{aligned}\sum_{n=0}^{40} i^n\cos(45+90n) &= \sum_{n=0,4,8,\cdots,40} i^n\cos(45+90n) + \sum_{n=1,5,9,\cdots,37} i^n\cos(45+90n)\\ &+ \sum_{n=2,6,10,\cdots,38} i^n\cos(45+90n) + \sum_{n=3,7,11,\cdots,39} i^n\cos(45+90n)\\ &= \frac{1}{\sqrt{2}}\left(\frac{40}{4}+1\right) + \frac{i}{\sqrt{2}}(-10) + \frac{-1}{\sqrt{2}}(-10) + \frac{-i}{\sqrt{2}}(+10)\\ &= \frac{1}{\sqrt{2}}\left(21 - 20i\right)\,.\end{aligned}$$

Problem 17

To have the numbers on the die be in an arithmetic progression with a common difference of one means that the three numbers must able to be ordered into one of the following sets

$$(1,2,3)\,,(2,3,4)\,,(3,4,5)\,,(4,5,6)\,.$$

These are 3! orderings of the three numbers in each tuple above. This means there are $4(3!) = 24$ possible orderings of the numbers on the die that are of the desired type. As we can have $6^3 = 216$ possible orderings of three face numbers the probability we get the ordering we want is

$$\frac{24}{216} = \frac{1}{9}\,.$$

Problem 18

For this we write y as

$$y = \left(\frac{\ln(3)}{\ln(2)}\right)\left(\frac{\ln(4)}{\ln(3)}\right)\cdots\left(\frac{\ln(32)}{\ln(31)}\right) = \frac{\ln(32)}{\ln(2)} = \log_2(32) = 5\,.$$

Problem 19

In each triangle, the center of the circumscribing circle is at the intersection of the perpendicular bisectors of the three sides. Note that one of the three sides of each of the four triangles is on the diagonal. Considering one of the two diagonals, note that the perpendicular bisectors on the two sides of E must be parallel. This means that these two sides of our quadrilateral must be parallel. In the same way, the other two sides of our quadrilateral are parallel since they must pass though the perpendicular bisectors of the other two parts of the other diagonal. Thus $PQRS$ is a parallelogram.

Problem 20

At the location of each letter in the given grid define the value of n to be the number of ways that we can *finish* the word CONTEST starting at that letter. Computing this value for the bottom row is easy. At each of these letters there is only *one* way to finish the word by walking from the outside inward. In the same way there is only *one* way to finish the word CONTEST starting at any location in the middle column (the tallest one that spells CONTEST vertically). Thus we have evaluated n at the following locations

```
C                             1
O                             1
```

```
N                          1
T                          1
E                          1
S                          1
T    1  1  1  1  1  1  1  1  1  1  1  1  1
     C  O  N  T  E  S  T  S  E  T  N  O  C
```

I've placed the word CONTEST along the horizontal and vertical axis to help orient the reader. Lets now consider the row before the last one. At the (S,S) location there are two ways to finish the word CONTEST, we could go down or right. This fills in the following two locations

```
C                          1
O                          1
N                          1
T                          1
E                          1
S                       2  1  2
T    1  1  1  1  1  1  1  1  1  1  1  1  1
     C  O  N  T  E  S  T  S  E  T  N  O  C
```

At the location (S,E) to complete the word CONTEST if we go right we have two paths where if we go down we have only one path. This means that we have a total of three ways to complete our word. This gives the following

```
C                          1
O                          1
N                          1
T                          1
E                          1
S                    3  2  1  2  3
T    1  1  1  1  1  1  1  1  1  1  1  1  1
     C  O  N  T  E  S  T  S  E  T  N  O  C
```

In general it seems the pattern is to add together the number on the right with the number below. Following this pattern we get for the second to last row

```
C                          1
O                          1
N                          1
T                          1
E                          1
S       6  5  4  3  2  1  2  3  4  5  6
```

```
T    1   1   1   1   1   1   1   1   1   1   1   1   1
     C   O   N   T   E   S   T   S   E   T   N   O   C
```

We can continue this pattern and we end up with the following

```
C                            1
O                        6   1   6
N                   15   5   1   5  15
T               20  10   4   1   4  10  20
E           15  10   6   3   1   3   6  10  15
S        6   5   4   3   2   1   2   3   4   5   6
T    1   1   1   1   1   1   1   1   1   1   1   1   1
     C   O   N   T   E   S   T   S   E   T   N   O   C
```

To compute the total number of ways to spell CONTEST we add up the numbers over each of the "C"s in the above grid. We find

$$1 + 2(6) + 2(15) + 2(20) + 2(15) + 2(6) + 2(1) = 127\,.$$

Problem 21

We first solve for a in the second equation given to get

$$a = x^2 - x\,.$$

Put this into the first equation to get

$$x^2 + (x^2 - x)x + 1 = 0\,,$$

or

$$x^3 + 1 = 0\,.$$

This has three solutions of which there is only one real solution. The real solution is $x = -1$ and so with that we have $a = 2$.

Problem 22

If we take $a = b = 0$ then the given relationship would give $2f(0) = 0$ and thus $f(0) = 0$ (and not $f(0) = 1$). If we take $a = 0$ then we get

$$f(b) + f(-b) = 0 + 2f(b) = 2f(b)\,,$$

or

$$f(-b) = f(b)\,,$$

which is one of the choices.

Problem 23

Let the roots of the expression

$$x^2 + mx + n = 0\,, \tag{261}$$

be given by r_1 and r_2. Then we can write the above in factored form as

$$(x - r_1)(x - r_2) = 0\,.$$

Expanding this and equating coefficients with Equation 261 we see that

$$r_1 r_2 = n \tag{262}$$

$$-(r_1 + r_2) = m\,. \tag{263}$$

The next part of the problem tells us that the roots of

$$x^2 + px + q = 0\,, \tag{264}$$

can be written as

$$(x - r_1^3)(x - r_2^3) = 0\,. \tag{265}$$

When we expand the above and compare coefficients with Equation 264 we must have

$$r_1^3 r_2^3 = q \tag{266}$$

$$-(r_1^3 + r_2^3) = p\,. \tag{267}$$

Now from Equations 262 and 266 we have $n^3 = q$. Next if we cube Equation 263 we get

$$-(r_1 + r_2)^3 = m^3\,,$$

or expanding

$$-(r_1^3 + 3r_1^2 r_2 + 3r_1 r_2^2 + r_2^3) = m^3\,.$$

Since we know that $r_1^3 + r_2^3 = -p$ from Equation 267 we can write the previous expression as

$$-(-p + 3r_1 r_2(r_1 + r_2)) = m^3\,.$$

Again using Equation 262 and 263 the above becomes

$$-(-p + 3n(-m)) = m^3\,.$$

If we solve the above for p we get $p = m^3 - 3mn$.

Problem 24

We will use partial fractions to write the terms in the sum as

$$\frac{1}{(2n-1)(2n+1)} = \frac{A}{2n-1} + \frac{B}{2n+1}\,.$$

Multiplying by $(2n+1)(2n-1)$ this means that

$$1 = A(2n+1) + B(2n-1)\,.$$

If we let $n = \frac{1}{2}$ then we get

$$1 = A(1+1) \quad \text{so} \quad A = \frac{1}{2}\,.$$

If we let $n = -\frac{1}{2}$ then we get

$$1 = B(-2) \quad \text{so} \quad B = -\frac{1}{2}\,.$$

Thus we have shown that

$$\frac{1}{(2n-1)(2n+1)} = \frac{1}{2(2n-1)} - \frac{1}{2(2n+1)}\,.$$

Thus the sum we want to evaluate can be written as

$$\begin{aligned}
\sum_{n=1}^{128} \frac{1}{(2n-1)(2n+1)} &= \frac{1}{2}\sum_{n=1}^{128} \frac{1}{2n-1} - \frac{1}{2}\sum_{n=1}^{128} \frac{1}{2n+1} \\
&= \frac{1}{2}\sum_{n=1}^{128} \frac{1}{2n-1} - \frac{1}{2}\sum_{n=2}^{129} \frac{1}{2(n-1)+1} \\
&= \frac{1}{2}\sum_{n=1}^{128} \frac{1}{2n-1} - \frac{1}{2}\sum_{n=2}^{129} \frac{1}{2n-1} \\
&= \frac{1}{2}\left(1 + \sum_{n=2}^{128} \frac{1}{2n-1} - \sum_{n=2}^{128} \frac{1}{2n-1} - \frac{1}{2(129)-1}\right) \\
&= \frac{1}{2}\left(1 - \frac{1}{257}\right) = \frac{128}{257}\,.
\end{aligned}$$

Problem 25

Using the prime factorization theorem we can write 1005! as the product of primes

$$1005! = 2^p 3^q 5^r 7^s 11^t \ldots\,.$$

The largest value of n that we can divide 1005! by 10^n will be the smaller of the two numbers p and r in the above prime factorization. We will now determine the number r. The numbers between one and 1005 that are divisible by five at least once are

$$5\,, 10\,, 15\,, 20\,, \ldots\,.$$

This is $\lfloor \frac{1005}{5} \rfloor = \lfloor 201 \rfloor = 201$ numbers. The numbers between one and 1005 that are divisable by five at least twice are the numbers

$$25\,, 50\,, 75\,, \ldots\,.$$

There are $\lfloor \frac{1005}{25} \rfloor = \lfloor 40.2 \rfloor = 40$ of these numbers. In the same way the numbers between one and 1005 that are divisable by five at least three times are $\lfloor \frac{1005}{125} \rfloor = \lfloor 8.04 \rfloor = 8$ and the numbers between one and 1005 that are divisable by five at least four times are $\lfloor \frac{1005}{625} \rfloor = 1$.

The power of five in the prime factorization of 1005! is then

$$r = 4(1) + 3(8-1) + 2(40-8) + 1(201-40) = 250\,.$$

This is the sum of one number that is divisable by 5^4. Of the eight numbers that are divisable by 5^3 (one is already counted) and the others give three powers of five each. Of the fourth numbers that are divisable by 5^2 (eight have already been counted) and the others give two powers of five each etc.

Another way to compute this same number is to recognize that we have 201 numbers that will contribute at least one power of five in the prime factorization, 40 numbers that contribute at least another power of five, 8 numbers that contribute yet another power of five and finally one number that will contribute one more power of five. This would give

$$r = 201 + 40 + 8 + 1 = 250\,.$$

As $\lfloor \frac{1005}{2} \rfloor = 502$ we have $p > 502$ so the number $n = \min(p, r) = r = 250$.

Problem 26

Draw the quadrilateral with vertices as $MNPQ$. Then by drawing the two diagonals we can write the area A of the full quadrilateral in terms of the two triangles (using the law of sines) that each diagonal divides it into. For example we have

$$\begin{aligned} A &= \triangle_{MNP} + \triangle_{PQM} \\ &= \frac{1}{2}ab\sin(N) + \frac{1}{2}cd\sin(Q)\,, \end{aligned}$$

and

$$\begin{aligned} A &= \triangle_{MNQ} + \triangle_{NPQ} \\ &= \frac{1}{2}ad\sin(M) + \frac{1}{2}bc\sin(P)\,. \end{aligned}$$

If we multiply both of these expressions for A by $\frac{1}{2}$ and then add (to get back A) we get

$$\begin{aligned} A &= \frac{1}{4}ab\sin(N) + \frac{1}{4}cd\sin(Q) + \frac{1}{4}ad\sin(M) + \frac{1}{4}bc\sin(P) \\ &\le \frac{1}{4}(ab + cd + ad + bc) = \frac{1}{4}(a+c)(b+d)\,. \end{aligned}$$

This inequality will be an equalty if and only if $\sin(N) = \sin(Q) = \sin(M) = \sin(P) = 1$ which means that all angles are 90 degrees and we have a rectangle.

Problem 27

The equation of a sphere with a radius of a that is tangent to the two vertical walls and the horizontal floor must have its center at a distance of a from each of these "planes". If we take the intersection of the walls and the floor as the origin of a three dimensional corridinate system the equation of the sphere is given by

$$(x-a)^2 + (y-a)^2 + (z-a)^2 = a^2 .$$

We are next told that the point $(5, 5, 10)$ is on each ball (we assume this means that it is on the surface of the sphere). Thus for both spheres we have

$$(5-a)^2 + (5-a)^2 + (10-a)^2 = a^2 .$$

If we expand the quadratics in this expression and simplify we get

$$75 - 20a + a^2 = 0 .$$

We can solve this quadradic equation for a to get

$$a = \frac{20 \pm \sqrt{400 - 4(75)}}{2} ,$$

which simplify to the two numbers 5 and 15. As these are radius of the two balls the sum of the two diameters is then $10 + 30 = 40$.

Problem 28

To start we note that we can sum the terms of $g(x)$ using the geometric series to get

$$g(x) = \frac{1-x^6}{1-x} .$$

From this we see that the *roots* of $g(x)$ will be the roots of the numerator $1 - x^6 = 0$ (which are the six roots of unity) but not the value $x = 1$ which is the zero of the denominator. These give five roots $x = r_i$ for $i = 0, 1, 2, 3, 4$ where four roots are complex and one root $x = -1$ is real. For each of these numbers we must have have $x^6 = 1$.

Next note that if we want to divide $g(x^{12})$ by $g(x)$ then we can write

$$g(x^{12}) = Q(x)g(x) + R(x) , \tag{268}$$

where $Q(x)$ is the quotient polynomial and $R(x)$ is the remainder polynomial in this division. As the degree of $g(x)$ is five and the degree of the remainder polynomial $R(x)$ must be less than that of $g(x)$ the degree of $R(x)$ will be four or less.

For the five roots of $g(x)$ we found above let $x = r_i$ and since for these roots $x^{12} = (x^6)^2 = 1^2 = 1$ Equation 268 becomes

$$g(1) = Q(1)g(1) + R(r_i) \quad \text{so} \quad 6 = R(r_i) .$$

As this holds true for the five roots $i = 0, 1, 2, 3, 4$ we see that $R(x)$ is a polynomial of degree four or less that equals 6 at five distinct points. Because of this $R(x)$ must in fact be the constant function and we have $R(x) = 6$ for all x.

Problem 29

If we expand the left-hand-side of the given expression we get

$$x^4 + x^2y^2 + x^2z^2 + x^2y^2 + y^4 + y^2z^2 + x^2z^2 + y^2z^2 + z^4\,,$$

or

$$x^4 + y^4 + z^4 + 2(x^2y^2 + x^2z^2 + y^2z^2)\,. \tag{269}$$

Using this in the left-hand-side of the inequality given for this problem we have

$$2(x^2y^2 + x^2z^2 + y^2z^2) \le (n-1)x^4 + (n-1)y^4 + (n-1)z^4\,.$$

On solving for $\frac{n-1}{2}$ we get

$$\frac{n-1}{2} \ge \frac{x^2y^2 + x^2z^2 + y^2z^2}{x^4 + y^4 + z^4}\,.$$

Note that if $x = y = z$ then the right-hand-side of the expression simplifies and we get

$$\frac{n-1}{2} \ge 1 \quad \text{so} \quad n \ge 3\,.$$

This means that for the given inequality to be true we must have $n \ge 3$. We can show that $n = 3$ is the smallest n can be as if we take $x = y = z$ then the left-hand-side of the given inequality becomes

$$9x^4\,,$$

while the right-hand-side of the given inequality becomes

$$3nx^4\,,$$

and these two expressions are equal if $n = 3$.

Problem 30

We first draw a circle through the ten corner points of the nonagon. We next draw segments (of length r) from the center of this circle to create isosceles triangles with sides a, b and d. Note that the top vertex angle of the isosceles triangle with the base of length a (and two equal sides of r) will have an angle of

$$\frac{360}{9} = 40\,,$$

as there are nine sides to a nonagon. In the same way, the top vertex angles of the isosceles triangles with base lengths of b and d are

$$2\left(\frac{360}{9}\right) = 80$$
$$4\left(\frac{360}{9}\right) = 160\,,$$

respectively. Next we need a result on isosceles triangles that relates the lengths of the two equal sides to that of the base. If we let the two equal sides of the isosceles triangle be of length r, the top vertex angle be θ, and the base be of length b then

$$\frac{b}{2} = r\cos\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = r\sin\left(\frac{\theta}{2}\right) .$$

Using this result for the three isosceles triangles introduced above we have

$$\begin{aligned} a &= 2r\sin(20) \\ b &= 2r\sin(40) \\ d &= 2r\sin(80) . \end{aligned}$$

From these three equations we want to eliminate r to get an equation that relates a, b, and d. To do that we will use

$$\sin(x) + \sin(y) = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) . \tag{270}$$

If we consider

$$\begin{aligned} a + b &= 2r\left(\sin(20) + \sin(40)\right) \\ &= 2r(2)\sin(30)\cos(10) . \end{aligned}$$

Recall that $\sin(30) = \frac{1}{2}$ and $\cos(10) = \sin(80) = \frac{d}{2r}$ to get

$$a + b = 4r\left(\frac{1}{2}\right)\left(\frac{d}{2r}\right) = d .$$

The 1978 Examination

Problem 1

We start by writing the given equation as

$$\left(\frac{2}{x}\right)^2 - 2\left(\frac{2}{x}\right) + 1 = 0\,.$$

Using the quadratic equation on the variable $\frac{2}{x}$ we get

$$\frac{2}{x} = \frac{2 \pm \sqrt{4 - 4(1)}}{2} = 1\,.$$

Problem 2

In this problem we are told that

$$\frac{4}{C} = d\,,$$

or if we write everything in terms of the radius r we get

$$\frac{4}{2\pi r} = 2r\,.$$

Solving for r we get $r = \pm\frac{1}{\sqrt{\pi}}$. Using this we have that the area is given by

$$A = \pi r^2 = \frac{\pi}{\pi} = 1\,.$$

Problem 3

If we put everything in terms of x. We have

$$\left(x - \frac{1}{x}\right)\left(y + \frac{1}{y}\right) = \left(x - \frac{1}{x}\right)\left(\frac{1}{x} + x\right) = x^2 - \frac{1}{x^2} = x^2 - y^2\,.$$

Problem 4

Let the given expression be denoted E. Then we have

$$\begin{aligned} E &= (a+b+c-d) + (a+b-c+d) + (a-b+c+d) + (-a+b+c-d) \\ &= 2a + 2b + 2c + 2d \\ &= 2(a+b+c+d) = 2(1111) = 2222\,. \end{aligned}$$

Problem 5

From the problem statement if we let b_1, b_2, b_3, and b_4 be the amount each of the boys payed. Then from the problem statement we have

$$\begin{aligned} b_1 + b_2 + b_3 + b_4 &= 60 \\ b_1 &= \frac{1}{2}(b_2 + b_3 + b_4) \\ b_2 &= \frac{1}{3}(b_1 + b_3 + b_4) \\ b_3 &= \frac{1}{4}(b_1 + b_2 + b_4)\,. \end{aligned}$$

These are four equations and four unknowns which we can write as follows

$$\begin{aligned} b_1 + b_2 + b_3 + b_4 &= 60 \\ 2b_1 - b_2 - b_3 - b_4 &= 0 \\ -b_1 + 3b_2 - b_3 - b_4 &= 0 \\ -b_1 - b_2 + 4b_3 - b_4 &= 0\,. \end{aligned}$$

We can solve such a system by hand or using `R` as follows

```
d = c( 1, 1, 1, 1, 2, -1, -1, -1, -1, 3, -1, -1, -1, -1, 4, -1 )
A = matrix( data=d, nrow=4, ncol=4, byrow=T )
b = c( 60, 0, 0, 0 )
solve( A, b )
```

This gives

```
[1] 20 15 12 13
```

Thus the fourth boy pays 13.

Problem 6

We are given the following two equations

$$x = x^2 + y^2 \tag{271}$$
$$y = 2xy\,. \tag{272}$$

If $y = 0$ then Equation 272 is satisfied for all x. If $y = 0$ then Equation 271 is $x = x^2$ which has the solutions $x = 0$ or $x = 1$. Thus $(0, 0)$ and $(1, 0)$ are two solutions to the above equations.

If we next assume that $y \neq 0$ then Equation 272 becomes $1 = 2x$ so $x = \frac{1}{2}$. Putting this value into Equation 271 we get

$$\frac{1}{2} = \frac{1}{4} + y^2 \quad \text{so} \quad y = \pm\frac{1}{2}\,.$$

Thus we have found two more solutions $\left(\frac{1}{2}, \pm\frac{1}{2}\right)$.

Problem 7

If we draw rays diagonally across the hexagon, connecting adjacent vertices, then all rays intercept at the center of the hexagon and will break the hexagon up into six equal triangles. The central angle in each triangles that is pointing into the center of the hexagon has a degree measure of

$$\frac{360}{6} = 60\,.$$

If we draw a vertical segment perpendicular to two sides of the hexagon and through the center we split this 60 degree angle in half to get a 30 degree angle. This later angle is in a right triangle with one edge length 6 which is one-half of the length 12 (the distance across the hexagon) and another leg of length one-half the side of the hexagon. Lets denote this last length by $\frac{s}{2}$ where s is the length of a side of the regular hexagon. Thus we have that

$$\tan(30^\circ) = \frac{s/2}{6}\,,$$

or

$$\frac{1}{\sqrt{3}} = \frac{s}{12} \quad \text{so} \quad s = 4\sqrt{3}\,.$$

Problem 8

As x, a_1, a_2, and y is an arithmetic sequence it takes the form

$$a_n = x + nh \quad \text{for} \quad n = 0, 1, 2, 3, \dots\,,$$

for some value of h. If we let $n = 3$ then we see that

$$3 = y = x + 3h \quad \text{so} \quad h = \frac{y - x}{3}\,.$$

In the same way as x, b_1, b_2, b_3, and y is an arithmetic sequence it takes the form

$$b_n = x + nk \quad \text{for} \quad n = 0, 1, 2, 3, \dots\,,$$

or some value of k. If we let $n = 4$ then we see that

$$4 = y = x + 4k \quad \text{so} \quad k = \frac{y - x}{4}\,.$$

Using this we can then compute

$$\frac{a_2 - a_1}{b_2 - b_1} = \frac{(x+2h)-(x+h)}{(x+2k)-(x+k)} = \frac{h}{k}$$
$$= \frac{y-x}{3}\left(\frac{4}{y-x}\right) = \frac{4}{3}\,.$$

Problem 9

If $x < 0$ then $x - 1 < -1$ and

$$\sqrt{(x-1)^2} = |x-1| = -(x-1)\,,$$

since $x - 1$ is negative. Thus in this case

$$|x - \sqrt{(x-1)^2}| = |x + (x-1)| = |2x - 1|\,.$$

As we are told that $x < 0$ we have $2x < 0$ so $2x - 1 < -1$ i.e. $2x - 1$ is negative and now know that taking the absolute value of this expression is the same as multiplying by negative one or

$$|2x - 1| = 1 - 2x\,.$$

Problem 10

If we draw a picture of the suggested situation we see that if A is located outside of the circle on the ray from P to B the it will be closer to B then to all the other points on the circle. We can even bring the point A inside the circle until it reached the center at P. If we pass through the center the points on the "bottom" of the circle will be closer to A than the point B and we won't have the desired condition hold. This gives the set of point that are on the ray starting at the point P and in the direction of B.

Problem 11

Our two equations are

$$x + y = r \tag{273}$$
$$x^2 + y^2 = r\,. \tag{274}$$

Note that the first equation is a diagonal line though the points $(r, 0)$ and $(0, r)$ and the second equation is a circle centered at the origin with a radius $\sqrt{r}$. If we draw the line first and then imagine increasing the radius of a circle centered at the origin until it touches the line we see that it must touch the line with a radius that is at an angle of 45 degrees with the x-axis. This means if T is the point of tangency then the point T has x and y coordinate given by

$$x = \sqrt{r}\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{r}}{\sqrt{2}}$$
$$y = \sqrt{r}\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{r}}{\sqrt{2}}.$$

If we put these values into Equation 273 we get.

$$\frac{\sqrt{r}}{\sqrt{2}} + \frac{\sqrt{r}}{\sqrt{2}} = r\,,$$

or $\sqrt{r} = \sqrt{2}$ and thus $r = 2$.

Problem 12

Drawing the suggested triangle, lets denote the angles $\angle DAE$ and $\angle AED$ with the variables x and y respectively. As the angles in a triangle must sum to 180 degrees we have that

$$x + 140 + y = 180 \quad \text{or} \quad x + y = 40\,.$$

This is one relationship between x and y. To find another one we will draw the other segments as described in the problem and relate the angles formed. As $DE = DC$ the triangle $\triangle CDE$ is isosceles and we have that $\angle DCE = y$. As $AB = BC$ the triangle $\triangle ABC$ is also isosceles and we have that $\angle ACB = x$. As the segment ACE is a straight line we have that

$$\angle ACB + \angle BCD + \angle DCE = 180\,.$$

As we know $\angle ACB + \angle DCE = x + y = 40$ this means that $\angle BCD = 140$. Next as $BC = CD$ we have that the triangle $\triangle BCD$ is isosceles and $\angle CBD = \angle CDE$. The fact that all three angles in this triangle must sum to 180 means that

$$\angle CBD + \angle CDE + \angle BCD = 180\,.$$

But $\angle BCD = 140$ so the above becomes

$$2\angle CBD = 40 \quad \text{so} \quad \angle CBD = \angle CDE = 20\,.$$

As the segment ABD is a straight line we have

$$\angle ABC = 180 - \angle CBD = 180 - 20 = 160\,.$$

Finally as the triangle $\triangle ABC$ is isosceles we have that

$$\angle BAC + \angle ACB + \angle ABC = 180 \quad \text{or} \quad 2x + 160 = 180\,.$$

Thus $x = 10$.

Problem 13

We are told that c and d satisfy

$$x^2 + ax + b = 0\,, \tag{275}$$

and that a and b satisfy

$$x^2 + cx + d = 0\,, \tag{276}$$

and we want to know the value of $a + b + c + d$. From Equation 275 we know that c and d must be given by

$$\frac{-a \pm \sqrt{a^2 - 4b}}{2}\,, \tag{277}$$

and from Equation 276 we know that a and b must be given by

$$\frac{-c \pm \sqrt{c^2 - 4d}}{2}\,, \tag{278}$$

If we now try to evaluate $a + b + c + d$ by using Equation 277 for c and d we get

$$a + b + \left[\frac{-a + \sqrt{a^2 - 4b}}{2}\right] + \left[\frac{-a - \sqrt{a^2 - 4b}}{2}\right] = b\,.$$

We could also seek to evaluate $a + b + c + d$ by using Equation 278 for a and b we get

$$\left[\frac{-c + \sqrt{c^2 - 4d}}{2}\right] + \left[\frac{-c - \sqrt{c^2 - 4d}}{2}\right] + c + d = d\,.$$

Thus as both b and d equal $a + b + c + d$ we have shown that $b = d$. Thus lets call the common value b i.e. we replace the variable d with the variable b in our problem statement. Restated in this way, we have that c and b are solutions to

$$x^2 + ax + b = 0\,,$$

and a and b are solutions to

$$x^2 + cx + b = 0\,.$$

Since b is a solution of both equations we have that

$$b^2 + ab + b = 0 \quad \text{and} \quad b^2 + cb + b = 0\,.$$

If we subtract these two equations we get $(a - c)b = 0$. Since $b \neq 0$ we must have $a = c$. Replacing all c's with a's our problem is now the statement that a and b are solutions to

$$x^2 + ax + b = 0\,.$$

As a and b are roots this means that the previous polynomial must factor as

$$(x - a)(x - b) = x^2 + ax + b\,.$$

If we multiply out the left-hand-side and equate the coefficients of x with the polynomial on the right-hand-side we get that

$$\begin{aligned} -(a + b) &= a \\ ab &= b\,. \end{aligned}$$

As we are told that $b \neq 0$ the second equation gives that $a = 1$. If we put that in the first equation we get $-1 - b = 1$ so $b = -2$. In summary then we have the values $a = 1$, $b = -2$, $c = 1$, and $d = -2$. With these values we find $a + b + c + d = -2$.

Problem 14

We are told that n is a solution to the given equation. This means that

$$n^2 - an + b = 0\,.$$

We are also told that a in base n is 18 so

$$a = n + 8\,.$$

Then the first equation gives

$$n^2 - (n+8)n + b = 0\,,$$

or

$$-8n + b = 0 \quad \text{or} \quad b = 8n\,.$$

This statement is the same as b in base n is 80.

Problem 15

Square the given expression to get

$$\sin^2(x) + 2\sin(x)\cos(x) + \cos^2(x) = \frac{1}{25}\,,$$

or

$$2\sin(x)\cos(x) = \frac{1}{25} - 1 = -\frac{24}{25}\,. \tag{279}$$

Note that when $0 \leq x < \pi$ we have that $\sin(x) > 0$. The above expression states that the product of $\sin(x)$ and $\cos(x)$ must be negative. This means that $\cos(x)$ must be negative (since we know that $\sin(x) > 0$) and we thus have that $\frac{\pi}{2} < x < \pi$. The left-hand-side of Equation 279 is equal to $\sin(2x)$ and we thus have

$$\sin(2x) = -\frac{24}{25}\,.$$

Using this we can compute

$$\cos(2x) = \pm\sqrt{1 - \sin^2(2x)} = \pm\sqrt{1 - \left(\frac{24}{25}\right)^2} = \pm\frac{7}{25}\,.$$

Now since we know that $\frac{\pi}{2} < x < \pi$ we know that $\pi < 2x < 2\pi$ and thus that $\cos(2x) < 0$. Thus we need to take the minus sign in the above. Using this we can conclude that

$$\tan(2x) = \frac{-\frac{24}{25}}{-\frac{7}{25}} = \frac{24}{7}\,.$$

If we then recall the double-angle formula for tan we have

$$\tan(2x) = \frac{2\tan(x)}{1 - \tan^2(x)}\,. \tag{280}$$

If we let $T \equiv \tan(x)$ then using what we know

$$\frac{24}{7} = \frac{2T}{1-T^2}\,.$$

This is a quadratic equation in T that has two solutions which are $T = -\frac{4}{3}$ and $T = \frac{3}{4}$. Given what quadrant the angle x is located the sign of the tangent must be negative and we have that $\tan(x) = -\frac{4}{3}$.

Problem 16

The answer to this problem cannot be N since otherwise every person would have shaken hands with every other person (and we are told that at least one person has not done this). The answer cannot also be $N-1$ for this would mean that only one other person has not shaken hands with everyone. As an example of how this statement is inconsistent assume that the person A has not shaken hands with person B. But this also means that B has not shaken hands with everyone for he has not shaken hands with A. The answer can be $N-2$ which we can get by singling out two people, say A and B from our N total who will not shake hands with each other. These two people will shake hands with everyone else and all other people will shake hands with everyone.

A simple example with $N = 4$ shows that this can hold true. Draw each person as a point in two dimensions and draw lines between points to represent the fact that the two people have shaken hands. For two of these four points draw lines from each of them to the other three. You will see that there are two points i.e. $N-2 = 4-2 = 2$ without a line connecting them.

Problem 17

If we are told that

$$\left[f(x^2+1)\right]^{\sqrt{k}} = k\,,$$

then we can simplify the expression we are given by writing the outermost exponent as

$$\sqrt{\frac{12}{y}} = 2\sqrt{\frac{3}{y}}\,.$$

The notice that we have

$$\left[f\left(\frac{9+y^2}{y^2}\right)\right]^{\sqrt{\frac{12}{y}}} = \left[f\left(\frac{9}{y^2}+1\right)\right]^{2\sqrt{\frac{3}{y}}} = \left\{\left[f\left(\frac{9}{y^2}+1\right)\right]^{\sqrt{\frac{3}{y}}}\right\}^2 = k^2\,.$$

Where we let $x^2 = \frac{9}{y^2}$.

Problem 18

We want the smallest n such that

$$\sqrt{n} - \sqrt{n-1} < 0.01\,.$$

If we multiply both sides by $\sqrt{n} + \sqrt{n-1}$ and simplify we get

$$1 < 0.01(\sqrt{n} + \sqrt{n-1})\,.$$

Thus we also want the smallest n such that

$$\sqrt{n} + \sqrt{n-1} > 100\,.$$

Now for all positive integer n we have the strict inequalities

$$2\sqrt{n-1} < \sqrt{n} + \sqrt{n-1} < 2\sqrt{n}\,. \tag{281}$$

If we want to guarantee that $\sqrt{n} + \sqrt{n-1}$ will be larger than 100 we can set the left-hand-side of the above to 100 and solve for n. When we do that we get $n = 2501$. Thus if $n = 2501$ we have shown that

$$2\sqrt{n-1} = 100 < \sqrt{n} + \sqrt{n-1}\,,$$

and our desired inequality holds for this value of n. We might wonder if there is a *smaller* value of n. To see that is not true let $n = 2500$ then from the right-hand-side of Equation 281 if $n = 2500$ then we have

$$\sqrt{n} + \sqrt{n-1} < 2\sqrt{2500} = 100\,,$$

and our inequality is not satisfied. Thus the smallest n is $n = 2501$.

Problem 19

The sum of all of the probabilities that we select each integer must equal one. Thus we need to have

$$\sum_{n=1}^{100} P\{N = n\} = 1\,,$$

or given what we know about the probability of an integer in each region we have

$$50p + 3p(100 - 51 + 1) = 1\,.$$

This means that $p = \frac{1}{200}$. Next we note that the perfect squares in the range of the integers $1 \le n \le 100$ are $1, 4, 9, 16, 25, 36, 49, 64, 81, 100$ and thus we have seven of them that are less than 50 and three of them that are greater than 50. Thus the probability of drawing a perfect square is

$$7p + 3(3p) = 16p = \frac{16}{200} = 0.08\,.$$

Problem 20

Let y equal the common expression given by each term, namely

$$
\begin{aligned}
y &= \frac{a+b-c}{c} = \frac{a-b+c}{b} = \frac{-a+b+c}{a} \\
&= \frac{a+b}{c} - 1 = \frac{a+c}{b} - 1 = \frac{b+c}{a} - 1 \,.
\end{aligned}
$$

From the above notice that we have

$$
\frac{a+b}{c} = \frac{a+c}{b} = \frac{b+c}{a} = y+1 \,. \tag{282}
$$

Thus the expression for x can be written

$$
x = \left(\frac{a+b}{c}\right)\left(\frac{b+c}{a}\right)\left(\frac{a+c}{b}\right) = (y+1)^3 \,. \tag{283}
$$

Now if we take the first equality from Equation 282 we get

$$
b(a+b) = c(a+c) \,.
$$

We can write this as

$$
a(b-c) = c^2 - b^2 \,.
$$

If we factor the right-hand-side we can write this as

$$
b - c = (c-b)\left(\frac{c+b}{a}\right) \,.
$$

If we assume that $c \neq b$ we can divide both sides by $c-b$ to get

$$
-1 = \frac{c+b}{a} \,.
$$

Looking again at Equation 282 we see that this means that $y+1 = -1$ or $y = -2$. Using this and Equation 283 we see that $x = (-1)^3 = -1$.

Lets check that this value for x is correct by considering a numerical example and making sure that everything is consistent. To do that we take $a = 2$, $b = -1$ and $c = -1$. Then with these numbers we have

$$
\begin{aligned}
\frac{a+b-c}{c} &= \frac{2-1-(-1)}{-1} = -2 \\
\frac{a-b+c}{b} &= \frac{2+1+1}{-1} = -2 \\
\frac{-a+b-c}{a} &= \frac{-2-1-1}{2} = -2 \,,
\end{aligned}
$$

so all expressions equal the same thing as they must. Next we have

$$
x = \left(\frac{a+b}{c}\right)\left(\frac{b+c}{a}\right)\left(\frac{a+c}{b}\right) = (-1)\left(\frac{-1-1}{2}\right)\left(\frac{2-1}{-1}\right) = (-1)^3 = -1 \,.
$$

Problem 21

To start we use $\log_a(b) = \frac{\log(b)}{\log(a)}$ on each of the three terms to get

$$\frac{\log(3)}{\log(x)} + \frac{\log(4)}{\log(x)} + \frac{\log(5)}{\log(x)} = \frac{\log(3(4)(5))}{\log(x)} = \frac{\log(60)}{\log(x)}\,.$$

Then using the first relationship in reverse, the previous expression is equal to

$$\frac{1}{\log_{60}(x)}\,.$$

Problem 22

Notice that by assuming that any two pairs of statements were true we would get a contradiction. This means that at most one statement from the four can be true. If we consider the assignment of the one true statement to each of the possible four statements it is only when we make the third statement true do we get a consistent set.

Problem 23

From the point F, drop a perpendicular towards the segment AB and let the point of intersection be denoted as P. Let the length AP be denoted x and the length PB be denoted y. Now as $AB = \sqrt{1+\sqrt{3}}$ we have that

$$x + y = \sqrt{1+\sqrt{3}}\,. \tag{284}$$

By construction, the triangle AEB is an equilateral triangle so the angle EAB is 60 degrees. As DB is a diagonal of the square the angle FBA is 45 degrees. The tangent of these two angles are related by

$$\begin{aligned}\tan(60^\circ) &= \frac{FP}{x}\\ \tan(45^\circ) &= \frac{FP}{y}\,.\end{aligned}$$

Since $\tan(60^\circ) = \sqrt{3}$ and $\tan(45^\circ) = 1$ the above becomes

$$x = \frac{FP}{\sqrt{3}} \quad \text{and} \quad y = FP\,.$$

Putting these expressions in Equation 284 we can solve for FP where we get

$$FP = \frac{\sqrt{3}}{\sqrt{1+\sqrt{3}}}\,.$$

Now to compute the area of the triangle ABF we use the formula "$\frac{1}{2}$ times the base times the height" where the base is the segment AB and the height is the segment FP and we compute

$$\frac{1}{2}\sqrt{1+\sqrt{3}}\left(\frac{\sqrt{3}}{\sqrt{1+\sqrt{3}}}\right) = \frac{\sqrt{3}}{2}.$$

Problem 24

Since the three expressions are in geometric progression we can write

$$\begin{aligned} x(y-z) &= a_0 \\ y(z-x) &= a_0 r \\ z(x-y) &= a_0 r^2 , \end{aligned}$$

for some values of $a_0 \neq 0$ and r. If we add the three equations the left-hand-side gives zero and we end with

$$a_0 + a_0 r + a_0 r^2 = 0 .$$

If we divide by a_0 we get $1 + r + r^2 = 0$.

Problem 25

In the `R` code `prob_25_1978.R` we draw the region specified by the given inequalities for $a = 4$. This plot is given in Figure 9. There we see that this figure has six sides.

Problem 26

To start with note that $\angle BCA$ is 90 because the triangle ABC is a 3-4-5 right triangle. Let the point where the circle is tangent to the side AB be denoted L and draw the segment LC. As we are told that at the point L the circle is tangent to AB we know that $\angle ALC$ is 90 and that $LC = 2r$ where r is the radius of the inscribed circle.

Let the angle $\angle LCA$ be denoted θ. Then using various right triangles we know that

$$\begin{aligned} \angle CAB &= \frac{\pi}{2} - \theta \\ \angle LCB &= \frac{\pi}{2} - \theta \\ \angle CBA &= \theta . \end{aligned}$$

These angles mean that triangle ABC is similar to triangle CBL and triangle ACL.

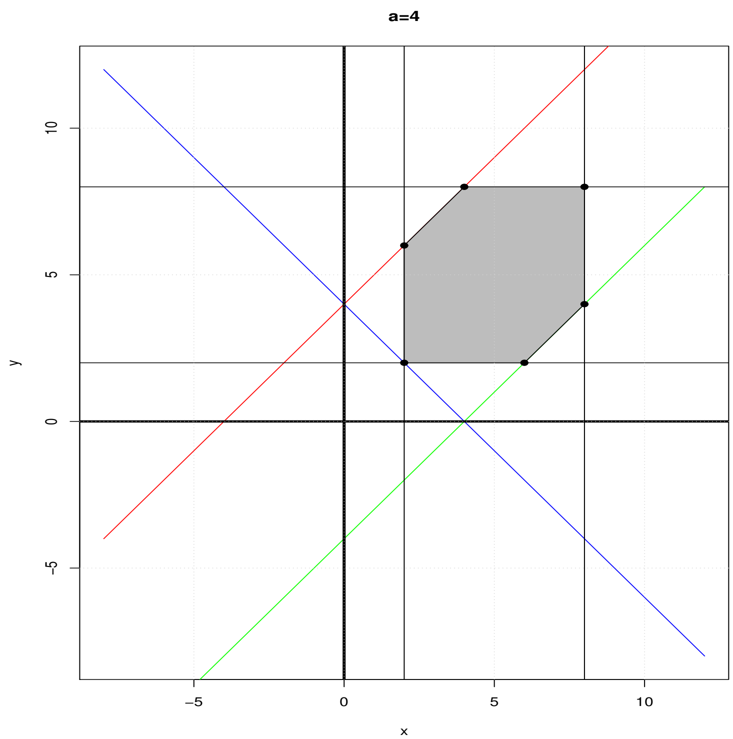

Figure 9: The polygon for Problem 25.

We now use the given lengths of the outer triangle to derive other line segment lengths. Using the right triangle ACL we have that

$$AC^2 = CL^2 + AL^2 \quad \text{or} \quad 64 = (2r)^2 + AL^2 \,.$$

This means that

$$AL = 2\sqrt{16 - r^2} \,. \tag{285}$$

Now using the fact that the triangle ACL is similar to the triangle ABC we have

$$\frac{AL}{AC} = \frac{AL}{8} = \frac{AC}{AB} = \frac{8}{10} \,.$$

If we put in the expression for Equation 285 into the above we can solve for r to get $r = \frac{12}{5}$.

Let the center of the circle be denoted by O. Now that we know the value of r, from the center of the circle draw segments (of length r) to the points R and Q. Now note that the triangle COQ is isosceles with a base of CQ and two equal sides of length r. In this triangle the two equal angles are θ and so we have

$$\begin{aligned} \tan(\theta) &= \frac{AC}{BC} = \frac{8}{6} = \frac{4}{3} \quad \text{in the same way} \\ \tan\left(\frac{\pi}{2} - \theta\right) &= \frac{BC}{AC} = \frac{6}{8} = \frac{3}{4} \,. \end{aligned}$$

In the triangle COQ drop a perpendicular of length h to the base CQ then using these right triangles we have

$$\tan(\theta) = \frac{h}{CQ/2} = \frac{4}{3}$$
$$\left(\frac{CQ}{2}\right)^2 + h^2 = r^2 \,.$$

Solving these two equations for h and CQ where we get $CQ = \frac{72}{25}$.

Next note that the triangle COR is isosceles with a base of CR and two equal sides of length r. In this triangle the two equal angles are $\frac{\pi}{2} - \theta$. In the triangle COR drop a perpendicular of length h (here I am just reusing a variable name and its value is different perhaps than the h above) to the base CR then using these right triangles we have

$$\tan\left(\frac{\pi}{2} - \theta\right) = \frac{h}{CR/2} = \frac{3}{4}$$
$$\left(\frac{CR}{2}\right)^2 + h^2 = r^2 \,.$$

Solving these two equations for h and CR where we get $CR = \frac{2(48)}{25}$.

Now that we know the lengths CQ and CR we have that

$$RQ = \sqrt{CQ^2 + CR^2} = \frac{2(12)}{5} = 4.8 \,,$$

when we put in the numbers above.

Problem 27

We want to find the smallest two numbers m_1 and m_2 (where $m_1 < m_2$) that are each divisible by all k such that $2 \le k \le 11$ with a remainder of one. This means that we can write these m_i as

$$m_i = q_i k + 1 \,,$$

for all k in the range above. Intuitively to have this property means that we can write m_i as

$$m_i = (2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11) q_i + 1 \,,$$

for some integer q_i. Note that a number of this form will have the needed divisibility property but it is not the smallest such number. For example, the multiple of 10 in the product above is already found in the product of $2 \cdot 5$ found earlier and thus there is no need to specify it explicitly. Dropping all "duplicate" factors we end up with a representation

$$m_i = (2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11) q_i + 1 \,.$$

The smallest number of this form would have $q_1 = 1$ and we get $m_1 = 27721$. The next smallest number of this form would have $q_2 = 2$ and we would get $m_2 = 55441$. Their difference is given by 27720.

Problem 29

I wish I had time to draw a nice figure but given constraints I'll just have to describe it as best as I can. I drew the quadrilateral $ABCD$ with corner labeling clockwise with A the southwest corner, followed by B as the northwest corner, C as the northeast corner, and D as the southeast corner. I then extended the edges AB to B', BC to C', etc the required lengths. In my drawing the segment BCC' was almost horizontal. Consider the triangle $BB'C'$ then in that figure. Notice that if we "fold" the segment BB' into the segment AB (they are the same length) across the segment BCC' this larger triangle will have the same height as the triangle ABC in the original quadrilateral. What is different however is that this larger triangle will have a base BCC' that is twice the base BC. Thus this triangle will have twice the area as the triangle ABC which is totally inside the original quadrilateral.

In the same way we have the following relationships between triangle areas

$$\begin{aligned} \triangle C'BB' &= 2\triangle CBA \\ \triangle A'DD' &= 2\triangle ADC \\ \triangle CC'D' &= 2\triangle CBD \\ \triangle B'A'A &= 2\triangle BAD\,. \end{aligned}$$

Notice also that the original quadrilateral can be split along each diagonal and has an area equal to the sum of

$$\triangle CBA + \triangle ADC\,,$$

or

$$\triangle CBD + \triangle BAD\,.$$

Thus the area of the larger quadrilateral is equal to the sum of all four triangles above plus the original quadrilateral. Adding the four triangles together in pairs such that in a pair the total area is twice the area of the two specific smaller triangles that complete the original quadrilateral we get that the larger quadrilateral has an area of

$$2(\text{area}ABCD) + 2(\text{area}ABCD) + \text{area}ABCD = 2(10) + 2(10) + 10 = 50\,.$$

Problem 30

There will be $\binom{3n}{2} = \frac{3n(3n-1)}{2}$ total games that must be played between all players. There will be $\binom{n}{2}$ matches played between two men, $\binom{2n}{2}$ matches played between two woman and $n(2n) = 2n^2$ matches played between a woman and a man. We can verify that these numbers are correct by checking that

$$\frac{3n(3n-1)}{2} = \frac{n(n-1)}{2} + \frac{2n(2n-1)}{2} + 2n^2\,,$$

which is a true statement. As the number of wins the women get to the number of wins that the men get are in proportion $7:5$ we can let the number of wins the woman get be $7q$ and the number wins the men get be $5q$ (for $q \geq 1$) then we must have

$$7q + 5q = \binom{3n}{2} \quad \text{or} \quad 3n(3n-1) = 24q\,.$$

If we factor $24 = 2^3 \cdot 3$ this means that

$$n(3n-1) = 2^3 q\,.$$

For the possible values of n none of the expressions $n(3n-1)$ is divisible by eight so the correct value of n is not given.

The 1979 Examination

Problem 1

The smaller rectangle will have an area 1/4 that of the larger rectangle or $\frac{72}{4} = 18$.

Problem 2

Divide the expression by xy to get

$$\frac{1}{y} - \frac{1}{x} = 1\,,$$

so the expression desired must be -1.

Problem 3

From the square $ABCD$ we know that $\angle DAB$ is 90 degrees. As $\triangle AEB$ is an equilateral we know that $\angle BAE$ is 60 degrees. Thus $\angle DAE$ is 150 degrees. As $\triangle AED$ is isosceles we have that

$$\angle AED = \frac{180 - 150}{2} = 15\,,$$

degrees.

Problem 4

Note that the sign of the x^3 term must be positive while the sign of the x^2 must be negative. Only one choice satisfies both of these conditions.

Problem 5

Let the number be written as $d_1 d_2 d_3$. Then this number must equal $d_3 d_2 d_1$ so that

$$100 d_1 + 10 d_2 + d_3 = 100 d_3 + 10 d_2 + d_1\,.$$

This simplifies to give $99(d_1 - d_3) = 0$ thus $d_1 = d_3$ so our three digit number is really $d_1 d_2 d_1$. To be the largest possible even number $d_1 = 8$ and $d_2 = 9$ so the number is 898. The sum of these digits is 25.

Problem 6

Notice that we can write the given sum as

$$1 + \frac{1}{2} + 1 + \frac{1}{4} + 1 + \frac{1}{8} + \cdots - 7 = 6 + \sum_{k=1}^{6} \left(\frac{1}{2}\right)^k - 7\,.$$

If we evaluate the given sum above we get

$$-1 + \left(\frac{1 - \left(\frac{1}{2}\right)^7}{1 - \frac{1}{2}} - 1 \right) = -\frac{1}{64}\,,$$

when we simplify.

Problem 7

We are told that $x = n^2$ for some n. The next larger perfect square is $(n+1)^2$ so expanding we get

$$n^2 + 2n + 1 = x + 2\sqrt{x} + 1\,,$$

is the next perfect square.

Problem 8

The graph of $x^2 + y^2 = 4$ is a circle of radius two. The graph of $y = |x|$ cuts a wedge with an interior angle of 90 degrees and angles of 45 degrees to the positive and negative x-axis. This means that its area is 1/2 of 1/2 of the area of the circle. The area is then $\frac{1}{4}(\pi 2^2) = \pi$.

Problem 9

The product we want to evaluate can be written as

$$\begin{aligned} 4^{1/3} \cdot 8^{1/4} &= 2^{2/3} \cdot 2^{3/4} = 2^{\frac{2}{3}+\frac{3}{4}} = 2^{\frac{17}{12}} = 2^{1+\frac{5}{12}} = 2(2^5)^{\frac{1}{12}} \\ &= 2(32)^{\frac{1}{12}} = 2\sqrt[12]{32}\,. \end{aligned}$$

Problem 10

One needs to draw the regular hexagon $P_1 \cdots P_6$ (six equal sides) and the points Q_i. I drew the points P_i clockwise. In that case one sees that the quadrilateral $Q_1Q_2Q_3Q_4$ occupies part of the area in the right one-half of the regular hexagon. Drawing segments from the center to Q_1, Q_2, Q_3, and Q_4 we get the quadrilateral broken up into three isosceles triangles with two edges having a length of two. Because the vertex of each of these triangles (the angle that corresponds with the center of the hexagon) has the angle $\frac{360}{6} = 60$ degrees these triangles are actually equilateral (have all three sides equal). Recalling that an equilateral triangle with side length s has an area of

$$\frac{\sqrt{3}}{4}s^2\,. \tag{286}$$

The area of the quadrilateral is

$$3\left(\frac{\sqrt{3}}{4}2^2\right) = 3\sqrt{3}\,.$$

Problem 11

Write the given fraction as

$$\frac{\sum_{k=1}^{2n} k - 2\sum_{k=1}^{n} k}{2\sum_{k=1}^{n} k} = \frac{115}{116}\,,$$

or

$$\frac{\sum_{k=1}^{2n} k}{2\sum_{k=1}^{n} k} - 1 = \frac{115}{116}\,.$$

If we evaluate these two sums we get

$$\frac{\frac{2n(2n+1)}{2}}{\frac{2n(n+1)}{2}} - 1 = \frac{115}{116}\,.$$

Simplifying this we get

$$1 - \frac{1}{n+1} = \frac{115}{116}\,,$$

which has a solution $n = 115$.

Problem 12

In the given figure draw the segment OB. Then as the triangle BOE is isosceles we have that

$$\angle OBE = \angle OEB = x\,.$$

As triangle ABO is also isosceles we have that

$$\angle BAO = \angle BOC = y\,.$$

Now in triangle OBE the angles must sum to 180 degrees so

$$2x + \angle BOE = 180\,. \tag{287}$$

Along a straight line $ACOD$ the angles must also sum to 180 degrees so

$$y + \angle BOE + 45 = 180\,. \tag{288}$$

Finally in triangle AEO the angles must sum to 180 degrees so

$$y + x + y + \angle BOE = 180\,. \tag{289}$$

This gives three equations and three unknowns which we can solve. If we use Equation 287 to solve for $\angle BOE$ and put that into Equation 288 and 289 we get

$$\begin{aligned} y - 2x &= -45 \\ 2y - x &= 0\,. \end{aligned}$$

Solving these give $x = 30$ and $y = 15$ so $\angle BAO = y = 15$ degrees.

Problem 13

We want to know when is $y - x < \sqrt{x^2}$ satisfied. Now as $\sqrt{x^2} = |x|$ the above is equal to

$$y - x < |x|\,,$$

or

$$y < x + |x|\,.$$

Now if $x < 0$ then $|x| = -x$ and the right-hand-side of the above is zero and we get $y < 0$. If $x > 0$ then $|x| = x$ and the right-hand-side of the above is $2x$. Thus the expression on the right-hand-side of the above inequality can be written

$$x + |x| = \begin{cases} 0 & x < 0 \\ 2x & x > 0 \end{cases}$$

Since we know how to evaluate the right-hand-side we can plot the y values that are less then this value. We see that this is all values of y that are less than zero and y values that are less than $2x$.

Note that the point $(x, y) = (-1, +1)$ shows that the second choice cannot be correct.

Problem 14

For this problem we are told that $a_1 = 1$ and that

$$\prod_{k=1}^{n} a_k = n^2\,,$$

for all n. If we take $n = 3$ this is

$$\prod_{k=1}^{3} a_k = 9\,.$$

This means that

$$\left(\prod_{k=1}^{2} a_k\right) a_3 = 9 \quad \text{or} \quad 2^2 a_3 = 9 \quad \text{or} \quad a_3 = \frac{9}{4}\,.$$

If we take $n = 5$ in this expression we get

$$\prod_{k=1}^{5} a_k = 25\,.$$

This means that

$$\left(\prod_{k=1}^{4} a_k\right) a_5 = 25 \quad \text{or} \quad 16 a_5 = 25 \quad \text{or} \quad a_5 = \frac{25}{16}\,.$$

Using these two values we can compute

$$a_3 + a_5 = \frac{61}{16}\,.$$

Problem 15

Let the volume of the two identical jars be x. Then in the first we have $\frac{p}{p+1}x$ volume of alcohol and $\frac{1}{p+1}x$ volume of water. In the second we have $\frac{q}{q+1}x$ volume of alcohol and $\frac{1}{q+1}x$ volume of water. When we mix we will have

$$\frac{p}{p+1}x + \frac{q}{q+1}x\,,$$

and

$$\frac{1}{p+1}x + \frac{1}{q+1}x\,,$$

for the volumes of alcohol and water respectively. The ratio of alcohol to water is the ratio of these two expressions. We can take that ratio and simplify to get $\frac{2pq+p+q}{p+q+2}$.

Problem 16

Let r be the radius of the smaller circle. We are told that $A_1 + A_2 = 9\pi$ thus

$$A_2 = 9\pi - \pi r^2\,.$$

If A_1, A_2, and $A_1 + A_2$ are in a arithmetic sequence the difference between any two sequential terms is the common difference d. Taking this difference between the last two and the first two terms of the sequence we have

$$\begin{aligned}(A_1 + A_2) - A_2 &= \pi r^2 = d \\ A_2 - A_1 &= A_2 - \pi r^2 = d\,.\end{aligned}$$

Using what we know about A_2 from before in the second of these expressions we get

$$9\pi - 2\pi r^2 = d\,.$$

Setting this equal to the first expression gives

$$9\pi - 2\pi r^2 = \pi r^2\,.$$

Solving for r we find $r = \sqrt{3}$.

Problem 17

Let P be the top point in the triangle. From the given information in the problem the side BP is of length x, the side BC is of length $y - x$, and the side PC is of length $z - y$. Now we will use the triangle inequality on each side of this triangle. This inequality states that the sum of the lengths of any two sides of a triangle must be longer than the other side.

If we apply this inequality in the form $PB + BC > PC$ we have

$$x + y - x > z - y \quad \text{or} \quad y > \frac{z}{2}\,.$$

This is in *contradiction* to III.

If we apply this inequality in the form $PC + CB > PB$ we have

$$z - x > x \quad \text{or} \quad x < \frac{z}{2}\,.$$

This is the statement I.

If we apply this inequality in the form $BP + PC > BC$ we have

$$z + z - y > y - z \quad \text{or} \quad y < x + \frac{z}{2}\,.$$

This is statement II. Thus only two statements are true.

Problem 18

Write $\log_5(10)$ as

$$\log_5(10) = \log_5(2(5)) = \log_5(2) + \log_5(5) = 1 + \log_5(2) = 1 + \frac{\log_{10}(2)}{\log_{10}(5)}\,.$$

Again using $\log_a(b) = \frac{1}{\log_b(a)}$ on the expression $\log_{10}(5)$ on the right-hand-side we get

$$\log_5(10) = 1 + \log_{10}(2)\log_5(10)\,.$$

We can solve this for $\log_5(10)$ to get

$$\log_5(10) = \frac{1}{1 - \log_{10}(2)} = \frac{1}{1 - 0.301} = \frac{1}{0.699} \approx \frac{1}{0.7} = \frac{10}{7}\,.$$

Problem 19

Using $256 = 2^8$ we write $x^{256} = x^{2^8}$ and $256^{32} = (2^8)^{32} = 2^{8(32)} = 2^{256} = 2^{2^8}$. Thus we have

$$x^{2^8} - 2^{2^8} = 0\,.$$

We can write this as

$$x^{256} = 2^{256}\,.$$

Taking the 1/256th root of both sides we get $|x| = 2$ so $x = \pm 2$ for the real roots. The sum of the squares of all the real roots is $(-2)^2 + 2^2 = 8$.

Problem 20

From the given expressions for a and b we find $b = \frac{1}{3}$. Recall that

$$\tan(\alpha + \beta) = \frac{\tan(\alpha) + \tan(\beta)}{1 - \tan(\alpha)\tan(\beta)}\,.$$

Thus

$$\tan(\arctan(a) + \arctan(b)) = \frac{a+b}{1-ab} = 1\,,$$

when we use the given values of a and b. This means that

$$\arctan(a) + \arctan(b) = \frac{\pi}{4}\,.$$

Problem 21

To start we draw the right triangle with a base length of a and a vertical length of b and the inscribed circle. Dropping perpendiculars from the center of the circle to each of the three sides we see that

$$\begin{aligned} a &= r + x \\ b &= r + y\,. \end{aligned}$$

In other words the lengths of the sides are equal to the radius r plus two other lengths x and y. By drawing edges from two of the vertices of the right triangle to the center of the circle we obtain two congruent triangles that show that the hypotenuse h can be written as

$$h = x + y\,, \tag{290}$$

From the Pythagorean theorem we also have that

$$h^2 = (x+r)^2 + (y+r)^2\,. \tag{291}$$

This gives two equations and two unknowns x and y. If we use Equation 290 for h in Equation 291 by expanding and simplifying we get

$$xy = r(x+y) + r^2 .$$

Again using Equation 290 for $x + y = h$ we get

$$xy = rh + r^2 .$$

Now the ratio that we want to compute is

$$\begin{aligned} R = \frac{\pi r^2}{\frac{1}{2}ab} &= \frac{2\pi r^2}{(r+y)(r+x)} \\ &= \frac{2\pi r^2}{r^2 + r(x+y) + xy} . \end{aligned}$$

As we know expressions for $x + y$ and xy in terms of r and h we put those in the above and simplify to get

$$R = \frac{\pi r}{r+h} .$$

Problem 22

It seems a common way to prove that there are no solutions in the *integers* is to show that one expression *is* divisible by a factor while the other expression *cannot* be divisible by that factor. That is the approach we will take for this problem. Note that the left-hand-side can be written as

$$m(m^2 + 6m + 5) = m(m+1)(m+5) .$$

This is not obviously divisible by anything but if we write it as

$$m(m+1)(m+2+3) = m(m+1)(m+2) + 3m(m+1) .$$

Note that for any integer m the expression $m(m+1)(m+2)$ will be divisible by three and so will the expression $3m(m+1)$. Thus the left-hand-side is divisible by three. Note that the right-hand-side can be written as

$$3n(9n^2 + 3n + 3) + 1 ,$$

and thus has a remainder of one when divided by three. Thus there can be no solutions in the integers to this equation.

Problem 23

In a 3d Cartesian coordinate system the vertices of a tetrahedron with an edge length of a are given by

$$\begin{aligned} A &= \left(-\frac{a}{2\sqrt{3}}, \frac{a}{2}, 0\right) \\ B &= \left(-\frac{a}{2\sqrt{3}}, -\frac{a}{2}, 0\right) \\ C &= \left(\frac{a}{\sqrt{3}}, 0, 0\right) \\ D &= \left(0, 0, \sqrt{\frac{2}{3}}a\right) . \end{aligned} \tag{292}$$

This puts the top of the vertex on the z-axis and the other vertices on the xy-plane.

Using this configuration the point $P = (x, y, 0)$ will have coordinates $x = -\frac{1}{2\sqrt{3}}a$ and a y value such that

$$-\frac{a}{2} \leq y \leq \frac{a}{2} .$$

The point Q will then be on a line from the point

$$\left(\frac{a}{\sqrt{3}}, 0, 0\right) ,$$

to the point

$$\left(0, 0, \sqrt{\frac{2}{3}}a\right) ,$$

We can parameterize this line by introducing t such that $0 \leq t \leq 1$ and then

$$\begin{aligned} x(t) &= \frac{a}{\sqrt{3}}(1-t) \\ y(t) &= 0 \\ z(t) &= \sqrt{\frac{2}{3}}at . \end{aligned}$$

The distance (squared) between the point P and Q as a function of t is then given by

$$\begin{aligned} d^2(t) &= (x_p - x_q)^2 + (y_p - y_q)^2 + (z_p - z_q)^2 \\ &= \left(-\frac{a}{2\sqrt{3}} - \frac{a}{\sqrt{3}}(1-t)\right)^2 + (y_p - 0)^2 + \left(0 - \sqrt{\frac{2}{3}}at\right)^2 . \end{aligned}$$

To make $d^2(t)$ as small as possible we should take $y_p = 0$. Thus we get

$$d^2(t) = \frac{a^2}{3}\left(\frac{3}{2} - t\right)^2 + \frac{2}{3}a^2t^2 .$$

For this problem we are told that $a = 1$. Next note that at the two end points of t we have

$$d^2(0) = \frac{1}{3}\left(\frac{3}{2}\right)^2 = \frac{3}{4}$$
$$d^2(1) = \frac{1}{3}\left(\frac{1}{4}\right) + \frac{2}{3} = \frac{3}{4}.$$

As there might be smaller value of $d^2(t)$ for a t such that $0 \le t \le 1$ we will take the derivative of $d^2(t)$ with respect to t, set the result equal to zero, and solve for t. The derivative set equal to zero is

$$\frac{d(d^2(t))}{dt} = \frac{2}{3}\left(\frac{3}{2} - t\right)(-1) + \frac{4}{3}t = 0.$$

Solving for t we find $t = \frac{1}{2}$. For this value of t we find that

$$d^2\left(\frac{1}{2}\right) = \frac{1}{3} + \frac{1}{6} = \frac{1}{2}.$$

As this is smaller than the values of $d^2(0)$ and $d^2(1)$ it is the minimum we seek. Thus the smallest distance is $d\left(\frac{1}{2}\right) = \frac{1}{\sqrt{2}}$.

Problem 24

We start by drawing the quadrilateral $ABCD$ and then extend the segments AB and CD until they meet at a point E. Next let β be the angle $\angle CBE$ and θ the angle $\angle BCE$. Then we have that

$$\cos(\beta) = -\cos(\angle ABC) = \sin(\angle BCD) = \sin(\theta).$$

As $\cos(\beta) = \sin(\theta)$ we have that $\beta + \theta = \frac{\pi}{2}$. Because BCE is a triangle this means that $\angle BEC$ is also $\frac{\pi}{2}$ so we have $\triangle BCE$ a right triangle with BC the hypotenuse. Using that fact we have that

$$BE = BC\cos(\beta) = 5\left(\frac{3}{5}\right) = 3.$$

Next using $BC^2 = BE^2 + CE^2$ we find $CE = 4$. The length AD is given by using the Pythagorean theorem as

$$AD^2 = (3+4)^2 + (4+20)^2 = 625.$$

Thus $AD = 25$.

Problem 25

The statements given mean that

$$x^8 = q_1(x)\left(x + \frac{1}{2}\right) + r_1$$
$$q_1(x) = q_2(x)\left(x + \frac{1}{2}\right) + r_2.$$

Using these we have that

$$x^8 = \left(x+\frac{1}{2}\right)\left(q_2(x)\left(x+\frac{1}{2}\right)+r_2\right)+r_1$$
$$= q_2(x)\left(x+\frac{1}{2}\right)^2 + r_2\left(x+\frac{1}{2}\right)+r_1 .$$

If we let $x = -\frac{1}{2}$ we get

$$\frac{1}{2^8} = r_1 .$$

Thus we have just shown that

$$x^8 = q_2(x)\left(x+\frac{1}{2}\right)^2 + r_2\left(x+\frac{1}{2}\right)+\frac{1}{2^8} ,$$

or

$$\frac{x^8-\frac{1}{2^8}}{x+\frac{1}{2}} = q_2(x)\left(x+\frac{1}{2}\right)+r_2 . \quad (293)$$

If we take $p = -\frac{1}{2}$ and recall that

$$\frac{x^8-p^8}{x-p} = \sum_{k=0}^{7} x^{7-k}p^k . \quad (294)$$

Now we want to evaluate Equation 293 at $x = -\frac{1}{2}$. To do that we use Equation 294 with $x = -\frac{1}{2} = p$ to evaluate the left-hand-side. We have

$$\sum_{k=0}^{7} p^{7-k}p^k = 8p^7 = 8\left(-\frac{1}{2}\right)^7 = -\frac{1}{16} .$$

Using that in Equation 293 we get

$$r_2 = -\frac{1}{16} .$$

Problem 26

In the given expression let $y = 1$ to get

$$f(x)+1 = f(x+1)-x-1 \quad \text{or} \quad f(x+1)-f(x) = x+2 . \quad (295)$$

In terms of the forward difference operator this is

$$\Delta f(x) = x+2 .$$

If we sum this from $x = 1$ to $x = X$ we get

$$\sum_{x=1}^{X}\Delta f(x) = \sum_{x=1}^{X}(x+2) = \sum_{x=1}^{X}x + 2\sum_{x=1}^{X}1 = \frac{X(X+1)}{2}+2X .$$

The sum of the left-hand-side is $f(X+1) - f(1)$ so that we get

$$f(X+1) = \frac{1}{2}X^2 + \frac{5}{2}X + 1\,.$$

This means that

$$f(X) = \frac{1}{2}(X-1)^2 + \frac{5}{2}(X-1) + 1 = \frac{1}{2}X^2 + \frac{3}{2}X - 1\,.$$

The above is a valid solution for f for all real x. We now want to find out when $f(n) = n$. Using the above expression this is

$$\frac{1}{2}n^2 + \frac{3}{2}n - 1 = n\,.$$

We can simplify and write this as

$$(n+2)(n-1) = 0\,.$$

This has two solutions only one of which is not $n = 1$.

Problem 27

As b and c are integers such that $|b| \leq 5$ and $|c| \leq 5$ the number of possible choices for b (or c) are $5 + 5 + 1 = 11$. Thus the total number of choices is given by $11^2 = 121$.

The roots of $x^2 + bx + c = 0$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}\,.$$

For the two roots to be distinct means that we must have $b^2 - 4c > 0$ or that $c < \frac{b^2}{4}$. Both roots will be positive if and only if the smaller of the two roots is positive or that

$$\frac{-b - \sqrt{b^2 - 4c}}{2} > 0\,.$$

This is equivalent to

$$-b > \sqrt{b^2 - 4c}\,.$$

This in tern is equivalent to

$$b^2 > b^2 - 4c \quad \text{and} \quad b < 0\,.$$

This is equivalent to

$$0 > -4c \quad \text{and} \quad b < 0\,.$$

This is finally equivalent to

$$c > 0 \quad \text{and} \quad b < 0\,.$$

Thus the region of interest is then

$$0 < c < \frac{b^2}{4} \quad \text{and} \quad b < 0\,.$$

We now count how many integer points are in this region. For various value of b we find

$$\begin{aligned} b &= -5 \quad \text{then} \quad c \in \{1, 2, 3, 4, 5\} \\ b &= -4 \quad \text{then} \quad c \in \{1, 2, 3\} \\ b &= -3 \quad \text{then} \quad c \in \{1, 2\} . \end{aligned}$$

For $b = -2$ and $b = -1$ there are no valid c values. Counting the above we see that there are a total of 10 points that satisfy the inequalities above. This means that the probability that we have two positive and distinct roots is $\frac{10}{121}$. The probability we *don't* have this is then one minus this number or $\frac{111}{121}$.

Problem 28

Lets introduce a Cartesian coordinate system with the point A at the origin. Then as ABC is an equilateral triangle with an edge length of two we have that $B = (2\sin(30), -2\cos(30)) = (1, -\sqrt{3})$ and $C = (-1, -\sqrt{3})$.

From the point A the points on the circle of radius r are the ones that satisfy

$$x^2 + y^2 = r^2 . \tag{296}$$

From the point B the points on the circle of radius r are the ones that satisfy

$$(x-1)^2 + (y+\sqrt{3})^2 = r^2 . \tag{297}$$

From the point C the points on the circle of radius r are the ones that satisfy

$$(x+1)^2 + (y+\sqrt{3})^2 = r^2 . \tag{298}$$

To find the point C' we want to find the intersection of Equation 296 and 298. If we expand Equation 298 and use Equation 296 to cancel quadratic terms finally end with

$$x = 2 + \sqrt{3}y .$$

If we use this to replace x in Equation 296 in terms of y we get

$$(2 + \sqrt{3}y)^2 + y^2 = r^2 .$$

Solving for y we get

$$y = \frac{-\sqrt{3} \pm \sqrt{r^2 - 1}}{2} = -\frac{\sqrt{3}}{2} \pm \frac{1}{2}\sqrt{r^2 - 1} .$$

The value $-\frac{\sqrt{3}}{2}$ is half-way from the origin at A to the line connecting the segment BC. As the point C' is *above* this line we would take the positive sign in the above expression and the (x, y) coordinate of C' is located at

$$\begin{aligned} x &= 2 + \sqrt{3}y = \frac{1}{2} + \frac{1}{2}\sqrt{3(r^2 - 1)} \\ y &= -\frac{\sqrt{3}}{2} + \frac{1}{2}\sqrt{r^2 - 1} . \end{aligned}$$

By symmetry the point B' is then located at

$$x = -\frac{1}{2} - \frac{1}{2}\sqrt{3(r^2-1)}$$
$$y = -\frac{\sqrt{3}}{2} + \frac{1}{2}\sqrt{r^2-1}\,.$$

Given these two points the length of $B'C'$ is then

$$B'C'^2 = \left(1+\sqrt{3(r^2-1)}\right)^2 + 0\,.$$

Thus

$$B'C' = 1+\sqrt{3(r^2-1)}\,.$$

Problem 29

Notice that

$$\left(x^3+\frac{1}{x^3}\right)^2 = x^6+2+\frac{1}{x^6}\,.$$

Thus we can write $f(x)$ as

$$f(x) = \frac{\left(x+\frac{1}{x}\right)^2 - \left(x^3+\frac{1}{x^3}\right)^2}{\left(x+\frac{1}{x}\right)^3 + \left(x^3+\frac{1}{x^3}\right)}\,.$$

Let

$$u = x+\frac{1}{x}$$
$$v = x^3+\frac{1}{x^3}\,.$$

Then f can be written as

$$f(x) = \frac{u^6-v^2}{u^3+v} = \frac{(u^3-v)(u^3+v)}{u^3+v} = u^3-v\,.$$

This means that we can write $f(x)$ as

$$\begin{aligned} f(x) &= \left(x+\frac{1}{x}\right)^3 - \left(x^3+\frac{1}{x^3}\right) \\ &= x^3+3x^2\left(\frac{1}{x}\right)+3x\left(\frac{1}{x}\right)^2+\frac{1}{x^3}-x^3-\frac{1}{x^3} \\ &= 3x+\frac{3}{x} = 3\left(x+\frac{1}{x}\right)\,. \end{aligned}$$

Now the smallest value for $x+\frac{1}{x}$ when $x>0$ is when

$$1-\frac{1}{x^2}=0 \quad \text{or} \quad x=1\,.$$

The function f at this value take the value of $f(1)=2$. Thus we have that

$$f(x) \geq 3(2) = 6\,.$$

The 1980 Examination

Problem 1

Besides simple division, we can simply see if the proposed values for n work. That is we first note that

$$7(14) = 98\,,$$

which is less than 100. Next note that

$$7(15) = 105\,,$$

which is greater than 100. Thus $n = 14$.

Problem 2

The two highest powers in the left-most and right-most products are x^8 and x^9 giving a polynomial with degree $8 + 9 = 17$.

Problem 3

We have

$$\frac{2x - y}{x + y} = \frac{2}{3}\,.$$

Divide the top and bottom of the fraction on the left-hand-side by y and let $r = \frac{x}{y}$ to get

$$\frac{2r - 1}{r + 1} = \frac{2}{3}\,.$$

Solving this for r gives $r = \frac{5}{4}$.

Problem 4

Since the triangle CDE is equilateral then each of its angles must be $60°$. We also know that each of the angles in the two squares are $90°$, namely $\angle DEF = \angle DCB = 90°$. Lets continue the line segment CE to the left and to the right forming a "base" for the given geometrical figure. Introduce points L "far to the left" and R "far to the right" on this line. Then knowing the angles in the triangle and the square we have $\angle FER = \angle BCL = 180 - 60 - 90 = 30°$.

Next draw a line parallel to $LCER$ and though the point D. Let the point where this line intersects the segment AB be denoted H and the point where this line intersects the segment GF be denoted I. By the two parallel lines we have that both $\angle HDC$ and $\angle IDE$ are $60°$

since they are equal $\angle DCE$ and $\angle DEC$ respectively. Note that $\angle HDC+\angle CDE+\angle EDI = 180^\circ$ as it should, as each angle in the sum is 60°.

Next note that

- As $\angle EDG = 90^\circ$ and $\angle EDI = 60^\circ$ we must have that $\angle IDG = 30^\circ$.
- As $\angle CDA = 90^\circ$ and $\angle CDH = 60^\circ$ we must have that $\angle HDA = 30^\circ$.

Using supplementary angles we must have

$$\angle HDA + \angle ADG + \angle IDG = 180^\circ \,.$$

Using what we know about the values of $\angle HDA$ and $\angle IDG$ we get

$$\angle ADG = 180 - 2(30) = 120^\circ \,.$$

Problem 5

Note that using the diagram that

$$\tan(60^\circ) = \frac{CQ}{PQ} = \frac{AQ}{PQ} \,.$$

The left-hand-side is

$$\tan(60^\circ) = \frac{\sin(60^\circ)}{\cos(60^\circ)} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3} \,.$$

Thus

$$\frac{PQ}{AQ} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \,.$$

Problem 6

The given expression is equal to

$$\sqrt{x} - 2x < 0 \,,$$

or

$$\sqrt{x}(1 - 2\sqrt{x}) < 0 \,.$$

As $\sqrt{x} > 0$ we can divide by it to get

$$1 - 2\sqrt{x} < 0 \quad \text{or} \quad x > \frac{1}{4} \,.$$

Problem 7

Draw a line segment from A to C. The length of this segment must be five as it forms a hypotenuse of a $3, 4, 5$ right triangle. Next note that

$$12^2 + 5^2 = 144 + 25 = 169 = 13^2 ,$$

Thus the other triangle is a $5, 12, 13$ right triangle and therefore $\angle ACD = 90°$. summing the area of these two right triangles the area A of this figure is given by

$$A = \frac{1}{2}(4)(3) + \frac{1}{2}(12)(5) = 36 .$$

Problem 8

Multiply the given expression by $a + b$ to get

$$\frac{a+b}{a} + \frac{a+b}{b} = 1 .$$

Simplifying some we get

$$\frac{b}{a} + \frac{a}{b} = -1 .$$

Let $x = \frac{a}{b}$ and our previous equation is

$$\frac{1}{x} + x = -1 .$$

If we multiply by x we get

$$x^2 + x + 1 = 0 .$$

Solving this with the quadratic equation gives

$$x = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{-3}}{2} .$$

Note that x is not real and thus a and b cannot both be real otherwise their ratio would be contradicting what we derived above. Thus there are no real solutions to this equation.

Problem 9

If we draw a picture of this situation we will find a triangle with edge lengths x, 3, and $\sqrt{3}$ and an internal angle of 30°. Using the law of cosigns on this triangle we have

$$(\sqrt{3})^2 = 3^2 + x^2 - 2(3)x\cos(30) .$$

Expanding and simplifying we get

$$x^2 - 3\sqrt{3}x + 6 = 0 .$$

Solving with the quadratic equation we get

$$x = \frac{3\sqrt{3} \pm \sqrt{9(3) - 4(6)}}{2},$$

which simplifies to give the numbers $2\sqrt{3}$ and $\sqrt{3}$. A condition that three numbers a, b, and c are valid sides of a triangle[5] is

$$\max(a, b, c) < \frac{1}{2}(a + b + c). \tag{299}$$

As both of these solutions satisfy this we don't have enough information to have a unique solution.

Problem 10

The angular speed ω is related to the linear speed v via. the radius r as $v = \omega r$. As all gears are touching they must all have the same linear speed or

$$\omega_A r_A = \omega_B r_B = \omega_C r_C.$$

We can write this as

$$\frac{\omega_A}{\frac{1}{r_A}} = \frac{\omega_B}{\frac{1}{r_B}} = \frac{\omega_C}{\frac{1}{r_C}}.$$

This means that the angular speeds are in proportion

$$\frac{1}{r_A} : \frac{1}{r_B} : \frac{1}{r_C}.$$

As we have that the radius of each disk is proportional to the number of teeth we have that the angular speeds are in proportion

$$\frac{1}{x} : \frac{1}{y} : \frac{1}{z}.$$

If we multiply this expression by the constant xyz we get

$$yz : xz : xy.$$

Problem 11

The terms of an arithmetic sequence are given by

$$a_n = a_0 + dn, \tag{300}$$

[5] https://en.wikipedia.org/wiki/Triangle_inequality

for $n = 0, 1, 2, \ldots$ and we are told that

$$\sum_{n=0}^{9} a_n = 100 \quad (301)$$

$$\sum_{n=0}^{99} a_n = 10\,. \quad (302)$$

For this problem we want to evaluate $\sum_{n=0}^{109} a_n$. Using Equation 300 in Equation 301 we can evaluate the left-hand-side to get

$$10a_0 + 45d = 100\,.$$

Doing the same thing in the left-hand-side of Equation 302 gives

$$100a_0 + 4950d = 10\,.$$

Solving these two equations for a_0 and d gives $a_0 = \frac{1099}{100}$ and $d = -\frac{11}{50}$. Using Equation 300 in the expression we want to evaluate we find

$$\sum_{n=0}^{109} a_n = 110a_0 + d\sum_{n=1}^{109} n = 110a_0 + d\left(\frac{109(110)}{2}\right) = 110a_0 + 5995d = -110\,.$$

Problem 12

From the description of the problem line one is given by $y = mx$ and line two by $y = nx$. Then as line one has four times the slope of line two we have that

$$m = 4n\,. \quad (303)$$

Let θ be the angle between the x-axis and line two. Then θ is also be the angle between line two and line one so that 2θ is the angle between line one and the x-axis. Using these angles we have that

$$\tan(\theta) = n$$
$$\tan(2\theta) = m\,.$$

Using the fact that

$$\tan(2\theta) = \frac{2\tan(\theta)}{1 - \tan^2(\theta)} = \frac{2n}{1 - n^2}\,,$$

we have that

$$\frac{2n}{1 - n^2} = m\,. \quad (304)$$

If we solve Equation 303 and 304 for m and n we have

$$m = \pm\frac{1}{\sqrt{2}}$$
$$n = \pm 2\sqrt{2}\,.$$

Using these we have that

$$mn = \left(\pm\frac{1}{\sqrt{2}}\right)\left(\pm 2\sqrt{2}\right) = 2\,.$$

Problem 13

The size of the step take a the nth step is given by

$$d_n = \left(\frac{1}{2}\right)^{n-1},$$

for $n \geq 1$. Lets enumerate the points that our bug goes to on each step.

- After one step $(1, 0)$
- After two steps $\left(1, \frac{1}{2}\right)$
- After three steps $\left(\frac{3}{4}, \frac{1}{2}\right)$
- After four steps $\left(\frac{3}{4}, \frac{3}{8}\right)$
- After five steps $\left(\frac{13}{16}, \frac{3}{8}\right)$
- After six steps $\left(\frac{13}{16}, \frac{13}{32}\right)$

Let (x^*, y^*) be the limiting point that the bug is working towards. Then we can argue that the location of this point is given by

$$x^* = 1 - \frac{1}{4} + \frac{1}{16} - \frac{1}{32} + \cdots = \sum_{k=0}^{\infty} \left(-\frac{1}{4}\right)^k = \frac{1}{1-(-1/4)} = \frac{4}{5},$$

and

$$\begin{aligned} y^* &= 0 + \frac{1}{2} - \frac{1}{8} + \frac{1}{32} + \cdots = \frac{1}{2}\left(1 - \frac{1}{4} + \frac{1}{16} - \cdots\right) \\ &= \frac{1}{2}\sum_{k=0}^{\infty}\left(-\frac{1}{4}\right)^k = \frac{1}{2}\left(\frac{4}{5}\right) = \frac{2}{5}. \end{aligned}$$

Note that our bug will get infinitely close to (x^*, y^*).

Problem 14

Consider $f(f(x))$ where we find

$$f(f(x)) = \frac{c\left(\frac{cx}{2x+3}\right)}{2\left(\frac{cx}{2x+3}\right)+3} = \frac{c^2x}{2cx+6x+9},$$

when we simplify. If we set this equal to x we get the following quadratic

$$c^2x = 2cx^2 + 6x^2 + 9x \quad \text{or} \quad 0 = 2(c+3)x^2 + (9-c^2)x.$$

This will hold true for all x if $c = -3$.

Problem 15

Let x be the price of the item in cents (and thus it is an integer) then from the problem we know that $1.04x = 100n$ where n is also an integer. If we multiply this expression by 100 we get

$$104x = 10^4 n\,.$$

As everything is an integer lets try to factor both sides into a product of primes. As $104 = 2^3 \cdot 13$ we can write the above as

$$2^3 \cdot 13 \cdot x = 2^4 \cdot 5^4 \cdot n\,.$$

If we divide by 2^3 this is

$$13x = 2 \cdot 5^4 \cdot n\,.$$

If we want n to be as small as possible then we must take x to be as small as possible such that the above expression holds. This leads us to take $n = 13$ and $x = 2 \cdot 5^4 = 1250$.

Problem 16

The surface area of the cube with a side of length one is $6(1^2) = 6$. The edge length of the tetrahedron is the length of the diagonal of one of the faces of the cube or $\sqrt{2}$. Thus we need to compute the surface area of a tetrahedron with that edge length. This tetrahedron has four sides each of which is a equilateral triangle with an edge length of $\sqrt{2}$. The area of such a triangle is

$$\frac{\sqrt{3}}{4}(\sqrt{2})^2 = \frac{\sqrt{3}}{2}\,.$$

Thus the surface area of the tetrahedron is then four times this value or $2\sqrt{3}$. The ratio we are asked for is then

$$\frac{6}{2\sqrt{3}} = \sqrt{3}\,.$$

Problem 17

Expand the given expression can be expanded as

$$\begin{aligned}
(n+i)^4 &= \sum_{k=0}^{4} \binom{4}{k} i^k n^{4-k} \\
&= i^0 n^4 + 4i^1 n^3 + \binom{4}{2} i^2 n^2 + 4i^3 n + i^4 \\
&= n^4 + 4n^3 i - \frac{4(3)}{2} n^2 - 4ni + 1 \\
&= 1 - 6n^2 + n^4 + 4n(n^2-1)i\,.
\end{aligned}$$

This will be an integer if $n(n^2-1) = 0$. This happens for $n \in \{-1, 0, +1\}$.

Problem 18

As we are told that $\log_b(\sin(x)) = a$ we can write this as

$$\frac{\ln(\sin(x))}{\ln(b)} = a \,.$$

This means that $\ln(\sin(x)) = a\ln(b) = \ln(b^a)$ or $\sin(x) = b^a$. Now

$$\cos(x) = \pm\sqrt{1-\sin^2(x)} = \pm\sqrt{1-b^{2a}} \,.$$

As we are told that $\cos(x) > 0$ we must take the positive sign above. Thus

$$\log_b(\cos(x)) = \log_b((1-b^{2a})^{1/2}) = \frac{1}{2}\log_b(1-b^{2a}) \,.$$

Problem 19

Draw a diameter to the circle and the three cords all parallel to each other. Then draw a perpendicular from the center of the circle through the diameter and the three cords. Draw segments of length r from the center of the circle to the points where the three cords meet the circle. Each of these is of length r (the radius of the circle). Let the vertical distance from the center of the circle to the first chord C_1 be x and the distance between C_1 and C_2 and C_2 and C_3 be d. Then from right triangles we have

$$x^2 + 10^2 = r^2 \tag{305}$$

$$(x+d)^2 + 8^2 = r^2 \tag{306}$$

$$(x+2d)^2 + 4^2 = r^2 \,. \tag{307}$$

If we subtract Equation 305 from Equation 306 we get

$$2xd + d^2 = 36 \,. \tag{308}$$

If we subtract Equation 306 from Equation 307 we get

$$2xd + 3d^2 = 48 \,. \tag{309}$$

These are two linear equations in the two unknowns xd and d^2. Subtracting Equation 308 from Equation 309 gives

$$2d^2 = 12 \quad \text{so} \quad d = \sqrt{6} \,.$$

If we put that value into Equation 308 we can solve for x to get $x = \frac{15}{\sqrt{6}}$. Then using Equation 305 we find

$$r = \frac{5\sqrt{11}}{\sqrt{2}} \,.$$

Problem 20

The probability we draw p pennies, n nickels, and $d = 6 - p - n$ dimes when drawing six coins is given by a multivariate hypergeometric distribution where we have

$$P(p, n, d \equiv 6 - p - n) = \frac{\left(\begin{array}{c} 2 \\ p \end{array}\right)\left(\begin{array}{c} 4 \\ n \end{array}\right)\left(\begin{array}{c} 6 \\ 6-p-n \end{array}\right)}{\left(\begin{array}{c} 12 \\ 6 \end{array}\right)},$$

for $0 \leq p \leq 2$, $0 \leq n \leq 4$, $0 \leq d \leq 6$, and $p + n + d = 6$. Note that $\left(\begin{array}{c} 12 \\ 6 \end{array}\right) = 924$. Only some combinations will give a monetary value greater than 50 cents. We see that unless $d \in \{4, 5, 6\}$ there is no way for the coins chosen to sum to the desired amount. The only way we can get a sum 50 cents or larger are for

- $d = 4$ with $n = 2$ and $p = 0$.
- $d = 5$ with $n = 1$ and $p = 0$ or with $n = 0$ and $p = 1$.
- $d = 6$ with $n = 0$ and $p = 0$.

Thus the probability that this happens is then

$$\begin{aligned} p &= \frac{1}{924}\left(\left(\begin{array}{c} 2 \\ 0 \end{array}\right)\left(\begin{array}{c} 4 \\ 2 \end{array}\right)\left(\begin{array}{c} 6 \\ 4 \end{array}\right) + \left(\begin{array}{c} 2 \\ 0 \end{array}\right)\left(\begin{array}{c} 4 \\ 1 \end{array}\right)\left(\begin{array}{c} 6 \\ 5 \end{array}\right) + \left(\begin{array}{c} 2 \\ 1 \end{array}\right)\left(\begin{array}{c} 4 \\ 0 \end{array}\right)\left(\begin{array}{c} 6 \\ 5 \end{array}\right) + \left(\begin{array}{c} 2 \\ 0 \end{array}\right)\left(\begin{array}{c} 4 \\ 0 \end{array}\right)\left(\begin{array}{c} 6 \\ 6 \end{array}\right)\right) \\ &= \frac{90 + 24 + 12 + 1}{924} = \frac{127}{924}. \end{aligned}$$

Problem 21

Let G be the midpoint of the segment CD. Then as G is the midpoint of CD and E is the midpoint of AC in triangle ACD we have that EG is parallel to AD.

First note that the area of triangle BEG is $\frac{2}{3}$ of that of the triangle BEC. This is because if the base is taken from BC the height of each triangle is from E to this base (i.e. the same) while the base of the triangle BEG (the segment BG) is $\frac{2}{3}$ as large as the base in the larger triangle BEC (the segment BC). Thus we have

$$\triangle BEG = \frac{2}{3} \triangle BEC .$$

Now in triangle BEG as D is the midpoint of BG and DF is parallel to EG we have that F must bisect the segment BE. This means that the area of the triangle BFD is one quarter that of BEG.

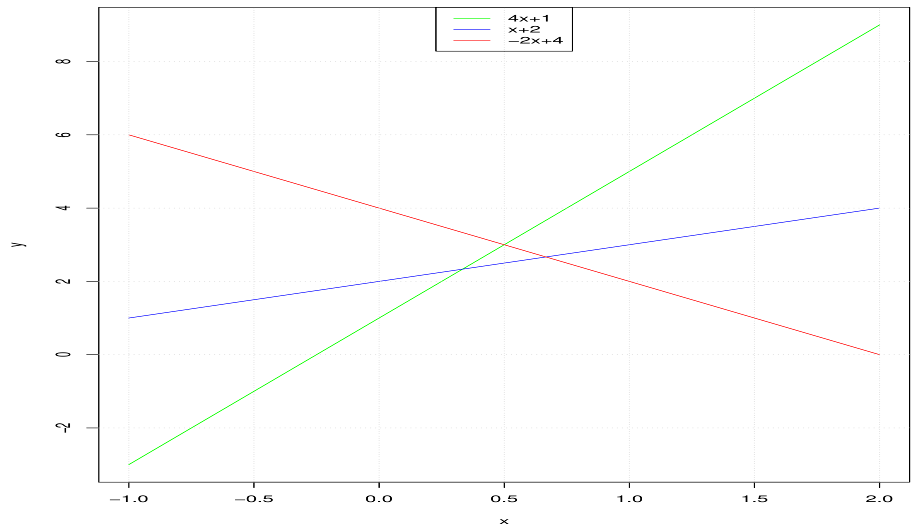

Figure 10: A plot of the three lines $4x + 1$, $x + 2$, and $-2x + 4$ as a function of x.

In symbols we have argued that

$$\begin{aligned}\triangle BFD &= \frac{1}{4}\left(\triangle BEG\right) \\ &= \frac{1}{4}\left(\frac{2}{3}\triangle BEC\right) = \frac{1}{6}\triangle BEC \\ &= \frac{1}{6}\left(\Box FECD + \triangle BFD\right) .\end{aligned}$$

Solving for $\triangle BFD$ we get

$$\triangle BFD = \frac{1}{5}\Box FECD \,.$$

Problem 22

We can plot the given functions (they are all lines) and then to compute f at a given value of x we take the minimum of the three functions pointwise. When we do that we get the plot given in Figure 10. From that plot the maximum value of the function f is at the intersection of the two lines $x + 2$ and $-2x + 4$. This point is $x = \frac{2}{3}$ and $y = \frac{8}{3}$.

Problem 23

From a trisected hypotenuse if we drop perpendiculars and horizontals to the two legs of the right triangle we divide these legs into three equal parts. Now let the horizontal leg have one third of its length be denoted a (so the total length of the leg of the original right triangle is

$3a$) and the vertical leg have one third of its length be denoted b (so that the length in the original triangle is $3b$). Then drawing the two triangles with hypotenuse lengths of $\cos(x)$ and $\sin(x)$ using the Pythagorean theorem on each we can write

$$(2b)^2 + a^2 = \sin^2(x)$$
$$b^2 + (2a)^2 = \cos^2(x)\,.$$

If we add these two equations we get

$$5b^2 + 5a^2 = 1\,.$$

This gives that $a^2 + b^2 = \frac{1}{5}$. Now the length (squared) of the hypotenuse h of the original triangle is equal to

$$h^2 = (3a)^2 + (3b)^2 = 9(a^2 + b^2) = \frac{9}{5}\,,$$

so the length we want is $h = \frac{3}{\sqrt{5}}$.

Problem 24

From what we are told we know that

$$8x^3 - 4x^2 - 42x + 45 = Q(x)(x-r)^2 = Q(x)(x^2 - 2rx + r^2)\,,$$

for some polynomial $Q(x)$ of degree one. Lets take the form of $Q(x)$ to be $Q(x) = 8x + t$ for some unknown value t. I choose the leading coefficient to be $8x$ so that it will cancel with the $8x^3$ on the left-hand-side when we expand. Expanding we get

$$\begin{aligned} 8x^3 - 4x^2 - 42x + 45 &= (8x + t)(x^2 - 2rx + r^2) \\ &= 8x^3 + (-16r + t)x^2 + 2r(4r - t)x + tr^2\,. \end{aligned}$$

This means that we must have

$$\begin{aligned} -4 &= -16r + t \\ -42 &= 2r(4r - t) \\ 45 &= tr^2\,. \end{aligned} \qquad (310)$$

In the first equation we have $t = -4 + 16r$ which we can put into the second equation to get

$$-42 = 2r(4r + 4 - 16r)\,,$$

or

$$-12r^2 + 4r + 21 = 0\,.$$

We can solve the above to find

$$r = \frac{-4 \pm \sqrt{1024}}{(-24)} = \frac{-4 \pm 32}{(-24)}\,,$$

which simplify to the two numbers $-\frac{7}{6}$ and $\frac{3}{2}$. From these two possible values of r using the fact that $t = -4 + 16r$ we get the two numbers $-\frac{68}{3}$ and 20. From Equation 310 we see that $t > 0$ so the second solution is the one we are interested in. This means that we have

$$t = 20 \quad \text{with} \quad r = \frac{3}{2} = 1.5\,.$$

The number from the choices closest to 1.5 is 1.52.

Problem 25

Note that from the problem statement we are told that

$$\lfloor \sqrt{n+c} \rfloor = \frac{a_n - d}{b},$$

is *constant* for ranges of n. For example when

- $2 \le n \le 4$ the above right-hand-side is the same value.
- $5 \le n \le 9$ the above right-hand-side is the same value

As there are an odd number of terms in each constant range and

$$1 + 3 + 5 + 7 + \cdots + (2N-1) + (2N+1) = (N+1)^2,$$

In general the right-hand-side will be constant for n such that

$$(N-1)^2 + 1 \le n \le N^2,$$

for the different values of $N \in \{1, 2, 3, \dots\}$.

This means that taking $n = 2$ and $n = 4$ (from the range $2 \le n \le 4$) we have that

$$\lfloor \sqrt{2+c} \rfloor = \frac{a_2 - d}{b} = \frac{a_4 - d}{b} = \lfloor \sqrt{4+c} \rfloor.$$

Note that $c = 0$ will not work as $\lfloor \sqrt{2} \rfloor \ne \lfloor 2 \rfloor = 2$. Also notice that $c = -1$ *will* work as as $\lfloor \sqrt{1} \rfloor = 1 = \lfloor \sqrt{3} \rfloor = 1$.

Taking $n = 5$ and $n = 9$ (from the range $5 \le n \le 9$) we have that

$$\lfloor \sqrt{5+c} \rfloor = \frac{a_5 - d}{b} = \frac{a_9 - d}{b} = \lfloor \sqrt{9+c} \rfloor.$$

Note that here too $c = -1$ will also work as

$$\lfloor \sqrt{5-1} \rfloor = \lfloor 2 \rfloor = 2 = \lfloor \sqrt{8} \rfloor.$$

Now in general for n in the range

$$(N-1)^2 + 1 \le n \le N^2,$$

then if we take $c = -1$ as we have

$$\lfloor \sqrt{(N-1)^2 + 1 + c} \rfloor = \lfloor \sqrt{N^2 + c} \rfloor,$$

which means that for n in the range above the fractions $\frac{a_n - d}{b}$ will all be equal.

To find values for b and d we need when $n = 1$ that

$$b\lfloor\sqrt{0}\rfloor + d = 1 \quad \text{so} \quad d = 1\,.$$

We also need that when $n = 2$ that

$$b\lfloor\sqrt{1}\rfloor + d = 3 \quad \text{so} \quad b = 2\,.$$

From all of this we have found that the sequence takes the form

$$a_n = 2\lfloor\sqrt{n-1}\rfloor + 1\,.$$

Thus

$$b + c + d = 2 - 1 + 1 = 2\,.$$

Problem 27

Let x be given by the given expression so

$$x = \sqrt[3]{5 + 2\sqrt{13}} + \sqrt[3]{5 - 2\sqrt{13}}\,.$$

Next using

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3\,.$$

We find

$$\begin{aligned}
x^3 &= 5 + 2\sqrt{13} + 3(5+2\sqrt{13})^{2/3}(5-2\sqrt{13})^{1/3} + 3(5+2\sqrt{13})^{1/3}(5-2\sqrt{13})^{2/3} + 5 - 2\sqrt{13} \\
&= 10 + 3[(5+2\sqrt{13})^2(5-2\sqrt{13})]^{1/3} + 3[(5+2\sqrt{13})(5-2\sqrt{13})^2]^{1/3} \\
&= 10 + 3[(5+2\sqrt{13})(25-2(13))]^{1/3} + 3[(25-52)(5-2\sqrt{13})]^{1/3} \\
&= 10 + 3[(5+2\sqrt{13})(-27)]^{1/3} + 3[(-27)(5-2\sqrt{13})]^{1/3} \\
&= 10 - 9(5+2\sqrt{13})^{1/3} - 9(5-2\sqrt{13})^{1/3} \\
&= 10 - 9x\,.
\end{aligned}$$

This means that x must satisfy

$$x^3 + 9x - 10 = 0\,.$$

Thus x must be is root of this polynomial. One root that is "easy" to find is $x = 1$. This means that we can factor the above into

$$x^3 + 9x - 10 = (x-1)(x^2 + x + 10) = 0\,.$$

The quadratic $x^2 + x + 10 = 0$ has a discriminant given by

$$b^2 - 4ac = 1^2 - 4(1)(10) = -39\,,$$

which is negative meaning that the two roots of this quadratic are complex. As the only real root is $x = 1$ this must be the value of x above.

The 1981 Examination

Problem 1

If we square this twice we get $(2^2)^2 = 16$.

Problem 2

The side of the square has a length given by $\sqrt{2^2 - 1} = \sqrt{3}$. This means that the area of the square is then $(\sqrt{3})^2 = 3$.

Problem 3

Write this as

$$\frac{1}{x}\left(1 + \frac{1}{2} + \frac{1}{3}\right) = \frac{1}{x}\left(\frac{6}{6} + \frac{3}{6} + \frac{2}{6}\right) = \frac{1}{x}\left(\frac{11}{6}\right).$$

Problem 4

Lets take our two numbers be denoted x and y with $x > y$. Then from the problem statement we are told are

$$3x = 4y \quad (311)$$
$$x - y = 8\,. \quad (312)$$

From Equation 311 we have that $y = \frac{3}{4}x$. If we put that into Equation 312 we can solve for x to get $x = 32$.

Problem 5

From the angles given in triangle BCD we know that $\angle CDB$ is 40 degrees. As CD and AB are parallel this means that $\angle DBA = \angle CDB = 40$. As triangle ABD is isosceles we know that $\angle DBA = \angle DAB = 40$. This means that $\angle ADB = 180 - 2(40) = 100$.

Problem 6

Let the right-hand-side of this expression be denoted by r then we have from the problem statement that

$$\frac{x}{x-1} = r\,.$$

If we solve for x in the above we get

$$x = \frac{r}{r-1}\,.$$

Using the expression we defined for r we compute that

$$r - 1 = \frac{1}{y^2 + 2y - 2}\,,$$

Thus evaluating $\frac{r}{r-1}$ we see that

$$x = y^2 + 2y - 1\,.$$

Problem 7

The numbers N that are divisible by these numbers take the form

$$N = 2^{2x} 3^y 5^z\,,$$

for positive integers x, y, and z such that $x \ge 2$, $y \ge 1$, and $z \ge 1$. To have $N \le 100 = 2^2 5^2$ we see that we could take $x = 2$, $y = 1$, $z = 1$ (which is the number $2^2 3^1 5^1 = 60$) but that $x = 2$, $y = 2$, $z = 1$ and $x = 2$, $y = 1$, $z = 2$ are too large. Larger values for x, y, and z will also have a N that is too large. This means that there is only one such number $N = 60$.

Problem 8

The given expression is equivalent to

$$\left(\frac{x^{-1} + y^{-1} + z^{-1}}{x + y + z}\right)\left(\frac{(xy)^{-1} + (yz)^{-1} + (zx)^{-1}}{xy + yz + zx}\right)\,.$$

In the first factor multiply the "top and bottom" by xyz to get

$$\left(\frac{yz + xz + xy}{(xyz)(x + y + z)}\right)\left(\frac{(xy)^{-1} + (yz)^{-1} + (zx)^{-1}}{xy + yz + zx}\right) = \frac{(xy)^{-1} + (yz)^{-1} + (zx)^{-1}}{(xyz)(x + y + z)}\,.$$

Next we multiply the "top and bottom" by

$$(xy)(yz)(zx) = x^2 y^2 z^2\,,$$

we get

$$\frac{(yz)(zx) + (xy)(zx) + (xy)(yz)}{(xyz)^3(x + y + z)} = \frac{(xyz)(z + x + y)}{(xyz)^3(x + y + z)} = \frac{1}{x^2 y^2 z^2}\,.$$

Problem 9

Let the side of the cube be denoted s. Then using the Pythagorean theorem the diagonal of a face has a length

$$\sqrt{s^2 + s^2} = \sqrt{2}s \,.$$

Again using the Pythagorean theorem we can write

$$PQ^2 = a^2 = 2s^2 + s^2 = 3s^2 \,.$$

This means that $s = \frac{a}{\sqrt{3}}$. The surface area of the cube is then

$$6s^2 = 6\left(\frac{a^2}{3}\right) = 2a^2 \,.$$

Problem 10

We can eliminate several solutions by considering the following special case. If we consider an arbitrary line *parallel* to $y = x$ say $y = x + b$ then the y-intercept on that line (i.e. $(0, b)$) is reflected to the point $(0, -b)$ and the symmetric line is $y = x - b$. In this special case we have $a = 1$ and the only line from the solutions that reproduces this case is $\frac{1}{a}x - \frac{b}{a}$.

We can derive the line symmetric to $y = ax + b$ (through $y = x$) in the following way. Due to the reflection one point on this line is the point $(b, 0)$. Another point on this line is where $y = ax + b$ and $y = x$ intersect. This location is

$$\left(\frac{b}{1-a}, \frac{b}{1-a}\right) .$$

The slope between these two points is then

$$\frac{\frac{b}{1-a} - 0}{\frac{b}{1-a} - b} = \frac{1}{a} \,.$$

The line that has that slope and goes thorough $(b, 0)$ is

$$y = \frac{1}{a}(x - b) \,.$$

which is one of the solutions.

Problem 11

One set of triples that will satisfy $a^2 + b^2 = c^2$ is the 3-4-5 multiples or numbers of the form

$$a = 3n$$
$$b = 4n$$
$$c = 5n \,,$$

for some integer $n \geq 1$. Notice that this means that

$$\begin{aligned} a &= 3n \\ b &= 3n + n \\ c &= 3n + 2n\,, \end{aligned}$$

which has the sides a, b, and c in an arithmetic progression where the common difference is n. Thus to have a triangle with this property we need one of the sides to be divisible by three, one of the sides to be divisible by four, and one of the sides divisible by five. Note that none of the numbers are divisible by four or five but that 81 is divisible by three and no other number is. In that case we see that $a = 81 = 3(27)$ so that $n = 27$. This means that the other sides of our right triangle have lengths of $b = 4(27) = 108$ and $c = 5(27) = 135$.

Problem 12

We want to know when

$$M\left(1 + \frac{p}{100}\right)\left(1 + \frac{q}{100}\right) > M\,.$$

If we cancel M, expand the above product, and simplify a bit we get

$$p - q - \frac{pq}{100} > 0\,.$$

We can solve this for p and using the fact that $100 - q < 0$ we get

$$p > \frac{100q}{100 - q}\,.$$

Problem 13

We want to find n such that

$$N(1 - 0.1)^n \leq 0.1N\,,$$

or

$$0.9^n \leq 0.1\,.$$

Taking the $\log_{10}$ of both sides we get

$$n\log_{10}\left(\frac{9}{10}\right) \leq \log_{10}\left(\frac{1}{10}\right)\,.$$

We can write the above as

$$n\left(\log_{10}(9) - 1\right) \leq -1\,,$$

or

$$n\left(2\log_{10}(3) - 1\right) \leq -1\,.$$

We are told that $2\log_{10}(3) = 0.954$ so that

$$n \geq \frac{1}{0.954} = 21.739\,.$$

Thus we should take $n = 22$ years.

Problem 14

Our sequence will take the form $a_n = a_0 r^n$ for $n \geq 0$. Then we are told that

$$a_0 + a_1 = a_0 + ra_0 = 7$$
$$\sum_{i=0}^{5} a_i = \sum_{i=0}^{1} a_i + a_0 \sum_{i=2}^{5} r^i = 91\,.$$

This means that

$$a_0 \sum_{i=2}^{5} r^i = 91 - 7 = 84\,,$$

or

$$a_0 r^2 \sum_{i=0}^{3} r^i = a_0 r^2 \left(\frac{1 - r^4}{1 - r}\right) = 84\,.$$

We want to know the value of

$$\begin{aligned}\sum_{i=0}^{3} a_0 r^i &= \sum_{i=0}^{1} a_i + a_0 \sum_{i=2}^{3} r^i = 7 + a_0 r^2 \sum_{i=0}^{1} r^i \\ &= 7 + a_0 r^2 (1 + r) = 7 + r^2 (a_0 + a_0 r) = 7 + 7r^2 = 7(1 + r^2)\,. \end{aligned} \tag{313}$$

Thus we need to solve

$$a_0(1 + r) = 7$$
$$a_0 r^2 \left(\frac{1 - r^4}{1 - r}\right) = 84\,,$$

for a_0 and r. If we take the second equation and multiply by one over the first equation we get

$$r^2 \left(\frac{1 - r^4}{1 - r^2}\right) = \frac{84}{7} = 12\,,$$

or

$$r^2(1 + r^2) = 12\,.$$

This is a quadradic equation for r^2. Solving that we find

$$r^2 = \frac{-1 \pm 7}{2}\,.$$

Taking the real root we see that $r^2 = 3$. Putting this in Equation 313 we see that the sum we seek is given by $7(1 + 3) = 28$.

Problem 15

We start with the problem statement of

$$(2x)^{\log_b(2)} - (3x)^{\log_b(3)} = 0\,,$$

which we write as
$$(2x)^{\log_b(2)} = (3x)^{\log_b(3)} \,.$$
If we take the $\log_b(\cdot)$ of each side we get
$$\log_b(2)\log_b(2x) = \log_b(3)\log_b(3x) \,,$$
or
$$\log_b(2)\left[\log_b(2) + \log_b(x)\right] = \log_b(3)\left[\log_b(3) + \log_b(x)\right] \,.$$
Solving for $\log_b(x)$ in this expression we get
$$\log_b(x) = -\frac{\log_b(3)^2 - \log_b(2)^2}{\log_b(3) - \log_b(2)} = -\log_b(3) - \log_b(2) = \log_b\left(\frac{1}{6}\right) \,,$$
which means that $x = \frac{1}{6}$.

Problem 16

Let the expression given be denoted by E and we start by writing the largest terms first. We have
$$\begin{aligned} E &= 1\cdot 3^{19} + 2\cdot 3^{18} + 1\cdot 3^{17} + 1\cdot 3^{16} + 2\cdot 3^{15} + \cdots \\ &= 1\cdot(3^2)^9\cdot 3 + 2\cdot(3^2)^9 + 1\cdot(3^2)^8\cdot 3 + 1\cdot(3^2)^8 + 2\cdot(3^2)^7\cdot 3 + \cdots \\ &= 3\cdot 9^9 + 2\cdot 9^9 + 3\cdot 9^8 + 1\cdot 9^8 + 6\cdot 9^7 + \cdots \\ &= 5\cdot 9^9 + 4\cdot 9^8 + 6\cdot 9^7 + \cdots \end{aligned}$$
The leading coefficient has a value of five.

Problem 17

If we start with the problem statement
$$f(x) + 2f\left(\frac{1}{x}\right) = 3x \,, \tag{314}$$
and let $x \to \frac{1}{x}$ we get
$$f\left(\frac{1}{x}\right) + 2f(x) = \frac{3}{x} \,.$$
If we solve for $f(x)$ in Equation 314 and put that in the previous expression we get
$$f\left(\frac{1}{x}\right) + 2\left(3x - 2f\left(\frac{1}{x}\right)\right) = \frac{3}{x} \,.$$
We can solve for $f\left(\frac{1}{x}\right)$ where we get
$$f\left(\frac{1}{x}\right) = 2x - \frac{1}{x} \,.$$

In this expression if we let $x \to \frac{1}{x}$ we get

$$f(x) = \frac{2}{x} - x\,.$$

This means that the expression $f(x) = f(-x)$ is equivalent to

$$\frac{2}{x} - x = -\frac{2}{x} + x\,,$$

which is equivalent to $x^2 = 2$ which has two solutions $x = \pm\sqrt{2}$.

Problem 18

If we plot $\sin(x)$ and $\frac{x}{100}$ we see that there will be an intersection of these two curves (and thus a solution to the given equation) for $x \geq 0$ approximately given by

$$0, \pi, 2\pi, 3\pi, 4\pi, \ldots\,.$$

In fact in each of the intervals $(2n\pi, (2n+1)\pi)$ for $n = 0, 1, 2, \ldots$ there is a root closer to the left-end-point (i.e. $2n\pi$) followed by a root closer to the right-end-point (i.e. $(2n+1)\pi$). As n increases the location of these two roots in each given interval "move inwards" and eventually the two roots migrate close to the x location where the $\sin(x)$ curve in that interval is largest or the x location

$$x = \frac{1}{2}(2n\pi + (2n+1)\pi) = 2n\pi + \frac{\pi}{2}\,.$$

At this point the $\sin(x)$ function takes the value one. Thus the *last* possible interval where we could have a root is the one where when $x = 2n\pi + \frac{\pi}{2}$ and we have

$$\frac{x}{100} = 1\,.$$

Solving this last equation we get

$$n = \frac{100 - \frac{\pi}{2}}{2\pi} = 15.66549\,.$$

The last interval where we have roots is then when $n = 15$. As we have two roots in each interval for $n = 0, 1, 2, \ldots, 14, 15$ we have $16 \times 2 = 32$ positive roots in all. We should also have 32 negative roots (corresponding to the roots when $x < 0$) but one of these roots is repeated i.e. $x = 0$. Thus we have in total

$$32 + 31 = 63\,,$$

real roots.

Problem 19

Extend the segment BN until it intersects the line AC. Let the point of intersection be denoted D. With that edge drawn we have that triangle ABN is congruent to triangle ADN. This is because these two triangles share the same side (the side AN) and have

$$\angle BAN = \angle DAN$$
$$\angle ANB = \angle AND\,.$$

This means that $BN = ND$. From the problem statement we are also told that $BM = MC$ by the fact that M is the midpoint of side BC. This means that MN is parallel to DC so triangle CBD is similar to triangle MBN. This means that

$$\frac{DC}{NM} = \frac{BD}{BN} = \frac{2BN}{BN} = 2\,.$$

Now $DC = 5$ from the above we have that $NM = \frac{5}{2}$.

Problem 20

If we start at the angle $\angle ADC$ and denote it by θ and then work "backwards" towards A we will arrive at a condition that will make starting at A not possible. As the points of reflection (in the text) are labeled R_3, R_2, and R_1 we will follow that convention and denote earlier points as R_0, R_{-1}, R_{-2} etc. From what we are told we compute

$$\angle BR_3D = 90 - \theta\,.$$

Then using

$$2\angle R_1R_3R_2 + \angle R_2R_3B = 180\,,$$

we get $\angle R_2R_3B = 2\theta$. Then $\angle R_3R_2B = 90 - 2\theta$. Using

$$2\angle R_3R_2B + \angle R_1R_2R_3 = 180\,,$$

we get $\angle R_1R_2R_3 = 4\theta$. Then

$$\angle R_2R_1R_3 = 180 - \angle R_1R_2R_3 - \angle R_2R_3R_1 = 180 - 4\theta - (90 - \theta) = 90 - 3\theta\,.$$

This then gives $\angle R_2R_1R_0 = 6\theta$.

At this point we see a pattern in the size of the angles. The size of the angle "between" the two equal angles at each reflection (working from A backwards) is given by

$$2m\theta\,,$$

for $m \geq 1$ and the two equal angles at each reflection is given by

$$90 - \frac{2m\theta}{2} = 90 - m\theta\,,$$

again for $m \geq 1$.

For example using these results, the point R_2 on the line AD will have its three angles ($m = 2$) given by

$$90 - 2\theta\,, 4\theta\,, 90 - 2\theta\,.$$

Starting at the point R_2 we would have $n = 1$ reflections until B.

The point R_1 on the line CD will have its three angles ($m = 3$) given by

$$90 - 3\theta\,, 6\theta\,, 90 - 3\theta\,.$$

Starting at the point R_1 we would have $n = 2$ reflections until B.

The point $A = R_0$ on the line AD will have its three angles ($m = 4$) given by

$$90 - 4\theta\,, 8\theta\,, 90 - 4\theta\,.$$

Starting at the point R_0 we would have $n = 3$ reflections until B.

In general then, the point R_{4-m} for $m \geq 1$ will have its three angles given by

$$90 - m\theta\,, 2m\theta\,, 90 - m\theta\,.$$

Starting at the point R_{4-m} we would have $m - 1$ reflections until B.

We can start this process from a point R_{4-m} as long as $90 - m\theta > 0$ or

$$m < \frac{90}{\theta} = \frac{90}{8} = 11.25\,.$$

Thus the largest m can be is $m = 11$. This would have $n = m - 1 = 10$ reflections until B. From this point the three angles are

$$2\,, 176\,, 2\,,$$

all in degrees.

Problem 21

From what we are told we know that

$$(a + b + c)(a + b - c) = (a + b)^2 - c^2 = 3ab\,,$$

which once we expand $(a + b)^2$ we can write as

$$a^2 - ab + b^2 - c^2 = 0\,.$$

For all triangles, the law of cosigns gives

$$c^2 = a^2 + b^2 - 2ab\cos(\theta)\,.$$

If we put this in the above (for c^2) we get

$$-ab + 2ab\cos(\theta) = 0\,.$$

This means that $\cos(\theta) = \frac{1}{2}$ so that $\theta = \frac{\pi}{3}$ or 60 degrees.

Problem 24

Note that if $x = e^{\pm i\theta}$ then $x^{-1} = e^{\mp i\theta}$ and that using Euler's formula we have

$$x + x^{-1} = (\cos(\theta) \pm i\sin(\theta)) + (\cos(\theta) \mp i\sin(\theta)) = 2\cos(\theta)\,.$$

In this case we have that $x^n = e^{\pm in\theta}$ and $x^{-n} = e^{\mp in\theta}$ so that in the same way

$$x^n + x^{-n} = 2\cos(n\theta)\,.$$

Problem 25

To start we will recall/prove **The Angle Bisector Theorem**. Draw the triangle ABC with sides opposite the angles of lengths a, b, and c. Here I am imagining A "at the top", followed by B, followed by C in a counterclockwise manner.

Then draw a segment from the vertex A *bisecting* the angle $\angle BAC$ to the side BC intersecting at D. Let the length of the segment to the left of D be given by d and the length of the segment to the right of D be denoted by e. Let the angle $\angle ADB$ be denoted θ. Then the law of signs in triangle ABD is

$$\frac{\sin(\alpha)}{d} = \frac{\sin(\theta)}{c} \quad \text{so} \quad \frac{\sin(\alpha)}{\sin(\theta)} = \frac{d}{c}\,.$$

Because $\angle ADC = \pi - \angle ADB = \pi - \theta$ we have

$$\sin(\angle ADC) = \sin(\pi - \theta) = \sin(\pi)\cos(-\theta) + \cos(\pi)\sin(-\theta) = \sin(\theta)\,.$$

This means that the law of signs in triangle ADC is given by

$$\frac{\sin(\alpha)}{e} = \frac{\sin(\pi - \theta)}{b} = \frac{\sin(\theta)}{b} \quad \text{so} \quad \frac{\sin(\alpha)}{\sin(\theta)} = \frac{e}{b}\,.$$

Equating these two expressions we get

$$\frac{d}{c} = \frac{e}{b}\,. \tag{315}$$

Now in the triangle given let the lengths of $AB = x$, $AD = y$, $AE = z$ and $AC = w$. Then using the angle bisector theorem on the triangle $\triangle BAE$ we find

$$\frac{2}{x} = \frac{3}{z} \quad \text{so} \quad z = \frac{3}{2}x\,.$$

Next using the angle bisector theorem on the triangle $\triangle DAC$ we find

$$\frac{3}{y} = \frac{6}{w} \quad \text{so} \quad y = \frac{1}{2}w\,.$$

If we let the common angle measure in $\angle BAD = \angle DAE = \angle EAC$ be denoted by α by using the law of cosigns to express $\cos(\alpha)$ we have

$$\cos(\alpha) = \frac{4 - x^2 - y^2}{2xy} = \frac{9 - y^2 - z^2}{2yz} = \frac{36 - z^2 - w^2}{2zw}.$$

We will use the above two expressions for z and y and write the first and second and the first and third equations as a system in the unknowns x and w. Note that these two unknowns are the unknown lengths of the two sides of the triangle $\triangle ABC$. When I do this I get

$$\begin{aligned}\frac{4 - x^2 - \frac{1}{4}w^2}{xw} &= \frac{9 - \frac{1}{4}w^2 - \frac{9}{4}x^2}{\frac{3}{2}xw}\\ \frac{4 - x^2 - \frac{1}{4}w^2}{xw} &= \frac{36 - \frac{9}{4}x^2 - w^2}{3xw}.\end{aligned}$$

I can write these two equations as

$$\begin{aligned}\frac{3}{2}x^2 - \frac{1}{4}w^2 &= 6\\ -\frac{3}{4}x^2 + \frac{1}{4}w^2 &= 24.\end{aligned}$$

If we add these two together we get $x^2 = 40$ or $x = 2\sqrt{10} = 6.324555$. Using this then gives that $w^2 = 216 = 6\sqrt{6} = 14.69694$. As the length of the side $BC = 11$ the smallest side has a length $AB = x = 2\sqrt{10}$.

Problem 26

For C to roll a six first A and B must not roll sixes. They each do not roll sixes with a probability of $\frac{5}{6}$. Thus C will roll a six first if the first six is rolled on the $3, 6, 9, 12, \ldots$ time. This will happen with a probability of

$$\begin{aligned}P(C) &= \left(\frac{5}{6}\right)^2 \frac{1}{6} + \left(\frac{5}{6}\right)^5 \frac{1}{6} + \left(\frac{5}{6}\right)^8 \frac{1}{6} + \cdots\\ &= \left(\frac{5}{6}\right)^2 \frac{1}{6} \sum_{k=0}^{\infty} \left(\frac{5}{6}\right)^{3k} = \frac{5^2}{6^3}\left(\frac{1}{1 - \left(\frac{5}{6}\right)^3}\right) = \frac{25}{91}.\end{aligned}$$

Problem 30

Since we are told that a, b, c, and d are solutions (i.e. roots) we can write

$$x^4 - bx - 3 = (x - a)(x - b)(x - c)(x - d).$$

Expanding the right-hand-side gives the expression

$$x^4 - (a + b + c + d)x^3 + (cd + (a + b)(c + d) + ab)x^2 - (cd(a + b) + ab(c + d))x + abcd.$$

This is still equal to the left-hand-side from before. Equating powers of x we see that since there is no term $O(x^3)$ in the left-hand-side we have that

$$a + b + c + d = 0\,.$$

Using this we see that

$$\begin{aligned} a + b + c &= -d \\ a + b + d &= -c \\ a + c + d &= -b \\ b + c + d &= -a\,. \end{aligned}$$

This means that the four ratios given are equal to

$$-\frac{1}{d}\,, -\frac{1}{c}\,, -\frac{1}{b}\,, -\frac{1}{a}\,.$$

Now the key observation to make is that if

$$x^4 - bx - 3 = 0\,,$$

has solutions $x \in \{a, b, c, d\}$ then taking $x \to -\frac{1}{v}$ this equation will have solutions

$$v \in \left\{-\frac{1}{a}, -\frac{1}{b}, -\frac{1}{c}, -\frac{1}{d}\right\}\,.$$

This equation is

$$\left(-\frac{1}{v}\right)^4 - b\left(-\frac{1}{v}\right) - 3 = 0\,.$$

Simplifying this is

$$3v^4 - bv^3 - 1 = 0\,.$$

The 1982 Examination

Problem 1

We want to find $r(x)$ such that we can write

$$x^3 - 2 = (x^2 - 2)q(x) + r(x)\,.$$

Using long-division we can show that $r(x) = 2x - 2$.

Problem 2

This would be

$$\frac{1}{4}(8x+2) = 2x + \frac{1}{2}$$

Problem 3

For this value of x note that $x^x = 2^2 = 4$ so the expression given is $4^4 = 256$.

Problem 4

Drawing this figure the perimeter P and area A would be related by

$$P = \frac{1}{2}(2\pi r) + 2r = A = \frac{1}{2}\pi r^2\,.$$

Solving for r in this expression we get

$$r = 2 + \frac{4}{\pi}\,.$$

Problem 5

From the problem statement we are told that

$$\frac{x}{y} = \frac{a}{b}\,.$$

Now as $a < b$ this means that $x < y$ and the smaller of the two numbers (between x and y) is x. From the fact that $x + y = c$ we have $y = c - x$. If we put this into the first expression and solve for x we get

$$x = \frac{ac}{a+b}\,.$$

Problem 6

The sum of the interior angles of a polygon with n sides is $180(n-2)$. Thus we know that n must be such that

$$180(n-2) = 2570 + \theta\,,$$

for some angle $\theta > 0$. Thus we need to have $180(n-2) \geq 2570$ which means that $n \geq 16.27778$. As n is a positive integer we will take $n = 17$ and then $180(n-2) =$ so that $\theta = 130$.

Problem 7

We can see that (A) and (D) are true for all x. If we consider (C) we see that the left-hand-side is given by

$$(x-1) \star (x+1) = x(x+2) - 1 = x^2 + 2x - 1\,,$$

while the right-hand-side is given by

$$\begin{aligned}(x \star x) - 1 &= ((x+1)(x+1) - 1) - 1 \\ &= (x^2 + 2x + 1 - 1) - 1 = x^2 + 2x - 1\,,\end{aligned}$$

which are equal.

If we consider (E) we see that the left-hand-side is given by

$$\begin{aligned}x \star (y \star z) &= x \star ((y+1)(z+1) - 1) \\ &= (x+1)((y+1)(z+1)) - 1 \\ &= (x+1)(y+1)(z+1) - 1\,,\end{aligned}$$

while the right-hand-side is given by

$$\begin{aligned}(x \star y) \star z &= ((x+1)(y+1) - 1) \star z \\ &= ((x+1)(y+1))(z+1) - 1 \\ &= (x+1)(y+1)(z+1) - 1\,,\end{aligned}$$

which are equal.

For (B) the left-hand-side is given by

$$x \star (y+z) = (x+1)(y+z+1) - 1\,,$$

while the right-hand-side is given by

$$\begin{aligned}(x \star y) + (x \star z) &= ((x+1)(y+1) - 1) + ((x+1)(z+1) - 1) \\ &= (x+1)(y+z+1) - 1 + x + 1 - 1 \\ &= (x+1)(y+z+1) - 1 + x\,,\end{aligned}$$

which are *not* equal.

Problem 8

An arithmetic progression means that the difference between two terms in the sequence is equal. For this sequence that means that

$$\binom{n}{2} - \binom{n}{1} = d\,,$$

and

$$\binom{n}{3} - \binom{n}{2} = d\,,$$

so that both differences equal a whole number (here d). Using the given expression for $\binom{j}{k}$ these are

$$\frac{n!}{2!(n-2)!} - \frac{n!}{1!(n-1)!} = d = \frac{n!}{3!(n-3)!} - \frac{n!}{2!(n-2)!}\,,$$

or

$$\frac{n(n-1)}{2} - n = \frac{n(n-1)(n-2)}{6} - \frac{n(n-1)}{2}\,.$$

If we divide by n and multiply by six we get

$$3(n-1) - 6 = (n-1)(n-2) - 3(n-1)\,.$$

This simplifies to

$$0 = n^2 - 9n + 14\,.$$

This has two solutions

$$n = \frac{9 \pm \sqrt{81 - 4(14)}}{2} = \frac{9 \pm \sqrt{81 - 56}}{2} = \frac{9 \pm 5}{2}\,,$$

or

$$n \in \{2, 7\}\,.$$

As we are told that $n > 3$ the solution to take is $n = 7$.

Problem 9

The area of the triangle can be computed from various parts in the figure. The first is the area of a rectangle with corners

$$(0,0), (9,0), (9,1), (0,1)\,,$$

which is

$$1(9) = 9\,.$$

From this we first subtract the triangle with vertices $(0,0)$, $(9,0)$, and $(9,1)$ which has area

$$\frac{1}{2}(1)(9) = \frac{9}{2}\,.$$

Next we also subtract the area of the triangle with vertices $(0,0)$, $(1,1)$, and $(0,1)$ which has area

$$\frac{1}{2}(1)(1) = \frac{1}{2}\,.$$

Using all of these results the area of original triangle is

$$9 - \frac{9}{2} - \frac{1}{2} = 9 - 5 = 4\,.$$

Next note that the line AB can be written as

$$y = \left(\frac{1-0}{9-0}\right) x = \frac{x}{9}\,.$$

Now if we cut this triangle into two parts using a vertical line at x then the triangle with vertices $\left(x, \frac{x}{9}\right)$, $(9,1)$, and $(x,1)$ will have an area

$$\frac{1}{2}(1 - \frac{x}{9})(9 - x)\,.$$

To have an area equal to one-half of the original area We want to find x such that

$$\frac{1}{2}\left(1 - \frac{x}{9}\right)(9 - x) = \frac{4}{2}\,.$$

This can be solved where we find $x = 3$ or $x = 15$. To have $0 < x < 9$ we need to take $x = 3$.

Problem 10

First note that the triangle AMN is similar to the triangle ABC. Let the smaller triangle have lengths that are that of the larger triangle but scaled by x where $x < 1$. This means that $MN = 24x$. Also along side AB we have

$$\begin{aligned} AM &= 12x \\ MB &= 12(1-x)\,. \end{aligned}$$

Finally along side AC we have

$$\begin{aligned} AN &= 18x \\ NC &= 18(1-x)\,. \end{aligned}$$

The perimeter of the triangle AMN is then

$$12x + 18x + 24x = 54x\,.$$

To evaluate this we need to determine the value of x. As MN is parallel to BC we know that

$$\begin{aligned} \angle CBO &= \angle BOM \\ \angle BCO &= \angle CON\,. \end{aligned}$$

Because BO and CO are angle bisectors. We then get two isosceles triangles (MBO and NCO) and thus have $MO = BM$ and $ON = NC$. As

$$MN = 24x = MO + ON = MB + NC = 12(1-x) + 18(1-x)\,,$$

we can solve for x and find $x = \frac{5}{9}$. Using this we find the perimeter of triangle AMN given by $54x = 30$.

n_4	number of valid choices for n_1
1	1
2	2
3	2
4	2
5	2
6	2
7	2
8	1
9	1

Table 8: The possible values for n_1 given the value of n_4.

Problem 11

Let our integer be written as $n_4n_3n_2n_1$ where each n_i is a digit in the appropriate range. For example we would require that $1 \le n_4 \le 9$. One condition on the numbers we want is that $|n_1 - n_4| = 2$. This is equivalent to

$$n_1 = n_4 \pm 2\,.$$

If we take $1 \le n_4 \le 9$ and then consider the possible choices we can get for n_1 (using the above formula if possible) we get Table 8. This gives a total of

$$1 + 2 \times 6 + 1 + 1 = 15\,,$$

ways to pick the digits n_1 and n_4. Now to pick the digits n_2 and n_3 for each we can select any digit in the range $0 - 9$ but excluding the two digits selected when we picked the digits n_1 and n_4. This gives $8 \times 7 = 56$ ways to pick these two digits and have them be distinct. In total then we have

$$15 \times 56 = 840\,,$$

ways to pick four digit numbers of the desired type.

Problem 12

Note that $f(x) = g(x) - 5$ with $g(x)$ an odd function. This means that

$$f(-7) = g(-7) - 5 = 7 \quad \text{so} \quad g(-7) = 12\,.$$

Using this and the fact that $g(x)$ is odd we know that $g(7) = -12$. This means that

$$f(7) = g(7) - 5 = -12 - 5 = -17\,.$$

Problem 13

From the given expression we can write

$$p\log_b(a) = \log_b(\log_b(a))\,,$$

or

$$\log_b(a^p) = \log_b(\log_b(a))\,.$$

Applying the function $f(x) = b^x$ to both sides we get

$$a^p = \log_b(a)\,.$$

Problem 14

As AG is tangent to the circle we know that $\angle PGA$ is a right angle. Lets draw a line through N and parallel to the segment PG. Let this segment/line intersect AG at the point M. Then $\angle AMN$ is also a right angle. By correspondence we see that $\triangle ANM$ is similar to $\triangle APG$. This means that

$$\frac{AN}{NM} = \frac{AP}{GP}\,,$$

or

$$\frac{3r}{MN} = \frac{5r}{r} \quad \text{so} \quad MN = \frac{3}{5}r\,.$$

Now draw the segment NF which will have length r. By the Pythagorean theorem we have

$$MF^2 + MN^2 = r^2\,.$$

Using the above expression for MN we get that

$$MF = \frac{4}{5}r\,.$$

Now the length of EF is

$$2MF = \frac{8}{5}r\,.$$

When $r = 15$ this is 24.

Problem 15

To start this problem lets assume that the $[\cdot]$ does not change the (x, y) solution and instead solve the system

$$\begin{aligned} y &= 2x + 3 \\ y &= 3(x - 2) + 5\,. \end{aligned}$$

For this we find $x = 4$ and $y = 11$. Note that these *are* integers. Let's see if a value of x "close" to four will also work. If we have something less than four i.e. if $3 < x < 4$ then $1 < x - 2 < 2$ so that $[x] = 3$ and $[x - 2] = 1$. Using these in the right-hand-sides of the two original expressions for y we get

$$\begin{aligned} y &= 2(3) + 3 = 9 \\ y &= 3(1) + 5 = 8 \,, \end{aligned}$$

which are not equal showing that no solutions for x in the range $3 < x < 4$ exist. Lets now consider the case where $4 < x < 5$. Then $[x] = 4$ and as $x - 2 > 2$ we have $[x - 2] = 2$. Using these in the right-hand-sides of the two original expressions for y we get

$$\begin{aligned} y &= 2(4) + 3 = 11 \\ y &= 3(2) + 5 = 11 \,, \end{aligned}$$

which is true. Thus we should take $4 < x < 5$ to have a solution. In this case $y = 11$ thus

$$4 + 11 < x + y < 5 + 11 \quad \text{so} \quad 15 < x + y < 16 \,.$$

Problem 16

The full surface of the cube without any holes is $6 \cdot 3^2$. Once we "cut" the center out we increase the surface area by the four sides of the 1×1 cube "inside" each face. That is by

$$6 \times 4 \times (1 \times 1) = 24 \,.$$

This also removes the surface area of the 1×1 square on the faces of the six sides. In total then the surface area is

$$6 \cdot 3^2 + 6 \cdot 4 - 6 \cdot 1 = 72 \,.$$

Problem 17

We write this as

$$3^2(3^x)^2 - 3^3(3^x) - 3^x + 3 = 0 \,,$$

or

$$9(3^x)^2 - 28(3^x) + 3 = 0 \,.$$

Then if we let $v = 3^x$ we have the equation

$$9v^2 - 28v + 3 = 0 \,.$$

This has solutions

$$v \in \left\{ \frac{1}{9}, 3 \right\} \,.$$

This means that x has solutions $x = -2$ and $x = 1$ so two real solutions.

Problem 18

Let the square $HGDC$ have dimensions $s \times s$ and the rectangle $HFBC$ have dimensions $d \times s$. Then as vectors we have $\overrightarrow{HB} = (d, 0, s)$ and $\overrightarrow{HD} = (0, s, s)$. Then the cosign of the angle between HB and HD is given by

$$\frac{\overrightarrow{HB} \cdot \overrightarrow{HD}}{||HB|| \cdot ||HD||} = \frac{s^2}{\sqrt{d^2 + s^2}\sqrt{s^2 + s^2}} = \frac{s}{\sqrt{2}\sqrt{d^2 + s^2}}\,.$$

Now from the given diagram we see that

$$\cos\left(\frac{\pi}{6}\right) = \frac{s}{\sqrt{d^2 + s^2}} = \frac{\sqrt{3}}{2}\,.$$

Using this in the above we find the cosign of $\angle BHD$ given by

$$\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{6}}{4}\,.$$

Problem 19

To start we write the function as

$$f(x) = |x - 2| - 2|x - 3| + |x - 4|\,.$$

If $-\infty < x < 2$ the function above is equal to

$$f(x) = -(x - 2) - 2(-x + 3) + (-x + 4) = 0\,.$$

If $2 < x < 3$ the function above is equal to

$$f(x) = x - 2 - 2(-x + 3) + (-x + 4) = 2x - 4\,.$$

If $3 < x < 4$ the function above is equal to

$$f(x) = x - 2 - 2(x - 3) + (-x + 4) = -2x + 8\,.$$

If $4 < x < \infty$ the function above is equal to

$$f(x) = x - 2 - 2(x - 3) + (x - 4) = 0\,.$$

Plotting each of these regions we see that the minimum of $f(x)$ is zero and the maximum of $f(x)$ is two giving a sum of two.

Problem 20

We want to find solutions to

$$x^2 + y^2 = x^3 \,,$$

with (x, y) both integers. To start if we let $x = y$ then we get $2x^2 = x^3$ so $x = 2$ i.e. we have at least one solution. If we take $x = 1$ then we get

$$1 + y^2 = 1 \quad \text{so} \quad y = 0 \,.$$

Thus we have no positive integer solutions when $x = 1$. Thus we must have $x \geq 2$ to have solutions. I next claim that we must have $y > x$ for if $y < x$ note that

$$x^2 + y^2 < x^2 + x^2 = 2x^2 \,.$$

This will be *smaller* than x^3 (and could not be equal to x^3) when $x > 2$. Based on this lets write $y = x + p$ with $p \geq 1$. Then we have

$$x^2 + (x + p)^2 = x^3 \,.$$

Expanding and simplifying we get

$$p^2 + 2px - x^3 + 2x^2 = 0 \,.$$

This is a quadratic equation for p. Solving we get

$$p = \frac{-2x \pm \sqrt{4x^2 - 4(-x^3 + 2x^2)}}{2} = -x \pm x\sqrt{x - 1} \,.$$

If we take the negative root we would see that $p < 0$ so we must take the positive root and have

$$p = -x + x\sqrt{x - 1} \,.$$

Using this we find that y is given by

$$y = x + p = x\sqrt{x - 1} \,.$$

This will not be an integer unless $x - 1$ is a perfect square say $x - 1 = n^2$ which means that $x = n^2 + 1$. Now that we have a representation for x this means that

$$y = n(1 + n^2) \,.$$

Lets check that the values of (x, y) found above give an integer solution to the original equation. We have

$$x^2 + y^2 = (n^2 + 1)^2 + n^2(1 + n^2)^2 = (n^2 + 1)^3 \,,$$

which is x^3 as it needs to be to be a solution. Thus we have an infinite number of solutions for $n \geq 1$. Note that $n = 1$ gives the first solution we found $x = y = 2$.

Problem 21

To solve this problem we remember that "medians are area bisectors" and their intersection at the centroid (denoted by O) of the triangle divides each median into portions that are 2/3 and 1/3 of the total median length. Let the length of AC be $2y$ (so that $CN = NA = y$) and the length of AB be $2x$ (so that $BM = MA = x$). Then as triangle ABC is a right triangle we can write that is area is given by

$$A_{\text{ABC}} = \frac{1}{2}s(2y) = sy\,.$$

Triangle BNC is also a triangle made by a median so by the "area bisector" theorem we have its area given by

$$A_{\text{BNC}} = \frac{1}{2}A_{\text{ABC}} = \frac{1}{2}sy\,.$$

But also since triangle BNC is a right triangle we have

$$A_{\text{BNC}} = \frac{1}{2}(m+n)p = \frac{1}{2}sy\,.$$

Here m is the distance from B to O (the centroid), n is the distance from O to N, and p is the distance from O to C. This means that solving for $m+n$ we have

$$m + n = s\left(\frac{y}{p}\right)\,.$$

Now

$$\frac{y}{p} = \frac{1}{\frac{p}{y}} = \frac{1}{\cos(\angle OCN)}\,.$$

Using the diagram given we can show that $\angle OCN = \angle OBC$ so that

$$\cos(\angle OCN) = \cos(\angle OBC) = \frac{m}{s}\,,$$

or

$$m + n = s\left(\frac{s}{m}\right)\,.$$

Because the centroid divides the median in the proportions $1:2$ we have

$$m = \frac{2}{3}(m+n)\,,$$

which when we put in the above gives

$$m + n = \frac{3s^2}{2(m+n)}\,.$$

Solving this for $m+n$ gives

$$m + n = BN = \frac{\sqrt{3}s}{\sqrt{2}} = \frac{\sqrt{6}}{2}s\,.$$

Problem 22

Let the left-most corner be denoted as A and the right-most corner denoted as B so the "base" of the figure is the segment APB. Now

$$\angle PRQ = 180 - \angle APR - \angle BPQ = 180 - 75 - 45 = 60\,.$$

Thus we see that $\triangle PQR$ is an equilateral triangle with a side length of a. Draw a line from R and parallel to APB. Let this intersect the segment BQ at the point C. Then $\angle RCQ$ is a right angle and

$$\angle RQC = 180 - \angle PQB - \angle PQR = 180 - 45 - 60 = 75\,.$$

Next we also have

$$\angle QRC = 90 - \angle RQC = 90 - 75 = 15 = \angle ARP\,.$$

Thus by angle-side-angle we see that $\triangle PRA$ is congruent to $\triangle QRC$ so that $RC = AR = h$. But $RC = AB = w$ so we have just shown that $w = h$.

Problem 23

Let the lengths of the triangle be $n-1$, n, and $n+1$. The largest angle must be opposite the largest side and the smallest angle must be opposite the smallest side. Introduce a triangle ABC with side lengths

$$\begin{aligned} AB &= n \\ BC &= n-1 \\ AC &= n+1\,, \end{aligned}$$

and angles

$$\begin{aligned} \angle A &= \theta \\ \angle B &= 2\angle A = 2\theta \\ \angle C &= 180 - \angle B - \angle A = 180 - 3\angle A = 180 - 3\theta\,. \end{aligned}$$

We want to know the value of $\cos(\angle A) = \cos(\theta)$. Using the law of cosigns (around $\angle A = \theta$) we can write

$$(n-1)^2 = (n+1)^2 + n^2 - 2n(n+1)\cos(\theta)\,.$$

Expanding and simplifying we can write this as

$$\cos(\theta) = \frac{n+4}{2(n+1)}\,.$$

Again using the law of cosigns (this time around $\angle B = 2\theta$) we can write

$$(n+1)^2 = n^2 + (n-1)^2 - 2n(n-1)\cos(2\theta)\,.$$

Expanding and simplifying we can write this as

$$\cos(2\theta) = \frac{n-4}{2(n-1)}\,.$$

Using the fact that

$$\cos(2\theta) = 2\cos^2(\theta) - 1\,,$$

we can relate these two expressions to get

$$\frac{n-4}{2(n-1)} = \frac{1}{2}\left(\frac{n+4}{n+1}\right)^2 - 1\,.$$

We can expand this to get

$$2n^3 - 7n^2 - 17n + 10 = 0\,.$$

We want an integer solution $n > 1$. The possible rational solutions $\frac{p}{q}$ of this expression will have p a factor of 10 or

$$p \in \{\pm 1, \pm 2, \pm 5, \pm 10\}\,,$$

and q a factor of two or

$$q \in \{\pm 1, \pm 2\}\,.$$

The only possible integers $n > 1$ would then be $\{2, 5, 10\}$. Trying each of these we find that $n = 5$ is a root and thus we can factor $n - 5$ from the above polynomial to get

$$2n^3 - 7n^2 - 17n + 10 = (n-5)(2n^2 + 3n - 2)\,.$$

This last quadratic has solutions $n = \frac{1}{2}$ and $n = -2$. Thus we have that $n = 5$ and thus

$$\cos(\theta) = \frac{n+4}{2(n+1)} = \frac{3}{4}\,,$$

when we simplify.

Problem 24

Let the length BD be equal to a, the length DE be equal to x, the length CE be equal to b, and the length AH be y. Then using the *Intersecting Secants Theorem*[6], which is a theorem about the lengths of segments drawn to a circle from an exterior point we can write

$$y(y+7) = 2(2+13) = 30\,.$$

This is equivalent to

$$y^2 + 7y - 30 = 0\,,$$

which we can factor as

$$(y-3)(y+10) = 0\,.$$

[6] https://www.mathopenref.com/secantsintersecting.html

The only positive solution to this equation is then $y = 3$. Knowing that length and because $\triangle ABC$ is an equilateral triangle we have the length of BJ given by

$$BJ = 16 - 7 - y = 16 - 7 - 3 = 6\,.$$

The intersecting secants theorem for the secants CD and CG gives

$$b(b + x) = 1(1 + 13) = 14\,,$$

The intersecting secants theorem for the secants BH and BE gives

$$a(a + x) = 6(6 + 7) = 78\,.$$

Finally because $\triangle ABC$ is an equilateral triangle we have

$$a + x + b = 16\,.$$

In summary we have three equations and three unknowns i.e. x, a, and b i.e.

$$\begin{aligned} bx + b^2 &= 14 & (316)\\ ax + a^2 &= 78 & (317)\\ x + a + b &= 16\,. & (318) \end{aligned}$$

From the last of these we have

$$x = 16 - a - b\,,$$

which if we put into the first two equations we get

$$\begin{aligned} 16b - ab &= 14\\ 16a - ab &= 78\,. \end{aligned}$$

If we multiply the second of these by minus one and add to the first we get

$$b = a - 4\,.$$

Putting this into $x + a + b = 16$ we get

$$x + a + (b - 4) = 16 \Rightarrow a = 10 - \frac{x}{2}\,.$$

If we put this expression for a in terms of x into Equation 317 we get

$$\left(10 - \frac{x}{2}\right)x + \left(10 - \frac{x}{2}\right)^2 = 78\,.$$

Expanding and simplifying this I find

$$x = \sqrt{88} = 2\sqrt{22}\,.$$

Problem 25

Label the intersections (i, j) for $0 \le i \le 3$ and $0 \le j \le 4$ with i increasing as we move East and j increasing as we move South. Let $p_{i,j}$ be the probability that we pass through intersection (i, j) moving from A to B. Then starting at the upper left corner and working towards the lower right corner (towards intersection B) we have that

$$
\begin{aligned}
p_{0,0} &= 1 \\
p_{1,0} &= p_{0,1} = \frac{1}{2} \\
p_{2,0} &= p_{0,2} = \frac{1}{4} \\
p_{3,0} &= p_{0,3} = \frac{1}{8} \,,
\end{aligned}
$$

since the only to get to these intersections is to sequentially step horizontally or vertically. We then compute

$$
p_{1,1} = \frac{1}{2} p_{0,1} + \frac{1}{2} p_{1,0} = \frac{1}{2} \,,
$$

since to get to the intersection $(1, 1)$ we must either come from intersection $(0, 1)$ or $(1, 0)$ each is equally likely. Using that same logic we can compute

$$
\begin{aligned}
p_{2,1} &= p_{1,2} = \frac{3}{8} \\
p_{1,3} &= \frac{1}{4} \\
p_{2,3} &= \frac{5}{16} \,.
\end{aligned}
$$

We cannot use that logic to compute $p_{3,j}$ for $j \ge 1$. For example to compute $p_{3,1}$ we have

$$
p_{3,1} = 1 p_{3,0} + \frac{1}{2} p_{2,1} = \frac{1}{8} + \frac{1}{2} \left(\frac{3}{8} \right) = \frac{5}{16} \,.
$$

This is because we will enter intersection $(3, 1)$ with certainty if we come from intersection $(3, 0)$ and one-half of the time if we come from intersection $(2, 1)$. In the same way we compute

$$
p_{3,2} = 1 p_{3,1} + \frac{1}{2} p_{2,2} = 1 \left(\frac{5}{16} \right) + \frac{1}{2} \left(\frac{3}{8} \right) = \frac{1}{2} \,,
$$

and

$$
p_{3,3} = 1 p_{3,2} + \frac{1}{2} p_{2,3} = 1 \left(\frac{1}{2} \right) + \frac{1}{2} \left(\frac{5}{16} \right) = \frac{16}{32} + \frac{5}{32} = \frac{21}{32} \,.
$$

Problem 26

We are told that

$$
(ab3c)_8 = a8^3 + b8^2 + 3 \cdot 8 + c = n^2 \,, \tag{319}
$$

for some integer n. Lets write n in base eight as

$$n = (de)_8 = d8 + e\,,$$

where $1 \leq d \leq 7$ and $0 \leq e \leq 7$. Then

$$n^2 = (d8 + e)^2 = d^2 8^2 + 2de8 + e^2\,.$$

Comparing this to Equation 319 we must have

$$a8^3 + b8^2 + 3 \cdot 8 + c = d^2 8^2 + 2de8 + e^2\,. \tag{320}$$

From this we see that d^2 must have a factor of eight or otherwise a would be zero. For $1 \leq d \leq 7$ we have that only $4^2 = 16$ has a factor of eight. This means that $d = 4$ and the above becomes

$$a8^3 + b8^2 + 3 \cdot 8 + c = 2 \cdot 8^3 + e8^2 + e^2\,. \tag{321}$$

From this we have that $a = 2$ and

$$b8^2 + 3 \cdot 8 + c = e8^2 + e^2\,.$$

As $0 \leq e \leq 7$ the term $e8^2$ does not have any powers of eight higher than two which means that $b = e$ and we are left with

$$3 \cdot 8 + c = e^2\,.$$

Taking $e = 0, 1, \cdots, 6, 7$ and computing $c = e^2 - 3 \cdot 8$ for the possible values of c I get

```
[1] -24 -23 -20 -15  -8   1  12  25
```

As we know that $0 \leq c \leq 7$ the only value in the above where that is true is if $c = 1$.

Problem 27

If we take the conjugate of the given equation we get

$$c_4\bar{z}^4 - ic_3\bar{z}^3 + c_2\bar{z}^2 - ic_1\bar{z} + c_0 = 0\,.$$

Note that we can introduce $-\bar{z}$ and write this as

$$c_4(-\bar{z})^4 + ic_3(-\bar{z})^3 + c_2(-\bar{z})^2 + ic_1(-\bar{z}) + c_0 = 0\,.$$

This means that $-\bar{z} = -a + bi$ is a solution to the original equation.

Problem 28

Recall that

$$1 + 2 + 3 + \cdots + (n-1) + n = \frac{1}{2}n(n+1)\,.$$

Then if we remove the number m where $1 \le m \le n$ from this sum we have

$$1 + 2 + 3 + \cdots + (n-1) + n - m = \frac{1}{2}n(n+1) - m\,,$$

for the new sum. To get the average value we divide that sum by the number of terms which is $n-1$ and we get

$$\frac{1}{n-1}\left(\frac{1}{2}n(n+1) - m\right) = 35\,\frac{7}{17} = \frac{602}{17}\,.$$

We can expand the left-hand-side we can write the above as

$$17(n(n+1) - 2m) = 1204(n-1)\,.$$

Note that everything is an integer and that $1204 = 2^2 \cdot 7 \cdot 43$ so we can write

$$17(n(n+1) - 2m) = 2^2 \cdot 7 \cdot 43 \cdot (n-1)\,.$$

From the form of the left-hand-side we must have at least one factor of seventeen in the product expression found in the right-hand-side. This means that

$$n - 1 = 17c \quad \text{so} \quad n = 17c + 1\,,$$

for some $c = 1, 2, 3, \ldots$. Putting this expression for n in the above gives

$$17((17c+1)(17c+2) - 2m) = 2^2 \cdot 7 \cdot 43 \cdot 17 \cdot c\,,$$

or

$$(17c+1)(17c+2) - 2^2 \cdot 7 \cdot 43 \cdot c = 2m\,,$$

or under further expansion

$$289c^2 - 1153c + 2 = 2m\,.$$

We can take $c = 1, 2, 3, \ldots$ and for each compute the value of m using the above formula. Once we know c we also know n as $n = 17c + 1$. Doing this in the following R code

```
cs = 1:10
ms = ( 289 * cs^2 - 1153 * cs + 2 ) / 2
ns = 17*cs + 1
R = rbind(cs, ns, ms)
rownames(R) = c('c', 'n', 'm')
print(round(R, 2))
```

gives

```
   [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8] [,9] [,10]
c     1    2    3    4    5    6    7    8    9    10
n    18   35   52   69   86  103  120  137  154   171
m -431 -574 -428    7  731 1744 3046 4637 6517  8686
```

From the problem statement we know that $1 \le m \le n$. The only column where this is true is when $c = 4$ where we have $n = 69$ and $m = 7$.

Problem 29

Let the minimum m_0 be given by $m_0 = x_0 y_0 z_0$ and assume that the variables are ordered as

$$x_0 \le y_0 \le z_0 .$$

We know that $z_0 < 2x_0$ from the problem statement but I claim that at the minimum of the product xyz over all feasible values for (x, y, z) we must have $z = 2x$. If this was not true consider a new product $x_1 y_1 z_1$ where $z_1 < 2x_1$ and such that

$$\begin{aligned} x_1 &= x_0 + h \\ y_1 &= y_0 \\ z_1 &= 2x_0 - h . \end{aligned}$$

Thus the point (x_1, y_1, z_1) is a feasible point (satisfies all of the constraint conditions) and perturbed from one where $z_0 = 2x_0$. Then

$$\begin{aligned} x_1 y_1 z_1 &= (x_0 + h) y_0 (2x_0 - h) = (x_0 + h)(y_0(2x_0) - h y_0) \\ &= x_0 y_0 (2x_0) - h x_0 y_0 + h y_0 (2x_0) - h^2 y_0 = m_0 + h x_0 y_0 - h^2 y_0 \\ &\approx m_0 + h x_0 y_0 , \end{aligned}$$

for $h \ll 1$. From this we see that the product $x_1 y_1 z_1$ would then be *larger* than the product when we take $h = 0$ and enforce $z = 2x$.

As $x + y + z = 1$ under the condition that $z = 2x$ we have that $y = 1 - x - z = 1 - 3x$. This means that

$$m = xyz = x(1 - 3x)(2x) = 2x^2(1 - 3x) .$$

Define the right-hand-side of the above to be equal to $f(x)$. The extreme points of $f(x)$ are given by solving

$$f'(x) = 4x(1 - 3x) + 2x^2(-3) = 0 .$$

This is equivalent to

$$f'(x) = 4x - 18x^2 = 0 .$$

This has solutions $x = 0$ or $x = \frac{2}{9} = 0.2222222$. Now from the ordering requirement $x_0 < y_0 < z_0$ specialized to what we have determined above we need to have

$$x_0 < 1 - 3x_0 < 2x_0 .$$

The "left-most" inequality gives that

$$x_0 < 1 - 3x_0 \quad \text{or} \quad x_0 < \frac{1}{4} = 0.25\,.$$

The "right-most" inequality gives that

$$1 - 3x_0 < 2x_0 \quad \text{or} \quad x_0 > \frac{1}{5} = 0.2\,.$$

Thus the valid ranges for x_0 are

$$0.2 = \frac{1}{5} \le x_0 \le \frac{1}{4} = 0.25\,.$$

There are three values for x_0 (the two endpoints and the value $\frac{2}{9}$) that we must evaluate to see which gives the smallest value for $f(x)$. We have

$$\begin{aligned}
f\left(\frac{2}{9}\right) &= 2\left(\frac{4}{81}\right)\left(1 - \frac{6}{9}\right) = \frac{8}{81}\cdot\frac{3}{9} = \frac{8}{27\cdot 9} = \frac{8}{243} = 0.03292181 \\
f\left(\frac{1}{5}\right) &= \frac{2}{25}\left(1 - \frac{3}{5}\right) = \frac{2}{25}\left(\frac{2}{5}\right) = \frac{4}{125} = 0.032 \\
f\left(\frac{1}{4}\right) &= \frac{2}{16}\left(1 - \frac{3}{4}\right) = \frac{1}{8}\left(\frac{1}{4}\right) = \frac{1}{32} = 0.03125\,.
\end{aligned}$$

Thus we see that the smallest value of $f(x)$ is when $x_0 = \frac{1}{4}$, $y_0 = 1 - 3x_0 = \frac{1}{4}$, $z_0 = 2x_0 = \frac{1}{2}$ and $m_0 = \frac{1}{32}$.

The 1983 AHSME Examination

Problem 1

Solving for y we have $y = \frac{x}{16}$. Using that in the first expression gives $\frac{x}{2} = \frac{x^2}{16^2}$. As $x \neq 0$ we can divide by it and then solve for x to get $x = \frac{16^2}{2} = 128$.

Problem 2

Draw a circle around P of radius 3. This second circle can intersect C in at most two points.

Problem 3

If we try enumerating primes such that $1 < p < q$ we start with $p = 2$ and $q = 3$ so that $r = 5$. For these numbers we see that all needed conditions are satisfied.

Problem 4

Draw the segment BE dividing the region into two equal parts. Then the height of $EBCD$ is

$$1 \sin(60°) = \frac{\sqrt{3}}{2} \,.$$

Thus the area of $EBCD$ is

$$\frac{1}{2} h(b_1 + b_2) = \frac{1}{2} \left(\frac{\sqrt{3}}{2} \right) (1 + 1) = \frac{\sqrt{3}}{2} \,.$$

As we have two of these the area of the full figure is twice this or $\sqrt{3}$.

Problem 5

If we draw this triangle we find the length of AC given by

$$\sqrt{9 - 4} = \sqrt{5} \,.$$

This then means that

$$\tan(B) = \frac{\sqrt{5}}{2} \,.$$

Problem 6

Write the given expression as

$$x^5 \left(x + \frac{1}{x}\right)\left(1 + \frac{2}{x} + \frac{3}{x^3}\right) = x(x^2+1)(x^3+2x^2+3)\,,$$

which we see will be a polynomial of degree $1+2+3=6$.

Problem 7

Alice's commission is

$$0.1(L-10)\,.$$

Bob's commission is

$$0.2(L-20)\,.$$

Setting these equal and solving for L gives $L=30$.

Problem 8

Note that

$$f(-x) = \frac{-x+1}{-x-1} = \frac{x-1}{x+1}\,,$$

which is one over $f(x)$.

Problem 9

Let W and M be the number of women and men. Then we are told that

$$\frac{W}{M} = \frac{11}{10} \quad \text{so} \quad W = \frac{11}{10}M\,.$$

The average A of the full sample would be given by

$$A = \frac{34W+32M}{W+M}\,.$$

Using the first expression for W in terms of M in the above for A (and canceling M) we get

$$A = \frac{34(11)+32(10)}{11+10} = \frac{694}{21} = 33\,\frac{1}{21}\,.$$

Problem 10

Draw the center of the circle on the segment AB and denote it as the point O. Draw a radius from O to E which is of length one (the length of the radius). As $\triangle ABC$ is equilateral we have $\angle ABC = 60$ degrees. Now $\triangle EOB$ has $OB = OE$ (both are radius of the circle) so that $\angle BEO$ is also 60 degrees and $\triangle EOB$ is also an equilateral triangle. This means that $\angle EOB = 60$ so that $\angle EOA = 120$ degrees or $\frac{2\pi}{3}$ radians. Using the law of cosigns on the triangle AOE we have

$$AE^2 = 1^2 + 1^2 - 2(1)(1)\cos\left(\frac{2\pi}{3}\right) = 2 + 1\,.$$

This means that $AE = \sqrt{3}$. Note in the original wording of this problem I originally thought the radius of the circle was two and not one.

Problem 11

Call this expression E. Then by expanding $\sin(x - y)$ and $\cos(x - y)$ we have

$$\begin{aligned} E &= (\sin(x)\cos(y) - \cos(x)\sin(y))\cos(y) + (\cos(x)\cos(y) + \sin(x)\sin(y))\sin(y) \\ &= \sin(x)\cos^2(y) - \cos(x)\cos(y)\sin(y) + \cos(x)\cos(y)\sin(y) + \sin(x)\sin^2(y) \\ &= \sin(x)\,. \end{aligned}$$

Problem 12

The given expression is equivalent to

$$\log_3(\log_2(x)) = 1\,,$$

or

$$\log_2(x) = 3\,,$$

or

$$x = 2^3 = 8\,.$$

This means that

$$x^{-1/2} = \frac{1}{8^{1/2}} = \frac{1}{2\sqrt{2}}\,.$$

Problem 13

From the definitions given we note that

$$\frac{(ab)^2}{abc} = \frac{ab}{c} = \frac{(xy)(xz)}{yz} = x^2\,,$$

and

$$\frac{(ac)^2}{abc} = \frac{ac}{b} = \frac{(xy)(yz)}{xz} = y^2\,,$$

and finally

$$\frac{(bc)^2}{abc} = z^2\,,$$

using the same logic. Thus the expression $x^2 + y^2 + z^2$ is given by (E).

Problem 14

Note that we can write the given number N as

$$N \equiv 3^{1001} \cdot 7^{1002} \cdot 13^{1003} = (3 \cdot 7 \cdot 13)^{1001} \cdot 7 \cdot 13^2\,,$$

and the value of the term in parenthesis is $3 \cdot 7 \cdot 13 = 273$. Determining the units digit of powers of 273 is simple. For example, if we let "ud" be the function that returns the "units digit" we see that

$$\begin{aligned}
\text{ud}(273^1) &= 3 \\
\text{ud}(273^2) &= 9 \\
\text{ud}(273^3) &= 7 \\
\text{ud}(273^4) &= 1 \\
\text{ud}(273^5) &= 3 \\
\text{ud}(273^6) &= 9\,,
\end{aligned}$$

and the pattern of units digits repeats. This means that after five exponents we are back to where we started and thus

$$\text{ud}(273^{1001}) = \text{ud}(273^1) = 3\,,$$

since $1001 = 200 \cdot 5 + 1$. Next note that as $13^2 = 169$ we have $\text{ud}(13^2) = 9$. Thus using what we have shown thus far we have

$$\text{ud}(N) = \text{ud}(3 \cdot 7 \cdot 9) = \text{ud}(21 \cdot 9) = 9\,.$$

Problem 15

If a draw is represented as the tuple (a, b, c) with each of the letters selected from the set $\{1, 2, 3\}$ then there are three ways to choose a, three ways to choose b, and three ways to choose c. This gives a total of $3^3 = 27$ possible valid tuples of three numbers.

The sum of the numbers drawn (i.e. $S = a + b + c$) will equal six if we draw the number two three times or the numbers 1, 2, 3 (in any order). There are $3! = 6$ ways to draw the numbers 1, 2, and 3 in any order. Thus there are $6 + 1 = 7$ ways to have our sum equal six. Thus the probability that we get a sum of six is

$$P(S = 6) = \frac{7}{27}\,.$$

The probability we are asked for is (using Bayes' rule) given by

$$P((2,2,2)|S=6) = \frac{P(2,2,2)}{P(S=6)} = \frac{1/27}{7/27} = \frac{1}{7}\,.$$

Problem 16

In the number described there will first be the digits $1-9$ (each taking one character/space) for a total of nine characters. Second, we will write the numbers $10-99$ each taking two characters/spaces. This is

$$99 - 10 + 1 = 90\,,$$

numbers and as each takes two spaces we have a total of $90 \times 2 = 180$ spaces used. Third we will put the numbers $100 - 999$ (each taking three characters/spaces) which is

$$999 - 100 + 1 = 900\,,$$

numbers for a total of $900 \times 3 = 2700$ spaces. To find the digit at the location 1983 note that the "ones-digit" numbers and the "twos-digit" numbers will take up

$$9 + 180 = 189\,,$$

spaces. We then need to "use up"

$$1983 - 189 = 1794\,,$$

more spaces. As $1793 = 589 \times 3$ this is 589 three digit numbers starting with the first three digit number which is 100. This means that the last number we will write down will be 597 so the digit at the 1983rd spot is a seven.

Problem 17

Let the point F be denoted in polar coordinates as $|F|e^{i\theta}$. From the fact that F is outside the unit circle and the angle we have $|F| > 1$ and $\theta > 0$. This means that one-over F (in polar) takes the form

$$\frac{1}{|F|}e^{-i\theta}\,.$$

We know that $\frac{1}{|F|} < 1$ and the negative angle means that one-over F is below the x-axis. Thus the point should be C.

Problem 18

We are told that

$$f(x^2+1) = x^4 + 5x^2 + 3 = (x^2)^2 + 5x^2 + 3\,.$$

Thus we see

$$f(x^2) = (x^2 - 1)^2 + 5(x^2 - 1) + 3\,.$$

Using this we have

$$f(x^2 - 1) = (x^2 - 2)^2 + 5(x^2 - 2) + 3 = x^4 + x^2 - 3\,,$$

when we expand and simplify.

Problem 19

Using the law of cosines to get the length BC we have

$$BC^2 = 6^2 + 3^2 - 2(6)(3)\cos\left(\frac{2\pi}{3}\right) = 36 + 9 - 36\left(\frac{1}{2}\right) = 63\,.$$

This means that $BC = \sqrt{63} = 3\sqrt{7}$. Note that AD bisects $\angle BAC$. Then by the "angle bisector theorem" we have

$$\frac{AB}{AC} = \frac{BD}{DC} \quad \text{so} \quad \frac{6}{3} = \frac{BD}{DC}\,,$$

Thus we have that $BD = 2DC$. We also know that

$$BC = BD + DC = 3\sqrt{7}\,,$$

or using what know about BD in terms of DC we have

$$2DC + DC = 3\sqrt{7}\,,$$

so $DC = \sqrt{7}$ and $BD = 2\sqrt{7}$. Now using the law of sines in $\triangle ABC$ gives

$$\frac{\sin(\angle BAC)}{\sin(\angle ABC)} = \frac{3\sqrt{7}}{3}\,.$$

As

$$\sin(\angle BAC) = \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}\,.$$

Using these two expression we have

$$\sin(\angle ABC) = \frac{\sqrt{3}}{2\sqrt{7}} = \frac{1}{2}\sqrt{\frac{3}{7}}\,.$$

Now using the law of sines in $\triangle ABD$ as

$$\frac{\sin(\angle BAD)}{\sin(\angle ABD)} = \frac{2\sqrt{7}}{AD}\,,$$

or

$$\frac{\sin\left(\frac{\pi}{3}\right)}{\sin(\angle ABC)} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}\sqrt{\frac{3}{7}}} = \frac{2\sqrt{7}}{AD}\,.$$

Using that expression we can solve for AD and find $AD = 2$.

Problem 20

We are told that $\tan(\alpha)$ and $\tan(\beta)$ are roots to $x^2 - px + q = 0$ which means that when we write this polynomial in factored form as

$$(x - \tan(\alpha))(x - \tan(\beta)) = 0\,,$$

and expand we will find that

$$\begin{aligned} p &= \tan(\alpha) + \tan(\beta) \\ q &= \tan(\alpha)\tan(\beta)\,. \end{aligned}$$

In the same way for the given roots of $x^2 - rx + s = 0$ we would have

$$\begin{aligned} r &= \cot(\alpha) + \cot(\beta) \\ s &= \cot(\alpha)\cot(\beta)\,. \end{aligned}$$

Using these if we consider the expression rs we find

$$\begin{aligned} rs &= \left(\frac{1}{\tan(\alpha)} + \frac{1}{\tan(\beta)}\right)\frac{1}{\tan(\alpha)\tan(\beta)} \\ &= \left(\frac{\tan(\alpha) + \tan(\beta)}{\tan(\alpha)\tan(\beta)}\right)\frac{1}{q} = \frac{p}{q}\left(\frac{1}{q}\right) = \frac{p}{q^2}\,. \end{aligned}$$

Problem 21

For the numbers A and B note that

$$\begin{aligned} 10^2 &= 100 \\ (3\sqrt{11})^2 &= 9 \cdot 11 = 99\,. \end{aligned}$$

Thus $A = 10 - 3\sqrt{11}$ is positive and $B = 3\sqrt{11} - 10$ is negative.

For C we have

$$\begin{aligned} 18^2 &= 324 \\ (4\sqrt{13})^2 &= 25 \cdot 13 = 325\,, \end{aligned}$$

so $C = 18 - 5\sqrt{13}$ is negative.

For the numbers D and E note that

$$\begin{aligned} 51^2 &= 2601 \\ (10\sqrt{26})^2 &= 100 \cdot 26 = 2600\,. \end{aligned}$$

Thus $D = 51 - 10\sqrt{26}$ is positive and $E = 10\sqrt{26} - 51$ is negative.

Thus there are only two positive numbers A and D to compare. We need to know which one is smaller. Note that

$$\frac{1}{A} = \frac{1}{10 - 3\sqrt{11}} \left(\frac{10 + 3\sqrt{11}}{10 + 3\sqrt{11}} \right) = \frac{10 + 3\sqrt{11}}{100 - 99} = 10 + 3\sqrt{11} \,,$$

and

$$\frac{1}{D} = \frac{1}{51 - 10\sqrt{26}} \left(\frac{51 + 10\sqrt{26}}{51 + 10\sqrt{26}} \right) = \frac{51 + 10\sqrt{26}}{1} = 51 + 10\sqrt{26} \,.$$

As "each number is larger" we have that

$$10 + 3\sqrt{11} < 51 + 10\sqrt{26} \quad \text{or} \quad \frac{1}{A} < \frac{1}{D} \quad \text{so} \quad D < A \,.$$

Thus the smallest positive number is D.

Problem 22

To find the points where these two curves intersect if we take $y = g(x) = 2ax + 2ab$ and put this in the expression for the other curve $y = f(x) = x^2 + 2bx + 1$ we get

$$2ax + 2ab = x^2 + 2bx + 1 \,.$$

We can simplify this to get

$$x^2 + 2(b - a)x + 1 - 2ab = 0 \,.$$

This will have no real solutions if

$$4(b - a)^2 - 4(1 - 2ab) < 0 \,.$$

If I expand this and simplify I get

$$a^2 + b^2 < 1 \,,$$

which is the interior of a circle with a radius of one. This has an area of $\pi \cdot 1^2 = \pi$.

Problem 23

If we assume that the ratio of the radius of consecutive circles is constant then we would have

$$\frac{r_i}{r_{i+1}} = c \,,$$

for $i \in \{1, 2, 3, 4\}$ and some constant c. This means that

$$\frac{r_1}{r_5} = \prod_{i=1}^{4} \frac{r_i}{r_{i+1}} = c^4 = \frac{8}{18} = \frac{4}{9} \,.$$

This means that $c = \sqrt{\frac{2}{3}}$. The radius of the middle circle can be obtained from

$$\frac{r_1}{r_3} = \prod_{i=1}^{2} \frac{r_i}{r_{i+1}} = c^2 = \frac{2}{3} \,.$$

Solving for r_3 I find $r_3 = 12$.

Problem 24

If the right triangle has legs a and b then the hypotenuse has a length of $\sqrt{a^2+b^2}$. The area is then

$$\frac{1}{2}ab \,,$$

and the perimeter is

$$a + b + \sqrt{a^2+b^2} \,.$$

If these two are to be equal we must have

$$a + b + \sqrt{a^2+b^2} = \frac{1}{2}ab \,.$$

Note if we can find one solution to this equation we have found an infinite number of solutions as ka and kb are also solutions. We can find one solution by taking $a = b$ and putting this in the above where we find

$$a = b = 2(2+\sqrt{2}) \,.$$

Thus the given condition has an infinite number of solutions.

Problem 25

We start with

$$60^a = 3 \quad (322)$$
$$60^b = 5 \,, \quad (323)$$

and want to evaluate

$$x = 12^{\frac{1-a-b}{2(1-b)}} \,.$$

Note that we can write x as

$$x = 12^{\frac{1}{2}\left(1-\frac{a}{1-b}\right)} = 12^{\frac{1}{2}} 12^{-\frac{a}{2(1-b)}} \,.$$

Now from Equation 323 we have

$$60^{-b} = 5^{-1} \quad \text{so} \quad 60^{1-b} = 60 \cdot 5^{-1} = \frac{60}{5} = 12 \,.$$

If we take the $1-b$th root of both sides of this last equation we get

$$60 = 12^{\frac{1}{1-b}}\,.$$

Next take the ath power of both sides to get

$$60^a = 12^{\frac{a}{1-b}}\,.$$

Using Equation 322 we get

$$12^{\frac{a}{1-b}} = 3\,.$$

Taking the square root of this gives

$$12^{\frac{a}{2(1-b)}} = 3^{\frac{1}{2}} \quad \text{so} \quad 12^{-\frac{a}{2(1-b)}} = 3^{-\frac{1}{2}}\,.$$

From the expression we derived for x we then have that

$$x = 12^{1/2} 3^{-1/2} = 2 \cdot 3^{1/2} \cdot 3^{-1/2} = 2\,.$$

Problem 26

Using

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)\,,$$

we have

$$P(A \cap B) = P(A) + P(B) - P(A \cup B)\,.$$

As $P(A \cup B) \le 1$ we know that $-P(A \cup B) \ge -1$ so using that with $P(A) = \frac{3}{4}$ and $P(B) = \frac{2}{3}$ we get

$$P(A \cap B) \ge P(A) + P(B) - 1 = \frac{3}{4} + \frac{2}{3} - 1 = \frac{5}{12}\,.$$

Next using

$$P(A \cup B) \ge \max(P(A), P(B)) = \frac{3}{4}\,,$$

we have that $-P(A \cup B) \le -\frac{3}{4}$ so that

$$P(A \cap B) \le \frac{3}{4} + \frac{2}{3} - \frac{3}{4} = \frac{2}{3}\,.$$

This means that $P(A \cap B) \in \left[\frac{5}{12}, \frac{2}{3}\right]$.

Problem 28

In the triangle we draw the line segment DE. Then we have

$$\text{Area}\triangle ABE = \text{Area}\triangle DBE + \text{Area}\triangle DEA\,,$$

and

$$\text{Area}BEFD = \text{Area}\triangle DBE + \text{Area}\triangle DEF\,.$$

From these two we have that

$$\text{Area}\triangle DEA = \text{Area}\triangle DEF\,.$$

These two triangles have a common base i.e. the segment DE and as they have the same area they must have the same heights. This means that the points A and F are the same distance from the segment DE. This in tern means that AFC is parallel to DE. As the segment DE divides the triangle $\triangle ABC$ with a segment parallel to its base we have

$$\frac{BE}{BC} = \frac{BD}{BA} = \frac{3}{5}\,.$$

Then

$$\text{Area}\triangle ABE = \frac{3}{5}\text{Area}ABC = \frac{3}{5}(10) = 6\,.$$

This is because if we drop a perpendiculars to the segment ADB from the points C and E (i.e. heights in $\triangle ABC$ and $\triangle DBE$) we see that fraction $\frac{BE}{BC}$ scales down the length of the hypotenuse CB to that of EB and correspondingly the lengths of the heights of the perpendiculars from C and E (by trigonometry) while the bases of the corresponding triangles remain the same.

Problem 29

Let the point A be at the origin of a Cartesian coordinate system. Then B would be the point $(1, 0)$, the point C would be located at $(1, 1)$, and D is located at the point $(0, 1)$. Let P be located at the point (x, y). Based on these coordinates the distance conditions in the problem become

$$\begin{aligned} x^2 + y^2 &= u^2 \\ (1-x)^2 + y^2 &= v^2 \\ (1-x)^2 + (1-y)^2 &= w^2\,. \end{aligned}$$

The condition $u^2 + v^2 = w^2$ then becomes

$$x^2 + y^2 + (1-x)^2 + y^2 = (1-x)^2 + (1-y)^2\,.$$

If we expand and simplify this we can write this constraint as

$$x^2 + 2y + y^2 - 1 = 0\,,$$

or as

$$x^2 + (y+1)^2 = 2\,.$$

Note that this constraint is that the point $P = (x, y)$ is on a circle of radius $\sqrt{2}$ with a center at $(0, -1)$.

Based on this observation ,the largest distance the point P can be from D is when P is located at the southern most point on that circle. This is the point $(0, -1-\sqrt{2})$. This point is located at a distance

$$1+1+\sqrt{2} = 2+\sqrt{2}\,,$$

from the point D.

Problem 30

From the given drawing we have that $CM = CN = CA = r$ the radius of the circle. We have

$$\angle ACP = 180^\circ - 40^\circ = 140^\circ\,.$$

Then using $\triangle CAP$ we have that

$$\angle APC = 180^\circ - 140^\circ - 10^\circ = 30^\circ\,.$$

Now in $\triangle CPB$ using the law of sines we have

$$\frac{\sin(\angle CPB)}{r} = \frac{\sin(10^\circ)}{CP}\,.$$

Now in $\triangle ACP$ using the law of sines we have

$$\frac{\sin(10^\circ)}{CP} = \frac{\sin(\angle APC)}{r} = \frac{\sin(30^\circ)}{r} = \frac{1}{2r}\,.$$

Setting these two expressions equal to each other gives that

$$\sin(\angle CPB) = \frac{1}{2}\,.$$

From the drawing we know that $\angle CPB > 90^\circ$ so we must have $\angle CPB = 150^\circ$. Then using $\triangle CPB$ we have

$$\angle BCN = 180^\circ - \angle CPB - \angle PBC = 180^\circ - 150^\circ - 10^\circ = 20^\circ\,.$$

The 1983 AIME Examination

Problem 1

We are told that

$$\log_x w = 24$$
$$\log_y w = 40$$
$$\log_{xyz} w = 12\,,$$

and we want to compute $\log_z w = \frac{\ln(w)}{\ln(z)}$. Lets write what we know in terms of the natural logarithm. We have

$$\frac{\ln(w)}{\ln(x)} = 24$$
$$\frac{\ln(w)}{\ln(y)} = 40$$
$$\frac{\ln(w)}{\ln(xyz)} = 12\,.$$

This last expression means

$$\frac{\ln(xyz)}{\ln(w)} = \frac{1}{12}\,,$$

or

$$\frac{\ln(x)}{\ln(w)} + \frac{\ln(y)}{\ln(w)} + \frac{\ln(z)}{\ln(w)} = \frac{1}{2}\,.$$

Using more of what we know this is

$$\frac{1}{24} + \frac{1}{40} + \frac{\ln(z)}{\ln(w)} = \frac{1}{12}\,,$$

so that

$$\frac{\ln(z)}{\ln(w)} = \frac{1}{60} \quad \text{so} \quad \frac{\ln(w)}{\ln(z)} = 60\,,$$

which is what we wanted to compute.

Problem 2

Our function can be written

$$f(x) = |x - p| + |x - 15| + |x - (p + 15)|\,.$$

Now if $x < p$ this becomes

$$\begin{aligned} f(x) &= -(x - p) - (x - 15) - (x - (p - 15)) \\ &= -x + p - x + 15 - x + p + 15 = -3x + 2p + 30\,. \end{aligned}$$

If $p < x < 15$ it becomes

$$f(x) = x - p - x + 15 - x + (p + 15) = -x + 30\,.$$

Finally if $15 < x < p + 15$ then it becomes

$$f(x) = x - p + x - 15 - x + (p + 15) = x\,.$$

Note that over $p < x < 15$ we have $f(x)$ is decreasing so the smallest value it will take will be when $x = 15$ to take a value of $f(15) = 15$.

Problem 3

Define $v \equiv x^2 + 18x + 30$, then the equation we are given can be written as

$$v = 2\sqrt{v + 15}\,. \tag{324}$$

If we square this we get $v^2 = 4(v + 15)$ which we can write

$$v^2 - 4v - 60 = 0\,.$$

This has two solutions $v \in \{-6, 10\}$. Now $v > 0$ from Equation 324 and so $v = -6$ is is not a valid solution. It follows that the only solution is $v = 10$. In that case the equation for x is

$$10 = x^2 = 18x + 30 \quad \text{or} \quad x^2 + 8x + 20 = 0\,. \tag{325}$$

We can solve for x to get

$$x = \frac{-18 \pm \sqrt{18^2 - 4(20)}}{2} = -9 \pm \sqrt{61}\,.$$

The product of the two real solutions is then

$$(-9 + \sqrt{61})(-9 - \sqrt{61}) = 81 - 61 = 20\,.$$

Note that the product of the two solutions is also the a_0 term (i.e. the constant) in the second expression in Equation 325.

Problem 4

Let point B be the origin of a Cartesian coordinate system with A at $(0, 6)$ and C at $(2, 0)$. We are told that A and C are on a circle of radius $\sqrt{50}$. Let the center of that circle be denoted by (p, q) so its equation is given by

$$(x - p)^2 + (y - q)^2 = 50\,.$$

Putting the values of (x, y) for the points A and C we get

$$\begin{aligned} p^2 + (6 - q)^2 &= 50 \\ (2 - p)^2 + q^2 &= 50\,. \end{aligned} \tag{326}$$

Expanding both sides and simplifying gives

$$\begin{aligned} p^2 + q^2 &= 14 + 12q \\ p^2 + q^2 &= 46 + 4p\,. \end{aligned}$$

If we set these equal to each other we have

$$14 + 12q = 46 + 4p \quad \text{so} \quad p = -8 + 3q\,.$$

If we put that expression for p into Equation 326 we can simplify and get

$$q^2 - 6q + 5 = 0\,.$$

This has roots $q = 1$ and $q = 5$. For these two values of q we have $p = -5$ and $p = 7$ respectively. We expect (p, q) to be in the second quadrant so we should take the first solution where $(p, q) = (-5, 1)$. This means that

$$p^2 + q^2 = 25 + 1 = 26\,.$$

Problem 5

We are told that

$$x^2 + y^2 = 7$$
$$x^3 + y^3 = 10\,.$$

To use this information we note that

$$(x+y)^2 = x^2 + y^2 + 2xy = 7 + 2xy \quad (327)$$
$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 = 10 + 3xy(x+y)\,. \quad (328)$$

Using Equation 327 have

$$xy = \frac{(x+y)^2 - 7}{2}\,.$$

If we put this in Equation 328 we get

$$(x+y)^3 = 10 + \frac{3}{2}(x+y)((x+y)^2 - 7)\,.$$

Note that this is an equation in $z \equiv x + y$ given by

$$z^3 = 10 + \frac{3}{2}z^3 - \frac{21}{2}z\,.$$

Note that we can write this as

$$z^3 - 21z + 20 = 0\,.$$

For this equation to have a rational root of the form $\frac{p}{q}$ then p must be a factor of $a_n = 20$ (unit coefficient) and q must be is a factor of $a_0 = 1$ (the coefficient of highest power of z). For this polynomial this means that

$$p \in \{\pm 1, \pm 2, \pm 4, \pm 5, \pm 20\} \quad \text{and} \quad q \in \{\pm 1\}\,.$$

By inspection $z = 1$ is one solution and we can factor to get

$$z^3 - 21z + 20 = (z-1)(z^2 + z - 20)\,.$$

Another solution is $z = 4$ so factoring again we have

$$z^3 - 21z + 20 = (z-1)(z-4)(z+5) = 0\,.$$

This has solutions $z = 1$, $z = 4$, or $z = -5$ the largest number is then $z = 4$.

Problem 6

We will write the expression for a_n as

$$\begin{aligned} a_n &= (7-1)^n + (7+1)^n \\ &= \sum_{k=0}^{n} \binom{n}{k} 7^k (-1)^{n-k} + \sum_{k=0}^{n} \binom{n}{k} 7^k 1^{n-k} \\ &= \sum_{k=0}^{n} \binom{n}{k} 7^k ((-1)^{n-k} + 1)\,. \end{aligned}$$

We could have expanded the terms $(7-1)$ and $(7+1)$ above "in the other order" and we would have gotten

$$a_n = \sum_{k=0}^{n} \binom{n}{k} 7^{n-k}((-1)^k + 1^k)\,.$$

Notice that when k is odd the term vanishes. Using this expression we have

$$\begin{aligned}
a_{83} &= \sum_{k=0}^{83} \binom{83}{k} 7^{83-k}((-1)^k + 1) \\
&= \sum_{k=0,2,4,\cdots}^{82} \binom{83}{k} 7^{83-k} \times (2) = 2\sum_{k=0}^{41} \binom{83}{2k} 7^{83-2k} \\
&= 14 \sum_{k=0}^{41} \binom{83}{2k} 7^{82-2k} \\
&= 14 \left(\binom{83}{0} 7^{82} + \binom{83}{2} 7^{80} + \binom{83}{4} 7^{78} + \binom{83}{6} 7^{76} + \cdots + \binom{83}{80} 7^2 + \binom{83}{82} \right)\,.
\end{aligned}$$

Note that each of the terms above (except potentially the last) is divisible by $49 = 7^2$ so the remainder of a_{83} when dividing by 49 will be the same as the remainder when dividing (the last term)

$$14 \times \binom{83}{82}\,,$$

by 49. Note that

$$\begin{aligned}
14 \times \binom{83}{82} &= 2 \times 7 \times 83 \\
&= 2 \times 7 \times (10 \times 7 + 13) \\
&= 2 \times 7 \times (11 \times 7 + 6) = 22 \times 7^2 + 2 \times 6 \times 7\,.
\end{aligned}$$

The first term is divisible by 49. The second term has a value of 84 which when dividing by 49 has a remainder of 35.

Problem 7

Consider the more general case where we have n knights. Then we can choose three knights to go on the quest in $\binom{n}{3}$ ways. There are n groups of three that can be found by linking a knight to the two to his or her right. These are the triples that have three neighboring knights in them. We now ask how many triples have two neighboring knights in them. To count these note that there are n pairs of knights and for each of these pairs we can create a set of three knights (of which only two were neighbors) by drawing another of the $n-2-2 = n-4$ knights from somewhere else around the table. This gives $n(n-4)$ groups of three knights of which two were neighbors in the original table. Using this we have the probability of interest given by

$$P = \frac{n + n(n-4)}{\binom{n}{3}} = \frac{6(n-3)}{(n-1)(n-2)}\,.$$

If we take $n = 25$ this becomes $P = \frac{11}{46}$.

Problem 9

Let $v \equiv x\sin(x)$ then our function is

$$f(v) = \frac{9v^2+4}{v} = 9v + \frac{4}{v}\,.$$

Now over the range of x given by $0 < x < \pi$ we see that $v(x)$ starts at zero goes to a maximum, and then falls back down to zero when $x = \pi$ all the time having $v(x) \geq 0$. In terms of $f(v)$ this means that the term $9v$ will start small, increase, and then decrease. The term $\frac{4}{v}$ will start large (when $x \approx 0$) and then decrease as $x \to \pi$. Lets look for the extremes of $f(v)$. Taking the derivative and setting it equal to zero gives

$$f'(v) = 9 - \frac{4}{v^2} = 0 \Rightarrow v^2 = \frac{4}{9}\,,$$

so

$$v = \pm\frac{2}{3}\,.$$

Only the solution $v = \frac{2}{3}$ is positive. For the second derivative I compute

$$f''(v) = \frac{8}{v^3} > 0\,,$$

so the value $v = +\frac{2}{3}$ is a *minimum* of $f(v)$. This function has a minimum value of $f(\frac{2}{3}) = 9(\frac{2}{3}) + 4(\frac{3}{2}) = 6 + 6 = 12$. The only thing we have to check is to make sure that there is an $0 < x < \pi$ such that

$$x\sin(x) = \frac{2}{3}\,.$$

This is equivalent to looking for a solution to

$$\sin(x) = \frac{2}{3x}\,.$$

Plotting these two functions for $0 < x < \pi$ we see that there does exist a solution x such that $v(x) = \frac{2}{3}$.

Problem 10

Consider a four digit number of the form $n_1n_2n_3n_4$. To count how many number we might have of the desired form we first pick which 2 digits are identical in

$$\left(\begin{array}{c}4\\2\end{array}\right) = \frac{4!}{2!2!} = \frac{4\cdot 3}{2} = 6\,,$$

ways. We then pick which specific digit $0-9$ to use in these two spots which can be done in ten ways. We then pick the other 2 digits in $9\cdot 8 = 72$ ways to get a total of $6\cdot 10\cdot 72$ ways to pick 4 digits with 2 digits repeating. The argument just given will over-count since the first digit n_1 can only be 1 and not any of $0, 2, \ldots, 9$. Thus to count the number we really want we take only $\frac{1}{10}$th of the previous number or $6\cdot 72 = 432$.

Problem 12

Let the length of AB be denoted by the two digit number nm. Then the length of CD is given by the two digit number mn. For both of these numbers to not have a leading zero we will need to take $1 \leq n \leq 9$ and $1 \leq m \leq 9$. As we know that $AB > CD$ we must have that $n > m$. Now using the Pythagorean theorem in the triangle $\triangle CHO$ we have that

$$OH^2 + HC^2 = OC^2 \,.$$

If we multiply this expression by four we get

$$(2OH)^2 + (2HC)^2 = (2OC)^2 \,.$$

In terms of the chords given and the length of OH (defined as r) this is

$$(2r)^2 + CD^2 = AB^2 \,.$$

In terms of the two digits m and n representing the lengths CD and AB we can write this as

$$(2r)^2 + (10m + n)^2 = (10n + m)^2 \,.$$

If we expand these and simplify we can write this as

$$(2r)^2 = 99(n^2 - m^2) = 3^2 \cdot 11 \cdot (n - m)(n + m) \,.$$

As the right-hand-side of this expression is an integer the left-hand-side must be an integer and so $2r$ must be an integer. In order that $2r$ be an integer we must have another factor of 11 on the right-hand-side. As $n > m$ we know that $n + m > n - m$ so let

$$n + m = \alpha \cdot 11 \,.$$

From the range of n and m the only possible value for α is one and we have $n + m = 11$. This then gives

$$(2r)^2 = 3^2 \cdot 11^2 (n - m) \,.$$

To have both sides be perfect squares we must have $n - m$ be a perfect square. Again from the range of n and m the only possible perfect squares for $n - m$ would be $\{1, 4, 9\}$.

If $n + m = 11$ and $n - m = 1$ we would have $n = 6$ and $m = 5$ which is a valid configuration. If $n + m = 11$ and $n - m = 4$ we would have $n = \frac{15}{2}$ which is not an integer and not a valid configuration. If $n + m = 11$ and $n - m = 9$ we would have $n = 10$ which again is not a valid configuration.

Thus it looks like $(n, m) = (6, 5)$ and the above expression for $2r$ becomes

$$(2r)^2 = 3^2 \cdot 11^2 \,.$$

This means that $r = \frac{33}{2}$ and the length AB is 65.

The 1984 AHSME Examination

Problem 1

Write this fraction as

$$\frac{1000^2}{252^2 - 248^2} = \frac{1000^2}{(252-248)(252+248)} = \frac{1000^2}{4(500)} = 500\,.$$

Problem 2

Multiply by a "form of one" given by $\frac{xy}{xy}$ to write this fraction as

$$\frac{x - \frac{1}{y}}{y - \frac{1}{x}} \times \frac{xy}{xy} = \frac{x^2y - x}{xy^2 - y} = \frac{x(xy-1)}{y(xy-1)} = \frac{x}{y}\,.$$

Problem 3

From the unique factorization theorem we have that we can write n as

$$n = 11^p \cdot 13^q \cdot 17^l \cdots\,.$$

Small value of n correspond to small values of p, q, l, etc. If we take $p = q = 1$ (and all other powers zero) we get $n = 143$. If we take $p = 2$, $q = 0$, and again all other powers zero we get $n = 121$. This is in the range $120 < n \leq 130$.

Problem 4

Draw perpendicular bisectors from the center of the circle bisecting the segments EF and BC. Let the intersection of these bisectors with EF be denoted Q and the intersection of the bisectors with BC be denoted P. Then as $BC = 5$ we have $BP = \frac{5}{2}$. Note that PQ is parallel AD and DQ is parallel to AP so we have

$$DQ = AP = 4 + 2.5 = 6.5\,,$$

and

$$DQ = DE + EQ = 3 + EQ\,.$$

Setting these two expressions equal gives $EQ = 3.5$. Using that we have $EF = 2EQ = 7$.

Problem 5

Apply the function $f(x) = \log_5(x)$ to both sides of the given expression to get

$$200 \log_5(n) < 300\,,$$

or

$$\log_5(n) < \frac{3}{2} = 1.5\,.$$

This means that we want the largest integer n such that

$$n < 5^{1.5} = 5^{1+\frac{1}{2}} = 5\sqrt{5}\,,$$

or

$$n^2 < 25 \cdot 5 = 125\,.$$

If $n = 10$ we have $n^2 = 100$. If $n = 11$ we have $n^2 = 121$. If $n = 12$ we have $n^2 = 144 > 125$. Thus we should take $n = 11$.

Problem 6

From the given statement we have $b = 3g$ and $g = 9t$. From these we have that $b = 3(9t) = 27t$. The requested sum in terms of teachers is then

$$b + g + t = 27t + 9t + t = 37t\,.$$

In terms of boys (using $t = \frac{b}{27}$) this is

$$\frac{37}{27}b\,.$$

Problem 7

Let s_d, l_d, and t_d be the speed (in steps per minute), length of a step, and total time walking for Dave. From the problem we have that

$$\begin{aligned} s_d &= 90 \quad \text{steps/minute} \\ l_d &= 75 \quad \text{cm} \\ t_d &= 16 \quad \text{minutes}\,. \end{aligned}$$

The same thing for Jack gives

$$\begin{aligned} s_j &= 100 \quad \text{steps/minute} \\ l_j &= 60 \quad \text{cm}\,, \end{aligned}$$

where t_j is unknown. Then since the distance to the school is the same for both students we must have

$$s_j t_j l_j = s_d t_d l_d\,.$$

Putting in numbers and solving I get $t_j = 18$ minutes.

Problem 8

Drop perpendiculars from A and B (called A' and B' respectively where they intersect the segment DC). Then the length of BB' is given by

$$BB' = 3\sqrt{2}\cos\left(\frac{\pi}{4}\right) = 3 = B'C = AA'$$

Lets now try to compute the length of DA'. From the given figure we have

$$\tan\left(\frac{\pi}{3}\right) = \frac{AA'}{DA'} = \frac{3}{DA'}\,.$$

Recalling that $\tan\left(\frac{\pi}{3}\right) = \sqrt{3}$ we can compute that $DA' = \frac{3}{\sqrt{3}} = \sqrt{3}$. Using what we have computed we can now find that

$$DC = DA' + A'B' + B'C = \sqrt{3} + 5 + 3 = 8 + \sqrt{3}\,.$$

Problem 9

Lets call this number v. Then by manipulating v we have

$$\begin{aligned} v &= 4^{16}\cdot 5^{25} = (4\cdot 5)^{16}\cdot 5^9 = (20)^{16}\cdot 5^9 = 2^{16}\cdot 10^{16}\cdot 5^9 \\ &= (2\cdot 5)^9\cdot 2^7\cdot 10^{16} = 2^7\cdot 10^{25}\,. \end{aligned}$$

Now recall that $2^7 = 128$ so

$$v = 128\cdot 10^{25}\,,$$

which is the number 128 followed by 25 zeros. This number will then have $3+25 = 28$ digits in it.

Problem 10

Plotting the given three points in the complex plane and we can see that the fourth point should be at $+2 - i$.

Problem 11

One step of the algorithm given is

$$f(x) = (x^2)^{-1} = x^{-2}\,.$$

Then n steps of this algorithm is $f(x)$ composed with itself n times. Two steps would be

$$f^{(2)}(x) = f(f(x)) = f(x^{-2}) = (x^{-2})^{-2} = x^{(-2)^2}\,.$$

Three steps would be

$$f^{(3)}(x) = f(f^{(2)}(x)) = f(x^{(-2)^2}) = (x^{(-2)^2})^{-2} = x^{(-2)^3}\,.$$

From the pattern above we see that n steps of this algorithm would be

$$f^{(n)}(x) = x^{(-2)^n}\,.$$

Problem 12

From the given recurrence we can write

$$a_{n+1} - a_n = 2n\,.$$

Summing both sides from $n = 1$ to $n = N$ and using the fact that the sum on the left-hand-side is telescoping we have

$$a_{N+1} - a_1 = 2\sum_{n=1}^{N} n = 2\left(\frac{(N+1)N}{2}\right) = N(N+1)\,.$$

If we take $N = 99$ we get

$$a_{100} = 2 + 99(100) = 9902\,.$$

Problem 13

Multiply by the "form of one" given by

$$\frac{(\sqrt{2}+\sqrt{3})-\sqrt{5}}{(\sqrt{2}+\sqrt{3})-\sqrt{5}}\,,$$

to write our fraction as

$$\begin{aligned}\frac{2\sqrt{6}}{\sqrt{2}+\sqrt{3}+\sqrt{5}} \times \frac{(\sqrt{2}+\sqrt{3})-\sqrt{5}}{(\sqrt{2}+\sqrt{3})-\sqrt{5}} &= \frac{2\sqrt{6}(\sqrt{2}+\sqrt{3}-\sqrt{5})}{(\sqrt{2}+\sqrt{3})^2-5} \\ &= \frac{2\sqrt{6}(\sqrt{2}+\sqrt{3}-\sqrt{5})}{2+3+2\sqrt{6}-5} \\ &= \sqrt{2}+\sqrt{3}-\sqrt{5}\,.\end{aligned}$$

Problem 14

If we take $f(x) = \log_{10}(x)$ of both sides we get

$$\log_{10}(x)^2 = 1\,.$$

This means that $\log_{10}(x) = \pm 1$ so $x = 10^{-1}$ or $x = 10$. The product of these two roots is one.

Problem 15

Write this expression as

$$\cos(2x)\cos(3x) - \sin(2x)\sin(3x) = 0\,,$$

or

$$\cos(2x + 3x) = 0\,.$$

This means that $\cos(5x) = 0$ so

$$5x = (2n+1)\frac{\pi}{2}\,,$$

for n an integer. With $n = 0$ and π as 180 degrees we have $x = 18$ degrees.

Problem 16

From the statement that $f(2 + x) = f(2 - x)$ we can conclude that the function $f(x)$ is symmetric across $x = 2$. If we take $x \to x - 2$ in that expression we see that

$$f(x) = f(2 - (x - 2)) = f(4 - x)\,.$$

From the above, if x is a root of f there will be another root with value $4 - x$. Thus if we have four roots of $f(x)$ they must be of the form x_1 and $4 - x_1$ and x_2 and $4 - x_2$. The sum of these four roots is eight.

Problem 17

Let the length of AH be denoted x, the length CH be denoted y, and the length of CB be denoted z. Then the area of $\triangle ABC$ is

$$\frac{1}{2}y(x + 16)\,.$$

Using the Pythagorean theorem in $\triangle AHC$ we have

$$x^2 + y^2 = 15^2\,, \tag{329}$$

in $\triangle HBC$ we have

$$y^2 + 16^2 = z^2\,, \tag{330}$$

and finally in $\triangle ABC$ we have

$$15^2 + z^2 = (x + 16)^2\,. \tag{331}$$

These are three equations and three unknowns. Putting Equation 330 into Equation 331 expanding and simplifying we get

$$y^2 = x^2 + 32x - 15^2\,.$$

If we put this into Equation 329 and simplify we get

$$x^2 + 16x - 15^2 = 0 .$$

This has the solutions $x = 9$ and $x = -25 < 0$. Only the positive root could be valid so taking $x = 9$ in Equation 329 gives $y^2 = 144$ so $y = 12$. Then the area of $\triangle ABC$ is

$$\frac{1}{2}(12)(9 + 16) = 6 \cdot 25 = 150 .$$

Problem 18

For the point (x, y) to be equally distant from the x and y axis means that it must be on the line $y = x$ or the line $y = -x$.

To be equidistant from the line $y = 2 - x$ means that the point (x, y) must also be on a line parallel to this line. These points must satisfy $y = b - x$ for some b. Using geometry where we draw the lines $y = 2 - x$ and $y = b - x$ we can determine that a point on that second line will be

$$\frac{|b - 2|}{\sqrt{2}} ,$$

away from the line $y = 2 - x$.

If we take $x = y$ in $y = b - x$ we have that $x = y = \frac{b}{2}$. Then as this is also the distance from that point to the x and y-axis to have this point be an equal distance to the line $y = 2 - x$ we must have

$$\frac{b}{2} = \frac{|b - 2|}{\sqrt{2}} .$$

To solve for b we first consider if $b > 2$. Then in that case we need to solve

$$\frac{b}{2} = \frac{b - 2}{\sqrt{2}} .$$

This has the solution

$$b = \frac{2\sqrt{2}}{\sqrt{2} - 1} .$$

Note that for this solution we have $b > 2$ so we have a consistent solution. This means that

$$x = y = \frac{b}{2} = \frac{\sqrt{2}}{\sqrt{2} - 1} .$$

If we multiply by the "form of one"

$$\frac{\sqrt{2} + 1}{\sqrt{2} + 1} ,$$

we can write x and y as

$$x = y = \frac{\sqrt{2}(\sqrt{2} + 1)}{2 - 1} = 2 + \sqrt{2} .$$

If instead we assumed that $b < 2$ then we must solve

$$\frac{b}{2} = \frac{-b+2}{\sqrt{2}}\,.$$

This has the solution

$$b = \frac{2}{\left(1 + \frac{1}{\sqrt{2}}\right)} = \frac{2\sqrt{2}}{1+\sqrt{2}}\,.$$

Note that from the above expression we have $b < 2$ and have found another consistent solution. With this solution we can write x and y as

$$x = y = \frac{b}{2} = \frac{\sqrt{2}(\sqrt{2}+1)}{2-1} = 2 + \sqrt{2}\,.$$

We can do the same arguments as above but requiring that $y = -x$. Doing this gives us two more solutions. In either case there are more than one solution and x is not uniquely determined.

Problem 19

In the integers given, there are six odd numbers i.e. $1, 3, 5, 7, 9, 11$ and five even numbers $2, 4, 6, 8, 10$. As sum of six balls will be odd if the number of odd balls drawn is itself odd. We can do this if we draw 1 odd ball (5 even balls), 3 odd balls (3 even balls), or 5 odd balls (1 even ball).

If we let the odd balls be denoted as "special" then the number of odd balls drawn d from eleven when there are six odd ball is given by a hypergeometric random variable

$$P(d) = \frac{\binom{6}{d}\binom{5}{6-d}}{\binom{11}{6}} \quad \text{for} \quad 1 \le d \le 6\,.$$

To have the sum be odd will happen if $d \in \{1, 3, 5\}$ which happens with a probability of

$$P(1) + P(3) + P(5) = \frac{118}{231}\,,$$

when we expand and simplify.

Problem 20

From the given expression we have that

$$x - |2x + 1| = \pm 3\,.$$

Now if $2x + 1 \geq 0$ (or $x \geq -\frac{1}{2}$) then the above expression is

$$x - (2x + 1) = \pm 3 \,.$$

This has two solutions $x = -4$ and $x = 2$. Only the solution $x = 2$ is larger than $-\frac{1}{2}$.

If $2x + 1 < 0$ (or $x < -\frac{1}{2}$) then the above expression is

$$x + (2x + 1) = \pm 3 \,.$$

This has two solutions $x = -\frac{4}{3}$ and $x = \frac{2}{3}$. Only the solution $x = -\frac{4}{3}$ is less than $-\frac{1}{2}$.

Thus this expression has two solutions.

Problem 21

Write these two equations as

$$\begin{aligned} b(a + c) &= 44 \\ c(a + b) &= 23 \,. \end{aligned}$$

Note that the number 23 is prime so that we must have $c = 23$ and $a + b = 1$ or $c = 1$ and $a + b = 23$. In the first case we cannot have $a + b = 1$ and have a and b be positive integers. As we have learned that $c = 1$ the first expression then gives

$$b(a + 1) = 44 \,.$$

Using this with $a + b = 23$ (or $b = 23 - a$) we can solve for a and b. The expression for a is quadratic and has solutions $a = 1$ and $a = 21$. Using $b = 23 - a$ if $a = 1$ then $b = 22$ and if $a = 21$ then $b = 2$. Thus the solutions (a, b, c) take the form

$$(1, 22, 1) \quad \text{or} \quad (21, 2, 1) \,.$$

Problem 22

The vertex of the parabola will happen when

$$\frac{dy}{dx} = 2ax + t = 0 \quad \text{so} \quad x_t = -\frac{t}{2a} \,.$$

Putting this in the expression for the parabola $y = ax^2 + tx + c$ we get

$$y_t = c - \frac{t^2}{4a} \,.$$

Now if we plot this point

$$\left(-\frac{t}{2a}, c - \frac{t^2}{4a} \right) \,,$$

as a function of t note that as $t = -2ax_t$ we have that $y_t = c - ax_t^2$ and so this point is

$$\left(x_t, c - ax_t^2 \right) \,,$$

which represents a parabola.

Problem 23

From the equations

$$\sin(x+y) = \sin(x)\cos(y) + \sin(y)\cos(x)$$
$$\sin(x-y) = \sin(x)\cos(y) - \sin(y)\cos(x)\,,$$

if we add these we get

$$\sin(x)\cos(y) = \frac{1}{2}(\sin(x+y) + \sin(x-y))\,.$$

Now if we take

$$u = x+y$$
$$v = x-y\,,$$

we have

$$x = \frac{1}{2}(u+v)$$
$$y = \frac{1}{2}(u-v)\,,$$

so we get

$$\sin(u) + \sin(v) = 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)\,. \tag{332}$$

Next using

$$\cos(x+y) = \cos(x)\cos(y) - \sin(y)\sin(x)$$
$$\cos(x-y) = \cos(x)\cos(y) + \sin(y)\sin(x)\,,$$

if we add these we get

$$\cos(x)\cos(y) = \frac{1}{2}(\cos(x+y) + \cos(x-y))\,.$$

In terms of u and v (defined above) this is

$$\cos(u) + \cos(v) = 2\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)\,. \tag{333}$$

Using Equation 332 we thus have

$$\sin(10) + \sin(20) = 2\sin(15)\cos(5)\,.$$

Using Equation 332 we thus have

$$\cos(10) + \cos(20) = 2\cos(15)\cos(5)\,.$$

Thus the ratio is given by

$$\frac{\sin(15)}{\cos(15)} = \tan(15)\,.$$

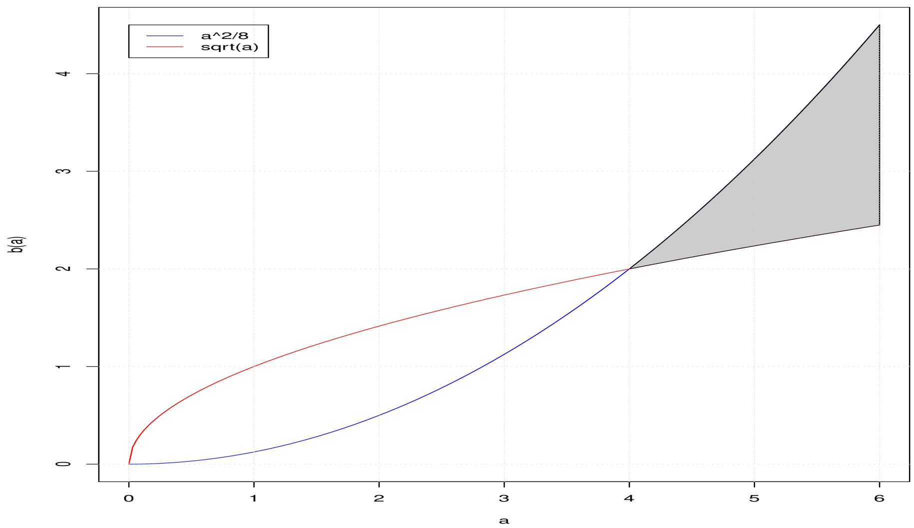

Figure 11: A plot of the region of admissible a and b.

Problem 24

To have real roots the discriminant must be positive. This means that

$$a^2 - 4(2b) > 0 \quad \text{or} \quad b < \frac{1}{8}a^2 \,,$$

and

$$(2b)^2 - 4a > 0 \quad \text{or} \quad b^2 > a \,.$$

This last inequality is equivalent to $|b| > \sqrt{a}$ but as b must be positive we must have $b > \sqrt{a}$.

Drawing these two regions in the (a, b) Cartesian plane we get Figure 11. Note that there is a region in the upper right that is satisfied by both inequalities. To make $a + b$ as small as possible we will make each number as small as possible. This means we need to find (a, b) at the intersection of the two curves above. Thus we need to solve

$$b = \frac{1}{8}a^2$$
$$b = \sqrt{a} \,.$$

Doing this we get $(a, b) = (4, 2)$ so that $a + b = 6$.

Problem 25

If we draw a general rectangular box with length, width, and height given by l, w, and h respectively then the first condition (the total area of all faces) is that

$$2hw + 2wl + 2lh = 22 \quad \text{or} \quad hw + wl + lh = 11 \,.$$

The second condition (the length of all edges) is

$$4l + 4w + 4h = 24 \quad \text{or} \quad l + w + h = 6\,.$$

If we square this last expression we have

$$36 = (l + w + h)^2 = l^2 + w^2 + h^2 + 2lw + 2lh + 2wh\,.$$

In terms of d the length of an internal diagonal this is

$$36 = d^2 + 2(lw + lh + wh) = d^2 + 2(11)\,.$$

Solving for d we get $d = \sqrt{14}$.

Problem 27

To start this problem let $AD = x$ and $BF = y$. Then using the Pythagorean theorem in $\triangle ABC$ we have

$$AB^2 + (x + 1)^2 = (y + 1)^2\,. \tag{334}$$

Using the Pythagorean theorem in $\triangle ABD$ we have

$$AB^2 + x^2 = 1^2\,.$$

If we subtract these two equations we get

$$(x + 1)^2 - x^2 = (y + 1)^2 - 1\,,$$

or

$$2x + 1 = y^2 + 2y\,. \tag{335}$$

Now using the Pythagorean theorem in $\triangle AFC$ we have

$$AF^2 + 1^2 = (1 + x)^2\,.$$

Using the Pythagorean theorem in $\triangle BFA$ we have

$$AF^2 + y^2 = AB^2\,.$$

If we subtract these two equations we get

$$1 - y^2 = (x + 1)^2 - AB^2\,.$$

If we use Equation 334 to replace AB we get

$$1 - y^2 = (1 + x)^2 - (1 - x^2) = 2x + 2x^2\,,$$

when we simplify.

As this point we have the two equations

$$\begin{aligned} y^2 + 2y &= 2x + 1 \\ -y^2 &= 2x^2 + 2x - 1 \,, \end{aligned} \tag{336}$$

which we can solve for x and y. To do that we add these two equations to get

$$y = x^2 + 2x \,,$$

and put that into Equation 336 which gives the polynomial

$$x^4 + 4x^3 + 6x^2 + 2x - 1 = 0 \,.$$

From the above if we try $x = -1$ we see that it is a solution so we can factor $x+1$ from the above to get

$$(x + 1)(x^3 + 3x^2 + 3x - 1) = 0 \,.$$

As we know that the solution we are looking for is not $x = -1$ we need to consider the cubic polynomial. It was difficult for me to find the roots of this polynomial by hand. One method to get a solution to the given problem however is to note that what we are looking for is the length $AC = 1 + AD = 1 + x$. Thus lets write the cubic polynomial above in terms of $v \equiv x + 1$. In that case we would be considering

$$(v - 1)^3 + 3(v - 1)^2 + 3(v - 1) - 1 = 0 \,.$$

Expanding and simplifying the left-hand-side we get

$$v^3 - 2 = 0 \,.$$

Thus $v = \sqrt[3]{2}$.

Problem 28

Write the second equation as

$$\sqrt{1984} - \sqrt{y} = \sqrt{x} \,.$$

Note that if $y = 1984$ and $x = 0$ then this equation is satisfied and that no $y \geq 1984$ is allowed as if $y \geq 1984$ the left-hand-side of this equation would be negative or zero and for the integer solutions we seek we must have $\sqrt{x} \geq 1$.

If we square both sides of the above we get

$$1984 - 2\sqrt{1984y} + y = x \,,$$

or

$$2\sqrt{1984y} = 1984 + y - x = 1984 + (y - x) > 1984 \,.$$

Note that for integer solutions x and y the expression $1984 + (y - x)$ will also be an integer. Now as

$$1984 = 2^6 \cdot 31 \,,$$

we have

$$\sqrt{1984} = 2^3 \cdot \sqrt{31}\,.$$

This means that we can write the condition above as

$$2^4\sqrt{31y} = 2^6 \cdot 31 + y - x\,.$$

In this the left-hand-side will be irrational while the right-hand-side will be an integer unless y is a multiple of 31 and a perfect square i.e. $y = 31\beta^2$ with β an integer and $\beta \geq 1$. In that case we have

$$2^4 \cdot 31\beta = 2^6 \cdot 31 + 31\beta^2 - x\,.$$

If we solve for x in the above we get

$$x = 31(\beta^2 - 16\beta + 64) = 31(\beta - 8)^2\,.$$

Now we also need to have $x < y$ which means that

$$31(\beta - 8)^2 < 31\beta^2\,.$$

We can expand this to get that $\beta > 4$.

From earlier we also know that we need to have $y < 1984 = 2^6 \cdot 31$ which means that

$$31\beta^2 < 2^6 \cdot 31\,.$$

This is equivalent to

$$\beta < 8\,.$$

The integer values for β satisfying both of these conditions i.e. $4 < \beta < 8$ are $\beta \in \{5, 6, 7\}$ and so there are three integer solutions. These are given by

$$(x, y) = (31(\beta - 8)^2, 31\beta^2)\,,$$

for the above β.

Problem 29

The largest value of $m = \frac{y}{x}$ will be when the line $y = mx$ is tangent to the circle in its North-West corner. By drawing this circle in the Cartesian coordinate plane we see that this line intersects the circle in an upwards sloping direction in the first quadrant. Let this point of intersection be denoted A, let the origin of the Cartesian coordinate plane be denoted C and let the center of the circle be denoted as O. Draw a segment from the center of the circle O to A and from O to C forming a triangle. As the line $y = mx$ is tangent to the circle we have that $\angle CAO = \frac{\pi}{2}$. Then as AO is a radius of the circle we have $|AO| = \sqrt{6}$. We also compute that

$$|CO| = \sqrt{3^2 + 3^2} = 3\sqrt{2}\,.$$

Given that we have a right triangle we can use the Pythagorean theorem to compute $|AC|$ we find

$$|AC|^2 = |CO|^2 - |AO|^2 = 18 - 6 = 12\,.$$

Thus $|AC| = 2\sqrt{3}$.

Let the Cartesian coordinates of the point A be denoted as (p, q). Then since A is on the original circle and at a known distance from the origin (the point O) we have that

$$\begin{aligned}(p-3)^2 + (q-3)^2 &= 6 \\ p^2 + q^2 &= 12\,.\end{aligned}$$

If I expand the first equation and then put in the second equation I get

$$p + q = 4 \quad \text{so} \quad p = 4 - q\,.$$

Putting this into $p^2 + q^2 = 12$ gives

$$q^2 - 4q + 2 = 0\,.$$

This has the two solutions $q = 2 \pm \sqrt{2}$. From the location of the point A we must have $q > 3$ so we take the positive sign and conclude that $q = 2 + \sqrt{2}$. With this we have $p = 4 - q = 2 - \sqrt{2}$. Using these expressions we have that

$$m = \frac{q}{p} = 3 + 2\sqrt{2}\,,$$

when we simplify.

Problem 30

The sum we want to evaluate is

$$\sum_{k=1}^{N} k w^k\,,$$

for $w = e^{i\frac{2\pi}{9}}$ and $N = 9$. Now from standard references we have that

$$\sum_{k=1}^{N} k w^{k-1} = \frac{1 - w^{N+1}}{(1-w)^2} - \frac{(N+1)w^N}{(1-w)}\,, \tag{337}$$

so that the sum we want is

$$\sum_{k=1}^{N} k w^k = \frac{w(1 - w^{N+1})}{(1-w)^2} - \frac{(N+1)w^{N+1}}{(1-w)}\,.$$

Now as $N = 9$ we have that $w^N = 1$ so $w^{N+1} = w$ and the right-hand-side of the above becomes

$$\sum_{k=1}^{9} k w^k = \frac{9w}{w-1}\,.$$

As $|w| = 1$ we have

$$\left|\sum_{k=1}^{9} k w^k\right| = \frac{9}{|w-1|} = \frac{9}{|\cos(40) - 1 + i\sin(40)|}$$
$$= \frac{9}{\sqrt{(\cos(40)-1)^2 + \sin^2(40)}} = \frac{9}{\sqrt{1 - 2\cos(40) + 1}}$$
$$= \frac{9}{\sqrt{2(1-\cos(40))}}\,.$$

Now using the fact that

$$\sin^2(\theta) = \frac{1}{2}(1 - \cos(2\theta)) \quad \text{or} \quad 1 - \cos(2\theta) = 2\sin^2(\theta)\,,$$

we have that

$$1 - \cos(40) = 2\sin^2(20)\,,$$

and we get

$$\left|\sum_{k=1}^{9} k w^k\right| = \frac{9}{2\sin(20)}\,,$$

from this we can determine the inverse of this expression and the answer.

The 1984 AIME Examination

Problem 1

From the problem description for a_n we know that $a_n = a_0 + n$ for $n \ge 0$ and a_0 unknown (for now). From the given sum we have

$$137 = a_1 + a_2 + \cdots + a_{97} + a_{98}$$
$$= 98a_0 + \sum_{k=1}^{98} k = 98a_0 + \frac{98(99)}{2}\,.$$

Solving the above for a_0 I find $a_0 = -\frac{4714}{98}$. Then

$$\sum_{k=1}^{49} a_{2k} = 49a_0 + \sum_{k=1}^{49} 2k = 40\left(-\frac{4714}{98}\right) + 2\left(\frac{50(49)}{2}\right) = 93\,,$$

when we simplify.

Problem 2

Note that $n = 15c = 3 \times 5 \times c$ for some c. As n is a multiple of five it must end in a zero or a five. As all of the digits in n are zero or eight it cannot end in a five and it must end in a zero. As three is a divisor of n the sum of the digits of n must be divisible by three. To make n as small as possible we would need to place at three eights as close to the units digit as possible. This gives the number 8880. Dividing that by fifteen we get 592.

Problem 4

Here $S = \{x_i\}_{i=1}^{n}$ and without loss of generality lets take $x_1 = 68$. Then we are told that

$$\frac{1}{n}\sum_{i=1}^{n} = 56\,,$$

or

$$\frac{1}{n}\left(68 + \sum_{i=2}^{n} x_i\right) = 56\,. \tag{338}$$

We are also told that

$$\frac{1}{n-1}\sum_{i=2}^{n} x_i = 55\,,$$

which means that $\sum_{i=2}^{n} x_i = (n-1)55$. Putting this into Equation 338 gives

$$\frac{1}{n}(68 + (n-1)55) = 56 \quad \text{so} \quad n = 13\,.$$

Thus we have now learned that

$$\sum_{i=1}^{13} x_i = 56(13) = 728$$
$$\sum_{i=2}^{13} x_i = 55(12) = 660\,.$$

As all of the numbers $\{x_i\}_{i=2}^{13}$ are $x_i \geq 1$ and their sum is 660 we can make one of these x_i as large as possible by making all of the others as small as possible. As there are twelve numbers in that sum we can do this if we take eleven of the numbers equal to one and then one number equal to

$$660 - 11(1) = 649\,.$$

Problem 5

Write the first equation in terms of a common logarithm as

$$\frac{\log(a)}{\log(8)} + 2\frac{\log(b)}{\log(4)} = 5\,.$$

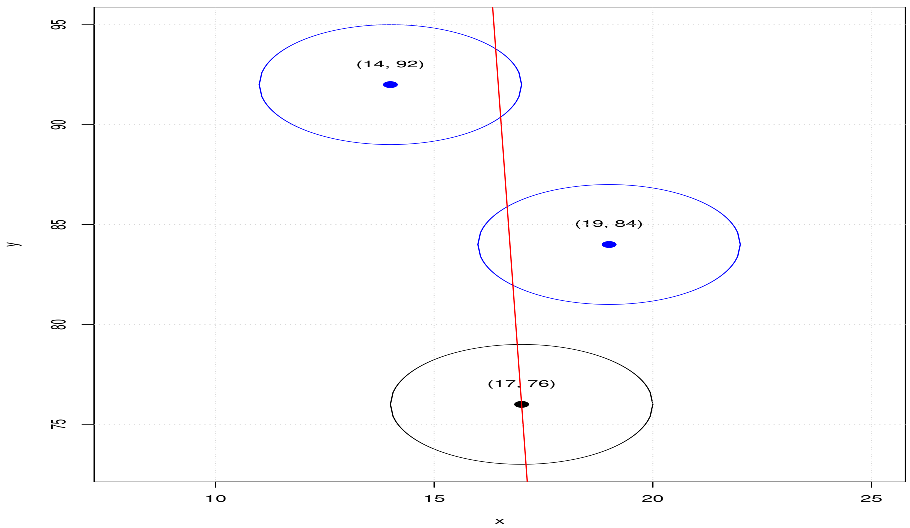

Figure 12: Plots of the three circles in this problem and a line that comes close to bisecting the desired areas.

Doing the same thing for the second equation I find

$$\frac{\log(b)}{\log(8)} + 2\frac{\log(a)}{\log(4)} = 7\,. \tag{339}$$

If we *subtract* these two equations we get

$$\frac{\log(a/b)}{\log(8)} + \frac{2\log(b/a)}{\log(4)} = -2\,.$$

Using the fact that $\log(b/a) = -\log(a/b)$ we can solve for $\log(a/b)$ in the above equation where I find

$$\log(a/b) = 3\log(2) = \log(8)\,.$$

This means that $\frac{a}{b} = 8$ or $a = 8b$. If we put this into Equation 339 I get

$$\frac{\log(b)}{\log(8)} + \frac{2(\log(8) + \log(b))}{\log(4)} = 7\,.$$

If we solve the above for $\log(b)$ I find $\log(b) = 3\log(2) = \log(8)$. Thus $b = 8 = 2^3$ and $a = 64 = 2^6$. Using this means that $ab = 2^9 = 512$.

Problem 6

In Figure 12 I have drawn the three circles from the problem and a line (in red) that looks like it might divide the areas of the two circles into two equal parts. This gives an idea about the type of the line we are looking for. Note that every line through $(17, 76)$ will divide the

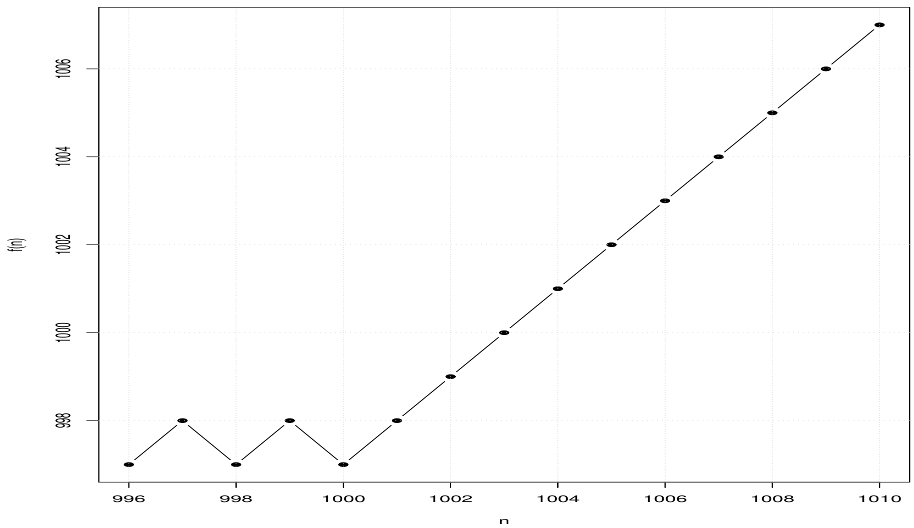

Figure 13: A plot of the function $f(n)$ vs. n.

area of the circle centered at $(17, 76)$ into two equal parts and so we don't need to worry about the area of that circle.

To make sure that the red line divides the areas of the other circles equally note that the center of mass of these two circles is the point $(16.5, 88)$ and that any line though this point will divide them into equal parts. Thus if the red line goes though the two points $(17, 76)$ and $(16.5, 88)$ it will have the desired property. A line of this type will have a slope of

$$m = \frac{88 - 76}{16.5 - 17} = -24\,.$$

Problem 7

As it seems the "form" of $f(n)$ changes for n around 1000 lets compute by hand several values of $f(n)$ in that region and see if we can determine a pattern. We have

$$\begin{aligned}
f(999) &= f(f(1004)) = f(1001) = 1001 - 3 = 998 \\
f(998) &= f(f(1003)) = f(1000) = 997 \\
f(997) &= f(f(1002)) = f(999) = 998 \\
f(996) &= f(f(1001)) = f(998) = 997 \\
f(995) &= f(f(1000)) = f(997) = 998 \\
f(994) &= f(f(999)) = f(998) = 997 \\
f(993) &= f(f(998)) = f(997) = 998\,.
\end{aligned}$$

From these results it looks like when $n \le 1000$ we have

$$f(n) = \begin{cases} 998 & n \text{ is odd} \\ 997 & n \text{ is even} \end{cases}\,. \tag{340}$$

A plot of this function is given in Figure 13. If this is true then we have $f(84) = 997$.

To prove that the above is true we start by asking for what values of n is $f(n) = 998$. Note that above we have shown that when

$$n \in \{993, 995, 997, 999\},$$

we have $f(n) = 998$.

If we also ask for what values of n is $f(n) = 997$. Note that above we have shown that when

$$n \in \{996, 998, 1000\},$$

we have $f(n) = 997$.

We will now prove the above expression for $f(n)$ in Equation 340 by induction for $n < 995$ (we computed it exactly when $993 \leq n \leq 1000$). Let n be odd and $n < 995$. In that case we have $n + 5 < 1000$ and if n is odd then $n + 5$ is even so

$$f(n) = f(f(\text{even})) = f(997) = 998\,.$$

Now let n be even and $n < 995$. In that case we have $n + 5 < 1000$ and $n + 5$ is odd so we have

$$f(n) = f(f(\text{odd})) = f(998) = 997\,.$$

Problem 8

Let $v = z^3$ to get $v^2 + v + 1 = 0$. This last equation has the solutions

$$v = \frac{1}{2}(-1 \pm i\sqrt{3})\,.$$

To write these two solutions in polar form first note that $|v| = 1$. Thus we have two sets of solutions

$$z^3 = \frac{1}{2}(-1 - i\sqrt{3}) = e^{\frac{4}{3}\pi i}$$
$$z^3 = \frac{1}{2}(-1 + i\sqrt{3}) = e^{\frac{2}{3}\pi i}\,.$$

To take the cube root of the first expression we write it as

$$z^3 = e^{\frac{4}{3}\pi i + 2\pi n i}\,,$$

for $n \in \{0, 1, 2\}$. Then we have

$$z = e^{\frac{4}{9}\pi i + \frac{2}{3}\pi n i}\,.$$

In the same way, for the other possible solution we have

$$z = e^{\frac{2}{9}\pi i + \frac{2}{3}\pi n i}\,,$$

again for $n \in \{0, 1, 2\}$. If we evaluate all of these angular exponents over this range of n and then list them in increasing order (without the i) we get

$$\frac{2}{9}\pi\,, \frac{4}{9}\pi\,, \frac{8}{9}\pi\,, \frac{10}{9}\pi\,, \frac{14}{9}\pi\,, \frac{16}{9}\pi\,.$$

As $\frac{1}{2}\pi$ is 90 degrees and π is 180 degrees the angle above that is between these two limits is $\frac{8}{9}\pi$. This is

$$\frac{8}{9}(180) = 160\,,$$

degrees.

Problem 9

The volume of a tetrahedron is $V = \frac{1}{3}A_0 h$ where A_0 is the area of a "base" and h is the height from the base to the vertex not in that base. If we take A_0 to be the area of face ABC we need to determine the value of h (since we are told $A_0 = 15$).

For visualization lets draw the tetrahedron with face ABC in the x-y plane such that the segment AB runs along the x-axis (A is at $(0, 0)$ and B is at $(3, 0)$), the point C is in the fourth quadrant (has a positive x coordinate and a negative y coordinate). With this configuration the point D is above the x-y plane.

As we are told that the area of triangle ABD is twelve and a possible "base" of this triangle is $AB = 3$ the height of this triangle from that base must be given by

$$\frac{1}{2}(3)h_{\text{ABD}} = 12 \quad \text{or} \quad h_{\text{ABD}} = 8\,.$$

If we draw this height on the face ABD (from the segment AB to the vertex at D) then since the angle between the faces ABC and ABD is thirty degrees when we drop a perpendicular from D to the face ABC we form a right triangle with a hypotenuse of length $h_{\text{ABD}} = 8$, a vertical leg (which is also the height of the tetrahedron), and an angle of 30 degrees between the hypotenuse and the leg of this triangle in the ABC face.

Using trigonometry we then have that

$$h = h_{\text{ABD}} \sin\left(\frac{\pi}{6}\right) = 8\left(\frac{1}{2}\right) = 4\,.$$

This means that the volume of the tetrahedron is given by

$$\frac{1}{3}(15)(4) = 20\,.$$

Value of c	Score	Range of n	Range of s
14	$70+n$	$0 \le n \le 16$	$70 \le s \le 86$
15	$75+n$	$0 \le n \le 15$	$75 \le s \le 90$
16	$80+n$	$0 \le n \le 14$	$80 \le s \le 94$
17	$85+n$	$0 \le n \le 13$	$85 \le s \le 98$
18	$90+n$	$0 \le n \le 12$	$90 \le s \le 102$
19	$95+n$	$0 \le n \le 11$	$95 \le s \le 106$
20	$100+n$	$0 \le n \le 10$	$100 \le s \le 110$
21	$105+n$	$0 \le n \le 9$	$105 \le s \le 114$
22	$110+n$	$0 \le n \le 8$	$110 \le s \le 118$
23	$115+n$	$0 \le n \le 7$	$115 \le s \le 122$
24	$120+n$	$0 \le n \le 6$	$120 \le s \le 126$
25	$125+n$	$0 \le n \le 5$	$125 \le s \le 130$
26	$130+n$	$0 \le n \le 4$	$130 \le s \le 134$
27	$135+n$	$0 \le n \le 3$	$135 \le s \le 138$
28	$140+n$	$0 \le n \le 2$	$140 \le s \le 142$
29	$145+n$	$0 \le n \le 1$	$145 \le s \le 146$
30	150	$n=0$	$s=150$

Table 9: The possible values for c and the corresponding values of Mary's score s.

Problem 10

The score on the math exam is defined as

$$s = 30 + 4c - w\,,$$

where c is the number of correctly answered questions and w is the number of wrong answers. As one can leave a question blank let n be the number of questions not answered. Then as there are thirty problems on the test we have

$$30 = c + w + n \quad \text{so} \quad w = 30 - c - n\,.$$

If we put this expression for w into the formula for s we find

$$s = 5c + n\,.$$

The numbers c and n are constrained to be integers such that $0 \le c \le 30$ and $0 \le n \le 30$. The upper bound of thirty is true as there are only thirty problems on the test. Tighter constraints on these numbers could be derived. We have now written Mary's score in terms of the sum of two nonnegative numbers. In addition, we are told that

$$s = 5c + n \ge 80\,.$$

From the above expression for s (in terms of c and n) we can gain some insight by letting c take different values and looking at the possible values of s for that range of n. Note that if we are given a value for c then the possible values for n are then $0 \le n \le 30 - c$. We do this in Table 9.

Using that table, we notice that for certain values of s if we know its value we can determine c exactly. For example, if $s = 150$ then we know that $c = 30$, if $145 \le s \le 146$ then we know that $c = 29$ and so forth. Also for some s a given score can be obtained in more than one way. For example if $s = 130$ we could have $c = 25$ with $n = 5$ or $c = 26$ with $n = 0$ and we can't determine c exactly. If one works towards smaller values of s one finds that for $s = 119$ there is a unique correct score $c = 23$ but for smaller s there is not.

Problem 11

The number of ways we can order all trees is given by

$$\frac{(3+4+5)!}{3!4!5!} = 27720\,,$$

which is obtained by imagining that each tree is unique and then recognizing that the three maple trees, the four oak trees, and the five birch trees are indistinguishable.

Next we will first count the number of ways we can get the desired situation (where no two birch trees are next to each other). To count this, note that we can put down the maple and the oak trees in

$$\frac{(3+4)!}{3!4!} = 35\,,$$

unique ways. Then with a given ordering of the seven trees just planted, we can place the birch trees in any of the eight locations (the spaces between each already planted tree and the spaces "at the end"). To make sure that we have no birch trees together once we "use up" a space by planting a tree there no other tree can go in that space. This gives

$$\frac{8\times 7\times 6\times 5\times 4}{5!} = 56\,,$$

ways to plant the five indistinguishable birch trees such that no two are next to each other. Thus the number of ways to plant all trees such that no two birch trees are adjacent is $35 \times 56 = 1960$.

The probability we seek is thus

$$\frac{1960}{27720} = \frac{7}{99}\,.$$

Thus we find $m + n = 99 + 7 = 106$.

Problem 12

This function $f(x)$ is symmetric across the lines $x = 2$ and $x = 7$. If we let $v = 2 + x$ (then $x = v - 2$) and the first relationship gives

$$f(v) = f(2-x) = f(2-(v-2)) = f(4-v)\,. \tag{341}$$

If we let $v = 7 + x$ (then $x = v - 7$) then in the same way the second relationship gives

$$f(v) = f(14 - v)\,. \tag{342}$$

It stands to reason that the "important" values of $f(x)$ will be when $2 < x < 7$. At least we can start by looking at the function $f(x)$ in that domain and see what conclusions we can draw. Note that if we take $x \in [2, 7]$ then using Equation 341 this interval maps to the interval $[-3, 2]$ and "flips". What I mean by this is that $x = 2$ maps to

$$4 - x = 4 - 2 = 2\,,$$

$x = 7$ maps to

$$4 - x = 4 - 7 = -3\,,$$

and as x moves from left to right in the domain $[2, 7]$ the variable $4 - x$ moves from right to left in the domain $[-3, 2]$. Thus we have concluded that $f(x)$ "looks the same" in the two intervals

$$[2, 7] \quad \text{and} \quad [-3, 2]\,.$$

If we map the interval $[-3, 2]$ under the transformation $x \to 4 - x$ we get back the interval $[2, 7]$.

In the same way, if we map the interval $[-3, 2]$ under the transformation $x \to 14 - x$ we get the interval $[7, 12]$. Thus we have concluded that $f(x)$ "looks the same" in the intervals

$$[2, 7] \quad \text{and} \quad [-3, 2] \quad \text{and} \quad [7, 12]\,.$$

Note that if we map the interval $[7, 12]$ under the transformation $x \to 14 - x$ we get back the interval $[2, 7]$.

In the above discussion each time we "map the interval" because we are applying a transformation of the form $x \to C - x$ the "direction" of $f(x)$ flips. What this means is that if $f(x)$ is increasing as x increases after the flip it is decreasing as x increases.

If we apply these transformations in alternative orders we can expand the interval $[2, 7]$ as far to the left and right as desired. For example, if we alternately apply $x \to 4 - x$ followed by $x \to 14 - x$ we get the intervals

$$[2, 7] \leftrightarrow [-3, 2] \leftrightarrow [12, 17] \leftrightarrow [-13, -8] \leftrightarrow \cdots\,.$$

If we alternately apply $x \to 14 - x$ followed by $x \to 4 - x$ we get the intervals

$$[2, 7] \leftrightarrow [7, 12] \leftrightarrow [-8, -3] \leftrightarrow [17, 27] \leftrightarrow \cdots\,.$$

We could apply these transformations "forever" effectively covering the entire real line with copies of the function for $x \in [2, 7]$. Based on the patterns above it looks like these intervals can all be written as

$$[2, 7] + 5m\,, \tag{343}$$

for $m \in \mathbb{Z}$. Because of the "flipping" of $f(x)$ if m is even, the direction of $f(x)$ is the same as in the original interval $[2, 7]$ while if m is odd the direction is the opposite.

Now if $x = 0$ is a root of $f(x)$ then this is in the interval given by Equation 343 for $m = -1$ and as we argued above this root will be in every interval for all m. The location of these mapped roots is then

$$5 + 5m\,,$$

for the same m as above $m \in \mathbb{Z}$. Two of these mapped zeros are the limits of the domain we are looking for roots over i.e. $x = \pm 1000$. When $m = 199$ the interval is

$$[997, 1002]\,,$$

and when $m = -201$ the interval is

$$[-1003, -998]\,.$$

Thus there are at least $199 - (-201) + 1 = 401$ roots in the domain $-1000 \leq x \leq 1000$. This is the minimum number of roots since there could be more roots than that number.

Problem 13

This problem is easy if one recalls

$$\cot(x + y) = \frac{\cot(x)\cot(y) - 1}{\cot(x) + \cot(y)}\,. \tag{344}$$

As the cotangent function is one-to-one this means that the above is equivalent to

$$x + y = \cot^{-1}\left(\frac{\cot(x)\cot(y) - 1}{\cot(x) + \cot(y)}\right)\,.$$

In this expression let

$$\begin{aligned} x &= \cot^{-1}(a) \\ y &= \cot^{-1}(b)\,, \end{aligned}$$

so that the above becomes

$$\cot^{-1}(a) + \cot^{-1}(b) = \cot^{-1}\left(\frac{ab - 1}{a + b}\right)\,. \tag{345}$$

Using this, we find the argument of cot in the expression we are given to be equal to

$$\cot^{-1}\left(\frac{21 - 1}{10}\right) + \cot^{-1}\left(\frac{273 - 1}{34}\right) = \cot^{-1}(2) + \cot^{-1}(8) = \cot^{-1}\left(\frac{16 - 1}{10}\right) = \cot^{-1}\left(\frac{3}{2}\right)\,.$$

This means that our expression is equal to

$$10\cot\left(\cot^{-1}\left(\frac{3}{2}\right)\right) = 15\,.$$

Problem 15

The problem statement is the same as the statement that

$$\frac{x^2}{t-1} + \frac{y^2}{t-9} + \frac{z^2}{t-25} + \frac{w^2}{t-49} = 1\,,$$

for $t \in \{4, 16, 36, 64\}$. If we clear the denominator we can write this as

$$x^2(t-9)(t-25)(t-49) + y^2(t-1)(t-25)(t-49) + z^2(t-1)(t-9)(t-49) + w^2(t-1)(t-9)(t-25)$$
$$= (t-1)(t-9)(t-25)(t-49)\,.$$

If we put everything on one side of the equality sign we get an expression

$$\begin{aligned} P(t) \equiv\, & (t-1)(t-9)(t-25)(t-49) \\ & - x^2(t-9)(t-25)(t-49) + y^2(t-1)(t-25)(t-49) \\ & - z^2(t-1)(t-9)(t-49) + w^2(t-1)(t-9)(t-25) = 0\,. \end{aligned}$$

This is a fourth order polynomial in t that vanishes at the four points listed above and thus by the uniqueness of polynomials must be *equal* to

$$(t-4)(t-16)(t-36)(t-64)\,.$$

If we subtract this expression from $P(t)$ (defined above) the result must be zero as these are two different forms for the same polynomial. The exact expression we get when we do this subtraction is helped by doing the algebra with `sympy` as

```
import sympy
from sympy import *

x, y, z, w, t = symbols('x y z w t')

# The first polynomial form:
P_t = (t-1)*(t-9)*(t-25)*(t-49) \
    - x**2 * (t-9)*(t-25)*(t-49) - y**2 * (t-1)*(t-25)*(t-49) \
    - z**2 * (t-1)*(t-9 )*(t-49) - w**2 * (t-1)*(t-9 )*(t-25)

# The second polynomial form:
PF = (t-4)*(t-16)*(t-36)*(t-64)

# Extract the coefficients of t (the right-hand-side) is zero:
p = poly(P_t - PF, t)
p.coeffs()
```

Running the above `python` code gives for the coefficient of t^3 the following expression

$$-w^2 - x^2 - y^2 - z^2 + 36\,.$$

As this must equal the value zero, we have just determined that the sum we seek has the value of 36.

The 1985 AHSME Examination

Problem 1

From this we have that $4x+2=16$ so that $4x+1=15$.

Problem 2

The fraction of the circle removed is $f=\frac{60}{360}=\frac{1}{6}$. The perimeter of the monster is then

$$C=2\pi r-\frac{1}{6}(2\pi r)+2r=2+\frac{5\pi}{3}\,.$$

when we take $r=1$ and simplify.

Problem 3

From the description we have that

$$MB=AB-AM=13-12=1\,,$$

and $BN=5$. Thus we see that $NM=BN-MB=5-1=4$.

Problem 4

Let p, d, and q be the number of each type of coin (for penny, dime, and quarter). Then we are told that

$$\begin{aligned} d&=2p \\ q&=3d=6p\,. \end{aligned}$$

The total amount of money M is then

$$M=p+10d+25q=p+20p+25(6)p=171p\,.$$

If we take $p=2$ we get 342.

Problem 5

Note that we can write the value of this sum S as

$$\frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}\right)=S\,.$$

This means that
$$2S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}\,.$$
The smallest common denominator of all of these fractions is
$$12 \cdot 5 = 60\,.$$
If we multiply by that number we get
$$2 \cdot 12 \cdot 5 \cdot S = 60 + 30 + 20 + 15 + 12 + 10 = 147\,.$$
This means that
$$S = \frac{147}{2 \cdot 5 \cdot 12} = \frac{49}{2 \cdot 5 \cdot 4} = \frac{7^2}{2^3 \cdot 5} = \frac{49}{40} = 1 + \frac{9}{40}\,.$$
Thus our sum S is $\frac{9}{40}$ larger than one. Note that
$$\frac{1}{4} + \frac{1}{5} = \frac{9}{20}\,.$$
So if we remove the terms
$$\frac{1}{2}\left(\frac{1}{4} + \frac{1}{5}\right) = \frac{1}{8} + \frac{1}{10}\,,$$
our sum should be one. Lets check that this is correct. Consider
$$\frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{6}\right) = 1\,,$$
as it should be.

Problem 6

Let b and g be the number of boys and girls in the class respectively. Then the probability we select a boy is
$$P_b = \frac{b}{b+g}\,,$$
and the probability we select a girl is
$$P_g = \frac{g}{b+g} = 1 - P_b\,.$$
We are told that that
$$P_b = \frac{2}{3}P_g = \frac{2}{3}(1 - P_b)\,.$$
Solving this for P_b we find $P_b = \frac{2}{5}$ which is the desired answer.

Problem 7

The given expression

$$a\boldsymbol{\nabla} \cdot b - c + d\,,$$

is evaluated first as

$$a\boldsymbol{\nabla} \cdot b - (c + d)\,,$$

and then as

$$\frac{a}{b - (c + d)}\,.$$

This is equivalent to $\frac{a}{b-c-d}$.

Problem 8

Starting with

$$ax + b = 0 \quad \text{and} \quad a'x + b' = 0\,,$$

we get two solutions

$$x = -\frac{b}{a} \quad \text{and} \quad x = -\frac{b'}{a'}\,.$$

For the problem statement we want

$$-\frac{b}{a} < -\frac{b'}{a'}\,,$$

or

$$\frac{b}{a} > \frac{b'}{a'}\,.$$

Problem 9

The fact that the even numbered rows are shifted one "column width" to the left does not matter in observing the pattern in that we can imagine each even numbered row shifted so that all columns of numbers "line up". From the pattern described in the problem the first row has the integers

$$2k + 1\,,$$

for $0 \le k \le 3$. The second row has numbers of this form for $4 \le k \le 7$ (but written from right to left). The third row has numbers of this form for $8 \le k \le 11$ and so on.

In general then for row r it looks like the values of k that make up that row are

$$4(r - 1) \le k \le 4r - 1\,.$$

The number 1985 has $k = \frac{1985-1}{2} = 992$. Using the above this will happen on row $r = 249$. As this number is odd the numbers in this row are increasing as we move to the right and 1985 will be in the first *position* from the right. As this is an odd row this is the second *column* of numbers (considering the shifting).

Problem 10

If we imagine making taking a circle very large then at any location on the circle the arc it cuts looks like a line which we can make intersect the graph $y = \sin(x)$ as many times as we like by making the circle arbitrarily large.

Problem 11

There are $2! = 2$ ways to order the two vowels O and E. The five other consonants (assumed distinct) can be ordered in $5! = 120$ ways. Thus the total number of ways to get words of the desired form (with the two Ts assumed distinct) is $2 \times 120 = 240$. As there are two Ts (assumed distinct that are actually not distinct) as distinct they can be arranged in $2! = 2$ ways this number over counts the total number of arrangements. Thus there are $\frac{240}{2} = 120$ ways to arrange the letters in the requested form.

Problem 12

The choices A and E are not perfect cubes. The choices B, C, and D have n as a divisor and D is the smallest of the three.

Problem 13

In the given figure draw a horizontal line connecting the left-most and right-most vertexes of the quadrilateral. Then dropping verticals to this horizontal line from the other two vertices I get the region divided up into four triangles which have bases and heights that are easy to determine. Denoting these triangles in the same way we would denote the four quadrants of the Cartesian axis I compute

$$\begin{aligned} A_I &= \frac{1}{2}(3)(2) = 3 \\ A_{II} &= \frac{1}{2}(1)(2) = 1 \\ A_{III} &= \frac{1}{2}(3)(1) = \frac{3}{2} \\ A_{IV} &= \frac{1}{2}(1)(1) = \frac{1}{2} \,. \end{aligned}$$

Adding these together I find the total area given by six.

Problem 14

If n are the number of sides of a polygon then the sum of the interior angles is $180(n-2)$. As we have three obtuse angles if we denote one of them by θ then we have that

$$90 < \theta < 180\,.$$

From our n interior angles we have $n-3$ of them that must be acute (i.e. less than 90 degrees). This means that we must have

$$180(n-2) < (n-3)90 + 3(180)\,.$$

This is equivalent to

$$n < 7\,.$$

Thus the largest n can be is six.

Problem 15

If we take the natural logarithm of both sides of the first equation we get

$$b\ln(a) = a\ln(b)\,.$$

Using the second equation $b = 9a$ to eliminate b from that expression gives

$$9a\ln(a) = a(\ln(a) + \ln(9))\,,$$

or

$$8a\ln(a) = a\ln(9)\,.$$

One solution to this equation is $a = 0$. If $a \neq 0$ then another solution is

$$a = 9^{1/8} = (9^{1/2})^{1/4} = 3^{1/4} = \sqrt[4]{3}\,.$$

Problem 16

Expand the given expression E to get

$$E \equiv 1 + \tan(A) + \tan(B) + \tan(A)\tan(B)\,.$$

Now recall that

$$\tan(x+y) = \frac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)}\,, \qquad (346)$$

so that

$$\tan(x) + \tan(y) = \tan(x+y)(1 - \tan(x)\tan(y))\,.$$

Using this above we find

$$E = 1 + \tan(A+B)(1 - \tan(A)\tan(B)) + \tan(A)\tan(B)\,.$$

Note that $A + B = 45$ degrees so $\tan(A+B) = 1$ and we get

$$E = 1 + (1 - \tan(A)\tan(B)) + \tan(A)\tan(B) = 2\,.$$

Problem 17

Moving from D to B, along the line segment DB I label the intersection of DB with the line L as the point E and the intersection of DB with the line L' as the point F. Along the segment DC I label the intersection with line L as the point G. I then denote the lengths $AD = h$, $DG = a$, and $GC = b$. Then using the **Pythagorean Theorem** we can derive several equations.

$$h^2 = 1^2 + AE^2 \quad \text{using the right triangle } AED \tag{347}$$
$$1^2 + EG^2 = a^2 \quad \text{using the right triangle } DEG \tag{348}$$
$$2^2 + FC^2 = (a+b)^2 \quad \text{using the right triangle } DFC \tag{349}$$
$$h^2 + a^2 = (AE + EG)^2 \quad \text{using the right triangle } ADG \tag{350}$$
$$(a+b)^2 + h^2 = 3^2 \quad \text{using the right triangle } DCB\,. \tag{351}$$

Noting that by the symmetry of the problem $FC = AE$ these become a system of five equations in the five unknowns

$$h\,, a\,, b\,, AE\,, EG\,.$$

If we use Equation 347 to eliminate h^2 from all equations we get

$$1 + EG^2 = a^2 \tag{352}$$
$$4 + AE^2 = (a+b)^2 \tag{353}$$
$$1 + a^2 = 2AEEG + EG^2 \tag{354}$$
$$(a+b)^2 + AE^2 = 8\,. \tag{355}$$

If we use Equation 352 to eliminate EG, since $EG = \sqrt{a^2 - 1}$, we get

$$4 + AE^2 = (a+b)^2 \tag{356}$$
$$1 = AE\sqrt{a^2 - 1} \tag{357}$$
$$(a+b)^2 + AE^2 = 8\,. \tag{358}$$

Using Equations 356 and 358 to eliminate AE^2 and solve for $(a+b)^2$ we find

$$a + b = \sqrt{6}\,.$$

Then using Equation 356 this means that $AE = \sqrt{2}$. Knowing this and using Equation 347 we have that $h = \sqrt{3}$.

From what we have computed thus far we can evaluate the area of $ABCD$ as

$$h(a+b) = \sqrt{3}\sqrt{6} = \sqrt{18} = 3\sqrt{2} = 4.242641\,.$$

Problem 18

To solve this problem we note that the number of marbles in the sets "Jane", "George", and "chipped" are distinct. Let C be the number of chipped marbles, J be the number of marbles received by Jane, and G be the number of marbles received by George. Then as the total number of marbles is 140 we have

$$J + G + C = 140\,.$$

We are told that $J = 2G$. If we put that into the above and solve for G we get

$$G = \frac{140 - C}{3}\,.$$

Now C must be one of the numbers listed i.e. $C \in \{18, 19, \ldots, 34\}$ and G must be an integer. If we try each of the possible values for C in the above fraction the only integer for G is when $C = 23$.

Problem 19

The first equation is

$$y = Ax^2\,, \tag{359}$$

and we can write the second equation as

$$y^2 - 4y + 3 = x^2\,. \tag{360}$$

If we solve Equation 359 for x^2 and put that in Equation 360 we get

$$y^2 - \left(4 + \frac{1}{A}\right) y + 3 = 0\,.$$

This equation will have two real solutions for y if and only if

$$\left(4 + \frac{1}{A}\right)^2 - 4(3) > 0\,.$$

This can be written as

$$\left|4 + \frac{1}{A}\right| > \sqrt{12} = 2\sqrt{3}\,.$$

The above will be true if

$$\frac{1}{A} + 4 > 2\sqrt{3} \quad \text{or} \quad \frac{1}{A} + 4 > -2\sqrt{3}\,.$$

These are

$$\frac{1}{A} > 2\sqrt{3} - 4 = -0.5358984 \quad \text{or} \quad \frac{1}{A} > -2\sqrt{3} - 4\,.$$

As $A > 0$ and both of the right-hand-sides of the above are negative both of these inequalities are true. Because of this there are two real solutions for y. Note that we solve the above for y we can show that both the two real solutions we have $y > 0$. Then using Equation 359 for each of these we have two solutions

$$x = \pm\sqrt{\frac{y}{A}}\,.$$

Thus we have $2 \times 2 = 4$ points where the graphs intersect.

Problem 20

We will ask how many of the smaller n^3 cubes will have only one black face. Note that

- The "corner" cubes have three black faces and there are eight of them.
- The "edge" cubes will have two black faces and there are $n-2$ of them per edge for a total of $12(n-2)$ of them.
- The "face" cubes will have one black face each and there are $n^2 - 2n - 2(n-2) = n^2 - 4n + 4$ per face for a total of $6(n^2 - 4n + 4)$ of them.

The number of cubes completely free of paint will then be

$$n^3 - (8 + 12(n-2) + 6(n^2 - 4n + 4)) = n^3 - 6n^2 + 12n - 8\,.$$

We want to set this equal to the number of cubes with just one face painted black to get the expression

$$n^3 - 6n^2 + 12n - 8 = 6(n^2 - 4n + 4)\,.$$

This gives the polynomial

$$n^3 - 12n^2 + 36n - 32 = 0\,.$$

This polynomial will have potential integer roots that are factors of -32 or

$$\{\pm 1, \pm 2, \pm 4, \pm 8, \pm 16, \pm 32\}\,.$$

I find roots to be $n = 2$ (twice) and $n = 8$.

Problem 21

If the exponent of this expression is zero (or $x = -2$) and the argument of the exponent is not zero then we have a solution. As the argument of the exponent when $x = -2$ is

$$4 + 4 - 1 = 7\,,$$

this is one integer solution.

If the argument of the exponent is one or

$$x^2 - x - 1 = 1 \quad \text{or} \quad x^2 - x - 2 = 0 \quad \text{or} \quad (x-2)(x+1) = 0\,.$$

This gives $x \in \{-1, 2\}$ and two more integer solutions.

If the argument of the exponent is -1 or

$$x^2 - x - 1 = -1\,,$$

or

$$x^2 - x = 0\,.$$

Then we have $x = 0$ and $x = 1$. To have powers of negative one equal one the exponent needs to be even. The exponent if $x = 0$ is two while the exponent if $x = 1$ is three. Thus we have one more integer solutions.

All together this gives a total of four integer solutions.

Problem 22

Now $\angle CAD = \frac{1}{2}\widehat{CD} = \frac{1}{2}(60) = 30$ degrees. This means that triangle ABO is a right triangle with $\angle BOA$ its right angle. From the "High School Exterior Angle Theorem" (HSEAT) we have that

$$\angle CBA = \angle BAO + \angle BOA = 30 + 90 = 120\,,$$

degrees. Next draw the line segment OC. Then as $OC = OA = r$ (the radius of the circle) we see that triangle OCA is an isosceles triangle and because of this that

$$\angle OCB = \angle OAB = 30\,,$$

degrees. Thus $\triangle CBO$ has angle measures given by

$$\begin{aligned}\angle CBO &= 120\\ \angle BCO &= 30\\ \angle COB &= 180 - 30 - 120 = 30\,,\end{aligned}$$

all in degrees. Thus triangle CBO is isosceles so that $CB = BO = 5$.

Problem 23

Notice as complex numbers we have $|x| = 1$ and $|y| = 1$. Writing them in polar form we have

$$\begin{aligned}x &= e^{\left(\pi - \frac{\pi}{3}\right)i} = e^{\frac{2\pi i}{3}}\\ y &= e^{\left(\pi + \frac{\pi}{3}\right)i} = e^{\frac{4\pi i}{3}}\,.\end{aligned}$$

The denominators (of three) in the angle part of the complex representation for x and y means that

$$x^3 = \left(e^{\frac{2\pi i}{3}}\right)^3 = e^{2\pi i} = 1\,,$$

with a similar derivation to show that $y^3 = 1$. This means that any power of x (or y) that is a multiple of three can be evaluated to one. This fact alone is enough to show us that $x^9 + y^9 = 1 + 1 = 2 \neq -1$.

Problem 24

We have

$$P(d=2) = \log_{10}(3) - \log_{10}(2) = \log_{10}\left(\frac{3}{2}\right).$$

Note that for any sequence of numbers of the form $\{k, k+1, k+2, \ldots, k+l-1, k+l\}$ we can show

$$P(\{k, k+1, \cdots, k+l-1, k+l\}) = \log_{10}(k+l-1) - \log_{10}(k) = \log_{10}\left(\frac{k+l+1}{k}\right).$$

Using this we compute

$$\begin{aligned}
P(\{2,3\}) &= \log_{10}\left(\frac{4}{2}\right) = \log_{10}(2) \\
P(\{4,5,6,7,8\}) &= \log_{10}\left(\frac{9}{4}\right) = \log_{10}\left(\left(\frac{3}{2}\right)^2\right) = 2\log_{10}\left(\frac{3}{2}\right) \\
&= 2P(d=2).
\end{aligned}$$

As this probability is the one we seek, we can stop.

Problem 25

Let the dimensions of the rectangular solid be a, b, and c. Then we are told that

$$abc = 8 \quad (361)$$
$$2ab + 2ac + 2bc = 32. \quad (362)$$

In addition as the lengths are in geometric progression we can take

$$\begin{aligned}
a &= a_0 \\
b &= a_0 r \\
c &= a_0 r^2.
\end{aligned}$$

If we put these expressions into Equation 361 we get

$$a_0^3 r^3 = 8 \quad \text{so} \quad a_0 r = 2. \quad (363)$$

If we put these expressions into Equation 362 we get

$$2a_0^2 r + 2a_0^2 r^2 + 2a_0^2 r^3 = 32.$$

We can simply this using Equation 363 to get

$$a_0 + 2r = 6.$$

Using $a_0 = \frac{2}{r}$ in the above we get $r^2 - 3r + 1 = 0$ which has solutions

$$r = \frac{3 \pm \sqrt{5}}{2}.$$

As $r > 0$ we must take the positive root so $r = \frac{3+\sqrt{5}}{2}$. Using this we have

$$a = a_0 = \frac{2}{r} = 3 - \sqrt{5},$$

when we simplify. We then also have

$$b = a_0 r = 2$$
$$c = a_0 r^2 = 2r = 3 + \sqrt{5}.$$

Now the sum of "all" the lengths is then

$$4a + 4b + 4c = 4(a + b + c).$$

We can compute part of the later as

$$a + b + c = (3 - \sqrt{5}) + 2 + (3 + \sqrt{5}) = 8.$$

This means that the sum of all the lengths is then $4 \times 8 = 32$.

Problem 26

The smallest value of n will be when we can factor out the smallest prime p from the numerator and the denominator. Thus for that value of n we will have

$$n - 13 = pC_n$$
$$5n + 6 = pC_d.$$

Solving both of these for n we get

$$n = 13 + pC_n$$
$$n = \frac{pC_d - 6}{5}.$$

Setting these two equal to each other we get

$$13 + pC_n = \frac{1}{5}(pC_d - 6).$$

We can "solve" for p to get

$$p(C_d - 5C_n) = 71.$$

As 71 is prime the above can be true if

$$p = 71 \quad \text{and} \quad C_d - 5C_n = 1.$$

For that value of p we would have

$$n - 13 = 71C_n$$
$$5n + 6 = 71C_d.$$

To have n be as small as possible we should take $C_n = 1$ so that $n = 84$.

Problem 27

Consider the sequence $x_1 = p$ and $x_n = (x_{n-1})^p$ for $n \geq 2$. If we iterate this we find

$$\begin{aligned} x_2 &= p^p \\ x_3 &= (p^p)^p = p^{p^2} \\ x_4 &= (p^{p^2})^p = p^{p^3} . \end{aligned}$$

The general solution looks to be

$$x_n = p^{p^{n-1}} ,$$

for $n \geq 1$. If $p = \sqrt[3]{3}$ then p^{n-1} will be an integer if $n - 1 = 3$. This means that $n = 4$.

Problem 28

The law of sines in this triangle gives

$$\frac{\sin(A)}{a} = \frac{\sin(3A)}{c} \quad \text{or} \quad \frac{\sin(A)}{27} = \frac{\sin(3A)}{48} .$$

Now note that

$$\begin{aligned} \sin(3A) &= \sin(A + 2A) = \sin(A)\cos(2A) + 2\cos(A)\sin(A)\cos(A) \\ &= \sin(A)(\cos^2(A) - \sin^2(A)) + 2\sin(A)(1 - \sin^2(A)) \\ &= \sin(A)(1 - 2\sin^2(A)) + 2\sin(A)(1 - \sin^2(A)) \\ &= 3\sin(A) - 4\sin^3(A) . \end{aligned} \tag{364}$$

Using this in the above we get

$$\frac{\sin(A)}{27} = \frac{1}{48}(3\sin(A) - 4\sin^3(A)) .$$

We can solve this for $\sin(A)$ and get $\sin(A) = \frac{\sqrt{11}}{6}$. Again using the law of sines gives

$$\frac{\sin(A)}{27} = \frac{\sin(B)}{b} = \frac{\sin(\pi - 4A)}{b} = \frac{\sin(4A)}{b} .$$

We can write

$$\sin(4A) = \sin(2(2A)) = 2\sin(2A)\cos(2A) = 4\sin(A)\cos(A)\cos(2A) .$$

Notice that since we know $\sin(A)$ we have

$$\begin{aligned} \cos(2A) &= 1 - 2\sin^2(A) = 1 - 2\left(\frac{11}{36}\right) = \frac{7}{18} \\ \cos(A) &= \sqrt{1 - \sin^2(A)} = \frac{5}{6} . \end{aligned}$$

Using all of this in the above we can solve for b where we find that

$$b = \frac{27\sin(4A)}{\sin(A)} = 35 ,$$

when we simplify.

Problem 29

Let $a = 8t$ and $b = 5t$ with t the base ten number made of 1985 ones. Then

$$9ab = 360t^2 .$$

Note that we can write t as

$$t = \sum_{i=0}^{1984} 10^i .$$

Thus

$$t^2 = \sum_{i,j=0}^{1984} 10^{i+j} .$$

If we change the variable in the summation such that $v \equiv i + j$ we can write the double sum above as

$$t^2 = \sum_{v=0}^{1984} \left(\sum_{i=0}^{v} 10^v \right) + \sum_{v=1985}^{2(1984)} \left(\sum_{i=v-1984}^{1984} 10^v \right) ,$$

or since the inner sums don't depend on i we can write this as

$$\begin{aligned} t^2 &= \sum_{v=0}^{1984} (v+1)10^v + \sum_{v=1985}^{2(1984)} (1984 - (v - 1984) + 1)10^v \\ &= \sum_{v=0}^{1984} (v+1)10^v + \sum_{v=1985}^{2(1984)} (3969 - v)10^v . \end{aligned}$$

If the coefficients in front of 10^v were always positive integers between zero and nine the above would be a base 10 representation of t^2 or close to one. As this is not true we would need to convert the terms above into powers of ten and rearrange the terms in the sums accordingly. As this seemed difficult to do I choose to work this problem in another way.

Note that the base ten number with n nines is given by

$$10^n - 1 ,$$

so the base ten number with n ones is given by

$$\frac{1}{9}(10^n - 1) .$$

The base ten number made of n digits d with $0 \le d \le 9$ is given by

$$\frac{d}{9}(10^n - 1) .$$

Thus a in this problem is of that form with $n = 1985$ and $d = 8$ and b in this problem is of that form with $n = 1985$ and $d = 5$. Thus we can write $9ab$ as

$$9ab = \frac{9 \cdot 5 \cdot 8}{9^2}(10^n - 1)^2 = \frac{40}{9}(10^n - 1)^2 .$$

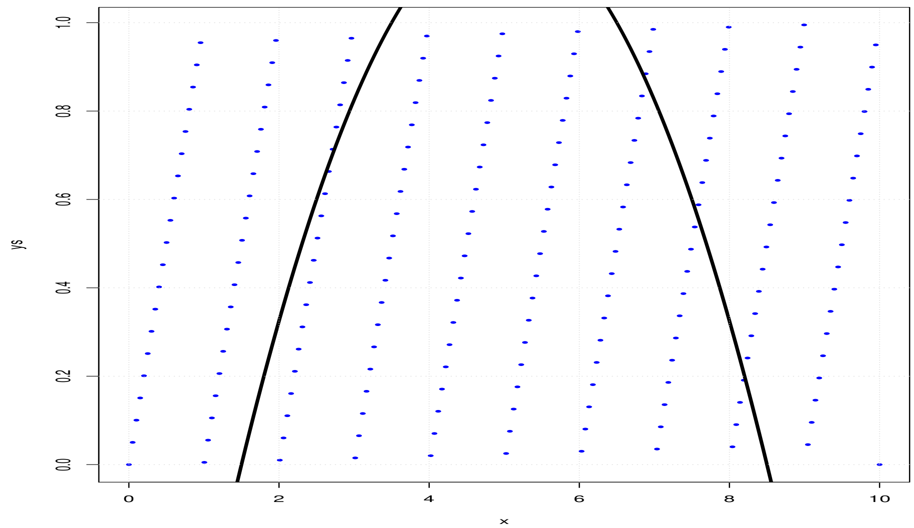

Figure 14: The left and right-hand-sides.

As the above product has 10 as a factor the last digit in the base 10 representation is zero. Thus sum of the digits of that number is the same as the sum of the digits of that number divided by 10 or

$$\frac{4}{9}(10^n - 1)^2 .$$

At this point we come to a very important point about problems like this one. The number $n = 1985$ in this problem seems very arbitrary i.e. it is obviously related to the year the test was given and not to any specific property that number might have. Thus we might be able to guess the answer by taking smaller values of n and looking for a pattern. For $n = 1$ the above number is 36 which have digits that sum to 9. For $n = 2$ the above number is 4356 which has digits that sum to 18. For $n = 3$ the above number is 443556 which has digits that sum to 27. From these few cases we might guess that the answer for general n is $9n$. Evaluating this for $n = 1985$ gives 17865.

Problem 30

Note that the function $\lfloor x \rfloor$ is monotonically increasing but the function

$$x - \lfloor x \rfloor ,$$

is *periodic*. Thus if we introduce this function into the given equation we can write it as

$$x - \lfloor x \rfloor = \frac{1}{40}(-4x^2 + 40x - 51) .$$

Plotting the left and right-hand-sides on the same graph using the following `R` code

```
xs = seq(0, 10, length.out=200)
y_1 = xs - trunc(xs)
y_2 = (-4*xs^2 + 40*xs - 51)/40

plot(xs, y_1, type='p', pch=19, cex=0.5, col='blue', xlab='x', ylab='ys')
lines(xs, y_2, type='l', lwd=3, col='black')
grid()
```

we get Figure 14. There we see that these two graphs intersect at four locations.

The 1985 AIME Examination

Problem 1

From the given expression for x_n we have $x_n x_{n-1} = n$ for $n > 1$. To use this, we write the desired product as

$$(x_8 x_7)(x_6 x_5)(x_4 x_3)(x_2 x_1) = 8 \cdot 6 \cdot 4 \cdot 2 = 384\,.$$

Problem 2

Let the right triangle we are considering have sides a and b with a hypotenuse of c such that the vertices are located in the x-y Cartesian plane at $(0, 0)$, $(b, 0)$, and $(0, a)$.

To solve this problem we will need the volume of the above triangle rotated about the y-axis. The line connecting the points $(b, 0)$ and $(0, a)$ is given by

$$y = a - \frac{a}{b}x \quad \text{so} \quad x = -\frac{b}{a}(y - a)\,.$$

Then the volume of rotation can then be given by integration

$$\begin{aligned} V &= \int_{y=0}^{a} (\pi x^2) dy = \pi \int_{y=0}^{a} \frac{b^2}{a^2}(y - a)^2 dy \\ &= \frac{\pi b^2}{a^2} \int_0^a (y - a)^2 dy = \frac{1}{3}(\pi b^2)a\,, \end{aligned}$$

when we integrate and simplify.

Using the above results if V_1 is the volume enclosed when we rotate this triangle about the vertical (i.e. the y-axis). We are told that

$$V_1 = 800\pi = \frac{1}{3}\pi b^2 a\,.$$

If V_2 is the volume enclosed when we rotate this triangle about the horizontal (i.e. the x-axis). We are told that

$$V_2 = 1920\pi = \frac{1}{3}\pi a^2 b\,.$$

We will then solve these two equations for a and b. Taking the ratio of these two volumes we have

$$\frac{V_1}{V_2} = \frac{b}{a} \quad \text{so} \quad b = \frac{V_1}{V_2} a = \frac{5}{12} a\,,$$

when I simplify. If I put this into the expression for V_2 and solve for a we find $a = 24$ and then $b = 10$. We then have

$$c = \sqrt{a^2 + b^2} = \sqrt{24^2 + 10^2} = 26\,.$$

Problem 3

Expanding the given expression we have

$$\begin{aligned} c &= (a + bi)^3 - 107i \\ &= a^3 + 3a^2(bi) + 3a(bi)^2 + (bi)^3 - 107i \\ &= a^3 - 3ab^2 + (3a^2 b - b^3 - 107)i\,. \end{aligned}$$

As c is a positive number we must have the imaginary part of the above equal to zero. This is equivalent to

$$b(3a^2 - b^2) = 107\,.$$

Now 107 is a prime integer so this means that $b = 1$ (so that $3a^2 - b^2 = 107$) or b $= 107$ (so that $3a^2 - b^2 = 1$).

In the first case (where $b = 1$) the other equation then gives

$$a^2 = 36\,,$$

the only positive solution is $a = 6$.

In the second case (where $b = 107$) the other equation then gives

$$3a^2 = 1 + 107^2 = 11450\,.$$

but 11450 is not divisible by three so in this case the solution for a would not be an integer.

Based on these arguments we have

$$c = a^3 - 3ab^2 = 198\,.$$

Problem 4

Let the small square in the center of the larger square have a side length of s. Let the segment that connects C with the segment AB intersect the segment AB at a point A'. Let the the segment that connects B with the segment AD intersect the segment AD at a point D'. Let the the segment that connects A with the segment CD intersect the segment CD at a point C'. From A' draw a segment parallel to BD' forming a small right triangle in the lower left corner of the square $ABCD$. This triangle has vertices at A, A', and a point on the segment AC' denote this last point A''. Then in the triangle $AA'A''$ let the angle $\angle A''AA' = \theta$. This triangle has lengths

$$AA' = \frac{1}{n}$$
$$A'A'' = s\,,$$

so that

$$\sin(\theta) = \frac{s}{\frac{1}{n}} = sn\,.$$

Now drop a perpendicular from C' to the segment AB and intersecting AB at the point E. The segment $C'E$ will have length one and the segment AE will have length $1 - \frac{1}{n}$. This means that

$$\sin(\theta) = \frac{1}{\sqrt{1 + \left(1 - \frac{1}{n}\right)^2}}\,.$$

Setting these two equal we have

$$sn = \frac{1}{\sqrt{1 + \left(1 - \frac{1}{n}\right)^2}}\,.$$

As we are told that the area of the small internal square is $\frac{1}{1}985$ we have that

$$s = \frac{1}{\sqrt{1985}}\,.$$

Using this in the above expression we have a single equation for n given by

$$\frac{n^2}{1985} = \frac{1}{1 + \left(1 - \frac{1}{n}\right)^2}\,.$$

We can simplify this to

$$n^2 - n - 992 = 0\,,$$

which has solutions $n = 32$ and $n = -31$. To have $n > 0$ we must take $n = 32$.

Problem 5

We can write this difference equation as

$$a_n - a_{n-1} + a_{n-2} = 0\,,$$

and try the solution $a_n = r^n$ to get

$$r^n - r^{n-1} + r^{n-2} = 0\,.$$

If we divide by r^{n-2} we get

$$r^2 - r + 1 = 0\,.$$

This has solutions

$$r = \frac{1 \pm \sqrt{3}i}{2} = \cos\left(\frac{\pi}{3}\right) \pm i \sin\left(\frac{\pi}{3}\right) = e^{\pm i \frac{\pi}{3}}\,.$$

Denote these two solutions r_- and r_+. Then the general solution is

$$a_n = C_- r_-^n + C_+ r_+^n\,.$$

Note that for these two roots we have some special properties that can help us simplify the results below. We have

$$\begin{aligned} r_-^{6n} &= r_+^{6n} = 1 \\ r_-^{3n} &= r_+^{3n} = (-1)^n\,, \end{aligned}$$

for n an integer. Using the above for $n = 1$ we have

$$r_-^3 = -1 \quad \text{so} \quad r_-^2 = -\frac{1}{r_-} = -r_+\,.$$

The same logic gives

$$r_+^2 = -r_-\,.$$

Now we are told that

$$\sum_{n=1}^{1492} a_n = 1985\,.$$

Using Equation 20 we can evaluate the left-hand-side of this to get

$$C_- r_- \left(\frac{1 - r_-^{1492}}{1 - r_-}\right) + C_+ r_+ \left(\frac{1 - r_+^{1492}}{1 - r_+}\right) = 1985\,.$$

Since $1492 = 248(6) + 4$ we have the above is equal to

$$C_- r_- \left(\frac{1 - r_-^{4}}{1 - r_-}\right) + C_+ r_+ \left(\frac{1 - r_+^{4}}{1 - r_+}\right) = 1985\,.$$

Then as $r_\pm^4 = -r_\pm$ we can write the above as

$$C_- r_- \left(\frac{1 + r_-}{1 - r_-}\right) + C_+ r_+ \left(\frac{1 + r_+}{1 - r_+}\right) = 1985\,. \tag{365}$$

We are also told that

$$\sum_{n=1}^{1985} a_n = 1492\,.$$

Again using Equation 20 we can evaluate the left-hand-side of this to get

$$C_-r_-\left(\frac{1-r_-^{1985}}{1-r_-}\right)+C_+r_+\left(\frac{1-r_+^{1985}}{1-r_+}\right)=1492\,.$$

Since $1985 = 330(6) + 5$ we have that the above is equal to

$$C_-r_-\left(\frac{1-r_-^{5}}{1-r_-}\right)+C_+r_+\left(\frac{1-r_+^{5}}{1-r_+}\right)=1492\,,$$

or

$$C_-r_-\left(\frac{1+r_-^{2}}{1-r_-}\right)+C_+r_+\left(\frac{1+r_+^{2}}{1-r_+}\right)=1492\,.$$

Now using $r_\pm^2 = -r_\mp$ this becomes

$$C_-r_-\left(\frac{1-r_+}{1-r_-}\right)+C_+r_+\left(\frac{1-r_-}{1-r_+}\right)=1492\,. \tag{366}$$

Now to solve these lets define

$$\begin{aligned} D_- &= \frac{C_-r_-}{1-r_-} \\ D_+ &= \frac{C_+r_+}{1-r_+}\,, \end{aligned}$$

and we have the two equations

$$\begin{aligned} D_-(1+r_-)+D_+(1+r_+) &= 1985 \\ D_-(1-r_+)+D_+(1-r_-) &= 1492\,. \end{aligned}$$

This is a system to solve for $D_\pm$. Doing so we get

$$\begin{aligned} D_- &= \frac{493-1985r_- - 1492r_+}{r_+^2-r_-^2} \\ D_+ &= \frac{-493+1492r_- + 1985r_+}{r_+^2-r_-^2}\,. \end{aligned}$$

The sum we want to evaluate is

$$\sum_{n=1}^{2001} a_n\,,$$

which can be done using Equation 20 to get

$$C_-r_-\left(\frac{1-r_-^{2001}}{1-r_-}\right)+C_+r_+\left(\frac{1-r_+^{2001}}{1-r_+}\right)\,.$$

Since $2001 = 333(6) + 3$ we have the above is equal to

$$C_-r_-\left(\frac{1-r_-^{3}}{1-r_-}\right)+C_+r_+\left(\frac{1-r_+^{3}}{1-r_+}\right)\,,$$

or using $r_\pm^3 = -1$ this is

$$C_-r_-\left(\frac{2}{1-r_-}\right) + C_+r_+\left(\frac{2}{1-r_+}\right).$$

From the definition of $D_\pm$ this is

$$2D_- + 2D_+ .$$

Putting in what we know for $D_\pm$ I find that this becomes

$$\begin{aligned}2(D_- + D_+) &= \frac{2}{r_+^2 - r_-^2}(-493r_- + 493r_+) = \frac{986}{r_+^2 - r_-^2}(-r_- + r_+)\\ &= \frac{986}{r_+ + r_-} = \frac{986}{2\cos\left(\frac{\pi}{3}\right)} = 986 .\end{aligned}$$

Problem 6 (triangles with equal heights)

For this problem we will use the fact that if two triangles have the same height then their areas are proportional to their bases. To start, we denote the point inside the triangle where the three internal segments intersect as P, the area of the left-most triangle as x, and the area of the right-most triangle as y.

Let the location where the segment from C intersects AB be denoted C' (for opposite C). Then as the triangles APC' and BPC' with areas 40 and 30 respectively have the same height (through the point P) their areas must be in proportion to their bases that is

$$\frac{40}{30} = \frac{AC'}{C'B} .$$

Also note that larger triangles ACC' and $CC'B$ have the same heights (this time through the point C) and thus their areas must be in proportion to their bases so

$$\frac{40+y+84}{30+35+x} = \frac{AC'}{C'B} .$$

Equating these two we get

$$\frac{40+y+84}{30+35+x} = \frac{40}{30} . \tag{367}$$

Doing the same thing for the smaller and larger triangles that have their base the segment AC we get

$$\frac{84+x+35}{y+40+30} = \frac{84}{y} . \tag{368}$$

Doing the same thing for the smaller and larger triangles that have their base the segment AC we get

$$\frac{y+84+x}{40+30+35} = \frac{x}{35} . \tag{369}$$

These together give three equations and two unknowns for x and y. Solving any two of them we find $x = 70$ and $y = 56$. Using these two numbers we can easily compute the area of the full triangle.

Problem 7

What what we are told we have $c = d^{2/3}$ and $a = b^{4/5}$ so that the expression $c - a = 19$ becomes

$$d^{2/3} - b^{4/5} = 19\,.$$

Note that we can factor the above as

$$(d^{1/3} - b^{2/5})(d^{1/3} + b^{2/5}) = 19\,.$$

If $d^{1/3}$ and $b^{2/5}$ are not integers then neither would c and a thus the above is an integer factorization of 19. As 19 is prime this means that either

$$\begin{aligned} d^{1/3} - b^{2/5} &= 1 \\ d^{1/3} + b^{2/5} &= 19\,. \end{aligned}$$

or

$$\begin{aligned} d^{1/3} - b^{2/5} &= 19 \\ d^{1/3} + b^{2/5} &= 1\,. \end{aligned}$$

As $d^{1/3} - b^{2/5} < d^{1/3} + b^{2/5}$ only the first condition is possible. In that case we can solve for $d^{1/3}$ and $b^{2/5}$ to get

$$\begin{aligned} d^{1/3} &= 10 \\ b^{2/5} &= 3\,. \end{aligned}$$

Thus we have that $d = 1000$ and $b = 243$. This means that $d - b = 1000 - 243 = 757$.

Problem 8

We would want to pick A_i to be the number "two" or "three" otherwise the value of $|A_i - a_i|$ could be made smaller by changing A_i to one of those. Note that if all $A_i = 2$ then $\sum_i A_i = 14$ (which is smaller than nineteen) and if all $A_i = 3$ then $\sum_i A_i = 21$ (which is larger than nineteen). Thus we expect to have some A_i equal two and some equal to three. If n is the number of A_i that equal two and m are the number of A_i that equal three we have

$$2n + 3m = 19\,.$$

To determine which i should have A_i equal two and which should have A_i equal three we compute

$$\begin{aligned} |2 - a_1| &= 0.56 \\ |2 - a_2| &= 0.61 \\ |2 - a_3| &= 0.65 \\ |2 - a_4| &= 0.71 \\ |2 - a_5| &= 0.79 \\ |2 - a_6| &= 0.82 \\ |2 - a_7| &= 0.86\,, \end{aligned}$$

and that

$$\begin{aligned}
|3 - a_1| &= 0.44 \\
|3 - a_2| &= 0.39 \\
|3 - a_3| &= 0.35 \\
|3 - a_4| &= 0.29 \\
|3 - a_5| &= 0.21 \\
|3 - a_6| &= 0.18 \\
|3 - a_7| &= 0.14 \,.
\end{aligned}$$

In general the "errors" are smaller when most of the A_i are three. To see how many A_i we can take to be three note that if

- If $m = 7$ and $n = 0$ then $\sum_i A_i = 21$ which is too large.
- If $m = 6$ and $n = 1$ then $\sum_i A_i = 20$ which is too large.
- If $m = 5$ and $n = 2$ then $\sum_i A_i = 19$ which is the desired number.
- If $m = 4$ and $n = 3$ then $\sum_i A_i = 18$ which is too small.
- If $m = 3$ and $n = 4$ then $\sum_i A_i = 17$ which is too small.
- If $m = 2$ and $n = 5$ then $\sum_i A_i = 16$ which is too small.
- If $m = 1$ and $n = 6$ then $\sum_i A_i = 15$ which is too small.
- If $m = 0$ and $n = 7$ then $\sum_i A_i = 14$ which is too small.

Thus we should take $m = 5$ and $n = 2$. To make M as small as possible we will take

$$A_i = 3 \quad \text{for} \quad 3 \le i \le 7 \,,$$

and

$$A_i = 2 \quad \text{for} \quad 1 \le i \le 2 \,.$$

Based on the absolute values above this gives

$$M = \max_i |A_i - a_i| = 0.61 \,.$$

Thus $100M = 61$.

Problem 9

Let r be the radius of the circle and imagine radii drawn from the center of the circle to the end points of the chords at the angles α, β, and $\alpha + \beta$. Next introduce a segment that is the perpendicular bisector of all three chords and that also bisects the three angles. Then from the problem description we have

$$\frac{2}{2} = 1 = r \sin\left(\frac{\alpha}{2}\right) \tag{370}$$

$$\frac{3}{2} = r \sin\left(\frac{\beta}{2}\right) \tag{371}$$

$$\frac{4}{2} = 2 = r \sin\left(\frac{\alpha+\beta}{2}\right) . \tag{372}$$

Notice that this has given us three equations for the three unknowns r, α, and β. We will try to eliminate two of the variables in order to get one equation in one variable which we can solve. To do that we start by expanding Equation 372 and then using the others we have

$$\begin{aligned} 2 &= r \sin\left(\frac{\alpha}{2}\right) \cos\left(\frac{\beta}{2}\right) + r \sin\left(\frac{\beta}{2}\right) \cos\left(\frac{\alpha}{2}\right) \\ &= \cos\left(\frac{\beta}{2}\right) + \frac{3}{2} \cos\left(\frac{\alpha}{2}\right) . \end{aligned}$$

Using Equations 370 and 371 we have

$$\begin{aligned} \cos\left(\frac{\beta}{2}\right) &= \sqrt{1 - \left(\frac{3}{2r}\right)^2} \\ \cos\left(\frac{\alpha}{2}\right) &= \sqrt{1 - \left(\frac{1}{r}\right)^2} . \end{aligned}$$

If we put this into the above we get

$$2 = \sqrt{1 - \left(\frac{3}{2r}\right)^2} + \frac{3}{2}\sqrt{1 - \left(\frac{1}{r}\right)^2} .$$

This is a single equation in the variable $\frac{1}{r^2}$ which we can solve. We find

$$\frac{1}{r^2} = \frac{15}{64} \quad \text{so} \quad \frac{1}{r} = \frac{\sqrt{15}}{8} .$$

Using Equation 370 and

$$\cos(\alpha) = 1 - 2\sin^2\left(\frac{\alpha}{2}\right) ,$$

we get

$$\cos(\alpha) = 1 - \frac{2}{r^2} = 1 - \frac{15}{32} = \frac{17}{32} .$$

Problem 11

The equation for an ellipse with the two foci given is

$$\sqrt{(9-x)^2 + (20-y)^2} + \sqrt{(49-x)^2 + (55-y)^2} = 2a , \tag{373}$$

where $2a$ is the length of the major axis (so that a is the length of the *semi-major* axis).

If the ellipse is tangent to the x-axis then there is a point on the ellipse $(x^*, 0)$ such that

$$\frac{dy}{dx}(x^*, 0) = 0\,.$$

Taking the derivative of Equation 373 with respect to x gives

$$\frac{2(9-x)(-1)+2(20-y)\left(-\frac{dy}{dx}\right)}{2\sqrt{(9-x)^2+(20-y)^2}} + \frac{2(49-x)(-1)+2(55-y)\left(-\frac{dy}{dx}\right)}{2\sqrt{(49-x)^2+(55-y)^2}} = 0\,.$$

Evaluating the above at the point $(x^*, 0)$ with the knowledge that the derivative is zero and calling the point $x^* = x$ for notational simplicity gives

$$\frac{-(9-x)}{\sqrt{(9-x)^2+20^2}} - \frac{(49-x)}{\sqrt{(49-x)^2+55^2}} = 0\,.$$

We can write this as

$$(9-x)\sqrt{(49-x)^2+55^2} = -(49-x)\sqrt{(9-x)^2+20^2}\,. \tag{374}$$

If we square both sides (and recognizing this might give us a spurious root) we get

$$(9-x)^2((49-x)^2+55^2) = (49-x)^2((9-x)^2+20^2)\,.$$

Expanding and simplifying gives us

$$21x^2 - 122x - 5723 = 0\,.$$

The two roots to this are given by

$$x \in \left\{-\frac{97}{7}, \frac{59}{3}\right\}\,.$$

If we put $x = -\frac{99}{7}$ into Equation 374 we get a left-hand-side and a right-hand-side that are of *opposite* sign (meaning it is a spurious root). If we take $x = \frac{59}{3}$ we see that Equation 374 is satisfied. Putting this value into Equation 373 we find $2a = 85$.

Problem 12

Let p_n be the vector with the ith component for $1 \le i \le 4$ representing the probability that the bug is at the location $i = 1$ (i.e. A), $i = 2$ (i.e. B), $i = 3$ (i.e. C), and $i = 4$ (i.e. D) at the nth step in the bugs walk. Then since the bug starts at A we have

$$p_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\,.$$

And in moving from the nth step to the $n+1$ step we have

$$p_{n+1} = \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix} p_n = \frac{1}{3} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} p_n \, .$$

If we call the matrix above E then the solution to the above iteration equations is

$$p_n = \left(\frac{1}{3}\right)^n E^n p_0 \, ,$$

for $n \geq 0$. There might be easier way to derive the answer needed but for me the easiest was to compute the needed matrix power i.e. E^7 by performing repeated matrix multiplications. We can do that in R with the following code

```
data = c(0, 1, 1, 1, 1, 0, 1, 1, 1, 1, 0, 1, 1, 1, 1, 0)
E = matrix(data, ncol=4, nrow=4, byrow=TRUE)
print(E %*% E %*% E %*% E %*% E %*% E %*% E)
```

for which we find E^7 is given by

```
     [,1] [,2] [,3] [,4]
[1,]  546  547  547  547
[2,]  547  546  547  547
[3,]  547  547  546  547
[4,]  547  547  547  546
```

This means that

$$p_7 = \frac{1}{3^7} E^7 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \, .$$

The solution to this problem is the first component of p_7. From the above we see that is given by

$$\frac{546}{3^7} = \frac{182}{729} \, .$$

The 1986 AHSME Examination

Problem 1

We would have

$$x - y + z - (x - y - z) = z - y + z - x + y + z = 2z\,.$$

Problem 2

This would be $y = \frac{1}{3}x + 8$.

Problem 3

Now $\angle ABC = 90 - 20 = 70$ and $\angle DBC = \frac{70}{2} = 35$, so $\angle BDC = 90 - 35 = 55$. All angles are in degrees.

Problem 4

The number 33 has digits that sum to six which is divisible by six but the number 33 is not divisible by six.

Problem 5

We can simplify as follows

$$\begin{aligned}\left(27^{1/6} - \left(6\frac{3}{4}\right)^{1/2}\right)^2 &= \left(27^{1/6} - \left(\frac{27}{4}\right)^{1/2}\right)^2 \\ &= 27^{1/3} + \frac{27}{4} - 2 \cdot 27^{1/6} \cdot \left(\frac{27}{4}\right)^{1/2} \\ &= 3 + \frac{27}{4} - 27^{\frac{1}{6}+\frac{1}{2}} = 3 + \frac{27}{4} - 27^{\frac{2}{3}} \\ &= 3 + \frac{27}{4} - 3^2 = \frac{3}{4}\,.\end{aligned}$$

Problem 6

Let w and h be the width (along the horizontal i.e. x-axis) and the height (along the vertical i.e. y-axis) of the block respectively. Let t be the table's height. Then from the configurations shown and the numbers given we have

$$r = h + (t - w) = 32$$
$$s = w + (t - h) = 28\,.$$

If we add these two equations together we get $2t = 60$ so that $t = 30$.

Problem 7

From the given solutions (most overlap largely with the domain $2 < x < 3$) if x is in that range we see that

$$\lfloor x \rfloor + \lceil x \rceil = 2 + 3 = 5\,,$$

is true. Now if $x = 2$ then we have

$$\lfloor x \rfloor + \lceil x \rceil = 2 + 2 = 4 \neq 5\,,$$

and if $x = 3$ then

$$\lfloor x \rfloor + \lceil x \rceil = 3 + 3 = 6 \neq 5\,.$$

Thus the solution set is $2 < x < 3$.

Problem 8

This would be the number

$$\text{FPP} = \frac{3615122 \times 5280^2}{226504825}\,.$$

Since the answers are all given with one the first digit distinct we should be able to use crude approximations to estimate this number. We have

$$\text{FPP} \approx \frac{3.6 \times 10^6 \times 5^2 \times 10^6}{226 \times 10^6} = \left(\frac{3.6 \times 25}{226}\right) 10^6\,.$$

In this lets take

$$\frac{3.6 \times 25}{226} \approx \frac{3.6 \times 5^2}{5^2 \times 9} = \frac{3.6}{9} = 0.4\,.$$

Thus I would estimate

$$\text{FPP} \approx 4 \times 10^5 = 400000\,.$$

Problem 9

We can write this product as

$$\begin{aligned}\prod_{k=2}^{10}\left(1-\frac{1}{k^2}\right) &= \prod_{k=2}^{10}\left(\frac{k^2-1}{k^2}\right) = \prod_{k=2}^{10}\left(\frac{(k-1)(k+1)}{k^2}\right) \\ &= \frac{\prod_{k=2}^{10}(k-1)\times\prod_{k=2}^{10}(k+1)}{\prod_{k=2}^{10}k^2} = \frac{\prod_{k=1}^{9}k\times\prod_{k=3}^{11}k}{\prod_{k=2}^{10}k^2} \\ &= \frac{1\left(\prod_{k=2}^{9}k\right)\times\left(\prod_{k=2}^{9}k\right)\left(\frac{11\cdot 10}{2}\right)}{\prod_{k=2}^{10}k^2} = \frac{11\cdot 10}{2\cdot 10^2} = \frac{11}{20}\,.\end{aligned}$$

Problem 10

Note that in the word AHSME there are no duplicated letters. Considering all of the 5! possible orderings when we place them in dictionary order we will place all words starting with the letter A first. These are the words A followed by the $4! = 24$ permutations of the letters E, H, M, and S. Next we would place all words starting with the letter E. This would be another $4! = 24$ words giving a total of $24 + 24 = 48$ words that we have placed. Next we would place all words that start with the letter H. This would be another 24 words for a total of $3 \times 24 = 72$. If we added all of the words that start with the next letter M we would have a total of 96 words which is larger than the target word number. Thus the target word starts with the letter M. After we have placed that letter we will place the letters A, E, H, and S. In placing these in dictionary we will first place the A followed by the other letters for a total of $3! = 6$ words. This brings us to $72 + 6 = 78$ words. Next we will place the E followed by the other letters for a total of six words for a total of $78 + 6 = 84$. Next we will place the H but we can't place all of the other letters without missing our target word. Thus the target word starts with MH and we need to place the A, E, and S starting with A. These we can do by hand. The 85th word will be $MHAES$ so the 86th word will be $MHASE$ which has a last letter of E.

Problem 11

The solution in the book is to quote the fact that the line segment from the right angle to the midpoint of the hypotenuse is of length one half that of the hypotenuse. Thus the length of MH would be $\frac{1}{2}AB = 6.5$. Here I present a very short proof of this using Cartesian coordinates.

Create a right triangle in the x-y plane by placing points at $(0, 0)$, $(a, 0)$ and $(0, b)$. Then the two legs are of length a and b and the hypotenuse has a length of $c = \sqrt{a^2 + b^2}$. The "line" of the hypotenuse is given by

$$y = -\frac{b}{a}(x-a) = -\frac{b}{a}x + b\,,$$

for $0 \leq x \leq a$. Let the midpoint of the hypotenuse be denoted by the point M which will be located at the Cartesian point $(\xi, y(\xi))$ such that the distances from $(0, b)$ and $(a, 0)$ are equal (i.e. at the midpoint of the hypotenuse). These two distances (squared) are given by

$$(0-\xi)^2 + (b - y(\xi))^2 = \xi^2 + \frac{b^2}{a^2}\xi^2$$
$$(a-\xi)^2 + (0 - y(\xi))^2 = (a-\xi)^2 + \frac{b^2}{a^2}(\xi - a)^2 \,.$$

If we set these two expressions equal we could expand everything and get a quadratic equation in ξ to solve to find the x location of the point M. Note from the form of the equations above if

$$\xi^2 = (a-\xi)^2 \,,$$

then both equations will be equal. As in this problem we have $\xi > 0$ and $a - \xi > 0$ so taking the square root and keeping these two roots gives

$$\xi = a - \xi \quad \text{so} \quad \xi = \frac{a}{2} \,.$$

Thus the point M is

$$\left(\frac{a}{2}, -\frac{b}{a}\left(\frac{a}{2}\right) + b\right) = \left(\frac{a}{2}, \frac{b}{2}\right) \,.$$

The distance from the right angle (here the origin) to this point is

$$\sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2} = \frac{1}{2}\sqrt{a^2 + b^2} \,,$$

showing the desired result.

Problem 12

Let c, w, and u be the number of correct, wrong, and unanswered problems that John had on this years AHSME. Under the new scoring system we would have

$$5c + 2u = 93 \,. \tag{375}$$

Under the old scoring system we would have

$$30 + 4c - w = 84 \quad \text{or} \quad 4c - w = 54 \,. \tag{376}$$

Finally as there are thirty problems on the AHSME test we have

$$c + u + w = 30 \,. \tag{377}$$

As the problem asks about the value of u we will create an equation for u. From Equation 375 we have

$$c = \frac{93 - 2u}{5} \,.$$

From Equation 376 we have

$$w = 4c - 54 = \frac{4}{5}(93 - 2u) - 54 \,.$$

If we put these two expressions into Equation 377 we can solve for u and find $u = 9$.

Problem 13

For the parabola $y = ax^2 + bx + c$ the vertex will satisfy

$$y'(x) = 2ax + b = 0 \quad \text{so} \quad x = -\frac{b}{2a}\,.$$

As we are told that

$$-\frac{b}{2a} = 4\,,$$

we get that

$$b = -8a\,. \tag{378}$$

As $y(4) = 2$ from the form of $y(x)$ we get that

$$16a + 4b + c = 2\,. \tag{379}$$

We are also told that $y(2) = 0$ or

$$4a + 2b + c = 0 \quad \text{so} \quad c = -4a - 2b\,,$$

which if we put this expression for c into Equation 379 and simplify we get

$$6a + b = 1\,.$$

Using Equation 378 in this we get $a = -\frac{1}{2}$, $b = 4$ and finally $c = -6$. Thus

$$abc = 12\,.$$

Problem 14

Let h, s, and j be lengths in a given unit (say meters) then we are told that

$$bh = cs \tag{380}$$
$$dj = eh \tag{381}$$
$$fj = g\,. \tag{382}$$

From the last of these and using the ones above we have that one meter is given by

$$1 = \frac{fj}{g} = \frac{f}{g}\left(\frac{eh}{d}\right) = \frac{f}{g}\left(\frac{e}{d}\right)\left(\frac{cs}{b}\right) = \left(\frac{fec}{gdb}\right)s\,.$$

Problem 15

The correct result is $A = \frac{1}{3}(x + y + z)$ which is to be compared with

$$\tilde{A} = \frac{1}{2}\left(\frac{1}{2}(x+y) + z\right) = \frac{1}{4}x + \frac{1}{4}y + \frac{1}{2}z\,.$$

Computing $\tilde{A} - A$ we find

$$\begin{aligned}\tilde{A} - A &= \left(\frac{1}{4} - \frac{1}{3}\right) x + \left(\frac{1}{4} - \frac{1}{3}\right) y + \left(\frac{1}{2} - \frac{1}{3}\right) z \\ &= -\frac{1}{12}(x+y) + \frac{z}{6} = -\frac{1}{12}(-x - y + 2z) \\ &= \frac{1}{12}(z - x + z - y) > 0\,,\end{aligned}$$

as both $z - x > 0$ and $z - y > 0$. Thus $\tilde{A}$ will always overestimate (be larger than) A.

Problem 16

From the given similar triangles we can write

$$\frac{AC}{AB} = \frac{CP}{PA} = \frac{PA}{BP}\,.$$

Using what we know about the lengths above we have

$$\frac{6}{8} = \frac{CP}{PA} = \frac{PA}{7 + CP}\,,$$

From the first equality above we have

$$PA = \frac{4}{3} CP\,.$$

If we put this into the second equality above gives

$$\frac{3}{4} = \frac{\frac{4}{3} CP}{7 + CP}\,.$$

Solving this for CP we get $CP = 9$.

Problem 17

To determine the smallest number of socks that we have to draw to guarantee we get p pairs we have to assume that on each draw the "universe" is working against us in such a way that each draw try's *not* to give us a pair when that socks color is considered with the colors of the other socks drawn.

For example we could draw four socks and have them all be different colors but on the fifth draw we are guaranteed to get at least one pair. Thus the number of draws to guarantee one pair is

$$d_1 = 5\,.$$

Lets assume that the color of the pair just produced is X. On the sixth draw we could get another pair but to guarantee that we get two pairs we have to assume that the sixth draw gives us the color X and we are assured that the seventh draw gives us a second pair or

$$d_2 = 7\,.$$

From this pattern we need two additional draws to get each additional pair so

$$\begin{aligned} d_3 &= 9 \\ d_4 &= 11\,, \end{aligned}$$

etc. It looks like the formula is $d_n = d_{n-1} + 2$. To get 10 pairs we need to know d_{10} from which using the above I compute $d_{10} = 23$.

Problem 18

If the plane were to cut the cylinder at a "right" angle the the intersection would be a circle with a radius of one. As we tilt the plane at an angle θ with respect to the horizontal the cut forms an ellipse. The minor axis will be of length two (twice the radius of the circle) and the major axis then must be of length

$$1.5 \times 2 = 3\,.$$

Problem 19

As we have a hexagon the number of sides is $n = 6$. Lets draw this hexagram with two horizontal sides, a top right corner denoted A, a middle right corner denoted B, a bottom right corner denoted C, and a bottom left corner denoted D. Assume that Alice start at A then she ends at a point P midway between C and D.

Draw the line segment AC forming $\triangle ABC$. Recall that the interior angle of a regular polygon is given by $\frac{180(n-2)}{n} = 120$. This means that $\angle BAC = \angle BCA = \frac{180-120}{2} = 30$. This then means that

$$AC = 2(2\cos(30)) = 2\left(2 \times \frac{\sqrt{3}}{2}\right) = 2\sqrt{3}\,.$$

Now to determine AP we use the Pythagorean identity

$$AP^2 = PC^2 + AC^2 \quad \text{or} \quad AP^2 = 1^2 + 4(3) = 13\,.$$

Thus $AP = \sqrt{13}$.

Problem 20

We are told that

$$x = \frac{A}{y},$$

and x increases by a fraction f (i.e. f is a number like 0.05) then the new value of x or x' is given by

$$x' = (1+f)x,$$

and

$$y' = \frac{A}{x'} = \frac{A}{(1+f)x} = \frac{y}{1+f}.$$

We have that y has decreased by an amount

$$\frac{y'-y}{y} = \frac{y'}{y} - 1 = \frac{1}{1+f} - 1 = -\frac{f}{1+f}.$$

In terms of $p\%$ we have $f = \frac{p}{100}$ so the above is

$$-\frac{\frac{p}{100}}{1+\frac{p}{100}} = -\frac{p}{100+p}.$$

This number is the fraction of the decrease in y. The percentage decrease would be that number multiplied by 100.

Problem 21

Note that triangle CAB is a right triangle. Thus the area of the left-most shaded area is

$$\frac{1}{2}(AB)(AC) - \pi(AC)^2\left(\frac{\theta}{2\pi}\right).$$

The area of the right-most shaded area is

$$\pi(AC)^2\left(\frac{\theta}{2\pi}\right).$$

Setting these two areas equal and simplifying gives

$$AB = 2\theta AC. \quad (383)$$

From the figure we have

$$\tan(\theta) = \frac{AB}{AC},$$

so if we solve for AB in that expression and put it into Equation 383 and simplify we get

$$2\theta = \tan(\theta).$$

Problem 22

There are $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5$ ways to select the six numbers. For the second smallest to be a three means that we need to select a one and a three or a two and a three from the set $\{1, 2, 3, \ldots, 10\}$. The number of sequences like that can be computed by recognizing that we can place the three in one of the six spots (in six ways). Then select a one or a two (in two ways). Then select the spot for the one or the two to go (in five ways). Then select numbers from $\{4, 5, 6, 7, 8, 9, 10\}$ to go in the four remaining spaces (in $7 \cdot 6 \cdot 5 \cdot 4$ ways). Thus the probability of selecting a number of the required form is

$$\frac{6 \cdot 2 \cdot 5 \cdot (7 \cdot 6 \cdot 5 \cdot 4)}{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5} = \frac{1}{3}\,.$$

Problem 23

If we let $x = 69$ we see that

$$N = x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1 = (x+1)^5 = 70^5\,.$$

If we factor 70 into primes we get $70 = 2 \cdot 5 \cdot 7$. This means that

$$N = 2^5 5^5 7^5\,.$$

From this representation the numbers that can divide N are made of taking the product of $2^m 5^n 7^l$ where m, n, and l are elements of $\{0, 1, 2, 3, 4, 5\}$. This gives $6 \cdot 6 \cdot 6 = 6^3 = 216$ possible divisors.

Problem 24

We can divide $x^4 + 6x^2 + 25$ by $p(x)$ (using polynomial long division) to show that

$$\frac{x^4 + 6x^2 + 25}{p(x)} = x^2 - bx + (6 - c + b^2) + \frac{b(2c - 6 - b^2)x + (c^2 - b^2 c - 6c + 25)}{p(x)}\,.$$

As we are told that $p(x)$ divides this polynomial we must have that

$$b(2c - 6 - b^2) = 0 \quad (384)$$

$$c^2 - b^2 c - 6c + 25 = 0\,. \quad (385)$$

One solution to Equation 384 is $b = 0$ which if we put this into Equation 385 we would require that c satisfy

$$c^2 - 6c + 25 = 0\,.$$

This last quadratic has complex roots so c would not be an integer if $b = 0$. Another solution to Equation 384 is

$$-b^2 + 2c - 6 = 0\,,$$

or

$$c = 3 + \frac{b^2}{2} \,. \tag{386}$$

If I put this into Equation 385 and simplify a bit I find

$$b^4 + 12b^2 - 64 = 0 \,.$$

This has solutions $b^2 = 4$ or $b^2 = -16$. To have b be an integer we need to take $b^2 = 4$ so that $b = \pm 2$. In either case using Equation 386 we have $c = 5$. Thus there are two possible polynomials that could be the solution to this problem

$$\begin{aligned} p_1(x) &= x^2 - 2x + 5 \\ p_2(x) &= x^2 + 2x + 5 \,. \end{aligned}$$

We will now see which of these two polynomials divides the second polynomial given. We find

$$\frac{3x^4 + 4x^2 + 28x + 5}{p_1(x)} = 3x^2 + 6x + 1 \,.$$

Thus we don't have to check $p_2(x)$. From this expression we find $p(1) = p_1(1) = 4$.

Problem 25

Using the fact that $\log_2(x)$ is an increasing function note that for $x = 1$ we have

$$\log_2(1) = 0 \quad \text{so} \quad \lfloor \log_2(1) \rfloor = 0 \,.$$

Next for $x = 2$ we have

$$\log_2(2) = 1 \quad \text{so} \quad \lfloor \log_2(2) \rfloor = 1 \,.$$

Next for $x = 4$ we have

$$\log_2(4) = 2 \quad \text{so} \quad \lfloor \log_2(x) \rfloor = 1 \,,$$

for $2 \le x \le 3$. Note that we can write this last range as $2^1 \le x \le 2^2 - 1$. Now for $x = 8$ we have

$$\log_2(8) = 3 \quad \text{so} \quad \lfloor \log_2(x) \rfloor = 2 \,,$$

for $4 \le x \le 7$. Note that we can write this last range as $2^2 \le x \le 2^3 - 1$. Now for $x = 16$ we have

$$\log_2(16) = 4 \quad \text{so} \quad \lfloor \log_2(x) \rfloor = 3 \,,$$

for $8 \le x \le 15$. Note that we can write this last range as $2^3 \le x \le 2^4 - 1$. This pattern continues and we can write our desired sum S as

$$\begin{aligned} S &= \sum_{N=2}^{1024} \lfloor \log_2(N) \rfloor \\ &= \sum_{N=2}^{2^2-1} \lfloor \log_2(N) \rfloor + \sum_{N=2^2}^{2^3-1} \lfloor \log_2(N) \rfloor + \sum_{N=2^3}^{2^4-1} \lfloor \log_2(N) \rfloor + \cdots + \sum_{N=2^9}^{2^{10}-1} \lfloor \log_2(N) \rfloor + \sum_{N=2^{10}}^{2^{10}} \lfloor \log_2(N) \rfloor \\ &= \sum_{k=1}^{9} \sum_{N=2^k}^{2^{k+1}-1} \lfloor \log_2(N) \rfloor + \lfloor \log_2(1024) \rfloor \,. \end{aligned}$$

As each term in the left-most sum has a value of k and there are

$$2^{k+1} - 1 - 2^k + 1 = 2^k \,,$$

terms we can write S as

$$S = \sum_{k=1}^{9} k2^k + 10 \,.$$

Using Equation 337 with $w = 2$ and $N = 9$ to evaluate this we find $S = 8204$.

Problem 27

Now as $AB \parallel DC$ we have that $\triangle ABC$ is similar to $\triangle CDE$. This means that

$$\frac{DE}{EB} = \frac{EC}{AE} = \frac{DC}{AB} \,. \tag{387}$$

Now let θ be the angle $\angle ABD$ then the area of $\triangle CDE$ is

$$\frac{1}{2} DC \times DE \times \sin(\theta) \,.$$

The area of $\triangle ABE$ is

$$\frac{1}{2} AB \times EB \times \sin(\theta) \,.$$

Thus the ratio of these two is

$$r = \left(\frac{DC}{AB}\right)\left(\frac{DE}{EB}\right) \,.$$

From Equation 387 these two ratios are equal.

Draw the segment AD. Now the angle of $\angle ADB$ is ninety degrees so

$$\cos(\alpha) = \frac{DE}{AE} \,.$$

Now $AE = EB$ as $\triangle AEB$ is an isosceles triangle so

$$\cos(\alpha) = \frac{DE}{EB} \,.$$

Using this, the ratio r is $r = \cos^2(\alpha)$.

Problem 30

Recall that the fixed points of the mapping

$$x = f(x) \equiv \frac{1}{2}\left(x + \frac{a}{x}\right) \,,$$

are the points $x = \pm\sqrt{a}$. Note that these fixed point iterates are the Newton root iterates when we seek to find the root to the nonlinear equation $g(x) \equiv x^2 - a = 0$. This means that if we start from an arbitrary point (x_0, y_0, z_0, w_0) and we iterate the vector mapping

$$\begin{bmatrix} y \\ z \\ w \\ x \end{bmatrix} = \frac{1}{2} \begin{bmatrix} x + \frac{a}{x} \\ y + \frac{a}{y} \\ z + \frac{a}{z} \\ w + \frac{a}{w} \end{bmatrix},$$

we expect each variable to converge to either $\sqrt{a}$ or $-\sqrt{a}$.

From the given equations we see that if $x > 0$ then $y > 0$ then $z > 0$ and $w > 0$ thus all variables are positive. In the same way if $x < 0$ then all the other variables are also negative. Thus there are only two solutions to the above system of equations and they are

$$(\sqrt{a}, \sqrt{a}, \sqrt{a}, \sqrt{a}),$$

and

$$(-\sqrt{a}, -\sqrt{a}, -\sqrt{a}, -\sqrt{a}).$$

In this problem $a = 17$.

The 1986 AIME Examination

Problem 1

Let $v = \sqrt[4]{x}$ then the given expression is

$$v = \frac{12}{7 - v}.$$

We can write this as $v^2 - 7v + 12 = 0$ or $(v - 3)(v - 4) = 0$. This means that

$$v = 3 \quad \text{or} \quad v = 4.$$

In terms of x this means that

$$x = 3^4 = 81 \quad \text{or} \quad x = 4^4 = 256.$$

The sum of these two values is 337.

Problem 2

Denote this expression as E then we have

$$\begin{aligned}E &= (\sqrt{5}+\sqrt{6}+\sqrt{7})(\sqrt{5}+\sqrt{6}-\sqrt{7})(\sqrt{5}-\sqrt{6}+\sqrt{7})(-\sqrt{5}+\sqrt{6}+\sqrt{7}) \\ &= (\sqrt{5}+\sqrt{5}(\sqrt{6}-\sqrt{7})+\sqrt{5}(\sqrt{6}+\sqrt{7})+(6-7)) \\ &\times(-5+\sqrt{5}(\sqrt{6}+\sqrt{7})+\sqrt{5}(\sqrt{6}-\sqrt{7})-(\sqrt{6}-\sqrt{7})(\sqrt{6}+\sqrt{7})) \\ &= (4+2\sqrt{30})(-5+2\sqrt{30}-(6-7)) \\ &= (4+2\sqrt{30})(-4+2\sqrt{30}) = 2^2(2+\sqrt{30})(-2+\sqrt{30}) \\ &= -4(2+\sqrt{30})(2-\sqrt{30}) = -4(4-30) = 104\,.\end{aligned}$$

Problem 3

Recall that

$$\tan(x+y) = \frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}\,.$$

Next multiply the second equation given by $\tan(x)\tan(y)$ to get

$$\tan(y)+\tan(x) = 30\tan(x)\tan(y)\,.$$

This means that

$$\tan(x)\tan(y) = \frac{25}{30} = \frac{5}{6}\,.$$

Using this value we can now compute

$$\tan(x+y) = \frac{25}{1-\frac{5}{6}} = 150\,.$$

Problem 4

Add all of the equations together to get

$$6(x_1+x_2+x_3+x_4+x_5) = 186\,,$$

or

$$x_1+x_2+x_3+x_4+x_5 = 31\,.$$

If we put this expression into each equation we get

$$\begin{aligned}x_1+31 &= 6 \\ x_2+31 &= 12 \\ x_3+31 &= 24 \\ x_4+31 &= 48 \\ x_5+31 &= 96\,.\end{aligned}$$

Each of these is easy to solve and we find $x_4 = 17$ and $x_5 = 65$. This means that $3x_4+2x_5 = 181$.

Problem 5

Using long division to divide $n^3 + 100$ by $n + 10$ we get

$$\frac{n^3 + 100}{n + 10} = n^2 - 10n + 100 - \frac{900}{n + 10} \,. \tag{388}$$

Now to have $n^3 + 100$ be divisible by $n + 10$ we need the right-hand-side of Equation 388 to be an integer. By looking at for the roots to $n^2 - 10n + 100 = 0$ we learn that $n^2 - 10n + 100$ is always a positive integer when n is an integer. The fraction $\frac{900}{n+10}$ is not an integer however when n is large. Thus when we subtract it the entire expression will not be an integer if n is too large. The largest n can be and have the entire expression be an integer is if

$$\frac{900}{n + 10} = 1 \quad \text{or} \quad n = 890 \,.$$

Problem 6

Recall that

$$1 + 2 + 3 + \cdots + n = \frac{n(n + 1)}{2} \,.$$

To this expression we add a number m where $1 \leq m \leq n$ and get

$$\frac{n(n + 1)}{2} + m = 1986 = 2 \cdot 3 \cdot 331 \,. \tag{389}$$

To solve this lets imagine that this page was *not* added then n must solve

$$n^2 + n - 2^2 \cdot 3 \cdot 331 = 0 \,.$$

Solving this gives $n \approx 62.5$. Now as n must be an integer if $n = 62$ we have

$$\frac{n(n + 1)}{2} = 1953 \,,$$

which is less than 1985. Next if $n = 63$ we have

$$\frac{n(n + 1)}{2} = 2016 \,,$$

which is larger than 1985. If $n = 62$ then the expression in Equation 389 gives $m = 33$ which could be a possible solution. If $n = 61$ in Equation 389 gives $m = 95$ which is greater than $n = 62$. Other values of n don't give consistent solutions for m.

Problem 7

Number of the given form look like

$$a_n = \sum_{k=0}^{\infty} b_k 3^k \,,$$

n	b_0	b_1	b_2	b_3	a_n
1	1	0	0	0	1
2	0	1	0	0	3
3	1	1	0	0	4
4	0	0	1	0	9
5	1	0	1	0	10
6	1	1	1	0	13
7	0	0	0	1	27

Table 10: The first few numbers a_n for $n \in \{1, 2, 3, 4, 5, 6, 7\}$.

where b_k is either zero or one. If we look at a few of these numbers we get the table 10. If we look at the pattern we see that the sequence of b_k that are turned to one from zero to produce a_n are the binary representation of n (flipped left to right). Or said another way, b_0 is the units digit in the binary representation of n, b_1 is the twos digit in the binary representation of n, b_2 is the fours digit in the binary representation of n etc. Thus for $n = 100$ the binary representation is

$$100 = (b_0\,, b_1\,, b_2\,, b_3\,, b_4\,, b_5\,, b_6)_2 = (0, 0, 1, 0, 0, 1, 1)_2\,,$$

and all other b_k's are zero. This means that

$$a_{100} = 3^2 + 3^5 + 3^6 = 981\,.$$

Problem 8

The sum we want to evaluate is

$$S = \sum_d \log_{10}(d)\,,$$

where d is a proper divisor of the number 10^6. Note that

$$10^6 = (2 \cdot 5)^6 = 2^6 \cdot 5^6\,.$$

This means that all the dividers of 10^6 are numbers of the form $d = 2^m \cdot 5^n$ where $0 \leq m \leq 6$ and $0 \leq n \leq 6$. For d to be a *proper* divisor it means that $(m, n) \neq (0, 0)$ and $(m, n) \neq (6, 6)$. Let I be the set of integers (m, n) that give a proper divisor d in the form above. Then we have

$$\begin{aligned}
S &= \sum_I \log_{10}(2^m \cdot 5^n) = \sum_I (m \log_{10}(2) + n \log_{10}(5)) \\
&= \sum_{m=0}^{6} \sum_{n=0}^{6} (m \log_{10}(2) + n \log_{10}(5)) - (0 + 0) - (6 \log_{10}(2) + 6 \log_{10}(5))\,.
\end{aligned}$$

Where in the last equation we have put the terms $(m, n) = (0, 0)$ and $(m, n) = (6, 6)$ into the sum and then subtracted them. We can evaluate the above sum as

$$
\begin{aligned}
S &= \sum_{m=0}^{6} (6+1) m \log_{10}(2) + \sum_{n=0}^{6} (6+1) n \log_{10}(5) - 6 \log_{10}(10) \\
&= 7 \log_{10}(2) \sum_{m=0} 6m + 7 \log_{10}(5) \sum_{n=0}^{6} n - 6 \\
&= 7 \log_{10}(2) \left(\frac{6(7)}{2} \right) + 7 \log_{10}(5) \left(\frac{6(7)}{2} \right) - 6 \\
&= 3 \cdot 7^2 \log_{10}(10) - 6 = 141 \,.
\end{aligned}
$$

Problem 9

Using various parallel lines in the problem we have that

$$
\begin{aligned}
DG &= 510 - AD - CG = 510 - IP - HP = 510 - d \\
EH &= 450 - CH - BE = 450 - GP - FP = 450 - d \\
FI &= 425 - AI - BF = 425 - DP - EP = 425 - d \,.
\end{aligned}
$$

Using the fact that $\triangle DPG$ is similar to $\triangle ABC$ we have

$$
\frac{DP}{AB} = \frac{DG}{AC} \,,
$$

or using the lengths we know and what we know for DG we get

$$
\frac{DP}{425} = \frac{510 - d}{510} \,. \tag{390}
$$

Using the fact that $\triangle PHE$ is similar to $\triangle ACB$ we have

$$
\frac{PE}{AB} = \frac{EH}{BC} \,,
$$

or using the lengths we know and what we know for EH we get

$$
\frac{PE}{425} = \frac{510 - d}{450} \,. \tag{391}
$$

Using the fact that $d = DP + PE$ with Equations 390 (solved for DP) and 391 (solved for PE) we get a single equation for d. Solving this we get $d = 306$.

Problem 10

To start we write down the forms of the number we start with $x = (abc)$ and the numbers we add together. We have

$$\begin{aligned}
(abc) &= a \cdot 10^2 + b \cdot 10 + c \\
(acb) &= a \cdot 10^2 + c \cdot 10 + b \\
(bac) &= b \cdot 10^2 + a \cdot 10 + c \\
(bca) &= b \cdot 10^2 + c \cdot 10 + a \\
(cab) &= c \cdot 10^2 + a \cdot 10 + b \\
(cba) &= c \cdot 10^2 + b \cdot 10 + a \,.
\end{aligned}$$

If we add all of these together the left-hand-side gives

$$(2a + 2b + 2c) \cdot 10^2 + (2a + 2b + 2c) \cdot 10 + (2a + 2b + 2c) \,,$$

or

$$2(a + b + c)(10^2 + 10 + 1) = 222(a + b + c) \,.$$

We are told this must equal $N - x$ and so we have

$$222(a + b + c) = N + x \,.$$

As $a + b + c$ is an integer the left-hand-side is a multiple of 222 say $222k$. As $x > 0$ the multiple must be such that $222k > N$. Thus $k > \frac{N}{222} = 14.38739$. We can consider multiples $k \geq 15$ compute the product $222k$ and then compute x using the above expression or

$$x = 222(a + b + c) - N \,.$$

If the x we find has only three digits that sum to k we have found a solution. Doing this in the following R code

```
ks = 15:19
N = 3194
xs = 222 * ks - N
print(data.frame(k=ks, x=xs))
```

we get

```
   k    x
1 15  136
2 16  358
3 17  580
4 18  802
5 19 1024
```

From this we see that $x = 358$.

Problem 11

Now we can write $p(x)$ as

$$p(x) = 1 - x + x^2 - x^3 + x4 - \cdots + x^{16} - x^{17} = \sum_{k=0}^{17} (-x)^k$$
$$= \frac{1 - (-x)^{18}}{1 + x} = \frac{1 - x^{18}}{1 + x}$$

Then when we $x = y - 1$ we would have $g(y) = p(y - 1)$

$$\begin{aligned} g(y) &= \frac{1 - (y-1)^{18}}{1 + (y-1)} = \frac{1 - (y-1)^{18}}{y} \\ &= \frac{1}{y}\left(1 - \sum_{k=0}^{18} \binom{18}{k} y^k (-1)^{18-k}\right) \\ &= \frac{1}{y}\left(1 - \left(1 + 18y(-1)^{17} + \binom{18}{2} y^2 (-1)^{16} + \binom{18}{3} y^3 (-1)^{15} + \sum_{k=4}^{18} \binom{18}{k} y^k (-1)^{18-k}\right)\right) \\ &= \frac{1}{y}\left(18y - \binom{18}{2} y^2 + \binom{18}{3} y^3 - \sum_{k=4}^{18} \binom{18}{k} y^k (-1)^{18-k}\right) \\ &= 18 - \binom{18}{2} y + \binom{18}{3} y^2 - \sum_{k=4}^{18} \binom{18}{k} y^{k-1} (-1)^{18-k}\,. \end{aligned}$$

From this we see that the coefficient of y^2 is $\binom{18}{3} = 816$.

The 1987 AHSME Examination

Problem 1

We have
$$(1+x^2)(1-x^3) = 1 - x^3 + x^2 - x^5 \,.$$

Problem 2

Notice that $\triangle BED$ is an equilateral triangle with sides of length one. Thus $DE = 1$ and $CE = AD = 2$. Using these, the perimeter of the quadrilateral is
$$3 + 2 + 1 + 2 = 8 \,.$$

Problem 3

Numbers less than 100 that end with a seven include
$$7\,, 17\,, 27\,, 37\,, 47\,, 57\,, 67\,, 77\,, 87\,, 97 \,.$$
Tests of divisibility by three eliminate many of these (and $77 = 7 \times 11$) leaving only six numbers.

Problem 4

Let E be the given expression so
$$E = \frac{2 + 1 + \frac{1}{2}}{\frac{1}{4} + \frac{1}{8} + \frac{1}{16}} \,.$$
Multiply this by $\frac{16}{16}$ and we get
$$\frac{32 + 16 + 8}{4 + 2 + 1} = \frac{56}{7} = 8 \,.$$

Problem 5

As N times the "percent frequency" must be an integer number of cases in that bin we have that
$$0.125N\,, 0.5N\,, 0.25N\,,$$
must all be integers. The smallest N can be to have this be true is $N = 8$.

Problem 6

Let the angle "below" the angle y be denoted by y' and let the angle "below" the angle z be denoted by z'. Then in the "outside" triangle summing all of the interior angles we have

$$x + (z + z') + (y + y') = 180 \, .$$

Summing all of the interior angles in the smaller inside triangle we have

$$w + z' + y' = 180 \quad \text{so} \quad z' + y' = 180 - w \, .$$

If we put this expression for $z' + y'$ into the first expression we get

$$x + y + z + (180 - w) = 180 \, ,$$

so solving for x we have

$$x = -y - z + w \, .$$

Problem 7

Let x be the common value of $a - 1 = b + 2 = c - 3 = d + 4$ then we have

$$\begin{aligned} a &= x + 1 \\ b &= x - 2 \\ c &= x + 3 \\ d &= x - 4 \, . \end{aligned}$$

From this we see that $d < b < a < c$.

Problem 8

Drop a perpendicular to AB from D. Let the point where that perpendicular intersects AB be called D'. Then $AD' = 13 - 3 = 10$. To compute the length AD we can use Pythagorean theorem in $\triangle AD'D$ as

$$AD = \sqrt{10^2 + 4^2} \, .$$

To compute BD again use the Pythagorean theorem in $\triangle DCB$ where we find

$$BD = \sqrt{3^2 + 4^2} = 5 \, .$$

This means that

$$AD + BD = 5 + \sqrt{100 + 16} = 5 + 10\sqrt{1 + \frac{16}{100}} \approx 5 + 10 \left(1 + \frac{8}{100}\right) = 15.8 \, .$$

Here we have used the fact that $(1 + x)^\alpha \approx 1 + \alpha x$.

Problem 9

An arithmetic sequence s_n with a common difference d takes the form $s_n = a + dn$ for n an integer. Taking the first term in the sequence to be $n = 1$ we are told that

$$s_1 = a + d = x \quad \text{so} \quad d = x - a\,.$$

Thus

$$s_n = a + (x - a)n\,.$$

Using this we have

$$s_2 = a + 2(x - a) = 2x - a = b\,,$$

and

$$s_3 = a + 3(x - a) = 3x - 2a = 2x\,.$$

This last expression means that $x = 2a$. The ratio we are looking for is

$$\frac{a}{b} = \frac{a}{2x - a} = \frac{a}{4a - a} = \frac{1}{3}\,.$$

Problem 10

From the statements in the problem we must have

$$a = bc \tag{392}$$
$$b = ac \tag{393}$$
$$c = ab\,. \tag{394}$$

If we put Equation 393 into Equation 392 we get

$$a = bc = ac^2 \quad \text{so} \quad a(1 - c^2) = 0\,.$$

Thus $a = 0$ or $c = \pm 1$. As we are told that $a \neq 0$ we can ignore the solution $a = 0$.

If $c = 1$ then Equation 392 and 393 give $a = b$ and Equation 394 gives

$$ab = 1 \quad \text{or} \quad a^2 = b^2 = 1\,.$$

This means that $a = b = +1$ or $a = b = -1$. Thus we have found the solutions

$$(a, b, c) = (-1, -1, 1) \quad \text{and} \quad (a, b, c) = (1, 1, 1)\,.$$

If $c = -1$ then Equation 392 and 393 give $a = -b$ and Equation 394 gives

$$ab = -1 \quad \text{with } a = -b \text{ this is} \quad b^2 = 1\,.$$

This means that $b = \pm 1$ and the "pairs" $(a, b) = (-1, 1)$ and $(a, b) = (1, -1)$. Thus we have found the solutions

$$(a, b, c) = (-1, 1, -1) \quad \text{and} \quad (a, b, c) = (1, -1, -1)\,.$$

This gives a total of four solutions.

Problem 11

Lets add these two equations to get

$$(c+1)x = 5 \quad \text{or} \quad x = \frac{5}{c+1}\,.$$

From $x - y = 2$ we have that

$$y = x - 2 = \frac{5}{c+1} - \frac{2(c+1)}{c+1} = \frac{3-2c}{c+1}\,.$$

To be in the first quadrant we must have $x > 0$ and $y > 0$. To have $x > 0$ we must have $c > -1$. To have $y > 0$ we need to have

$$3 - 2c > 0 \quad \text{or} \quad c < \frac{3}{2}\,.$$

Combining these two we have

$$-1 < c < \frac{3}{2}\,.$$

Problem 12

The choice D is not possible for once the secretary types the fourth letter her box must have letters $1, 2, 3$. If the fifth letter comes before she can type another letter her box would then look like $1, 2, 3, 5$ and she would type the fifth letter next (as there are only a total of five letters). After that she would type the third letter.

Problem 13

Following the hint the total length is

$$L = \sum_{i=0}^{599} 2\pi r_i\,,$$

with

$$r_i = 1 + \frac{4}{599} i\,,$$

for $0 \le i \le 599$. Thus the total length is given by

$$\begin{aligned} L &= 2\pi \sum_{i=0}^{599} \left(1 + \frac{4}{599} i\right) = 2\pi \left(600 + \frac{4}{599} \sum_{i=1}^{599} i\right) \\ &= 2\pi \left(600 + \frac{4}{599} \left(\frac{599(600)}{2}\right)\right) = 3600\pi\,, \end{aligned}$$

when we simplify. To get meters we divide by 100 to get 36π.

Problem 14

Method 1: Let the side of the square be of length $2m$ and draw the segment MN. Then using the Pythagorean theorem we have

$$|MN| = \sqrt{2}m$$
$$|AM| = \sqrt{(2m)^2 + m^2} = \sqrt{5}m\,.$$

Next draw a line from A to C bisecting the line segment MN at the point P. Then

$$|MP| = \frac{1}{2}|MN| = \frac{m}{\sqrt{2}}\,.$$

From this we have that

$$\sin\left(\frac{\theta}{2}\right) = \frac{|MP|}{|AM|} = \frac{1}{\sqrt{10}}\,,$$

and

$$\cos\left(\frac{\theta}{2}\right) = \sqrt{1 - \sin\left(\frac{\theta}{2}\right)^2} = \sqrt{1 - \frac{1}{10}} = \frac{3}{\sqrt{10}}\,.$$

From these two we have

$$\sin(\theta) = 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) = 2\left(\frac{3}{10}\right) = \frac{3}{5}\,.$$

Method 2: Using the law of sines in the triangle $\triangle AMN$ we would have

$$\frac{\sin(\theta)}{|MN|} = \frac{\sin(\angle AMN)}{|AN|}\,.$$

Now as $|AM| = |AN|$ we have that

$$\angle AMN = \frac{\pi - \theta}{2} = \frac{\pi}{2} - \theta\,.$$

Thus

$$\sin(\angle AMN) = \sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = \cos\left(\frac{\theta}{2}\right)\,.$$

This means that the law of sines gives

$$\frac{\sin(\theta)}{\sqrt{2}m} = \frac{\cos\left(\frac{\theta}{2}\right)}{\sqrt{5}m} = \frac{3}{\sqrt{50}m}\,.$$

Solving for $\sin(\theta)$ we get $\frac{3}{5}$ as before.

Problem 15

Consider the expression
$$x^2y + xy^2 + x + y = 63\,,$$
and replace xy using $xy = 6$ in the above to get
$$6x + 6y + x + y = 63\,,$$
or
$$x + y = 9\,.$$
If we square this we get
$$(x + y)^2 = x^2 + 2xy + y^2 = 81\,,$$
so
$$x^2 + y^2 = 81 - 2xy = 81 - 2(6) = 81 - 12 = 69\,.$$

Problem 16

If we let the smallest integer be denoted n then we are told that
$$\begin{aligned} n &= VYZ_5 = V \cdot 5^2 + Y \cdot 5 + Z \\ n + 1 &= VYX_5 = V \cdot 5^2 + Y \cdot 5 + X \\ n + 2 &= VVW_5 = V \cdot 5^2 + V \cdot 5 + W\,. \end{aligned}$$
Note that each of the numbers $\{V, W, X, Y, Z\}$ must be a number that is between zero and four. In going from $n+1$ to $n+2$ note that the "fifth" place changes from Y to V. This will only happen if $X = 4$, $W = 0$, and $V = Y + 1$. Also in going from n to $n+1$ we must have $Z + 1 = X$. As $X = 4$ this means that $Z = 3$. Thus the only digits remaining for Y and V to be are $\{1, 2\}$ and to have $V = Y + 1$ we must have $Y = 1$ and $V = 2$. This means that
$$XYZ_5 = 4 \cdot 5^2 + 1 \cdot 5 + 3 = 100 + 5 + 3 = 108\,.$$

Problem 17

Let A, B, C, and D be the scores of the four people taking the test. Then we are told that
$$B + D = A + C\,. \tag{395}$$
We are also told that
$$A + C > B + D\,,$$
when B and C change scores. This means that the relationship in terms of the original variables is
$$A + B > C + D\,. \tag{396}$$

Finally we are told that

$$D > B + C\,. \tag{397}$$

Now using Equation 397 as all numbers are positive we can conclude that $D > B$ and $D > C$. Thus if $D > A$ then D is the largest score. We will now use Equation 395 to eliminate A from Equation 396. Now from equation 395 we have $A = B + D - C$ and putting that into Equation 396 we get

$$B + D - C + B > C + D \quad \text{which becomes} \quad B > C\,.$$

Thus we now know that $D > B > C$. This alone can be used to eliminate some of the answers for this problem. The question we now face is where does A fall in the above rankings. Using Equation 395 we have $A = B + D - C$ and then using the fact that $D > C$ we have

$$A > B + C - C = B\,,$$

using the fact that $B > C$ we get

$$A > C + D - C = D\,.$$

Thus the order is $A > D > B > C$.

Problem 18

Let a be the length of an algebra book and g the length of a geometry book. Let L be the shelf-length. Then we are told that

$$\begin{aligned} Aa + Hg &= L \\ Sa + Mg &= L \\ Ea &= L\,. \end{aligned}$$

From this last equation we have $E = \frac{L}{a}$. Using Cramer's rule to solve the first two equations above for a I find

$$a = \frac{\begin{vmatrix} L & H \\ L & M \end{vmatrix}}{\begin{vmatrix} A & H \\ S & M \end{vmatrix}} = \frac{LM - HL}{AM - SH} = L\left(\frac{M - H}{AM - SH}\right)\,.$$

This means that

$$E = \frac{AM - SH}{M - H}\,.$$

Problem 19 (what number is closest?)

Write the given expression as

$$\begin{aligned}
\sqrt{65} - \sqrt{63} &= \sqrt{64+1} - \sqrt{64-1} \\
&= 8\left(1 + \frac{1}{64}\right)^{1/2} - 8\left(1 - \frac{1}{64}\right)^{1/2} \\
&\approx 8\left(1 + \frac{1}{128} - \frac{1}{512} + \frac{1}{4096}\right) - 8\left(1 - \frac{1}{128} - \frac{1}{512} - \frac{1}{4096}\right) \qquad (398) \\
&= 2\left(\frac{8}{128}\right) + 2\left(\frac{8}{4096}\right) = \frac{33}{256} = 0.12890625\,.
\end{aligned}$$

Here in Equation 398 we have used the Taylor series

$$(1+x)^\alpha = 1 + \alpha x + \left(\begin{array}{c}\alpha \\ 2\end{array}\right) x^2 + \left(\begin{array}{c}\alpha \\ 3\end{array}\right) x^3 + O(x^4)\,. \qquad (399)$$

Note that we have to compute at least the cubic term in the Taylor series above as using only up to the quadratic term would give

$$\sqrt{65} - \sqrt{63} \approx \frac{1}{8} = 0.125\,,$$

which does not allow us to choose between the choices (A) and (B).

Problem 20

Write this expression as

$$\sum_{k=1}^{45} \log_{10}\left(\frac{\sin(k)}{\cos(k)}\right) + \sum_{k=46}^{89} \log_{10}\left(\frac{\sin(k)}{\cos(k)}\right)\,.$$

Recall that $\sin(x) = \cos(90 - x)$ and $\cos(x) = \sin(90 - x)$ so the above can be written as

$$\sum_{k=1}^{45} \log_{10}\left(\frac{\sin(k)}{\cos(k)}\right) + \sum_{k=46}^{89} \log_{10}\left(\frac{\cos(90-k)}{\sin(90-k)}\right)\,.$$

If we let $l = 90 - k$ this last sum can be written as

$$\sum_{l=1}^{44} \log_{10}\left(\frac{\cos(l)}{\sin(l)}\right)\,.$$

This means that the full sum we have is

$$\sum_{k=1}^{45} \log_{10}\left(\frac{\sin(k)}{\cos(k)}\right) + \sum_{l=1}^{44} \log_{10}\left(\frac{\cos(l)}{\sin(l)}\right)\,,$$

or

$$\log_{10}\left(\frac{\sin(45)}{\cos(45)}\right)+\sum_{k=1}^{44}\log_{10}\left(\frac{\sin(k)}{\cos(k)}\cdot\frac{\cos(k)}{\sin(k)}\right),$$

or

$$\log_{10}(1)+\sum_{k=1}^{44}\log_{10}(1)=0\,.$$

Problem 21

Let the side of the inscribed square in the Figure 1 triangle be denotes by s. Then we are told that $s^2 = 441$ so $s = 21$. Also in that triangle let A' be the point where the square's corner intersects the segment BC, let B' be the point where the square's corner intersects the segment AC, and finally let C' be the point where the square's corner intersects the segment AB.

As $\triangle ABC$ is an isosceles triangle we have that $\angle A = \angle C = 45$ degrees. We also have that $\angle A'B'C = \angle C'B'A = 45$ degrees. Thus $\triangle A'CB'$ and $\triangle AC'B'$ are also isosceles right triangles and $A'C = A'B' = s = C'B' = AC'$.

From the fact that $A'B'$ is parallel to AB and A' bisects BC we have that B' bisects AC.

We also have

$$BC = 2A'B = 2s = AB = 42\,,$$

so that

$$AC = \sqrt{4s^2 + 4s^2} = \sqrt{2}(2s) = 42\sqrt{2}\,.$$

Finally the area of the full triangle is then

$$\frac{1}{2}(2s)(2s) = 2s^2 = 882\,.$$

Now in Figure 2 let the side of the inscribed square be given the length s'. Now as the small triangles with corners A and C are congruent and isosceles we have

$$s' = AC - s' - s' = 42\sqrt{2} - 2s'\,.$$

Solving this gives $s' = 14\sqrt{2}$ so that the area of this inscribed triangle is ${s'}^2 = 392$.

Problem 22

If we imagine this situation if we drop a perpendicular from the center of the sphere O to the bottom of the spherical "bowel" region it will "puncture" the original plane of the water at the center of a circle $\mathcal{C}$ with a diameter 24. Let this center point be denoted by C. The distance from O to C is $r-8$ if r is the radius of the sphere. If we draw the radius of the sphere from the center O to a point on the circle $\mathcal{C}$ (denoted A) that segment will have a length r. Thus in a two dimensional plane the three points O, C, and A will form a right triangle with sides of length $OC = r-8$, $CA = \frac{24}{2} = 12$, and $AO = r$. Then using the Pythagorean theorem we have

$$OC^2 + CA^2 = AO^2\,,$$

or

$$(r-8)^2 + 12^2 = r^2\,.$$

Solving this for r we get $r = 13$.

Problem 23

We can write this expression as

$$x^2 = p(x-444)\,.$$

This means that p divides x^2 and as p is prime this means that p divides x. Thus we can write $x = np$ for some n. Putting this back into the above gives

$$n^2p^2 = p(444 - np) \quad \text{or} \quad n(n+1)p = 444 = 2^2 \cdot 3 \cdot 37\,.$$

This means that $p \in \{2, 3, 37\}$. We can take each of these values for p, put it in the original equation, and then solve for x. Doing this in the following R code

```
ps = c(2, 3, 37)
for( p in ps ){
  print(polyroot(c(-444*p, p, 1)))
}
```

We see that only $p = 37$ gives integer roots for x i.e. we have $x \in \{-148, 111\}$.

Problem 24

Let our function be denoted

$$f(x) = \sum_{n=0}^{N} a_n x^n\,,$$

where we must have $a_N \neq 0$ or else the polynomial is not Nth degree (it would be of degree less than N). For this expression we have

$$f(x^2) = \sum_{n=0}^{N} a_n x^{2n}\,. \tag{400}$$

We also have

$$f(x)^2 = \left(\sum_{n=0}^{N} a_n x^n\right)\left(\sum_{m=0}^{N} a_m x^m\right)$$
$$= \sum_{n=0}^{2N}\left(\sum_{k=0}^{n} a_k a_{n-k}\right) x^n \,. \quad (401)$$

In the above we have followed the convention that $a_k = 0$ if $k > N$.

As an aside, lets check that the above formula gives the correct coefficient for x^{2N}. In Equation 401 this coefficient is

$$\left(\sum_{k=0}^{2N} a_k a_{2N-k}\right) \,.$$

Now as $a_k = 0$ if $k > N$ so the above simplifies to

$$\left(\sum_{k=0}^{N} a_k a_{2N-k}\right) \,.$$

Now also $a_{2N-k} = 0$ if $2N - k > N$ or $k < N$. This means that the only nonzero element of the sum is when $k = N$ and we get the coefficient of a_N^2 (as we should).

Now if $f(x^2) = f(x)^2$ then this means that

$$\sum_{n=0}^{N} a_n x^{2n} = \sum_{n=0}^{2N}\left(\sum_{k=0}^{n} a_k a_{n-k}\right) x^n \,. \quad (402)$$

We will now equate the coefficients of different powers of x in the above expression. Equating the coefficients of x^{2N} on both sides of the above gives

$$a_N = a_N^2 \quad \text{or} \quad a_N(1 - a_N) = 0 \,.$$

Now $a_N \neq 0$ so we must have $a_N = 1$.

Next equating the coefficients of x^{2N-1} on both sides of the above gives

$$0 = \sum_{k=0}^{2N-1} a_k a_{2N-1-k} \,.$$

Using the fact that $a_k = 0$ for $k > N$ the right-hand-side of this becomes

$$0 = a_{N-1} a_N + a_N a_{N-1} = 2 a_N a_{N-1} = 2 a_{N-1} \,.$$

This means that $a_{N-1} = 0$.

Next equating the coefficients of x^{2N-2} on both sides of the above gives

$$a_{N-1} = \sum_{k=0}^{2N-2} a_k a_{2N-2-k} \,.$$

Using the fact that $a_k = 0$ for $k > N$ the right-hand-side of this becomes

$$a_{N-1} = a_{N-2}a_N + a_{N-1}^2 + a_N a_{N-2}\,.$$

Using the fact that $a_N = 1$ and $a_{N-1} = 0$ in the above we have that $a_{N-2} = 0$.

Next equating the coefficients of x^{2N-3} on both sides of the above gives

$$0 = \sum_{k=0}^{2N-3} a_k a_{2N-3-k}\,.$$

Using the fact that $a_k = 0$ for $k > N$ the right-hand-side of this becomes

$$0 = a_{N-3}a_N + a_{N-2}a_{N-1} + a_{N-1}a_{N-2} + a_N a_{N-3}\,.$$

Using the fact that $a_N = 1$ and $a_{N-2} = a_{N-1} = 0$ in the above we have that $a_{N-3} = 0$.

At this point it looks like when we study the coefficients of x^{2N-p} we learn that $a_{N-p} = 0$. We have shown this for $p \in \{1, 2, 3\}$. If we assume this holds true for all p the largest value we can take for p is $p = N$ where we would conclude that $a_0 = 0$. This means that we have shown that

$$f(x) = x^N\,.$$

Notice that for any N we have

$$f(x^2) = (x^2)^N = f(x)^2 = (x^N)^2\,,$$

is true. As N can take any natural number we have an infinite number of such functions. Note this is a different result than the back of the book obtains. If anyone sees anything wrong/correct with what I have done please contact me.

Problem 25

The area of this triangle can be written as

$$A = \frac{1}{2}\left\| \begin{vmatrix} 0 & 0 & 1 \\ 36 & 15 & 1 \\ x & y & 1 \end{vmatrix} \right\| = \frac{1}{2}|36y - 15x| = \frac{3}{2}|12y - 5x|\,.$$

Now to make this as small as possible where x and y are integers we need the expression $|12y - 5x|$ to be equal to one. One way we can do this is to take $x = 5$ and $y = 2$. Thus the smallest area is $\frac{3}{2}$.

Problem 26

Let X be a uniform random variable drawn from $[0, 2.5]$ and let $Y = 2.5 - X$. To solve this problem we will make a simple table of the ranges of X (and thus Y) such that the given condition will hold. For example if

$$0 < X < 0.5 \quad \text{so} \quad 2 < Y < 2.5\,,$$

then $[X] = 0$ and $[Y] = 2$. Here $[\cdot]$ is the "round" operation. Here $[X] + [Y] \neq 3$. Next if

$$0.5 < X < 1.0 \quad \text{so} \quad 1.5 < Y < 2.0\,,$$

then $[X] = 1$ and $[Y] = 2$. Here $[X] + [Y] = 3$. Next if

$$1.0 < X < 1.5 \quad \text{so} \quad 1.0 < Y < 1.5\,,$$

then $[X] = 1$ and $[Y] = 1$. Here $[X] + [Y] \neq 3$. Next if

$$1.5 < X < 2.0 \quad \text{so} \quad 0.5 < Y < 1.0\,,$$

then $[X] = 2$ and $[Y] = 1$. Here $[X] + [Y] = 3$. Finally if

$$2.0 < X < 2.5 \quad \text{so} \quad 0.0 < Y < 0.5\,,$$

then $[X] = 2$ and $[Y] = 0$. Here $[X] + [Y] \neq 3$.

Thus we can get a sum of three if

$$0.5 < X < 1.0 \quad \text{or} \quad 1.5 < X < 2.0\,.$$

This will happen with a probability of

$$\int_{0.5}^{1.0} \frac{dx}{2.5} + \int_{1.5}^{2.0} \frac{dx}{2.5} = \frac{2}{5}\,.$$

Problem 28

As the coefficients of the given polynomial are real all complex roots must come in complex conjugate pairs. Thus we can write the four roots as z_1, $\bar{z}_1$, z_2, and $\bar{z}_2$. As we are told that $|z_i| = 1$ we know that $z_i = e^{i\theta_i}$ and thus $\bar{z}_i = e^{-i\theta_i} = \frac{1}{z_i}$ for $i \in \{1, 2\}$. Given all of this when we write our polynomial in root factored form we would have

$$(z - e^{i\theta_1})(z - e^{-i\theta_1})(z - e^{i\theta_2})(z - e^{-i\theta_2}) = 0\,.$$

If we expand this and compare our result to the original polynomial given in the problem we will find that

$$-(e^{i\theta_1} + e^{-i\theta_1} + e^{i\theta_2} + e^{-i\theta_2}) = a\,,$$

or

$$-\left(\frac{1}{\bar{z}_1} + \frac{1}{z_1} + \frac{1}{\bar{z}_2} + \frac{1}{z_2}\right) = a\,.$$

Thus the desired sum we see is $-a$.

Problem 29

For the given sequence we want to know the value of n such that $t_n = \frac{19}{87}$. To start this problem lets evaluate a t_n for a few n. We have

$$\begin{aligned}
t_1 &= 1 \\
t_2 &= 1 + t_1 = 2 \\
t_3 &= \frac{1}{t_2} = \frac{1}{2} \\
t_4 &= 1 + t_2 = 1 + 2 = 3 \\
t_5 &= \frac{1}{t_4} = \frac{1}{3} \\
t_6 &= 1 + t_3 = 1 + \frac{1}{2} = \frac{3}{2}\,.
\end{aligned} \tag{403}$$

From the above we notice that if n is *even* have

$$t_n > 1\,,$$

and for n *odd*

$$t_n < 1\,.$$

These are is easily proved using

$$t_n = 1 + t_{\left(\frac{n}{2}\right)}\,,$$

when n is even and mathematical induction. Because of that as in this problem we are looking for the n such that $t_n = \frac{19}{87} < 1$ we know that n is odd. Because we now know that n is odd the remaining part of this problem will be to take the given recursion relationships and work backwards using them getting expressions for t_X for somewhat complicated values for X until we reach an identity such as $t_X = 1$ from which we can conclude that $X \equiv 1$.

To start this process as we know that n is odd we have

$$t_n = \frac{1}{t_{n-1}} = \frac{19}{87} \quad \text{so} \quad t_{n-1} = \frac{87}{19}\,.$$

As n is odd $n-1$ is even so

$$1 + t_{\left(\frac{n-1}{2}\right)} = \frac{87}{19} \quad \text{so} \quad t_{\left(\frac{n-1}{2}\right)} = \frac{68}{19}\,.$$

Here we see that $t_{\left(\frac{n-1}{2}\right)} > 1$ so we know that $\frac{n-1}{2}$ is even and we have

$$t_{\left(\frac{n-1}{2}\right)} = 1 + t_{\left(\frac{n-1}{4}\right)} = \frac{68}{19} \quad \text{so} \quad t_{\left(\frac{n-1}{4}\right)} = \frac{49}{19} > 1\,.$$

This means that $\frac{n-1}{4}$ is also even and so

$$t_{\left(\frac{n-1}{4}\right)} = 1 + t_{\left(\frac{n-1}{8}\right)} = \frac{49}{19} \quad \text{so} \quad t_{\left(\frac{n-1}{8}\right)} = \frac{30}{19}\,.$$

As this is larger than one know that $\frac{n-1}{8}$ is also even and that $t_{\left(\frac{n-1}{8}\right)}$ must satisfy

$$1 + t_{\left(\frac{n-1}{16}\right)} = \frac{30}{19},$$

thus

$$t_{\left(\frac{n-1}{16}\right)} = \frac{11}{19} < 1.$$

As this is less than one we know that $\frac{n-1}{16}$ is odd and must satisfy

$$t_{\left(\frac{n-1}{16}\right)} = \frac{1}{t_{\left(\frac{n-1}{16}-1\right)}} = \frac{11}{19}.$$

Which means that

$$t_{\left(\frac{n-17}{16}\right)} = \frac{19}{11} > 1.$$

This means that $\frac{n-17}{16}$ is even. At this point the pattern of what we are doing is clear and we will finish the calculations with less comments. The above means that

$$1 + t_{\frac{n-17}{32}} = \frac{19}{11} \quad \text{so} \quad t_{\frac{n-17}{32}} = \frac{8}{11} < 1.$$

Thus $\frac{n-17}{32}$ is odd so that

$$\frac{1}{t_{\left(\frac{n-17}{32}-1\right)}} = \frac{8}{11} \quad \text{so} \quad t_{\left(\frac{n-49}{32}\right)} = \frac{11}{8} > 1.$$

Thus $\frac{n-49}{32}$ is even so that

$$1 + t_{\left(\frac{n-49}{64}\right)} = \frac{11}{8} \quad \text{so} \quad t_{\frac{n-49}{64}} = \frac{3}{8} < 1.$$

So $\frac{n-49}{64}$ is odd so that

$$\frac{1}{t_{\left(\frac{n-49}{64}-1\right)}} = \frac{3}{8} \quad \text{so} \quad t_{\left(\frac{n-113}{64}\right)} = \frac{8}{3} > 1.$$

So $\frac{n-113}{64}$ is even so that

$$1 + t_{\left(\frac{n-113}{128}\right)} = \frac{8}{3} \quad \text{so} \quad t_{\left(\frac{n-113}{128}\right)} = \frac{5}{3} > 1.$$

So $\frac{n-113}{128}$ is even so that

$$1 + t_{\left(\frac{n-113}{256}\right)} = \frac{5}{3} \quad \text{so} \quad t_{\left(\frac{n-113}{256}\right)} = \frac{2}{3} < 1.$$

So $\frac{n-113}{256}$ is odd so that

$$\frac{1}{t_{\left(\frac{n-369}{256}\right)}} = \frac{2}{3} \quad \text{so} \quad t_{\left(\frac{n-369}{256}\right)} = \frac{3}{2} > 1.$$

So $\frac{n-369}{256}$ is even so that

$$1 + t_{\left(\frac{n-369}{512}\right)} = \frac{3}{2} \quad \text{so} \quad t_{\left(\frac{n-369}{512}\right)} = \frac{1}{2}.$$

At this point from Equation 403 we know that $t_3 = \frac{1}{2}$ so we must have

$$\frac{n-369}{512} = 3 \Rightarrow n = 1905.$$

This means that the sum of digits in n is 15.

Problem 30

We first consider if the point E is on the segment AC or BC. Let h_E be the vertical height/distance from the point E to the segment AB and h_C the same same thing from the point C. Then

$$\frac{\text{Area}\triangle ADE}{\text{Area}\triangle ACB} = \frac{h_E AD}{h_C AB}\,.$$

Now using the angles in the triangle ADE (namely angles $\angle EAD$ and $\angle EDA$) we have

$$AD = h_E + h_E \frac{1}{\tan(60^\circ)} = h_E\left(1 + \frac{1}{\sqrt{3}}\right)\,.$$

Now using the angles in the triangle ABC (namely angles $\angle CAB$ and $\angle ABC$) we have

$$AB = h_C + h_C \cot(30^\circ) = h_c\left(1 + \sqrt{3}\right)\,.$$

This means that

$$\frac{\text{Area}\triangle ADE}{\text{Area}\triangle ACB} = \frac{h_E^2\left(1 + \frac{1}{\sqrt{3}}\right)}{h_C^2\left(1 + \sqrt{3}\right)} = \frac{h_E^2}{h_C^2\sqrt{3}}\,.$$

If we consider "sliding" the point E to C from where it is located in the original drawing note that triangles ADE and ABC then have the same height

$$h_E = h_C\,,$$

This means that

$$\frac{\text{Area}\triangle ADE}{\text{Area}\triangle ACB} = \frac{1}{\sqrt{3}} > \frac{1}{2}\,.$$

This means that the point E must be on the segment AC as in the original drawing.

From that argument we know that

$$h_E = \alpha h_C\,,$$

with $\alpha < 1$. Now if

$$\text{Area}\triangle ADE = \frac{1}{2}\text{Area}\triangle ABC\,,$$

we must have

$$\frac{1}{2} AD h_E = \frac{1}{4} AB h_C\,,$$

or using the angles in the two triangles as above we have

$$\frac{1}{2}\left(1 + \frac{1}{\sqrt{3}}\right) h_E^2 = \frac{1}{4}\left(1 + \sqrt{3}\right) h_C^2\,,$$

or

$$\alpha^2\left(1 + \frac{1}{\sqrt{3}}\right) = \frac{1}{2}(1 + \sqrt{3})\,.$$

If we solve for α we get

$$\alpha = \left(\frac{3}{4}\right)^{1/4}\,.$$

We then have

$$\frac{AD}{AB} = \frac{\left(1 + \frac{1}{\sqrt{3}}\right) \alpha h_C}{(1+\sqrt{3})h_C} = \frac{1}{\sqrt[4]{12}},$$

when we simplify.

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Problem 1

To get the desired number when we don't allow any carrying the units digit of m can be

- zero and then the units digit of n would then need to be two or
- one and then the units digit of n would then need to be one or
- two and then the units digit of n would then need to be zero.

This gives three possible choices for the units digits for m and n such that their sum has a units digit of two.

In the same way, to get the desired sum the tens digit of m can be

- zero and then the tens digit of n would then need to be nine or
- one and then the tens digit of n would then need to be eight or
- two and then the tens digit of n would then need to be seven or
- three and then the tens digit of n would then need to be six or
- four and then the tens digit of n would then need to be five or
- five and then the tens digit of n would then need to be four or
- six and then the tens digit of n would then need to be three or
- seven and then the tens digit of n would then need to be two or
- eight and then the tens digit of n would then need to be one or
- nine and then the tens digit of n would then need to be zero.

This gives ten possible choices for the tens digits for m and n such that their sum has a tens digit of nine.

In the same way, there are five ways to specify the hundreds digits of m and n and two ways to specify the thousands digit for m and n. This gives a total number of "simple" numbers with this sum of

$$3 \times 10 \times 5 \times 2 = 300\,.$$

Problem 2

The largest distance between two points on two spheres will be the points on the spheres where the segment connecting their two centers (when extended if needed) would intersect the spheres. This largest distance would then go from the farthest point of the first sphere to that sphere's center (say C_1), from that center to the other spheres center (say C_2), and then from C_2 to the farthest point on the second sphere. This gives a maximum distance of

$$r_1 + d(C_1, C_2) + r_2 = 19 + \sqrt{(12+2)^2 + (8+10)^2 + (-16-5)^2} + 87 = 137\,.$$

Problem 3

If we write down the divisors of a natural number k in increasing order we have

$$1\,, d_1\,, d_2\,, \ldots\,, d_{n-1}\,, d_n\,, k\,.$$

For example, the divisors of twelve are

$$1\,, 2\,, 3\,, 4\,, 6\,, 12\,.$$

When we write down the divisors of a number in this way if n is we always have the fact that the "outer" products of the divisors equal k or

$$1 \cdot k = d_1 \cdot d_n = d_2 \cdot d_{n-1} = d_3 \cdot d_{n-2} = \cdots\,.$$

From this if our number k is "nice" then we can only have two divisors and $n = 2$ since otherwise the product of these numbers would be larger than k and we have

$$k = d_1 \cdot d_2\,.$$

By the Fundamental Theorem of Arithmetic the divisors of k will be products of powers of prime numbers. This means that d_1 will be the smallest prime that divides k and d_2 will be the next smallest prime that divides k or d_1^2 (if d_1^2 is a divisor of k). We can find the first ten "nice" numbers by computing them using the above rules. The first few prime numbers are

$$\{2\,, 3\,, 5\,, 7\,, 11\,, 13\,, 17\,, 19\,, 23\,, 29\}\,.$$

Then the numbers that are the product of the first two distinct primes are

$$
\begin{aligned}
2 \cdot 3 &= 6 \\
2 \cdot 5 &= 10 \\
2 \cdot 7 &= 14 \\
2 \cdot 11 &= 22 \\
2 \cdot 13 &= 26 \\
2 \cdot 17 &= 34 \\
2 \cdot 19 &= 38 \\
3 \cdot 5 &= 15 \\
3 \cdot 7 &= 21 \\
3 \cdot 11 &= 33 \\
3 \cdot 13 &= 39 \\
5 \cdot 7 &= 35 \\
5 \cdot 11 &= 55 \,.
\end{aligned}
$$

The numbers that are the product of a prime and second power of that prime are

$$
\begin{aligned}
2 \cdot 2^2 &= 8 \\
3 \cdot 3^2 &= 27 \\
5 \cdot 5^2 &= 125 \,.
\end{aligned}
$$

We could go further but these are all of the numbers we need to find the first ten "nice" numbers. Selecting and then ordering them from the above numbers we find the sum we need is given by

```
sum(c(6, 8, 10, 14, 15, 21, 22, 26, 27, 33))
```

This evaluates to 182.

Problem 4

If $x > 60$ then this expression is

$$|y| = -\frac{3}{4}x + 60 \,.$$

Now as $|y| > 0$ the valid x values for the right-hand-side are when

$$60 - \frac{3}{4}x > 0 \quad \text{or} \quad x < 80 \,.$$

Thus when $60 < x < 80$ we have the two lines

$$y = \pm \left(60 - \frac{3}{4}x \right) \,.$$

When $x = 60$ we have $|y| = 15$ and when $x = 80$ we have $|y| = 0$. Now if $0 < x < 60$ this expression is

$$|y| = \frac{5}{4}x - 60\,.$$

To have $|y| > 0$ the values of x must be constrained as

$$\frac{5}{4}x - 60 > 0 \quad \text{or} \quad x > 48\,.$$

When $x = 48$ we have $|y| = 0$ and we can draw a picture in the (x, y) plane of what this shape looks like. We can also show that if $0 < x < 48$ or $x < 0$ then our equation has no solution. From a drawing of the four lines that make up this shape we can break the total area up into the area of four triangles to get the total area is

$$A = 2\left(\frac{1}{2}(60 - 48)15 + \frac{1}{2}20(15)\right) = 15 \cdot 32 = 480\,.$$

Problem 5

Lets "complete-the-square" involving the x and y variables. To do this lets write this expression as

$$y^2 + 3x^2y^2 - 30x^2 = 517\,.$$

Then lets write the left-hand-side as

$$(y^2 + A)(Bx^2 + C) = Bx^2y^2 + Cy^2 + ABx^2 + AC\,.$$

To have this equal the right-hand-side we must take $A = -10$, $B = 3$, and $C = 1$ so that the original expression can be written as

$$(y^2 - 10)(3x^2 + 1) + 10 = 517\,,$$

or

$$(y^2 - 10)(3x^2 + 1) = 507\,,$$

If we perform a prime factorization of 507 we get

$$(y^2 - 10)(3x^2 + 1) = 3 \cdot 13^2\,.$$

As x and y are integers the left-hand-side will also be an integer thus we have two factorizations. This can happen only if

$$y^2 - 10\,,$$

is equal to one of 1, 3, 13, $3 \cdot 13$, 13^2, or $3 \cdot 13^2$. Solving for y in each of these the only one that has an integer solution for y is when

$$y^2 - 10 = 3 \cdot 13 = 39 \quad \text{so} \quad y = \pm 7\,.$$

If this is true then we must have

$$3x^2 + 1 = 13 \quad \text{so} \quad x = \pm 2\,.$$

Using these we can compute

$$3x^2y^2 = 3 \cdot 4 \cdot 49 = 588\,.$$

Problem 6

Let h be the height of the trapezoid $XPQY$ so that its area can also be written as

$$\text{Area}(XPQY) = \frac{1}{2}h(PQ + XY)\,. \tag{404}$$

As $BC = 19$ the area of the trapezoid $ZWPQ$ i.e. the one "above" the trapezoid $XPQY$ can be written as

$$\frac{1}{2}(9 - h)(PQ + WZ)\,.$$

As we are told that all areas are equal by setting these two expressions equal to each other and using the fact that $WZ = XY$ we get

$$h(PQ + XY) = (19 - h)(PQ + XY)\,.$$

This means that $h = 19 - h$ so we see that $h = \frac{19}{2}$.

The area of the trapezoid $XPQY$ is $\frac{1}{4}$ the area of the rectangle $ABCD$ so

$$\text{Area}(XPQY) = \frac{1}{4}AB \cdot BC\,. \tag{405}$$

Next note that XY is $\frac{1}{4}$ the perimeter of the rectangle $ABCD$ so

$$XY = \frac{1}{4}(2AB + 2BC) = \frac{1}{2}(AB + 19)\,.$$

Using Equations 404 and 405 we can write area of the trapezoid $XPQY$ as

$$\frac{1}{2}h(PQ + XY) = \frac{1}{4}AB \cdot BC\,.$$

or using what we know about h, PQ, the expression for XY, and BC we get

$$\frac{1}{2}\left(\frac{19}{2}\right)\left(87 + \frac{1}{2}(AB + 19)\right) = \frac{1}{4}AB \cdot 19\,.$$

If we solve for AB we find $AB = 193$.

Problem 7

From the Fundamental Theorem of Arithmetic we have that a, b, and c can be written as

$$\begin{aligned}
a &= 2^{a_2}3^{a_3}5^{a_5}7^{a_7}\cdots \\
b &= 2^{b_2}3^{b_3}5^{b_5}7^{b_7}\cdots \\
c &= 2^{c_2}3^{c_3}5^{c_5}7^{c_7}\cdots\,.
\end{aligned}$$

From this we can conclude that

$$[a, b] = 2^{\max(a_2,b_2)}3^{\max(a_3,b_3)}5^{\max(a_5,b_5)}7^{\max(a_7,b_7)}\cdots\,,$$

and as we are told that

$$[a, b] = 1000 = 2^3 \cdot 5^3\,,$$

The only prime factors of a and b must be the numbers two and five and we have that

$$\begin{aligned}\max(a_2, b_2) &= 3\\ \max(a_5, b_5) &= 3\,.\end{aligned}$$

In the same way from

$$[b, c] = 2^4 \cdot 5^3\,,$$

we conclude that

$$\begin{aligned}\max(b_2, c_2) &= 4\\ \max(b_5, c_5) &= 3\,.\end{aligned}$$

From

$$[c, a] = 2^4 \cdot 5^3\,,$$

we conclude that

$$\begin{aligned}\max(a_2, c_2) &= 4\\ \max(a_5, c_5) &= 3\,.\end{aligned}$$

The powers of two and five decouple and for the variables a_2, b_2, and c_2 we must satisfy

$$\max(a_2, b_2) = 3 \quad (406)$$

$$\max(b_2, c_2) = 4 \quad (407)$$

$$\max(a_2, c_2) = 4\,. \quad (408)$$

We would like to count how many solutions to the above equations there are. From Equation 406 we see that $a_2 = 3$ (with $0 \le b_2 \le 3$) or $b_2 = 3$ (with $0 \le a_2 \le 3$).

In the first case if $a_2 = 3$, with $0 \le b_2 \le 3$, then from Equation 408 we see that $c_2 = 4$. This means that

$$\begin{aligned}a &\propto 2^3\\ b &\propto 2^{b_2} \quad \text{where} \quad 0 \le b_2 \le 3\\ c &\propto 2^4\,.\end{aligned}$$

There are four "numbers" of this form.

In the second case if $b_2 = 3$, with $0 \le a_2 \le 3$, then from Equation 407 we see that $c_2 = 4$. This means that

$$\begin{aligned}a &\propto 2^{a_2} \quad \text{where} \quad 0 \le a_2 \le 3\\ b &\propto 2^3\\ c &\propto 2^4\,.\end{aligned}$$

There are four "numbers" of this form. Note that in this last set if $a_2 = 3$ we have a "form" of a number that is found in the previous set. This gives $4 + 3 = 7$ total numbers that can be the "base two factorization" of a, b, and c.

The powers five imply that for the variables a_5, b_5, and c_5 we must satisfy

$$\max(a_5, b_5) = 3 \quad (409)$$
$$\max(b_5, c_5) = 3 \quad (410)$$
$$\max(a_5, c_5) = 3\,. \quad (411)$$

We now would like to count how many solutions to the above equations there are. From Equation 409 we see that $a_5 = 3$ (with $0 \le b_5 \le 3$) or $b_5 = 3$ (with $0 \le a_5 \le 3$).

In the first case, if $a_5 = 3$ with $0 \le b_5 \le 3$, then

- if $0 \le b_5 \le 2$ then from Equation 410 we have $c_5 = 3$.
- if $b_5 = 3$ then from Equation 410 we can have $0 \le c_5 \le 3$.

Both of these satisfy all of the required maximization conditions on a_5, b_5, and c_5. This means that

$$\begin{aligned} & a \propto 5^3 \\ & b \propto 5^{b_5} \quad \text{with} \quad 0 \le b_5 \le 2 \quad \text{and} \quad c \propto 5^3 \\ & b \propto 5^3 \quad \text{with} \quad c = 5^{c_5} \quad \text{with} \quad 0 \le c_5 \le 3\,. \end{aligned}$$

If we write the "thing" that a, b, and c must be proportional to based on the above conditions as a three term column vector the above conditions give the following factors

$$\begin{bmatrix} 5^3 \\ 5^0 \\ 5^3 \end{bmatrix}, \begin{bmatrix} 5^3 \\ 5^1 \\ 5^3 \end{bmatrix}, \begin{bmatrix} 5^3 \\ 5^2 \\ 5^3 \end{bmatrix}, \begin{bmatrix} 5^3 \\ 5^3 \\ 5^0 \end{bmatrix}, \begin{bmatrix} 5^3 \\ 5^3 \\ 5^1 \end{bmatrix}, \begin{bmatrix} 5^3 \\ 5^3 \\ 5^2 \end{bmatrix}, \begin{bmatrix} 5^3 \\ 5^3 \\ 5^3 \end{bmatrix}.$$

These are seven distinct things.

In the second case, if $b_5 = 3$ with $0 \le a_5 \le 3$, then

- if $0 \le a_5 \le 2$ then from Equation 411 we have $c_5 = 3$.
- if $a_5 = 3$ then from Equation 411 we can have $0 \le c_5 \le 3$.

Both of these satisfy all of the required maximization conditions on a_5, b_5, and c_5. This means that

$$\begin{aligned} & b \propto 5^3 \\ & a \propto 5^{a_5} \quad \text{with} \quad 0 \le a_5 \le 2 \quad \text{and} \quad c \propto 5^3 \\ & a \propto 5^3 \quad \text{with} \quad c = 5^{c_5} \quad \text{with} \quad 0 \le c_5 \le 3\,. \end{aligned}$$

Again if we write the "thing" that a, b, and c must be proportional to based on the above conditions as a three term column vector the above conditions give the following factors

$$\begin{bmatrix} 5^0 \\ 5^3 \\ 5^3 \end{bmatrix}, \begin{bmatrix} 5^1 \\ 5^3 \\ 5^3 \end{bmatrix}, \begin{bmatrix} 5^2 \\ 5^3 \\ 5^3 \end{bmatrix}, \begin{bmatrix} 5^3 \\ 5^3 \\ 5^0 \end{bmatrix}, \begin{bmatrix} 5^3 \\ 5^3 \\ 5^1 \end{bmatrix}, \begin{bmatrix} 5^3 \\ 5^3 \\ 5^2 \end{bmatrix}, \begin{bmatrix} 5^3 \\ 5^3 \\ 5^3 \end{bmatrix}.$$

There are seven items in this list and of these we get four of them (the last four) that are duplicates of items in the previous list giving $7 + 7 - 4 = 10$ unique ways we can have a "five" factor for a, b, and c.

Combining the seven ways we can have a "two" factor and the ten ways we can have a "five" factor for a, b, and c we get a total of $7 \times 10 = 70$ tuples of the form (a, b, c).

Problem 8

The given inequality can be written as two others. One of them is

$$8(n + k) < 15n\,,$$

which we can simplify to

$$8k < 7n\,. \tag{412}$$

The second one is

$$13n < 7(n + k)\,,$$

which can be simplified to

$$6n < 7k\,. \tag{413}$$

The above two inequalities can be "solved" for k to get

$$\frac{6n}{7} < k < \frac{7n}{8}\,. \tag{414}$$

Lets see what restrictions this inequality has on possible values for k. If we take $n = 1$ in Equation 414 we get

$$\frac{6}{7} < k < \frac{7}{8}\,,$$

from which we see that no integer $k \geq 1$ satisfies this.

Taking $n = 2$ Equation 414 gives

$$\frac{12}{7} < k < \frac{14}{8} \quad \text{or} \quad 1\,\frac{5}{7} < k < 1\,\frac{3}{4}\,,$$

from which we again see that no integer $k \geq 1$ satisfies this.

Taking $n = 3$ Equation 414 gives

$$\frac{18}{7} < k < \frac{21}{8} \quad \text{or} \quad 2\,\frac{4}{7} < k < 2\,\frac{5}{8}\,,$$

from which we see that no integer $k \geq 1$ satisfies this.

Notice that we can write Equation 414 as

$$\frac{48n}{56} < \frac{56k}{56} < \frac{49n}{56}\,.$$

or

$$48n < 56k < 49n\,. \tag{415}$$

Again taking a few small n we see that no k exists that will satisfy this condition. We would like to make n as large as possible so that there is only one value of k that satisfies the above. In between (and not including them) the two integers $48n$ and $49n$ there are

$$49n - 48n - 1 = n - 1\,,$$

integers. Thus if we pick a n such that there are *two* multiples of 56 i.e.

$$n - 1 = 2(56) = 112\,,$$

Then for that value of $n = 113$ we will have two integer solutions for k. For example with $n = 113$ we find the bounds in Equation 415 given by

$$5424 < 56k < 5537\,.$$

In the above $k \in \{97, 98\}$ satisfy this. If we take $n = 112$ then we find the bounds in Equation 415 given by

$$5376 < 56k < 5488\,.$$

In the above only $k = 97$ satisfy this. Thus $n = 112$ is the largest n with a unique k.

Problem 9

As all angles at P are equal we have

$$\angle APB = \angle BPC = \angle APC = \frac{360}{3} = 120^\circ\,.$$

Recall that

$$\cos(120^\circ) = -\frac{1}{2}\,.$$

As this expression is so "clean" lets use the law of cosigns in the triangle APB as

$$\begin{aligned} AB^2 &= AP^2 + BP^2 - 2AP \cdot BP\cos(120^\circ) \\ &= 10^2 + 6^2 - 2(10)(6)\left(-\frac{1}{2}\right) = 196\,. \end{aligned}$$

Using the Pythagorean theorem in the triangle ABC we have

$$196 + BC^2 = AC^2 \tag{416}$$

Next using the law of cosigns in the triangle BPC as

$$BC^2 = 36^2 + CP^2 - 2(6)(CP)\left(-\frac{1}{2}\right) = 36 + CP^2 + 6CP\,. \tag{417}$$

Using the law of cosigns in the triangle APC as

$$AC^2 = 10^2 + CP^2 - 2(10)(CP)\left(-\frac{1}{2}\right) = 100 + CP^2 + 10CP\,. \tag{418}$$

As I want to solve for CP we will use Equation 417 (to replace BC^2) and 418 (to replace AC^2) in Equation 416 to get

$$196 + 36 + CP^2 + 6CP = 100 + CP^2 + 10CP\,,$$

which solving gives $CP = 33$.

Problem 10

Let v_a be the velocity of Al and v_b be the velocity of Bob (in steps per unit time). Then we are told that

$$v_a = 3v_b\,.$$

Now let T_a and T_b be the time that Al and Bob take to do their walk. Then we are told that

$$v_aT_a = 150$$
$$v_bT_b = 75\,.$$

In terms of v_b the first equation above is

$$3v_bT_b = 150 \quad \text{or} \quad v_bT_b = 50\,.$$

This means that

$$\frac{T_a}{T_b} = \frac{50}{75} = \frac{2}{3}\,.$$

If we let L be the length of the escalator in steps then we also have

$$L = (v_a + v_e)T_a \tag{419}$$
$$L = (v_b - v_e)T_a\,. \tag{420}$$

Here v_e is the velocity of the escalator. We can write Equation 419 as

$$v_eT_a = L - 150\,,$$

and Equation 419 as

$$v_eT_b = 75 - L\,.$$

If we take the ratio of these two and use the result above we get

$$\frac{T_a}{T_b} = \frac{L - 150}{75 - L} = \frac{2}{3}\,.$$

Solving for L I find $L = 120$.

Problem 11

Lets write 3^{11} as the sum of k terms starting at a value of s in the following way

$$\begin{aligned}
3^{11} &= s + (s+1) + (s+2) + \cdots + (s+k-1) \\
&= \sum_{l=0}^{k-1}(s+l) = sk + \sum_{l=0}^{k-1} l = sk + \sum_{l=1}^{k-1} l \\
&= sk + \frac{1}{2}(k-1)k = k\left(s + \frac{k-1}{2}\right) .
\end{aligned} \tag{421}$$

To make sure that both sides are of this expression are integers we need to write the above like

$$2 \cdot 3^{11} = k(2s + k - 1) .$$

Based on how s and k are defined to maximize k we would want to minimize s at the same time. Based on the above factorization and that we want to maximize k (we could put the factor of two with the factor $2s + k - 1$) we will propose that

$$\begin{aligned}
k &= 2 \cdot 3^p \\
2s + k - 1 &= 3^{11-p} .
\end{aligned}$$

for some p where $0 \le p \le 11$. Now for p's in this range the above will be an identity. Based on this we have that

$$s = \frac{3^{11-p} - k + 1}{2} = \frac{3^{11-p} - 2 \cdot 3^p + 1}{2} .$$

Now we can make k large by taking p large. The largest we can take p is $p = 11$ but then if we do that from the above we see that $s < 0$. Thus we want to take p large such that $s > 0$. This last condition is

$$3^{11-p} - 2 \cdot 3^p + 1 > 0 . \tag{422}$$

We can multiply this by 3^p to get a quadratic equation is 3^p that we can solve to get approximately

$$3^p \approx 3^{11/2} .$$

This means that p should be "close" to $p \approx \frac{11}{2} = 5.5$. We will take $p = 5$ and see if Equation 422 is satisfied. We find that for $p \le 5$ it is while for $p \ge 6$ it is not. This means that

$$\begin{aligned}
p &= 5 \\
k &= 2 \cdot 3^p = 486 \\
s &= 122 .
\end{aligned}$$

Problem 12

For this problem we want to find the smallest m such that

$$n < \sqrt[3]{m} = n + r < n + 10^{-3} . \tag{423}$$

This is equivalent to finding the smallest m such that

$$n^3 < m < (n + 10^{-3})^3 \,. \tag{424}$$

Now as n is fixed in Equation 424 from all possible integers m that could satisfy the above the *smallest* one would be $m = n^3 + 1$ (i.e. one more than the left-hand-end point). Because of that argument we will find the smallest n (this will then make m the smallest) such that

$$n^3 + 1 < (n + 10^{-3})^3 \,.$$

If we expand the right-hand-side of this we get

$$n^3 + 1 < n^3 + 3n^2(10^{-3}) + 3n(10^{-3})^2 + 10^{-9} \,,$$

which is equivalent to

$$n^2 + \frac{n}{10^3} - \frac{1}{3}\left(\frac{10^9 - 1}{10^6}\right) > 0 \,. \tag{425}$$

Obviously for n large this will be true but for small n (say $n = 0$) it will not be true. Thus lets find the values of n where the left-hand-side is *equal* to zero. To simplify the algebra lets approximate the above quadratic with

$$n^2 + \frac{n}{10^3} - \frac{10^3}{3} > 0 \,. \tag{426}$$

Using the quadratic formula on the left-hand-side (set equal to zero) and simplifying we find

$$\begin{aligned} n &= \frac{-\frac{1}{10^3} \pm \sqrt{\frac{1}{10^6} - 4\left(-\frac{10^3}{3}\right)}}{2} = \frac{-10^{-3} \pm \sqrt{10^{-3} + \frac{4}{3}10^3}}{2} \\ &\approx \pm\frac{1}{2}\sqrt{\frac{4}{3}10^3} = \pm\frac{1}{\sqrt{3}}10^{3/2} \,. \end{aligned}$$

Evaluating this approximately (since $n > 0$) we have

$$n \approx \frac{1}{\sqrt{3}}9^{3/2} = \frac{1}{\sqrt{3}}3^3 = \sqrt{3} \cdot 3^2 = \sqrt{3} \cdot 9 \approx 1.7 \cdot 9 \approx 18 \,.$$

While there are a lot of approximations in the above we can start with $n = 18$ and then evaluate the right-hand-side of Equation 425 to see if it is satisfied and then move n up or down by one until it is. We can do that in the following `R` code

```
ns = c(18, 19)
rhs = function(n){
  n^2 + n/1000 - (1/3)*(10^9 - 1)/10^6
}
sapply(ns, rhs)
```

Running the above gives

```
[1] -9.315333 27.685667
```

Thus $n = 18$ is "too small" but $n = 19$ is large enough to make the left-hand-side positive. With this value of n we have $m = n^3 + 1$. We can check that these numbers satisfy the constraints of the original problem. In R we have

```
n = 19
m = n^3 + 1
print(c(n, m^(1/3), n+10^(-3)))
```

This gives the output

```
[1] 19.00000 19.00092 19.00100
```

indicating that we have found a solution. One can check that for smaller n Equation 423 is not satisfied.

Problem 13

From the given "bubble" procedure as we progress during the first pass when we consider exchanging two numbers at locations i and $i+1$ (for $1 \le i \le n-1$) before the exchange the number at location i is the largest of the r_j numbers for $j \le i$. Another way to say this is to note that when we compare r_1 and r_2 and exchange if needed when we end we have

$$r_2' = \max(r_1, r_2)\,.$$

Comparing the value r_2 with the unseen r_3 when finished we see that

$$r_3' = \max(r_2', r_3) = \max(r_1, r_2, r_3)\,.$$

Comparing the value r_3' with the unseen r_4 when finished we see that

$$r_4' = \max(r_3', r_4) = \max(r_1, r_2, r_3, r_4)\,.$$

Thus if during the first pass when we compare r_{19}' to r_{20} the only way r_{20} will stay where it is will be if

$$r_{20} > \max(r_1, r_2, \ldots, r_{18}, r_{19})\,.$$

Then we need the numerical value of r_{20} to be large enough that it would move to the 30th location so this means that it must be larger than all of r_{21}, r_{22}, up to r_{29} and r_{30} or

$$r_{20} > \max(r_{21}, r_{22}, \ldots, r_{29}, r_{30})\,,$$

but it can't be larger than r_{31} or else it would exchange locations and move to the thirty-first spot.

From these arguments in the original ordering of the first thirty-one numbers r_{31} must be the largest and r_{20} must be the second largest. As there are 31! ways to place 31 numbers in 31 spots and $(31-1-1)! = 29!$ ways to order 31 numbers where we require the largest in the 31st spot and the second largest in the 20th spot the probability this happens is

$$\frac{29!}{31!} = \frac{1}{31 \times 30} = \frac{1}{930}\,.$$

Thus $p + q = 931$.

Problem 14

To start we recognize that $324 = 2^2 \cdot 3^4 = 4 \cdot 3^4$. Then the expression we are given can be written as

$$\mathcal{E} = \frac{(10^4 + 4\cdot 3^4)(22^4 + 4\cdot 3^4)(34^4 + 4\cdot 3^4)(46^4 + 4\cdot 3^4)(58^4 + 4\cdot 3^4)}{(4^4 + 4\cdot 3^4)(16^4 + 4\cdot 3^4)(28^4 + 4\cdot 3^4)(40^4 + 4\cdot 3^4)(52^4 + 4\cdot 3^4)}\,.$$

Now based on the form of this expression consider

$$\begin{aligned}
x^4 + 4y^4 &= x^4 + 4x^2y^2 + 4y^4 - 4x^2y^2 \\
&= (x^2 + 2y^2)^2 - 4x^2y^2 \\
&= (x^2 + 2y^2 + 2x^2y^2)(x^2 + 2y^2 - 2x^2y^2) \\
&= (x^2 + 2xy + y^2 + y^2)(x^2 - 2xy + y^2 + y^2) \\
&= ((x+y)^2 + y^2)((x-y)^2 + y^2)\,.
\end{aligned}$$

The first factor in the above product is the "up" factor while the second factor is the "down" factor. Using this we can write the product we are given with all "up" factors first and then all "down" factors second as

$$\begin{aligned}
\mathcal{E} &= \frac{(13^2+3^2)(25^2+3^2)(37^2+3^2)(49^2+3^2)(61^2+3^2)\times(7^2+3^2)(19^2+3^2)(31^2+3^2)(43^2+3^2)(55^2+3^2)}{(7^2+3^2)(19^2+3^2)(31^2+3^2)(43^2+3^2)(55^2+3^2)\times(1^2+3^2)(13^2+3^2)(25^2+3^2)(37^2+3^2)(49^2+3^2)} \\
&= \frac{61^2+3^2}{1^1+3^2} = \frac{3730}{10} = 373\,,
\end{aligned}$$

once we cancel many of the common factors and simplify.

Problem 15

The sides of the two squares are given by

$$\begin{aligned}
s_1 &= \sqrt{441} = 21 \\
s_2 &= \sqrt{440} = \sqrt{2^3\cdot 5\cdot 11} = 2\sqrt{2\cdot 5\cdot 11} = 2\sqrt{110}\,.
\end{aligned}$$

Lets let the angles at the points A and B be denoted by the *angles* A and B. The usage should be clear from context. In each triangle the location of the point C will be taken in the Cartesian coordinate plane as $(0,0)$, the location of the point A will be $(0,v)$, and the location of the point B will be $(h,0)$. Thus part of this problem is to determine

$$AC + CB = v + h\,.$$

From the given triangles and the definitions above we have

$$\begin{aligned}\tan(A) &= \frac{h}{v}\\ \tan(B) &= \frac{v}{h}\,,\end{aligned}$$

plus all other trigonometric relations on the angles A and B can be written in terms of the variables h and v.

Based on the first figure (with the square S_1) the point $(21,21)$ is on the line connecting AB. This line is given by

$$y = v - \frac{v}{h}x\,. \tag{427}$$

If we put the point $(21,21)$ in that expression we get

$$\frac{21}{v} + \frac{21}{h} = 1\,. \tag{428}$$

This is one relationship between h and v.

Moving to the second figure (with the square S_2) let the left-most corner of the square S_2 be denoted by the point D, the northern-most corner of the square S_2 be denoted by the point E, the east-most corner by F, and the southern-most corner by G. Then from the right angles involved we have

$$\begin{aligned}\angle ADE &= \angle B\\ \angle CDG &= \angle A\\ \angle CGD &= \angle B\\ \angle FGB &= \angle A\,,\end{aligned}$$

and all other angles are right-angles. Using these triangles we will compute the x and y coordinate of the point F. First the point G is located at

$$x = \sqrt{440}\cos(B)\,,$$

and $y = 0$. Then the point F is located at

$$x = \sqrt{440}\cos(B) + \sqrt{440}\cos(A)\,,$$

and

$$y = \sqrt{440}\sin(A)\,.$$

Putting the coordinates for F into Equation 427 we get

$$\sqrt{440}\sin(A) = v - \frac{v}{h}\sqrt{440}(\cos(B) + \cos(A))\,,$$

or

$$\sin(A) = \frac{v}{\sqrt{440}} - \frac{1}{h}(\cos(B) + \cos(A)),$$

or as $\cos(B) = \sin(A)$ this is

$$\frac{\sin(A)}{v} = \frac{1}{\sqrt{440}} - \frac{1}{h}(\sin(A) + \cos(A)),$$

or

$$\left(\frac{1}{v} + \frac{1}{h}\right)\sin(A) = \frac{1}{\sqrt{440}} - \frac{\cos(A)}{h}.$$

Using Equation 428 we can simplify the left-hand-side of this to get

$$\frac{\sin(A)}{21} = \frac{1}{\sqrt{440}} - \frac{\cos(A)}{h},$$

or

$$\frac{\sin(A)}{21} + \frac{\cos(A)}{h} = \frac{1}{\sqrt{440}}.$$

From the original triangle we have that

$$\begin{aligned}
\sin(A) &= \frac{h}{\sqrt{v^2 + h^2}} \\
\cos(B) &= \frac{h}{\sqrt{v^2 + h^2}} = \sin(A) \\
\cos(A) &= \frac{v}{\sqrt{v^2 + h^2}},
\end{aligned}$$

thus we can write the above as a single equation in terms of h and v as

$$\frac{h}{21} + \frac{v}{h} = \frac{\sqrt{v^2 + h^2}}{\sqrt{440}}. \tag{429}$$

At this point we have two equations 428 and 429 and two unknowns h and v which we could attempt to solve for. The algebra for these remaining steps seems to be complicated.

The 1988 AHSME Examination

Problem 1

We have
$$\sqrt{8} + \sqrt{18} = 2\sqrt{2} + 3\sqrt{2} = 5\sqrt{2}\,.$$

Problem 2

By similar triangles we will have
$$\frac{ZY}{BC} = \frac{XY}{AB}\,.$$
From what we are told this means that
$$\frac{YZ}{4} = \frac{5}{3}\,.$$
Thus $YZ = \frac{20}{3} = 6\frac{2}{3}$.

Problem 3

The vertical bars (ignoring overlaps) will take up an area
$$10 \times 1 \times 2 = 20\,.$$
The horizontal bars will take up an area of the same amount but that area also includes the area of four squares that the vertical bars take up. Thus with out these the horizontal bars take up
$$20 - 4 \times 1 \times 1 = 16\,.$$
The total area covered is then $20 + 16 = 36$.

Problem 4

Write this expression as
$$y = 2 - \frac{2}{3}x\,,$$
then the slope is $-\frac{2}{3}$.

Problem 5

Expanding we have

$$(x+2)(x+b) = x^2 + (2+b)x + 2b = x^2 + cx + 6\,.$$

This means that $2b = 6$ so that $b = 3$. Then $c = 2 + b = 5$.

Problem 6

This is a rectangle.

Problem 7

This would be

$$\frac{60 \times 512}{120} = \frac{1}{2}(512) = 256\,,$$

seconds. If we divide this by 60 to get the time in minutes as four minutes is 240 seconds the answer is four minutes.

Problem 8

We are told that

$$\frac{b}{a} = 2 \quad \text{and} \quad \frac{c}{b} = 3\,,$$

We want to compute

$$\frac{a+b}{b+c} = \frac{\frac{a}{b}+1}{1+\frac{c}{b}} = \frac{\frac{1}{2}+1}{1+3} = \frac{3}{8}\,,$$

when we simplify.

Problem 9

The side S must be larger than the "diagonal of the table". This means that

$$S > \sqrt{10^2 + 8^2} = \sqrt{164}\,.$$

Recalling that $12^2 = 144$, $13^2 = 169$, and $14^2 = 196$. Thus for S to be a the smallest integer such that $S^2 > 164$ we should take $S = 13$.

Problem 10

The given statement means that

$$-0.00312 < C - 2.43865 < +0.00312\,,$$

or solving for C this means that

$$2.43553 < C < 2.44177\,.$$

Thinking about rounding the numbers on either side of C above we see that when both are rounded to the second digit to the right of the decimal point we have $C = 2.44$.

Problem 11

Working out the percentage increases I find

$$\begin{aligned}
\frac{\Delta A}{A} &= \frac{10}{40} = \frac{1}{4} = 0.25 \\
\frac{\Delta B}{B} &= \frac{20}{50} = \frac{2}{5} = 0.4 \\
\frac{\Delta C}{C} &= \frac{30}{70} = \frac{3}{7} \approx 0.42 \quad \text{as} \quad \frac{1}{7} \approx 0.14 \\
\frac{\Delta D}{D} &= \frac{30}{100} = \frac{3}{10} = 0.3 \\
\frac{\Delta E}{E} &= \frac{40}{120} = \frac{1}{3} = 0.3333\,.
\end{aligned}$$

Based on these the largest is C.

Problem 12

To solve this problem create a grid with nine rows representing the value of the first draw and nine columns representing the value of the second draw. At the intersection of the row and the column place the sum of the two draws. We can then count the number of times that each digit appears as the units digit of each sum. I find that the digit 0 appears nine times and each other digit appears eight times. Thus 0 appears the most.

Problem 13

The given expression means that

$$\tan(x) = 3\,.$$

The "triangle" that identity would indicate is one with a base leg of length one, a vertical leg of length three, and a hypotenuse of length $\sqrt{1+9} = \sqrt{10}$. This is the triangle in the x-y coordinate plane with vertices $A = (0, 0)$, $B = (1, 0)$ and $C = (1, 3)$ and x the angle $\angle BAC$. Then from that configuration we have

$$\begin{aligned} \sin(x) &= \frac{3}{\sqrt{10}} \\ \cos(x) &= \frac{1}{\sqrt{10}} \,. \end{aligned}$$

This means that

$$\sin(x)\cos(x) = \frac{3}{10} \,.$$

Problem 14

From the definition of $\binom{a}{k}$ the denominator of both $\binom{-\frac{1}{2}}{100}$ and $\binom{\frac{1}{2}}{100}$ is the same so we have

$$\frac{\binom{-\frac{1}{2}}{100}}{\binom{\frac{1}{2}}{100}} = \frac{n(n-1)(n-2)\cdots(n-98)(n-99)}{p(p-1)(p-2)\cdots(p-98)(p-99)} \,,$$

for $n = -\frac{1}{2}$ and $p = \frac{1}{2}$. The numerator of this fraction can be written

$$\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{197}{2}\right)\left(-\frac{199}{2}\right) \,.$$

The denominator of this fraction can be written

$$\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{195}{2}\right)\left(-\frac{197}{2}\right) \,.$$

Thus the desired ratio is then

$$\frac{\left(-\frac{199}{2}\right)}{\frac{1}{2}} = -199 \,.$$

Problem 15

We do long division of $ax^3 + bx^2 + 1$ by $x^2 - x - 1$ to show that

$$\frac{ax^3 + bx^2 + 1}{x^2 - x - 1} = ax + (b + a) + \frac{(2a + b)x + a + b + 1}{x^2 - x + 1} \,.$$

Then to be a factor means that

$$\begin{aligned} 2a + b &= 0 \\ a + b + 1 &= 0 \,. \end{aligned}$$

These two equations have the solution $a = 1$ and $b = -2$.

Problem 16

Let h be the height of the triangle ABC. Let a be the length of a side of the triangle ABC. Note that $\angle ACB$ is 60 degrees. This mean that

$$a\cos\left(\frac{\pi}{3}\right) = h \quad \text{so} \quad h = \frac{a}{2}\,.$$

This means that for an equilateral triangle with a side of length a and a height h we have an area given by

$$\frac{1}{2}ah = \frac{1}{2}(2h)h = h^2\,.$$

Let the height of the inside triangle be denoted by h'. We want to determine h' in terms of h. Note that

$$h' = h - AA' - \frac{h}{6}\,,$$

so we need to determine AA'. Now as $\angle C'CB$ is thirty degrees we have that

$$CC' = AA' = \frac{h/6}{\sin\left(\frac{\pi}{6}\right)} = \frac{h/6}{1/2} = \frac{h}{3}\,.$$

This means that

$$h' = h - \frac{h}{3} - \frac{h}{6} = \frac{h}{2}\,.$$

Then as the inside triangle is also an equilateral triangle the ratio of their two areas is

$$\frac{(h')^2}{h^2} = \frac{(h/2)^2}{h^2} = \frac{1}{4}\,.$$

Problem 17

If $x > 0$ and $y > 0$ then these two equation are

$$\begin{aligned} 2x + y &= 10 \\ x &= 12\,. \end{aligned}$$

This has the solutions $x = 12$ and $y = -14$. This solution is *not* consistent with the assumption that $x > 0$ and $y > 0$ and thus is not an actual solution to the original equations.

If $x > 0$ and $y < 0$ then these two equation are

$$\begin{aligned} 2x + y &= 10 \\ x - 2y &= 12\,. \end{aligned}$$

This has the solutions $x = \frac{32}{5}$ and $y = -\frac{14}{5}$. This solution is consistent with the assumption that $x > 0$ and $y < 0$ and thus is a solution to the original equations. In this case we have $x + y = \frac{18}{5}$.

If $x < 0$ and $y > 0$ then these two equation are

$$y = 10$$
$$x = 12 \,.$$

This solution is not consistent with the initial assumptions and thus is not an actual solution to the original equations.

Finally, if $x < 0$ and $y < 0$ then these two equation are

$$y = 10$$
$$x = 32 \,.$$

This solution is not consistent with the initial assumptions and thus is not an actual solution to the original equations.

Problem 18

If there are n people in the tournament there are two ways that the outcome of the nth and the $n-1$th game can happen. There are two possible outcomes of the game between the winner of this last game and the person in the $n-2$ spot. Following this logic there are

$$2^{n-1} \,,$$

possible outcomes with n people. When $n = 5$ this is sixteen.

Problem 19

Note that we can write the numerator of the given fraction as

$$(bx + ay)(a^2x^2 + b^2y^2) + 2a^2bxy^2 + 2ab^2x^2y \,.$$

When we divide this by $bx + ay$ we get

$$a^2x^2 + b^2y^2 + \frac{2abxy(ay + bx)}{bx + ay} = a^2x^2 + b^2y^2 + 2abxy = (ax + by)^2 \,.$$

Problem 20

Let the side of the square be denoted "s" then we see that

$$AF = \frac{s}{2} = FD = BE = EC \,,$$

and

$$DE = \sqrt{s^2 + \frac{s^2}{4}} = \frac{s\sqrt{5}}{2}\,.$$

Now

$$XY = DE + DE = s\sqrt{5}\,.$$

Now the area of the rectangle must equal the area of the original square thus we must have

$$YZ \cdot XY = YZ \cdot s\sqrt{5} = s^2 \quad \text{so} \quad YZ = \frac{s}{\sqrt{5}}\,.$$

This means that

$$\frac{XY}{YZ} = \frac{s\sqrt{5}}{\frac{s}{\sqrt{5}}} = 5\,.$$

Problem 21

As $z = a + bi$ we have

$$z + |z| = a + bi = \sqrt{a^2 + b^2} = 2 + 8i\,.$$

This means that $b = 8$ and $a + \sqrt{a^2 + b^2} = 2$. This last equation is

$$\sqrt{a^2 + 64} = 2 - a\,,$$

which we can solve for a to get $a = -15$. In this case we have

$$|z|^2 = a^2 + b^2 = 289\,.$$

Problem 22

From the law of cosigns we have under the standard labeling of a triangle we have

$$a^2 = b^2 + c^2 - 2bc\cos(A)\,.$$

Note that if $|A| < \frac{\pi}{2}$ then $\cos(A) > 0$ so that

$$a^2 < b^2 + c^2\,.$$

Lets call this the "quadratic triangle inequality". If we apply this to the three given lengths 10, 24, and x we must enforce

$$\begin{array}{lllll} 10^2 < 24^2 + x^2 & \Rightarrow & x^2 > -476 & & \\ 24^2 < 10^2 + x^2 & \Rightarrow & x^2 > 476 & \Rightarrow & |x| > 21.81742 \\ x^2 < 10^2 + 24^2 & \Rightarrow & x^2 > 676 & \Rightarrow & |x| < 26\,. \end{array}$$

From these inequalities for x we can have $x \in \{22, 23, 24, 25\}$.

Problem 23

Recall that a tetrahedron is a 3D object with a triangular base and triangular sides. In that solid we label the four vertices of our tetrahedron with the letters A, B, C, and D. Now on any face of the tetrahedron we have a triangle so the triangle inequality must hold. For example, as $AB = 41$ and if the other sides of the triangle with that edge are AD and BD (so that the vertex not in the plane of $\triangle ABD$ is C) then we must have

$$AD + BD > 41 \,.$$

Now as $AB = 41$ the other sides AC, BC, AD, and BD must be drawn from the lengths $\{7, 13, 18, 27, 36\}$. Intuitively, as 41 is the largest of all of the side lengths we will be able to enforce the triangle inequality only if the four edges that "connect" to AB are taken from the largest lengths available i.e. the numbers $\{13, 18, 27, 36\}$. We might try

$$AC = 36 \quad \text{and} \quad BC = 13 \,,$$

with

$$AD = 27 \quad \text{and} \quad BD = 18 \,.$$

In that configuration CD would have the smallest length of seven.

If we draw that tetrahedron we would have a face $\triangle ACD$ with lengths $AD = 27$, $AC = 36$, and $CD = 7$. This cannot be a valid configuration as

$$CD + AD = 7 + 27 = 34 \not> 36 = AC \,,$$

meaning that the triangle inequality is not satisfied in this triangle. This means that we need to increase the length of the segment CD. If we make it the next larger number from 7 we would take $CD = 13$ (and then take $BC = 7$) we can check that the triangle inequality is satisfied for all the faces of the tetrahedron.

Problem 24

An isosceles trapezoid means that the two "legs" (not the bases) are of the same length.

Method 1: Let the angle between the bottom base and each leg be denoted by α. Then we are told that

$$\sin(\alpha) = 0.8 \quad \text{so} \quad \cos(\alpha) = \sqrt{1 - 0.8^2} = 0.6 \,.$$

Note that $0.6 : 0.8 : 1.0 = 3 : 4 : 5$ and thus the two base triangles are three-four-five triangles. Let the legs have a lengths x and the top base have a length of y. Then dropping perpendiculars from the corers on the top base to the bottom base and using the three-four-five triangles we see that the height is

$$h = 0.8x \,.$$

Projecting the legs onto the horizontal gives a length $0.6x$ and thus the base of the trapezoid has a length

$$0.6x + y + 0.6x = y + 1.2x\,.$$

If we know the theorem that a circumscribing rectangle has sides that sum to the same value we can write

$$2x = (y + 1.2x) + y = 2y + 1.2x\,.$$

As the bottom base has a length of 16 we also have

$$16 = 1.2x + y\,.$$

These give two equations with a solution $(x, y) = (10, 4)$. This means that the area is then

$$A = \frac{1}{2}(0.8x)(1.2x + y + y) = 80\,.$$

Method 2: In this method lets explicitly draw the circle inside the isosceles trapezoid. Let its center be denoted as O and then from the vertices on the bottom base draw segments to this center. The angle with these segments and the bottom base is then $\frac{\alpha}{2}$. The radius of this circle is given by

$$\tan\left(\frac{\alpha}{2}\right) = \frac{r}{8}\,.$$

Recalling that

$$\tan(2\theta) = \frac{2\tan(\theta)}{1 - \tan^2(\theta)}\,,$$

with $2\theta = \alpha$ we get

$$\tan(\alpha) = \frac{\sin(\alpha)}{\cos(\alpha)} = \frac{0.8}{0.6} = \frac{4}{3} = \frac{2\tan(\theta)}{1 - \tan^2(\theta)}\,.$$

Solving this for $\tan(\theta)$ and noting that $\tan(\theta) > 0$ we get $\tan(\theta) = \frac{1}{2}$. This means that

$$r = 8\tan(\theta) = 4\,.$$

The height of the trapezoid is $h = 2r = 8$.

Next we need to determine the length of the top base. To do that we next draw line segments from O to two tangents. One is drawn to the leg of the trapezoid OL and the other is drawn to the horizontal base OT. Using segments from O to the left-most corner of the trapezoid and the fact that the angles of triangle sum to 180 degrees we can show that the angle between these two segments LO and OT is also α and that if the upper corner of the trapezoid is denoted U then $\angle UOT = \theta$. This means

$$\frac{1}{2}\text{top length} = r\tan(\theta) = 4\left(\frac{1}{2}\right) = 2\,.$$

Thus the length of the top is four and that the area is then

$$A = \frac{1}{2}a(a + b) = \frac{1}{2}(8)(16 + 4) = 80\,.$$

Problem 25

Let the number of elements in X, Y, and Z be denoted by n_x, n_y, and n_y. Let the sum of the samples in X, Y, and Z be denoted by S_x, S_y, and S_z i.e.

$$S_x = \sum_{i=1}^{n_x} x_i \,,$$

and the same for S_y and S_z. From the information given in the problem we have

$$\begin{aligned}
\frac{1}{n_x} S_x &= 37 \\
\frac{1}{n_y} S_y &= 23 \\
\frac{1}{n_z} S_z &= 41 \\
\frac{1}{n_x + n_y}(S_x + S_y) &= 29 \\
\frac{1}{n_x + n_z}(S_x + S_z) &= 39.5 \\
\frac{1}{n_y + n_z}(S_y + S_z) &= 33 \,.
\end{aligned}$$

We can use the first three equations to solve the above for S in terms of n and put these into the last three equations. When I do that and simplify I get

$$\begin{aligned}
4n_x - 3n_y &= 0 \\
-5n_x + 3n_z &= 0 \\
-5n_y + 4n_z &= 0 \,.
\end{aligned}$$

Notice that this is a system of three equations and three unknowns. From the first equation we have $n_x = \frac{3}{4} n_y$ which if you put into the other two equations gives

$$\begin{aligned}
-\frac{15}{4} n_y + 3n_z &= 0 \\
-5n_y + 4n_z &= 0 \,.
\end{aligned}$$

This second equation multiplied by $\frac{3}{4}$ is the first equation. Thus there are only two independent equations

$$\begin{aligned}
4n_x + 3n_y &= 0 \\
-5n_x + 3n_z &= 0 \,.
\end{aligned}$$

If we Let n_z be arbitrary these two equations can be written as

$$\begin{aligned}
n_x &= \frac{3}{5} n_z \\
n_y &= \frac{4}{5} n_z \,.
\end{aligned}$$

Using these we have

$$S_x = 37\left(\frac{3}{5}n_z\right) = \frac{111}{5}n_z$$
$$S_y = 23\left(\frac{4}{5}n_z\right) = \frac{92}{5}n_z$$
$$S_z = 41n_z = \frac{205}{5}n_z\,,$$

and

$$n_x + n_y + n_z = \frac{12}{5}n_z\,,$$

when we simplify. The average of the elements in the set $X \cup Y \cup Z$ is then

$$\frac{S_x + S_y + S_z}{n_x + n_y + n_z} = \frac{\frac{408}{5}n_z}{\frac{12}{5}n_z} = 34\,.$$

Problem 26

Let x be the common value then we have

$$x = \log_9(p) \quad \text{or} \quad p = 9^x \tag{430}$$
$$x = \log_{12}(q) \quad \text{or} \quad q = 12^x \tag{431}$$
$$x = \log_{16}(p+q) \quad \text{or} \quad p + q = 16^x\,. \tag{432}$$

From Equations 430 and 431 we have

$$\frac{q}{p} = \frac{12^x}{9^x} = \frac{4^x}{3^x} = \left(\frac{4}{3}\right)^x\,.$$

From Equations 432 and 431 we have

$$1 + \frac{p}{q} = \frac{16^x}{12^x} = \left(\frac{4}{3}\right)^x = \frac{q}{p}\,.$$

If we let $r = \frac{q}{p}$ then this equation is

$$1 + \frac{1}{r} = r\,.$$

If we multiply by r this can be converted to the quadratic

$$r^2 - r - 1 = 0\,,$$

which has solutions

$$r = \frac{1 \pm \sqrt{5}}{2}\,.$$

As p and q are positive we must take the positive sign above.

Problem 27

The area we seek is given by the area of the bottom rectangle plus the top triangle. We can write this as

$$\begin{aligned}\text{Area} &= BC \cdot CD + \frac{1}{2} BC \cdot (AB - CD) \\ &= \frac{BC}{2}(CD + AB)\,.\end{aligned}$$

Drop a perpendicular from O and onto BC and let O' be that intersection. Let $\angle DOO' = \theta$. Then as $AB \parallel CD$ we have that $\angle OAB = \theta$ also. Let the radius of the circle be denoted by r. Then from the diagram we have

$$BC = 2r\sin(\theta)$$
$$AB = r + r\cos(\theta) = r(1+\cos(\theta)) \quad (433)$$
$$CD = r - r\cos(\theta) = r(1-\cos(\theta))\,. \quad (434)$$

Then, using these, the area can be written as

$$\text{Area} = \frac{1}{2}(2r\sin(\theta))(2r) = 2r^2\sin(\theta)\,.$$

We now need to solve for r and θ in terms of AB and CD. These are given by solving Equation 433 and 434 where we find

$$r = \frac{AB + CD}{2}\,,$$

and

$$\cos(\theta) = \frac{AB - CD}{AB + CD}\,.$$

From this last equation we have

$$\sin(\theta) = \sqrt{1 - \sin^2(\theta)} = \frac{2\sqrt{AB \cdot CD}}{AB + CD}\,.$$

Using these we have

$$\text{Area} = (AB + CD)\sqrt{AB \cdot CD}\,.$$

Now $AB + CD$ will be an integer if AB and CD are. To have the area be an integer we have to make sure that the square root gives an integer. From the choices given (D) has $AB \cdot CD = 36$ which is a perfect square.

Problem 28

For a binomial random variable like this one we have

$$w = \binom{5}{3} p^3(1-p)^2\,.$$

We are told that $w = \frac{144}{625} = \frac{2^4 3^2}{5^4}$ and compute that $\left(\begin{array}{c} 5 \\ 3 \end{array} \right) = 10$. Thus we have

$$p^3(1-p)^2 = \frac{2^3 3^2}{5^5} = \left(\frac{2}{5}\right)^3 \left(\frac{3}{5}\right)^2 .$$

By inspection this is true if $p = \frac{2}{5}$. Thus there is at least one root. To show that there are at least two roots $0 < p < 1$ let

$$f(p) \equiv p^3(1-p)^2 - \frac{2^3 3^2}{5^5} .$$

We note that $f(0) < 0$ and $f(1) < 0$ while $f\left(\frac{1}{2}\right) > 0$. Thus there should be at least one real root p such that $0 < p < \frac{1}{2}$ and another such that $\frac{1}{2} < p < 1$.

Problem 29

Let the "best fit" line be denoted by $y(x) = b + mx$. Then we want to find b and m such that

$$L(b, m) = \sum_{i=1}^{3} (y_i - b - mx_i)^2 ,$$

is as small as possible. The minimum will happen when the first order conditions hold i.e. $\frac{\partial L}{\partial b} = \frac{\partial L}{\partial m} = 0$. We find

$$\begin{aligned}
\frac{\partial L}{\partial b} &= \sum_{i=1}^{3} 2(y_i - b - mx_i)(-1) = 0 \\
\frac{\partial L}{\partial m} &= \sum_{i=1}^{3} 2(y_i - b - mx_i)(-x_i) = 0 .
\end{aligned}$$

We can write these two equations as

$$\begin{aligned}
3b + mS_x &= S_y \\
S_x b + mS_{xx} &= S_{xy} ,
\end{aligned}$$

where we have defined

$$\begin{aligned}
S_x &= \sum_{i=1}^{3} x_i \\
S_y &= \sum_{i=1}^{3} y_i \\
S_{xx} &= \sum_{i=1}^{3} x_i^2 \\
S_{xy} &= \sum_{i=1}^{3} x_i y_i .
\end{aligned}$$

To solve this problem we need to solve the system above for m (we don't need the value of b).

To simplify things first note that from conditions given in the problem we have

$$S_x = x_1 + x_2 + x_3 = x_1 + x_2 + (2x_2 - x_1) = 3x_2 \, .$$

To solve for only m we will use Cramer's rule where we have

$$m = \frac{\left|\begin{array}{cc} 3 & S_y \\ S_x & S_{xy} \end{array}\right|}{\left|\begin{array}{cc} 3 & S_x \\ S_x & S_{xx} \end{array}\right|} = \frac{3S_{xy} - S_x S_y}{3S_{xx} - S_x^2} \, .$$

For the numerator N I find

$$N = 3S_{xy} - S_x S_y = 3(x_1y_1 + x_2y_2 + x_3y_3) - 3x_2(y_1 + y_2 + y_3) = 3(x_1 - x_2)y_1 + 3(x_3 - x_2)y_3 \, .$$

For the denominator D I find

$$D = 3S_{xx} - S_x^2 = 3(x_1^2 + x_2^2 + x_3^2) - (3x_2)^2 = 3(x_1^2 - 2x_2^2 + x_3^2) \, .$$

To further simplify things let $h \equiv x_2 - x_1 = x_3 - x_2$ and we can write

$$\begin{aligned} x_1 &= x_2 - h \\ x_3 &= x_2 + h \\ x_3 - x_1 &= 2h \, , \end{aligned}$$

in the above to get

$$\begin{aligned} N &= -3hy_1 + 3hy_3 \\ D &= 3((x_2 - h)^2 - 2x_2^2 + (x_2 + h)^2) = 6h^2 \, , \end{aligned}$$

when I simplify. This means that

$$m = \frac{y_3 - y_1}{2h} = \frac{y_3 - y_1}{x_3 - x_1} \, .$$

Problem 30

If we pick a value for x_0 such that $x_0 = f(x_0)$ then the given sequence will take on only *one* value i.e. x_0. To see what values for x_0 cause this we should solve

$$x = f(x) = 4x - x^2 \, .$$

This has solutions $x = 0$ and $x = 3$. And if we start our iteration with this value of x we get the constant sequence. We now ask for what values of x will one mapping of $f(x)$ bring us to either $\{0, 3\}$. For the first we need to solve

$$0 = 4x - x^2 \quad \text{so} \quad x \in \{0, 4\} \, .$$

Thus we have a new value in our set of points that have a finite number of items in their iterated sequence. For the second we need to solve

$$3 = 4x - x^2\,,$$

which has complex roots. At this point, the set $\mathcal{S}$ giving rise to a finite number of iterations is

$$\mathcal{S} = \{0, 3\} \cup \{4\}\,.$$

Next we want to see for what value of x will we have $f(x) = 4$ or

$$4 = 4x - x^2 \quad \text{so} \quad x \in \{2\}\,.$$

and our set is

$$\mathcal{S} = \{0, 3\} \cup \{4\} \cup \{2\}\,.$$

Next we want to see for what value of x will we have $f(x) = 2$ or

$$2 = 4x - x^2 \quad \text{so} \quad x \in \{0.5857864, 3.4142136\}\,.$$

We see that the number of points that are mapped to points already in $\mathcal{S}$ is increasing at each step (in the above case by two). Thus there seem to be an infinite number of them.

The 1988 AIME Examination

Problem 1

The total number of combinations for this lock under the redesign would be

$$N = \sum_{k=1}^{9} \binom{10}{k}\,,$$

while the number of combinations for this lock under the original design would have been

$$O = \binom{10}{5}\,.$$

We can write N in terms of O as

$$\begin{aligned}
N &= \binom{10}{1} + \binom{10}{2} + \binom{10}{3} + \binom{10}{4} \\
&+ \binom{10}{5} \\
&+ \binom{10}{6} + \binom{10}{7} + \binom{10}{8} + \binom{10}{9} \\
&= O + 2\binom{10}{1} + 2\binom{10}{2} + 2\binom{10}{3} + 2\binom{10}{4} \\
&= O + 2(10) + 2\left(\frac{10!}{8!2!}\right) + 2\left(\frac{10!}{7!3!}\right) + 2\left(\frac{10!}{6!4!}\right) \\
&= O + 20 + 10 \cdot 9 + 10 \cdot 3 \cdot 8 + 10 \cdot 3 \cdot 2 \cdot 7 \\
&= O + 770\,.
\end{aligned}$$

Where in the above I have used the fact that

$$\binom{n}{k} = \binom{n}{n-k},$$

for $0 \leq k \leq n$. Thus there are 770 more combinations.

Problem 2

When asked for the value of a "function" at a large integer argument one of the only ways to solve such a problem (without the use of a computer) is to find that the sequence you are looking at is periodic. Thus one think you should always try is to compute as many terms as possible looking for the numbers to repeat. This is the method of solution for this problem.

From the definition of f_n we have

$$\begin{aligned}
f_1(11) &= 2^2 = 4 \\
f_2(11) &= f_1(f_1(11)) = f_1(4) = 16 \\
f_3(11) &= f_1(f_2(11)) = f_1(16) = (1+6)^2 = 49 \\
f_4(11) &= (4+9)^2 = 13^2 = 169 \\
f_5(11) &= (1+6+9)^2 = 16^2 = 256 \\
f_6(11) &= (2+5+6)^2 = 13^2 = 169 \\
f_7(11) &= 256 \\
f_8(11) &= 169 \\
f_9(11) &= 256 \quad \text{etc.}
\end{aligned}$$

Thus it looks like

$$f_n(11) = \begin{cases} 169 & n \text{ even} \\ 256 & n \text{ odd} \end{cases},$$

when $n \geq 4$. Based on this we have

$$f_{1988}(11) = 169\,.$$

When I originally worked this problem I *mistakenly* thought that the definition of f_1 was the sum of the squared digits (i.e. square the digits and then sum vs. sum the digits and then square the result). Interestingly that problem also has a periodic solution. In that case using this modified definition of f_n we have

$$\begin{aligned}
f_1(11) &= 2 \\
f_2(11) &= f_1(f_1(11)) = f_1(2) = 4 \\
f_3(11) &= f_1(f_2(11)) = f_1(4) = 16 \\
f_4(11) &= 1 + 36 = 37 \\
f_5(11) &= 9 + 49 = 58 \\
f_6(11) &= 25 + 64 = 89 \\
f_7(11) &= 64 + 81 = 145 \\
f_8(11) &= 1 + 16 + 25 = 42 \\
f_9(11) &= 16 + 4 = 20 \\
f_{10}(11) &= 4 \\
f_{11}(11) &= 16 \quad \text{etc.}
\end{aligned}$$

Note that

$$\begin{aligned}
f_{10}(11) &= 4 = f_2(11) \\
f_{11}(11) &= 16 = f_3(11)\,,
\end{aligned}$$

thus we expect

$$f_{n+8}(11) = f_n(11)\,,$$

for $n \geq 2$. Then using this and as

$$1988 = 248 \times 8 + 4\,,$$

we have

$$f_{1988}(11) = f_4(11) = 37\,.$$

Problem 3

We can write the given expression in terms of $\log(x)$ i.e. the logarithm to base ten as

$$\frac{\log(\log_8 x)}{\log 2} = \frac{\log(\log_2 x)}{\log 8} = \frac{\log(\log_2 x)}{3\log 2}\,,$$

or

$$\log\left(\frac{\log x}{\log 8}\right) = \frac{1}{3}\log\left(\frac{\log x}{\log 2}\right)\,.$$

This is equivalent to

$$\log(\log(x)) - \log(\log 8) = \frac{1}{3}\left(\log(\log(x)) - \log(\log(2))\right)\,.$$

We can solve the above for $\log(\log(x))$ to find

$$\log(\log(x)) = \frac{3}{2}\log(3) + \log(\log(2)) = \log\big(3^{3/2}\log(2)\big)\,.$$

This means that

$$\log(x) = 3^{\frac{3}{2}} \log(2) .$$

If we divide both sides by $\log(2)$ we get

$$\frac{\log(x)}{\log(2)} = 3^{\frac{3}{2}} ,$$

or converting this back into $\log_2(x)$ we get

$$\log_2(x) = 3^{\frac{3}{2}} .$$

If we square this I get

$$(\log_2(x))^2 = 3^3 = 27 .$$

Problem 4

As the absolute value is always a non-negative number the right-hand-side of this expression is bounded below as

$$19 + |x_1 + x_2 + \cdots + x_n| \geq 19 .$$

This means that to have a solution we must have

$$|x_1| + |x_2| + \cdots + |x_n| \geq 19 .$$

Now

$$\sum_{i=1}^{n} |x_i| < \sum_{i=1}^{n} 1 = n .$$

This means that the left-hand-side is smaller than n. From this if we were to have $n = 19$ then the left-hand-side is smaller than 19 while the right-hand-side is greater than or equal to 19. Thus we see that $n > 19$ and the smallest value we can have for n is $n = 19 + 1 = 20$.

Problem 5

The "target" number we want to find divisors for can be written as $2^{99} \cdot 5^{99}$. All divisors of this number come in the form of $2^i \cdot 5^j$ where

$$0 \leq i \leq 99 \quad \text{and} \quad 0 \leq j \leq 99 .$$

This gives a total of $100 \times 100 = 10^4$ possible divisors of our target number. To be a multiple of $10^{88} = 2^{88} \cdot 5^{88}$ means that our divisor must be of the form $2^{88+m} \cdot 5^{88+n}$ for

$$\begin{aligned} 0 \leq & \quad m \quad \leq (99 - 88) = 11 \\ 0 \leq & \quad n \quad \leq (99 - 88) = 11 . \end{aligned}$$

There are $12 \times 12 = 144$ numbers of this form. This means that the probabilities is

$$\frac{144}{10000} = \frac{9}{625} .$$

The answer we want is then

$$m + n = 9 + 625 = 634 .$$

$4a$				
$3a$	74			
$2a$	b			186
a	$2b - 74$	103		
0				

Table 11: Our initial grid (with variables).

Problem 6

In the grid given if we put a a in the first column above the zero then as the first column (moving upwards) is an arithmetic sequence starting with zero and a first element equal to a the rest of the elements moving upwards are

$$2a\,,3a\,,4a\,.$$

Next we introduce a variable b in the second column below the 74. Then in this second column moving upwards from the b to the 74 we have a common difference of $74 - b$. Thus all entries in this column can be determined by adding or subtracting $74 - b$ to the previous entry. This means that the value in the spot below b is given by

$$b - (74 - b) = 2b - 74\,.$$

Our grid now looks like that given in Table 11.

Using that grid the common difference in the second row from the bottom can be computed in two ways. We have

$$2b - 74 - a = 103 - (2b - 74)\,.$$

Which we can write

$$2(2b - 74) - a = 103\,.$$

Now in the third row from the bottom the common difference is $b - 2a$ and thus as 186 is three "steps" from b we have

$$b + 3(b - 2a) = 186\,.$$

These gives two equations and two unknowns a and b which we can write as

$$-a + 4b = 257$$
$$-6a + 4b = 186\,.$$

Solving these gives $a = 13$ and $b = 66$. Thus the common difference in the second column is $74 - b = 74 - 66 = 8$. With all of this we can fill the first and second columns with numbers to get Table 12.

Using this table the common difference in the top row is given by

$$82 - 52 = 30\,.$$

52	82			
39	74			
26	66			186
13	58	103		
0	50			

Table 12: Our grid with the first two columns completed.

Thus the location with the star has a value of

$$52 + 30 \cdot 3 = 142\,.$$

Problem 7

If we draw this triangle with BC along an x-axis and put the point A above the segment BC. Then we drop an altitude from A onto the segment BC, let the intersection point be denoted by P and the length of AP be h. Now let the angle CAB be broken up into

$$\angle CAB = \angle BAC = \angle BAP + \angle PAC = x + y\,,$$

where I have defined the variables $x = \angle BAP$ and $y = \angle PAC$. Next we recall that

$$\tan(x+y) = \frac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)}\,.$$

From the right triangles in the diagram we see that

$$\tan(x) = \frac{3}{h}$$
$$\tan(y) = \frac{17}{h}\,,$$

As we know that

$$\tan(\angle CAB) = \frac{22}{7}\,,$$

we can write the above as

$$\frac{22}{7} = \frac{\frac{3}{h} + \frac{17}{h}}{1 - \frac{3 \cdot 17}{h^2}}\,.$$

So solve this we $v = \frac{1}{h}$ then the above can be written as

$$561v^2 + 70v - 11 = 0\,.$$

Using the quadratic equation we find

$$v = \frac{-70 \pm \sqrt{70^2 - 4(561)(-11)}}{2(561)} = \frac{-70 \pm 172}{2(561)}\,.$$

As $v = \frac{1}{h}$ must be positive we must take the positive sign above and find that

$$v = \frac{51}{561} = \frac{1}{11},$$

thus

$$h = 11.$$

This means that the area of $\triangle ABC$ is given by

$$\frac{1}{2}bh = \frac{1}{2}(11)(3+17) = 110.$$

Problem 8

From the given expressions we have

$$f(x, x+y) = \left(\frac{x+y}{y}\right) f(x, y).$$

In this we will take $z \equiv x + y$ so that $y = z - x$. This means that

$$f(x, z) = \left(\frac{z}{z-x}\right) f(x, z-x). \tag{435}$$

Using this expression we can start with what we want to evaluate and sequentially make the arguments "smaller" until we get something we can evaluate directly. At each step to use the above in evaluating $f(x, z)$ we need $z > x$. If not we will use the fact that $f(x, z) = f(z, x)$ to get an equivalent expression where it is true. With this discussion we have

$$\begin{aligned}
f(14, 52) &= \left(\frac{52}{52-14}\right) f(14, 38) = \left(\frac{52}{38}\right) f(14, 38) \\
&= \left(\frac{52}{38}\right)\left(\frac{38}{24} f(14, 24)\right) = \left(\frac{52}{24}\right) f(14, 24) \\
&= \left(\frac{52}{24}\right)\left(\frac{24}{10} f(14, 10)\right) = \frac{52}{10} f(14, 10) \\
&= \frac{52}{10} f(10, 14) \\
&= \frac{52}{10}\left(\frac{14}{4} f(10, 4)\right) = \frac{52}{10} \cdot \frac{14}{4}\left(\frac{10}{6} f(4, 6)\right) \\
&= \frac{52}{10} \cdot \frac{14}{4} \cdot \frac{10}{6} \cdot \frac{6}{2} f(4, 2) \\
&= \frac{52}{10} \cdot \frac{14}{4} \cdot \frac{10}{6} \cdot \frac{6}{2} f(2, 4) \\
&= \frac{52}{10} \cdot \frac{14}{4} \cdot \frac{10}{6} \cdot \frac{6}{2}\left(\frac{4}{2} f(2, 2)\right) \\
&= \frac{52}{10} \cdot \frac{14}{4} \cdot \frac{10}{6} \cdot \frac{6}{2} \cdot \frac{4}{2} \cdot 2.
\end{aligned}$$

If we simplify this we get 364. Notice that we first used $f(x, z) = f(z, x)$ in going from the third line to the fourth.

Problem 9

To have the cube of a number n end in the digit eight means that n must end in a two. Thus $n = 10k + 2$ for some integer $k \geq 0$. With that form we have

$$\begin{aligned} n^3 &= (10k+2)^3 = (10k)^3 + 3(10k)^2(2) + 3(10k)(2^2) + 8 \\ &= 1000k^3 + 600k^2 + 120k + 8 \,. \end{aligned}$$

The terms in the above are decreasing in size from right to left. For the above to end with the two digits 88 means that $100k$ must end with the digits of 80 or that $12k$ must end with the digit 8. This then means that k must end in a 4 or a 9. This means that k takes the form $k = 5m + 4$ where $m \geq 0$. This means that n^3 takes the form

$$\begin{aligned} n^3 &= (10(5m+4)+2)^3 = (50m+42)^3 \\ &= 125000m^3 + 315000m^2 + 264600m + 74088 \,. \end{aligned}$$

From the above this number will end with the two digits 88. To end with the three digits 888 we need $m \times 6$ to end in an 8. The smallest m that does this is $m = 3$. This means that $k = 15 + 4 = 19$ so that $n = 10k + 2 = 190 + 2 = 192$.

Problem 11

Write the given expression as

$$\sum_{k=1}^{n} z_k = \sum_{k=1}^{n} w_k \,.$$

From the given numbers w_k we can write the above as

$$\sum_{k=1}^{n} z_k = (32 - 7 - 9 + 1 - 14) + (170 + 64 + 200 + 27 + 43)i = 3 + 504i \,.$$

As we are told that z_k is on the line $y = mx + 3$ we have that

$$z_k = x_k + iy_k = x_k + i(mx_k + 3) \,,$$

for $1 \leq k \leq n$. Using this the above is

$$\sum_{k=1}^{n} x_k + i \sum_{k=1}^{n} (mx_k + 3) = 3 + i504 \,.$$

This means that

$$\sum_{k=1}^{n} x_k = 3 \,,$$

and so by equating the imaginary part of the above and using this with $n = 5$ we have

$$504 = m \sum_{k=1}^{n} x_k + 3(5) = 3m + 15 \,.$$

This means that $m = 163$.

Problem 13

If $x^2 - x - 1$ is a factor of $ax^17 + bx^16 + 1$ we must be able to write

$$ax^{17} + bx^{16} + 1 = (x^2 - x - 1)(Ax^{15} + Bx^{14} + Cx^{13} + \cdots),$$

for various coefficients A, B, C, etc. This means that any root of the right-hand-side is also a root of left-hand-side. Some roots of the right-hand-side are the solutions to

$$x^2 - x - 1 = 0,$$

or

$$x = \frac{1 \pm \sqrt{1 - 4(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}.$$

Lets define $p \equiv \frac{1+\sqrt{5}}{2}$ and $q \equiv \frac{1-\sqrt{5}}{2}$. For the numbers p and q to be roots of the left-hand-side means that

$$\begin{aligned} ap^{17} + bp^{16} + 1 &= 0 \quad (436) \\ aq^{17} + bq^{16} + 1 &= 0. \quad (437) \end{aligned}$$

If we multiply Equation 436 by q^{16} and Equation 437 by p^{16} we get

$$\begin{aligned} ap(pq)^{16} + b(pq)^{16} + q^{16} = 0 \\ aq(pq)^{16} + b(pq)^{16} + p^{16} = 0. \end{aligned}$$

To evaluate this note that $pq = \frac{1-5}{4} = -1$ and we then get

$$\begin{aligned} ap(-1)^{16} + b(-1)^{16} + q^{16} &= 0 \\ aq(-1)^{16} + b(-1)^{16} + p^{16} &= 0, \end{aligned}$$

or

$$\begin{aligned} ap + b + q^{16} &= 0 \Rightarrow ap + b = -q^{16} \\ aq + b + p^{16} &= 0 \Rightarrow aq + b = -p^{16} \end{aligned}$$

Using this we can solve for a and b. If we subtract these two equations we get

$$ap - aq = -q^{16} + p^{16},$$

or

$$a = \frac{-q^{16} + p^{16}}{p - q} = \frac{p^{16} - q^{16}}{p - q}$$

The right-hand-side of this can be factored. We find

$$\begin{aligned} a &= \frac{(p^8 - q^8)(p^8 + q^8)}{p - q} = \frac{(p^4 - q^4)(p^4 + q^4)(p^8 + q^8)}{p - q} \\ &= \frac{(p^2 - q^2)(p^2 + q^2)(p^4 + q^4)(p^8 + q^8)}{p - q} = (p + q)(p^2 + q^2)(p^4 + q^4)(p^8 + q^8). \end{aligned}$$

For these p and q we have $p + q = 1$. This means that

$$\begin{aligned} p^2 + q^2 &= (p + q)^2 - 2pq = 1^2 - 2(7) = 3 \\ p^4 + q^4 &= (p^2 + q^2)^2 - 2p^2q^2 = 9 - 2(-1)^2 = 7 \\ p^8 + q^8 &= (p^4 + q^4)^2 - 2(p^4q^4)^2 = 49 - 2(-1)^4 = 47. \end{aligned}$$

Thus we find that $a = 1 \cdot 3 \cdot 7 \cdot 47 = 987$.

Problem 14

If we draw the curve $xy = 1$ (or $y = \frac{1}{x}$) and the line $y = 2x$ in the x-y plane. Now for a point on $C = (u, v)$ the "mapped" point on $C^* = (x, y)$ will be on the line perpendicular to the line $y = 2x$. That means it will be on a line with a slope $-\frac{1}{2}$. This means that

$$\frac{y - v}{x - u} = -\frac{1}{2}\,.$$

At the same time the *midpoint* between (x, y) and (u, v) is on the line $y = 2x$ this means that

$$\frac{y + v}{2} = 2\left(\frac{x + u}{2}\right)\,.$$

We cans solve these for u and v to get

$$u = \frac{4y - 3x}{5}$$
$$v = \frac{4x + 3y}{5}\,.$$

Now as the point (u, v) is on the curve $uv = 1$ we have

$$\left(\frac{4y - 3x}{5}\right)\left(\frac{4x + 3y}{5}\right) = 1\,.$$

If we expand this and simplify we get

$$12x^2 - 7xy - 12y^2 + 25 = 0\,.$$

Thus $b = -7$ and $c = -12$ so $bc = 84$.

1983 – 1988 Dropped AHSME Problems

Problem 1

We start with the given equation

$$\sqrt{x^2 - 7ax + 10a^2} - \sqrt{x^2 + ax - 6a^2} = x - 2a\,. \tag{438}$$

Note that we can factor each of the quadratics in the square roots to get

$$\sqrt{(x-2a)(x-5a)} - \sqrt{(x+3a)(x-2a)} = x - 2a\,. \tag{439}$$

If we are lucky enough we might notice that $x = 2a$ is a root i.e. it makes both sides of this expression zero regardless of the value of a. In the following we will "sort of" assume $x \neq 2a$. The assumption that $x \neq 2a$ is not really needed but when we take the intersection of two domains below there is a "hidden" domain intersection at the single point $x = 2a$ and it is often easier to be "thinking" $x \neq 2a$ in the manipulations below. We just need to be sure to remember that in the end $x = 2a$ is a solution to the original equation for all a. See below for a bit more detail on this.

The ordering of the "roots" in the arguments of the square root depend on the sign of a. Now if $a > 0$ then these roots in increasing ordered are

$$-3a < 0 < 2a < 5a\,, \tag{440}$$

while if $a < 0$ then these roots in increasing order are

$$5a < 2a < 0 < -3a\,. \tag{441}$$

We will start by considering the case when $a > 0$ and Equation 440 holds. In that case as the right-hand-side of Equation 439 is a real number, the square roots in the left-hand-side of that expression are only defined in a domain of x where their arguments are positive. Then from the first square root term on the left-hand-side of Equation 439 we must have

$$x \le 2a \quad \text{or} \quad x \ge 5a\,.$$

and from the second square root term on the left-hand-side of Equation 439 we must have

$$x \le -3a \quad \text{or} \quad x \ge 2a\,.$$

To have the left-hand-side of Equation 439 be real we need to consider the intersection of these two regions. One "intersection" of these two domains is the single point $x = 2a$. As noted above, this point is also a solution to the original equation. A more nontrivial intersection of these two domains is

$$x \le -3a \quad \text{or} \quad x \ge 5a\,. \tag{442}$$

We start by assuming that $x \le -3a$. Then in that case $x - 2a \le -5a < 0$ and we can write Equation 439 as

$$\sqrt{-(x-2a)(5a-x)} - \sqrt{-(x-2a)(-x-3a)} = x - 2a\,.$$

Now as $x - 2a \le 0$ we can "factor" $\sqrt{2a-x}$ from each of the three terms above to get

$$\sqrt{2a-x}\left(\sqrt{5a-x} - \sqrt{-x-3a}\right) = -\sqrt{2a-x}\cdot\sqrt{2a-x}\,.$$

From this expression we see that $x = 2a$ is a solution (but unfortunately its not in the domain $x \le -3a$). If $x \ne 2a$ we can "divide' out" $\sqrt{2a-x}$ to get

$$\sqrt{5a-x} - \sqrt{-x-3a} = -\sqrt{2a-x}\,. \tag{443}$$

For real $a > 0$ and $x < -3a$ all of the above radicands are positive. If we square this (keeping in mind we might be introducing spurious solutions in doing so) we get

$$(5a-x) - 2\sqrt{(5a-x)(-x-3a)} + (-x-3a) = (2a-x)\,.$$

If we simplify this we get

$$x = -2\sqrt{(5a-x)(-x-3a)}\,.$$

Squaring this (keeping the comment above about spurious solutions in mind) and simplifying we get

$$3x^2 - 8ax - 60a^2 = 0\,. \tag{444}$$

Using the quadratic equation we find roots to the above given by

$$x = -\frac{10}{3}a \quad \text{and} \quad x = 6a\,. \tag{445}$$

Only the first root is in the domain we are considering here of $x \le -3a$. If we put this root back into the original expression given by Equation 443 we get

$$5\sqrt{\frac{a}{3}} - \sqrt{\frac{a}{3}} = 4\sqrt{\frac{a}{3}} = -4\sqrt{\frac{a}{3}}\,,$$

which is not true. Thus we have found no solutions when $a > 0$ and $x \le -3a$.

We now assume the other "side" for x where we have nonnegative square roots when $a > 0$ that is $x \ge 5a$. In that case $x - 2a \ge 3a > 0$ and we can write Equation 439 by "factoring" $\sqrt{x-2a}$ from each of the three terms to get

$$\sqrt{x-2a}\left(\sqrt{x-5a} - \sqrt{x+3a}\right) = \sqrt{x-2a}\cdot\sqrt{x-2a}\,.$$

From this expression we see that $x = 2a$ is a solution (but not in the domain $x \ge 5a$). Assuming $x \ne 2a$ we can "divide out" $\sqrt{x-2a}$ to get

$$\sqrt{x-5a} - \sqrt{x+3a} = \sqrt{x-2a}\,. \tag{446}$$

If I square both sides of the above I get

$$(x-5a) - 2\sqrt{(x-5a)(x+3a)} + (x+3a) = x - 2a\,.$$

Simplifying gives

$$x = 2\sqrt{(x-5a)(x+3a)}\,.$$

If I square both sides and simplify I again get Equation 444 which again has solutions given by Equation 445. In this case only the second root is in the domain $x \ge 5a$. If we put this root back into the original expression given by Equation 446 we get

$$\sqrt{a} - \sqrt{9a} = \sqrt{4a}\,,$$

which is not true. Thus we have found no solutions when $a > 0$ and $x \ge 5a$.

In summary then when $a > 0$ we have only the solution $x = 2a$.

We now considering the case when $a < 0$ and Equation 441 holds. As the right-hand-side of Equation 439 is a real number, the square roots in the left-hand-side are only defined in a domain of x where their arguments are positive. From the first square root term on the left-hand-side of Equation 439 we must have

$$x \le 5a \quad \text{or} \quad x \ge 2a\,.$$

and from the second square root term on the left-hand-side of Equation 439 we must have

$$x \le 2a \quad \text{or} \quad x \ge -3a\,.$$

Again note that one of the intersections of these two domains is the point $x = 2a$. Thus we have one root/solution when $a < 0$ for $x = 2a$. A "larger" nontrival domain where both radicands are positive is

$$x \le 5a \quad \text{or} \quad x \ge -3a\,. \tag{447}$$

For $a < 0$ and based on this domain where the solutions x must lie to have positive radicands we first consider the case that $x \le 5a$. In that case $x - 2a \le 3a < 0$ and we can write Equation 439 by "factoring" $\sqrt{-(x-2a)}$ from each of the three terms to get

$$\sqrt{-(x-2a)(5a-x)} - \sqrt{-(x-2a)(-x-3a)} = -\sqrt{2a-x}\cdot\sqrt{2a-x}\,.$$

When $x \ne 2a$ we can divide by $\sqrt{-(x-2a)}$ to get

$$\sqrt{5a-x} - \sqrt{-x-3a} = -\sqrt{2a-x}\,. \tag{448}$$

For real $a < 0$ when $x < 5a$ all of the above radicands are positive. If we square this (keeping in mind we might be introducing spurious roots) we get

$$(5a-x) - 2\sqrt{(5a-x)(-x-3a)} + (-x-3a) = (2a-x)\,.$$

If we simplify this we get

$$x = -2\sqrt{(5a-x)(-x-3a)}\,.$$

If I square both sides and simplify I again get Equation 444 which again has solutions given by Equation 445. In this case only the second root is in the domain $x \le 5a$. If we put this root back into the original expression given by Equation 448 we get

$$\sqrt{-a} - \sqrt{-9a} = -\sqrt{-4a}\,,$$

which is true. Thus when $a < 0$ and $x \leq 5a$ we have found the solution $x = 6a$.

Finally, we consider $a < 0$ and $x \geq -3a$. In that case $x - 2a \geq -5a > 0$ so Equation 439 becomes

$$\sqrt{x-5a} - \sqrt{x+3a} = \sqrt{x-2a}\,. \tag{449}$$

For real $a < 0$ when $x \geq -3a$ all of the above radicands are positive. If we square this we get

$$(x-5a) - 2\sqrt{(x-5a)(x+3a)} + (x+3a) = (x-2a)\,.$$

If we simplify this we get

$$x = 2\sqrt{(x-5a)(x+3a)}\,.$$

If I square both sides and simplify I again get Equation 444 which again has solutions given by Equation 445. In this case only the first root is in the domain $x \geq -3a$. If we put this root back into the original expression given by Equation 449 we get

$$\sqrt{-\frac{10}{3}a - 5a} - \sqrt{-\frac{10}{3}a + 3a} = \sqrt{-\frac{10}{3}a - 2a}\,,$$

which simplifies to

$$\sqrt{-\frac{25}{3}a} - \sqrt{-\frac{1}{3}a} = \sqrt{-\frac{16}{3}a}\,,$$

or

$$5\sqrt{-\frac{1}{3}a} - \sqrt{-\frac{1}{3}a} = 4\sqrt{-\frac{1}{3}a}\,,$$

which is true. Thus when $a < 0$ and $x \geq -3a$ we have found the solution $x = -\frac{10}{3}a$.

In summary then when $a < 0$ we have the solutions

$$x \in \left\{6a, 2a, -\frac{10}{3}a\right\}\,.$$

Thus if $a > 0$ the sum of the solutions is $2a$ while if $a < 0$ the sum of the solutions is $\frac{14}{3}a$.

Problem 2

In Figure 15 we plot the function $g(x)$. As $h(x)$ is $g(x)$ shifted upwards by an amount ϵ from that figure we see that the root of $g(x)$ at $x = -3$ becomes two roots of $h(x)$, the root of $g(x)$ at $x = -2$ becomes two roots of $h(x)$, the root of $g(x)$ at $x = 1$ stays one root of $h(x)$, and the root of $g(x)$ at $x = 3$ is not a root of $h(x)$. This gives five roots for $g(x)$.

Problem 3

Given the "traditional formula" of

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\,,$$

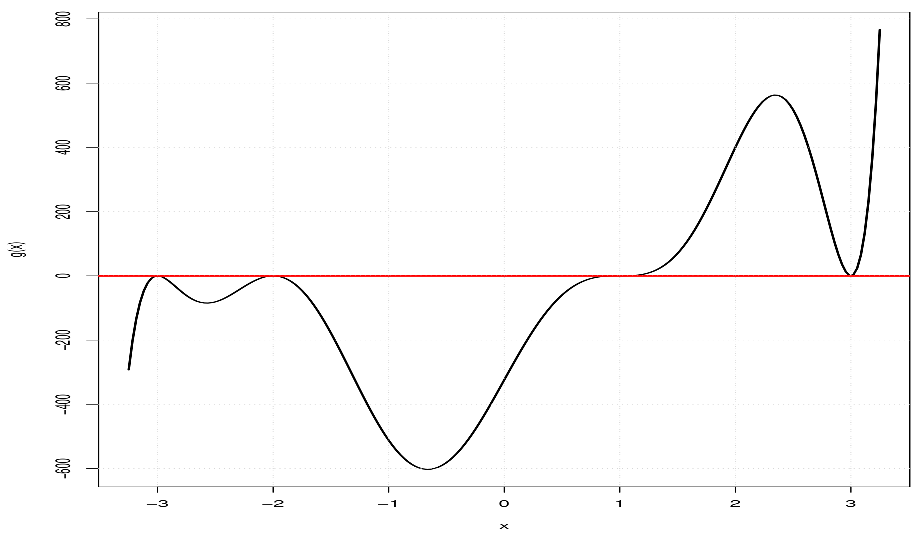

Figure 15: The left and right-hand-sides.

we can multiply by a "form of one" given by

$$O \equiv \frac{-b \mp \sqrt{b^2 - 4ac}}{-b \mp \sqrt{b^2 - 4ac}},$$

to get

$$x \times O = \frac{b^2 - (b^2 - 4ac)}{2a(-b \mp \sqrt{b^2 - 4ac})} = \frac{2c}{-b \mp \sqrt{b^2 - 4ac}}.$$

Problem 4

Note that we can write the given equations as

$$c = \frac{a + e}{2} \tag{450}$$

$$c = \frac{b + d}{2} \tag{451}$$

$$d = \frac{c + e}{2}. \tag{452}$$

This means that c is "between" a and e and also between b and d as the first two equations above state that it is the midpoint of the line segments ae and bd. The third equation states that d is the midpoint of the line segment ce. Thus the smallest/largest number from this set cannot be the points c or d.

From Equation 450 we can place c on a number line and "surround" it as

$$a < c < e \quad \text{or} \quad e < c < a.$$

To start let assume the first of these or that $a < c < e$. We can "draw" this situation as follows

```
a          c          e
```

where there are exactly the same number of spaces (10) between the a and the c as between the c and the e. From Equation 452 we know where d must go in this figure (exactly between the c and the e) or

```
a          c     d     e
```

We have yet to place the number b. Now given Equation 451 we can write

$$
\begin{aligned}
b &= 2c - d \\
&= c + (c - d) = c - (d - c) \,.
\end{aligned}
$$

The strange way that we wrote the above is indicative of how we will find the location the point b in terms of its location relative to c and distances we know. That is we know from the above that b is located exactly in between a and c and we get

```
a     b     c     d     e
```

In this case then the largest and smallest of the given numbers are $\{e, a\}$.

Next if we assume the other inequality or that $e < c < a$. We can "draw" this situation as follows

```
e          c          a
```

where there are exactly the same number of spaces (10) between the e and the c as between the c and the a. From Equation 452 we know where d must go in this figure (exactly between the c and the e) or

```
e     d     c          a
```

Again we have yet to place the number b. Now given Equation 451 we can write

$$
\begin{aligned}
b &= 2c - d \\
&= c + (c - d) \,.
\end{aligned}
$$

Again this indicates how we will find the location the point b in terms of its location relative to c and distances we know. That is we know from the above that b is located exactly in between c and a and we get

```
e    d    c    b    a
```

Again we see that the largest and smallest of the given numbers are $\{a, e\}$.

Problem 5

Recall that the discriminant D is given by

$$D = b^2 - 4ac\,.$$

If b is even then it takes the form $b = 2n$ and D is

$$D = 4n^2 - 4ac\,.$$

This means that D is divisible by four. If b is odd then it takes the form $b = 2m + 1$ and D is then

$$\begin{aligned} D &= (2m+1)^2 - 4ac = 4m^2 + 4m + 1 - 4ac \\ &= 4(m^2 + m - ac) + 1\,. \end{aligned}$$

This means that $D - 1$ is dividable by four. Thus as b must be either even or odd this means that D or $D - 1$ must be dividable by four.

From the numbers given we see that for $D' = 23$ since neither D' or $D' - 1 = 22$ is dividable by four it cannot be the value for a discriminant of a quadratic equation.

Problem 7

Draw the given figure with the bus at a point B at the center of a Cartesian coordinate system, the road extending at an angle of 30° from the x-axis towards the North-East, a point A on that road towards which the runner will run. Finally on the x-axis is the point P where the runner is currently. We assume that if she runs towards A she will meet the bus at that location. Let v be the velocity of the bus. This means that the time it takes for the bus to get to A must be more than the time it takes the runner to get there. This means that

$$\frac{BA}{v} \geq \frac{PA}{15}\,,$$

or

$$v \leq \frac{15BA}{PA}\,.$$

This gives an upper bound on the buses speed. If the bus was going any faster then it would not possible for the runner to catch it meaning that

$$v = 15\left(\frac{BA}{PA}\right)_{\max}\,,$$

where we have expressed v in terms of the maximal possible value for the ratio $\frac{BA}{PA}$.

Let $\angle BPA = \theta$ then by the law of sines we have

$$\frac{BA}{\sin(\theta)} = \frac{PA}{\sin(30)} = 2PA\,,$$

or

$$\frac{BA}{PA} = 2\sin(\theta)\,.$$

This means that

$$\left(\frac{BA}{PA}\right)_{\max} = 2\sin(90^\circ) = 2\,,$$

so that $v = 15(2) = 30$.

Problem 8

From the definition of $f_{n+1}(x)$ we see that

$$f_{1981}(x) = f_{1982}(x)\,,$$

is equal to

$$f_{1981}(x) = |1 - f_{1981}(x)|\,.$$

Thus if we let $v \equiv f_{1981}(x)$ the above is equal to

$$v = |1 - v|\,.$$

To solve the above we have

$$v = \pm(1 - v)\,.$$

The plus sign gives

$$v = 1 - v \quad \text{where} \quad v = \frac{1}{2}\,.$$

The minus sign gives

$$v = -1 + v\,,$$

which has no solution. Thus to solve this problem we now need to determine the number of solutions to

$$f_{1981}(x) = \frac{1}{2}\,.$$

Lets consider the question of how many solutions to

$$f_n(x) = \frac{1}{2}\,,$$

there are. For $n = 1$ this is

$$f_1(x) = |1 - x| = \frac{1}{2}\,,$$

this has two solutions $x \in \left\{\frac{1}{2}, \frac{3}{2}\right\}$. For $n = 2$ this is

$$f_2(x) = |1 - |1 - x|| = \frac{1}{2},$$

which based on $f_1(x) = \frac{1}{2}$ has two solutions

$$|1 - x| = \frac{1}{2} \quad \text{or} \quad |1 - x| = \frac{3}{2}.$$

The first of these two equations gives $x \in \left\{\frac{1}{2}, \frac{3}{2}\right\}$ while the second of these gives

$$x \in \left\{-\frac{1}{2}, \frac{5}{2}\right\}.$$

This gives in total four solutions to $f_2(x) = \frac{1}{2}$. For $n = 3$ we have

$$f_3(x) = |1 - f_2(x)| = \frac{1}{2},$$

Thus we need $f_2(x) = \frac{1}{2}$ which has four solutions (by the above) or $f_2(x) = \frac{3}{2}$ which is

$$|1 - |1 - x|| = \frac{3}{2},$$

or

$$1 - |1 - x| = \pm\frac{3}{2},$$

Splitting this into two equations (for the plus and the negative sign) the plus sign has no solutions while the negative sign has two given by $x \in \left\{-\frac{3}{2}, \frac{7}{2}\right\}$ giving two more solutions for a total of six solutions to $f_3(x) = \frac{1}{2}$. These solutions are

$$x \in \left\{-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}\right\}.$$

The pattern seems clear. The number of solutions to $f_n(x) = \frac{1}{2}$ is $2n$. Thus the number of solutions to $f_{1981}(x) = \frac{1}{2}$ is $2(1981) = 3962$.

Problem 10

Let $n^2 - 440 = b^2$ so that

$$n^2 - b^2 = 440.$$

This means that

$$(n - b)(n + b) = 440 = 2^3 \cdot 5 \cdot 11.$$

As the product of $(n - b)(n + b)$ is even if n was even and b was odd or vice-versa then the left-hand-side of the above would be the product of two odd numbers and hence odd which is a contradiction. Thus both n and b must be even. From the above factorization $n - b$

must have a factor of two or four and $n+b$ must have a factor or two or four. Thus we have to consider two cases

$$2|n-b \quad \text{and} \quad 4|n+b\,, \tag{453}$$

or

$$4|n-b \quad \text{and} \quad 2|n+b\,, \tag{454}$$

If Equation 453 is true then the possible "factors" are

$$\begin{aligned} n-b &\in \{2\cdot 5\cdot 11\,, 2\cdot 1\cdot 11\,, 2\cdot 5\cdot 1\,, 2\cdot 1\cdot 1\} \quad \text{paired with} \\ n+b &\in \{4\cdot 1\cdot 1\,, 4\cdot 5\cdot 1\,, 4\cdot 1\cdot 11\,, 4\cdot 5\cdot 11\}\,. \end{aligned}$$

If Equation 454 is true then the possible "factors" are

$$\begin{aligned} n-b &\in \{4\cdot 5\cdot 11\,, 4\cdot 1\cdot 11\,, 4\cdot 5\cdot 1\,, 4\cdot 1\cdot 1\} \quad \text{paired with} \\ n+b &\in \{2\cdot 1\cdot 1\,, 2\cdot 5\cdot 1\,, 2\cdot 1\cdot 11\,, 2\cdot 5\cdot 11\}\,. \end{aligned}$$

These are $4+4=8$ systems of equations for n and b. If we solve these we get the following solutions

```
   x1 x2 x3  x4    x5  x6 x7 x8
n  57 21 27 111   111  27 21 57
b -53 -1 17 109 -109 -17  1 53
```

As we are looking for the number of $n \geq 1$ we see that there are *four* distinct numbers.

Problem 12

Lets assume that on the current month (month "zero") the amount of sugar we can buy if we have D_0 dollars and sugar costs S_0 (per gram) is

$$A_0 = \frac{D_0}{S_0}\,,$$

grams. Now if in the next month (month "one") we have

$$\begin{aligned} D_1 &= \left(1 + \frac{q}{100}\right) D_0 \\ S_1 &= \left(1 + \frac{p}{100}\right) S_0\,, \end{aligned}$$

then we can now buy

$$A_1 = \frac{D_1}{S_1} = \frac{\left(1+\frac{q}{100}\right)}{\left(1+\frac{p}{100}\right)} \left(\frac{D_0}{S_0}\right)\,.$$

grams. This means that

$$\frac{A_1}{A_0} - 1 = \frac{1+\frac{q}{100}}{1+\frac{p}{100}} - 1\,. \tag{455}$$

Now from Equation 455 we get

$$\begin{aligned}\frac{A_1}{A_0}-1&=\frac{1+\frac{q}{100}-1-\frac{p}{100}}{1+\frac{p}{100}}\\&=-\frac{p-q}{100+p}\,.\end{aligned}$$

Now

$$\frac{p-q}{100+p}<\frac{p-q}{100}\,.$$

Thus the *decrease* is less than $p-q\%$.

Problem 13

Lets call this expression E. Then with some algebra we have

$$\begin{aligned}E&=\binom{\frac{1}{2}}{k}\\&=\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\cdots\left(\frac{1}{2}-k+1\right)}{k(k-1)(k-2)\cdots2\cdot1}\,.\end{aligned}$$

The numerator of the above has k terms. Lets multiply the top and bottom by 2^k (and then simplify) to get

$$\begin{aligned}E&=\frac{1(1-2)(1-4)(1-6)\cdot(1-2k+2)}{2^k k!}\\&=\frac{1(-1)(-3)(-5)\cdots(3-2k)}{2^k k!}\\&=\frac{(-1)^{k-1}1\cdot3\cdot5\cdots(2k-5)(2k-3)}{2^k k!}\\&=\frac{(-1)^{k-1}1\cdot2\cdot3\cdot4\cdot5\cdots(2k-5)\cdot(2k-4)\cdot(2k-3)\cdot(2k-2)}{2^k k!(2\cdot4\cdot6\cdots(2k-4)\cdot(2k-2))}\\&=\frac{(-1)^{k-1}(2k-2)!}{2^k k!2^{k-1}(1\cdot2\cdot3\cdots(k-2)\cdot(k-1))}\\&=\frac{(-1)^{k-1}(2k-2)!}{2^{2k-1}k!(k-1)!}\\&=\frac{(-1)^{k-1}(2k-2)!}{2^{2k-1}k(k-1)!(k-1)!}\,.\end{aligned}$$

Next note that

$$\binom{2k-2}{k-1}=\frac{(2k-2)!}{(k-1)!(2k-2-k+1)!}=\frac{(2k-2)!}{(k-1)!(k-1)!}\,.$$

This means that we have

$$
\begin{aligned}
E &= \frac{(-1)^{k-1}}{2^{2k-1}k}\binom{2k-2}{k-1} = \frac{(-1)^{k-1}}{4^k 2^{-1}k}\binom{2k-2}{k-1} \\
&= \frac{(-1)^{k-1}}{4^{k-1}\cdot 4^1 \cdot 2^{-1}k}\binom{2k-2}{k-1} \\
&= \left(-\frac{1}{4}\right)^{k-1}\frac{1}{2k}\binom{2k-2}{k-1}.
\end{aligned}
$$

Problem 14

Lets call this expression E. Then we have that

$$
\begin{aligned}
E &= \sqrt[3]{1000 + \frac{3}{10^8}} - \sqrt[3]{1000 + \frac{15}{10^9}} \\
&= \sqrt[3]{10^3\left(1 + \frac{3}{10^{11}}\right)} - \sqrt[3]{10^3\left(1 + \frac{15}{10^{12}}\right)} \\
&= 10\sqrt[3]{1 + \frac{3}{10^{11}}} - 10\sqrt[3]{1 + \frac{15}{10^{12}}}.
\end{aligned}
$$

Now using the fact that

$$
(1+x)^\alpha \le 1 + \alpha x \quad \text{when} \quad \alpha < 1.
$$

Using the above inequality as an approximate equality we have

$$
\begin{aligned}
E &\approx 10\left(1 + \frac{1}{10^{11}}\right) - 10\left(1 + \frac{5}{10^{11}}\right) \\
&= \frac{1}{10^{10}} - \frac{5}{10^{11}} = \frac{10-5}{10^{11}} = \frac{5}{10^{11}}.
\end{aligned}
$$

This means

$$
E \approx 0.000, 000, 000, 05.
$$

In the above I have placed commas to separate groups of zeros and make the number easier to read.

Problem 16

One difficulty with this problem is that when the first person says that "I use base" 10 what that means is that he uses the number

$$
(10)_b = b,
$$

as his base. It seems too easy to assume that he means that he uses *ten* as his base. The insight into this problem is to understand that we don't know what the bases are that the two speakers are using.

If we let the first speakers base be b and the set of people that use this base as B so that the number that use this base can be written as $|B|$ then we know that

$$|B| = (26)_b = 2b + 6 .$$

From this first speakers comment the *other* base (which we denote as c) is

$$c = (14)_b = b + 4 .$$

Finally, if we let C be the set of people that speak this second base then we have

$$|C| = (22)_b = 2b + 2 .$$

We know that the second speaker must use the second base c as the quoted total number of residents N at 25 could not be a number in base b as it is smaller than the first persons statement that $(26)_b$ people use his base. Thus from this second speakers comments we know that

$$N = (25)_c = 2c + 5 ,$$

and

$$|B \cap C| = (13)_c = c + 3 .$$

Then from the inclusion-exclusion identity we have

$$N = 1_c + |B| + |C| - |B \cap C| ,$$

which using what we know becomes

$$2c + 5 = 1 + (2b + 6) + (2b + 2) - (c + 3) .$$

Simplifying this we get

$$3c = 1 + 4b .$$

Putting in what we know about c in terms of b we can solve for b and find $b = 11$. From that we have $c = 11 + 4 = 15$ so that $N = 30 + 5 = 35$.

Problem 17

In working this problem I found two solutions that are equivalent to the correct one but have a much different looking form. This is very consistent with the comments given in the book for reasons why this problem was rejected.

Method 1: As x is acute we know that both $\sin(x)$ and $\cos(x)$ are positive so we can take

$$\sin(x) = \sqrt{1 - \cos^2(x)} .$$

If we put this into the given expression we get

$$\sqrt{1 - \cos^2(x)} + \cos(x) = \frac{4}{3} .$$

Squaring both sides gives

$$1 + 2\cos(x)\sqrt{1-\cos^2(x)} = \frac{16}{9}\,,$$

or

$$\cos(x)\sqrt{1-\cos^2(x)} = \frac{7}{18}\,.$$

If we square again we get

$$\cos^2(x)(1-\cos^2(x)) = \frac{7^2}{18^2}\,.$$

We can write this as

$$\cos^4(x) - \cos^2(x) + \frac{7^2}{18^2} = 0\,.$$

If we let $v = \cos^2(x)$ this is a quadratic equation for v and has a solution given by

$$v = \frac{1}{2}\left(1 \pm \sqrt{1 - 4\left(\frac{49}{18^2}\right)}\right) = \frac{1}{2}\left(1 \pm \frac{4\sqrt{2}}{9}\right)\,.$$

As $\sqrt{2} < 2$ both of these expressions are positive. As $\cos(x)$ is a decreasing function for $0 < x < \frac{\pi}{2}$ to have x be the largest angle we want the smaller of the two expressions above and thus we take the minus sign. This means that

$$\cos^2(x) = \frac{1}{2}\left(\frac{9-4\sqrt{2}}{9}\right)\,.$$

Note that we can write

$$9 - 4\sqrt{2} = (-1+2\sqrt{2})^2\,.$$

The argument of the square above is positive so taking the square root we have

$$\sqrt{9-4\sqrt{2}} = -1 + 2\sqrt{2}\,.$$

Using this we have

$$\cos(x) = \frac{-1+2\sqrt{2}}{3\sqrt{2}}\,,$$

or

$$x = \cos^{-1}\left(\frac{-1+2\sqrt{2}}{3\sqrt{2}}\right)\,.$$

If we numerically compute this we find that it numerically matches the solution (D).

Method 2: If we draw a right triangle in the Cartesian plane with an acute angle x, a leg along the positive x-axis of length a, an vertical leg of length o, and hypotenuse of length h then we can write

$$\sin(x) = \frac{o}{h}$$
$$\cos(x) = \frac{a}{h}\,.$$

We can write the given equation as

$$\frac{o}{h} + \frac{a}{h} = \frac{4}{3},$$

or

$$o + a = \frac{4}{3}h.$$

As we have a right triangle we have that $h = \sqrt{o^2 + a^2}$. If we put that into the above and square we get

$$(o + a)^2 = \frac{16}{9}(o^2 + a^2).$$

We can expand and simplify to get

$$7o^2 - 18oa + 7a^2 = 0.$$

If we then divide by a^2 we get

$$7\left(\frac{o}{a}\right)^2 - 18\left(\frac{o}{a}\right) + 7 = 0.$$

Note that the above is a quadratic for $\frac{o}{a} = \tan(x)$. Numerically we can solve it and then take the $\tan^{-1}$ to determine the value of x. Doing this I get

```
> atan(polyroot(c(7, -18, 7)))
[1] 0.4455613+0i 1.1252351-0i
```

The larger of these two numbers (the second) matches the solution (D).

Problem 18

We are told that

$$a + b = 1 \tag{456}$$

and that $a^2 + b^2 = 2$ and want to find the value of $a^3 + b^3$. If we square the first equation we have

$$(a + b)^2 = 1.$$

Expanding this we get

$$a^2 + 2ab + b^2 = 1.$$

If we then use the second equation in this we get

$$2 + 2ab = 1.$$

This means that

$$ab = -\frac{1}{2}. \tag{457}$$

Next consider

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

Using Equation 456 and 457 we can write this as

$$1 = a^3 + b^3 + 3\left(-\frac{1}{2}\right)a + 3\left(-\frac{1}{2}\right)b\,.$$

We can write this as

$$1 = a^3 + b^3 - \frac{3}{2}(a+b)\,.$$

Again using Equation 456 we find

$$a^3 + b^3 = \frac{5}{2}\,.$$

Problem 19

By symmetry, the circle will need to sit above the parabola $y = x^2$ and have its center on the y-axis. Thus it will need to have it center at $(0, r)$ and an equation given by

$$x^2 + (y-r)^2 = r^2\,.$$

This means that

$$y = r \pm \sqrt{r^2 - x^2}\,.$$

To be above the parabola we needed to have the circle above $y = x^2$ or

$$r \pm \sqrt{r^2 - x^2} \geq x^2\,.$$

for all $-r \leq x \leq +r$. The expression using the plus sign in the above will always be above the expression we get when we use the minus sign and thus the condition we actually need to enforce is

$$r - \sqrt{r^2 - x^2} \geq x^2\,.$$

We can write this as

$$\sqrt{r^2 - x^2} \leq r - x^2\,,$$

If we square this we get

$$r^2 - x^2 \leq r^2 - 2rx^2 + x^4\,,$$

or simplifying

$$x^4 + (1-2r)x^2 \geq 0\,.$$

Dividing by x^2 we get

$$x^2 \geq 2r - 1\,.$$

Recall that this has to hold for all $-r \leq x \leq +r$. The smallest the left-hand-side happens when $x = 0$ and thus r must satisfy

$$0 \geq 2r - 1\,.$$

This means that $r \leq \frac{1}{2}$. The largest value of r is thus $r = \frac{1}{2}$.

Problem 20

Write this function as

$$y = \frac{\log(x+1)}{\log(x)}\,.$$

We need $x > 0$ for $\log(x)$ to be defined. We also need $x \neq 1$ so that the denominator of the above is not zero.

Now if $0 < x < 1$ then $\log(x) < 0$ and over that range we have

$$-\infty < \log(x) < 0^-\,.$$

Over this range of x we have $\log(x+1) > 0$ and so

$$0^+ < \log(x+1) < \log(2)\,.$$

This means that over $0 < x < 1$ we have

$$0^- < y < -\infty\,,$$

with a vertical asymptote at $x = 1$.

Next when $x > 1$ we have

$$0^+ < \log(x) < \infty\,,$$

while

$$\log(2) < \log(x+1) < \infty\,.$$

Using these results, the limit of this ratio for $x \to 1^+$ tends to $+\infty$ and the limit of this ratio for large x tends to

$$\lim_{x\to\infty} \frac{\log(x+1)}{\log(x)} = 1\,.$$

Thus over $x > 1$ we have

$$1 < y < +\infty\,.$$

Combining these two results we get a plot that looks like (E).

Problem 21

Recall that to evaluate $\log_b(x)$ for any base b we must have $x > 0$. Thus we must have $x > 0$ so that $\log_{1/2}(x)$ is defined. Next we must have $\log_{1/2}(x) > 0$ so that

$$\log_2(\log_{1/2}(x))\,,$$

defined. Now to have

$$\log_{1/2}(x) > 0\,.$$

If we let $p = \log_{1/2}(x)$ then equivalently we have

$$\left(\frac{1}{2}\right)^p = x\,.$$

When $p > 0$ we see that we will have $0 < x < 1$. Finally we need to have

$$\log_2(\log_{1/2}(x)) > 0\,,$$

so that $\log_{1/2}(\log_2(\log_{1/2}(x)))$ will be defined. Now to have

$$\log_2(\log_{1/2}(x)) > 0\,,$$

We need to have

$$\log_{1/2}(x) > 1\,.$$

This means that

$$\left(\frac{1}{2}\right)^{\log_{1/2}(x)} < \left(\frac{1}{2}\right)^1\,,$$

or

$$x < \frac{1}{2}\,.$$

Thus we need to take the intersection of all of these domains or

$$\{x|x > 0\} \cap \{x|x < 1\} \cap \left\{x|x < \frac{1}{2}\right\}\,.$$

This is the set

$$\left\{x|0 < x < \frac{1}{2}\right\}\,.$$

Problem 22

Let the shorter arm have a length of l so that the longer arm will have a length $1.05l$. Then under one weighing we have

$$210(1.05l) = lW_1\,.$$

Under the second weighing we have

$$W_2(1.05l) = 210l\,.$$

We can solve the above for W_1 and W_2 we get $W_1 = 220.5$ and $W_2 = 200$. Thus $W_1 + W_2 = 420.5$.

Problem 23

Let the side of the equilateral triangle be denoted s. Let $AF = a$ and $FB = b$ so that $a + b = s$. We are told in the problem that $AF = pFB$ or $a = pb$. Since this problem has a tangent to a circle and two secants given by AC and BC we will use the *Tangent-Secant Theorem*. Using this theorem for the tangent-secant AF and AC we have

$$ADs = a^2\,.$$

Using this theorem for the tangent-secant BF and BC we have

$$BEs = b^2\,.$$

If we divide these two expressions we get

$$\frac{AD}{BE} = \frac{a^2}{b^2}\,.$$

As we know that $\frac{a}{b} = p$ we have that

$$\frac{AD}{BE} = p^2\,.$$

Problem 24

From the definition of r we have $0 < r < 1$ so that $\lfloor r \rfloor = 0$ and thus (B) and (E) always only give the integer zero and thus cannot be correct.

Lets consider the expression $\lfloor 10r \rfloor$. Note that we have

$$\begin{aligned} P\{\lfloor 10r \rfloor = 0\} &= P\{0 \le 10r < 1\} \\ &= P\left\{0 \le r < \frac{1}{10}\right\} = \frac{1}{10}\,, \end{aligned}$$

which is not good as the uniform probability we get an integer between $0-10$ should be $\frac{1}{11}$. To see the problem we can compute

$$\begin{aligned} P\{\lfloor 10r \rfloor = 10\} &= P\{10 \le 10r < 11\} \\ &= P\left\{1 \le r < \frac{10}{11}\right\} = 0\,. \end{aligned}$$

Thus we will never get the integer 10. By both of these arguments (A) cannot be correct.

Lets consider (C). Then we compute

$$
\begin{aligned}
P\{\lfloor 10r + 0.5 \rfloor = n\} &= P\{n \leq 10r + 0.5 < n + 1\} \\
&= P\left\{n - \frac{1}{2} \leq 10r < n + \frac{1}{2}\right\} \\
&= P\left\{\frac{1}{10}\left(n - \frac{1}{2}\right) \leq r < \frac{1}{10}\left(n + \frac{1}{2}\right)\right\} \\
&= \int_{\max\left(\frac{1}{10}\left(n-\frac{1}{2}\right),0\right)}^{\min\left(\frac{1}{10}\left(n+\frac{1}{2}\right),1\right)} dr \,.
\end{aligned}
$$

If we take $n = 0$ the above is

$$
\int_0^{\frac{1}{20}} = \frac{1}{20} \,.
$$

If we take $n = 10$ the above is

$$
\int_{\frac{1}{10}\left(\frac{19}{2}\right)}^{1} = 1 - \frac{19}{20} = \frac{1}{20} \,.
$$

If $n \in \{0, 10\}$ then the integral above evaluates to

$$
\frac{1}{10}\left(n + \frac{1}{2}\right) - \frac{1}{10}\left(n - \frac{1}{2}\right) = \frac{1}{10} \,.
$$

Note that these probabilities are not all equal and thus (C) cannot be correct.

All of these indicate that (D) is correct.

Problem 26

The number of heads H is a binomial random variable with parameters $(p = 0.5, n)$. Thus we have

$$
\begin{aligned}
\mu &= np = \frac{n}{2} \\
\sigma^2 &= npq = \frac{n}{4} \,.
\end{aligned}
$$

This means that $\sigma = \frac{\sqrt{n}}{2}$. Then we know that

$$
P\left\{|H - \mu| < 2\sigma\right\} \approx 0.95 \,,
$$

or using the above we have

$$
P\left\{\frac{n}{2} - \sqrt{n} < H < \frac{n}{2} + \sqrt{n}\right\} \approx 0.95 \,.
$$

This is the statement given. We can write this as

$$
P\left\{-1 < \frac{H - 0.5n}{\sqrt{n}} < +1\right\} = 0.95 \,. \tag{458}
$$

Now for this problem we want to find n such that

$$P\{0.4n \leq H \leq 0.6n\} = 0.95\,.$$

We can write this as

$$P\left\{-\frac{0.1n}{\sqrt{n}} \leq \frac{H - 0.5n}{\sqrt{n}} \leq +\frac{0.1n}{\sqrt{n}}\right\} = 0.95\,.$$

This will match Equation 458 if

$$\frac{0.1n}{\sqrt{n}} = 1 \quad \text{or} \quad n = 100\,.$$

Problem 27

Lets call this expression E. We start by expanding this and then grouping terms. We have

$$\begin{aligned} E &= ab(c^2 + d^2) + cd(a^2 + b^2) \\ &= abc^2 + abd^2 + cda^2 + cdb^2\,. \end{aligned}$$

If we group the first and third terms together and the second and fourth terms together we get

$$\begin{aligned} E &= (abc^2 + cda^2) + (abd^2 + cdb^2) \\ &= ac(bc + ad) + bd(ad + cb) \\ &= (ac + bd)(ad + cb)\,. \end{aligned}$$

Thus we see that $ac+bd$ is a factor of E. Note that we can eliminate some choices by selecting values for a, b, c, and d computing the value of E and looking which of the suggested solutions are *not* factors of E. For example we might try

```
a = 1
b = 3
c = 2
d = 5
a*b*(c^2 + d^2) + c*d*(a^2 + b^2) ##= 187 this is our 'target' we want to factor
```

The number 187 factors as

```
factor 187
187: 11 17
```

The possible factors (from the solutions) take the form

```
c( a*b, c^2 + d^2, a*b + c*d, a*c + b*d )
```

which are

```
[1]  3 29 13 17
```

Only (D) is correct in this list.

1983 – 1988 Dropped AIME Problems

Problem 1

Note that

$$
\begin{aligned}
(37abc) &= 37000 + (abc) \\
(37bca) &= 37000 + (bca) \\
(37cab) &= 37000 + (cab)\,.
\end{aligned}
$$

Thus for 37 to divide the left-hand-side is must divide the numbers (abc), (bca), and (cab).

Let $x = (abc)$, $y = (bca)$, and $z = (cab)$ then note

$$
\begin{aligned}
10x - y &= a \cdot 10^3 + b \cdot 10^2 + c \cdot 10 - b \cdot 10^2 - c \cdot 10 - a \\
&= a \cdot 10^3 - a = a(10^3 - 1) = 999a\,. \qquad (459)
\end{aligned}
$$

In the same way we have

$$
\begin{aligned}
10y - z &= 999b \\
10z - x &= 999c\,.
\end{aligned}
$$

Now $999 = 27 \cdot 37$ so we see that 37 divides 999. From these equations we have that if any one of x, y, or z is divisible by 37 then the others are also and we have a valid solution to our problem. Thus this problem now reduces to finding how many numbers of the form $x = (abc)$ are divisible by 37. Numbers like this are the "multiples of 37" or

$$000\,, 037\,, 074\,, 111\,,$$

etc. There are

$$\frac{999}{37} = 27\,,$$

non-zero multiples numbers like that giving $27 + 1 = 28$ total numbers of the desired form.

Problem 2

If x is an integer then

$$x - \lfloor x \rfloor = 0\,,$$

and thus the second equation becomes

$$y = 18.4\,.$$

If we put that into the first equation we get

$$\lfloor x \rfloor = 7.4\,,$$

which is not possible so we know that x is not an integer. This means that

$$x - \lfloor x \rfloor ,$$

is the fractional part of x and as x is not an integer we know that

$$0 < x - \lfloor x \rfloor < 1 ,$$

with both inequities strict (i.e. not equal). Then using the second equation we have

$$y = 18.4 - (x - \lfloor x \rfloor) .$$

Using the above we have that

$$-1 < -(x - \lfloor x \rfloor) < 0 ,$$

so adding 18.4 to this we get that

$$17.4 < y < 18.4 .$$

This means that $\lfloor y \rfloor \in \{17, 18\}$.

Lets assume that $\lfloor y \rfloor = 18$. Then from the first equation given we have

$$y = 25.8 - \lfloor x \rfloor .$$

If we put this into the second equation given we get an expression that we can write as

$$\lfloor x \rfloor - (x - \lfloor x \rfloor) = 7.4 .$$

Now $\lfloor x \rfloor$ is an integer and $x - \lfloor x \rfloor$ is a floating point number positive and less than one. This will be satisfied if $\lfloor x \rfloor = 8$ and $x - \lfloor x \rfloor = 0.6$ so that $x = 8.6$. Putting this into the second equation given we get

$$8.6 + y - 8 = 18.4 \quad \text{so} \quad y = 17.8 .$$

This solution has $\lfloor y \rfloor = 17$ not $\lfloor y \rfloor = 18$ as was assumed at the start. Thus we can't have $\lfloor y \rfloor = 17$.

Thus we now assume that $\lfloor y \rfloor = 17$. Then from the first equation given we have

$$y = 26.8 - \lfloor x \rfloor .$$

If we put this into the second equation given we get an expression that we can write as

$$\lfloor x \rfloor - (x - \lfloor x \rfloor) = 8.4 .$$

Following the same arguments as above this will be satisfied if $\lfloor x \rfloor = 9$ and $x - \lfloor x \rfloor = 0.6$ so that $x = 9.6$. Putting this into the second equation given we get

$$9.6 + y - 9 = 18.4 \quad \text{so} \quad y = 17.8 .$$

This value of y has $\lfloor y \rfloor = 17$ as assumed.

At this point we have $(x, y) = (9.6, 17.8)$ Lets check that this solution solves the given two equations. We have

$$9 + 17 + 17.8 = 43.8$$
$$9.6 + 17.8 - 9 = 18.4\,.$$

Showing that it is a solution.

This means that

$$100(x + y) = 100(27.4) = 274\,.$$

Problem 4

From the fact that DE is parallel to AB we have

$$\triangle APB \sim \triangle EPD\,.$$

This means that

$$\frac{\text{Area}(\triangle APB)}{\text{Area}(\triangle EPD)} = \alpha^2\,,$$

for some number α. From the problem we are told the value of this ratio

$$\alpha^2 = \frac{36}{25} \quad \text{so} \quad \alpha = \frac{6}{5}\,.$$

Now as $\triangle APB$ and $\triangle ACB$ have the same base (the segment AB) their areas are proportional to their heights and we have

$$\frac{\text{Area}(\triangle ACB)}{\text{Area}(\triangle APB)} = \frac{h_{ACB}}{h_{APB}}\,.$$

Let h_1 be the height of $\triangle APB$, h_2 the height of $\triangle EPD$, and h_3 the height of $\triangle DCE$. Then

$$\frac{\text{Area}(\triangle ACB)}{\text{Area}(\triangle APB)} = \frac{h_1 + h_2 + h_3}{h_1}\,. \tag{460}$$

As $\triangle APB \sim \triangle EPD$ we have

$$h_1 = \frac{6}{5} h_2 \quad \text{or} \quad h_2 = \frac{5}{6} h_1\,,$$

and also that

$$AB = \frac{6}{5} DE\,.$$

Now as $\triangle ACB \sim \triangle DCE$ and using the previous expression we see that the "expansion factor" from $\triangle ACB$ to $\triangle DCE$ is $\frac{5}{6}$. Thus

$$h_1 + h_2 + h_3 = \frac{6}{5} h_3\,.$$

Putting in what we know about h_2 in terms of h_1 and solving this for h_3 we get

$$h_3 = \frac{55}{6} h_1 \,.$$

This all means that

$$h_1 + h_2 + h_3 = h_1 + \frac{5}{6} h_1 + \frac{55}{6} h_1 = 11 h_1 \,.$$

Using Equation 460 we have that

$$\text{Area}(\triangle ACB) = 11 \text{Area}(\triangle APB) = 11(36) = 396 \,.$$

Problem 5

Adding these two equations together to get

$$x^3 + y^3 = 16(x + y) \,.$$

Factoring the left-hand-side gives

$$(x + y)(x^2 - xy + y^2) = 16(x + y) \,.$$

As $x + y \neq 0$ we have

$$x^2 - xy + y^2 = 16 \,. \tag{461}$$

Subtracting these two equations together gives

$$x^3 - y^3 = 10x - 10y = 10(x - y) \,.$$

Factoring the left-hand-side gives

$$(x - y)(x^2 + xy + y^2) = 10(x - y) \,.$$

As $x - y \neq 0$ we have

$$x^2 + xy + y^2 = 10 \,. \tag{462}$$

If we add Equation 461 and 462 we get

$$x^2 + y^2 = 13 \,.$$

If we subtract Equation 461 from 462 we get

$$2xy = -6 \quad \text{or} \quad xy = -3 \,.$$

Now the expression we want to evaluate can be written in terms of $x^2 + y^2$ and xy as

$$\begin{aligned}(x^2 - y^2)^2 &= x^4 - 2x^2y^2 + y^4 = x^4 + 2x^2y^2 + y^4 - 4x^2y^2 \\ &= (x^2 + y^2)^2 - 4x^2y^2 \\ &= 13^2 - 4(9) = 133 \,.\end{aligned}$$

Problem 7

We are told that

$$y^2 - zx = -103 \quad (463)$$
$$z^2 - xy = 22\,. \quad (464)$$

If we subtract Equation 463 from 464 we get

$$z^2 - y^2 - xy + zx = 125\,,$$

or

$$z^2 - y^2 - x(y - z) = 125\,,$$

or

$$(z + y)(z - y) + x(z - y) = 125\,,$$

or

$$(z - y)(x + y + z) = 125\,.$$

Now 125 factors as 5^3 so we have that

$$x + y + z \in \{1, 5, 5^2, 5^3\}\,.$$

Now $x + y + z$ cannot equal one or five as we are told that x, y, and z are positive and distinct. Thus we only have

$$x + y + z \in \{25, 125\}\,.$$

In the first case we would have

$$x + y + z = 25\,,$$

and the other "factor" must be five so

$$z - y = 5 \quad \text{so} \quad z = 5 + y\,.$$

Thus

$$x + y + z = 25 \quad \text{becomes} \quad x + 2y = 20 \quad \text{or} \quad x = 20 - 2y\,.$$

As $2y$ is even and 20 is even x must be even. Lets put these two expressions for z and x in terms of y into Equation 463 to get

$$3y^2 - 10y + 3 = 0\,,$$

when we simplify. This has solutions $y \in \left\{\frac{1}{3}, 3\right\}$ of which only $y = 3$ is an integer. If $y = 3$ then we would have $x = 14$ and $z = 8$ so that the expression we want to evaluate is given by

$$x^2 - yz = 172\,.$$

In the second case we would have

$$x + y + z = 125\,,$$

and the other "factor" must be one so

$$z - y = 1 \quad \text{so} \quad z = 1 + y\,.$$

Thus

$$x + y + z = 25 \quad \text{becomes} \quad x + 2y + 1 = 125 \quad \text{or} \quad x = 124 - 2y\,.$$

Lets put these two expressions for z and x in terms of y into Equation 463 to get

$$3y^2 - 122y - 21 = 0\,,$$

when we simplify. This has non-integer roots so the only possible value for $x^2 - yz$ is 172.

Problem 8

Let the lengths $AD = a$ and $BE = b$. Then from the problem we are told that $AC = 3AD = 3a$ so that

$$CD = AC - AD = 3a - a = 2a\,,$$

and that $BC = 4BE = 4b$ so that

$$CE = BC - BE = 4b - b = 3b\,.$$

Next draw the segments AE and BD. Now as AB is a diameter of the circle we have $AE \perp BC$ and $BD \perp AC$. Using the Pythagorean theorem in the right triangles $\triangle ADB$ and $\triangle AEB$ we have

$$a^2 + BD^2 = 900 \tag{465}$$

$$b^2 + AE^2 = 900\,. \tag{466}$$

Now using the Pythagorean theorem in the right triangles $\triangle BDC$ and $\triangle CEA$ we get

$$(2a)^2 + BD^2 = (4b)^2 \tag{467}$$

$$(3b)^2 + AE^2 = (3a)^2\,. \tag{468}$$

Using Equation 465 and 466 to get expressions for BD^2 and AE^2 we put these into Equations 467 and 468 to get

$$3a^2 - 16b^2 = -900 \tag{469}$$

$$9a^2 - 8b^2 = 900\,. \tag{470}$$

If we use Equation 469 to solve for $3a^2$ and put that into 470 we get one equation for b that we can solve to find $b = 3\sqrt{10}$. Then using Equation 469 we get $a = 6\sqrt{5}$. Using Equation 466 we get

$$AE = \sqrt{900 - b^2} = 9\sqrt{10}\,.$$

To find the area of the triangle $\triangle ACB$ we can sum the area of two right triangles. We have

$$\begin{aligned}
\text{Area}(\triangle ACB) &= \text{Area}(\triangle AEB) + \text{Area}(\triangle AEC) \\
&= \frac{1}{2} AE \cdot b + \frac{1}{2} AE(3b) \\
&= 2b \cdot AE = 2(3\sqrt{10})(9\sqrt{10}) = 540\,.
\end{aligned}$$

Problem 9

For N of the given form we would have

$$\begin{aligned} N &= (abc) + (cba) \\ &= (100a + 10b + c) + (100c + 10b + a) \\ &= 100(a + c) + 20b + a + c \\ &= 101(a + c) + 20b\,. \end{aligned}$$

We are told that $a \neq 0$ and $c \neq 0$ which means that

$$\begin{aligned} 1 &\leq a \leq 9 \\ 0 &\leq b \leq 9 \\ 1 &\leq c \leq 9\,. \end{aligned}$$

To count how many Ns are of the above form we need to ask when $1 \leq a \leq 9$ and $1 \leq c \leq 9$ how many distinct values of $a+c$ are there. The smallest this sum can be is when $a = c = 1$ (so $a + c = 2$) and the largest this sum can be is when $a = c = 9$ (so $a + c = 18$). This means we have $18 - 2 + 1 = 17$ possible distinct values for $a + c$. Next as b can be any of the numbers $0 \leq b \leq 9$ we have ten choices for b. This means that the number of possible Ns are given by

$$17 \times 10 = 170\,.$$

Problem 10

It helps to draw this triangle with AB along an x-axis and the vertex C above the segment AB. The point P is in the interior of the triangle creating three smaller triangles $\triangle PAB$, $\triangle PBC$, and $\triangle PCA$.

Since we are given the length of three sides of this triangle to start this problem we will first use Heron's formula to compute its area. To do that we need to compute the semi-perimeter s as

$$s = \frac{1}{2}(13 + 14 + 15) = 21\,.$$

Then Heron's formula states

$$A^2 = s(s - a)(s - b)(s - c) = 3^2 \cdot 7^2 \cdot 2^4\,.$$

Taking the square root gives $A = 84$.

Since three angles are all equal we can define them as θ so that

$$\angle PAB = \angle PBC = \angle PCA = \theta\,.$$

Next lets introduce the notation $x = AP$, $y = BP$, and $z = CP$. Then using the law of cosigns in the triangles $\triangle PAB$, $\triangle PBC$, and $\triangle PCA$ to "compute" the sides y, z and x

respectively we have

$$\begin{aligned}
y^2 &= x^2 + 13^2 - 26x\cos(\theta) \\
z^2 &= y^2 + 14^2 - 28y\cos(\theta) \\
x^2 &= z^2 + 15^2 - 30z\cos(\theta)\,.
\end{aligned}$$

If we add these three equations together (and simplify a bit) we get

$$13^2 + 14^2 + 15^2 = 2(13x + 14y + 15z)\cos(\theta)\,. \tag{471}$$

Notice from the drawing we can compute the areas of the three smaller triangles as

$$\begin{aligned}
13x\sin(\theta) &= 2\text{Area}(\triangle PAB) \\
14y\sin(\theta) &= 2\text{Area}(\triangle PBC) \\
15z\sin(\theta) &= 2\text{Area}(\triangle PCA)\,.
\end{aligned}$$

So the right-hand-side of Equation 471 can be written

$$\begin{aligned}
\text{RHS} &= 2\left[\frac{2\text{Area}(\triangle PAB)}{\sin(\theta)} + \frac{2\text{Area}(\triangle PBC)}{\sin(\theta)} + \frac{2\text{Area}(\triangle PCA)}{\sin(\theta)}\right]\cos(\theta) \\
&= 4\frac{\cos(\theta)}{\sin(\theta)}\text{Area}(\triangle ABC) = 4\cdot 84\frac{\cos(\theta)}{\sin(\theta)}\,.
\end{aligned}$$

This means that

$$\tan(\theta) = \frac{4\cdot 84}{13^2 + 14^2 + 15^2} = \frac{168}{295}\,.$$

The 1989 AHSME Examination (AHSME 40)

Problem 1

$$(-1)^{25} + 1 = -1 + 1 = 0\,.$$

Problem 2

For this we have

$$\sqrt{\frac{1}{9} + \frac{1}{16}} = \sqrt{\frac{16+9}{9 \cdot 16}} = \frac{5}{3 \cdot 4} = \frac{5}{12}\,.$$

Problem 3

Let s be the dimension of the side of the square. The the perimeter of one of the vertical rectangles can be written in terms of s as

$$24 = 2s + 2\left(\frac{s}{3}\right)\,.$$

Solving for s we get $s = 9$ so that the area is given by $s^2 = 81$.

Problem 4

Let the perpendicular from A to the segment CD intersect CD at the point called A'. Then denoting the distance from D to A' as x and by "dropping" the length of the segment AB onto DF we have that

$$2x + 4 = 10\,,$$

so that $x = 3$. Let the height of the trapezoid be denoted by h. Then in the right triangle DAA' we have

$$5^2 = x^2 + h^2 = 3^2 + h^2\,,$$

so $h = 4$.

Now again using right triangles we have

$$DB^2 = (4+x)^2 + h^2 = 7^2 + 4^2 = 65\,,$$

this means that $DB = \sqrt{65}$.

Now $DE = 2DB = 2\sqrt{65}$. As AB is parallel to DC and intersects the midpoint of DE it also intersects the midpoint of EF. Thus

$$EF = 2h = 8\,.$$

In the right triangle DEF we have

$$EF^2 + DF^2 = DE^2 \,,$$

or

$$8^2 + (10 + CF)^2 = 4 \cdot 65 = 260 \,.$$

Expanding this becomes

$$CF^2 + 20CF - 96 = 0 \,.$$

Solving for CF we find that the only positive solution is $CF = 4$.

Problem 5

There are $10 + 1 = 11$ vertical lines with 20 toothpicks in each and there are $20 + 1 = 21$ horizontal lines with 10 toothpicks in each for a total of

$$11 \cdot 20 + 21 \cdot 10 = 430 \,,$$

toothpicks.

Problem 6

From the given line if we take $x = 0$ we get $y = \frac{6}{b}$ and if we take $y = 0$ we get $x = \frac{6}{a}$. This means that this graph is a right triangle with a "base" of length $\frac{6}{a}$ and a "height" of length $\frac{6}{b}$. This means that the area of the triangle is

$$\frac{1}{2}\left(\frac{6}{a}\right)\left(\frac{6}{b}\right) = \frac{18}{ab} \,.$$

If this is equal to six then solving for ab we have $ab = 3$.

Problem 7

Now $\angle B = 50°$ and $\angle BHA = 90°$ so $\angle BAH = 40°$. As $\angle A = 100°$ we have

$$\angle HAC = 100 - \angle BAH = 100 - 40 = 60 \,.$$

This means that triangle AHC is a $30 - 60 - 90$ right triangle. Then as $\angle C = 30°$ we have

$$AH = AC \sin(30°) = \frac{1}{2} AC = AM \,.$$

Thus the triangle HAM has $AM = AH$ so it is an isosceles triangle with a vertex angle of $60°$. This means that

$$\angle AHM = \angle AMH = \frac{180 - 60}{2} = 60 \,.$$

This means that $\triangle AHM$ is actually equilateral. Finally we have

$$\angle MHC = \angle AHC - \angle AHM = 90 - 60 = 30 \,.$$

Problem 8

Lets factor this polynomial as

$$x^2 + x - n = (x + a)(x + b)\,,$$

which means that

$$\begin{aligned} a + b &= 1 \\ ab &= -n\,. \end{aligned}$$

From the first expression we have that $b = 1 - a$. When we put this in the second we get

$$n = a(a - 1)\,.$$

As a and b must be integers from how n is calculated it is also an integer. Note that if

$$\begin{aligned}
a = 1 &\quad \text{then} \quad n = 0 \\
a = 2 &\quad \text{then} \quad n = 2 \\
a = 3 &\quad \text{then} \quad n = 6 \\
a = 4 &\quad \text{then} \quad n = 12 \\
a = 5 &\quad \text{then} \quad n = 20 \\
a = 6 &\quad \text{then} \quad n = 30 \\
a = 7 &\quad \text{then} \quad n = 42 \\
a = 8 &\quad \text{then} \quad n = 56 \\
a = 9 &\quad \text{then} \quad n = 72 \\
a = 10 &\quad \text{then} \quad n = 90 \\
a = 11 &\quad \text{then} \quad n = 110\,.
\end{aligned}$$

Larger values of a will have $n > 100$. Counting the number of values of a where n is in the desired range we see that there are $10 - 2 + 1 = 9$ of them.

Problem 9

The last name of this child will be the same as its parents and thus will start with a Z. Now we can select two different letters to use for the first letter in the first name and the first letter in middle name in $(26 - 1) \times (26 - 2) = 25 \times 24 = 600$ ways. This is because there are $26 - 1 = 25$ ways to select the first letter of the first name (excluding the letter Z) and then $25 - 1 = 24$ ways to select the first letter of the second name excluding Z and the letter picked for the first name. If we want these two letters to be in alphabetical order we will need to ignore one-half of these 600 giving a total of 300 ways to select the monogram.

Problem 10

Now from the formula given we have

$$
\begin{aligned}
u_{n+1} &= -\frac{1}{u_n+1} = -\frac{1}{-\frac{1}{u_{n-1}+1}+1} \\
&= \frac{-(u_{n-1}+1)}{-1+u_{n-1}+1} = \frac{-1-u_{n-1}}{u_{n-1}} \\
&= -\frac{1}{u_{n-1}} - 1 = -\frac{1}{\left(-\frac{1}{u_{n-2}+1}\right)} - 1 \\
&= u_{n-2}+1-1 = u_{n-2}\,.
\end{aligned}
$$

We can write the above $u_{n+1} = u_{n-2}$ as $u_n = u_{n-3}$ or $u_n = u_{n+3}$. Thus

$$u_1 = u_4 = u_7 = u_{10} = u_{13} = u_{16} = u_{19}\,.$$

Problem 11

From $a < 2b$ and $b < 3c$ we have

$$a < 2(3c) = 6c\,.$$

Then using $c < 4d$ in the above we get

$$a < 6(4d) = 24d\,.$$

Thus the largest a can be is bounded by how large d can be. Working backwards then if $d < 100$ the largest d can be is $d = 99$. Then from $c < 4d$ we get that $c < 396$. Thus the largest c can be is $c = 395$. Then from $b < 3c$ we have that $b < 1185$ so the largest b can be is $b = 1184$. Finally from $a < 2b$ we have $a < 2368$ thus the largest a can be is $2368-1 = 2367$.

Problem 12

The velocity of westbound cars observed by the eastbound driver is 120 miles per hour. Now five minutes is

$$\frac{1}{12}\,,$$

of an hour. Thus the eastbound car will "pass" $120\left(\frac{1}{12}\right) = 10$ miles of westbound cars. If he passes 20 cars spaced W apart we must have

$$(20-1)W = 10\,\text{miles}\,.$$

This means that

$$W = \frac{10}{20-1} = \frac{1}{2-\frac{1}{10}}\,.$$

In 100 miles there would be $\frac{100}{W}$ cars or

$$100\left(2 - \frac{1}{10}\right) = 200 - 10 = 190\,.$$

This is closest to 200.

Problem 13

If $\alpha = 0$ the area of the shaded part of the given figure is infinite which means that choice A cannot be the correct answer (as it gives an area of zero).

Note that the given figure is a parallelogram. Recall that the area S of a parallelogram is given by

$$S = ab\sin(\alpha')\,,$$

where a and b are the lengths of the two sides of the parallelogram and α' is the angle between them. We will let a be the length of the horizontal side and b be the length of the "vertical" side. Then for the horizontal side length a as the distance between the two vertical parallel lines is one we have

$$1 = a\sin(\alpha) \quad \text{so} \quad a = \frac{1}{\sin(\alpha)}\,.$$

In the same way for the "vertical" side length b as the distance between the two horizontal parallel lines is one we have

$$b\sin(\alpha) = 1 \quad \text{so} \quad b = \frac{1}{\sin(\alpha)}\,.$$

This means that

$$S = \frac{\sin(\alpha')}{\sin(\alpha)^2} = \frac{\sin(180^\circ - \alpha)}{\sin(\alpha)^2} = \frac{\sin(\alpha)}{\sin(\alpha)^2} = \frac{1}{\sin(\alpha)}\,.$$

Problem 14

We want to simplify

$$E = \cot(10) + \tan(5) = \frac{\cos(10)}{\sin(10)} + \frac{\sin(5)}{\cos(5)}\,.$$

Using the double angle formulas for cos(10) and sin(10) we get

$$
\begin{aligned}
E &= \frac{\cos^2(5) - \sin^2(5)}{2\sin(5)\cos(5)} + \frac{\sin(5)}{\cos(5)} \\
&= \frac{\cos(5)}{2\sin(5)} - \frac{\sin(5)}{2\cos(5)} + \frac{\sin(5)}{\cos(5)} \\
&= \frac{\cos(5)}{2\sin(5)} + \frac{\sin(5)}{2\cos(5)} = \frac{\cos^2(5) + \sin^2(5)}{2\sin(5)\cos(5)} \\
&= \frac{1}{\sin(10)} = \csc(10)\,.
\end{aligned}
$$

Problem 15

Method 1: Using the law of cosigns in $\triangle ABC$ gives

$$\cos(\angle A) = \frac{7^2 - 5^2 - 9^2}{-2(9)(5)} = \frac{19}{30}\,.$$

We next drop a perpendicular from the point B to the segment AC and denote its intersection with AC as the point AH_b. Then

$$AH_b = AB\cos(\angle A) = 5\left(\frac{19}{30}\right) = \frac{19}{6}\,.$$

Next note that the segment BH_b is a height in the isosceles triangle ABD and thus

$$AD = 2AH_b = \frac{19}{3}\,.$$

As $AC = 9$ we have $CD = 9 - \frac{19}{3} = \frac{8}{3}$ from which we see that

$$AD : DC = \frac{19}{3} : \frac{8}{3} = 19 : 8\,.$$

Method 2: In this method we also drop a perpendicular from the point B to the segment AC and denote its intersection with AC as the point AH_b. Then the Pythagorean theorem in $\triangle ABH_b$ we have

$$BH_b^2 = AB^2 - AH_b^2 = 25 - AH_b^2\,. \tag{472}$$

The Pythagorean theorem in $\triangle BH_bC$ gives

$$BH_b^2 = BC^2 - H_bC^2 = 49 - (AC - AH_b)^2 = 49 - (9 - AH_b)^2\,. \tag{473}$$

Setting these two expressions equal to each other gives

$$25 - AH_b^2 = 49 - (9 = AH_b)^2\,.$$

We can solve this for AH_b and find

$$AH_b = \frac{19}{6}\,,$$

as before. Using this in Equation 472 gives

$$BH_b^2 = 25 - \left(\frac{19}{6}\right)^2 = \frac{539}{36}.$$

Thus

$$BH_b = \frac{\sqrt{7^2 \cdot 11}}{6} = \frac{7\sqrt{11}}{6}.$$

The Pythagorean theorem in $\triangle BH_bD$ gives

$$H_bD^2 = BD^2 - BH_b^2 = 25 - \frac{539}{36} = \frac{361}{36},$$

so

$$H_bD = \frac{19}{6}.$$

Thus

$$AD = AH_b + H_bD = \frac{19}{6} + \frac{19}{6} = \frac{19}{3},$$

and

$$DC = AC - AD = 9 - \frac{19}{3} = \frac{8}{3}.$$

So the proportion we are interested in is given by

$$AD : DC = \frac{19}{3} : \frac{8}{3} = 19 : 8.$$

Problem 16

The slope of the line between the two points given is

$$m = \frac{281 - 17}{48 - 3} = \frac{264}{45} = \frac{88}{15}.$$

This means that the line connecting these two points is given by

$$y - 17 = \frac{88}{15}(x - 3).$$

We can manipulate this to write it in the form

$$-88x + 15y = -9.$$

Thus our problem is to determine the number of integer solutions to the above equation where

$$3 \leq x \leq 48 \quad \text{and} \quad 17 \leq y \leq 281.$$

To simplify this a bit let

$$\begin{aligned} x &= 3 + p \\ y &= 17 + q, \end{aligned}$$

where the domain of p and q are

$$0 \le p \le 45 \quad \text{and} \quad 0 \le q \le 264 \, .$$

Putting these expressions in the above linear equation gives

$$-88(3+p) + 15(17+9) = -9 \, ,$$

or

$$-88p + 15q = 0 \, . \tag{474}$$

As p increases the left-hand-side decreases by 88 and we will need q to increase by a multiple to offset this so that the total sum is still zero. This leads us to consider the least-common-multiple of 88 and 15. We find

$$\text{lcm}(88, 15) = 1320 \, .$$

Note that

$$\frac{1320}{88} = 15 \quad \text{and} \quad \frac{1320}{15} = 88 \, .$$

This means that as I increase p by 15 I have to increase q by 88. The number of times I can do this and stay in the bounds above for p and q are given by

$$15 n_p \le 45 \Rightarrow n_p \le 3 \, ,$$

for p and

$$88 n_q \le 264 \Rightarrow n_q \le 3 \, ,$$

for q. This means that the integer solutions to Equation 474 are given by

$$(p, q) \in \{(0,0) \, , (15, 88) \, , (30, 176) \, , (45, 264)\} \, ,$$

giving four integer solutions.

Problem 17

Let the side of the equilateral triangle be a and the side length of the square be s. Then the perimeters of the triangle and the square are given by

$$P_{\text{triangle}} = 3a$$
$$P_{\text{square}} = 4s \, .$$

We are told that

$$P_{\text{triangle}} - P_{\text{square}} = 1989 \, ,$$

and that $a = s + d$. This means that

$$3(s+d) - 4s = 1989 \, ,$$

or

$$3d - s = 1989 \, .$$

We must have $s > 0$. If $s = 0$ (corresponding to the smallest that d could be i.e. d would need to be larger than that number) we would have

$$3d = 1989 = 3^2 \cdot 13 \cdot 17 \, .$$

This means that $d = 3 \cdot 13 \cdot 17 = 663$. Thus we must have $d > 663$ and d cannot be in the range $1 \le d \le 663$.

Problem 18

Call this expression E. Then we can write E as

$$\begin{aligned} E &= x+\sqrt{x^2+1} - \frac{1}{x+\sqrt{x^2+x}}\left(\frac{x-\sqrt{x^2+1}}{x-\sqrt{x^2+1}}\right) \\ &= x+\sqrt{x^2+1} - \left(\frac{x-\sqrt{x^2+1}}{x^2-(x^2+1)}\right) \\ &= x+\sqrt{x^2+1} - \left(-x+\sqrt{x^2+1}\right) = 2x\,. \end{aligned}$$

So for this expression to be rational x must be rational.

Problem 19

Let the vertices of the triangle be denoted by A, B, and C with AC along the x-axis and B above the segment AC. Let the center be denoted O, the radius of the circle be denoted R and the arcs AB, BC, and CA be three, four, and five respectively. Draw segments from A, B, and C to the circles center O forming three isosceles triangles $\triangle AOB$, $\triangle BOC$, and $\triangle AOC$.

From the given arc lengths we have that

$$2\pi R = 3+4+5 = 12\,.$$

Thus we have $R = \frac{6}{\pi}$. The central angles are given by

$$\begin{aligned} \angle AOB &= \frac{3}{R} = \frac{\pi}{2} \\ \angle BOC &= \frac{4}{R} = \frac{2\pi}{3} \\ \angle AOC &= \frac{5}{R} = \frac{5\pi}{6}\,. \end{aligned}$$

Now the area of the triangle $\triangle ABC$ is the sum of the three isosceles triangles. Recall that the area of an isosceles triangle with a vertex angle θ and legs of length l is given by

$$A_{\text{isosceles}} = \frac{l^2}{2}\sin(\theta)\,.$$

Thus what we want is given by

$$\begin{aligned} \text{Area}(\triangle ABC) &= \frac{1}{2}R^2\sin\left(\frac{\pi}{2}\right) + \frac{1}{2}R^2\sin\left(\frac{2\pi}{3}\right) + \frac{1}{2}R^2\sin\left(\frac{5\pi}{6}\right) \\ &= \frac{1}{2}\left[1+\left(\frac{\sqrt{3}}{2}\right)+\left(\frac{1}{2}\right)\right]R^2 \\ &= \frac{9}{\pi^2}(3+\sqrt{3})\,, \end{aligned}$$

when we simplify.

Problem 20

Let A be the event where we are told that

$$\lfloor \sqrt{x} \rfloor = 12 .$$

This means that

$$12 \leq \sqrt{x} < 13 ,$$

or

$$144 \leq x < 169 .$$

Now let B be the event where

$$\lfloor \sqrt{100x} \rfloor = 120 .$$

This means that

$$120 \leq \sqrt{100x} < 120 ,$$

or

$$14400 \leq 100x < 14641 ,$$

or

$$144 \leq x < 146.41 .$$

To find the probability we seek we have

$$\begin{aligned} P(B|A) &= \frac{P(B \cap A)}{P(A)} = \frac{P(144 \leq x < 146.41)}{P(144 \leq x < 169)} \\ &= \frac{\frac{146.41-144}{100}}{\frac{169-144}{100}} = \frac{241}{2500} , \end{aligned}$$

when we simplify.

Problem 21

To start we let the square flag have a sides of length one. Next we let the inner blue square (denoted B) have sides of length b. Next, in each of the four red arms create rectangles (denoted R) by creating four isosceles triangles in each corner. Let the width of each R rectangles be denoted r so that each rectangle is of dimension $r \times b$. Let these isosceles triangles have a side of length l.

Then we can decompose the area of the entire cross as the sum of the square B the four rectangles R, and the four corner isosceles triangles. This is the statement

$$b^2 + 4br + 4\left(\frac{1}{2}l^2\right) = 0.36 . \tag{475}$$

Next in each isosceles right-triangle we have

$$l^2 + l^2 = b^2 \quad \text{so} \quad b = l\sqrt{2}\,. \tag{476}$$

As a side of the square has a length of one and each isosceles triangle has a edge length of l the "middle" portion of a side is of length

$$1 - 2l\,.$$

This is also the length of the hypotenuse of a right triangle with two legs of length r. This means that

$$2r^2 = (1 - 2l)^2 \quad \text{so} \quad r = \frac{1 - 2l}{\sqrt{2}}\,. \tag{477}$$

If we put Equation 476 (for b) and 477 (for r) into Equation 475 we

$$2l^2 + 4(l\sqrt{2})\left(\frac{1 - 2l}{\sqrt{2}}\right) + 2l^2 = 0.36\,.$$

When we simplify this we get

$$l^2 - l + 0.09 = 0\,.$$

This has two solutions $l = 0.1$ or $l = 0.9$. Now $2l < 1$ by the geometry of this problem and so we have that $l = 0.1$. This means that $b^2 = 2l^2 = 2(0.01) = 0.02$ or 2%.

Problem 22

We want blocks that differ from the given one in only two ways. There are $\binom{4}{2} = 6$ ways that can happen. For example by changing the material and size or changing the material and color etc. Once we pick the two characteristics that will be different we then look at the number of possible choices for each that would result in a different selection. For example if we change the material (of which there are two choices and we have used one) and size (of which there are three choices and we have used one) then we have

$$(2 - 1) \times (3 - 1) = 2\,,$$

additional blocks that differ in the two attributes suggested.

Following this procedure if we change

- material and color we get $1 \times 3 = 3$ additional blocks.
- material and shape we get $1 \times 3 = 3$ additional blocks.
- size and color we get $2 \times 3 = 6$ additional blocks.
- size and shape we get $2 \times 3 = 6$ additional blocks.

- color and shape we get $3 \times 3 = 9$ additional blocks.

Together this gives

$$2 + 3 + 3 + 6 + 6 + 9 = 29\,,$$

additional blocks.

Problem 23

If we look at the path drawn we see that it is a "single step" followed by a walk half-way around a square of size 1×1, followed by another "single step" followed by by a walk half-way around a square of size 2×2, followed by another "single step" followed by by a walk half-way around a square of size 3×3 etc. The length walked after n of these "single step followed by 1/2 of a square" units is

$$\begin{aligned}
L_n &= 1 + \frac{1}{2}(4) \\
&+ 1 + \frac{1}{2}(4 \cdot 2) \\
&+ 1 + \frac{1}{2}(4 \cdot 3) \\
&\vdots \\
&+ 1 + \frac{1}{2}(4n) \\
&= n + \sum_{k=1}^{n} \frac{1}{2}(4k) = n + 2\sum_{k=1}^{n} k \\
&= n + n(n+1) = n^2 + 2n\,.
\end{aligned}$$

Lets check this function L_n for some simple inputs n. We compute

$$\begin{aligned}
L_1 &= 1 + 2 = 3 \\
L_2 &= 4 + 4 = 8 \\
L_3 &= 9 + 6 = 15\,.
\end{aligned}$$

These agree with the diagram given in the text. Overriding notation we note that if n is odd, the final location of L_n steps from the origin is on the y-axis at the location $(0, n)$. In the same way if n is even the final location of L_n steps from the origin is on the x-axis at $(n, 0)$.

Next we ask for what value of n is $L_n \approx 1989$. If we solve

$$n^2 + 2n - 1989 = 0\,,$$

we have $n \approx 43.6$. Note that

$$\begin{aligned}
L_{43} &= 1935 < 1989 \\
L_{44} &= 2024 > 1989\,.
\end{aligned}$$

From the above argument we see that after taking L_{43} steps our particle is located at $(0, 43)$ and we need to walk

$$1989 - 1935 = 54\,,$$

more steps. The first step will be a step "up" to $(0, 44)$. The next 53 steps will start by going around a square with sides 44×44. Thus after 44 steps along this path our particle are will be at $(44, 44)$ and have $53 - 44 = 9$ more to go. These will be downwards ending at the point

$$(44, 44 - 9) = (44, 35)\,.$$

Problem 24

As a general problem solving strategy note that as the answers to this problem are relatively small numbers we should be able to explicitly *enumerate* all of the solutions if we can't come up with a more general way of computing (f, m). Thus we consider the following cases

- Imagine we have zero females and five males. Then $(f, m) = (0, 5)$.
- Next assume we have only one female and four males. If we draw that table we see that $(f, m) = (2, 5)$.
- Next assume we have two females and three males. In this case we can have the two females sitting next to each other (or not). If they sit next to each other we see that $(f, m) = (4, 5)$. If they do not we have $(f, m) = (3, 4)$.
- If we have three females we then have two males. By symmetry with the two female and three male case above we have $(f, m) = (5, 4)$ or $(f, m) = (4, 3)$.
- If we have four females and one male then again by symmetry with the one female four male case we have $(f, m) = (5, 2)$.
- Finally, if we have five females we have no males and by symmetry with the above we have $(f, m) = (5, 0)$.

Counting up the choices above we get

$$(f, m) \in \{(0, 5)\,, (2, 5)\,, (4, 5)\,, (3, 4)\,, (5, 4)\,, (4, 3)\,, (5, 2)\,, (5, 0)\}\,.$$

Thus there are eight choices.

Problem 25

The sum of all the ranks of all ten runners is

$$1 + 2 + 3 + \cdots + 9 + 10 = \frac{1}{2}(10)(11) = 55\,.$$

Now every *winning* score must be an integer less than half of this

$$\frac{1}{2}(55) = 27.5\,,$$

and greater than the lowest possible score which is

$$1 + 2 + 3 + 4 + 5 = 15\,.$$

This gives $27 - 15 + 1 = 13$ winning scores. Note that for a winning score s in the range $15 \le s \le 27$ the loosing score must be $l = 55 - s$ and is in the range $28 \le 55 - s \le 40$. Note that there are thirteen scores in that range (as there must be).

Problem 26

One should first attempt to draw this figure. Notice that this regular octahedron is composed of two pyramids "on top of each other" each pyramid has a square base. As we are told that this is a regular octahedron we know that all of edge lengths are equal.

Let the side length of the cube be denoted l. Then imagining the square base projected onto the x-y plane we note that it has a side length that is the hypotenuse of a right triangle with legs $\frac{l}{2}$. This means that it has a length of

$$\left(\frac{l}{2}\right)\sqrt{2} = \frac{l}{\sqrt{2}}\,.$$

The heights of each pyramid are that of one-half of l or $\frac{l}{2}$. This gives a pyramidal volume of

$$\frac{1}{3}\left(\frac{l}{\sqrt{2}}\right)^2\left(\frac{l}{2}\right) = \frac{l^3}{12}\,.$$

The volume of the regular octahedron twice this or

$$\frac{2l^3}{12} = \frac{l^3}{6}\,.$$

The ratio of the two volumes requested is then

$$\frac{\frac{l^3}{6}}{l^3} = \frac{1}{6}\,.$$

Problem 27

Write this expression as

$$x + y = \frac{n - z}{2}\,.$$

Note that for x and y to have integer solutions if $n - z$ must be even. This means that if n then z must be even also and if n is odd then z must be odd also.

To solve this problem we can start with $n = 14$ and see what conclusions we might be able to reach. From the above we have

$$x + y = \frac{14 - z}{2},$$

and we must have z even. The smallest choice for z is then $z = 2$ and we have

$$x + y = 6.$$

The only values for (x, y) that solve the above are

$$(x, y) \in \{(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)\},$$

for five solutions. We could also have $z = 4$ and have

$$x + y = 5.$$

The only values for (x, y) that solve the above are

$$(x, y) \in \{(1, 4), (2, 3), (3, 2), (4, 1)\},$$

for four solutions. We could have $z = 6$ and have

$$x + y = 4.$$

The only values for (x, y) that solve the above are

$$(x, y) \in \{(1, 3), (2, 2), (3, 1)\},$$

for three solutions. We could have $z = 8$ and have

$$x + y = 3.$$

The only values for (x, y) that solve the above are

$$(x, y) \in \{(1, 2), (1, 2)\},$$

for two solutions. Finally we could have $z = 10$ and have

$$x + y = 2.$$

The only values for (x, y) are $(1, 1)$ for one more solution. This gives a total of

$$5 + 4 + 3 + 2 + 1 = 15,$$

total solutions. As we are looking for 28 solutions this value of n is "too low".

Generalizing a bit it looks like when n is even the choices for z are

$$[n - 4, n - 6, \ldots, 4, 2].$$

For each of these the number of solutions (x, y) are

$$\left[1, 2, \ldots, \frac{n}{2} - 3, \frac{n}{2} - 2\right].$$

The total number of solutions in this case is then

$$T_n \equiv \sum_{k=1}^{\frac{n}{2}-2} k = \frac{1}{2}\left(\frac{n}{2}-2\right)\left(\frac{n}{2}-1\right).$$

Using this formula we find $T_{16} = 21$ and $T_{18} = 28$. Thus $n = 18$ works.

Using the same logic as above if n is odd the choices for z are

$$[n-4, n-6, \ldots, 3, 1].$$

For each of these the number of solutions (x, y) are

$$\left[1, 2, \ldots, \frac{n-1}{2} - 2, \frac{n-1}{2} - 1\right].$$

The total number of solutions in this case is then

$$T_n \equiv \sum_{k=1}^{\frac{n-1}{2}-1} k = \frac{1}{2}\left(\frac{n-1}{2}-1\right)\left(\frac{n-1}{2}\right).$$

Using this formula we find $T_{17} = 28$ and $T_{19} = 36$. Thus $n = 17$ also works and we have shown that $n \in \{17, 18\}$.

Problem 28

Let $v = \tan(x)$ then the equation for v is a quadratic with solutions

$$v = \frac{9 \pm \sqrt{77}}{2}.$$

Then $x = \tan^{-1}(v)$. The standard range of the $x = \tan^{-1}(v)$ function is such that $-\frac{\pi}{2} < x < \frac{\pi}{2}$, where if $v > 0$ then $x > 0$ and if $v < 0$ then $x < 0$. Because $77 < 81 = 9^2$ both of the v numbers above are positive. For each positive v there will be a root x_I in the first quadrant (from the standard range of $\tan^{-1}(v)$) and another in the third quadrant given by

$$x_{III} = x_I + \pi.$$

Let the two value of v above be given by v_- and v_+ and their corresponding standard range values by x_- and x_+. Then we want to evaluate

$$x_- + x_+ + (x_- + \pi) + (x_+ + \pi) = 2(x_- + x_+) + 2\pi. \tag{478}$$

Now

$$x_- + x_+ = \tan^{-1}(v_-) + \tan^{-1}(v_+).$$

We can evaluate this using trigonometric identities. Using

$$\tan(\theta_1 + \theta_2) = \frac{\tan(\theta_1) + \tan(\theta_2)}{1 - \tan(\theta_1)\tan(\theta_2)},$$

we have (changing $\theta_i \to x_i$) that

$$x_1 + x_2 = \tan^{-1}\left(\frac{\tan(x_1) + \tan(x_2)}{1 - \tan(x_1)\tan(x_2)}\right).$$

Using the fact that $\tan(x_1)$ and $\tan(x_2)$ are two roots of a quadratic equation we have

$$\tan(x_1) + \tan(x_2) = -(-9) = 9$$
$$\tan(x_1)\tan(x_2) = 1.$$

This means that

$$x_1 + x_2 \to \lim_{t\to 1}\left(\tan^{-1}\left(\frac{9}{1-t}\right)\right).$$

The limit on the right-hand-side would depend on whether $t \to 1^-$ or $t \to 1^+$ in the former the limit would be $-\frac{\pi}{2}$ while in the later it would be $\frac{\pi}{2}$. As $x_i > 0$ the limit should be $\frac{\pi}{2}$ and we have

$$x_1 + x_2 = \frac{\pi}{2}.$$

Using this in Equation 478 we get the sum we want given by

$$2\left(\frac{\pi}{2}\right) + 2\pi = 3\pi.$$

Problem 29

Note that using the binomial theorem we have

$$(1+i)^{99} = \sum_{k=0}^{99}\binom{99}{k}1^k i^{99-k} = -i\sum_{k=0}^{99}\binom{99}{k}i^{-k}.$$

In the above we have used

$$i^{99} = i^{100-1} = i^{100}i^{-1} = \frac{1}{i} = -i.$$

The above also shows that $i^{-k} = (-1)^k i^k$ so we can write

$$\begin{aligned}
(1+i)^{99} &= -i\left[\sum_{k \text{ even}}^{98}\binom{99}{k}(-1)^k i^k + \sum_{k \text{ odd}}^{99}\binom{99}{k}(-1)^k i^k\right] \\
&= -i\left[\sum_{k=0}^{49}\binom{99}{2k}(-1)^{2k} i^{2k} + \sum_{k=1}^{49}\binom{99}{2k+1}(-1)^{2k+1} i^{2k+1}\right] \\
&= -i\left[\sum_{k=0}^{49}\binom{99}{2k}(-1)^k - i\sum_{k=1}^{49}\binom{99}{2k+1}(-1)^k\right] \\
&= -\sum_{k=1}^{49}\binom{99}{2k+1}(-1)^k - i\sum_{k=0}^{49}\binom{99}{2k}(-1)^k.
\end{aligned}$$

Notice that the sum we want to evaluate is proportional to the imaginary term above. Thus if we can evaluate the imaginary part of

$$(1+i)^{99},$$

we have our desired sum. Note that

$$1+i=\sqrt{2}e^{\frac{\pi}{4}i}.$$

Thus

$$\begin{aligned}(1+i)^{99} &= 2^{99/2}e^{\frac{99\pi}{4}i} = 2^{99/2}e^{\frac{100\pi}{4}i}e^{-\frac{\pi}{4}i} = 2^{99/2}e^{25\pi i}e^{-\frac{\pi}{4}i} \\ &= 2^{99/2}e^{\pi i}e^{-\frac{\pi}{4}i} = -2^{99/2}\left(\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right) \\ &= -2^{49}+2^{49}i.\end{aligned}$$

Equating the two terms and extracting the imaginary part we get

$$-\sum_{k=0}^{49}\binom{99}{2k}(-1)^k = 2^{49}.$$

The 1989 AIME Examination

Problem 1

We want to evaluate $E \equiv \sqrt{n(n+1)(n+2)(n+3)+1}$ for $n=28$. If we expand everything under the radicand we get

$$\sqrt{n^4+6n^3+11n^2+6n+1}.$$

We now ask (hope) that this factors into a perfect square or that we can write the above as

$$\sqrt{(n^2+An+1)(n^2+An+1)},$$

for some value of A. If we expand the argument of the radicand above we get

$$\sqrt{n^4+2An^3+(A^2+2)n^2+2An+1}.$$

This will match what we are given above if we take $A=3$. Thus we have shown that we can write E as

$$\sqrt{(n^2+3n+1)^2}=n^2+3n+1.$$

Taking $n=28$ this gives $E=869$.

The 1990 AHSME Examination (AHSME 41)

Problem 1

This is

$$\frac{x^2}{8} = 8\,,$$

or

$$x^2 = 64 \quad \text{so} \quad x = \pm 8\,.$$

Problem 2

This is

$$\left(\frac{1}{4}\right)^{-\frac{1}{4}} = 4^{1/4} = (4^{1/2})^{1/2} = 2^{1/2} = \sqrt{2}\,.$$

Problem 3

If we let a_i be the measure of angle i with $i = 1$ the smallest and $i = 4$ the largest angle. Then from what we are told we have

$$\begin{aligned} a_1 &= 75 \\ a_2 &= 75 + d \\ a_3 &= 75 + 2d \\ a_4 &= 75 + 3d\,. \end{aligned}$$

This means that

$$360 = \sum_{i=1}^{4} a_i = 4(75) + (0 + d + 2d + 3d) = 300 + 6d\,.$$

Solving we find $d = 10$ so that $a_4 = 105$.

Problem 4

As $ABCD$ is a parallelogram we have $DC = AB = 16$ which means that $EC = DC + ED = 16 + 4 = 20$. Next note that

$$\triangle FED \sim \triangle BEC\,,$$

so we can write

$$\frac{ED}{EC} = \frac{FD}{BC} \quad \text{so} \quad \frac{4}{20} = \frac{FD}{10}\,,$$

so $FD = 2$.

Problem 5

Write each of these in the same "way" as

$$
\begin{aligned}
(5^{1/3} \cdot 6^{1/3})^{1/2} &= 5^{1/6} \cdot 6^{1/6} = (5 \cdot 6)^{1/6} = 30^{1/6} \\
(6^{1/2} \cdot 5^{1/6})^{1/2} &= 5^{1/6} \cdot 6^{1/2} = 5^{1/6}(6^3)^{1/6} = (1080)^{1/6} \\
(5^{1/2} \cdot 6^{1/6})^{1/2} &= (5^3 \cdot 6)^{1/6} = (750)^{1/6} \\
(5^{1/3} \cdot 6^{1/6})^{1/2} &= (5^2 \cdot 6)^{1/6} = (150)^{1/6} \\
(6^{1/3} \cdot 5^{1/6})^{1/2} &= (6^2 \cdot 5)^{1/6} = (180)^{1/6} \,.
\end{aligned}
$$

The second of these is the largest.

Problem 6

A line will be d from a given point if it is tangent to the circle of radius d at that point. Thus we next draw the point A and then the point B with circles of radius two and three around each. Next we imagine tangents at each of the points on the circle drawn around A of radius two. When we do that we see that *three* lines will also be tangent to the circle drawn around B of radius three.

Problem 7

Let s_1, s_2, and s_3 be the integer length of the sides of the given triangle. With out loss of generality lets take $s_1 \leq s_2 \leq s_3$. Now we must have

$$\frac{1}{3}(s_1 + s_2 + s_3) \leq s_3 \,, \tag{479}$$

for if *not* we would have $3s_3 < s_1 + s_2 + s_3$ and three multiples of s_3 (the largest side) is smaller than the perimeter of the triangle which is a contradiction. We must also have that

$$\frac{1}{2}(s_1 + s_2 + s_3) > s_3 \,, \tag{480}$$

for if *not* we would have

$$\frac{1}{2}(s_1 + s_2 + s_3) \leq s_3 \,.$$

This inequality is equivalent to

$$s_1 + s_2 \leq s_3 \,,$$

which is a violation of the triangle inequality and would be another contradiction. Thus we need an integer s_3 such that

$$\frac{1}{3}(8) \leq s_3 < \frac{1}{2}(8) \quad \text{or} \quad 2\,\frac{2}{3} \leq s_3 < 4 \,.$$

To be an integer this means that $s_3 = 3$. Thus we have

$$3 \geq s_2 \geq s_1 \, .$$

We also need $s_1 + s_2 = 5$. This can happen only if $s_2 = 3$ and $s_1 = 2$. This is an isosceles triangle with equal legs of length three and a base of length two. This has a height given by

$$h^2 = 3^2 - \left(\frac{2}{2}\right)^2 = 8 \, .$$

Thus the area of this triangle is

$$\frac{1}{2}(2)\sqrt{8} = 2\sqrt{2} \, .$$

Problem 8

If we assume that $x \geq 3$ then this equation is

$$x - 2 + x - 3 = 1 \quad \text{or} \quad 2x - 5 = 1 \, .$$

This has a solution $x = 3$ which is in the domain $x \geq 3$ and so there is at least one solution to this equation. If $2 < x < 3$ then this equation is

$$x - 2 - (x - 3) = 1 \, ,$$

which is an identity for all x. Thus there are an infinite number of solutions to this equation.

Problem 9

To have "few" black edges we want most of the cube to be composed of red edges. If we start by considering a cube with only one black edge then we have two faces that have that edge in common. To have fewer black edges the next black edge should be placed adjacent to a face *different* than the previous two. This will then "cover" two additional faces. As a cube has six total faces we need to place one more black edge to cover the $6-2-2=2$ remaining uncovered faces. This gives three black edges to perform the requested construction.

An example of this covering in the unit cube $[0,1]^3$ would be to color black the segments

- $(0,0,0) \leftrightarrow (1,0,0)$
- $(0,0,1) \leftrightarrow (0,1,1)$
- $(1,1,0) \leftrightarrow (1,1,1)$

Problem 10

From a "corner" one should be able to see all cubes on three faces which would be

$$3 \times 11^2 = 363 \,.$$

Some of the cubes on the different faces are "the same" and the above number has "double counted" the cubes along common edges and "triple counted" the corner cube. Along any two faces that are adjacent there will be 11 cubes that are common between the two (along the edge). Excluding the "corner" cube there are 10 cubes common between each pair of faces. There are three pairs of faces that can be seen from one corner. Removing the double counted common edge cubes and then tripled counted corner cube gives

$$363 - 3 \times 10 - 2 = 331 \,,$$

cubes that can be seen from one location.

If we think of the three faces as the "sets" A, B, and C with elements equal to the cubes on that face that can be seen from a location then we can use the inclusion-exclusion identity to determine $|A \cup B \cup C|$ we have

$$|A\cup B\cup C| = |A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C|+|A\cap B\cap C| = 3\times 11^2-11-11-11+1 = 331 \,.$$

Here $|A| = |B| = |C| = 11^2 = 121$ as we can see the full face of cubes. The intersections i.e. $|A \cap B| = 11$ are the number of cubes we can see in common between faces A and B etc.

Problem 11

Let our integer be denoted N and a divisor of N by d. Then the number $\frac{N}{d}$ is another divisor of N. Thus if

$$d \neq \frac{N}{d} \,,$$

each divisor comes with another distinct one and there will be an even number of divisors for N. This is true unless

$$d = \frac{N}{d} \quad \text{or} \quad N = d^2 \,,$$

i.e. N is a perfect square. The perfect squares less than 50 are

$$1^2 \,, 2^2 \,, 3^2 \,, 4^2 \,, 5^2 \,, 6^2 \,, 7^2 \,.$$

There are seven of these numbers.

Problem 12

From the definition of $f(x)$ we have

$$f(\sqrt{2}) = 2a - \sqrt{2} \,.$$

Thus
$$f(f(\sqrt{2})) = a(2a - \sqrt{2})^2 - \sqrt{2} = a(4a^2 - 4\sqrt{2}a + 2) - \sqrt{2}\,.$$
Setting this equal to $-\sqrt{2}$ means that
$$a(4a^2 - 4\sqrt{2} + 2) = 0\,.$$
This will be true if $a = 0$ or $2a^2 - 2\sqrt{2}a + 1 = 0$. This last quadratic equation has the single solution
$$a = \frac{\sqrt{2}}{2}\,.$$

Problem 13

If we follow this procedure for a bit we see that each new pair of the variables X and S is given by
$$\begin{aligned}(X, S) &= (3, 0)\\ (X, S) &= (5, 5)\\ (X, S) &= (7, 12)\\ (X, S) &= (9, 21)\,,\end{aligned}$$
etc. The changes to X are simple. If we let $n = 0$ correspond to the first values for (X, S) we have $X_0 = 3$ and
$$X_n = X_{n+1} + 2\,.$$
The above means that
$$X_n = 3 + 2n\,,$$
for $n \geq 0$.

The update of S depends on the value of X and we have $S_0 = 0$ with
$$S_n = S_{n-1} + X_n\,,$$
for $n \geq 1$. This means that
$$S_n = S_{n-1} + 3 + 2n\,.$$
In terms of the Δ notation this is
$$\Delta S_n = 3 + 2n\,,$$
or summing both sides this is
$$\sum_{n=1}^{N} \Delta S_n = \sum_{n=1}^{N} (3 + 2n)\,,$$
or
$$S_N - S_0 = 3N + 2\left(\frac{N(N+1)}{2}\right) = 3N + N^2 + N = N^2 + 4N + 4 - 4 = (N+2)^2 - 4\,.$$

As $S_0 = 0$ the above is an expression for S_N. The first value of N where $S_N \geq 10000$ or

$$(N+2)^2 \geq 10004\,,$$

or

$$N \geq 98.02\,.$$

If $N = 98$ I find $S_{98} = 9996$ and $S_{99} = 10197$. We exit the loop when $N = 99$ and find $X_{99} = 201$.

Problem 14

Recalling some angle to arc length identities we have that

$$x = \frac{1}{2}\widehat{CB} \tag{481}$$

$$\angle ABC = \frac{1}{2}\widehat{AC} \tag{482}$$

$$\angle ACB = \frac{1}{2}\widehat{AB}\,. \tag{483}$$

As these two later angles are equal we have $\widehat{AC} = \widehat{AB}$. Using this in

$$\widehat{AB} + \widehat{BC} + \widehat{AC} = 360\,,$$

we have

$$2\widehat{AB} + \widehat{BC} = 360\,. \tag{484}$$

Next we have

$$\angle CDB = \angle D = \frac{1}{2}(\widehat{CAB} - \widehat{CB}) = \frac{1}{2}(2\widehat{AB} - \widehat{CB})\,. \tag{485}$$

In the problem we are told that $\angle D = \frac{1}{2}\angle ACB$ so using Equation 483 we in Equation 485 we get

$$\frac{1}{2}\left(\frac{1}{2}\widehat{AB}\right) = \widehat{AB} - \frac{1}{2}\widehat{CB}\,,$$

or

$$\widehat{CB} = \frac{3}{2}\widehat{AB}\,. \tag{486}$$

If we put that into Equation 484 we get

$$2\widehat{AB} + \frac{3}{2}\widehat{AB} = 360 \quad \text{so} \quad \widehat{AB} = \frac{720}{7}\,.$$

Using Equation 486 we get $\widehat{CB} = \frac{1080}{7}$ so that

$$x = \frac{1}{2}\widehat{CB} = \frac{540}{7}\,.$$

In radians this is given by

$$x = \frac{540}{7} \times \frac{\pi}{180} = \frac{3\pi}{7}\,.$$

Problem 15

Let our four numbers be a, b, c, and d. Considering the sums we would have when we drop a variable each time we get

$$a+b+c = 180 \tag{487}$$
$$a+b+d = 197 \tag{488}$$
$$a+c+d = 208 \tag{489}$$
$$b+c+d = 222\,. \tag{490}$$

This is a system of four equations and four unknowns. Adding all of the equations together we get

$$3(a+b+c+d) = 807\,,$$

so

$$a+b+c+d = 269\,. \tag{491}$$

If from this we subtract Equation 487 above we find $d = 89$. If from this we subtract Equation 488 we find $c = 72$. If from this we subtract Equation 489 we find $b = 61$. Finally, if from Equation 491 we subtract Equation 490 we find $a = 47$. The largest of these numbers is 89.

Problem 16

If everyone shook hands there would be

$$\binom{26}{2}\,,$$

handshakes. This would include the 13 husband-wife handshakes and the $\binom{13}{2}$ handshakes between the women. Thus the total number of handshakes is given by

$$\binom{26}{2} - 13 - \binom{13}{2} = 234\,,$$

when we simplify.

Problem 17

Lets assume our number is of the form abc with a, b, and c digits with $1 \le a \le 9$, $0 \le b \le 9$, and $0 \le c \le 9$. To count the increasing numbers lets start with $a = 1$. Then we are considering numbers of the form $1bc$ with $1 < b < c$. We can have

- $b = 2$ so that our number is $12c$ and to have $2 < c$ there are seven choices

- $b = 3$ so that our number is $13c$ and to have $3 < c$ there are six choices
- $b = 4$ so that our number is $14c$ and to have $4 < c$ there are five choices
- Continuing this logic
- $b = 8$ so that our number is $18c$ and to have $8 < c$ there is one choice

Thus there are
$$1 + 2 + 3 + \cdots + 6 + 7 = \frac{8(7)}{2} = 28\,,$$
"increasing" numbers of the form $1bc$.

Next lets start with $a = 2$. Then we are considering numbers of the form $2bc$ with $2 < b < c$. We can have

- $b = 3$ so that our number is $23c$ and to have $3 < c$ there are six choices
- $b = 4$ so that our number is $24c$ and to have $4 < c$ there are five choices
- Continuing this logic
- $b = 8$ so that our number is $28c$ and to have $8 < c$ there is one choice

Thus there are
$$1 + 2 + 3 + \cdots + 6 = \frac{7(6)}{2} = 21\,,$$
"increasing" numbers of the form $2bc$.

If this pattern continues we expect to have
$$1 + 2 + 3 + \cdots + 5 = \frac{6(5)}{2} = 15\,,$$
"increasing" numbers of the form $3bc$.

Numbers of the form $\frac{n(n+1)}{2}$ are called triangular numbers T_n

`https://en.wikipedia.org/wiki/Triangular_number`

Using that definition the number of "increasing" numbers of the form $1bc$ is T_7. The number of "increasing" numbers of the form $2bc$ is T_6. The number of "increasing" numbers of the form $3bc$ is T_5. Continuing this pattern the number of "increasing" numbers of the form $7bc$ is $T_{8-7} = T_l = l$. Thus the total number of "increasing" numbers abc is
$$\sum_{k=1}^{7} T_k\,.$$

We can evaluate this to find its value given by 84.

Next we look for "decreasing" numbers of the form abc. To do that lets start with $a = 9$. Then we are considering numbers of the form $9bc$ with $9 > b > c$. We can have

- $b = 8$ so that our number is $98c$ and to have $8 > c$ there are eight choices for c i.e. $c \in \{7, 6, 5, 4, 3, 2, 1, 0\}$
- $b = 7$ so that our number is $97c$ and to have $7 > c$ there are seven choices
- $b = 6$ so that our number is $96c$ and to have $6 > c$ there are six choices
- Continuing this logic
- $b = 2$ so that our number is $92c$ and to have $2 > c$ there is one choice

Thus there are

$$1 + 2 + 3 + \cdots + 7 + 8 = \frac{9(8)}{2} = 36\,,$$

"decreasing" numbers of the form $9bc$. Note that this number is T_8. We could now do the same for numbers of the form $8bc$ and would find that there are T_7 of them. The logic at this point is the same as that in the previous part. In total, the number of "decreasing" numbers abc are

$$\sum_{k=1}^{8} T_k\,.$$

We find this number to be 120.

Adding this number to the number of "increasing" numbers we get a total of $84 + 120 = 204$ total numbers of the required type.

Problem 18

The first few powers of three are

$$\begin{aligned} 3^1 &= 3 \\ 3^2 &= 9 \\ 3^3 &= 27 \\ 3^4 &= 81 \\ 3^5 &= 243\,. \end{aligned}$$

Thus the units value for powers of three take the form $\{3, 9, 7, 1\}$ and then repeat. This means that when a is drawn from the given set of 100 numbers i.e. $\{1, 2, 3, \ldots, 99, 100\}$ there are $\frac{100}{4} = 25$ of them where 3^a will have a units digit of each of the choices $\{3, 9, 7, 1\}$.

The first few powers of seven are

$$\begin{aligned}
7^1 &= 7 \\
7^2 &= 49 \\
7^3 &= 343 \\
7^4 &= 2401 \\
7^5 &= 16807\,.
\end{aligned}$$

Thus the units value for powers of seven take the form $\{7, 9, 3, 1\}$ and then repeat. This also means that when b is drawn from the given set of 100 numbers i.e. $\{1, 2, 3, \ldots, 99, 100\}$ there are $\frac{100}{4} = 25$ of them where 7^b will have a units digit of each of the choices $\{7, 9, 3, 1\}$.

To get a units digit of an eight in a sum of the form $3^a + 7^b$ we need to have the units digit of 3^a and 7^b both nine, or the units digit of one be one and the other be seven.

Let $\mathcal{U}(n)$ be a function that returns the units digit of the number n, thus $\mathcal{U}(123) = 3$ and $\mathcal{U}(654321) = 1$. Then the number of ways we could get a units digit of an eight is the size of the set

$$[(\mathcal{U}(3^a) = 9) \cap (\mathcal{U}(7^b) = 9)] \cup [(\mathcal{U}(3^a) = 1) \cap (\mathcal{U}(7^b) = 7)] \cup [(\mathcal{U}(3^a) = 7) \cap (\mathcal{U}(7^b) = 1)]\,.$$

In the union above the sets are distinct and we can simply sum the sizes of the three sets above. We find

$$|(\mathcal{U}(3^a) = 9) \cap (\mathcal{U}(7^b) = 9)| = 25^2\,,$$

as the draws of a and b are independent. The same calculation can be done for the other two sets. The total number of possible units digits is 100^2. Thus the probability we seek is

$$\frac{3(25^2)}{100^2} = \frac{3}{16}\,,$$

when we simplify.

Problem 19

Notice that we can write this fraction $f(N)$ as

$$\begin{aligned}
f(N) \equiv \frac{(N+4-4)^2 + 7}{N+4} &= \frac{(N+4)^2 - 8(N+4) + 16 + 7}{N+4} \\
&= N + 4 - 8 + \frac{23}{N+4} = N - 4 + \frac{23}{N+4}\,.
\end{aligned}$$

Now I claim that this fraction f will not be in lowest terms if $N + 4$ is a multiple of 23 i.e. when $N + 4 = 23k$ for some k. Lets check this for $k = 1$. If $N + 4 = 23$ then our fraction is

$$f(19) = \frac{19^2 + 7}{19 + 4} = \frac{368}{23} = \frac{2^4 \cdot 23}{23}\,,$$

which is not in lowest terms. For $k = 2$ we would have $N + 4 = 2(23)$ or $N = 42$ we find

$$f(42) = \frac{42^2 + 7}{42 + 4} = \frac{1771}{46} = \frac{7 \cdot 11 \cdot 23}{2 \cdot 23},$$

which is also not in lowest terms.

We now ask for how many N such that $1 \leq N \leq 1990$ or $4 \leq N + 4 \leq 1994$ are multiples of 23. In terms of the multiples k these are

$$\frac{4}{23} \leq \frac{N+4}{23} \leq \frac{1994}{23},$$

or

$$0.173913 \leq k \leq 86.6957\,.$$

Thus there are 86 possible values for k.

Problem 20

From the given problem note that

$$\angle FBC + \angle FCB = 90^\circ \tag{492}$$
$$\angle FCB + \angle FCD = 90^\circ \tag{493}$$
$$\angle FCD + \angle EDC = 90^\circ\,. \tag{494}$$

If we subtract Equation 493 from 492 we get $\angle FBC = \angle FCD$. If we subtract Equation 494 from 493 we get $\angle FCB = \angle EDC$. Together these mean that

$$\triangle BFC \sim \triangle CED\,,$$

so

$$\frac{FC}{ED} = \frac{BF}{EC} = \frac{BC}{DC} \quad \text{or} \quad \frac{FC}{5} = \frac{BF}{7}\,. \tag{495}$$

In the same way as the above (using three sets of complementary angles) we can argue that $\angle ADE = \angle BAF$ and $\angle DAE = \angle ABF$ and so that $\triangle AED \sim \triangle BFA$ so

$$\frac{AF}{ED} = \frac{BF}{AE} = \frac{AB}{AD} \quad \text{or} \quad \frac{3 + EF}{5} = \frac{BF}{3}\,. \tag{496}$$

Using the fact that $FC = EC - EF = 7 = EF$ in Equation 495 we have that

$$\frac{7 - EF}{5} = \frac{BF}{7}\,. \tag{497}$$

This with Equation 496 gives two equations in the two unknowns EF and BF. Solving these we get $EF = 4$ and $BF = \frac{21}{5} = 4.2$.

Problem 21

Let the base of the pyramid be located at the points $A = \left(-\frac{1}{2}, \frac{1}{2}, 0\right)$, $B = \left(\frac{1}{2}, -\frac{1}{2}, 0\right)$, $C = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$, and $D = \left(-\frac{1}{2}, \frac{1}{2}, 0\right)$. Note that the area of the base of this pyramid is one and by symmetry the vertex of the pyramid is located at $P = (0, 0, h)$ for some h. The volume of the pyramid is then $\frac{h}{3}$.

The angle 2θ is related to the points of the pyramid as from the dot product of

$$\overrightarrow{PC} \cdot \overrightarrow{PB} = ||\overrightarrow{PC}||\,||\overrightarrow{PB}|| \cos(2\theta)\,. \tag{498}$$

Note that

$$\overrightarrow{PC} = \left(\frac{1}{2} - 0\right)\hat{i} + \left(\frac{1}{2} - 0\right)\hat{j} + (0 - h)\hat{k} = \frac{1}{2}\hat{i} + \frac{1}{2}\hat{j} - h\hat{k}$$
$$\overrightarrow{PB} = \left(\frac{1}{2} - 0\right)\hat{i} + \left(-\frac{1}{2} - 0\right)\hat{j} + (0 - h)\hat{k} = \frac{1}{2}\hat{i} - \frac{1}{2}\hat{j} - h\hat{k}\,.$$

These mean that

$$\overrightarrow{PC} \cdot \overrightarrow{PB} = \frac{1}{4} - \frac{1}{4} + h^2 = h^2\,,$$

and

$$||\overrightarrow{PC}|| = \sqrt{\frac{1}{4} + \frac{1}{4} + h^2} = \sqrt{\frac{1}{2} + h^2} = ||\overrightarrow{PB}||\,.$$

Using these results Equation 498 becomes

$$h^2 = \left(\frac{1}{2} + h^2\right)\cos(2\theta)\,.$$

Solving for h^2 we get

$$h^2 = \frac{\cos(2\theta)}{2(1 - \cos(2\theta))}\,.$$

If we recall the "double angle formula for cosign" or Equation 214 we can write

$$1 - \cos(2\theta) = 2\sin^2(\theta)\,,$$

and so

$$h^2 = \frac{\cos(2\theta)}{4\sin^2(\theta)} \quad \text{so} \quad h = \frac{\sqrt{\cos(2\theta)}}{2\sin(\theta)}\,,$$

and the volume of our pyramid is then given by $\frac{1}{3}$ of this or

$$\frac{\sqrt{\cos(2\theta)}}{6\sin(\theta)}\,.$$

Problem 22

For x we have
$$x^6 = -64 = -2^6 = 2^6 e^{i\pi + 2\pi i k},$$
for $k \in \mathbb{Z}$. This means that the six solutions for x_k are
$$x_k = 2e^{\frac{\pi i}{6} + \frac{2\pi i}{6}k} = 2e^{\frac{\pi i}{6} + \frac{\pi i}{3}k}.$$
for $k \in \{0, 1, 2, 3, 4, 5\}$. If we compute these we find
$$\begin{aligned}
x_0 &= 2e^{\frac{i\pi}{6}} = 2\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right) = 2\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right) \\
x_1 &= 2e^{\frac{i\pi}{6} + \frac{i\pi}{3}} = 2e^{\frac{i\pi}{2}} = 2(0 + i) \\
x_2 &= 2e^{\frac{i\pi}{6} + \frac{i2\pi}{3}} = 2e^{\frac{i5\pi}{6}} = 2\left(-\frac{\sqrt{3}}{2} + \frac{i}{2}\right) \\
x_3 &= 2e^{\frac{i\pi}{6} + i\pi} = 2e^{\frac{i7\pi}{6}} = 2\left(-\frac{\sqrt{3}}{2} - \frac{i}{2}\right) \\
x_4 &= 2e^{\frac{i\pi}{6} + \frac{i4\pi}{3}} = 2e^{\frac{i3\pi}{2}} = 2(0 - i) \\
x_5 &= 2e^{\frac{i\pi}{6} + \frac{i5\pi}{3}} = 2e^{\frac{i11\pi}{6}} = 2\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right).
\end{aligned}$$
The product desired is then given by
$$x_0 x_5 = 2e^{\frac{i\pi}{6}} \cdot 2e^{\frac{i11\pi}{6}} = 4e^{\frac{i2\pi}{6}} = 4e^{i2\pi} = 4.$$

Problem 23

Write the first equation as
$$\frac{\ln(x)}{\ln(y)} + \frac{\ln(y)}{\ln(x)} = \frac{10}{3}.$$
If we let $v = \frac{\ln(x)}{\ln(y)}$ then the above can be written as
$$v^2 - \frac{10}{3}v + 1 = 0.$$
Solving this we see that the two solutions are
$$v \in \left\{\frac{1}{3}, 3\right\}.$$
This means that
$$\frac{\ln(x)}{\ln(y)} = \frac{1}{3} \quad \text{or} \quad \frac{\ln(x)}{\ln(y)} = 3.$$

These mean that

$$x = y^{1/3} \quad \text{or} \quad x = y^3 \,.$$

Putting these into $xy = 144$ we get

$$y^{4/3} = 144 \quad \text{or} \quad y^4 = 144 \,.$$

Solving these and then solving for x we find

$$\begin{array}{lcl} y = 144^{3/4} & \text{and} & x = 144^{1/4} \\ y = 144^{1/4} & \text{and} & x = 144^{3/4} \,. \end{array}$$

In either case we have

$$\frac{1}{2}(x+y) = \frac{1}{2}(144^{3/4} + 144^{1/4}) = 13\sqrt{3} \,,$$

when we simplify.

Problem 24

In Adams high school let b_A be the number of boys and g_A be the number of girls. Then from the averages given in that school we have

$$\frac{71b_A + 76g_A}{b_A + g_A} = 74 \,,$$

or

$$71b_A + 76g_A = 74b_A + 74g_A \,,$$

or

$$-3b_A + 2g_A = 0 \,. \qquad (499)$$

In Baker high school let b_B be the number of boys and g_B be the number of girls. Then from the averages given in that school we have

$$\frac{81b_B + 90g_B}{b_B + g_B} = 84 \,,$$

or

$$-b_B + 2g_B = 0 \,. \qquad (500)$$

Now for the average of the boys in Adams and Baker we have

$$\frac{71b_A + 81b_B}{b_A + b_B} = 79 \,,$$

or multiplying by $b_A + b_B$ and simplifying we get

$$-4b_A + b_B = 0 \,. \qquad (501)$$

We want to evaluate

$$\frac{76 g_A + 90 g_B}{g_A + g_B}.$$

In this we will use Equation 499 to replace g_A with b_A and Equation 500 to replace g_B with b_B to get

$$\frac{76\left(\frac{3}{2}b_A\right) + 90\left(\frac{1}{2}b_B\right)}{\frac{3}{2}b_A + \frac{1}{2}b_B},$$

which simplifies to

$$\frac{228 b_A + 90 b_B}{3 b_A + b_B}.$$

Use Equation 501 to replace b_B with b_A to get

$$\frac{228 b_A + 90(4 b_A)}{3 b_A + (4 b_A)} = 84\,.$$

Problem 25

Lets imagine the cube in a three dimensional Cartesian coordinate system. We can place the corners of the cube at the (x, y, z) locations

$$\left(\frac{1}{2}, \frac{1}{2}, \pm\frac{1}{2}\right), \left(\frac{1}{2}, -\frac{1}{2}, \pm\frac{1}{2}\right), \left(-\frac{1}{2}, \frac{1}{2}, \pm\frac{1}{2}\right), \left(-\frac{1}{2}, -\frac{1}{2}, \pm\frac{1}{2}\right).$$

Then one sphere is located with its center at $(0, 0, 0)$. This sphere is surrounded by four spheres "above" and "below" it. If we imagine the four spheres "above" this center sphere note that if they have a radius of r then their centers must be located offset from the corner of the square by (r, r, r) and so at the points

$$\left(\frac{1}{2} - r, \frac{1}{2} - r, \frac{1}{2} - r\right), \left(\frac{1}{2} - r, -\frac{1}{2} - r, \frac{1}{2} - r\right), \left(-\frac{1}{2} - r, \frac{1}{2} - r, \frac{1}{2} - r\right), \left(-\frac{1}{2} - r, -\frac{1}{2} - r, \frac{1}{2} - r\right).$$

The distance from each of these four points to the center of the central sphere located at $(0, 0, 0)$ is given by $2r$. Selecting the point

$$\left(\frac{1}{2} - r, \frac{1}{2} - r, \frac{1}{2} - r\right),$$

this means that

$$(2r)^2 = \left(\frac{1}{2} - r\right)^2 + \left(\frac{1}{2} - r\right)^2 + \left(\frac{1}{2} - r\right)^2.$$

If we expand and simplify this we get

$$r^2 + 3r - \frac{3}{4} = 0\,.$$

Solving this we find

$$r = \frac{-3 \pm 2\sqrt{3}}{2},$$

of which only the plus sign gives a positive value for r.

Problem 26

Let the number each person picks be given by x_i for $1 \le i \le 10$. Here $i = 1$ is the "Northern most" person and the people are enumerated clockwise around this circle. From the averages given that are announced we have

$$2(1) = x_{10} + x_2 \quad (502)$$
$$2(2) = x_1 + x_3 \quad (503)$$
$$2(3) = x_2 + x_4 \quad (504)$$
$$2(4) = x_3 + x_5 \quad (505)$$
$$2(5) = x_4 + x_6 \quad (506)$$
$$2(6) = x_5 + x_7 \quad (507)$$
$$2(7) = x_6 + x_8 \quad (508)$$
$$2(8) = x_7 + x_9 \quad (509)$$
$$2(9) = x_8 + x_{10} \quad (510)$$
$$2(10) = x_9 + x_1 \,. \quad (511)$$

This problem asks us to find the value of x_6. Using Equations 506, 504 502, 510, and 508 (in that order) we get a single equation involving x_6. We have

$$\begin{aligned} x_6 &= 10 - x_4 \\ &= 10 - 6 + x_2 = 4 + x_2 \\ &= 4 + (2 - x_{10}) = 6 - x_{10} \\ &= 6 - (18 - x_8) = -12 + x_8 \\ &= -12 + (14 - x_6) = 2 - x_6 \,. \end{aligned}$$

We can solve this for x_6 where we find $x_6 = 1$.

Problem 27

Given the ith height h_i (altitude) for $1 \le i \le 3$ the area of the triangle is given by

$$A = \frac{1}{2} b_i h_i \,,$$

where b_i is the "base" (i.e. side) of the triangle that the ith height impinges on. Thus

$$b_i = \frac{2A}{h_i} \,,$$

for $1 \le i \le 3$. The triangle inequality means that

$$b_1 + b_2 > b_3 \,,$$

for any ordering of the sides. In terms of h_i this is

$$\frac{1}{h_1} + \frac{1}{h_2} > \frac{1}{h_3} \,.$$

If we are looking for a violation of this fact then lets order the three heights given as $h_1 \geq h_2 \geq h_3$ which is *opposite* the order given. We can then check this inequality for the given altitudes. We find

- $\frac{1}{2} + \frac{1}{\sqrt{3}} = 1.07735 > \frac{1}{1} = 1$ which is true.
- $\frac{1}{5} + \frac{1}{4} = 0.45 > \frac{1}{3} = 0.333333$ which is true.
- $\frac{1}{13} + \frac{1}{12} = 0.160256 > \frac{1}{5} = 0.2$ which is *not* true.
- $\frac{1}{\sqrt{113}} + \frac{1}{8} = 0.219072 > \frac{1}{7} = 0.142857$ which is true.
- $\frac{1}{17} + \frac{1}{15} = 0.12549 > \frac{1}{8} = 0.125$ which is true.

Thus (C) is not possible for a set of altitudes.

Problem 30

Note that for the given values for a and b we have

$$ab = (3 + 2\sqrt{2})(3 - 2\sqrt{2}) = 9 - 8 = 1\,,$$

and

$$a + b = 2 \cdot 3 = 6\,.$$

Next lets try to get a recurrence relationship for R_n. If we multiplying the definition of R_n by $a + b$ we get

$$R_n(a + b) = \frac{1}{2}(a^{n+1} + ab^n + ba^n + b^{n+1}) = R_{n+1} + \frac{1}{2}(ab^n + ba^n)\,.$$

As $ab = 1$ the above becomes

$$R_n(a + b) = R_{n+1} + \frac{1}{2}(b^{n-1} + a^{n-1}) = R_{n+1} + R_{n-1}\,.$$

Now using the fact that $a + b = 6$ we can write the above as

$$R_{n+1} = 6R_n - R_{n-1}\,. \tag{512}$$

From the definition of R_n we have

$$R_0 = 1\,,$$

and

$$R_1 = \frac{1}{2}(a + b) = 3\,.$$

Using these two values and Equation 512 we have

$$R_2 = 6R_1 - 1 = 6(3) - 1 = 17\,,$$

and

$$R_3 = 6(17) - 3 = 99\,.$$

At this point the units digits of R_n for $0 \le n \le 3$ are

$$1\,,3\,,7\,,9\,.$$

We will use Equation 512 to see if we can find a pattern in the units digit. Using the following R code

```
Rs = c(1, 3)
for( n in seq(2, 15) ){
  n = length(Rs)
  Rs = c(Rs, 6*Rs[n] - Rs[n-1])
}
Rs = data.frame(n=seq(0, length(Rs)-1), Rn=Rs, Units=(Rs %% 10))
print(Rs)
```

we get the following table

```
    n           Rn Units
1   0            1     1
2   1            3     3
3   2           17     7
4   3           99     9
5   4          577     7
6   5         3363     3
7   6        19601     1
8   7       114243     3
9   8       665857     7
10  9      3880899     9
11 10     22619537     7
12 11    131836323     3
13 12    768398401     1
14 13   4478554083     3
15 14  26102926097     7
16 15 152139002499     9
```

From this we see that the units digit repeats at a period of six so

$$\text{UnitsDigit}(R_{n+6}) = \text{UnitsDigit}(R_n)\,.$$

If we "divide" 12345 by six we find that

$$12345 = 2057 \times 6 + 3\,.$$

This means that

$$\text{UnitsDigit}(R_{12345}) = \text{UnitsDigit}(R_3) = 9\,.$$

The 1991 AHSME Examination (AHSME 42)

Problem 1

For this we have

$$(1, -2, -3) = \frac{-3+1}{-3+2} = 2\,.$$

Problem 2

This is $|3 - \pi| = \pi - 3$.

Problem 3

This would be

$$(4^{-1} - 3^{-1})^{-1} = \left(\frac{3}{12} - \frac{4}{12}\right)^{-1} = \left(-\frac{1}{12}\right)^{-1} = -12\,.$$

Problem 4

An isosceles triangle has two equal sides. A scalene triangle has all sides of unequal length. An obtuse triangle has one angle greater than ninety degrees. Thus an obtuse right triangle would have a right angle and another angle greater than ninety degrees. The sum of these two would be larger than 180 degrees which is not possible.

Problem 5

We can write the area of this polygon in terms of the "rectangle" part (first) and the "triangle" part (second) as

$$A = 10(20) + \frac{1}{2}bh = 200 + 10(10) = 300\,.$$

Here the area of the "triangle" part is the area of a isosceles triangle with a base $b = 5 + 10 + 5 = 20$ and angles $\angle ABC = \angle AGF = 45°$. This means that this triangle has a height given by

$$\frac{h}{10} = \tan(45°) = 1 \quad \text{thus} \quad h = 10\,.$$

Problem 6

Call this expression E. We can simplify the given expression as

$$\begin{aligned} E &= \sqrt{x\sqrt{x\sqrt{x}}} = \sqrt{x\sqrt{xx^{\frac{1}{2}}}} \\ &= \sqrt{x\sqrt{x^{\frac{3}{2}}}} = \sqrt{x - x^{\frac{3}{4}}} \\ &= \sqrt{x^{\frac{7}{4}}} = x^{\frac{7}{8}} \,. \end{aligned}$$

Problem 7

We have

$$\frac{a+b}{a-b} = \frac{\frac{a}{b}+1}{\frac{a}{b}-1} = \frac{x+1}{x-1} \,.$$

Problem 8

The volume of the liquid X in the box is

$$6 \cdot 3 \cdot 12 \,,$$

centimeters cubed. The volume of liquid X when pored out is

$$(\pi r^2) 0.1 \,,$$

in the same units. If we set these equal and solve for r we get

$$r = \sqrt{\frac{2160}{\pi}} \,.$$

Problem 9

If we start with a population P at time $t = 0$ then the *fractional* change at $t = 2$ is given by

$$\frac{P\left(1 + \frac{i}{100}\right)\left(1 + \frac{j}{100}\right) - P}{P} = \frac{i}{100} + \frac{j}{100} + \frac{ij}{10^4} \,.$$

The *percent* change is this multiplied by 100 or

$$i + j + \frac{ij}{100} \,.$$

Problem 10

For this problem I drew two concentric circles (centered on C) with the smaller one having a radius $r = 9$ and the larger one having a radius of $R = 15$. Then P is a point on the inner circle and let the two endpoints of the chord be denoted A and B. Then as the endpoints of the chord AB are on the outer circle, the distance from the circle center C to each of A and B is fifteen. The distance AB can be computed with the law of cosigns as

$$AB^2 = R^2 + R^2 - 2R^2 \cos(\angle ACB) \,,$$

or

$$AB^2 = 2R^2(1 - \cos(\angle ACB)) \,.$$

With $R = 15$ this is

$$AB = 15\sqrt{2(1 - \cos(\angle ACB))} = 3 \cdot 5\sqrt{2(1 - \cos(\angle ACB))} \,. \tag{513}$$

Now if we think about the possible values for the angle $\angle ACB$ we see that the largest $\angle ACB$ can be is when the chord AB passes through C so that $\angle ACB = \pi$. In this case the length of the chord is $2R = 30$.

The smallest angle is when the chord AB is perpendicular (tangent) to the smaller circle so that $CP \perp AB$. If we draw this situation the triangle $\triangle ACB$ is isosceles with its vertex angle $\angle ACB$, with two equal sides of length $R = 15$ and a height $h = CP = r = 9$. This means that each half of this isosceles triangle is a right triangle with its other leg of length

$$\sqrt{15^2 - 9^2} = \sqrt{144} = 12 \,.$$

This means the shortest cord has a length of $2 \times 12 = 24$.

Note that the possible angles given above are for the segment AB to *one* side of the segment CP. There are another symmetric set of angles for $\angle ACB$ on the other side.

As the longest chord is of length 30 and the smallest chord is of length 24 as we change the angle $\angle ACB$ we move continuously between these two values. There are $30 - 24 + 1 = 7$ integers inclusive between 30 and 24. Counting the symmetry above we would have $2 \times 7 = 14$ integer length chords but that would double count the cords of length 30 and 24 and thus we have $14 - 2 = 12$ cords of integer length.

Problem 11

After the ten minute (1/6th of an hour) head start Jack is at the location

$$\frac{1}{6}(15) = \frac{5}{2} \,,$$

kilometers from the start while Jill is still at the starting location.

Jack will take

$$\frac{5 - 5/2}{15} = \frac{1}{6},$$

of an hour more to get to the top of the hill five kilometers away. After this amount of time Jill will be at the location

$$\frac{1}{6}(16) = \frac{8}{3},$$

kilometers from the start. At this point Jack starts coming back down the hill. For t in hours the location of Jill (from their joint starting point) is given by

$$x_{\text{Jill}} = \frac{8}{3} + 16t\,,$$

while the location of Jack (from their joint starting point) is at

$$x_{\text{Jack}} = 5 - 20t\,.$$

They will meet when these two expressions are equal. Setting these equal and then solving for t gives $t = \frac{7}{108}$ hours. At this time Jack will be $5 - x_{\text{Jack}}$ from the top or

$$20 \times \frac{7}{108} = \frac{35}{27},$$

kilometers.

Problem 12

Recall that the sum of the interior angles in a n sided polygon is given by Equation 5. If $n = 6$ this gives the value $720°$. Lets denote the values of the internal angles by the sequence

$$m\,, m - d\,, m - 2d\,, m - 3d\,, m - 4d\,, m - 5d\,.$$

Here m is the largest internal angle. Then the sum of these is given by

$$6m - d\sum_{i=1}^{5} i = 6m - d\left(\frac{5(6)}{2}\right) = 6m - 15d\,.$$

From the above we must have

$$6m - 15d = 720\,,$$

or dividing by three we have

$$2m - 5d = 240\,. \tag{514}$$

As $5 \mid 240$ and $5 \mid 5d$ we must have that $5 \mid m$ where the symbol $\mid$ means "divides". To be convex means that $m < 180°$. The smallest integer less than 180 and divisible by five is 175. Taking this number in Equation 514 we get $d = 22$ and the internal angles are

$$175\,, 153\,, 131\,, 109\,, 87\,, 65\,.$$

Problem 13

The probability that X wins is $\frac{1}{4}$. The probability that Y wins is $\frac{3}{5}$. The probability that Z wins is then

$$1 - \frac{1}{4} - \frac{3}{5} = \frac{3}{20}.$$

The probability that Z looses is then $\frac{17}{20}$. The odds against Z winning are then 17-to-3.

Problem 14

Let n be the number we cube and consider the prime factorization of n so that

$$n^3 = (p_1^{n_1} p_2^{n_2} \cdots p_{k-1}^{n_{k-1}} p_k^{n_k})^3 = p_1^{3n_1} p_2^{3n_2} \cdots p_{k-1}^{3n_{k-1}} p_k^{3n_k}.$$

where $p_1, p_2, \ldots p_{k-1}, p_k$ are distinct primes and $n_i \geq 1$ for $1 \leq i \leq k$.

To count the number of divisors of the number n^3, note that from the first factor any number of the form

$$p_1^{t_1},$$

for t_1 and integer in the range $0 \leq t_1 \leq 3n_1$ will be a factor of n^3. There are $3n_1 + 1$ integers in this range and thus $3n_1 + 1$ numbers of the form $p_1^{t_1}$.

From the second factor any number of the form

$$p_2^{t_2},$$

for t_2 and integer in the range $0 \leq t_2 \leq 3n_2$ will also be a factor of n^3. There are $3n_2 + 1$ integers in this range and thus $3n_2 + 1$ numbers of the form $p_2^{t_2}$. The product of two numbers of this form will also be a divisor of n^3. Following this pattern we see that in general all divisors of n^3 will be numbers of the form

$$p_1^{t_1} p_2^{t_2} \cdots p_{k-1}^{t_{k-1}} p_k^{t_k},$$

with $0 \leq t_i \leq 3n_i$ for all $1 \leq i \leq k$. There are $3n_i + 1$ numbers for each i and thus

$$\prod_{i=1}^{k} (3n_k + 1),$$

total divisors of n^3. A number of this form can be written as $3n + 1$ for some integer n. The only numbers (from the choices possible) that is of this form is 202.

Problem 15

If we space the people out with a gap of two between each person then placing another person in any of those gap seats will force that new person to be sitting next to an existing person. When you diagram this you are specifying the sequence: filled, empty, empty, filled, empty, empty, filled, empty, empty, etc. These are "groups" of three chairs with one person placed per group. There are

$$\frac{60}{3} = 20\,,$$

groups/persons that we can place at a 60 seat table. Thus $N = 20$.

Problem 16

Let m be the number of seniors taking the AHSME and n the number of non-seniors taking the AHSME. Then we are told that

$$m + n = 100\,, \tag{515}$$

and

$$n = 1.5m\,. \tag{516}$$

Solving these two equations we find $m = 40$ and $n = 60$. Next let S be the sum of the seniors' scores and O (for other) be the sum of the non-seniors scores on the AHSME. We are told that

$$\begin{aligned} \frac{1}{100}(S + O) &= 100 \\ \frac{1}{m}S &= 1.5\left(\frac{1}{n}O\right)\,. \end{aligned}$$

Since we know m and n we can solve the above for S and O and find $S = O = 5000$. This means that the average of the seniors is given by $\frac{1}{m}S = \frac{5000}{40} = 125$.

Problem 17

All two digit palindromes must look like dd for some digit d where $1 \leq d \leq 9$. Trying different values for d we see that the only two digit prime palindromes is 11.

Next the range of possible choices for the three digit prime factor n must be such that

$$1000 \leq 11n \leq 2000\,,$$

or

$$90.9091 \leq n \leq 181.818\,.$$

As n is an integer we have that

$$91 \leq n \leq 181\,.$$

As we need n to be three digits we are further restricted to

$$100 \leq n \leq 181\,.$$

n	Prime or Not
101	Yes
111	No
121	No
131	Yes
141	No
151	Yes
161	No
171	No
181	Yes

Table 13: Numbers of the form $1m1$.

Our three digit palindrome n in terms of its digits takes the form dmd. We cannot have $d \in \{0, 2, 4, 5, 6, 8\}$ or else dmd will not be a prime number. Thus $d \in \{1, 3, 7\}$. From the above considerations we must have $d = 1$. We can then enumerate all numbers of the form $1m1$ from the above range and determine if they are prime. This is done in Table 13.

Using the above table we find the numbers with the properties requested are

$$11 \times 101\,, 11 \times 131\,, 11 \times 151\,, 11 \times 181\,,$$

or four numbers.

Problem 18

Let our real number be r. Then our condition is that

$$(3 + 4i)z = r\,.$$

Then solving for z we have

$$z = \frac{r}{3 + 4i} = \frac{r(3 - 4i)}{9 - 16} = \frac{r(3 - 4i)}{-5} = \frac{r}{5}(-3 + 4i)\,.$$

This means

$$z = x + iy = -\frac{3r}{5} + \frac{4r}{5}\,.$$

Thus

$$x = -\frac{3r}{5}$$
$$y = \frac{4r}{5}\,.$$

Solving for r in the first equation we have $r = -\frac{5}{3}x$. Putting this into the second equation gives

$$y = -\frac{4}{3}x\,,$$

which is a line through the origin.

Problem 19

As we are told that DE is parallel to AC we have that $\angle DEC = 90^\circ$. Using the Pythagorean theorem in the right triangle $\triangle ACB$ we get that $AB = 5$. Using the Pythagorean theorem in the right triangle $\triangle DAB$ we get that

$$DB^2 = 12^2 + 5^2 = 169 \quad \text{so} \quad DB = 13\,.$$

Next recall that the sum of the interior angles in a n sided polygon is given by Equation 5 thus when $n = 4$ we find this sum to be 360°. As $\angle C = \angle E = \angle DAB = 90^\circ$ this becomes

$$3 \times 90 + \angle CAB + \angle ADE = 360\,.$$

Which is equivalent to

$$\angle ADE = 90 - \angle CAB = \angle CBA\,,$$

where in the last step we have used the fact that triangle $\triangle ACB$ is a right triangle.

Note that the expression we want or $\frac{DE}{DB}$ is equal to $\cos(\angle BDE)$ which we can write as

$$\begin{aligned} \cos(\angle BDE) &= \cos(\angle EDA - \angle ADB) \\ &= \cos(\angle EDA)\cos(\angle ADB) + \sin(\angle EDA)\sin(\angle ADB)\,. \end{aligned} \tag{517}$$

AS $\angle EDA = \angle CBA$ using $\triangle ACB$ we see that

$$\begin{aligned} \cos(\angle EDA) &= \frac{4}{5} \\ \sin(\angle EDA) &= \frac{3}{5}\,. \end{aligned}$$

Thus Equation 517 becomes

$$\frac{DE}{DB} = \frac{4}{5}\cos(\angle ADB) + \frac{3}{5}\sin(\angle ADB)\,.$$

Using $\triangle DAB$ we see that

$$\begin{aligned} \cos(\angle ADB) &= \frac{12}{13} \\ \sin(\angle ADB) &= \frac{5}{13}\,, \end{aligned}$$

Using these Equation 517 becomes

$$\frac{DE}{DB} = \frac{4}{5} \cdot \frac{12}{13} + \frac{3}{5} \cdot \frac{5}{13} = \frac{63}{65}\,.$$

This means that $m = 63$ and $n = 65$ so that $m + n = 128$.

Problem 20

Let

$$v = 2^x - 4$$
$$w = 4^x - 2\,,$$

then the given expression is

$$v^3 + w^3 = (v + w)^3\,.$$

We can factor the left-hand-side of this to get

$$(v + w)(v^2 + vw + w^2) = (v + w)^3\,.$$

One solution to this is if $v + w = 0$. In terms of x this means that

$$4^x + 2^x - 6 = 0\,,$$

or factoring

$$(2^x + 3)(2^x - 2) = 0\,.$$

To make the above true we need $2^x = -3$ (which has no real solutions) or $2^x = 2$ which has the solution $x = 1$.

To find other solutions we assume that $v + w \neq 0$ and divide by it to get

$$v^2 + vw + w^2 = (v + w)^2\,.$$

If we expand the right-hand-side and then simplify we get

$$wv = 0\,.$$

This means that

$$(2^x - 4)(4^x - 2) = 0\,.$$

The two solutions to this are $2^x = 4$ (or $x = 2$) and $4^x = 2$ (or $x = \frac{1}{2}$). Thus the sum of all real solutions is given by

$$1 + 2 + \frac{1}{2} = \frac{7}{2}\,.$$

Problem 21

In the given expression let

$$v = \frac{x}{x - 1}\,,$$

and solve for x in terms of v to get

$$x = \frac{v}{v - 1}\,.$$

This means that

$$f(v) = \frac{1}{\frac{v}{v-1}} = \frac{v - 1}{v} = 1 - \frac{1}{v}\,.$$

From this we find that

$$f(\sec^2(\theta)) = 1 - \cos^2(\theta) = \sin^2(\theta)\,.$$

Problem 22

By symmetry, we can draw a line from the center of the larger circle (denoted O) through the center of the smaller circle (denoted o) and then through the point P. Draw the segments OB and oA which by tangents will both be perpendicular to the segment BP. Let the larger circle have a radius R, the smaller circle have a radius r, and the distance along the line OoP from the "edge" of the smaller circle to the point P by l.

First using the Pythagorean theorem in the right triangle $\triangle oAP$ we have

$$4^2 + r^2 = (r + l)^2 \,. \tag{518}$$

Next by the similar triangles $\triangle OBP \sim \triangle oAP$ we have

$$\frac{OB}{oA} = \frac{BP}{AP} \quad \text{or} \quad \frac{R}{r} = \frac{8}{4} = 2 \,,$$

thus $R = 2r$.

Using these two similar triangles again we have

$$\frac{OP}{oP} = \frac{R + 2r + l}{r + l} = \frac{BP}{AP} = \frac{8}{4} = 2 \,,$$

so we end with

$$R + 2r + l = 2r + 2l \,,$$

so $l = R = 2r$.

If we use what we know in Equation 518 to write everything in terms of r we get

$$16 + r^2 = (3r)^2 \quad \text{or} \quad r^2 = 2 \,.$$

This means that $\pi r^2 = 2\pi$.

Problem 23

Let $A = (0, 0)$, $C = (2, 0)$, $D = (2, 2)$, and $A = (0, 2)$ so that $F = (1, 0)$ and $E = (0, 1)$. Then the line ED is

$$y - 1 = \left(\frac{2 - 1}{2 - 0}\right)(x - 0) \quad \text{or} \quad y = 1 + \frac{x}{2} \,.$$

The line BE is $y = x$. The line AF is

$$y - 2 = \left(\frac{0 - 2}{1 - 0}\right) x \quad \text{or} \quad y = 2 - 2x \,.$$

Now point I is the intersection of AF and ED which has an x coordinate of

$$2 - 2x = 1 + \frac{x}{2} \quad \text{or} \quad x = \frac{2}{5}\,.$$

For this x we have $y = \frac{6}{5}$ and thus $I = \left(\frac{2}{5}\,\frac{6}{5}\right)$.

Now point H is the intersection of BD and AF which has an x coordinate of

$$x = 2 - 2x \quad \text{or} \quad x = \frac{2}{3}\,.$$

For this x we have $y = \frac{2}{3}$ and thus $H = \left(\frac{2}{3}\,\frac{2}{3}\right)$.

We now compute several areas. Note that

$$\begin{aligned}
\text{Area}\,\triangle AIE &= \frac{1}{2}(1)\left(\frac{2}{5}\right) = \frac{1}{5} \\
\text{Area}\,\triangle BHF &= \frac{1}{2}(1)\left(\frac{2}{3}\right) = \frac{1}{3} \\
\text{Area}\,\triangle ABF &= \frac{1}{2}(2)(1) = 1\,.
\end{aligned}$$

Using these areas we have

$$\text{Area}\,[EIHB] = \text{Area}\,\triangle ABF - \text{Area}\,\triangle AIE - \text{Area}\,\triangle BHF = 1 - \frac{1}{5} - \frac{1}{3} = \frac{7}{15}\,.$$

Problem 24

The mapping between the original space and the rotated space is best derived/remembered by using polar coordinates. For a counter-clockwise rotation the new point $z' = r'e^{i\theta'}$ gets mapped to the point

$$z = re^{i\theta} = r'e^{i\left(\theta' - \frac{\pi}{2}\right)}\,,$$

as the new angle θ' must be larger than the original angle θ by $\frac{\pi}{2}$. Note that we can write the above as

$$z = r'e^{i\theta'}(-i) = -iz'\,.$$

This means that the new location (x', y') relative to the old location (x, y) is given by

$$x + iy = -i(x' + iy') = y' - ix'\,,$$

or

$$\begin{aligned}
x &= y' \\
y &= -x'\,.
\end{aligned}$$

Using these in the equation $y = \log_{10}(x)$ we get

$$-x' = \log_{10}(y') \quad \text{or} \quad y' = 10^{-x'}\,.$$

Problem 25

From the definition of T_n and a summation identity we have

$$T_n = \frac{1}{2}n(n+1)\,.$$

This means that

$$\frac{T_n}{T_n - 1} = \frac{\frac{1}{2}n(n+1)}{\frac{1}{2}n(n+1) - 1} = \frac{n^2+n}{n^2+n-2} = \frac{n(n+1)}{(n-1)(n+2)}\,.$$

Note that this fraction is larger than one for $n \geq 2$. Using this the expression for P_n can be written as

$$\begin{aligned} P_n &= \prod_{k=2}^{n} \frac{T_k}{T_k - 1} = \prod_{k=2}^{n} \frac{k(k+1)}{(k-1)(k+2)} \\ &= \frac{\prod_{k=2}^{n} k(k+1)}{\prod_{k=2}^{n}(k-1)(k+2)} = \frac{\left(\prod_{k=2}^{n} k\right)\left(\prod_{k=3}^{n+1} k\right)}{\left(\prod_{k=1}^{n-1} k\right)\left(\prod_{k=4}^{n+2} k\right)} \\ &= \frac{\left(n \cdot \prod_{k=2}^{n-1} k\right)\left(3 \cdot \prod_{k=4}^{n+1} k\right)}{\left(1 \cdot \prod_{k=2}^{n-1} k\right)\left((n+2) \cdot \prod_{k=4}^{n+1} k\right)} = \frac{3n}{n+2}\,. \end{aligned}$$

As we want to evaluate this for $n = 1991$ (a relatively large value for n) we write the above as

$$P_n = 3\left(\frac{n+2-2}{n+2}\right) = 3\left(1 - \frac{2}{n+2}\right)\,.$$

Then we find

$$P_{1991} = 3\left(1 - \frac{2}{1993}\right) \approx 3\left(1 - \frac{2}{2000}\right) = 3(1 - 0.001) = 3 - 0.003 = 2.997\,.$$

Problem 26

Let the six digits of this number be given by d_1, d_2, d_3, d_4, d_5, d_6. Then as every digit is divisible by one the first digit d_1 can be anything. As we need $d_1d_2d_3d_4d_5$ divisible by five we have that $d_5 = 5$.

As we need d_1d_2, $d_1d_2d_3d_4$, and $d_1d_2d_3d_4d_5d_6$ divisible by two, four, and six respectively we need to have d_2, d_4, and d_6 all be even numbers. Using the digits $\{1, 2, 3, 4, 5, 6\}$ this means that the possible values of these digits are given by the templates in Table 14.

Now for the number $d_1d_2d_3d_4$ to be divisible by four means that the number d_3d_4 must be

d_1	d_2	d_3	d_4	d_5	d_6
	2		4	5	6
	2		6	5	4
	4		2	5	6
	4		6	5	2
	6		2	5	4
	6		4	5	2

Table 14: Possible choices for the digits $\{2, 4, 6\}$ in a six digit cute number.

divisible by four. Some multiples of four that have two digits are given by

$$\begin{aligned}
4 \times 3 &= 12 \\
4 \times 4 &= 16 \\
4 \times 5 &= 20 \\
4 \times 6 &= 24 \\
4 \times 8 &= 28 \\
4 \times 9 &= 32 \\
4 \times 10 &= 36 \\
4 \times 11 &= 40 \\
4 \times 12 &= 44 \\
4 \times 13 &= 48 \\
4 \times 14 &= 52 \\
4 \times 15 &= 56 \\
4 \times 16 &= 60 \\
4 \times 17 &= 64 \\
4 \times 18 &= 68 \\
4 \times 19 &= 72 \,.
\end{aligned}$$

Note that for larger multiples of four, the product will include the digits seven, eight, nine or have three digits and thus is not a valid two digit number for this problem. Now some of the two digit numbers above are actually not possible for this problem. For example the two digit number $d_3 d_4$ cannot have the digit zero, or have repeated digits, or include the digit five. Because of this, the valid the multiples of four that are two digits and end in a two are then

$$12\,, 32\,.$$

The valid multiples of four that end in a four are then

$$24\,, 64\,.$$

The valid multiples of four that end in a six are then

$$16\,, 36\,.$$

Next we place these possible two digit numbers into the locations in Table 14. For example, one of the multiples of four that end in a two is 12 and this could possibly be placed into rows three and five to give the numbers

$$?41256 \quad \text{and} \quad ?61254\,.$$

Using the final missing digit these would need to become the numbers

$$341256 \quad \text{and} \quad 361254\,.$$

Each of these numbers is not cute as three does not divide the first three digits of each i.e. 341 or 361.

The other multiples of four that end in a **two** is 32 and again this could possibly be placed into rows three and five to give the numbers

$$?43256 \quad \text{and} \quad ?63254\,.$$

Using the final missing digit (of one) these would need to become the numbers

$$143256 \quad \text{and} \quad 163254\,.$$

Each of these numbers is not cute as three does not divide 143 or 163.

Next, one of the multiples of four that end in a **four** is 24 and this could possibly be placed into rows one and six to give the numbers

$$?22456 \quad \text{and} \quad ?62452\,.$$

Each of these numbers has a duplicate two's digit and is thus not cute.

The other multiples of four that end in a four is 64 and again this could possibly be placed into rows one and six to give the numbers

$$?26456 \quad \text{and} \quad ?66452\,.$$

Each of these numbers has a duplicate six's digit and is thus not cute.

Next, one of the multiples of four that end in a **six** is 16 and this could possibly be placed into rows two and four to give the numbers

$$?21654 \quad \text{and} \quad ?41652\,.$$

Using the final missing digit (of three) these would need to become the numbers

$$321654 \quad \text{and} \quad 341652\,.$$

This first number is cute. Three does not divide 341 and so the second number is not cute.

The other multiples of four that end in a six is 36 and again this could possibly be placed into rows two and four to give the numbers

$$?23654 \quad \text{and} \quad ?43652\,.$$

Using the final missing digit (of one) these would become the numbers

$$123654 \quad \text{and} \quad 143652\,.$$

This first number is cute. Three does not divide 143 and so the second number is not cute.

In summary then we have found *two* cute six digit numbers

$$321654 \quad \text{and} \quad 123654\,.$$

Problem 27

Multiply the second fraction on the left-hand-side by the "form of one" given by

$$\frac{x+\sqrt{x^2-1}}{x+\sqrt{x^2-1}}\,,$$

to get

$$x+\sqrt{x^2-1}+\frac{x+\sqrt{x^2-1}}{x-(x^2-1)}=20\,,$$

or

$$x+\sqrt{x^2-1}+x+\sqrt{x^2-1}=20\,,$$

or

$$x+\sqrt{x^2-1}=10\,,$$

or

$$\sqrt{x^2-1}=10-x\,.$$

If we square this we get

$$x^2-1=100-20+x^2\,.$$

Solving for x we get $x=\frac{101}{20}$. Now let the expression we want to evaluate be denoted by E. Then applying the same trick i.e. by multiplying the second fraction on the left-hand-side by the "form of one" this time given by

$$\frac{x^2-\sqrt{x^4-1}}{x^2-\sqrt{x^4-1}}\,,$$

we get

$$E=x^2+\sqrt{x^4-1}++\frac{x^2-\sqrt{x^4-1}}{x^4-(x^4-1)}=2x^2\,.$$

For the value of x found above we have

$$E=2\left(\frac{101}{20}\right)^2=51.005\,.$$

Problem 28

For this problem we repeatedly draw three marbles from the urn. On each draw, we look at the three marbles and then depending on which of the four cases from the problem we have drawn we make the given substitution.

In all cases, we start with three marbles and remove them and replace them with either one, two, two, or two marbles. Thus each application of this procedure will reduce the total number of marbles in the urn by one or two. Thus, if it is possible to end with the given configurations we must imagine the three marbles in the urn before the last application of this procedure. Some "final configurations" can be eliminated using this logic. For example, it is not possible to start with the final three marbles (of any color) and end up with either two black marbles or one white marble. Thus choices (A) and (E) are not possible.

Next notice that the *net* change in colors for each of the given transformations is given by

- $-3B + B = -2B$
- $-2B - W + B + W = -B$
- $-B - 2W + 2W = -B$
- $-3W + B + W = -2W + B$

Notice that we only ever decrease the white marbles by two. Thus given that we start with an even number we will always have an even number of white marbles. This means that the final three marbles could *not* be

- three white or
- one white with 2 black

Both of these "map to" choice (D) meaning that it is not possible.

Finally, notice that every set that removes any white marbles places some white marbles back in the urn. Thus we cannot ever end this procedure with zero white marbles. This means that choice (C) is not possible. These considerations mean that only choice (B) is possible.

Problem 29

From the way the figure is constructed we have $\angle BAC = \angle ABC = \angle ACB = \angle PA'Q = 60°$, $\angle APQ = \angle QPA'$, $\angle AQP = \angle PQA'$, $AP = PA'$, and $AQ = QA'$. Thus we have that

$$\triangle APQ \cong \triangle A'PQ\,.$$

Summing the three angles in the triangle $\triangle A'PB$ we have

$$\angle BA'P + \angle A'PB + 60 = 180\,.$$

Considering the supplementary angles at A' we have

$$\angle BA'P + 60 + \angle QA'C = 180\,.$$

If we subtract these two equations we get that $\angle A'PB = \angle QA'C$. Using this with the fact that $\angle QCB = \angle ABC = 60°$ we have

$$\triangle BA'P \sim \triangle CQA'\,,$$

from this since $BA' = 1$ and $A'C = 2$ we have

$$\frac{BP}{2} = \frac{1}{QC} = \frac{PA'}{A'Q}\,. \tag{519}$$

Using the fact that $BP = 3 - PA$, $QC = 3 - AQ$, $PA' = PA$, and $A'Q = AQ$ we can write the above as

$$\frac{3 - PA}{2} = \frac{1}{3 - AQ} = \frac{PA}{AQ}\,.$$

Let $x = PA$ and $y = AQ$ and the above are the two equations

$$\begin{aligned}(3 - x)(3 - y) &= 2\\ y &= x(3 - y)\,.\end{aligned}$$

Expanding we get

$$9 - 3y - 3x + xy = 2 \tag{520}$$

$$y = 3x - xy\,. \tag{521}$$

From the last equation we get $xy = 3x - y$ which if we put into Equation 520 and simplify gives

$$7 - 4y = 0 \quad \text{so} \quad y = \frac{7}{4}\,.$$

Using this in Equation 521 will give $x = \frac{7}{5}$. As we now know the lengths of PA' and QA' we can use the law of cosigns to compute PQ^2. We have

$$\begin{aligned}PQ^2 &= PA'^2 + A'Q^2 - 2PA'A'Q\cos(\angle PA'Q)\\ &= \left(\frac{7}{5}\right)^2 + \left(\frac{7}{4}\right)^2 - 2\left(\frac{7}{5}\right)\left(\frac{7}{4}\right)\cos(60) = \frac{49 \cdot 21}{16 \cdot 25}\,.\end{aligned}$$

Thus $PQ = \frac{7\sqrt{21}}{20}$.

Problem 30

Recalling that $n(S) = 2^{|S|}$ we can write the given expression as

$$2^{100} + 2^{100} + 2^{|C|} = 2^{|A \cup B \cup C|} ,$$

or

$$2^{101} + 2^{|C|} = 2^{|A \cup B \cup C|} ,$$

or

$$1 + 2^{|C|-101} = 2^{|A \cup B \cup C|-101} . \quad (522)$$

Now the left-hand-side of the above is larger than one. Based on that the right-hand-side must be larger than one and thus

$$|A \cup B \cup C| > 101 .$$

Also $|A \cup B \cup C|$ is an integer so the right-hand-side of Equation 522 will be a power of two. The only way this can happen is if

$$|C| = 101 \quad \text{and} |A \cup B \cup C| = 102 .$$

The inclusion-exclusion formula states that

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| .$$

Using what we know we can write

$$102 = 100 + 100 + 101 - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| ,$$

or solving for what we want $|A \cap B \cap C|$ gives

$$|A \cap B \cap C| = |A \cap B| + |A \cap C| + |B \cap C| - 199 .$$

Lets use $|X \cap Y| = |X| + |Y| - |X \cup Y|$ three times in the above to write

$$\begin{aligned} |A \cap B \cap C| &= |A| + |B| - |A \cup B| \\ &\quad + |A| + |C| - |A \cup C| \\ &\quad + |B| + |C| - |B \cup C| - 199 \\ &= 200 + 200 + 202 - 199 - |A \cup B| - |A \cup C| - |B \cup C| \\ &= 403 - (|A \cup B| + |A \cup C| + |B \cup C|) . \end{aligned}$$

As $A \cup B$, $A \cup C$, and $B \cup C$ are subsets of $A \cup B \cup C$ they must have sizes that are smaller than $A \cup B \cup C$ or 102. This means that

$$|A \cup B| + |A \cup C| + |B \cup C| \leq 306 ,$$

so that

$$|A \cap B \cap C| \geq 403 - 306 = 97 .$$

This is the smallest possible size for $A \cap B \cap C$.

The 1992 AHSME Examination (AHSME 43)

Problem 1

Write what we are given as

$$6(8x + 10\pi) = 2^2 \cdot 3(4x + 5\pi) = 2^2 P = 4P\,.$$

Problem 2

From the statement that 20% of the objects are beads we know that 80% of the objects must be coins. From the statement that 40% of the coins are silver we know that 60% of the coins are gold. This means that

$$0.8 \times 0.6 = 0.48\,,$$

or 48% of the objects are gold coins.

Problem 3

The equation for m as described is

$$m = \frac{m-3}{1-m} \quad \text{or} \quad m^3 = 3\,.$$

The positive solution is $m = \sqrt{3}$.

Problem 4

The expression 3^a is always odd regardless of the value of a. The expression $b-1$ is even if b is odd so $(b-1)^2$ is even. Then $(b-1)^2c$ is even no mater what c is. Thus the sum

$$3^a + (b-1)^2 c\,,$$

will be an odd number plus an even number and so is odd.

Problem 5

This is $6 \cdot 6^6 = 6^7$.

Problem 6

Write the given expression as

$$\frac{x^y y^x}{y^y x^x} = \frac{x^y}{x^x} \cdot \frac{y^x}{y^y} = x^{y-x} y^{x-y} = \left(\frac{y}{x}\right)^{x-y} = \left(\frac{x}{y}\right)^{y-x} .$$

Problem 7

The given statements mean that

$$\frac{w}{x} = \frac{4}{3} \quad (523)$$
$$\frac{y}{z} = \frac{3}{2} \quad (524)$$
$$\frac{z}{x} = \frac{1}{6} . \quad (525)$$

We want to evaluate $\frac{w}{y}$. If we divide equation 523 by Equation 524 we get

$$\frac{w}{x} \cdot \frac{z}{y} = \frac{4}{3} \cdot \frac{2}{3} ,$$

or

$$\frac{w}{y} \left(\frac{z}{x}\right) = \frac{8}{9} .$$

Using Equation 525 we can write this as

$$\frac{w}{y} = 6 \left(\frac{8}{9}\right) = \frac{16}{3} .$$

This means that $w : y = 16 : 3$.

Problem 8

Considering a $n = 5$ square floor there are 5 black squares on each diagonal and so $2 \times 5 - 1$ black squares in both the diagonals avoiding “double counting” the central square. For a $n \times n$ square floor there will be $2n - 1$ diagonal squares. For this problem this means

$$2n - 1 = 101 \quad \text{so} \quad n = 51 .$$

The total number of tiles is then $n^2 = 2601$.

Problem 9

If we just add up the area of the five equilateral triangle we will double count the area of overlap. If $A(s)$ is the formula for the area of an equilateral triangle with side length s i.e.

$$A(s) = \frac{\sqrt{3}}{4}s^2\,,$$

then as the overlapping triangles are also equilateral triangles with sides of length $s = \frac{2\sqrt{3}}{2} = \sqrt{3}$ the area requested would be

$$5A(2\sqrt{3}) - 4A(\sqrt{3}) = \frac{\sqrt{3}}{4}(5(4\cdot 3) - 4\cdot 3) = 12\sqrt{3}\,,$$

when we simplify

Problem 10

As k and x are integer if we write the given expression as

$$12 = kx - 3k = k(x-3)\,.$$

We see that both k and $x-3$ must be integer factors of 12. As $k \geq 1$ this means that

$$k \in \{1, 2, 3, 4, 6, 12\}\,.$$

We can then compute $x - 3 = \frac{12}{k}$ to find that

$$x - 3 \in \{12, 6, 4, 3, 2, 1\}\,.$$

so the solutions x are

$$x \in \{15, 9, 7, 6, 5, 4\}\,.$$

Thus there are six solutions.

As an *alternative* but similar problem we can be given the equation

$$kx - 12 = 3x\,,$$

and want to know for how many positive integers k does the above have integer solutions for x. Note that in the above the right-hand-side is $3x$ and not $3k$.

To solve this version of the problem we write the above as

$$(k-3)x = 12\,.$$

If $k \neq 3$ then we have

$$x = \frac{12}{k-3}\,.$$

As we are told that we want k to be a positive integer we know that $k \geq 1$. In addition x will not be an integer if

$$k - 3 \geq 13 \quad \text{or} \quad k \geq 16\,.$$

Thus we can just enumerate all of the possible k's between $1 \leq k \leq 15$ (with $k \neq 3$) and see which ones give integer solutions for x. We could do this by hand as there are not that many numbers to compute but using R we have

```
ks = c(1:2, 4:15)
xs = 12/(ks-3)
print(xs)
```

The outputs of this are

```
 [1]  -6.000000 -12.000000  12.000000   6.000000   4.000000   3.000000
 [7]   2.400000   2.000000   1.714286   1.500000   1.333333   1.200000
[13]   1.090909   1.000000
```

From the above we see that x is an integer for eight of these value for k.

Problem 11

Let r be the radius of the small circle and R be the radius of the larger circle. Then we are told that $R = 3r$. Let O be the common circles center and let T be the point of tangent of the segment BC with the smaller circle. Then OT is perpendicular to TC so the Pythagorean theorem gives

$$TC^2 + r^2 = (3r)^2\,,$$

so $TC = \sqrt{8}r$. As T is on the perpendicular bisector of BC we have that $BT = TC = \sqrt{8}r$, so that $BC = 2TC = 2\sqrt{8}r$.

Next note that $\angle ABC = 90$ as B is on the circle and AC is a diameter. Another application of the Pythagorean theorem gives

$$AB^2 + BC^2 = AC^2\,,$$

or using what we know this is

$$12^2 + (2\sqrt{8}r)^2 = (6r)^2\,.$$

If we expand and simplify this we find $r = 6$. This means that $R = 3r = 18$.

Problem 12

The given line goes though the following two points $\left(0, \frac{11}{3}\right)$ and $(-11, 0)$. Reflecting these two points thought the x-axis we see that the point $(-11, 0)$ will not change (it is its own reflection) and the point $\left(0, \frac{11}{3}\right)$ will go to the point $\left(0, -\frac{11}{3}\right)$. This means that the new line must satisfy

$$y - 0 = \left(\frac{-\frac{11}{3} - 0}{0 + 11}\right)(x + 11)\,,$$

or

$$y = -\frac{x}{3} - \frac{11}{3}\,.$$

In this form we see that $m = -\frac{1}{3}$ and $b = -\frac{11}{3}$ so that

$$m + b = -\frac{12}{3} = -4\,.$$

Problem 13

The given expression is equivalent to

$$a^2 + b^{-1} = 13(a^{-1} + b)\,,$$

or multiplying by ab we get

$$a^2 b + a = 13(b + ab^2)\,,$$

which we can write as

$$a(ab + 1) = 13b(1 + ab)\,,$$

or

$$a = 13b\,.$$

Now to have $a + b \leq 100$ means that

$$13b + b \leq 100 \quad \text{so} \quad b \leq \frac{100}{14} = 7.14286\,.$$

For b to be a positive integer in this range means that $b \in \{1, 2, 3, 4, 5, 6, 7\}$. Thus there are seven solutions.

Problem 14

The expression in I is the equation for of line. The expression II is equivalent to the line in I but not at the point $x = -2$ where the right-hand-side is undefined. Thus II is the same as the line in I but with the point $(-2, -4)$ from that line "removed". When $x \neq -2$ the expression III is equivalent to II but at $x = -2$ the value of y can be anything and the expression evaluates is true. Thus none of these three are exactly equivalent.

Problem 15

From the given relationship we have that

$$
\begin{aligned}
z_1 &= 0\\
z_2 &= i\\
z_3 &= -1+i\\
z_4 &= (-1+i)^2 + i = 1 - 2i - 1 + i = -i\\
z_5 &= (-i)^2 + i = -1 + i = z_3\\
z_6 &= z_4 = -i\\
z_7 &= z_5 = z_3\,.
\end{aligned}
$$

Thus we see that we have $z_{2n+1} = z_3$ for $n \geq 1$ and $z_{2n} = -i$ for $n \geq 2$. As $2n + 1 = 111$ when $n = 55$ we have that $z_{111} = -1 + i$. Thus $|z_{111}| = \sqrt{1^2 + 1^2} = \sqrt{2}$.

Problem 16

Let $u = \frac{x}{y}$ and $v = \frac{z}{y}$ then write the given expression as

$$
\frac{1}{u-v} = \frac{u+1}{v} = u\,.
$$

These are two equations in the two unknowns u and v given by

$$
\begin{aligned}
\frac{1}{u-v} &= u\\
\frac{u+1}{v} &= u\,.
\end{aligned}
$$

We can write these as

$$
\begin{aligned}
1 &= u^2 - uv\\
u + 1 &= uv\,.
\end{aligned}
$$

If we put the second equation into the first we get

$$
u^2 - u - 2 = 0\,.
$$

Solving this we get $u \in \{-1, 2\}$. As we are told that x and y are positive we have that $u = 2$.

Problem 17

We can sum the digits of the number N and see if N is divisible by three or nine. Summing the digits requires us to keep track of the "pattern" in how the the digits of the number are produced. Note that there is a 19 followed by a "pattern" and then followed by the digits

909192. Note that for the groupings in the middle there is a simplified way to express the sum of the digits. For example when we put down the numbers

$$20\,,21\,,22\,,\cdots,29\,,$$

the sum of the digits will be

$$2\times 10+\sum_{1=0}^{9}1=2\times 10+\frac{9(10)}{2}=2\times 10+45\,.$$

This observation can help sum the digits in the "middle" groups.

Thus grouping the terms like suggested above the sum of the digits in N (called D) can be written as

$$\begin{aligned}De &= 1+9+\sum_{d=2}^{8}(d\times 10+45)+(9(3)+(0+1+2))\\ &= 10+27+3+10\sum_{d=2}^{8}d+45\times(8-2+1)\\ &= 40+315+10(35)=705\,.\end{aligned}$$

The sum of the digits in this number is 12 indicating that our number N is divisible by three (but not nine), thus k cannot be larger than two and so $k=1$.

Problem 18

Working "backwards" we find that

$$\begin{aligned}a_7 &= 120=a_5+a_6\\ &= 2a_5+a_4=2(a_4+a_3)+a_4=3a_4+2a_3\\ &= 3(a_3+a_2)+2a_3=5a_3+3a_2\\ &= 5(a_2+a_1)+3a_2\\ &= 8a_2+5a_1\,.\end{aligned}$$

Now a_1 and a_2 are positive integers with $a_2>a_1$. Because of that let $a_2=a_1+A$ for $A\geq 1$. Then we have

$$120=8(a_1+A)+5a_1=13a_1+8A\,. \tag{526}$$

From the fact that $a_1\geq 1$ and $A\geq 1$ we have that

$$a_1\leq\frac{120-8}{13}=8.61538\,,$$

and

$$A\leq\frac{120-13}{8}=13.375\,.$$

From the above if we take $a_1 \in \{1, 2, 3, 4, 5, 6, 7, 8\}$ we can solve for A using Equation 526 to get

$$A = \frac{120 - 13a_1}{8}.$$

Doing this we can see how many solutions give A a positive integer. For the range of a_1 above I find the corresponding A given by

```
[1] 13.375 11.750 10.125  8.500  6.875  5.250  3.625  2.000
```

Thus only $a_1 = 8$ gives an integer for A of $A = 2$. This means that working "forward" we have

$$\begin{aligned}
a_2 &= a_1 + A = 8 + 2 = 10 \\
a_3 &= a_2 + a_1 = 10 + 8 = 18 \\
a_4 &= 18 + 10 = 28 \\
a_5 &= 28 + 18 = 46 \\
a_6 &= 46 + 28 = 74 \\
a_7 &= 74 + 46 = 120 \\
a_8 &= 120 + 74 = 194\,,
\end{aligned}$$

for the value of a_8 asked for.

Problem 19

Let the the original cube have an edge length of s and imagine cutting off a "corner" of the cube at the midpoints of s. This corner is a tetrahedron with three equal faces and an unequal "base". In the tetrahedron that is cut off three of its "edge" lengths will equal $\frac{s}{2}$. The "base" triangle of this tetrahedron will be an equilateral triangle with lengths a given by

$$a^2 = \left(\frac{s}{2}\right)^2 + \left(\frac{s}{2}\right)^2,$$

since the corners of each face of the square meet at right angles. This means that $a = \frac{s}{\sqrt{2}}$.

Using the formula for the area of an equilateral triangle with edge length of a given by

$$\frac{\sqrt{3}}{4}a^2,$$

we get that the "base" of one of the tetrahedron has an area A_0 of

$$A_0 = \frac{\sqrt{3}s^2}{8}.$$

To determine the volume of this tetrahedron we need to determine its height.

To determine this tetrahedron's height we drop a perpendicular from the corner cube vertex to the equilateral base triangle. Then using symmetry, this height should intersect the equilateral triangle at the common angle bisectors (the incenter) and the common perpendicular bisectors (the circumcenter). This height h forms one leg of a triangle with hypotenuse of length $\frac{s}{2}$. The other leg of this right triangle involving the height h is the hypotenuse of the right triangle in the base equilateral triangle with a corner angle of

$$\frac{60}{2} = 30^\circ ,$$

and a leg length of

$$\frac{1}{2}\left(\frac{s}{\sqrt{2}}\right) = \frac{s}{2\sqrt{2}} .$$

This means that the hypotenuse from the corner of the base to the circumcenter/incenter of the base has a length of

$$\frac{\frac{s}{2\sqrt{2}}}{\cos(30^\circ)} = \frac{s}{\sqrt{6}} .$$

Using this we can determine the length of the height using the Pythagorean theorem as

$$h^2 = \left(\frac{s}{2}\right)^2 - \left(\frac{s}{\sqrt{6}}\right)^2 = \frac{s^2}{12} .$$

The volume V of one tetrahedron is now given by

$$V = \frac{1}{3}A_0 h = \frac{1}{3}\left(\frac{\sqrt{3}s^2}{8}\right)\left(\frac{s}{\sqrt{12}}\right) = \frac{s^3}{48} .$$

There are eight tetrahedron's with this volume for a total volume of the cuboctahedron given by

$$s^3 - 8\left(\frac{s^3}{48}\right) = \frac{5s^3}{6} .$$

The fraction of the original volume (of s^3) this represents from the original cube is then $\frac{5}{6} = 0.833333$.

Problem 20

Connect the "points" A_1 to A_2, A_3 to A_4, A_4 to A_5 and so on until you have connected the "tops" of the spikes forming a regular n sided regular polygon. Recall that the sum of the interior angles of any polygon is

$$180(n-2) ,$$

this means that for a regular n sided polygon each interior angle has a measure of

$$\frac{180(n-2)}{n} .$$

Treating the angle names as "points" note that in the triangle $\triangle A_1B_1A_2$ since $B_1A_1 = B_1A_2$ we have an isosceles triangle with a vertex angle $B_1 = \angle A_1B_1A_2$. Let α be the measure of one of these base angles. Then

$$2\alpha + A_1 = \frac{180(n-2)}{n}\,. \tag{527}$$

Using the fact that the triangle $\triangle A_1B_1A_2$ is an isosceles triangle we have

$$2\alpha + B_1 = 180\,.$$

so that

$$\alpha = \frac{180 - B_1}{2}\,.$$

If we put this into Equation 527 we get

$$180 - B_1 + A_1 = \frac{180(n-2)}{n}\,. \tag{528}$$

We are told that $B_1 - 10 = A_1$ so $-B_1 + A_1 = -10$. Putting this into Equation 528 gives

$$-10 + 180 = \frac{180(n-2)}{n} = 180 - \frac{360}{n}\,.$$

Solving this for n gives $n = 36$.

Problem 21

From the definition of the Cesàro sum of A we are told that

$$\frac{S_1 + S_2 + S_3 + \cdots + S_{98} + S_{99}}{99} = 1000\,,$$

or using the definition of the partial sums we have

$$\frac{99a_1 + 98a_2 + 97a_3 + \cdots + 2a_{98} + a_{99}}{99} = 1000\,.$$

The sum we are asked to evaluate can be written as

$$\begin{aligned}\frac{100(1) + 99a_1 + 98a_2 + 97a_3 + \cdots + 2a_{98} + a_{99}}{1000} &= 1 + \frac{99}{100}\left(\frac{99a_1 + 98a_2 + 97a_3 + \cdots + 2a_{98} + a_{99}}{99}\right) \\ &= 1 + \frac{99}{100}(1000) = 991\,.\end{aligned}$$

Problem 23

Consider the sets of elements from the given set that have a remainder of $0, 1, 2, 3, 4, 5, 6$ when divided by seven Using the `python` code

```
for remainder in range(0, 7):
    print(f'remainder= {remainder}:', end='')
    elts_in_set = []
    for elt in range(1, 51):
        if elt % 7 == remainder:
            elts_in_set.append(elt)
    print(f'{len(elts_in_set):2d} elements= {elts_in_set}')
```

we get

```
remainder= 0: 7 elements= [7, 14, 21, 28, 35, 42, 49]
remainder= 1: 8 elements= [1, 8, 15, 22, 29, 36, 43, 50]
remainder= 2: 7 elements= [2, 9, 16, 23, 30, 37, 44]
remainder= 3: 7 elements= [3, 10, 17, 24, 31, 38, 45]
remainder= 4: 7 elements= [4, 11, 18, 25, 32, 39, 46]
remainder= 5: 7 elements= [5, 12, 19, 26, 33, 40, 47]
remainder= 6: 7 elements= [6, 13, 20, 27, 34, 41, 48]
```

The benefit of this decomposition is that the sum of an element from the set (that has a remainder of i when divided by seven) with an element in the set with remainder of j will have a remainder of

$$(i + 3) \bmod 7 \,.$$

This means that we can take all elements of the sets above such that the pairwise sum of the remainders is not divisible by seven. As all but one of the sets above has seven elements to get the maximal set we should take the set with a remainder of one (giving eight elements). Then we cannot take any point in the set with a remainder of six. We can then take all elements in the sets with remainders two and three giving a set of size

$$8 + 7 + 7 = 22 \,.$$

To this we can add a single element from the set with a remainder of zero to give a 23 element set.

Problem 24

To start, let the parallelogram $ABCD$ be drawn in the x-y Cartesian coordinate system with the "base" segment AD along the x-axis and the segment BC above AD and shifted rightwards. Towards that end we can place $A = (0, 0)$ and $D = (5, 0)$. The coordinates of the other points will be determined shortly.

From the given area of the parallelogram $ABCD$ we have that the height h must be equal to

$$h = \frac{10}{AD} = \frac{10}{5} = 2 \,.$$

If we drop a perpendicular from B to the x-axis (and denote that intersection with the x-axis as the point B') it will be of length h and we can determine the length AB' using the Pythagorean theorem as

$$AB^2 = AB'^2 + B'B^2 \quad \text{or} \quad 3^2 = AB'^2 + 2^2 \,,$$

so that $AB' = \sqrt{5}$. This means that B is located at $B = (\sqrt{5}, 2)$. In addition

$$\begin{aligned} \cos(\angle BAB') &= \frac{\sqrt{5}}{3} \\ \sin(\angle BAB') &= \frac{2}{3} \,. \end{aligned}$$

The point E is on the same line as AB but two units from A. Using the angle $\angle BAB'$ its location is given by

$$E = (2\cos(\angle BAB'), 2\sin(\angle BAB')) = \left(\frac{2\sqrt{5}}{3}, \frac{4}{3} \right) \,.$$

The point C is located five units to the right of B or at

$$C = B + (5, 0) = (\sqrt{5} + 5, 2) \,.$$

The point F is located two units to the right of B or at

$$F = B + (2, 0) = (\sqrt{5} + 2, 2) \,.$$

The point G is located two units to the right of A or at

$$G = A + (2, 0) = (2, 0) \,.$$

Finally, the point H is located five units from E or

$$H = \left(5 + \frac{2\sqrt{5}}{3}, \frac{4}{3} \right) \,.$$

Now in the x-y Cartesian coordinate plane the area of a triangle denoted by the three points E, F, and G is given by

$$\frac{1}{2} \left| \left| \overrightarrow{EG} \times \overrightarrow{EF} \right| \right| \,,$$

which is the norm of the vector cross product of the two vectors in the plane. Here I have "constructed" the cross product so that its value will be positive (pointing out of the page). The vectors needed for this computation are

$$\begin{aligned} \overrightarrow{EG} &= (2, 0) - \left(\frac{2\sqrt{5}}{3}, \frac{4}{3} \right) = \left(2 - \frac{2\sqrt{5}}{3}, -\frac{4}{3} \right) \\ \overrightarrow{EF} &= (\sqrt{5} + 2, 2) - \left(\frac{2\sqrt{5}}{3}, \frac{4}{3} \right) = \left(2 + \frac{\sqrt{5}}{3}, \frac{2}{3} \right) \,. \end{aligned}$$

Next we compute

$$\overrightarrow{EG} \times \overrightarrow{EF} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 - \frac{2\sqrt{5}}{3} & -\frac{4}{3} & 0 \\ 2 + \frac{\sqrt{5}}{3} & \frac{2}{3} & 0 \end{vmatrix} = \hat{k}\left(\frac{2}{3}\left(2 - \frac{2\sqrt{5}}{3}\right) + \frac{4}{3}\left(2 + \frac{\sqrt{5}}{3}\right)\right) = 4\hat{k}\,.$$

when we simplify. This means that the area of the triangle $\triangle EGF$ is $\frac{4}{2} = 2$.

In the same way the area of the triangle $\triangle GHF$ is given by

$$\frac{1}{2}\left|\left|\overrightarrow{GH} \times \overrightarrow{GF}\right|\right| \,,$$

The vectors needed for this computation are

$$\overrightarrow{GH} = \left(3 + \frac{2\sqrt{5}}{3}, \frac{4}{3}\right)$$
$$\overrightarrow{GF} = (\sqrt{5} + 2, 2) - (2, 0) = (\sqrt{5}, 2)\,.$$

Next we compute

$$\overrightarrow{GH} \times \overrightarrow{GF} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 + \frac{2\sqrt{5}}{3} & \frac{4}{3} & 0 \\ \sqrt{5} & 2 & 0 \end{vmatrix} = 6\hat{k}\,.$$

when we simplify. This means that the area of the triangle $\triangle GHF$ is $\frac{6}{2} = 3$.

The total area of the quadrilateral $EFHG$ is the sum of the area of these two triangles or $2 + 3 = 5$.

Problem 25

To start, let place the segments in an x-y Cartesian coordinate system with the "base" segment BC on the x-axis. We can take $B = (0, 0)$ and $C = (4, 0)$. From the given angle $\angle ABC$ the point A is located at

$$A = (3\cos(120°), 3\sin(120°)) = \left(-\frac{3}{2}, \frac{3\sqrt{3}}{2}\right)\,.$$

Now the slope of the segment AD will be $-\frac{1}{m}$ where m is the slope of the segment AB. This slope is given by

$$m = \frac{\frac{3\sqrt{3}}{2} - 0}{-\frac{3}{2} - 0} = -\sqrt{3}\,.$$

This means that the slope of the segment AD will be $\frac{1}{\sqrt{3}}$ and the line obtained by extending AD takes the form

$$y = \frac{1}{\sqrt{3}}\left(x + \frac{3}{2}\right) + \frac{3\sqrt{3}}{2}\,.$$

As the perpendicular to BC thought the point C will have $x = 4$ fixed the location of the point D will be $D = (4, y)$ with y given by the value of the line AD evaluated at $x = 4$. I find that value of y to be $y = \frac{10\sqrt{3}}{3}$ and the point D is thus

$$D = \left(4, \frac{10\sqrt{3}}{3}\right).$$

Using the distance formula we can then compute that the distance CD is

$$\frac{10\sqrt{3}}{3} = \frac{10}{\sqrt{3}}.$$

Problem 26

The area of the "smile" (denoted S) is given by adding areas of various parts and subtracting areas of parts we are not interested in. For example we have

$$\begin{aligned} S &= \text{Area of circular sector } BEA - \text{Area of circular sector } BDA \\ &+ \text{Area of circular sector } AFB - \text{Area of circular sector } ADB \\ &+ \text{Area of circular sector } EDF. \end{aligned}$$

We can compute these in tern. First note that

$$BE = AB = AF = 2,$$

and

$$AC = BC = CD = 1.$$

Now as $CD \perp AB$ using the Pythagorean theorem we have that

$$DB^2 = BC^2 + DC^2 = 1^2 + 1^2 = 2,$$

thus $DB = \sqrt{2}$. This means that

$$DE = BE - BD = 2 - \sqrt{2}.$$

As $CD = BC = AC$ we have $S\angle ABD = \angle BAD = 45°$ finally $\angle ADB = 90°$. Using all of this and with symmetry we have

$$\begin{aligned} \text{Area of circular sector } BEA &= \text{Area of circular sector } AFB \\ &= \left(\frac{45}{360}\right)\pi \times 2^2 = \frac{\pi}{2}, \end{aligned}$$

and

$$\begin{aligned} \text{Area of circular sector } BDA &= \text{Area of circular sector } ADB \\ &= \text{Area of right triangle } BCD\text{S} + \text{Area of circular sector } CDA \\ &= \frac{1}{2}BC \times CD + \frac{1}{4}\pi CD^2 \\ &= \frac{1}{2} + \frac{\pi}{4}. \end{aligned}$$

Finally we have

$$\begin{aligned}\text{Area of circular sector } EDF &= \left(\frac{90}{360}\right)\pi DE^2 \\ &= \frac{\pi}{4}(2-\sqrt{2})^2\,.\end{aligned}$$

All of this together gives

$$S = 2\left(\frac{\pi}{2}\right) - 2\left(\frac{1}{2}+\frac{\pi}{4}\right) + \frac{\pi(2-\sqrt{2})^2}{4} = 2\pi - \pi\sqrt{2} - 1\,,$$

when we simplify

Problem 27

Using the "Power of a Point Theorem (Case 1)" (see below for a proof) we have that

$$PA \cdot PB = PD \cdot PC\,,$$

or using what we know this is

$$18 \cdot 8 = (PC + 7)PC\,.$$

Solving this for PC we find the only positive solution is $PC = 9$. If we are lucky enough to then note that $PA = PB + BA = 18 = 2PC$ and $\angle APC = 60°$ we have that $\angle PCA = 90°$ so that triangle $\triangle PCA$ is a right triangle. This means that we can compute AC and find

$$AC = 18\sin(60°) = 9\sqrt{3}\,.$$

Now as $\angle PCA = 90°$ we have $\angle ACD = 90°$ so AD must be a diameter of the circle. Using the fact that $CD = 7$ and the Pythagorean theorem we have

$$AD^2 = AC^2 + CD^2 = (9\sqrt{3})^2 + 7^2 = 292\,.$$

As $AD = 2r$ we have $AD^2 = 4r^2$ so $r^2 = \frac{292}{4} = 73$.

A Proof of the Power of a Point Theorem:

Here we prove this theorem. A small write up (with diagrams that match the descriptions below) is given

`https://artofproblemsolving.com/wiki/index.php/Power_of_a_Point_Theorem`

Case 1: The interior intersection of two chords: Consider a circle with an interior point at E. Through this point E draw two chords. Let the first chord run "North-East" from points C to A. Let the second chord run "Eastwards" from points D to B. When these points are drawn moving clockwise around the circle we have the points $ABCD$ with the segments AC and BD intersecting at E. Draw the chords AD and BC. Then we have that

$$\triangle DEA \sim \triangle CEB\,.$$

This is because angles $\angle DEA = \angle CEB$ (opposite angles are equal) and $\angle DAC = \angle DBC$ as they both equal $\frac{1}{2}\widehat{DC}$. Because of this similarity we have

$$\frac{DE}{AE} = \frac{CE}{EB} \quad \text{so} \quad DE \cdot EB = AE \cdot EC\,,$$

which is a second version of the power of a point theorem.

Case 2: The exterior intersection of a tangent and a secant: Consider a circle with an external point at B (say "above" the circle). From this point B draw a tangent and a secants making an inverted "V". Let the left-most segment be the tangent segment intersecting the circle at a point A and let the right-most secant intersect the circle at the points C and then D so that moving from left to right we have the points $ABCD$. Draw the chords AC and AD.

Note that we can relate the angles $\angle BDA$ and $\angle BAC$ though the arc $\widehat{AC}$ as

$$\angle BDA = \frac{1}{2}\widehat{AC} = \angle BAC\,. \tag{529}$$

Using Equation 529 for one angle equivalence and the fact that $\angle B$ is common in both triangles we have that $\triangle CAB \sim \triangle ADB$. Using this we have

$$\frac{BC}{AB} = \frac{AB}{BD} \quad \text{or} \quad AB^2 = BC \cdot BD\,,$$

as we were to show.

Case 3: The exterior intersection of two secants: Consider a circle with an external point at C (say "above" the circle). From this point C draw two secants making an inverted "V". Let the left-most intersections with the circle be the points A and B and the right-most intersections with the circle be the points D and E so that moving from left to right we have the points $ABCDE$. Draw the chords AD and BE. Then we have that

$$\triangle ACD \sim \triangle ECA\,.$$

This is because $\angle C$ is common between the two triangles and $\angle CAD = \angle CEB$ as they both equal $\frac{1}{2}\widehat{BD}$. Because of this similarity we have

$$\frac{AC}{CE} = \frac{CD}{CB} \quad \text{so} \quad AC \cdot CB = CE \cdot CD\,,$$

which is one version of the power of a point theorem.

Problem 28

We can write this quadratic as

$$z^2 - z - (5 - 5i) = 0\,.$$

Using the quadratic formula gives roots of

$$z = \frac{1 \pm \sqrt{1 + 4(5 - 5i)}}{2} = \frac{1 \pm \sqrt{21 - 20i}}{2}\,.$$

We need to evaluate the square root of the complex number $21 - 20i$ above. To do that we will write it in polar as

$$21 - 20i = \sqrt{21^2 + 20^2}e^{-i\theta} = \sqrt{841}e^{-i\theta} = 29e^{-i\theta}\,,$$

where

$$\tan(\theta) = \frac{20}{21}\,. \tag{530}$$

This means that

$$\sqrt{21 - 201i} = \sqrt{29}e^{-i\theta/2} = \sqrt{29}(\cos(\theta/2) - i\sin(\theta/2))\,.$$

Our two roots are thus

$$z = \frac{1 \pm \sqrt{29}(\cos(\theta/2) - i\sin(\theta/2))}{2}\,.$$

The product of the two real parts of the roots is given by

$$\frac{1}{4}(1 + \sqrt{29}\cos(\theta/2))(1 - \sqrt{29}\cos(\theta/2)) = \frac{1}{4}(1 - 29\cos^2(\theta/2))\,. \tag{531}$$

Now Equation 530 indicates that θ is an acute angle in a right triangle with legs 20, 21, and a hypotenuse of $\sqrt{20^2 + 21^2} = 29$ thus we have

$$\cos(\theta) = \frac{21}{29}\,.$$

Next using this in the identity

$$\cos^2(\theta) = \frac{1}{2}(1 + \cos(2\theta))\,,$$

when written as

$$\cos^2(\theta/2) = \frac{1}{2}(1 + \cos(\theta))\,,$$

we find

$$\cos^2(\theta/2) = \frac{50}{58}\,.$$

Using this in Equation 531 we get the desired product given by

$$\frac{1}{4}\left(1 - 29\left(\frac{50}{58}\right)\right) = -6\,.$$

Problem 29

The probability we get h heads in 50 flips is given by

$$\binom{50}{h}\left(\frac{2}{3}\right)^h\left(\frac{1}{3}\right)^{50-h},$$

for $0 \le h \le 50$. The probability we get an even number of flips is then the sum

$$e = \sum_{h \text{ is even}} \binom{50}{h}\left(\frac{2}{3}\right)^h\left(\frac{1}{3}\right)^{50-h}.$$

The probability we get a odd number of flips would be the sum

$$o = \sum_{h \text{ is odd}} \binom{50}{h}\left(\frac{2}{3}\right)^h\left(\frac{1}{3}\right)^{50-h}.$$

We must have $e + o = 1$. Consider the value of $e - o$. We have

$$\begin{aligned}
e - o &= \sum_{0 \le k \le 50 \text{ and even}} \binom{50}{k}\left(\frac{2}{3}\right)^k\left(\frac{1}{3}\right)^{50-k} - \sum_{1 \le k \le 49 \text{ and odd}} \binom{50}{k}\left(\frac{2}{3}\right)^k\left(\frac{1}{3}\right)^{50-k} \\
&= \sum_{0\le k\le 50} \binom{50}{k}(-1)^k\left(\frac{2}{3}\right)^k\left(\frac{1}{3}\right)^{50-k} \\
&= \sum_{0\le k\le 50} \binom{50}{k}\left(-\frac{2}{3}\right)^k\left(\frac{1}{3}\right)^{50-k}.
\end{aligned}$$

Using the binomial theorem we can write this as

$$\left(-\frac{2}{3}+\frac{1}{3}\right)^{50} = \left(-\frac{1}{3}\right)^{50} = \frac{1}{30}^{50}.$$

Thus we have that

$$\begin{aligned}
e + o &= 1 \\
e - o &= \frac{1}{30^{50}}.
\end{aligned}$$

If we solve these for e by adding these two equations we get

$$e = \frac{1}{2}\left(1 + \frac{1}{30^{50}}\right).$$

Problem 30

Here I draw the segment AB along the x-axis of a Cartesian coordinate system with $A = (-46, 0)$ and $B = (46, 0)$. The segment DC is "above" the segment AB with $D = (-9.5, h)$ and $C = (9.5, h)$ where h is the isosceles trapezoids height (and is currently unknown).

By symmetry, the center of the circle will need to be located at the center of the segment AB which in the way I have drawn this figure is located at $O = (0, 0)$.

From the center of this circle draw a segment to the side BC of the isosceles trapezoid that is tangent to the side BC. Call that point of intersection P so that this segment is OP. The slope of the segment BC is given by

$$m_{BC} = \frac{h - 0}{9.5 - 46} = -\frac{2h}{73}.$$

As OP is tangent to BC at P the slope of the segment OP is

$$-\frac{1}{m_{BC}} = \frac{73}{2h}.$$

This means that the "line" OP is given by the equation

$$y = \frac{73}{2h}x.$$

The "line" BC is given by the equation

$$y - 0 = -\frac{2h}{73}(x - 46).$$

Next we determine where this line intersects the segment CB i.e. what are the coordinates of the point P. Thus we need to solve

$$\frac{73}{2h}x = -\frac{2h}{73}(x - 46).$$

Solving for x I find

$$x = \frac{184h^2}{4h^2 + 73^2}, \tag{532}$$

so that y is given by

$$y = \frac{6716h}{4h^2 + 73^2}. \tag{533}$$

Now in order that the length of AD (and BC) be as small as possible we need to have $P = C$ that is the point of tangent is the point C. Equating the y-coordinates of these two points gives

$$\frac{6716h}{4h^2 + 73^2} = h.$$

Solving this for h^2 gives

$$h = \frac{\sqrt{1387}}{2}.$$

Putting this value of h back into Equations 532 and 533 gives $x = \frac{19}{2}$ and $y = h$ as it should.

The minimum distance (squared) of the length BC is then given by

$$\left(\frac{19}{2} - 46\right)^2 + \left(\frac{\sqrt{1387}}{2} - 0\right)^2 = 1679,$$

when we simplify.

The 1993 AHSME Examination (AHSME 44)

Problem 1

For this we compute

$$E = a^b - b^c + c^a = 1^{-1} - (-1)^2 + 2^1 = 1 - 1 + 2 = 2\,.$$

Problem 2

We compute

$$\angle B = 180 - 55 - 75 = 50\,.$$

Using the fact that $DB = BE$ we have that $\triangle BDE$ is isosceles so that

$$\angle BDE = \frac{180 - \angle B}{2} = \frac{130}{2} = 65\,.$$

Problem 3

Call this expression E then we have

$$E = \frac{15^{30}}{45^{15}} = \frac{3^{30} \cdot 5^{30}}{5^{15} \cdot 9^{15}} = \frac{3^{30} \cdot 5^{15}}{30^{30}} = 5^{15}\,.$$

Problem 4

If we evaluate

$$3 \circ y = 4 \cdot 3 - 3y + 3y = 12\,,$$

so $3 \circ y = 12$ is satisfied for all y.

Problem 5

Last year the combined cost was

$$C_0 = 160 + 40 = 200\,.$$

This year the combined cost is

$$C_1 = 1.05(160) + 1.1(40) = 212\,.$$

The percentage increase in cost is then

$$\frac{C_1 - C_0}{C_0} = \frac{C_1}{C_0} - 1 = 1.06 - 1 = 0.06\,,$$

or 6%.

Problem 6

Call this expression E. Then we have

$$\begin{aligned} E &= \sqrt{\frac{8^{10} + 4^{10}}{8^4 + 4^{11}}} = \sqrt{\frac{2^{30} + 2^{20}}{2^{12} + 2^{22}}} = \sqrt{\frac{2^{20}(2^{10} + 1)}{2^{12}(1 + 2^{10})}} \\ &= \sqrt{2^8} = 2^4 = 16\,. \end{aligned}$$

Problem 7

Note that from how R_k is defined we have

$$\begin{aligned} R_3 &= 100 + 10 + 1 \\ R_4 &= 10^3 + 10^2 + 10 + 1\,, \end{aligned}$$

etc. Thus the expression we see to evaluate can be written as

$$Q = \frac{R_{24}}{R_4} = \frac{\sum_{k=0}^{23} 10^k}{\sum_{k=0}^{3} 10^k}\,.$$

We can write this as

$$\sum_{k=0}^{23} 10^k = Q \sum_{k=0}^{3} 10^k = Q(10^3 + 10^2 + 10 + 1)\,.$$

Expanding the left-hand-side term-by-term we get

$$10^{23} + 10^{22} + 10^{21} + \cdots + 10^3 + 10^2 + 10 + 1 = Q(10^3 + 10^2 + 10 + 1)\,.$$

In this form it is easier (I think) to determine the form of Q. We see that Q should look like

$$Q = 10^{20} + 10^{16} + 10^{12} + 10^8 + 10^4 + 1\,.$$

Just to be complete we can "check" this by multiplying it by $10^3 + 10^2 + 10 + 1$ to get

$$\begin{aligned} Q(10^3 + 10^2 + 10 + 1) &= (10^{23} + \cdots + 10^{20}) + (10^{19} + \cdots + 10^{16}) + (10^{15} + \cdots + 10^{12}) \\ &+ (10^{11} + \cdots + 10^8) + (10^7 + \cdots + 10^4) + (10^3 + \cdots + 1)\,, \end{aligned}$$

which is the left-hand-side of the above.

From this expression for Q note that it is 21 digits long with six ones and so has $21 - 6 = 15$ zeros.

Problem 9

Let W be the wealth of "the world". Then the wealth of A and B are

$$W_A = \left(\frac{d}{100}\right) W$$
$$W_B = \left(\frac{f}{100}\right) W \, .$$

Let N the the worlds population. Then the population of A and B are given by

$$N_A = \left(\frac{c}{100}\right) N$$
$$N_B = \left(\frac{e}{100}\right) N \, .$$

The wealth of a citizen of A and B are then given by

$$c_A = \frac{W_A}{N_A} = \frac{dW}{cN}$$
$$c_B = \frac{W_B}{N_B} = \frac{fW}{eN} \, .$$

Then the ratio requested is given by

$$\frac{c_A}{c_B} = \frac{de}{cf} \, .$$

Problem 10

We are told that $r = (3a)^{3b}$ and also that $r = a^b x^b$. If we set these two equal to each other we get

$$[(3a)^3]^b = [ax]^b \, ,$$

or

$$3^3 a^3 = ax \quad \text{so} \quad x = 27a^2 \, .$$

Problem 11

Taking the 2^x of both sides gives

$$\log_2(\log_2(x)) = 2^2 = 4 \, .$$

Doing this again gives

$$\log_2(x) = 2^4 = 16 \, .$$

Doing this again gives

$$x = 2^{16} \, .$$

Note that $2^{10} = 1024$ and $2^6 = 32$ so $2^{16} = 1024 \times 32 = 32768$ which has five digits.

Problem 12

If we take $v = 2x$ in the expression we are given we have $x = \frac{v}{2}$ and we get

$$f(v) = \frac{2}{2 + \frac{v}{2}} = \frac{4}{4 + v}.$$

This means that

$$2f(x) = \frac{8}{4 + x}.$$

Problem 13

For this problem for the smaller square to be inscribed in the larger square I imagine the larger square in the first quadrant of a Cartesian plane (so that its bottom left corner is at the origin) and the smaller square then has to be "tilted" or "rotated" so that its corners touch the edges of the larger square.

Now the larger square has a side of length $S = \frac{28}{4} = 7$ and the smaller one a side of length $s = \frac{20}{4} = 5$. Let one of the corners of the smaller square be located at $(x, 0)$ along the bottom edge of the larger square. Then by symmetry the corners of the inner square will divide the edges of the larger square (walking counter clockwise around the outer square) into the lengths

$$x\,, 7 - x\,, x\,, 7 - x\,, x\,, 7 - x\,, x\,, 7 - x\,.$$

This means that we form a small right triangle in the corner of the coordinate axis with vertices $(0, 0)$, $(x, 0)$ and $(0, 7 - x)$. Thus using the Pythagorean theorem we have that

$$x^2 + (7 - x)^2 = s^2 = 5^2\,.$$

Expanding and simplifying this we get

$$(x - 3)(x - 4) = 0 \quad \text{so} \quad x \in \{3, 4\}\,.$$

Thus

$$7 - x \in \{4, 3\}\,.$$

These two solutions are equivalent in the distances their vertices would be to the vertices of the outer square. Thus without loss of generality lets take $x = 3$. If we draw that square, the distance that would be the largest between two vertices would be either the distance between the points $(0, 4)$ and $(7, 7)$ or

$$d_1 = \sqrt{(0 - 7)^2 + (4 - 7)^2} = \sqrt{58}\,,$$

or the distance between the points $(0, 4)$ and $(7, 0)$ or

$$d_2 = \sqrt{(0 - 7)^2 + (4 - 0)^2} = \sqrt{65}\,.$$

This second number is the largest.

Problem 14

To solve this problem we will compute the area of the quadrilateral $EABC$ and then add it to the area of the triangle DEC.

For the first part we start by drawing a line segment connecting E and C. Then drop a vertical from E and C onto the line that connects A and B. Let the vertical from E intersect the line connecting A and B at a point E' and the vertical from C intersect the line connecting A and B at a point C'.

As we know that $\angle B = 120$ the supplementary angle is $180 - 60 = 120$. This means that

$$E'A = BC' = 2\cos(60) = 1$$
$$EE' = CC' = 2\sin(60) = \sqrt{3}\,.$$

Using these the rectangle $EE'C'C$ has an area of

$$CC' \cdot EE' = \sqrt{3}(E'A + AB + BC') = \sqrt{3}(1 + 2 + 1) = 4\sqrt{3}\,.$$

From this we want to subtract the area of the two triangles $\triangle EE'A$ and $\triangle CC'B$ each of which is

$$\frac{1}{2}AE' \cdot EE' = \frac{1}{2}(1)(\sqrt{3}) = \frac{\sqrt{3}}{2}\,.$$

Using this the area of the quadrilateral $EABC$ is

$$4\sqrt{3} - 2\frac{\sqrt{3}}{2} = 3\sqrt{3}\,.$$

Next we need to compute the area of the triangle $\triangle ECD$. In this triangle the base EC is of length four and the sides are of length four. This is thus an equilateral triangle and has an area of

$$\frac{\sqrt{3}}{4}(4^2) = 4\sqrt{3}\,.$$

The area of the total figure is then

$$3\sqrt{3} + 4\sqrt{3} = 7\sqrt{3}\,.$$

Problem 15

The sum of all the interior angles in an n-sided polygon is

$$180(n-2)\,.$$

For a regular n-sided polygon each angle will be

$$\theta = \frac{180(n-2)}{n}\,.$$

We want to know for how many values of n will this be an integer. We can write the above as

$$\theta = 180 - \frac{360}{n}.$$

For θ to be an integer means that $\frac{360}{n}$ must be an integer. If we factor 360 we get

$$360 = 2^3 \cdot 3^2 \cdot 5^1.$$

The integer values of n that we can divide 360 by and get an integer then take the form $n = 2^p \cdot 3^q \cdot 5^r$ for $0 \le p \le 3$, $0 \le q \le 2$, and $0 \le r \le 1$. This is

$$4 \times 3 \times 2 = 24,$$

different values for n. As we need $n > 3$ we can't have $n = 2$ or $n = 3$ and thus find $24 - 2 = 22$ solutions.

Problem 16

Let s_n be the location in the sequence where the first n is written. Then we have that

$$s_n = \sum_{k=1}^{n-1} k + 1 = \frac{1}{2}(n-1)n + 1 = \frac{1}{2}(n^2 - n + 2).$$

Lets evaluate this for some simple n. We have

$$\begin{aligned} s_1 &= 1 \\ s_2 &= 2 \\ s_3 &= 4 \\ s_4 &= 7, \end{aligned}$$

all of which are correct. We write the first n at the location s_n and then write $n-1$ additional n's afterwards.

To find what the 1993rd term is we first find the largest n such that

$$s_n \le 1993,$$

or

$$\frac{1}{2}(n^2 - n - 2) \le 1993,$$

or

$$n^2 - n - 3984 \le 0.$$

As $\sqrt{3984} = 63.1189$ note that

$$\begin{aligned} s_{63} &= 1954 \\ s_{64} &= 2017. \end{aligned}$$

Thus at the location 1954 we start writing the number 63 and then write it again 62 more times. Thus the number at the position 1993 is 63. The number we seek is

$$63 \bmod 5 = 3.$$

Problem 17

Let the side of the square be denoted by s. Now the angle between any two consecutive outgoing segments is

$$\frac{360}{12} = 30\,.$$

For the right triangle with its lower base along the segment from the center of the clock to the "three" (and of length $b = \frac{s}{2}$) and its vertical along the vertical face of the clock we have the length of this vertical given by

$$h = \frac{s}{2}\tan(30^\circ) = \frac{s}{2}\left(\frac{1/2}{\sqrt{3}/2}\right) = \frac{s}{2\sqrt{3}}\,.$$

This means that the area of the triangles is given by

$$t = \frac{1}{2}bh = \frac{1}{2}\left(\frac{s}{2}\right)\left(\frac{s}{2\sqrt{3}}\right) = \frac{s^2}{8\sqrt{3}}\,.$$

Now we can compute q by taking the area of the full square and subtracting the areas of the triangles and then dividing by the number of quadrilateral regions (four) as

$$q = \frac{1}{4}\left(s^2 - 8t\right) = \frac{(\sqrt{3}-1)}{4\sqrt{3}}s^2\,,$$

when we simplify. This means that

$$\frac{q}{t} = 2(\sqrt{3}-1)\,,$$

when we simplify.

Problem 18

Note that Al follows a four day cycle and Barb follows a ten day cycle and that

$$\begin{aligned}\text{Least Common Multiple}(4,10) &= \text{Least Common Multiple}(2^2, 2\cdot 5)\\ &= 2^{\max(1,2)}\cdot 5^{\max(1)} = 20\,.\end{aligned}$$

Thus every 20 days the cycle of Al and Barbs work-rest days repeats. In the first 20 days Al rests on days

$$4\,,8\,,12\,,16\,,20\,,$$

while Barb rests on the days

$$8\,,9\,,10\,,18\,,19\,,20\,,$$

during which there are two overlap days i.e eight and twenty. As there are $\frac{1000}{20} = 50$ cycles of 20 days there are $50 \times 2 = 100$ overlapping rest days.

Problem 19

Multiply this equation by mn to get

$$4n + 2m = mn\,. \tag{534}$$

From this we have that

$$mn - 2m = 4n\,,$$

or

$$m(n-2) = 4n\,,$$

or solving for m we have

$$m = \frac{4n}{n-2} = \frac{4(n-2+2)}{n-2} = 4 + \frac{8}{n-2}\,.$$

- Now in the above if $n = 1$ we would have $m < 0$ and so no positive integer solutions
- The right-hand-side is undefined if $n = 2$.
- If $n = 3$ we find $m = 12$.
- If $n = 4$ we find $m = 4 + \frac{8}{2} = 8$.
- If $n = 5$ then m is not an integer.
- If $n \in \{7, 8, 9\}$ then m is not an integer.
- If $n = 10$ then $m = 4 + 1 = 5$.
- If $n \geq 11$ then m is not an integer.

Thus from the above we see that the choices are $(n, m) \in \{(3, 12), (4, 8), (6, 6), (10, 5)\}$ for a total of four solutions of the desired type.

Problem 20

We are given

$$10z^2 - 3iz - k = 0\,.$$

This will have solutions given by

$$\begin{aligned} z &= \frac{3i \pm \sqrt{(-3i)^2 - 4(10)(-k)}}{2(10)} = \frac{3i \pm \sqrt{9(-1) + 40k}}{20} \\ &= \frac{3i \pm \sqrt{40k - 9}}{20}\,. \end{aligned}$$

- From this we see that if $k = \frac{9}{40}$ there will only be one imaginary root so choice (A) is not true.
- If $k < 0$ then both roots are pure imaginary and (B) is true.
- If k is pure imaginary then in general $\sqrt{40k - 9}$ will be complex so both roots will be complex and (C) and (D) are not true.
- To show choice (E) is false we ask if we can I find a complex k such that $3i = \sqrt{40k - 9}$. If you square this we get $-9 = 40k - 9$ which has a solution $k = 0$. This means that the trivial complex number $k = 0$ gives a root of zero so (E) is false.

Problem 21

We are told that

$$a_4 + a_7 + a_{10} = 17 \quad (535)$$

$$\sum_{k=4}^{14} a_k = 77\,. \quad (536)$$

An arithmetic sequence takes the form $a_k = a_1 + d(k-1)$ for $k \geq 1$. From Equation 535 this would give

$$3a_1 + d(3) + d(6) + d(9) = 17\,,$$

or

$$3a_1 + 18d = 17\,.$$

From Equation 536 this would give

$$(14 - 4 + 1)a_1 + \sum_{k=4}^{14} d(k-1) = 77\,,$$

which we can simplify to

$$11a_1 + 88d = 77\,.$$

Solving these two systems for a_1 and d gives $a_1 = \frac{5}{3}$ and $d = \frac{2}{3}$. Now we want to know the value of k when $a_k = 13$. This means that

$$\frac{5}{3} + \frac{2}{3}(k-1) = 13\,.$$

Solving for k we get $k = 18$.

Problem 22

Let v be value of the sum assigned to the top block. Let x_1, x_2, x_3 be value of the sums assigned to the three blocks one from the top when looking down they are in the "order"

```
      x1
x2        x3
```

This means that the top block is assigned a value

$$v = x_1 + x_2 + x_3\,.$$

Let $y_1, y_2, y_3, y_4, y_5, y_6$ be the values assigned to as the sums on the blocks two levels from the top where looking from the top the "order" of these block is

```
         y1
      y2        y3
y4        y5        y6
```

This means that we have

$$\begin{aligned} x_1 &= y_1 + y_2 + y_3 \\ x_2 &= y_2 + y_4 + y_5 \\ x_3 &= y_3 + y_5 + y_6\,. \end{aligned}$$

Finally, let $z_1, z_2, \cdots, z_9, z_{10}$ be values assigned to block at the base where looking from the top the "order" of these blocks is

```
            z1
         z2        z3
      z4        z5        z6
z7        z8        z9        z10
```

This means that

$$\begin{aligned} y_1 &= z_1 + z_2 + z_3 \\ y_2 &= z_2 + z_4 + z_5 \\ y_3 &= z_3 + z_5 + z_6 \\ y_4 &= z_4 + z_7 + z_8 \\ y_5 &= z_5 + z_8 + z_9 \\ y_6 &= z_6 + z_9 + z_{10}\,. \end{aligned}$$

We can now start with y_i and compute x_i as

$$\begin{aligned} x_1 &= z_1 + 2z_2 + 2z_3 + z_4 + 2z_5 + z_6 \\ x_2 &= z_2 + z_4 + z_5 + z_4 + z_7 + z_8 + z_5 + z_8 + z_4 \\ &= z_2 + 2z_4 + 2z_5 + z_7 + 2z_8 + z_9 \\ x_3 &= z_3 + z_5 + z_6 + z_5 + z_8 + z_9 + z_6 + z_9 + z_{10} \\ &= z_3 + 2z_5 + 2z_6 + z_8 + 2z_9 + z_{10}\,. \end{aligned}$$

Finally we can take these x_i and derive an expression for v as

$$\begin{aligned} z &= z_1 + 2z_2 + 2z_3 + z_4 + 2z_5 + z_6 \\ &+ z_2 + 2z_4 + 2z_5 + z_7 + 2z_8 + z_9 \\ &+ z_3 + 2z_3 + 2z_6 + z_8 + 2z_9 + z_{10} \\ &= z_1 + 3z_2 + 3z_3 + 3z_4 + 6z_5 + 3z_6 + z_7 + 3z_8 + z_9 + z_{10}\,. \end{aligned}$$

For this expression to make v as small as possible we must take $z_5 = 1$ and z_1, z_7, z_{10} drawn from the set $\{10, 9, 8\}$. In that case we have

$$v = 6 + 27 + 3(z_2 + z_3 + z_4 + z_6 + z_8 + z_9)\,.$$

To minimize this we take $z_2, z_3, z_4, z_6, z_8, z_9$ from the set of $\{7, 6, 5, 4, 3, 2\}$. This means that

$$v = 33 + 3(13 + 9 + 5) = 33 + 3(27) = 114\,.$$

Problem 23

Draw the segment BD. Then as AD is a diameter of the circle we have that $\angle ABD = 90°$. Thus using the right triangle $\triangle ABD$ we have

$$\begin{aligned} AB = AD \cos \angle DAB &= 1 \cos\left(\frac{1}{2}\angle BAC\right)\,. \\ &= 1 \cos\left(\frac{1}{2}(12°)\right) = \cos(6°)\,. \end{aligned}$$

Now by symmetry

$$\angle BXD = \frac{1}{2}\angle BXC = \frac{1}{2}(36°) = 18°\,.$$

This means that

$$\angle AXB = 180° - \angle BED = 180° - 18° = 162°\,,$$

and

$$\angle ABX = 180° - \angle BAX - \angle AXB = 180° - 6° - 162° = 12°\,.$$

From these calculations we know all the angles in the triangle $\triangle ABX$ and the length of one side. Using the law of sines we have

$$\frac{AB}{\sin(162)} = \frac{AX}{\sin(12)}\,.$$

Recall that $AB = \cos(6)$ and that

$$\begin{aligned} \sin(162) = \sin(180 - 18) &= \sin(180)\cos(18) - \sin(18)\cos(180). \\ &= -\sin(18)(-1) = \sin(18)\,, \end{aligned}$$

We get

$$\frac{\cos(6)}{\sin(18)} = \frac{AX}{\sin(12)}\,,$$

or

$$AX = \cos(6)\sin(12)\csc(18)\,.$$

Problem 24

Let D be a random variable that indicates the number of draws needed to get three shiny pennies. Then we have $3 \le D \le 7$. We have

$$P(D=3) = \frac{3}{7} \cdot \frac{2}{6} \cdot \frac{1}{5} = \frac{1}{35},$$

and by conditioning on when the single dull penny is drawn we have

$$\begin{aligned} P(D=4) &= \frac{4}{7}\left(\frac{3}{6} \cdot \frac{2}{5} \cdot \frac{1}{4}\right) + \frac{3}{7}\left(\frac{4}{6}\right)\left(\frac{2}{5}\right)\left(\frac{1}{4}\right) + \frac{3}{7}\left(\frac{2}{6}\right)\left(\frac{4}{5}\right)\left(\frac{1}{4}\right) \\ &= \frac{1}{35} + \frac{1}{35} + \frac{1}{35} = \frac{3}{35}. \end{aligned}$$

Using these results we see that

$$P(D \ge 5) = 1 - P(D=3) - P(D=4) = 1 - \frac{4}{35} = \frac{31}{35}.$$

From this expression we see that $a = 31$ and $b = 35$. This means that $a + b = 66$.

Problem 26

Note that we can "complete the square" to write

$$\begin{aligned} 8x - x^2 &= -(x^2 - 8x + 16 - 16) \\ &= 16 - (x-4)^2, \end{aligned}$$

and

$$\begin{aligned} 14x - x^2 - 48 &= -(x^2 - 14x) - 48 \\ &= -(x^2 - 14x + 49 - 49) - 48 \\ &= -(x-7)^2 + 49 - 48 = 1 - (x-7)^2. \end{aligned}$$

This means that we can write $f(x)$ as

$$f(x) = \sqrt{16 - (x-4)^2} - \sqrt{1 - (x-7)^2}.$$

In order to have $f(x)$ *not* defined we would need to have one of the arguments of the square roots negative. If this happens with the first square root then we have

$$16 - (x-4)^2 < 0,$$

which is true if $x > 8$ or $x < 0$. If this happens with the second square root then we have

$$1 - (x-7)^2 < 0,$$

which is true if $x > 8$ or $x < 6$.

Taken together the only regions where both square roots are defined is when $6 \leq x \leq 8$. By plotting the above quadratics we see that for all x in this region $\sqrt{16-(x-4)^2}$ is decreasing as we go from $x=6$ to $x=8$ and thus is the largest when $x=6$. In this region $\sqrt{1-(x-7)^2}$ increasing from $x=6$ to $x=7$ and then decreases to when $x=8$. Note that by symmetry and the fact that the maximum of this square root is when $x=7$ the smallest this can be is when $x=6$ (which is the same value as when $x=8$). To make the first square root be as large as possible we need to take $x=6$. This means that the largest our function $f(x)$ can be is

$$f(6)=\sqrt{16-(6-4)^2}-\sqrt{1-(6-7)^2}=\sqrt{12}-\sqrt{0}=2\sqrt{3}\,.$$

Problem 27

When we think about moving the circle from corner to corner as it rolls around inside the given triangle we see that it will have to "change direction" at the points where it entirely fills each corner. When the circle fills each corner its center will be offset by an amount from the two edges forming the corner. The amount of offset will depend on which corner the circle is occupying. We will determine these offset distances and then subtract them from the lengths AB, BC, and CA to determine the length of the path the point P has traveled.

The easiest corner to evaluate is corner B. At that point when the "circle changes direction" the center is exactly a unit distance from the segment AB and a unit distance from the segment BC.

Next if we consider the corner A then when the circle "changes direction" the circle will be tangent to the two sides AB and AC. This means if we drop a perpendicular from P to the side AB (intersecting at a point P') and a perpendicular from P to the side AC (intersecting at a point P'') we will form two congruent right triangles

$$\triangle AP'P \cong \triangle AP''P\,.$$

We now seek to determine the offset distances AP' and AP'' as these will need to be subtracted from AB and AC in determining the length of the distance the center of the circle traveled.

From the sides of the right triangle $\triangle ABC$ we have

$$\tan(\angle CAB)=\frac{6}{8}=\frac{3}{4}\,.$$

Now by symmetry we have

$$\angle PAP'=\frac{1}{2}\angle CAB\,.$$

As $PP'=1$ if we knew the value of $\tan(\angle PAP')$ we could use

$$\tan(\angle PAP')=\frac{PP'}{AP'}=\frac{1}{AP'}\,,$$

to find AP'. To find this tangent recall the tangent half-angle formula

$$\tan(2\theta) = \frac{\sin(2\theta)}{\cos(2\theta)} = \frac{2\sin(\theta)\cos(\theta)}{\cos^2(\theta) - \sin^2(\theta)}$$
$$= \frac{2\tan(\theta)}{1 - \tan^2(\theta)} . \quad (537)$$

Then if we take

$$2\theta = \angle CAB \quad \text{so that} \quad \theta = \frac{1}{2}\angle CAB = \angle PAP' ,$$

the above is

$$\frac{3}{4} = \frac{2\tan(\theta)}{1 - \tan^2(\theta)} .$$

The above reduces to a quadratic equation for $\tan(\theta)$. Solving it we find $\tan(\theta) = \frac{1}{3}$ or $\tan(\theta) = -3$. As we expect $\tan(\theta) > 0$ the first solution is the correct one and we have that

$$AP' = \frac{1}{\tan(\theta)} = 3 .$$

By symmetry we also have that $AP'' = 3$.

Next if we consider the corner C then when the circle "changes direction" the circle will be tangent to the two sides AB and AC. This means if we drop a perpendicular from P to the side BC (intersecting at a point Q') and a perpendicular from P to the side AC (intersecting at a point Q'') we will form two congruent right triangles

$$\triangle CQ'P \cong \triangle CQ''P .$$

We now seek to determine the offset distances CQ' and CQ'' as these will need to be subtracted from BC and AC in determining the length of the distance the center of the circle traveled.

From the sides of the right triangle $\triangle ABC$ we have

$$\tan(\angle ACB) = \frac{8}{6} = \frac{4}{3} .$$

Now by symmetry we have

$$\angle PCQ' = \frac{1}{2}\angle ACB .$$

As $PQ' = 1$ if we knew the value of $\tan(\angle PCQ')$ we could use

$$\tan(\angle PCQ') = \frac{PP'}{CQ'} = \frac{1}{CQ'} ,$$

to find CQ'. To find this tangent we again use the half-angle formula where we take

$$2\theta = \angle ACB \quad \text{so that} \quad \theta = \frac{1}{2}\angle ACB = \angle PCQ' ,$$

the above is

$$\frac{4}{3} = \frac{2\tan(\theta)}{1 - \tan^2(\theta)} .$$

The above reduces to a quadratic equation for $\tan(\theta)$. Solving it we find $\tan(\theta) = \frac{1}{2}$ or $\tan(\theta) = -2$. As we expect $\tan(\theta) > 0$ the first solution is the correct one and we have that

$$CQ' = \frac{1}{\tan(\theta)} = 2\,.$$

By symmetry we also have that $CQ'' = 2$.

Now with these lengths determined we can determine the length of the path point P travels. Moving from A to B to C this length would be

$$(AB - AP' - 1) + (BC - 1 - CQ') + (CA - CQ'' - AP'') = (8 - 3 - 1) + (6 - 1 - 2) + (10 - 2 - 3) = 12\,.$$

The 1994 AHSME Examination (AHSME 45)

Problem 1

Call this expression E. Then we have

$$\begin{aligned} E &= 2^8 \cdot 3^8 \cdot 2^{18} \cdot 3^{18} = 2^{26} \cdot 3^{26} = (2 \cdot 3)^{26} \\ &= 6^{26} = (6^2)^{13} = 36^{13} \,. \end{aligned}$$

Problem 2

From the given diagram if we let the two horizontal lengths be denoted as x_1 and x_2 and the two vertical lengths be denoted as y_1 and y_2 then from the areas given we have

$$\begin{aligned} x_2 y_1 &= 35 \\ x_1 y_2 &= 6 \\ x_2 y_2 &= 14 \,. \end{aligned}$$

We want to evaluate $x_1 y_1$. From the above we have that $x_1 = \frac{6}{y_2}$, $y_1 = \frac{35}{x_2}$, and $x_2 = \frac{14}{y_2}$. This means that

$$y_1 = \frac{35}{14} y_2 \,,$$

so

$$x_1 x_2 = \left(\frac{6}{y_2}\right)\left(\frac{35 y_2}{14}\right) = 15 \,.$$

Problem 3

We can write this as $2x^x$ which is only equal to one of the given expressions.

Problem 4

From the two points on the diameter of the circle we get that the diameter must have a length of $D = 25 - (-5) = 30$ and thus a radius of $R = \frac{D}{2} = 15$. Again from the points on the diameter the center of the circle must be located at

$$\left(\frac{1}{2}(25 - 5), 0\right) = (10, 0) \,.$$

This means that the equation for the circle is given by

$$(x - 10)^2 + y^2 = 15^2 \,.$$

Putting in the value $(x, 15)$ gives

$$(x - 10)^2 + 15^2 = 15^2 \quad \text{so} \quad x = 10 \,.$$

Problem 5

If x is the unknown number then Pat performs

$$\frac{x}{6} - 14 = 16\,,$$

so $x = 180$. Pat should have computed

$$6x + 14 = 1094\,.$$

Problem 6

From the rule given to generate the sequence by working "backwards" we must have

$$d + 0 = 1 \quad \text{so} \quad d = 1\,.$$

Next

$$c + d = 0 \quad \text{or} \quad c + 1 = 0 \quad \text{so} \quad c = -1\,.$$

Next

$$b + c = 0 \quad \text{or} \quad b - 1 = 1 \quad \text{so} \quad b = 2\,.$$

Finally

$$a + b = c \quad \text{or} \quad a + 2 = -1 \quad \text{so} \quad a = -3\,.$$

Problem 7

Given that G is the center of the square $ABCD$ we have that the triangle ABG has a base of length $AB = 10$ and a height of length five. Thus the total area covered by these squares T is

$$\begin{aligned} T &= 2 \times \text{Area of square } ABCD - \text{Area of } \triangle GAB \\ &= 2 \times 10^2 - \frac{1}{2}(10)5 = 175\,. \end{aligned}$$

Problem 8

From the given value for the perimeter and the number of sides the length of each side is

$$\frac{56}{28} = 2\,.$$

Now there is one square in the left-most "column" in this figure, three squares in the adjacent column to its right, five in the next adjacent column, and seven squares in the central column. From that point on-wards the number of squares in the remaining columns (moving rightward) is given by

$$5\,,3\,,1\,.$$

Thus there are a total of

$$2(1) + 2(3) + 2(5) + 7 = 25\,,$$

squares in the figure. As the area of one square is $2^2 = 4$ we find the total area of the figure to be $4 \times 25 = 100$.

Problem 9

From the problem statement we are told that

$$\angle A = 4\angle B$$
$$90 - \angle B = 4(90 - \angle A)\,.$$

If we solve these two equations we find $\angle B = 18$.

Problem 10

From the ordering given we have that $m(b,c) = b$ and $m(a,e) = a$ so that the expression we are asked to evaluate can be written as

$$M(M(a,b), m(d,a)) = M(b,a) = b\,.$$

Problem 11

For cubes with volumes V given we can compute the side length s and from that the surface area A. We have

- If $V = 27$ then $s = 3$ so $A = 6s^2 = 54$.
- If $V = 8$ then $s = 2$ so $A = 24$.
- If $V = 1$ then $s = 1$ so $A = 6$.

To minimize the surface area we want as many blocks to have faces that overlap. Placing the $s = 3$ block first we will place the $s = 2$ block on top of the $s = 3$ block. This figure has a surface area of

$$(54 - 4) + (24 - 4) = 70\,.$$

To place the third $s = 1$ block we will place it next to the $s = 2$ block and on top of the $s = 3$ block. This will remove two $s^2 = 1^2 = 1$ faces of area from the above two block configuration and add four more for a total surface area of

$$70 - 2 + 4 = 72\,.$$

Problem 12

We have

$$(i - i^{-1})^{-1} = \frac{1}{i - i^{-1}}\,.$$

If we multiply this by $\frac{i}{i}$ we get

$$\frac{i}{i^2 - 1} = \frac{i}{-1 - 1} = -\frac{i}{2}\,.$$

Problem 13

From the diagram as $AB = AC$ we have $\angle ABC = \angle ACB$. Lets define that angle as θ. In addition define $\angle A = \angle BAC = \phi$. Next as $BC = CP$ we have $\angle ABC = \angle BPC = \theta$ also. Now as $PA = PC$ we have

$$\angle PAC = \angle PCA = \phi\,,$$

also. Using the exterior angle theorem we have that

$$\angle BPC = \theta = 2\phi\,.$$

Using triangle $\triangle BPC$ we have

$$\angle PCB = 180 - 2\theta\,.$$

Then using the fact that $\angle ACB = \angle BCP + \angle PCA$ we have

$$\theta = (180 - 2\theta) + \phi\,,$$

or $3\theta - \phi = 180$. Using $\theta = 2\phi$ we get

$$6\phi - \phi = 180 \quad \text{so} \quad \phi = 36\,.$$

Problem 14

The given series is the sum of the arithmetic sequence with terms $a_0 + ih$ for $i = 0$ to $i = N$ where $a_0 = 20$ and $h = \frac{1}{5}$. The final value of this sequence must satisfy

$$40 = a_0 + Nh\,,$$

or

$$40 = 20 + \frac{N}{5}\,.$$

This gives $N = 100$. Now we want to evaluate

$$\begin{aligned}\sum_{i=0}^{N}(a_0 + hi) &= a_0(N+1) + h\sum_{i=1}^{N} i \\ &= a_0(N+1) + \frac{hN(N+1)}{2}\,.\end{aligned}$$

Using what we know this evaluates to 3030.

Problem 16

Let R and B be the initial number of red and blue marbles. Then the first condition given tells us that

$$\frac{1}{7}(R-1+B) = R-1\,. \tag{538}$$

The second condition tells us that

$$\frac{1}{5}(R+B-2) = R\,. \tag{539}$$

These are two equations in the two unknowns R and B. Solving them gives $R = 4$ and $B = 18$. Thus the total number of marbles initially is $R + B = 22$.

Problem 17

Let this rectangle be placed in a Cartesian coordinate plane with the vertices located at $(0,0)$, $(8,0)$, $(8,2\sqrt{2})$, and $(0,2\sqrt{2})$. The center of this rectangle is located at $(4,\sqrt{2})$. The circle with a radius of two with this center has an equation of

$$(x-4)^2 + (y-\sqrt{2})^2 = 2^2\,.$$

When $y = 0$ this is

$$(x-4)^2 + 2 = 4 \quad \text{so} \quad x = 4 \pm \sqrt{2}\,.$$

These are the locations of the circles intersection on the x-axis. Lets draw segments from the center of the circle to each of the two points $(4-\sqrt{2},0)$ and $(4+\sqrt{2},0)$ and a perpendicular from the circle center to the point $(4,0)$. Notice that the triangle formed from the points $(4-\sqrt{2},0)$, $(4,0)$, and $(4,\sqrt{2})$ is a right triangle with two sides of length $\sqrt{2}$ and thus is an isosceles right triangle with a vertex angle of $45°$. Thus the triangle formed by the points $(4-\sqrt{2},0)$, $(4,0)$, and $(4,\sqrt{2})$ is a right triangle with an area of

$$2\left(\frac{1}{2}(\sqrt{2})^2\right) = 2\,.$$

Note that there are two such triangles (above and below the mid-line). Now the area of the circle that overlaps with the rectangle that is not in these two triangle is then

$$2\frac{90}{360}(\pi 2^2) = 2\pi\,.$$

The "two" is because there are two angular sections (to the right and left of the medial line). Thus the total overlapping area is

$$2 \times 2 + 2\pi = 4 + 2\pi\,.$$

Problem 18

As

$$\angle A + \angle B + \angle C = 180 \,,$$

using what we know about the angles we have

$$\angle A + 4\angle A + 4\angle A = 180 \,,$$

which gives $\angle A = 20$. As $\angle A$ is on a circle the *arc* measure $\widehat{BC}$ is then

$$\widehat{BC} = 2\angle A = 40 \,.$$

As we are told that B and C are adjacent points of a regular polygon with n sides this is $\frac{1}{n}$ of the full arc length of 360. Thus

$$\frac{360}{n} = 40 \,,$$

so $n = 9$.

Problem 19

To guarantee that we have at least 10 disks of the same number we can imagine selecting disks and having the environment give us disks in an order in which it is impossible (or as hard as possible) to have 10 of the same number. This means that we can have the environment give us back

$$D = \sum_{i=1}^{9} i + \sum_{i=10}^{50} 9 \,,$$

disks before the next one will need to produce at least one grouping of the same 10 numbers. In the above we have the environment give us as many disk with different numbers as possible refusing to give us 10 disks all with the same number. Evaluating this sum we get

$$D = \frac{9(10)}{2} + 9(50 - 10 + 1) = 414 \,.$$

If we draw one more disk or 415 we will have at least one set with 10 of the same number.

Problem 20

We are told that

$$x \,, y \,, z \,, \tag{540}$$

is a geometric sequence while

$$x \,, 2y \,, 3z \,, \tag{541}$$

is an arithmetic sequence. From Equation 540 we have that

$$\frac{y}{x} = r = \frac{z}{y} \,.$$

and from Equation 541 we have that

$$2y - x = d = 3z - 2y\,. \tag{542}$$

Lets divide Equation 542 by x to get

$$\frac{2y}{x} - 1 = \frac{3z}{x} - \frac{2y}{x}\,.$$

In terms of the common ratio r this is

$$2r - 1 = 3r^2 - 2r\,.$$

Solving this quadratic equation gives $r = \frac{1}{3}$ or $r = 1$. As $x \neq y$ we must have $r = \frac{1}{3}$.

Problem 21

To be odd means that our number must end with one of the digits

$$1\,,3\,,5\,,7\,,9\,.$$

If the sum of the digits is four our number can only end in a one or a three. If our number ends with a one then it takes the form

$$D1\,,$$

where the sum of the digits in the D part must be three. Thus the front of the number can thus be (zero is excluded)

$$3\,,12\,,21\,,111\,.$$

These are the numbers 31, 121, 211, and 1111. Now 31 is prime. Note $121 = 11 \times 11$ and is not prime. For the number 211 we can divide by smaller primes to eventually conclude that it is prime. For 1111 note that $1111 = 11 \times 101$ and so is not prime.

If our number ends with a three then the only form it can take 13 (which is prime). Thus there are two counterexamples the numbers 121 and 1111.

Problem 23

We need to first decide if the line has a value of $y(3)$ larger than three, a value between one and three or a value less than one. To decide this, if we consider the line that goes though the point $(3, 3)$ (where $m = 1$) the area to the left of this line is

$$A_l = \frac{1}{2}(3^2) = \frac{9}{2}\,,$$

while the area to the right of this line is

$$A_r = \frac{9}{2} + 2\,.$$

As $A_l < A_r$ we need more area to the left to make the areas equal and thus our line needs to be less step and we need to decrease the value of m.

If we consider the line that goes though the point $(3, 1)$ (where $m = \frac{1}{3}$) the area to the left of this line is

$$A_l = 6 + \frac{3}{2},$$

while the area to the right of this line is

$$A_r = \frac{3}{2} + 2.$$

As $A_l > A_r$ we need more area to the right to make the areas equal and thus our line needs to be more step and we need to increase the value of m.

These arguments indicate that the line should have its $y(3) = 3m$ value such that

$$1 < 3m < 3.$$

If we denote the lines value at $x = 3$ as just y then the area to the left of this line is

$$A_l = 3(3 - y) + \frac{1}{2}(3y),$$

while the area to the right of this line is

$$A_r = \frac{1}{2}(3y) + 1(2).$$

Setting these two equal and solving for y gives $y = \frac{7}{3}$. To go through the point $(3, \frac{7}{3})$ this means that the line $y = mx$ must satisfy

$$\frac{7}{3} = 3m \quad \text{so} \quad m = \frac{7}{9}.$$

Problem 25

We are given

$$|x| + y = 3 \tag{543}$$
$$|x|y + x^3 = 0. \tag{544}$$

If we assume that $x > 0$ then Equation 544 gives

$$y + x^2 = 0,$$

so $y = -x^2$. If we put this into Equation 543 we get

$$-x^2 + x - 3 = 0.$$

Solving this quadratic equation gives $x = \frac{1 \pm \sqrt{1-4(3)}}{2}$ which is complex. Thus we cannot have $x > 0$ and we must have $x < 0$. In that case Equation 544 gives

$$-xy + x^3 = 0\,,$$

or dividing by x

$$-y + x^2 = 0\,,$$

or $y = x^2$. If we put this into Equation 543 we get

$$-x + x^2 = 3\,.$$

Solving this quadratic equation gives $x = \frac{1 \pm \sqrt{13}}{2}$. To have $x < 0$ we need to take the negative sign so

$$x = \frac{1 - \sqrt{13}}{2}\,,$$

and

$$y = x^2 = \frac{1 + 13 - 2\sqrt{13}}{4} = \frac{7 - \sqrt{13}}{2}\,.$$

Using these two forms when I subtract I get

$$x - y = -3\,.$$

Problem 26

The *interior* angle of a regular m polygon is given by

$$180\left(\frac{m-2}{m}\right)\,. \tag{545}$$

The *exterior* angle is then

$$360 - \frac{180(m-2)}{m} = 180\left(\frac{m+2}{m}\right)\,.$$

Now the regular polygon we place "outside" of this central polygon will have an interior angle $\frac{1}{2}$ of the exterior angle above. This is

$$90\left(\frac{m+2}{m}\right)\,. \tag{546}$$

If $m = 10$ then this equals 108. To find the regular n polygon that has an *interior* angle equal to this angle we set this equal to Equation 545 (with m replaced with n). This is

$$108 = \frac{180(n-2)}{n}\,.$$

Solving for n gives $n = 5$.

Problem 27

In the problem statement we are told that $P(\text{White}) = \frac{2}{3}$ and $P(\text{Yellow}) = \frac{1}{3}$. Next whether a kernel will pop depends on its color as

$$\begin{aligned} P(\text{Pop}|\text{White}) &= \frac{1}{2} \\ P(\text{Pop}|\text{Yellow}) &= \frac{2}{3}. \end{aligned}$$

We are asked to find $P(\text{White}|\text{Pop})$. Bayes' rule gives

$$\begin{aligned} P(\text{White}|\text{Pop}) &= \frac{P(\text{Pop}|\text{White})P(\text{White})}{P(\text{Pop})} \\ &= \frac{P(\text{Pop}|\text{White})P(\text{White})}{P(\text{Pop}|\text{White})P(\text{White}) + P(\text{Pop}|\text{Yellow})P(\text{Yellow})} \\ &= \frac{\frac{1}{2}\cdot\frac{2}{3}}{\frac{1}{2}\cdot\frac{2}{3} + \frac{2}{3}\cdot\frac{1}{3}} = \frac{3}{5}, \end{aligned}$$

when we simplify.

Problem 28

Let our line be given by $y = mx + b$. We are told that when $x = p$ we have $y = 0$ so

$$mp + b = 0 \quad \text{so} \quad b = -mp .$$

This means our line takes the form

$$y = mx - mp = m(x - p) . \tag{547}$$

To have the y intercept positive taking $x = 0$ in the above we have that $-mp > 0$ so that $mp < 0$ so $m < 0$ (since $p \geq 2$). To have the y intercept an integer means that mp is an integer.

To pass though the point $(x, y) = (4, 3)$ means that

$$m(4 - p) = 3 \quad \text{or} \quad 4 - p = \frac{3}{m} .$$

Now the left-hand-side $4 - p$ is an integer and the only way the right-hand-side can be an integer (with $m < 0$) is if m is a factor of three so $m \in \{-1, -3\}$. If $m = -1$ we have

$$4 - p = -3 \quad \text{so} \quad p = 7 ,$$

which is prime. If $m = -3$ we have

$$4 - p = -1 \quad \text{so} \quad p = 5 ,$$

which is also prime. Thus there are two lines of this type.

Problem 29

In the given figure draw the center of the circle as a point O. Then draw the segments OB, OC, and OA each of which is of length r. We are told that the length of the minor arch BC is r. Note that we can write that length as a fraction of the circle circumference as

$$2\pi r\left(\frac{\angle BOC}{2\pi}\right).$$

Setting this equal to r gives $\angle BOC = 1$ in radians. As $AB = AC$ we have arcs AB and AC equal. Thus by symmetry we have $\angle AOB = \angle AOC$. Using this and summing all three angles centered at O gives

$$2\angle AOB + \angle BOC = 2\pi\,,$$

or

$$\angle AOB = \pi - \frac{\angle BOC}{2} = \pi - \frac{1}{2}.$$

Now using the law of cosigns twice we have

$$\begin{aligned} BC^2 &= 2r^2 - 2r^2\cos(\angle BOC) = 2r^2(1-\cos(1)) \\ AB^2 &= 2r^2 - 2r^2\cos\left(\pi - \frac{1}{2}\right) = 2r^2\left(1-\cos\left(\pi-\frac{1}{2}\right)\right). \end{aligned}$$

As $\cos(\pi-\theta) = -\cos(\theta)$ this means that

$$AB^2 = 2r^2\left(1+\cos\left(\frac{1}{2}\right)\right).$$

Thus we have

$$\frac{AB^2}{BC^2} = \frac{1+\cos\left(\frac{1}{2}\right)}{1-\cos(1)}.$$

To simplify this recall that

$$\begin{aligned} \cos^2(\theta) &= \frac{1+\cos(2\theta)}{2} \\ \sin^2(\theta) &= \frac{1-\cos(2\theta)}{2}, \end{aligned}$$

which mean that we can write

$$\begin{aligned} 1+\cos(\theta) &= 2\cos^2\left(\frac{\theta}{2}\right) \\ 1-\cos(\theta) &= 2\sin^2\left(\frac{\theta}{2}\right). \end{aligned}$$

Using this last expression we get

$$\frac{AB^2}{BC^2} = \frac{1+\cos\left(\frac{1}{2}\right)}{2\sin^2\left(\frac{1}{2}\right)} = \frac{1}{2}\left(\frac{1+\cos\left(\frac{1}{2}\right)}{1-\cos^2\left(\frac{1}{2}\right)}\right) = \frac{1}{2}\left(\frac{1}{1-\cos\left(\frac{1}{2}\right)}\right).$$

Using the expression for $1 - \cos(\theta)$ again this is

$$\frac{AB^2}{BC^2} = \frac{1}{2}\left(\frac{1}{2\sin^2\left(\frac{1}{4}\right)}\right),$$

thus

$$\frac{AB}{BC} = \frac{1}{2\sin\left(\frac{1}{4}\right)} = \frac{1}{2}\csc\left(\frac{1}{4}\right).$$

The 1995 AHSME Examination (AHSME 46)

Problem 1

The average of the first three scores is

$$A_3 = \frac{87+83+88}{3} = \frac{170+88}{3} = \frac{258}{3} = 86\,.$$

The average of her four scores is

$$A_4 = \frac{258+90}{4} = \frac{348}{4} = 87\,.$$

This is an increase by one.

Problem 2

Squaring this we get

$$2+\sqrt{x} = 9\,,$$

so $\sqrt{x} = 7$. Squaring this again we get $x = 49$.

Problem 3

The television advertiser price can be written as

$$\begin{aligned} TA &= 3(29.98) + 9.98 = 3(30-0.02) + 10 - 0.02 \\ &= 90 - 0.006 + 10 - 0.02 \\ &= 100 - 0.08 = 99.99 - 0.07\,. \end{aligned}$$

Problem 4

In order we are told that $M = 0.3Q$, $Q = 0.2P$, and $N = 0.5P$. Thus

$$\frac{M}{N} = \frac{0.3(0.2P)}{0.5P} = \frac{0.06}{0.5} = \frac{6}{50} = \frac{3}{25}\,.$$

Problem 5

Three ants per square inch is equivalent to

$$\frac{3\,\text{ants}}{\text{inch}^2} \times \frac{12^2\,\text{inch}^2}{1\text{ft}^2} = 432\,\frac{\text{ants}}{\text{ft}^2}\,.$$

Our field area is

$$300 \times 400\,\text{ft}^2 = 120000\,\text{ft}^2\,.$$

This means that the number of ants N should be given by

$$\begin{aligned} N &= 12 \cdot 10^4 \cdot 432 = 5148 \cdot 10^4 = 5.184 \cdot 10^3 \cdot 10^4 \\ &= 5.184 \cdot 10^7 = 51.8 \cdot 10^6\,. \end{aligned}$$

This is closest to 50 million.

Problem 6

To form the cube we will fold the x face "up" (i.e. out of the page towards the reader) the face B "up", the face C sideways (so that it shares a side with face A). This would put face D on top of the cube and E on the front (bottom) face. From this x will be opposite the face C.

Problem 7

We are told the earths radius R_e is $R_e = 4000$ miles. The distance traveled is then $C = 2\pi R_e$ and the time traveled (in hours) would be given by

$$H = \frac{C}{500} = \frac{2\pi(4000)}{500} = 16\pi\,.$$

We can approximate this as

$$H \approx 16 \times 3.145 \approx 48 + 1.6 = 49.6\,.$$

Problem 8

By the fact that DE is parallel to AC we have $\triangle BDE \sim \triangle BAC$. This means that

$$\frac{DE}{AC} = \frac{BD}{BA} = \frac{BD}{\sqrt{AC^2 + BC^2}} = \frac{BD}{\sqrt{6^2 + 8^2}} = \frac{BD}{10}\,.$$

We know that

$$\frac{DE}{AC} = \frac{4}{6}\,,$$

so using this in the above we get

$$\frac{2}{3} = \frac{BD}{10} \Rightarrow BD = \frac{20}{3}\,.$$

Problem 9

I count 8 “small” triangles, four isosceles triangles that are in the “North”, “East”, “South”, and “West” positions, and four “corner” right triangles that are in the “North-East”, “South-East”, “South-West”, and “North-West” corners. This gives a total of $8+4+4 = 16$ triangles.

Problem 10

The lines $y = x$ and $y = -x$ cut an X in the Cartesian plane. The line $y = 6$ is horizontal that cuts a triangle in the top 1/2 of the Cartesian plane. This triangle has a height of $h = 6$ and a base of $2h = 12$ to give an area of

$$\frac{1}{2}(6)(12) = 36\,.$$

Problem 11

From the given conditions a can be either four or five and d can be either zero or five. For the numbers b and c if $b = 3$ then $c \in \{4, 5, 6\}$. If $b = 4$ then $c \in \{5, 6\}$ and if $b = 5$ then $c = 6$. Thus we can pick that pair bc in

$$3 + 2 + 1 = 6\,,$$

ways. We can pick the a and c numbers in $2 \times 2 = 4$ ways. Thus in total we can pick the number $abcd$ in

$$4 \times 6 = 24\,,$$

ways.

Problem 12

We are told that $f(x)$ is linear so

$$f(x) = ax + b\,.$$

We are told that $f(1) \leq f(2)$ which means that

$$a + b \leq 2a + b \Rightarrow a \geq 0\,.$$

We are told that $f(3) \geq f(4)$ which means that

$$3a + b \geq 4a + b \Rightarrow 0 \geq a\,.$$

The only way these both can be true is if $a = 0$. Thus $f(x) = b = f(5) = 5$ and $f(0) = 5$ also.

Problem 13

Starting in the ones column by changing the six will not change the addition result for larger place holders there are no other sixes in the number. We can't change the zero in the ones column or the total with six will not be six. Similar logic indicates that we can't change the 1, 3, or 8 in the tens location. Similar logic indicates that we can't change the 4 or 5 in the hundreds location and the 0 can not be changed from arguments made earlier. These arguments mean that the only digits that could be changed are 2 or 9.

In the thousands position then we have the sum (with a one from a carry from the hundreds position sum) of

```
  1
  2
+ 9
---
  2
```

This sum as written will "stand" but if so then we have not replaced any digits and the total sum is incorrect. Notice that we can change the 2 digit to any digit we like and the column sum will be correct. Lets replace it with a where $1 \leq a \leq 9$. Then replacing all 2's with a's the addition problem looks like (with 1 for carry)

```
       1 1 1
     7 4 a 5 8 6
+    8 a 9 4 3 0
----------------
   1 a 1 a 0 1 6
```

From the 10 thousands place we see that we need

$$(1 + 4 + a) \bmod 10 = 1\,.$$

If $a = 6$ this is will be true. In that case the problem looks like

```
     1 1 1 1
     7 4 a 5 8 6
+    8 6 9 4 3 0
----------------
   1 6 1 a 0 1 6
```

which is a true addition. Thus we change the digit $d = 2$ with the digit $e = 6$ to get a valid problem. Thus $d + e = 8$.

Problem 14

We are told that $f(x) = ax^4 - bx^2 + x + 5$ and that

$$f(-3) = a3^4 - b3^2 + (-3 + 5) = 2\,. \quad (548)$$

From the form of $f(x)$ we have

$$f(3) = a3^4 - b3^2 + (3 + 5) = 8\,.$$

From Equation 548 we see that

$$a3^4 - b3^2 = 0\,.$$

Using this in $f(3)$ we find

$$f(3) = 3 + 5 = 8\,.$$

Problem 15

Lets follow the bugs steps. For a few steps he will be at the locations

$$5 \to 1 \to 2 \to 4 \to 1 \to 2 \to 4 \to 1 \to \cdots\,.$$

Notice that the sequence $1 \to 2 \to 4$ will repeat as long as needed. If we let c be the number of these three jump "cycles" that the bug jumps we want to decompose 1995 as

$$1995 = 1 + 3c + r \quad \text{or} \quad 1994 = 3c + r\,.$$

The one above is for the first jump from five to one. The above gives $c = 664$ and $r = 2$. Thus after the first jump and then $c = 664$ cycles the bug is at the position four. Two more jumps places the bug at position two.

Problem 16

We are given two estimates of the attendance numbers at Atlanta $\hat{A}$ and Boston $\hat{B}$. We are told that $\hat{A} = 50000$ and that the actual attendance A is within 10% of this number or

$$50000(0.9) < A < 50000(1.1) \quad \text{or} \quad 45000 < A < 55000\,.$$

Next we are told from Bob's estimate $\hat{B} = 60000$ that

$$0.9B < \hat{B} < 1.1B \quad \text{or} \quad 0.9B < 60000 < 1.1B\,.$$

From $0.9B < 60000$ we get that $B < 66666.67$ and from $1.1B > 60000$ we get that $B > 54545.45$ thus

$$54545.45 < B < 66666.67\,.$$

We are asked for the largest possible value of $B - A$. This would be $66666.67 - 45000 = 21666.67$ which is closest to (E).

Problem 17

This pentagon $ABCDE$ has $n = 5$ sides and so the sum of the internal angles using Equation 5 is $180(n-2) = 540°$. Thus any internal angles of the *regular* pentagon $ABCDE$ has an angular measure of

$$\frac{180(n-2)}{n} = \frac{540}{5} = 108° \,.$$

In the given figure draw the center of the circle as the point O and connect O to the two points A and D each of length r (the radius r of the circle). As the segments AB and DC are tangent to the circle we have $\angle OAB = \angle ODC = 90°$. As $\angle AED$ is an internal angle of the pentagon we have

$$\angle EAO = 108 - 90 = 18° \,.$$

The same argument can be used to show that $\angle EDO = 18°$.

Using Equation 5 again the sum of the internal angles in *four* sided figure $AEDO$ is

$$180(4-2) = 360° \,,$$

so the internal angle

$$\angle AOD = 360 - 108 - 2(18) = 216° \,.$$

This means that the exterior angle $\angle AOD$ is $360 - 216 = 144°$ which is also the measure of the minor arc $\overset{\frown}{AD}$.

Problem 18

Imagine our two rays OA and OB starting from the origin of a Cartesian coordinate systems with O at $(0,0)$ with B on the x-axis "to the right" of O, and A "above" the segment OB such that $\angle AOB = 30°$. Then using the "law of sines" in the triangle $\triangle AOB$ we have

$$\frac{\sin(30°)}{AB} = \frac{\sin(\angle OBA)}{OA} \quad \text{or} \quad \frac{1}{2} = \frac{\sin(\angle OBA)}{OB} \,,$$

when we recall that $AB = 1$. This means that

$$OB = 2\sin(\angle OBA) \,.$$

The largest that OB can be is when $\angle OBA = 90°$ and then $OB = 2 \times 1 = 2$.

Problem 19

Let the side of the smaller equilateral triangle have a length of s so that $s = FD = DE = EF$. From the right angle at D as $\angle BCA = 60°$ we have that $\triangle EDC$ is a $30 - 90 - 60$ right triangle with $ED = s$. In that triangle we have

$$\sin(60°) = \frac{ED}{EC} \,,$$

or

$$\frac{\sqrt{3}}{2} = \frac{s}{EC} \quad \text{so} \quad EC = \frac{2s}{\sqrt{3}} .$$

As $\angle CED = 30°$ and $\angle FED = 60°$ we have $\angle FEA = 180 - 60 - 30 = 90°$ so triangle $\triangle FEA$ is $30 - 90 - 60$ right triangle. In that triangle we can write

$$\tan(60°) = \frac{FE}{AE} ,$$

or

$$\sqrt{3} = \frac{s}{AE} \quad \text{so} \quad AE = \frac{s}{\sqrt{3}} .$$

From these two results we have that $AC = AE + EC = \frac{s}{\sqrt{3}} + \frac{2s}{\sqrt{3}} = \sqrt{3}s$ so the ratio of areas of triangle $\triangle DEF$ to that of triangle $\triangle ABC$ is

$$\frac{\frac{\sqrt{3}}{4}s^2}{\frac{\sqrt{3}}{4}AC^2} = \frac{s^2}{AC^2} = \frac{s^2}{3s^2} = \frac{1}{3} .$$

Problem 20

In Table 15 I enumerate all possible outcomes for the even/oddness of a, b, and c along with the even/oddness of the product ab and finally of the expression $ab + c$. Thus we see that only under certain conditions will $ab + c$ be even. The probability we are in any given row of the above table is the product of the probability that a, b, and c take on the given even/oddness.

If X is one of the numbers a, b, or c then from the set from which these numbers are drawn we see that the probability X is even/odd is given by

$$P(X\,\text{even}) = \frac{2}{5}$$
$$P(X\,\text{odd}) = \frac{3}{5} .$$

Using this result and the table above we see that the event E that the expression $ab + c$ is even is given by

$$P(E) = \left(\frac{2}{5}\right)^3 + \left(\frac{2}{5}\right)^2\left(\frac{3}{5}\right) + \left(\frac{2}{5}\right)^2\left(\frac{2}{5}\right) + \left(\frac{3}{5}\right)^3 = \frac{59}{125} ,$$

when we simplify.

Problem 21

The first diagonal connects the points given and is thus along the line

$$y - 3 = \frac{3}{4}(x - 4) \quad \text{or} \quad y = \frac{3}{4}x .$$

a	b	c	ab	$ab+c$
Even	Even	Even	Even	Even
Even	Even	Odd	Even	Odd
Even	Odd	Even	Even	Even
Odd	Even	Even	Even	Even
Even	Odd	Odd	Even	Odd
Odd	Even	Odd	Even	Odd
Odd	Odd	Even	Odd	Odd
Odd	Odd	Odd	Odd	Even

Table 15: Even or odd values for a, b, c and the even or oddness of $ab+c$.

Notice that $(0,0)$ is the midpoint of the given segment/diagonal. The length of this diagonal is d and is given by

$$d = \sqrt{8^2 + 6^2} = \sqrt{64 + 36} = 10\,.$$

For a rectangle the other diagonal must, go though the same point $(0,0)$, and be of the same length d. This means we need to consider opposite points on a circle of radius $\frac{d}{2} = 5$ that have integer (x,y) coordinates. Thus (x,y) must satisfy

$$x^2 + y^2 = 5^2\,.$$

For $x = 0$ we get $y = \pm 5$ giving one rectangle. For $x = \pm 3$ we get $y = \pm 4$ giving one rectangle with a "new" diagonal connecting $(-3,4)$ and $(3,-4)$ and one rectangle with a "new" diagonal connecting $(-3,-4)$ and $(3,4)$. For $x = \pm 4$ we get $y = \pm 3$ giving one rectangle with a "new" diagonal connecting $(-4,3)$ and $(4,-3)$. Finally for $x = \pm 5$ we get $y = 0$ for one more rectangle. This is a total of five rectangles.

Problem 22

The longest side of the original rectangle must be the largest length given or 31. Let w be the rectangles width. Let the right triangle that is cut out of the corner of the rectangle have legs of length $31 - h$ and $w - d$ so that the two sides of the pentagon will have lengths h and d. The final remaining side of the pentagon will have a length given by c where

$$c^2 = (w-d)^2 + (31-h)^2\,. \tag{549}$$

Thus from the given numbers $\{13, 19, 20, 25\}$ we need to assign values to w, h, d, and c such that $w > d$ and Equation 549 for c is satisfied. We could do that "by hand" but I would probably make a mistake. We can do this in the following `python` code

```
from itertools import permutations
for (w, h, d, c) in permutations([13, 19, 20, 25]):
    if w < d: continue
    if c**2 == ((w-d)**2 + (31-h)**2):
        print(w, h, d, c)
```

Running that code we get $w = 25$, $h = 19$, $d = 20$, and $c = 13$. The area of the pentagon is the area of the rectangle minus the area of the right triangle cut out or

$$31w - \frac{1}{2}(31 - h)(w - d) = 31(25) - \frac{1}{2}(31 - 19)(5) = 745\,.$$

Problem 23

This triangle must satisfy the triangle inequality applied to each of its sides or

$$\begin{aligned} 11 + 15 > k \quad &\text{or} \quad k < 26 \\ k + 15 > 11 \quad &\text{or} \quad k > -4 \\ k + 11 > 15 \quad &\text{or} \quad k > 4\,. \end{aligned}$$

Thus we have learned that $4 < k < 26$. For k to be an integer means that we must have $5 \le k \le 25$.

We now ask for which ks in that range will our triangle be obtuse? Note that obtuse triangles will satisfy $c^2 > a^2 + b^2$ where c is the larger of the three sides.

- If $5 \le k \le 11$ then the largest side is the side of length 15 and our triangle will be obtuse if
$$k^2 + 11^2 < 15^2 \quad \text{or} \quad k^2 < 104\,.$$
For the range of ks above only when $5 \le k \le 10$ will the above be true. These are six numbers.
- If $12 \le k < 15$ then the largest side is the side of length 15 and our triangle will be again obtuse if
$$11^2 + k^2 < 15^2 \quad \text{or} \quad k^2 < 104\,.$$
None of the ks in this range satisfy this.
- If $15 \le k \le 25$ then the largest side is the side of length k and our triangle will be obtuse if
$$11^2 + 15^2 < k^2 \quad \text{or} \quad k^2 > 346\,.$$
For the range of ks above only when $19 \le k \le 25$ will the above be true. These are seven numbers.

Thus in the total number of triangles is $6 + 0 + 7 = 13$.

Problem 24

We are told that

$$A \log_{200} 5 + B \log_{200} 2 = C\,.$$

If we multiply by ln(200) this is equal to

$$A\ln(5) + B\ln(2) = C\ln(200)\,.$$

Note that

$$200 = 2^3 \cdot 5^2\,,$$

so that

$$\ln(200) = 3\ln(2) + 2\ln(5)\,.$$

Using this in the above we get

$$A\ln(5) + B\ln(2) = 3C\ln(2) + 2C\ln(5)$$

As A, B, and C are integers we must have $A = 2C$ and $B = 3C$. This would imply that C is a common factor of A, B, and C unless $C = 1$. Thus

$$A + B + C = 2C + 3C + C = 6C = 6\,.$$

Problem 26

Draw the segment CF. Then as CD is a diameter of the circle we have that $\angle CFD = 90°$. Let $\angle CDF = \theta$. In the right triangle $\triangle DFC$ we have

$$DF = DC\cos(\theta)\,,$$

or

$$6 + 2 = 2r\cos(\theta) \quad \text{so} \quad r\cos(\theta) = 4\,. \tag{550}$$

In the right triangle $\triangle DOE$ we have

$$\cos(\theta) = \frac{DO}{DE} = \frac{r}{6}\,.$$

Putting this into Equation 550 gives

$$\frac{r^2}{6} = 4 \quad \text{so} \quad r^2 = 24\,.$$

The area of the circle is then $\pi r^2 = 24\pi$.

Problem 27

To start we will evaluate $f(n)$ for several values of n. We find

$$\begin{aligned}
f(1) &= 0 = 2 - 2 \\
f(2) &= 2 = 4 - 2 \\
f(3) &= 2 \cdot 2 + 2 = 6 = 8 - 2 \\
f(4) &= 2 \cdot 3 + 2 \cdot 4 = 6 + 8 = 14 = 16 - 2 \\
f(5) &= 2 \cdot 4 + 2 \cdot 7 + 8 = 8 + 14 + 8 = 30 = 32 - 2 \\
f(6) &= 2 \cdot 5 + 2 \cdot 11 + 2 \cdot 15 = 10 + 22 + 30 = 62 = 64 - 2\,.
\end{aligned}$$

In each of the above calculations I evaluated $f(n)$ and then wrote the value obtained $f(n)$ in the form

$$2^n - 2 .$$

If this is true then we need to evaluate the remainder when dividing by 100 of

$$f(100) = 2^{100} - 2 .$$

To evaluate this remainder lets see if we can do it for simpler/smaller values of n. Note that

$$2^{10} = 1024 = 10 \times 100 + 24 .$$

Thus 2^{10} has a remainder of 24 when divided by 100. Next consider

$$2^{20} = 1024^2 = 1048576 ,$$

has a remainder of 76 when divided by 100. Next consider

$$2^{30} = 1073741824 ,$$

which has a remainder of 24 when divided by 100. Next

$$2^{40} = 1099511627776 ,$$

which has a remainder of 76 when divided by 100. If we assume that this pattern continues we have argued that

$$(2^{10})^n \bmod 100 = 24 ,$$

when n odd and

$$(2^{10})^n \bmod 100 = 76 ,$$

when n is even. Thus $2^{100} = (2^{10})^{10}$ should have a remainder of 76 when divided by 100 so $2^{100} - 2$ will have a remainder of $76 - 2 = 74$.

Problem 28

Let this circle have a radius of r. Next draw a line perpendicular to all of the three parallel lines and going through the center of the circle. This segment will be on the perpendicular bisector of each of the parallel segments. Assume that the center of the circle is between the "top" and the "middle" horizontal segments (if it is not our equations should result in a contradiction). Let h_T, h_M, and h_B be the distances from the center of the circle to the "top", "middle", and "bottom" horizontal lines respectively.

Then considering the right triangles (all with a hypotenuse of r) that result from connecting the center of the circle to an endpoint of the horizontal segment we find for the "top" segment

$$h_T^2 + 7^2 = r^2 , \tag{551}$$

for the "middle" segment

$$h_M^2 + \left(\frac{\sqrt{a}}{2}\right)^2 = r^2 \,, \tag{552}$$

and for the "bottom" segment

$$h_B^2 + 5^2 = r^2 \,. \tag{553}$$

Also from the facts that total distance between the "top" and the "bottom" chords is six and that the "middle" chord is midway between the "top" and the "bottom" we have

$$h_T + h_B = 6 \tag{554}$$

$$h_T + h_M = \frac{6}{2} = 3 \,. \tag{555}$$

This system represents five equations in five unknowns h_T, h_M, a, r, and h_B that we can hopefully solve. As we are only asked for the value of a we will work to get a single equation with just that unknown.

Lets use Equation 554 and 555 to eliminate h_M and h_B in terms of h_T. When we do that the three equations earlier become

$$h_T^2 + 49 = r^2 \tag{556}$$

$$(3 - h_T)^2 + \frac{a}{4} = r^2 \tag{557}$$

$$(6 - h_T)^2 + 25 = r^2 \,. \tag{558}$$

If we subtract Equation 556 from 558 we get a single equation in h_T given by

$$12 - 12h_T = 0 \quad \text{so} \quad h_T = 1 \,.$$

Using that in Equation 556 we find $r^2 = 50$. Using both of these in Equation 557 gives $a = 184$.

Problem 30

I was unable to work this problem the first time I saw it. Upon a second attempt (later in life) I made a bit more progress and with a little peak at the solutions in the back was able to solve it.

This is the solution I came up with. If we place the groups of subcubes (subcubes are the smaller cubes inside the larger cube) in the "corner" (octant) of an x-y-z Cartesian coordinate plane such that one corner of the larger cube is at $(0, 0, 0)$ and the other corner is at $(3, 3, 3)$ then a vector $\mathbf{N}$ representing that one of the internal diagonal is given by

$$\mathbf{N} = (1, 1, 1) \,,$$

or normalized as

$$\hat{\mathbf{N}} = \frac{1}{\sqrt{3}}(1, 1, 1) \,.$$

A point on this plane is certainly the centroid of the larger cube which is the point $\mathbf{r}_0 = \left(\frac{3}{2}, \frac{3}{2}, \frac{3}{2}\right)$ then the equation of the plane in question is given by

$$\hat{\mathbf{N}} \cdot (\mathbf{r} - \mathbf{r}_0) = 0 \quad \text{or} \quad \frac{1}{\sqrt{3}}(1,1,1) \cdot \left(x - \frac{3}{2}, y - \frac{3}{2}, z - \frac{3}{2}\right) = 0 \,.$$

We can simplify this to

$$x + y + z = \frac{9}{2} \,.$$

One side of the plane is thus defined by points that satisfy

$$x + y + z < \frac{9}{2} \,,$$

and the other side of the plane is defined by

$$x + y + z > \frac{9}{2} \,.$$

Now for this problem we want to count the number of subcubes which have points on both sides of this plane as that will be the number of subcubes that this plane intersects.

We can denote a unique "corner" of each of the 27 subcubes as (x_c, y_c, z_c) where each variable is drawn from $\{0, 1, 2\}$. Then we can get all other vertices of that subcube by adding to this corner the eight "offsets"

$$(0,0,0)\,, (1,0,0)\,, (0,1,0)\,, (0,0,1)\,, (1,1,0)\,, (0,1,1)\,, (1,0,1)\,, (1,1,1)\,.$$

To solve this problem for each subcube "corner" we systematically test if all of the vertices of that subcube are on the same side of the plane above and count up the number of subcubes where this is not true. This is done in the `python` code `1995_AHSME_prob_30.py`. The use of python can help in solving this problem for larger cubes. For example if the larger cube has a side of length n (so n^3 subcubes) the equation of the plane is

$$x + y + z = \frac{3n}{2} \,,$$

and we can the same arguments as above to count the number of subcubes that the plane intersects. This extension is done in the above `python` code where we find

```
large_cube_side_length=    3; number_split_cubes= 19 from total_sucubes=    27
large_cube_side_length=    4; number_split_cubes= 44 from total_sucubes=    64
large_cube_side_length=    5; number_split_cubes= 55 from total_sucubes=   125
```

The first row is the answer for this problem.

The 1996 AHSME Examination (AHSME 47)

Problem 1

The addition of the numbers in the second column is not correct. The largest number in that column is seven. If we change that seven to a six the total addition is correct.

Problem 2

Let n be the number of "normal" pay days and w the number of "exceptionally well" pay days. Then we are told that

$$3n + 5w = 36\,.$$

In addition we know that $n+w = 10$ and we want to know the value of w. Putting $n = 10-w$ into the above and solving for w gives $w = 3$.

Problem 3

Call this expression E. Then using the fact that $3! = 3 \cdot 2 \cdot 1 = 6$ we have

$$E = \frac{(3!)!}{3!} = \frac{6!}{6} = 5! = 120\,.$$

Problem 4

The median is a measure of central tendency. If we want this to be as large as possible we need to add three more integers that are as large as possible i.e. in this example greater than nine. Adding these three numbers would give a sorted list of

$$3\,,5\,,5\,,7\,,8\,,9\,,x\,,y\,,z\,.$$

The median is the fifth element in this list which is eight.

Problem 5

The sum $c+d$ is the largest two term sum and the sum $a+b$ is the smallest two term sum. Thus the largest expression is $\frac{c+d}{a+b}$.

Problem 6

From the formula given we can compute

$$\begin{aligned} f(0) &= 0 \\ f(-1) &= (-1)^0(1)^2 = 1 \\ f(-2) &= 0 \\ f(-3) &= (-3)^{-2}(-1)^0 = \frac{1}{9}\,. \end{aligned}$$

Thus the sum we want to evaluate is given by $1 + \frac{1}{9} = \frac{10}{9}$.

Problem 7

Let t be the age of the twins and y the age of the younger child (in integer years) so that $t > y$. The father must pay

$$4.95 + 2(0.45)t + 0.45y = 9.45\,,$$

or

$$0.9t + 0.45y = 4.5\,.$$

If we take $t \in \{1, 2, 3, 4, 5\}$ we would find that $y \in \{8, 6, 4, 2, 0\}$. In order to have $t > y$ we can only have

$$(t, y) \in \{(4, 2), (5, 0)\}\,.$$

The second solution represents an "unborn" child. Thus the only possible choice is $t = 4$ where $y = 2$.

Problem 8

Since $15 = 3 \times 5$ we have

$$15 = (k \cdot 2^r) \times 5 = k \cdot 4^r$$

Dividing by k gives

$$2^r \cdot 5 = 2^{2r} \quad \text{or} \quad 2^r = 5\,.$$

This means that $r = \log_2(5)$.

Problem 9

From the lengths of AB, BP, and PA we have that triangle $\triangle BPA$ must be a right triangle since

$$AB^2 = BP^2 + PA^2 \quad \text{or} \quad 5^2 = 3^2 + 4^2\,.$$

Using this point P and its right angle as the center of a Cartesian coordinate system with PA the x-axis and PB the y axis we have that $B = (0, 4, 0)$ and $A = (3, 0, 0)$. Then the point D is located at $D = (3, 0, 5)$. From these locations we have that

$$PD = \sqrt{3^2 + 0^2 + 5^2} = \sqrt{34}\,.$$

Problem 10

There are $n = 8$ corners in the cube. The number of "pairs" for this value of n is given by $\binom{n}{2} = \frac{8\times 7}{2} = 28$.

Problem 11

The points that are exactly one unit from the center of the segment (which is tangent to the circle) will be at a distance of R where

$$1^2 + 2^2 = R^2 \quad \text{so} \quad R = \sqrt{5}\,.$$

All of these points are then in an annular ring between two circles of radius $r = 2$ and $R = \sqrt{5}$. Thus the area of the points we are considering is given by

$$\pi R^2 - \pi(2^2) = 5\pi - 4\pi = \pi\,.$$

Problem 12

As k is odd we know that $f(k) = k + 3$. This later number is then even so we have $f(f(k)) = \frac{k+3}{2}$. We don't know if $\frac{k+3}{2}$ is even or odd. If we assume its odd then we have

$$f(f(f(k))) = \frac{k+3}{2} + 3 = 27\,.$$

This gives $k = 45$. Notice that for this value of k we have

$$\begin{aligned} f(k) &= 48 \\ f(f(k)) &= 24 \\ f(f(f(k))) &= 12 \neq 27\,. \end{aligned}$$

On the other hand if $\frac{k+3}{2}$ is even then we have

$$f(f(f(k))) = \frac{k+3}{4} = 27\,.$$

This gives $k = 105$. For this value of k we have

$$\begin{aligned} f(k) &= 108 \\ f(f(k)) &= 54 \\ f(f(f(k))) &= 27 \,. \end{aligned}$$

The sum of the digits in k is six.

Problem 13

Let Sunny's running rate be r. Then the position of Sunny after the head start is given by

$$x_s(t) = h + rt \,.$$

The position of Moonbeam is given by

$$x_m(t) = 0 + mrt \,.$$

Moonbeam will catch Sunny at the time t when $x_m(t) = x_s(t)$. Solving this gives

$$t = \frac{h}{(m-1)r} \,.$$

At this time Moonbeam will have run a distance of

$$mr \left(\frac{h}{(m-1)r} \right) = \frac{mh}{m-1} \,.$$

Problem 14

Note that we can compute some sums "easily". We have

$$\sum_{d=1}^{9} E(d) = 2 + 4 + 6 + 8 = 20 \,.$$

Let the sum we want to compute be denoted by S. Then we can write S as

$$S = E(100) + \sum_{i=1}^{9} \sum_{d=0}^{9} E(id) + \sum_{d=1}^{9} E(d) \,.$$

Here id is the two digit number with i as the tens digit and d as the units digit. Now when i is odd we have

$$\sum_{d=0}^{9} E(id) = 2 + 4 + 6 + 8 = 20 \,,$$

and when i is even we have

$$\sum_{d=0}^{9} E(id) = 10i + (2+4+6+8) = 10i + 20\,.$$

This means that we can write S above as

$$\begin{aligned} S &= 0 + \sum_{i=1,3,5,7,9} 20 + \sum_{i=2,4,6,8} (10i + 20) + 20 \\ &= 20(5) + 4(20) + 10\sum_{i=1}^{4} 2i + 20 = 400\,, \end{aligned}$$

when we simplify.

Problem 15

Let the height of the rectangle be given by H and the width of the rectangle be given by W. Then as the altitude of the A triangle is $\frac{W}{2}$ we have the area of the A triangle given by

$$\frac{1}{2}\left(\frac{H}{n}\right)\left(\frac{W}{2}\right)\,.$$

Then as the altitude of the B triangle is $\frac{H}{2}$ we have the area of the B triangle given by

$$\frac{1}{2}\left(\frac{W}{m}\right)\left(\frac{H}{2}\right)\,.$$

This means that the ratio of these two numbers is $\frac{m}{n}$.

Problem 16

Let X_i be the face value seen on the ith die toss. We can compute the probability that $X_1 + X_2 = X_3$ by conditioning on the value of X_3. Now each specific value of X_3 will occur with a probability of $\frac{1}{6}$. Given the value of X_3 we can compute the probability that $X_1 + X_2 = X_3$ by summing up the individual cases. We have

- If $X_3 = 1$ then $X_1 + X_2 \neq X_3$ for all possible X_1 and X_2.
- If $X_3 = 2$ then we must have $(X_1, X_2) \in \{(1,1)\}$ to have $X_1 + X_2 = X_3$.
- If $X_3 = 3$ then we must have $(X_1, X_2) \in \{(1,2),(2,1)\}$.
- If $X_3 = 4$ then we must have $(X_1, X_2) \in \{(1,3),(2,2),(3,1)\}$.
- If $X_3 = 5$ then we must have $(X_1, X_2) \in \{(1,4),(2,3),(3,2),(4,1)\}$.

- If $X_3 = 6$ then we must have $(X_1, X_2) \in \{(1,5), (2,4), (3,3), (4,2), (5,1)\}$.

The probabilities of each of the the above six events are given by

$$0\,,\frac{1}{36}\,,\frac{2}{36}\,,\frac{3}{36}\,,\frac{4}{36}\,,\frac{5}{36}\,,$$

respectively. Using these we have that if we let E be the event that $X_1 + X_2 = X_3$ we have that

$$P(E) = \frac{1}{6}(0) + \frac{1}{6}\left(\frac{1}{36}\right) + \frac{1}{6}\left(\frac{2}{36}\right) + \frac{1}{6}\left(\frac{3}{36}\right) + \frac{1}{6}\left(\frac{4}{36}\right) + \frac{1}{6}\left(\frac{5}{36}\right) = \frac{5}{72}\,,$$

when we simplify.

Next let F be the event that at least one of X_1, X_2, or X_3 is a two. The probability we seek to compute is $P(F|E)$ or

$$P(F|E) = \frac{P(F \cap E)}{P(E)} = \frac{0 + \frac{1}{6}\left(\frac{1}{36}\right) + \frac{1}{6}\left(\frac{2}{36}\right) + \frac{1}{6}\left(\frac{1}{36}\right) + \frac{1}{6}\left(\frac{2}{36}\right) + \frac{1}{6}\left(\frac{2}{36}\right)}{P(E)}\,.$$

When we evaluate this and simplify we get $P(F|E) = \frac{8}{15}$.

Problem 17

As $\angle DCF = \angle FCE = \angle ECB = \frac{90}{3} = 30$. In the right triangle $\triangle CBE$ we have

$$\tan(30) = \frac{6}{BC} \quad \text{so} \quad BC = 6\sqrt{3}\,.$$

Now $AD = BC = 6\sqrt{3}$ so $DF = AD - AF = 6\sqrt{3} - 2$. In the right triangle $\triangle CDF$ we have

$$\tan(30) = \frac{1}{\sqrt{3}} = \frac{DF}{CD} \quad \text{so} \quad CD = \sqrt{3}DF = 18 - 2\sqrt{3}\,.$$

This means that the area $[ABCD]$ is given by

$$[ABCD] = BC \cdot CD = 6\sqrt{3}(18 - 2\sqrt{3}) = 36(3\sqrt{3} - 1)\,.$$

Taking $\sqrt{3} \approx 1.7$ we get $[ABCD] \approx 147.6$.

Problem 18

Draw the circles with $C_{(2,0)}$ and $C_{(5,0)}$ the circles centered at $(2,0)$ and $(5,0)$ respectively. Next draw the common external tangent to these two circles as described in the problem. As the first circle has a larger radius than the second circle this line slants “downwards” and eventually will intersect the x-axis. Denote the point of tangency of this line with the circle

$C_{(2,0)}$ as A, the point of tangency of this line with the circle $C_{(5,0)}$ as B, this lines x-intercept as the point C, and take $D = (5, 0)$ and $E = (2, 0)$.

Then as $\angle EAC = \angle DBC = 90°$ we have the right triangles $\triangle EAC \sim \triangle DBC$. This means that

$$\frac{AE}{BC} = \frac{EC}{DC} \quad \text{or} \quad \frac{2}{1} = \frac{3 + DC}{DC}\,.$$

Solving we get $DC = 3$. This means that the point C is located at $C = (2+3+3, 0) = (8, 0)$.

Now in the right triangle $\triangle DBC$ we have

$$BC = \sqrt{CD^2 - BD^2} = \sqrt{9-1} = 2\sqrt{2}\,.$$

This means that $\tan(\angle BDC) = 2\sqrt{2}$. This is the slope of any segment that is perpendicular to the segment ABC so the slope of ABC is

$$-\frac{1}{2\sqrt{2}}\,.$$

Then ABC considered as a line has the slope above and goes through the point $C = (8, 0)$ and thus has the equation

$$y = -\frac{1}{2\sqrt{2}}(x-8) = -\frac{x}{2\sqrt{2}} + 2\sqrt{2}\,.$$

This line has a y-intercept of $2\sqrt{2}$.

Problem 19

Recall that the sum of the internal angles in a polygon is given by Equation 5. When $n = 6$ we get a angle sum of 720°. As our polygon is a *regular* hexagon this means that each internal angle has a measurement of $\theta = \frac{720}{6} = 120°$.

Let the length of the side of the larger *outer* hexagon be denoted by S. Then when we connect the midpoints of two sides of this hexagon we form a isosceles triangle with a vertex angle of θ and equal sides of length $\frac{S}{2}$. Let the length of the base of this isosceles triangle be s (which is also the length of the side of the internal hexagon). Using the law of cosigns we have that

$$s^2 = \left(\frac{S}{2}\right)^2 + \left(\frac{S}{2}\right)^2 - 2\left(\frac{S}{2}\right)^2 \cos(\theta) = \frac{S^2}{2} - \frac{S^2}{2}\left(-\frac{1}{2}\right) = \frac{3S^2}{4}\,.$$

This means that the internal hexagon has a side of length

$$s = \frac{\sqrt{3}}{2}S\,.$$

We now ask what is the area of a regular hexagon with a side of length w. As a regular hexagon is composed of six equilateral triangles with a side of length w its area is

$$6 \times \frac{\sqrt{3}}{4} w^2 = \frac{3\sqrt{3}}{2} w^2 .$$

Thus the given ratio is given by

$$\frac{A_{\text{inner}}}{A_{\text{outer}}} = \left(\frac{s}{S}\right)^2 = \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{4} .$$

Problem 20

Let $O = (0,0)$ be the origin of a Cartesian coordinate system let $O' = (6,8)$ be the origin of the circle and let P be the point $(12,16)$. Note that the line from O to P is given by

$$y = \frac{16}{12} x = \frac{4}{3} x .$$

Notice that the point $O' = (6,8)$ is *on* that line. This means that by symmetry, a path that goes "above" the circle has a symmetric path that goes "below" the circle that is the same arc length. Let the line $y = \frac{4}{3}x$ intersect the circle at the two points A and B with A closer to O than B. These would be points with x coordinates given by

$$(x-6)^2 + \left(\frac{4}{3}x - 8\right)^2 = 25 .$$

Solving this for x I find $x = 3$ or $x = 9$ with $y = 4$ or $y = 12$. This means that the points A and B are given by $A = (3,4)$ and $B = (9,12)$.

One might think that the shortest path from O to P would go from O to A in a straight line then around the circle to B and then from B to P in a straight line. We will now compute the length of that path. First we have that

$$\begin{aligned} |OA| &= \sqrt{3^2 + 4^2} = 5 \\ |BP| &= \sqrt{(12-9)^2 + (16-12)^2} = \sqrt{9+16} = 5 , \end{aligned}$$

and the length around the semicircle arc $\widehat{AB}$ is

$$\frac{2\pi(5)}{2} = 5\pi .$$

This gives a total length of $10 + 5\pi > 25.5$.

Another path would go from O to the point of tangency (denoted C) such that OC is tangent to the circle, then along an arc of the circle to a second point D such that the segment DP is also tangent to the circle. Now for point C to be tangent to our circle means that $OC \perp CO'$. This means that $\triangle OCO'$ is a right triangle. One of the legs of that triangle has a length

$|O'C| = 5$. Next as $|OA| = 5$ and $|AO'| = 5$ we have that the hypotenuse of this triangle has a length of $|OO'| = 10$. This means that

$$|OC| = \sqrt{OO'^2 - O'C^2} = \sqrt{100 - 25} = 5\sqrt{3}\,.$$

From these lengths we have that in that triangle that

$$\sin(\angle O'OC) = \frac{|O'C|}{|OO'|} = \frac{5}{10} = \frac{1}{2}\,,$$

thus $\angle O'OC = 30°$ and so $\angle OO'C = 90 - 30 = 60°$.

Lets do the same calculation for the right triangle $\triangle O'CP$. From before we had that $|BP| = 5$ and $|O'B| = |O'D| = 5$ so $|O'P| = |O'B| + |BP| = 5 + 5 = 10$. Using the Pythagorean theorem in the right triangle $\triangle O'DP$ we have

$$|DP| = \sqrt{O'P^2 - O'D^2} = \sqrt{100 - 25} = 5\sqrt{3}\,.$$

Like before we also have

$$\sin(\angle O'PD) = \frac{|O'D|}{|O'P|} = \frac{5}{10} = \frac{1}{2}\,,$$

so $\angle O'PD = 30°$ and so $\angle DO'P = 90 - 30 = 60°$.

This means that

$$\angle CO'D = 180 - \angle OO'C - \angle DO'P = 180 - 60 - 60 = 60\,.$$

With this angle we can compute the arc length of $\widehat{CD}$. In fact using what we know we have that the distance $|OC| + \widehat{CD} + |DP|$ is given by

$$|OC| + \widehat{CD} + |DP| = 5\sqrt{3} + \frac{60}{360}(2\pi(5)) + 5\sqrt{3} = 10\sqrt{3} + \frac{5\pi}{3} \approx 22.5565\,,$$

which is a smaller length than the first length we calculated.

Problem 21

Define the angles $x = \angle ABD$ and $y = \angle BAC$. As the triangle $\triangle ABD$ is isosceles we have

$$\angle D = \frac{180° - x}{2}\,.$$

As the triangle $\triangle BAC$ is isosceles we have

$$\angle C = \frac{180° - y}{2}\,.$$

This means that

$$\angle D + \angle C = 180° - \frac{1}{2}(x + y)\,.$$

From the right triangle $\triangle BEA$ we have $x + y = 90°$. This means that $\angle D + \angle C = 180 - 45 = 135°$.

Problem 23

Let our rectangular box have dimensions $h \times w \times l$. Then from the sum of the edge lengths we have that

$$4h + 4w + 4l = 140 \quad \text{or} \quad h + w + l = \frac{70}{2} = 35\,. \tag{559}$$

We are also told that

$$21^2 = w^2 + l^2 + h^2\,. \tag{560}$$

The total surface area is given by $2hw + 2wl + 2hl$. If we square Equation 559 we get

$$(h + w + l)^2 = 35^2\,,$$

or

$$h^2 + w^2 + l^2 + 2hw + 2hl + 2wl = 35^2\,.$$

Using Equation 560 in this we get

$$2hw + 2wl + 2hl = 35^2 - 21^2 = 784\,.$$

Problem 24

If there were *no* ones in the sequence of the first 1234 terms we would have a sum of

$$1234 \times 2 = 2468\,.$$

We now need to count the number of ones in the first n terms of the sequence. Note that from the sequence given they are located at the indexes

$$1\,, 3\,, 6\,, 10\,, 15\,, 21\,, \ldots\,.$$

Let p_n be the position (index location) in the sequence of the nth one. Then from the given sequence we have $p_1 = 1$ and

$$p_n = p_{n-1} + \left(\sum_{i=1}^{n-1} 1\right) + 1 = p_{n-1} + (n-1) + 1 = p_{n-1} + n\,. \tag{561}$$

Lets check that this formula is correct. Using it we find

$$\begin{aligned}
p_1 &= 1 \\
p_2 &= 1 + 2 = 3 \\
p_3 &= 3 + 3 = 6 \\
p_4 &= 6 + 4 = 10 \\
p_5 &= 10 + 5 = 15 \\
p_6 &= 21\,,
\end{aligned}$$

all of which are correct. Lets see if we can find a general solution to Equation 561 which we can write as

$$p_{n+1} = p_n + n + 1 \quad \text{for} \quad n \geq 1\,.$$

In terms of a forward difference this is

$$\Delta p_n = n + 1\,.$$

Summing both sides from $n = 1$ to $n = N - 1$ gives

$$p_N - p_1 = \sum_{n=1}^{N-1}(n+1) = \sum_{n=2}^{N} n = -1 + \sum_{n=1}^{N} n = -1 + \frac{N(N+1)}{2}\,.$$

This means that

$$p_N = \frac{N(N+1)}{2}\,.$$

We can check that this gives the correct values for p_N for $N \in \{1, 2, 3, 4, 5, 6\}$. We would now like to know the value of N such that $p_N \leq 1234$ or

$$\frac{N(N+1)}{2} \leq 1234\,.$$

Lets start looking for N's that solve $N^2 = 2468$ where we find $N \approx 50$. We then find for values around $N = 50$ we first have $p_N \leq 1234$. We find

$$\begin{aligned} p_{50} &= \frac{50(51)}{2} = 1275 \\ p_{49} &= 1225\,. \end{aligned}$$

Thus there are 49 ones in the given sequence up to the 1234's term. This means that the sum of all of the terms is

$$2468 - 49 = 2419\,.$$

Problem 25

Write the "constraint" as

$$x^2 - 14x + y^2 - 6y = 6\,,$$

or completing the square we have

$$(x-7)^2 - 49 + (y-3)^2 - 9 = 6\,,$$

or

$$(x-7)^2 + (y-3)^2 = 64\,.$$

This is a circle centered at $(7, 3)$ with a radius of $\sqrt{64} = 8$.

Now if we consider the line $3x + 4y = c$ for various values of c (say for example $c = 12$) we see that it is a line running "South-West". As we move this line in the "North-East" direction the value of the constant c will increase. This means that to maximize this linear function we need to move this line in a perpendicular direction until it is just tangent to the circle. This line could be written as

$$y = -\frac{3}{4}x + \frac{c}{4},$$

which means that the slope of the line perpendicular to this one (and in the direction we need to move the line) is given by

$$-\frac{1}{-\frac{3}{4}} = \frac{4}{3}.$$

We now ask what is the line with the slope of the above and passing though the center of the circle $(7, 3)$. This line would take the form

$$y - 3 = \frac{4}{3}(x - 7) \quad \text{or} \quad y = \frac{4}{3}x - \frac{19}{3}.$$

We then ask where does this line intersect our circle. Replacing $y - 3$ in the equation for the circle with the expression above gives

$$(x - 7)^2 + \frac{16}{9}(x - 7)^2 = 64.$$

Solving this for x we find

$$x = \frac{59}{5}.$$

Then y is given by

$$y = \frac{4}{3}\left(\frac{59}{5}\right) - \frac{19}{3} = \frac{47}{5}.$$

This point (x, y) has a value for $3x + 4y$ given by

$$3x + 4y = 3\left(\frac{59}{5}\right) + 4\left(\frac{47}{5}\right) = 73.$$

Problem 26

Let r, w, b, and g be the number of marbles of each "color". Let $n = r + w + b + g$ be the total number of marbles. Then the probabilities of each event are given by

$$P_a = \frac{\binom{r}{4}}{\binom{n}{4}} \tag{562}$$

$$P_b = \frac{\binom{w}{1}\binom{r}{3}}{\binom{n}{4}} \tag{563}$$

$$P_c = \frac{\binom{w}{1}\binom{b}{1}\binom{r}{2}}{\binom{n}{4}} \tag{564}$$

$$P_d = \frac{\binom{w}{1}\binom{b}{1}\binom{r}{1}\binom{g}{1}}{\binom{n}{4}}\,. \tag{565}$$

Setting Equation 562 equal to Equation 563 we can conclude

$$\frac{r(r-1)(r-2)(r-3)}{4!} = \left(\frac{w}{1}\right)\left(\frac{r(r-1)(r-2)}{3!}\right),$$

or

$$\frac{r-3}{4} = w \quad \text{or} \quad r = 3 + 4w\,. \tag{566}$$

Setting Equation 562 equal to Equation 564 we can conclude

$$\frac{r(r-1)(r-2)(r-3)}{4!} = \left(\frac{w}{1}\right)\left(\frac{b}{1}\right)\left(\frac{r(r-1)}{2!}\right),$$

or

$$\frac{(r-2)(r-3)}{4\cdot 3} = wb\,.$$

Using Equation 566 to replace w and by simplifying we can write this as

$$\frac{r-2}{3} = b \quad \text{or} \quad r = 2 + 3b\,. \tag{567}$$

Setting Equation 562 equal to Equation 565 we can conclude

$$\frac{r(r-1)(r-2)(r-3)}{4!} = \left(\frac{w}{1}\right)\left(\frac{b}{1}\right)\left(\frac{r}{1}\right)\left(\frac{g}{1}\right).$$

Using Equation 566 and 567 to replace w and b we get

$$\frac{r-1}{2} = g \quad \text{or} \quad r = 1 + 2g \,. \tag{568}$$

Each expression above is equal to r so

$$r = 3 + 4w = 2 + 3b = 1 + 2g \,.$$

We want to find the smallest w, b, and g such that the above is true. If we take $w \in \{1, 2, 3, 4, 5\}$ we get that r would be

$$\{7, 11, 15, 19, 23\} \,.$$

If we take $b \in \{1, 2, 3, 4, 5\}$ we get that r would be

$$\{5, 8, 11, 14, 17\} \,.$$

If we take $g \in \{1, 2, 3, 4, 5\}$ we get that r would be

$$\{3, 5, 7, 9, 11\} \,.$$

The first place each of these are equal is when $r = 11$ where $w = 2$, $b = 3$, and $g = 5$. Thus $n = r + w + b + g = 21$.

Problem 27

If we draw the two spheres described in this problem and look at the highest location "reached" for the bottom sphere ($z = 5.5$) and the lowest location "reached" for the upper sphere ($z = 4.5$) we see that any points in the region of overlap between the two spheres will need to have

$$4.5 \le z \le 5.5 \,.$$

As z must be an integer will we need to have $z = 5$. If we put this value of z into the two equations that represent the "upper" and "lower" sphere respectively we get the region of overlap must be the intersection of the points (x, y) such that

$$\begin{aligned} x^2 + y^2 + (5 - 10.5)^2 &\le 6^2 \\ x^2 + y^2 + 4^2 &\le 4.5^2 \,, \end{aligned}$$

or if we expand and simplify these we get

$$x^2 + y^2 \le 5.75 \tag{569}$$

$$x^2 + y^2 \le 4.25 \,. \tag{570}$$

The *intersection* of these two conditions is just Equation 570. Now notice that $\sqrt{4.25} \approx 2.06155$ so Equation 570 is

$$x^2 + y^2 \le 2.06155^2 \,.$$

The *integer* solutions to this can be enumerated and are for (x, y) in the set

$$\{(2,0)\,, (1,1)\,, (1,0)\,, (1,-1)\,, (0,2)\,, (0,1)\,, (0,0)\,, (0,-1)\,, (0,-2)\,, (-1,1)\,, (-1,0)\,, (-1,-1)\,, (-2,0)\} \,.$$

There are thirteen points in this set.

Problem 28

This is the problem of computing the distance from a plane and a point not on that plane. We can solve this in the general case using vector algebra. Let the points be located in a three dimensional Cartesian coordinate system where $B = (0,0,0)$, $C = (4,4,0)$, $D = (0,4,0)$, and $A = (0,4,3)$. Then two vectors in the plane spanned by the three points B, C, and A are

$$\overrightarrow{\mathbf{CB}} = (0,0,0) - (4,4,0) = -4\mathbf{i} - 4\mathbf{j}$$
$$\overrightarrow{\mathbf{CA}} = (0,4,3) - (4,4,0) = -4\mathbf{i} + 3\mathbf{k}\,.$$

Then a normal to this plane $\mathbf{N}$ is given by

$$\begin{aligned}
\mathbf{N} = \overrightarrow{\mathbf{CB}} \times \overrightarrow{\mathbf{CA}} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & -4 & 0 \\ -4 & 0 & 3 \end{vmatrix} \\
&= \mathbf{i}\begin{vmatrix} -4 & 0 \\ 0 & 3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} -4 & 0 \\ -4 & 3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} -4 & -4 \\ -4 & 0 \end{vmatrix} \\
&= \mathbf{i}(-12) - \mathbf{j}(-12) + \mathbf{k}(16) = -12\mathbf{i} + 12\mathbf{j} + 16\mathbf{k}\,.
\end{aligned}$$

A unit normal $\mathbf{n}$ is then given by

$$\mathbf{n} = \frac{\mathbf{N}}{||\mathbf{N}||} = \frac{1}{\sqrt{34}}(-3\mathbf{i} + 3\mathbf{j} + 4\mathbf{k})\,.$$

A vector from the plane to the point D is given by

$$\overrightarrow{\mathbf{BD}} = (0,4,0) - (0,0,0) = 4\mathbf{j}\,.$$

The distance we seek is then given by $|\overrightarrow{\mathbf{BD}} \cdot \mathbf{n}|$ or

$$\frac{12}{\sqrt{34}} = 2.05798\,,$$

which is closest to answer (C).

The 1997 AHSME Examination (AHSME 48)

Problem 1

To have $3a$ end in a nine we need $a = 3$. Then to have $ab = 3b$ end in a two we need $b = 4$ thus $a + b = 7$.

Problem 2

As the "left" side must move vertically up from the bottom to the top it must take $8+2 = 10$ units to do so. As the "top" must move horizontally a distance of 12. Thus the perimeter is $2(10) + 2(12) = 44$.

Problem 3

For this sum to be zero each term must be zero. Thus $x = 3$, $y = 4$, and $z = 5$. Thus we have $x + y + z = 12$.

Problem 4

We are told that $a = 1.5c = \frac{3}{2}c$ and

$$b = 1.25c = \left(1 + \frac{1}{4}\right) c = \frac{5}{4}c\,.$$

This last expression gives $c = \frac{4}{5}b$. Thus

$$a = \frac{3}{2} \cdot \frac{4}{5}b = \frac{6}{5}b = 1.2b\,,$$

which is 20% larger.

Problem 5

For each of the five congruent smaller rectangles let the shorter side of the rectangle have a length w and the longer side of the rectangle have a length h. Then from the placement of the five smaller rectangles in the larger rectangle it looks like the length of the "top" of the larger rectangle can be written in terms of lengths in the smaller rectangle as

$$3w\,,$$

and the "bottom" of the larger rectangle can be written as

$$2h\,.$$

Setting these two expressions equal gives $h = \frac{3}{2}w$.

In a similar way the perimeter of the larger rectangle can be written as

$$5w + 4h = 5w + 6w = 11w\,.$$

Setting this equal to 176 we find $w = 16$. With that we have that $h = 24$. The perimeter of a smaller rectangle is then given by

$$2w + 2h = 80\,.$$

Problem 6

This sum of the first 200 numbers can be written

$$\begin{aligned}
S &= \sum_{n=1}^{200}(-1)^{n+1}n \\
&= \sum_{n \text{ odd}} (-1)^{n+1}n + \sum_{n \text{ even}} (-1)^{n+1}n \\
&= \sum_{k=0}^{99}(-1)^{2k+1+1}(2k+1) + \sum_{k=1}^{100}(-1)^{2k+1}(2k) \\
&= \sum_{k=0}^{99}(2k+1) - \sum_{k=1}^{100}(2k) = 1 + \sum_{k=1}^{99}(2k+1-2k) - 200 \\
&= 1 + \sum_{k=1}^{99} 1 - 200 = 1 + 99 - 200 = -100\,.
\end{aligned}$$

This means that the average of the first 200 numbers is

$$\frac{-100}{200} = -\frac{1}{2}\,.$$

Problem 7

We would have

$$a + b + c + d + e + f + g = -1\,.$$

Now if all seven numbers were larger than 13 the sum would be also and could not equal -1. If six of the numbers were larger than 13 the last one could be taken to be

$$g = -1 - (a + b + c + d + e + f)\,,$$

and the sum would be -1. Thus we can have at most six numbers larger than 13.

Problem 8

The only places where there could be a pair n and m such that $n < m$ but $C(n) > C(m)$ would be if n and m were in different "regions" i.e. $1 \leq n \leq 24$ with $25 \leq m \leq 48$ etc. It might just be easiest to compute $C(n)$ for a number of values n and look for the required condition. We find

$$\begin{aligned} C(21) &= 252 \\ C(22) &= 264 \\ C(23) &= 276 \\ C(24) &= 288 \\ &\vdots \\ C(25) &= 275 \\ C(26) &= 286 \\ C(27) &= 297 \\ C(28) &= 308 \\ &\vdots \\ C(44) &= 484 \\ C(45) &= 495 \\ C(46) &= 506 \\ C(47) &= 517 \\ C(48) &= 528 \\ &\vdots \\ C(49) &= 490 \\ C(50) &= 500 \\ C(51) &= 510 \\ C(52) &= 520 \\ C(53) &= 530 \end{aligned}$$

From the above we see that for $n \in \{23, 24, 45, 46, 47, 48\}$ there is at least one m where $m > n$ and $C(n) > C(m)$. Thus there are six value of n.

Problem 9

The area we seek can be obtained from

$$[CDEF] = [ABCD] - [CFB] - [BAE] = 2^2 - [CFB] - \frac{1}{2}(2)(1) = 3 - [CFB]\,.$$

Thus we need to compute the area $[CFB]$. Note that

$$\tan \angle ABE = \frac{1}{2} \quad \text{and} \quad \tan \angle AEB = \frac{2}{1} = 2\,.$$

	1	2	~~3~~ 4	4	5	6
1	2	3	5	5	6	7
2	3	4	6	6	7	8
3	4	5	7	7	8	9
~~4~~ 3	4	5	7	7	8	9
5	6	7	9	9	10	11
6	7	8	10	10	11	12

Table 16: In the above table the rows represent possible values for the first die and columns represent possible values for the second die. Note the two adjustments to the faces of the two die as specified by the problem are show using the notation $\not{x}$. The values in the grid are the sum of the two faces.

As

$$\angle ABE + \angle AEB = 90^\circ$$
$$\angle ABE + \angle FBC = 90^\circ\,,$$

by subtracting these two we get $\angle FBC = \angle AEB$ so $\tan\angle FBC = \tan\angle AEB = 2$. This means that in the right triangle $\triangle BFC$ we have

$$\frac{FC}{FB} = 2 \quad \text{so} \quad FC = 2FB\,.$$

The Pythagorean theorem in that right triangle is

$$BF^2 + FC^2 = BC^2 \quad \text{or} \quad BF^2 + 4BF^2 = 4\,.$$

Thus $BF = \frac{2}{\sqrt{5}}$. This means that

$$[BFC] = \frac{1}{2}BF \cdot FC = \frac{1}{2}BF(2FB) = BF^2 = \frac{4}{5}\,,$$

and

$$[CDEF] = 3 - [CFB] = 3 - \frac{4}{5} = \frac{11}{5}\,.$$

Problem 10

In Table 16 we show the possible sums that can come from two die that are as specified in the problem statement. Note that there are $6 \times 6 = 36$ possible outcomes of which 20 are odd to give a probability of $\frac{20}{36} = \frac{5}{9}$.

Problem 11

Let the scores on each of the ten games be denoted s_i for $1 \le i \le 10$. Then we are told that

$$\frac{1}{9}\sum_{i=1}^{9} s_i > \frac{1}{5}\sum_{i=1}^{5} s_i\,,$$

or using the numbers given that

$$\frac{1}{9}\left(\sum_{i=1}^{5} s_i + 23 + 14 + 11 + 20\right) > \frac{1}{5}\sum_{i=1}^{5} s_i\,.$$

We can solve the above for $S_5 \equiv \sum_{i=1}^{5} s_i$ to get

$$S_5 < 85\,. \tag{571}$$

Next we are told that

$$\frac{1}{10}\left(S_5 + (23 + 14 + 11 + 20) + s_{10}\right) > 18\,.$$

Solving this for s_{10} we find

$$s_{10} > 112 - S_5\,. \tag{572}$$

For s_{10} to be as small as possible must have S_5 as large as possible. From Equation 571 this means that we need $S_5 = 84$. From Equation 572 we get that $s_{10} > 112 - 84 = 28$. Thus $s_{10} = 29$.

Problem 12

If $mb > 0$ then m and b must be of the same sign. The given line will have an x-intercept when $y = 0$ at the point $x = -\frac{b}{m}$. As m and b are the same sign this expression is always negative. Notice that (E) has a positive x-intercept and thus is not possible.

Problem 13

Let our integer N be written as $N = 10t + u$ with $1 \le t \le 9$ and $0 \le u \le 9$. Then the reversed digits number N' is $N' = 10u + t$. Then

$$N + N' = 10(t + u) + t + u = 11(t + u)\,.$$

We want $11(t + u)$ to be a perfect square. This means that it must equal $11^2 n^2$ for $n \ge 1$ so that $t + u = 11n^2$. If $n = 2$ then this is

$$t + u = 44\,,$$

which is not possible with the values for t and u. Thus $n = 1$ and we must have

$$t + u = 11\,.$$

There are solutions to this when

$$(t, u) \in \{(2, 9)\,, (3, 8)\,, (4, 7)\,, (5, 6)\}\,,$$

and solutions exchanging t and u. This gives a total of $4 + 4 = 8$ solutions.

Problem 14

If P_n is the population in year n then we are told that

$$P_{n+2} - P_n = CP_{n+1}\,, \tag{573}$$

for some constant C. If we take $n = 1994$ then Equation 573 is

$$P_{1996} - P_{1994} = CP_{1995} \quad \text{or} \quad P_{1996} - 39 = 60C\,. \tag{574}$$

If we take $n = 1995$ then Equation 573 is

$$P_{1997} - P_{1995} = CP_{1996} \quad \text{or} \quad 123 - 60 = 60P_{1996}\,. \tag{575}$$

This last equation is equal to $CP_{1996} = 63$. If we multiply Equation 574 by P_{1996} we get

$$P_{1996}^2 - 39P_{1996} = 60CP_{1996} = 60 \cdot 63 = 3780\,.$$

This is a quadratic equation for P_{1996} with solutions $\{-45, 84\}$. We need a positive solution so $P_{1996} = 84$.

Problem 15

Recall that medians of a triangle divide the triangle into two triangles with equal areas and that the the point where the medians intersect divide those segments in the ratio of $2:1$. The statement about the are means that

$$[ABC] = 2[ADB] = 2[CEB]\,,$$

and the statement about the lengths means that

$$BG = \frac{2}{3}BD = \frac{2}{3}(8) = \frac{16}{3} \quad \text{and} \quad GD = \frac{1}{3}BD = \frac{1}{3}(8) = \frac{8}{3}\,.$$

and

$$EG = \frac{1}{3}EC = \frac{1}{3}(12) = 4 \quad \text{and} \quad CG = \frac{2}{3}EC = \frac{2}{3}(12) = 8\,.$$

Using these lengths we have that

$$[BEC] = [BEG] + [BGC] = \frac{1}{2}EG \cdot GB + \frac{1}{2}GB \cdot GC = \frac{1}{2}(4)\left(\frac{16}{3}\right) + \frac{1}{2}\left(\frac{16}{3}\right)(8) = \frac{12 \cdot 16}{2 \cdot 3}\,.$$

Thus

$$[ABC] = 2[BEC] = \frac{12 \cdot 16}{3} = 64\,.$$

Problem 17

When $x = k$ the y values of these two graphs are $\log_5(k)$ and $\log_5(k+4)$. Thus from what we are told about the vertical distance between these two points we have

$$\log_5(k+4) - \log_5(k) = \frac{1}{2},$$

or

$$\log_5\left(\frac{k+4}{k}\right) = \frac{1}{2},$$

or

$$1 + \frac{4}{k} = 5^{1/2} = \sqrt{5}.$$

Thus

$$\frac{4}{k} = \sqrt{5} - 1,$$

or

$$\frac{k}{4} = \frac{1}{\sqrt{5}-1} = \frac{1}{\sqrt{5}-1}\left(\frac{\sqrt{5}+1}{\sqrt{5}+1}\right) = \frac{\sqrt{5}+1}{5-1}.$$

Thus $k = 1 + \sqrt{5}$ so $a = 1$ and $b = 5$ and $a + b = 6$.

Problem 18

Assume that there are N numbers in our data set $\{x_i\}_{i=1}^N$ then from the fact about the mean we have

$$\frac{1}{N}\sum_{i=1}^{N} x_i = 22.$$

From the information we are given 32 is the most common element so there must be at least two of them in our dataset, and the number 10 is also in our dataset, and the number m is in our dataset (this means that N is odd). Let the index of the median m be denoted as i^*. If m is replaced by $m + 10$ our new mean is 24 means that

$$\frac{1}{N}\left(\sum_{i=1;i\neq i^*}^{N} x_i + (m+10)\right) = 24.$$

Using the above this means that

$$\frac{22N + 10}{10} = 24 \quad \text{so} \quad N = 5.$$

This means that $i^* = 3$ and at this point our ordered list of numbers looks like

$$\{10, x_2, m, 32, 32\}. \tag{576}$$

As the median of the list when m is replaced by $m+10$ is $m+10$ that means that $m+10 \leq 32$ so $m \leq 22$.

If m is replaced by $m-8$ our ordered list of numbers would look like

$$\{10\,,m-8\,,x_2\,,32\,,32\} \quad \text{or} \quad \{m-8,10\,,x_2\,,32\,,32\}\,,$$

depending on whether $m-8$ is less than 10 or not. In either case the median is x_2 which we are told equals $m-4$ and thus the original ordered set of numbers in Equation 576 looks like

$$\{10\,,m-4\,,m\,,32\,,32\}\,.$$

The average of these numbers being 22 means that

$$\frac{1}{5}(10+m-4+m+32+32)=22 \quad \text{so} \quad m=20\,.$$

Problem 19

Denote the center of the circle as O, then dropping a perpendicular from O to the x-axis denote the intersection of that perpendicular with the x-axis as the point O_x. In the same way denote the intersection of the horizontal line from O and perpendicular to the y-axis as O_y. Finally denote the tangent point of the segment BC with the circle as T.

From the angles given in the triangle (and the length of the segment AB) we have

$$AB = BC\cos(60^\circ) = BC\left(\frac{1}{2}\right) \quad \text{so} \quad BC = 2AB = 2$$
$$AC = BC\cos(30^\circ) = BC\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}\,.$$

Next note that

$$BO_x = BT \quad \text{and} \quad CO_y = CT\,, \tag{577}$$

as both segments (in each pair) are tangents to a circle from the same external point (and are thus equal in length). Now $OO_y = OO_x = OT = R$ the radius of the circle. Then from the given figure and the above we have

$$R = AC + CO_y = \sqrt{3} + CT$$
$$R = AB + CO_x = 1 + BT\,.$$

If we add these two equations we get

$$2R = 1 + \sqrt{3} + (CT + BT)\,. \tag{578}$$

As $CT + BT = BC = 2$ the above is

$$2R = 1 + \sqrt{3} + 2 \quad \text{so} \quad R = \frac{3+\sqrt{3}}{2}\,.$$

To evaluate R note that $\sqrt{3} \approx 1.73$ so $R \approx 2.365 \approx 2.37$.

Problem 20

Let the integer we sum start at x and thus the terms in the series can all be written as $x+i$ for $0 \le i \le 99$. This means that our sum S is given by

$$S = \sum_{i=0}^{99}(x+i) = 100x + \sum_{i=0}^{99} i = 100x + \frac{99(100)}{2} = 100\left(x + \frac{99}{2}\right) = 100x + 4950\,.$$

This means that any such sum must end with 50. Only choice (A) does that.

Problem 21

Note that

$$\log_8(n) = \frac{\log_2(n)}{\log_2(8)} = \frac{1}{3}\log_2(n)\,.$$

The later expression will be rational if and only if n is a power of two. Now we have

$$\begin{aligned}
2^1 &= 2\\
2^2 &= 4\\
2^3 &= 8\\
2^4 &= 16\\
2^5 &= 32\\
2^6 &= 64\\
2^7 &= 128\\
2^8 &= 256\\
2^9 &= 512\\
2^{10} &= 1024\\
2^{11} &= 2048 > 1997\,.
\end{aligned}$$

Thus the desired sum S is

$$S = \sum_{n=1}^{1997} f(n) = \sum_{p=1}^{10} f(2^p) = \frac{1}{3}\sum_{p=1}^{10} p = \frac{1}{3}\left(\frac{10(11)}{2}\right) = \frac{55}{3}\,.$$

Problem 22

Let the first letter in each persons name represent the amount of dollars held when they went shopping. Then we are told that

$$A + B + C + D + E = 56\,,$$

and

$$
\begin{aligned}
|A - B| = 19 \quad &\text{or} \quad A - B = 19s_1 \\
|B - C| = 7 \quad &\text{or} \quad B - C = 7s_2 \\
|C - D| = 5 \quad &\text{or} \quad C - D = 5s_3 \\
|D - E| = 4 \quad &\text{or} \quad D - E = 4s_4 \\
|E - A| = 7 \quad &\text{or} \quad E - A = 11s_5 \,.
\end{aligned}
$$

Here $s_i \in \{-1, +1\}$ for $1 \leq i \leq 5$ is the "sign" needed to "evaluate" the absolute value expressions above. Now if we add all of these equations together we get

$$
(A + B + C + D + E) - (B + C + D + E + A) = 19s_1 + 7s_2 + 5s_3 + 4s_4 + 11s_5 \,,
$$

or

$$
19s_1 + 7s_2 + 5s_3 + 4s_4 + 11s_5 = 0 \,. \tag{579}
$$

We need to determine what settings of signs s_i above will make the above true. As the right-hand-side is zero any positive/negative settings for s_i will have another solution where we take the negative of each s_i i.e. if $\{s_i\}_{i=1}^{5}$ is a solution to the above equation then another solution is $\{-s_i\}_{i=1}^{5}$.

Now by inspection if s_i is positive for all i we cannot have a solution. In the same way, taking only one s_i negative and all of the others positive it is not possible to produce a solution. In the case where two of s_i are negative, by trying various possible combinations for s_i we find a solution with $s_1 = s_4 = -1$ and the other $s_i = 1$. By the symmetry argument above there can be no solutions with three, four, or five negative s_i.

When $s_1 = s_4 = -1$ we need to solve the system

$$
\begin{aligned}
A + B + C + D + E &= 56 \\
A - B &= -19 \\
B - C &= 7 \\
C - D &= 5 \\
D - E &= -4 \\
E - A &= 11 \,.
\end{aligned}
$$

To solve this system we will write everything in terms of a single variable (say A). To do this note that from these equations we have $A = -19 + B$ and $B = 7 + C$ so $A = -19 + 7 + C = -12 + C$. Next $C = 5 + D$ so $B = 12 + D$ and $A = -7 + D$. Next $D = -4 + E$ so $C = 1 + E$, $B = 8 + E$, and $A = -11 + E$. Next $E = 11 + A$ so $D = 7 + A$, $C = 12 + A$, $B = 19 + A$. Putting these expressions (with everything in terms of A) into the first equation gives

$$
A + (19 + A) + (12 + A) + (7 + A) + (11 + A) = 56 \,.
$$

This has the solution $A = \frac{7}{5}$ which is not an integer and cannot be a solution to our problem.

As mentioned above another solution corresponds to the negation of the one we found above namely $s_1 = s_4 = +1$ and the other $s_i = -1$. In that case we need to solve the system

$$\begin{aligned} A + B + C + D + E &= 56 \\ A - B &= 19 \\ B - C &= -7 \\ C - D &= -5 \\ D - E &= 4 \\ E - A &= -11 \,. \end{aligned}$$

To solve this system we will again write everything in terms of a single variable (say E). To do this note that from these equations we have $A = 19 + B$ and $B = -7 + C$ so $A = 12 + C$. Next $C = -5 + D$ so $B = -12 + D$ and $A = 7 + D$. Next $D = 4 + E$ so $C = -1 + E$, $B = -8 + E$, and $A = 11 + E$. Putting these expressions (with everything in terms of E) into the first equation gives

$$(11 + E) + (-8 + E) + (-1 + E) + (4 + E) + E = 56 \,.$$

This has the solution $E = 10$. Using that we find that

$$\begin{aligned} A &= 21 \\ B &= 2 \\ C &= 9 \\ D &= 14 \,. \end{aligned}$$

Problem 23

When we imagine this folded it looks to be a unit cube with a corner cut off. Thus if V_c is the volume of this corner the volume we are asked to find is $1^3 - V_c$. By symmetry, the volume of the corner cut off is the same as the volume in the positive octant and below the plane with points $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. This is a pyramid with a "base" given by a isosceles right triangle with legs of length one and a hypotenuse of length $\sqrt{2}$ and a height of one. Thus this pyramid has a volume

$$V_c = \frac{1}{3}\left(\frac{1}{2}1^2\right)1 = \frac{1}{6} \,.$$

This means that the volume we seek is $1 - \frac{1}{6} = \frac{5}{6}$.

Problem 24

Let $R_{.....}$ be the number of rising numbers with unspecified digits in the five locations. Then we are told that

$$R_{.....} = \binom{9}{5} = 126 \,.$$

We can compute

$$\begin{aligned}
R_{1\ldots.} &= \begin{pmatrix} 8 \\ 4 \end{pmatrix} = 70 \\
R_{2\ldots.} &= \begin{pmatrix} 7 \\ 4 \end{pmatrix} = 35 \\
R_{3\ldots.} &= \begin{pmatrix} 6 \\ 4 \end{pmatrix} = 15 \\
R_{4\ldots.} &= \begin{pmatrix} 5 \\ 4 \end{pmatrix} = 5 \\
R_{5\ldots.} &= \begin{pmatrix} 5 \\ 4 \end{pmatrix} = 1\,.
\end{aligned}$$

As $70+35=105>97$ the 97th rising number starts with a two and does not have a one. All rising numbers that start with a two will be made up of the digits $\{3,4,5,6,7,8,9\}$. Thus we can compute that

$$\begin{aligned}
R_{23\ldots} &= \begin{pmatrix} 6 \\ 3 \end{pmatrix} = 20 \\
R_{24\ldots} &= \begin{pmatrix} 5 \\ 3 \end{pmatrix} = 10\,.
\end{aligned}$$

As $70+20+10=100>97$ the 97th rising number starts with 24 and does not contain a one or a three. All rising numbers that start with 24 will be made up of the digits $\{5,6,7,8,9\}$. Thus we can compute that

$$\begin{aligned}
R_{245\ldots} &= \begin{pmatrix} 4 \\ 2 \end{pmatrix} = 6 \\
R_{246\ldots} &= \begin{pmatrix} 3 \\ 2 \end{pmatrix} = 3\,.
\end{aligned}$$

As $70+20+6=96<97$ the 97th rising number starts with 246. These three numbers are

$$24678\,,24679\,,24689\,.$$

Thus the 97th rising number is 24678 and does not have a one, three, or a five in it.

Problem 26

Define $\theta = \angle ACB = \frac{1}{2}\angle APB$. Then using the fact that $AP = BP = 3$ and the law of cosigns we get

$$\begin{aligned}
AB^2 &= AP^2 + BP^2 - 2AP \cdot BP\cos(2\theta) \\
&= 9 + 9 - 2\cdot 9\cos(2\theta) = 18 - 18\cos(2\theta) = 18(1-\cos(2\theta)) \\
&= 18(1 - 2\cos^2(\theta) + 1) = 36(1-\cos^2(\theta)) \\
&= 36\sin^2(\theta)\,.
\end{aligned}$$

This means that $AB = 6\sin(\theta)$.

Now if we drop a perpendicular from P to the segment AB (intersecting at the point P') we have $\angle APP' = \theta = \angle BPP'$ and

$$AP' = P'B = PB\sin(\theta) = 3\sin(\theta) \quad \text{and} \quad PP' = PB\cos(\theta) = 3\cos(\theta)\,.$$

Let the segment PP' intersect the segment AD at the point E. Note that with that point we have

$$\triangle PED \sim \triangle CBD\,,$$

so

$$\frac{PD}{DC} = \frac{ED}{BD} \quad \text{or} \quad \frac{2}{DC} = \frac{ED}{1} \quad \text{or} \quad ED = \frac{2}{DC}\,.$$

Next using the angle bisector theorem in triangle $\triangle APD$ we have

$$\frac{AP}{AE} = \frac{PD}{DE} \quad \text{or} \quad \frac{2}{AE} = \frac{2}{DE} \quad \text{or} \quad AE = \frac{3}{2}ED\,.$$

Now

$$AD = AE + ED + \frac{3}{2}ED + ED = \frac{5}{2}ED\,.$$

This means that

$$AD \cdot DC = \left(\frac{5}{2}ED\right) DC = \frac{5}{2}\left(\frac{2}{DC}\right) DC = 5\,.$$

Problem 27

This expression is

$$f(x) = f(x+4) + f(x-4)\,. \tag{580}$$

If we add four to x we get

$$f(x+4) = f(x+8) + f(x) = f(x+8) + f(x+4) + f(x-4)\,,$$

where we have used Equation 580 to replace $f(x)$. The above is equivalent to

$$f(x-4) = -f(x+8)\,.$$

If we add four to x this is

$$f(x) = -f(x+12)\,. \tag{581}$$

Now using the above expression twice we have

$$f(x+24) = f((x+12)+12) = -f(x+12) = -(-f(x)) = f(x)\,.$$

Thus $p = 24$.

Problem 28

We seek to find solutions to

$$|a+b|+c=19 \tag{582}$$
$$ab+|c|=97\,. \tag{583}$$

In complicated equations (like this one) it can be helpful to first look for *symmetries* which might simplify the problem and guide solutions. In the above notice that if (a,b) is a solution then so is $(-a,-b)$. Another symmetry is to note that if $(a,b)=(x,y)$ is a solution to the above then so is $(a,b)=(y,x)$.

Note that we can write the above as

$$|a+b|=19-c \tag{584}$$
$$|c|=97-ab\,. \tag{585}$$

Now as $|x|\geq 0$ for all x using the above we have that

$$19-c\geq 0 \quad \text{so} \quad c\leq 19\,, \tag{586}$$

and that

$$97-ab\geq 0 \quad \text{so} \quad ab\leq 97\,.$$

As $c\leq 19$ we have $|c|\leq 19$ so from Equation 585 we have that

$$97-ab=|c|\leq 19 \quad \text{so} \quad ab\geq 78\,.$$

Taken together we get

$$78\leq ab\leq 97\,. \tag{587}$$

As ab must then be positive we have either $a<0$ and $b<0$ or $a>0$ and $b>0$ i.e. they must be of the same sign and we have $a+b<0$ or $a+b>0$ depending. Without loss of generality lets assume that $a>0$ and $b>0$. Now in this case if $c>0$ then Equations 582 and 583 give

$$a+b+(97-ab)=19\,,$$

or

$$ab-a-b=78\,,$$

or

$$ab-a-b+1=79\,,$$

or

$$(a-1)(b-1)=79\,.$$

Now as 79 is prime the only solutions to this are

$$a-1=1 \quad \text{and} \quad b-1=79 \quad \text{or} \quad a-1=79 \quad \text{and} \quad b-1=1\,.$$

Solving these we get $(a,b)=(2,80)$ and $(a,b)=(80,2)$. Using these in Equation 582 we find that $c=-63$ which is not positive as was assumed in the beginning. Thus there are no solutions with $c>0$.

If we assume that $c < 0$ then Equations 582 and 583 give

$$\begin{aligned} a + b + c &= 19 \\ ab - c &= 97 \,. \end{aligned}$$

Adding these we get

$$ab + a + b = 116 \,,$$

or

$$(a+1)(b+1) = 117 \,.$$

Now as $117 = 3^2 \cdot 13$ the solutions to this are when

$$\begin{aligned} a + 1 &= 1 \\ a + 1 &= 3 \\ a + 1 &= 3^2 \\ a + 1 &= 13 \\ a + 1 &= 3 \cdot 13 \\ a + 1 &= 3^2 \cdot 13 \,. \end{aligned}$$

with $b + 1 = \frac{117}{a+1}$ in each case. Solving these we find

$$\begin{aligned} (a, b) &= (0, 116) \\ (a, b) &= (2, 38) \\ (a, b) &= (8, 12) \\ (a, b) &= (12, 8) \\ (a, b) &= (38, 2) \\ (a, b) &= (116, 0) \,. \end{aligned}$$

For these solutions we find

$$a + b \in \{116, 40, 20, 20, 40, 116\} \,.$$

and thus

$$c \in \{-97, -21, -1, -1, -21, -97\} \,,$$

all of which are negative as they must be under our assumptions. There are six solutions found here. There are another six solutions if we consider $-a$ and $-b$. This gives a total of 12 solutions.

The 1998 AHSME Examination (AHSME 49)

Problem 1

In the grid the location II shares three sides with other rectangles. By trial and error this could be the rectangle D since it has three sides it could "share" with other rectangles. Once we have placed rectangle D, to its left we place E, to its right we place A, below it we place C and in the South-West direction we place B. Thus the rectangles are placed

```
EDA
BC
```

The rectangle at position I is then E.

Problem 2

To make the fraction as large as possible we want $A + B$ to be as large as possible while $C + D$ is a small as possible. To make $C + D$ as small as possible we would sum the two smallest digits so $C + D = 0 + 1 = 1$. Thus $\frac{A+B}{C+D} = A + B$ which will be an integer. This will be largest when $A + B$ is the sum the two largest digits or $8 + 9 = 17$.

Problem 3

To subtract b from a 2 and get a 3 means that we must borrow from the a digit in the 10s place to get

$$12 - b = 3 \quad \text{so} \quad b = 9 .$$

To next subtract 8 from $a-1$ and get a 7 we will have to borrow from the 7 in the hundredths place (making it a 6) and we have

$$1(a - 1) - 8 = 7 .$$

Here $1(a - 1)$ is the two digit number with its first digit a one and its second digit the value $a - 1$. This will be true if $1(a - 1) = 15$ so that $a - 1 = 5$ so $a = 6$.

Finally we have the subtraction of the hundreds digits to have $6 - 4 + 2 = c$. This means that

$$a + b + c = 6 + 9 + 2 = 17 .$$

a	b	$\|a-b\|$
$3^2 = 9$	$2 \cdot 3 \cdot 37 = 222$	213
$3^3 = 27$	$2 \cdot 37 = 74$	47
$2 \cdot 3^3 = 54$	37	17

Table 17: Choices of a and b where $ab = 1998$.

Problem 4

From the definition of $[a, b, c]$ we see that

$$[60, 30, 90] = \frac{90}{90} = 1$$
$$[2, 1, 3] = 1$$
$$[10, 5, 15] = 1\,.$$

Thus the expression we want to evaluate is equivalent to $[1, 1, 1] = \frac{1+1}{1} = 2$.

Problem 5

Call this expression E. The we can write E as

$$E = 2^{1995}(2^3 - 2^2 - 2^1 + 1) = (8 - 4 - 2 + 1)2^{1995} = 3 \cdot 2^{1995}\,.$$

Problem 6

Write 1998 as $1998 = 2 \cdot 3^3 \cdot 37$. Then for different integers a and b we have Table 17. From that table we see the smallest difference is 17.

Problem 7

Call this expression E. We have

$$E = \sqrt[3]{N\sqrt[3]{N\sqrt[3]{N}}} = \sqrt[3]{N\sqrt[3]{N^{4/3}}} = \sqrt[3]{NN^{4/9}} = \sqrt[3]{N^{\frac{13}{9}}} = N^{\frac{13}{27}}\,.$$

Problem 8

The area of each region must be $\frac{1}{3}(1^2) = \frac{1}{3}$. Setting that equal to the area of the North-East trapezoid we get

$$\frac{1}{3} = \frac{1}{2} \cdot \frac{1}{2}\left(\frac{1}{2} + x\right)\,.$$

Solving for x gives $x = \frac{5}{6}$.

Problem 9

Let N be the number of members in the audience. Then $0.2N$ heard the entire talk and $0.1N$ heard none of the talk. Of the remaining

$$1N - 0.3N = 0.7N\,,$$

audience members one-half heard $\frac{1}{3}$ and the other one-half heard $\frac{2}{3}$. These are $0.5(0.7N) = 0.35N$ people each. Thus the average fraction of the talk heard is

$$\frac{0.2N(1) + 0.1N(0) + 0.35N\left(\frac{1}{3}\right) + 0.35N\left(\frac{2}{3}\right)}{N} = 0.55\,.$$

Multiplying this by the length of the talk (60 minutes) gives an average length of 33 minutes.

Problem 10

Let the outer square have a length denoted by S. Let the long side of the rectangle have a length of H and the short side have a length of W. Then from how the rectangles are placed in the square we have $S = H + W$. As we are told the perimeter of each rectangle we have

$$2H + 2W = 2(H + W) = 14 \quad \text{so} \quad H + W = 7\,.$$

Thus the area of the larger square is $S^2 = (H + W)^2 = 7^2 = 49$.

Problem 11

There are $\left(\begin{array}{c}4\\2\end{array}\right) = 6$ pairs of points from the rectangle that could be diagonals of a circle. Two of these pairs are the two diagonals of the rectangle. Recall that the circumscribing circle of this rectangle has its diameter the diagonal (either one) of the rectangle. Thus there are $6 - 1 = 5$ unique circles.

Problem 12

We can write this as

$$\log_3(\log_5(\log_7(N))) = 2^{11}\,.$$

Define $a = 2^{11}$ then the above is equivalent to

$$\log_5(\log_7(N)) = 3^a\,.$$

Define $b = 3^a$ then the above is

$$\log_7(N) = 5^b \,.$$

Define $c = 5^b$ then the above is

$$N = 7^c \,.$$

Thus N has only seven in its prime factorization.

Problem 13

We are told that

$$X_1 X_2 X_3 X_4 = 144 = 2^4 \cdot 3^2 \,,$$

for X_i the value shown on the upper face of the ith die. From this we see that we can have at most two sixes and no fives.

If we have no sixes then we must have two threes and two fours to give a sum of

$$3 + 3 + 4 + 4 = 14 \,.$$

If we have one six then we must have one three, one four, and one two for a sum of

$$6 + 3 + 4 + 2 = 15 \,.$$

If we have two sixes we have reduced our product by $6^2 = 2^2 \cdot 3^2$ and thus have 2^2 left in our product. This means that the two remaining die rolls can be a four or a one or two twos. These gives sums of

$$\begin{aligned} 6 + 6 + 4 + 1 &= 17 \\ 6 + 6 + 2 + 2 &= 16 \,. \end{aligned}$$

Thus it looks like the possible values for the sum are given by $\{14, 15, 16, 17\}$ thus 18 is not possible.

Problem 14

For this y we have

$$\begin{aligned} y = ax^2 + bx + c &= a\left(x^2 + \frac{b}{a}x\right) + c \\ &= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) - \frac{b^2}{4a} + c \\ &= a\left(x + \frac{b}{a}\right)^2 - \frac{b^2}{4a} + c \,. \end{aligned} \quad (588)$$

As the vertex is at $(4, -5)$ we know that $\frac{b}{a} = -4$ and $-\frac{b^2}{4a} + c = -5$. The first expression gives

$$b = -4a\,. \tag{589}$$

Putting that in the second expression we get

$$-\frac{(4a)^2}{4a} + c = -5 \quad \text{so} \quad c = 4a - 5\,. \tag{590}$$

Both of these conditions in Equation 588 give

$$y = a(x-4) - 5\,.$$

To have two positive roots we must have $a > 0$. From Equation 589 we see that $b < 0$. From Equation 590 we don't have enough information to determine the sign of c.

Problem 15

To start we recall that the area of an equilateral triangle with a side length s is given by 198. For this problem let the side of the equilateral triangle be given by t (with an area of T) and the side of the regular hexagon be given by h (with an area of H). Thus we have

$$T = \frac{\sqrt{3}}{4}t^2\,.$$

As a regular hexagon is composed of six equilateral triangles (of side length h) we have

$$H = 6\left(\frac{\sqrt{3}}{4}h^2\right)\,.$$

As these are equal we have $T = H$. In that expression we can solve for $\frac{t}{h}$ and find $\frac{t}{h} = \sqrt{6}$.

Problem 16

To get the area of the shaded region we can take one-half the area of the largest/outer circle and add one-half of the area of the smallest/left-most circle and then subtract one-half of the area of the medium/right-most circle. This would be the expression

$$A_s = \frac{1}{2}\pi(a+b)^2 + \frac{1}{2}\pi a^2 - \frac{1}{2}\pi b^2 = \pi a(a+b)\,,$$

when we simplify. To get the area of the unshaded region we can take one-half the area of the largest/outer circle and add one-half of the area of the medium/right-most circle and then subtract one-half of the area of the smallest/left-most circle. This would be the expression

$$A_u = \frac{1}{2}\pi(a+b)^2 + \frac{1}{2}\pi b^2 - \frac{1}{2}\pi a^2 = \pi b(a+b)\,,$$

when we simplify. The ratio we seek is then

$$\frac{A_s}{A_u} = \frac{\pi a(a+b)}{\pi b(a+b)} = \frac{a}{b}\,.$$

Problem 17

Let $y = 0$ to get $f(x) = x + f(0)$. As $f(0) = 2$ if we take $x = 0$ we must have $f(0) = 2$. Thus

$$f(x) = x + 2\,.$$

This means that $f(1998) = 2000$.

Problem 18

Let r be the common radius. Then if h_A and h_M are the heights of the right circular cone and the right circular cylinder respectively then we have

$$\begin{aligned} A &= \frac{1}{3}h_A(\pi r^2) \\ M &= h_M(\pi r^2) \\ C &= \frac{4}{3}\pi r^3\,. \end{aligned}$$

We are then told that $h_A = h_M = 2r$ so the above become

$$\begin{aligned} A &= \frac{2}{3}\pi r^3 \\ M &= 2\pi r^2 \\ C &= \frac{4}{3}\pi r^3\,. \end{aligned}$$

From these notice that $A - M = -\frac{4}{3}\pi r^3 = -C$.

Problem 19

We let $A = (-5, 0)$ and $B = (+5, 0)$ and have the third point be denoted C. Then when drawing this triangle in the x-y Cartesian coordinate plane (with $\theta > 0$ for example) if we associate the "base" with the segment from $(-5, 0)$ to $(+5, 0)$ and the height with the vertical from the point C to the x-axis then we have the area of the triangle given by

$$A = \frac{1}{2}bh = \frac{1}{2}(10)(5\sin(\theta)) = 25\sin(\theta)\,.$$

Setting this equal to 10 gives

$$\sin(\theta) = \frac{2}{5}\,.$$

There will be four values of θ which have $\sin(\theta) = \pm\frac{2}{5}$ (one in each quadrant).

Problem 21

Let v_s and v_w be the velocity of Sunny and Windy when they run. On the first race in an amount of time T Sunny ran a length h or

$$v_s T = h \,,$$

while Windy ran a length $h - d$ or

$$v_w T = h - d \,.$$

Dividing these two we get

$$\frac{v_w}{v_s} = \frac{h - d}{h} \,.$$

Now in the second race Sunny will finish in a time T' given by

$$T' = \frac{h + d}{v_s} \,.$$

At this time T' Windy will be at the location

$$v_w T' = \frac{v_w}{v_s}(h + d) = \frac{h^2 - d^2}{h} \,.$$

The distance between the two runners is then

$$h - v_w T' = h - \frac{h^2 - d^2}{h} = \frac{d^2}{h} \,.$$

Problem 22

Recall that $\log_b n = \frac{\ln(n)}{\ln(b)}$ where ln is the natural log. Using this we can write our sum as

$$S = \sum_{k=1}^{100} \frac{\ln(k)}{\ln(100!)} = \frac{1}{\ln(100!)} \sum_{k=1}^{100} \ln(k) = \frac{1}{\ln(100!)} \ln\left(\prod_{k=1}^{100} k\right) = \frac{1}{\ln(100!)} \ln(100!) = 1 \,.$$

Problem 23

Lets "complete-the-square" in each of these given expressions. For the first we have

$$x^2 - 12x + 36 - 36 + y^2 - 6y + 9 - 9 = 4 \quad \text{so} \quad (x - 6)^2 + (y - 3)^2 = 49 \,. \tag{591}$$

For the second we have

$$x^2 - 4x + 4 + y^2 - 12y + 36 = 4 + 36 + k \quad \text{so} \quad (x - 2)^2 + (y - 6)^2 = 40 + k \,. \tag{592}$$

The first is the equation of a circle with a center at $(6, 3)$ and a radius $\sqrt{49} = 7$. The second is the equation of a circle with a center at $(2, 6)$ and a radius $\sqrt{40 + k}$. Drawing both of these circles in a Cartesian coordinate plane we note that if k is "small" this second circle will be entirely "inside" the first circle. If k is "large" then this second circle will be entirely outside the first one.

Note that from the left-hand-side of Equation 592 we know that $40 + k > 0$ so $k > -40$.

If we subtract the two original expressions we get

$$0 = 4 - k + 8x - 6y \quad \text{so} \quad 8x - 6y = k - 4 \,.$$

Which we can write as

$$8(x - 6) - 6(y - 3) = k - 34 \,. \tag{593}$$

If we multiply Equation 591 by 8^2 we get

$$(8(x - 6))^2 + 8^2(y - 3)^2 = 7^2 \cdot 8^2 \,. \tag{594}$$

Solving Equation 593 for $8(x - 6)$ and squaring we get

$$(8(x - 6))^2 = 6^2(y - 3)^2 + 12(k - 34)(y - 3) + (k - 34)^2 \,.$$

If we put this into Equation 594 and simplify we get

$$100(y - 3)^2 + 12(k - 34)(y - 3) + [(k - 34)^2 - 7^2 \cdot 8^2] = 0 \,.$$

This will only have real solutions if the discriminant is nonnegative or if

$$12^2(k - 34)^2 - 4(100)[(k - 34)^2 - 7^2 \cdot 8^2] \geq 0 \,.$$

This simplifies to

$$-36 \leq k \leq 104 \,.$$

Then $b - a = 104 - (-36) = 140$.

Problem 24

There are $10 \times 10 \times 10 = 1000$ ways to choose the first three digits $d_1d_2d_3$ of the phone number. Once these three digits are specified we can make a memorable number by assigning the digits $d_4d_5d_6$ *or* the digits $d_5d_6d_7$ to these (now specified) first three digits. If we assign $d_4d_5d_6$ to $d_1d_2d_3$ we then have 10 choices for the digit d_7. If we assign $d_5d_6d_7$ to $d_1d_2d_3$ we have 10 choices for the digit d_4. Thus we have $10 \times 1000 = 10000$ ways a memorable number can be formed where the first three digits equals the digits $d_4d_5d_6$ and another 1000 ways a memorable number can be made where the first three digits equals the last three digits. This seems to indicate that we have a total of 20000 total memorable numbers.

We will have less than 20000 memorable numbers because some of the numbers in the two sets of numbers above will be the same. That means we need to determine how many numbers are in the overlap of the above two sets. If two numbers are in the overlap it means that

$$d_1d_2d_3 = d_4d_5d_6 = d_5d_6d_7 \,.$$

Setting the first digits equal we have that $d_1 = d_4 = d_5$. Setting the second digits equal means that $d_2 = d_5 = d_6$. Setting the third digits equal means that $d_3 = d_6 = d_7$. The summary of these equations means that all digits are equal. Thus there are ten numbers of this form and the total number of memorable numbers is $20000 - 10 = 19990$.

Problem 25

Denote the points in this problem as $A = (0, 2)$, $B = (4, 0)$, $C = (7, 3)$ and $D = (m, n)$. The fold of the paper must be a perpendicular bisector of the segments AB and CD. The midpoint of the segment AB is the point

$$\frac{1}{2}\left((0, 2) + (4, 0)\right) = (2, 1) \,.$$

The slope of the segment AB is

$$m_{AB} = \frac{0-2}{4-0} = -\frac{1}{2} \,.$$

The fold then must have a slope of $-\frac{1}{m_{AB}} = 2$. As the fold must also go through the midpoint of AB we have its equation given by

$$y - 1 = 2(x - 2) \quad \text{or} \quad y = 2x - 3 \,. \tag{595}$$

The slope of the segment CD must be the same as the slope of the segment AB or

$$m_{CD} = \frac{3-n}{7-m} = -\frac{1}{2} \,.$$

We can solve for m and write this as

$$m = 13 - 2n \,. \tag{596}$$

The midpoint of the segment CD is

$$\frac{1}{2}\left((7, 3) + (m, n)\right) = \left(\frac{1}{2}(m + 7), \frac{1}{2}(n + 3)\right) \,,$$

and must be on the fold and so satisfies the line given by Equation 595.

$$\frac{1}{2}(n + 3) = (m + 7) - 3 \,.$$

Using the above expression for m in terms of n we can solve for n to find $n = \frac{31}{5}$. Using Equation 596 we find $m = \frac{3}{5}$. This means that $m + n = \frac{34}{5} = \frac{35-1}{5} = 7 - \frac{1}{5} = 6.8$.

Problem 26

I drew this figure with point A at the origin of an x-y Cartesian coordinate plane. The point D 46 units to the right of A. The point C perpendicularly above the point D, and finally the point B at the angle of 120° from the AD segment and 13 units away from A.

Extend BC "backwards" to intersect the segment AD at a point B'. Then triangle $\triangle B'BA$ is a right triangle with $\angle BAB' = 60^\circ$ and $AB = 13$. From this we have that

$$B'A\cos(60^\circ) = AB \quad \text{so} \quad B'A\left(\frac{1}{2}\right) = 13 \quad \text{so} \quad B'A = 26\,,$$

and

$$B'A\sin(60^\circ) = B'B \quad \text{so} \quad B'B = 13\sqrt{3}\,.$$

Now $\triangle ABC$ is a right triangle and so we have

$$13^2 + BC^2 = AC^2\,. \tag{597}$$

In the right triangle $\triangle B'DC$ we have

$$B'D^2 + DC^2 = B'C^2 \quad \text{or} \quad (26+46)^2 + DC^2 = (13\sqrt{3} + BC)^2\,,$$

or

$$72^2 + DC^2 = (13\sqrt{3} + BC)^2\,. \tag{598}$$

In the right triangle $\triangle ADC$ we have

$$AD^2 + DC^2 = AC^2 \quad \text{or} \quad 46^2 + DC^2 = AC^2\,. \tag{599}$$

We will use Equations 598 and 599 to eliminate DC^2 to get

$$72^2 + AC^2 - 46^2 = (13\sqrt{3} + BC)^2\,.$$

Expanding and simplifying this becomes

$$2561 + AC^2 = 26\sqrt{3}BC + BC^2\,. \tag{600}$$

Lets us Equations 600 and 597 to eliminate AC^2 to get

$$2561 + 13^2 + BC^2 = 26\sqrt{3}BC + BC^2 \quad \text{so} \quad BC = \frac{105}{\sqrt{3}}\,.$$

Using that value in Equation 597 we find $AC = \sqrt{3844} = 62$.

Problem 28

I drew this figure with point A at the origin of an x-y Cartesian coordinate plane. The point C is "to the right" of A. The point B is perpendicularly above the point C, and finally the point D is on the segment BC. I let $\theta = \angle BAD$ so that from the problem statement we have $\angle DAC = 2\theta$.

To simplify notation we let $AC = x$. Then from what we are told in the problem statement we have $AD = \frac{3}{2}x$. From the right triangle $\triangle ACD$ we have

$$DC^2 = AD^2 - AC^2 = \frac{9}{4}x^2 - x^2 = \frac{5}{4}x^2 \quad \text{so} \quad DC = \frac{\sqrt{5}x}{2}.$$

This means that

$$\tan(\angle DAC) = \tan(2\theta) = \frac{CD}{AC} = \frac{\sqrt{5}}{2}.$$

Lets now consider $\tan(\angle BAC)$. Note that

$$\tan(\angle BAC) = \tan(\theta + 2\theta) = \frac{BC}{AC} = \frac{BD + \frac{\sqrt{5}x}{2}}{x} = \frac{BD}{x} + \frac{\sqrt{5}}{2}. \tag{601}$$

Using Equation 346 to evaluate $\tan(\theta + 2\theta)$ we get

$$\tan(\theta + 2\theta) = \frac{\tan(\theta) + \tan(2\theta)}{1 - \tan(\theta)\tan(2\theta)} = \frac{BD}{x} + \frac{\sqrt{5}}{2}. \tag{602}$$

From the above we know the value of $\tan(2\theta)$. Lets compute the value of $\tan(\theta)$. To do that in Equation 346 let $x = y = \theta$ to get

$$\tan(2\theta) = \frac{\sqrt{5}}{2} = \frac{2\tan(\theta)}{1 - \tan^2(\theta)}.$$

This is a quadratic equation in $\tan(\theta)$. Solving we get

$$\tan(\theta) \in \left\{ -\sqrt{5}, \frac{1}{\sqrt{5}} \right\}.$$

If we assume that $\tan(\theta) = -\sqrt{5}$ then Equation 602 would say that $\tan(3\theta) < 0$. Thus we know that $\tan(\theta) = \frac{1}{\sqrt{5}}$ and using Equation 602 we have that

$$\tan(3\theta) = \frac{7}{5}\sqrt{5}.$$

Using this value in the above we find BD given by

$$BD = \frac{9\sqrt{5}}{10}x.$$

Using that I find

$$\frac{CD}{BD} = \frac{\frac{\sqrt{5}}{2}x}{\frac{9\sqrt{5}}{10}x} = \frac{5}{9}.$$

This means that $m = 5$ and $n = 9$ so $m + n = 14$.

The 1999 AHSME Examination (AHSME 50)

Problem 1

Denote this sum as S. Then we can write S as

$$
\begin{aligned}
S &= \sum_{i=1}^{99} i(-1)^{i+1} = \sum_{i \text{ odd}} i(-1)^{i+1} + \sum_{i \text{ even}} i(-1)^{i+1} \\
&= \sum_{k=0}^{49} (2k+1)(-1)^{2k+1+1} + \sum_{k=1}^{49} (2k)(-1)^{2k+1} \\
&= \sum_{k=0}^{49} (2k+1) + (-1) \sum_{k=1}^{49} (2k) \\
&= 1 + \sum_{k=1}^{49} (2k+1-2k) = 1 + \sum_{k=1}^{49} 1 = 1 + 49 = 50\,.
\end{aligned}
$$

Problem 2

Choice A is false. All equilateral triangles are similar to each other. If two equilateral triangles have different lengths they are not congruent but similar.

Problem 3

This would be

$$
\frac{1}{2}\left(\frac{1}{8} + \frac{1}{10}\right) = \frac{9}{80}\,,
$$

when we simplify.

Problem 4

For this problem we will first form the numbers $4n+1$ for $1 \le n \le 25$ and the numbers $5m-1$ for $1 \le m \le 21$ which will generate the proposed numbers that could be primes. The primes we seek must be numbers in the intersection of these two sets of numbers. We find these numbers to be

$$
\{9\,, 29\,, 49\,, 69\,, 89\}\,.
$$

The only actual primes in this list are 29 and 89. The sum of these two numbers is 118. Note that the answer in the back of the book seems to be for a slightly different problem.

Problem 5

Let R be the retail price, M the mark price, and A the price Alice paid. Then we are told that $M = 0.7R$ and $A = 0.5M$. Thus

$$A = 0.5(0.7R) = 0.35R\,.$$

This is 35% of the retail price.

Problem 6

Call this expression E. Then

$$E = 2^{1999} \cdot 5^{2001} = (2 \cdot 5)^{1999} \cdot 5^2 = 25 \cdot 10^{1999}\,.$$

This is 25 with 1999 zeros. The sum of these digits is 7.

Problem 7

The sum of the interior angles of a hexagon (with the number of sides $n = 6$) is given by $S = 180(n-2) = 720$ and there are $n = 6$ interior angles. In addition, for the polygon to be convex each interior angle must be less than 180°. If we had four acute angles then the sum of them must be less than $4(90) = 360$ and the sum of the two other angles must then be larger than

$$720 - 360 = 360\,,$$

which means that one of them is larger than 180 and the hexagon would not be convex. Thus we have "too many" acute angles.

If we have three acute angles their sum must be less than $3(90) = 270$ and the other three interior angles must sum to at least $720 - 270 = 450$. This means that at least one must be larger than

$$\frac{450}{3} = 150\,,$$

which is possible without violating convexity.

Problem 8

Let Walters age be W and his grandmothers age be G. From the problem statement at the end of 1994 we have $W = \frac{1}{2}G$. Next note that $1994 - W$ is the year that Walter was born and thus $1994 - G$ is the year that his grandmother was born. We are told that

$$(1994 - W) + (1994 - G) = 3838\,.$$

This simplifies to

$$W + G = 150\,.$$

Using $W = \frac{1}{2}G$ we find $G = 100$ and $W = 50$. Thus in 1999 (in five years) Walter will be $50 + 5 = 55$.

Problem 9

Ashley could have driven a maximum of $75(2) = 150$ miles. Let $m_3m_2m_1$ be the digits in the number representing the number of miles that Ashley drove. Then we need to know what numbers

$$27972 + m_3m_2m_1\,,$$

where $1 \le m_3m_2m_1 \le 150$ is also a palindrome and the individual digits are bounded as $0 \le m_3 \le 1$, $0 \le m_2 \le 5$, and $0 \le m_1 \le 9$.

Now let o_i be the digit in the ith location in the "original" number i.e. $27972 = o_5o_4o_3o_2o_1$ so $o_5 = 2$, $o_4 = 7$, etc. In addition we let r_i be the digit in the ith location of the "result" of adding 27972 to $m_3m_2m_1$. Because $27972 + 150 = 28122$ we must have $r_5 = 2$. Because this result must also be a palindrome we must have $r_5 = 2 = r_1$ so $m_1 = 0$.

As $o_2 = 7$ if $m_2 \in \{0, 1, 2\}$ there will be no "carry" and we would have $r_2 \in \{7, 8, 9\}$ in each case. We will have $r_2 = r_4 = 8$ if $m_3 = 1$ and the number $m_3m_2m_1 = 110$. To give $27972 + 110 = 28082$ as the "other" palindrome.

If $m_2 \in \{3, 4, 5\}$ then $r_2 \in \{0, 1, 2\}$ in each case and our total number is not a palindrome.

This means that Ashley's average speed is $\frac{110}{2} = 55$ miles-per-hour. Note that the answer in the back of the book seems to be for a slightly different problem (does Ashley do a two or a three hour drive?).

Problem 10

If three of these statements are true and one is false we can assume each statement in tern is false and see if a consistent set of results is obtained. For example,

- If I is false, then II, III, and IV can all be true and have a consistent set of results.
- If II is false, then I, III, and IV would be inconsistent.
- If III is false, then I, II, and IV can all be true and have a consistent set of results.
- If IV is false, then I, II, and III can all be true and have a consistent set of results.

Taken together in all cases II must be true.

Problem 11

Let n be the total number of lockers. Note that

- It will cost $0.02(9) = 0.18$ to label the lockers $1, 2, \ldots, 8, 9$ leaving $137.94 - 0.18 = 137.76$ yet to spend.
- It will cost $0.04(90) = 3.6$ to label the lockers $10, 11, \ldots, 98, 99$ leaving $137.76 - 3.6 = 134.16$ yet to spend.
- It will cost $0.06(900) = 54$ to label the lockers $100, 101, \ldots, 998, 999$ leaving $134.16 - 54 = 80.16$ yet to spend.
- Each locker from that point onwards will cost 0.08 and there will be $\frac{80.16}{0.08} = 1002$ of them.

This gives a total of

$$n = 9 + 90 + 900 + 1002 = 2001\,.$$

Problem 12

The expression $p(x) - q(x)$ will be a third degree polynomial and will have at most three real roots.

Problem 13

We can write this as $a_{n+1} = 99^{1/3} a_n$. By iterating we have that

$$a_n = (99^{(n-1)/3})a_1 = 99^{(n-1)/3}\,,$$

since $a_1 = 1$. We thus have that $a_{100} = 99^{99/3} = 99^{33}$.

Problem 14

Let N be the number of songs sung in trios i.e. the total number of songs sung. Let m, a, t, and h be the number of songs sung by Mary, Alina, Tina, and Hanna respectively. Then we are told that $h = 7$ and $m = 4$. If we add up m, a, t, and h then we get that

$$m + a + t + h = 3N\,,$$

as the songs are sung in trios. Using what we know about h and m this is

$$3N = 11 + a + t\,,$$

a	t	$11+a+t$	Is $11+a+t$ a multiple of three	N
5	5	21	Yes	7
5	6	22	No	-
6	5	22	No	-
6	6	23	No	-

Table 18: Possible values for a and t and whether or not $11+a+t$ is a multiple of three.

and we know that $5 \le a \le 6$ and $5 \le t \le 6$. The above means that the sum of $11+a+t$ must be a multiple of three. In Table 18 we tabulate the possible values of a and t and then ask if the given sum is a multiple of three. If it is then we can compute a value for N. The given table shows that $N = 7$ is the only consistent solution.

Problem 15

Lets square the given expression to get

$$\sec^2(x) - 2\sec(x)\tan(x) + \tan^2(x) = 4\,.$$

Use $\tan^2(x) + 1 = \sec^2(x)$ to replace $\tan^2(x)$ in the above to get

$$2\sec^2(x) - 2\sec(x)\tan(x) = 5\,,$$

or

$$2\sec(x)(\sec(x) - \tan(x)) = 5\,.$$

Using the given expression in the problem this becomes

$$4\sec(x) = 5\,.$$

This means that $\sec(x) = \frac{5}{4}$. From the equation given this means that $\tan(x) = -\frac{3}{4}$. Thus we have

$$\sec(x) + \tan(x) = \frac{1}{2}\,.$$

Problem 16

Recall that a rhombus is a parallelogram with four equal sides. The diagonal of a rhombus are perpendicular and bisect each other. If we draw our rhombus (and its diagonals) we see that it is made up of four congruent right triangles with legs of length $\frac{24}{2} = 12$ and $\frac{10}{2} = 5$. Each of these right triangles has its hypotenuse as the sides of the rhombus and thus the side of the rhombus is of length

$$h = \sqrt{12^2 + 5^2} = 13\,.$$

Now the inscribed circle will be tangent to each of the four sides of the rhombus. Thus the radius of the inscribed circle will be the perpendicular distance from the center of the rhomboid to one of the sides of the rhomboid. This is also the length of the altitude a to the hypothesis in any of the four internal right triangles. We can compute the length of this altitude by computing the area of an internal right triangle in two ways. One uses the product of the legs and the other using the length of the hypotenuse times its altitude or

$$A = \frac{1}{2}(5)(12) = \frac{1}{2}(13)a \quad \text{so} \quad a = \frac{60}{13}.$$

Problem 17

From what we are told we have that

$$P(x) = (x-19)f(x) + 99 \tag{603}$$
$$P(x) = (x-99)g(x) + 19\,, \tag{604}$$

for polynomials $f(x)$ and $g(x)$. Now if I set $x = 99$ in Equation 603 I must have

$$P(99) = 80f(99) + 99 = 19 \quad \text{so} \quad f(99) = -1\,.$$

This means that $f(x)$ has a remainder of -1 when we divide it by $x - 99$ and thus we can write $f(x)$ as

$$f(x) = (x-99)h(x) - 1\,,$$

for some $h(x)$. Putting this into Equation 603 gives us that

$$P(x) = (x-19)(x-99)h(x) - (x-19) + 99 = (x-19)(x-99)h(x) - x + 118\,.$$

This means that the remainder when we divide $P(x)$ by $(x-19)(x-99)$ is $-x+118$.

Problem 18

As $\log(x)$ ranges from $-\infty < \log(x) < 0$ as x ranges from $0 < x < 1$ the function $\cos(\log(x))$ will have an infinite number of zeros on this interval.

Problem 19

Using the Pythagorean theorem in triangle $\triangle ADB$ we have

$$BD^2 + DA^2 = AB^2 \quad \text{or} \quad 57 + DA^2 = AB^2\,.$$

We write this as

$$AB^2 - DA^2 = 57 \quad \text{or} \quad (AB - DA)(AB + DA) = 57\,.$$

As AB is an integer we must have AD an integer also. Thus we are looking for the factors of 57. One factorization is $57 = 1 \times 57$. As $AB - DA < AB + DA$ this means that

$$\begin{aligned} AB - DA &= 1 \\ AB + DA &= 57 \,. \end{aligned}$$

Solving for AB gives $AB = AC = 29$.

Another factorization is $57 = 3 \times 19$. This means that

$$\begin{aligned} AB - DA &= 3 \\ AB + DA &= 19 \,. \end{aligned}$$

Solving for AB gives $AB = AC = 11$.

This last value is the smallest possible value for AC.

Problem 20

We are told that

$$a_n = \frac{1}{n-1} \sum_{k=1}^{n-1} a_k \,, \tag{605}$$

for $n \geq 3$. Given a_1 and a_2 using this formula we find that a_3 would be given by

$$a_3 = \frac{1}{2}(a_1 + a_2) \,.$$

The formula for a_4 is then given by

$$a_4 = \frac{1}{3}(a_1 + a_2 + a_3) = \frac{1}{3}\left(a_1 + a_2 + \frac{1}{2}(a_1 + a_2)\right) = \frac{1}{2}(a_1 + a_2) \,.$$

when we simplify. Based on this it looks like it might be that $a_n = \frac{1}{2}(a_1 + a_2)$ for all $n \geq 3$ i.e. it is the same value for all n. To prove this, we consider Equation 605 where we have

$$a_n = \frac{1}{n-1}\left(\sum_{k=1}^{n-2} a_k + a_{n-1}\right) = \frac{1}{n-1}\left((n-2)a_{n-1} + a_{n-1}\right) = a_{n-1} \,.$$

This means that

$$a_9 = 99 = \frac{1}{2}(a_1 + a_2) = \frac{1}{2}(19 + a_2) \,.$$

Solving for a_2 we get $a_2 = 179$.

Problem 21

As

$$20^2 + 21^2 = 29^2 \,,$$

the given triangle is a right triangle which means that the circumscribing circle will have the triangle's hypotenuse as its diameter. Thus the radius of the circumscribing circle is then $r = \frac{29}{2}$. Let T be the area of the triangle then we have

$$T = \frac{1}{2}(21)(20) = 210 \,.$$

In addition, the largest external area C will be the area of the semicircle and thus equal to the sum of the other three areas or

$$C = A + B + T = A + B + 210 \,.$$

Problem 22

If we draw these two curves the first is concave down and the second is concave up. The points where they intersect will be when

- $x < a$ and $x < c$ or
- $x > b$ and $x > c$.

The first condition is the intersection of the pair of lines

$$\begin{aligned} y &= +(x - a) + b \\ y &= -x + c + d \,. \end{aligned}$$

Taking $(x, y) = (2, 5)$ in the above and simplifying we get

$$a - b = -3 \tag{606}$$
$$c + d = 7 \,. \tag{607}$$

The second condition is the intersection of the pair of lines

$$\begin{aligned} y &= -(x - a) + b \\ y &= x - c + d \,. \end{aligned}$$

Taking $(x, y) = (8, 3)$ in the above and simplifying we get

$$a + b = 11 \tag{608}$$
$$c - d = 5 \,. \tag{609}$$

If we add Equations 606 and 608 we get $a = 4$. If we add Equations 607 and 609 we get $c = 6$. Together these give $a + c = 10$.

Problem 23

As this hexagon is equiangular each angle is given by

$$\frac{180°(n-2)}{n} = 120°\,,$$

when $n = 6$ as it is for a hexagon. We place A at the original of an x-y Cartesian plane such that $A = (0,0)$ and $B = (1,0)$. Then as $\angle ABC = 120°$ we have

$$C = (1 + 4\cos(60), 0 + 4\sin(60)) = (3, 2\sqrt{3})\,.$$

If we draw a horizontal though C we can determine that the segment CD is $60°$ from this horizontal and thus as $CD = 2$ we have

$$D = (3 - 2\cos(60), 2\sqrt{3} + 2\sin(60)) = (2, 3\sqrt{3})\,.$$

Continuing by drawing E we find that DE is parallel to the x axis and so

$$E = (2 - 4, 3\sqrt{3}) = (-2, 3\sqrt{3})\,.$$

We now need to determine the location of the point F such that $\angle FED = 120° = \angle BAF$. If $AF = b$ (as its the bottom of the two segments AF and FE) then the point F can be given by "walking" from A as

$$F = (-b\cos(60), b\sin(60)) = \left(-\frac{b}{2}, \frac{b\sqrt{3}}{2}\right)\,.$$

If $EF = t$ (for top) then the point F can be given by "walking" from E as

$$F = (-2 - t\cos(60), 3\sqrt{3} - t\sin(60)) = \left(-2 - \frac{t}{2}, 3\sqrt{3} - \frac{t\sqrt{3}}{2}\right)\,.$$

Setting these two expressions equal gives two equations for the unknowns b and t. Solving I find $t = 1$ and $b = 5$. This means that the point F is given by

$$F = \left(-\frac{5}{2}, \frac{5\sqrt{3}}{2}\right)\,.$$

These points are labeled in Figure 16.

Now that we have all of the points labeled we can compute the area of the figure. The simplest method seems to compute the area of the bounding rectangle and then subtract the four "corner" triangles. The bounding rectangle has an area of (in terms of the components of the points defined above)

$$R = (C_x - F_x)(E_y - 0) = \left(3 + \frac{5}{2}\right) 3\sqrt{3} = \frac{33\sqrt{3}}{2}\,.$$

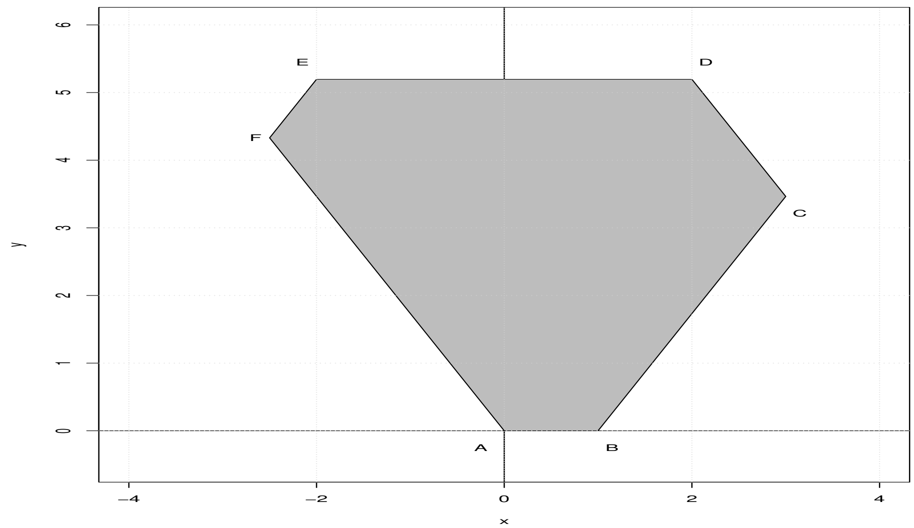

Figure 16: The hexagonal region for Problem 23.

To denote the four "corner" triangles we will let C' and F' be the vertical projections of the points C and F onto the x axis and C'' and F'' be the vertical projections of the points C and F onto the horizontal line though DE. Then the four triangles have areas given by

$$
\begin{aligned}
[\triangle BC'C] &= \frac{1}{2}(2)(2\sqrt{3}) = 2\sqrt{3} \\
[\triangle DC''C] &= \frac{1}{2}(1)(\sqrt{3}) = \frac{\sqrt{3}}{2} \\
[\triangle EF''F] &= \frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{8} \\
[\triangle FF'A] &= \frac{1}{2}\left(\frac{5}{2}\right)\left(\frac{5\sqrt{3}}{2}\right) = \frac{25\sqrt{3}}{8} .
\end{aligned}
$$

These four areas sum to $\frac{23\sqrt{3}}{4}$. Thus the area we seek is given by

$$
\frac{33\sqrt{3}}{2} - \frac{23\sqrt{3}}{4} = \frac{43\sqrt{3}}{4} .
$$

Problem 24

The numbers used in the problem and in the solution seem different. Here I assume there are p points on the circle and we are looking for a convex n-sided polygon with $p > n$. There are $\begin{pmatrix} p \\ n \end{pmatrix}$ convex n-gon's. This is because from the p points we need to select n of them to select a n sided figure. Once these points are select the convex polygon is obtained by connecting the points with chords in a clockwise (or counterclockwise) manner. Thus there are $\binom{p}{n}$ positive examples.

The total number of chords we can select can be determined in the following way. There are $\binom{p}{2}$ total chords and from this number we must select n. This is the number

$$\binom{\binom{p}{2}}{n}.$$

Thus the probability P desired is the ratio of these two numbers or

$$P = \frac{\binom{p}{n}}{\binom{\binom{p}{2}}{n}}.$$

In the *problem* we have $p = 5$ and $n = 4$ to get

$$P = \frac{\binom{5}{4}}{\binom{\binom{5}{2}}{4}} = \frac{5}{\binom{10}{4}} = \frac{5}{210} = \frac{1}{42}.$$

In the *answer* we have $p = 6$ and $n = 4$ to get

$$P = \frac{\binom{6}{4}}{\binom{\binom{6}{2}}{4}} = \frac{15}{\binom{15}{4}} = \frac{15}{1365} = \frac{1}{91}.$$

Problem 25

Multiply both sides by 7! to get

$$5 \cdot 6! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot a_2 + 7 \cdot 6 \cdot 5 \cdot 4 \cdot a_3 + 7 \cdot 6 \cdot 5 \cdot a_4 + 7 \cdot 6 \cdot a_5 + 7a_6 + a_7\,,$$

or

$$3600 = 2520a_2 + 840a_3 + 210a_4 + 42a_5 + 7a_6 + a_7\,.$$

This means that

$$3600 - a_7 = 7(360a_2 + 120a_3 + 30a_4 + 6a_5 + a_6)\,.$$

This means that $3600 - a_7 \equiv 0 \pmod 7$ (i.e. the number $3600 - a_7$ must be divisible by seven). As $3600 \equiv 2 \pmod 7$ this means that $a_7 = 2$. Using that value the above we get that

$$\frac{3600-2}{7} = 514 = 360a_2 + 120a_3 + 30a_4 + 6a_5 + a_6\,,$$

or

$$514 - a_6 = 6(60a_2 + 20a_3 + 5a_4 + a_5)\,.$$

By the same reasoning as above this means that $514 - a_6 \equiv 0 \pmod 6$. As $514 \equiv 4 \pmod 6$ when $a_6 \in \{0, 1, 2, 3, 4, 5\}$ we must take $a_6 = 4$ and we get

$$\frac{514-4}{6} = 85 = 60a_2 + 20a_3 + 5a_4 + a_5\,,$$

or

$$85 - a_5 = 5(12a_2 + 4a_3 + a_4)\,.$$

This means that $85 - a_5 \equiv 0 \pmod 5$. As $85 \equiv 0 \pmod 5$ when $a_5 \in \{0, 1, 2, 3, 4\}$ we must have $a_5 = 0$. Using that we get

$$\frac{85}{5} = 17 = 12a_2 + 4a_3 + a_4\,,$$

or

$$17 - a_4 = 4(3a_2 + a_3)\,.$$

Thus $17 - a_4 \equiv 0 \pmod 4$. As $17 \equiv 1 \pmod 4$ when $a_4 \in \{0, 1, 2, 3\}$ we have $a_4 = 1$. Using that we get

$$\frac{16}{4} = 3a_2 + a_3\,,$$

or

$$4 - a_3 = 3a_2\,.$$

Thus $4 - a_3 \equiv 0 \pmod 3$. As $4 \equiv 1 \pmod 3$ when $a_3 \in \{0, 1, 2\}$ we have $a_3 = 1$ and the above gives $3 = 3a_2$ so $a_2 = 1$ also. Given the numbers above we compute

$$a_2 + a_3 + a_4 + a_5 + a_6 + a_7 = 1 + 1 + 1 + 0 + 4 + 2 = 9\,.$$

Problem 27

If we square each of these equations we get

$$9\sin^2(A) + 24\sin(A)\cos(B) + 16\cos^2(B) = 36 \quad (610)$$
$$16\sin^2(B) + 24\sin(B)\cos(A) + 9\cos^2(A) = 1\,. \quad (611)$$

If we add these two equations we get

$$9(1) + 16(1) + 24(\sin(A)\cos(B) + \sin(B)\cos(A)) = 37\,.$$

This simplifies to

$$\sin(A+B) = \frac{1}{2}\,.$$

This means that $A+B = 30°$ or $A+B = 150°$. If the first of these is true then certainly we have $A < 30°$ and thus

$$3\sin(A) + 4\cos(B) < 3\left(\frac{1}{2}\right) + 4 = \frac{11}{2} < 6\,,$$

which is a contradiction. Thus we must have $A + B = 150°$ and thus $C = 180 - 150 = 30°$.

Problem 28

It seems like the solution to this problem involves a technique that could be useful in other places. Notice that if x_i is restricted to be from a restricted set of numbers say $\{A, B, C\}$ then sums of the form $\sum_i x_i^p$ are easy to evaluate in terms of the *number* of times each of the x_i takes values from the set above. For example we have

$$\sum_i x_i^p = N_A A^p + N_B B^p + N_C C^p\,,$$

where N_A are the number of times that $x_i = A$. The same for N_B and N_C.

Given this comment let a, b, and c be the number of times that x_i are -1, $+1$ and $+2$ respectively. Then we seek to optimize

$$\sum_{i=1}^{n} x_i^3 = -a + b + 8c\,, \tag{612}$$

subject to the constraints that

$$\sum_{i=1}^{n} x_i = -a + b + 2c = 19 \tag{613}$$

$$\sum_{i=1}^{n} x_i^2 = a + b + 4c = 99\,. \tag{614}$$

If we view these two equations as a system for a and b and solve the above we get solutions for a and b in terms of c. We find

$$a = 40 - c \tag{615}$$

$$b = 59 - 3c\,. \tag{616}$$

If we put these two expressions into Equation 612 we seek to optimize

$$-(40 - c) + (59 - 3c) + 8c = 19 + 6c\,,$$

when we simplify.

Now to minimize $\sum_{i=1}^{n} x_i^3$ we would take c as small as possible which would be $c = 0$ to give $m = 19$.

Now to maximize $\sum_{i=1}^n x_i^3$ we would take c as large as possible subject to the constraints that $a \geq 0$, $b \geq 0$, and $c \geq 0$. From $a \geq 0$ and Equation 615 we have that $c \leq 40$. From $b \geq 0$ and Equation 616 we have that $c \leq \frac{59}{3} < 20$. Thus the largest c can be is when $c = 19$. This means that $M = 19 + 6(19) = 7(19) = 133$.

Thus we have $\frac{M}{m} = \frac{7(19)}{19} = 7$.

The 1999 Sample AMC 10 Examination

Problem 1

This would be the number

$$\frac{1}{2}\left(\frac{1}{6} + \frac{1}{4}\right) = \frac{5}{24}\,.$$

Problem 2

Let R be the retail price, M the mark price, and A the price Alice paid. Then we are told that $M = 0.6R$ and $A = 0.5M$. Thus

$$A = 0.5(0.6R) = 0.3R\,.$$

This is 70% of the retail price.

Problem 3

Let the three numbers be $a \leq b \leq c$. Then we are told that

$$\frac{1}{3}(a + b + c) = a + 10 \tag{617}$$

$$\frac{1}{3}(a + b + c) = c - 15\,, \tag{618}$$

and the median (which is b) is five. Putting $b = 5$ in the above and solving for a and c we get $a = 0$ and $c = 25$. Thus $a + b + c = 30$.

Problem 4

The sum of the interior angles of a quadrilateral must equal $S = 180(n - 2) = 360$ when $n = 2$. If we consider $o = 3$ obtuse angles then the sum of these o angles must be larger than $3(90) = 270$ and thus the fourth interior angle in our quadrilateral will be smaller than $360 - 270 = 90$ and will be acute. Thus this number of obtuse angles is possible.

If we try for a larger number of obtuse angles say $o = 4$ then the sum of these interior angles will be larger than $90(4) = 360$ and is thus not possible.

Problem 5

Sum these numbers in pairs as

$$S = (1 - 2) + (3 - 4) + (5 - 6) + \cdots + (199 - 200) = \sum_{i=1}^{50} (-1) = -100\,.$$

Thus the average is $\frac{S}{200} = -\frac{1}{2}$.

Problem 6

Call this expression E. Then write E as

$$E = 2^{1999} \cdot 5^{2000} = 5(2 \cdot 5)^{1999} = 5 \cdot 10^{1999}\,.$$

This is the number "five" followed by 1999 zeros. The sum of these digits is five. Note that the answer in the back of the book seems to be for a slightly different problem.

Problem 7

Numbers of the form $5n + 1$ will have a units digit of a six or a one. Note that any number that has a units digit of a six is even and is not prime.

Numbers of the form $6m - 1$ have a units digit of a five, a one, a seven, a three, or a nine.

The only common ending between these two sets of numbers are numbers that end in a one. The possible numbers (less than 100) that end in a one are

$$11\,, 21\,, 31\,, 41\,, 51\,, 61\,, 71\,, 81\,, 91\,.$$

For each of these numbers the value of m in the representation $6m - 1$ will be an integer for the subset

$$11\,, 41\,, 71\,.$$

The sum of these numbers is 123. Note that the answer in the back of the book seems to be for a slightly different problem.

Problem 8

The book's solution is much simpler than the one given here.

I first draw the two rectangles. Lets denote the corner points of the "left-most" rectangle as $A = (-2, 0)$, $B = (0, 0)$, $C = (0, 4)$, and $D = (-2, 4)$. Lets denote the corner points of the "right-most" rectangle as $E = (1, 0)$, $F = (5, 0)$, $G = (5, 12)$, and $H = (1, 12)$. Lets take our bisecting line to have an equation of $y = mx + b$ and let it intersect the verticals of our rectangles at the points $P = (-2, -2m+b)$, $Q = (0, b)$, $R = (1, m+b)$, and $S = (5, 5m+b)$.

Now by symmetry in order for this line to bisect each rectangle we must have $AP = CQ$ or

$$-2m + b = 4 - b\,, \tag{619}$$

and $ER = GS$ or

$$m + b = 12 - (5m + b)\,. \tag{620}$$

These give two equation for the two unknowns m and b. Solving them we find $m = 1$ and $b = -3$.

Problem 9

The two-inch cube has a volume of $2^3 = 8$ cubic inches. Silver is thus valued at $\frac{200}{8} = 25$ dollars per cubic inch. The three inch cube has a volume of $3^3 = 27$ cubic inches. The value of this is then

$$27(25) = 675\,,$$

dollars.

Problem 10

The only blocks that will have one face painted will be the ones that are "interior" to each face. As we need to remove one corner and one edge from each we would have that that 4×6 face will be reduced to a $(4-2) \times (6-2) = 2 \times 4$ region with 8 cubes that have paint on one face. For the faces of the other dimensions, we have the 4×8 face reduced to a 2×6 region and thus get 12 more blocks. Finally, we have the 6×8 face reduced to a 4×6 region and thus get 24 more blocks. As each region appears twice this is

$$2(8) + 2(12) + 2(24) = 88\,,$$

blocks.

Problem 11

The vertical runs have a length given by

$$8 + 2 = 10 = 3 + 6 + 1\,,$$

While the horizontal runs must have a length of 12 (top and bottom). Thus the perimeter is $2(10) + 2(12) = 44$.

Problem 12

Call this number N. As $8^2 = 64 > 50$ there cannot be any digit larger than an eight in this number. If the largest digit is a seven then as

$$50 - 7^2 = 1\,,$$

the only other digit can be a one and the number is $N = 17$.

If the largest digit is a six then as

$$50 - 6^2 = 14 = 9 + 4 + 1\,,$$

this number is $N = 1236$.

If the largest digit is a five then as

$$50 - 5^2 = 25 = 16 + 9\,,$$

and this number is $N = 345$.

If the largest digit is a four then as

$$50 - 4^2 = 34 = 9 + 4 + 1 + 20\,,$$

which is inconsistent with the type of numbers we are looking for. The largest number above is $N = 1236$ which has a digit product of 36.

Problem 13

Let Walters age be W and his grandmothers age be G. From the problem statement at the end of 1994 we have $W = \frac{1}{2}G$. Next note that $1994 - W$ is the year that Walter was born and thus $1994 - G$ is the year that his grandmother was born. We are told that

$$(1994 - W) + (1994 - G) = 3844\,.$$

This simplifies to

$$W + G = 144\,.$$

Using $W = \frac{1}{2}G$ we find $G = 96$ and $W = 48$. Thus in 1999 (in five years) Walter will be $48 + 5 = 54$.

Problem 14

The units digit of a product depends on the product of the units digits of the factors. For the number we are given we have

$$(2 \cdot 4 \cdot 6 \cdot 8)(12 \cdot 14 \cdot 16 \cdot 18)(22 \cdot 24 \cdot 26 \cdot 28)(32 \cdot 34 \cdot 36 \cdot 38) \cdots (92 \cdot 94 \cdot 96 \cdot 98)\,.$$

There are 10 "groups" that are products of numbers ending with a two, a four, a six, and an eight. The product of these digits is

$$2 \cdot 4 \cdot 6 \cdot 8 = 384\,,$$

and thus has a units digit of a four.

The product of 10 numbers that end if a four will have the same units digit as 4^{10}. Note that

$$\begin{aligned} 4^2 &\equiv 6 \pmod{10} \\ 4^3 &\equiv 4 \pmod{10} \\ 4^4 &\equiv 6 \pmod{10}\,. \end{aligned}$$

Thus we have that

$$4^{10} = 4^2 \cdot 4^2 \cdot 4^2 \cdot 4^2 \cdot 4^2 \equiv 6 \pmod{10}\,,$$

and the product ends in a six.

Problem 15

The sum of three elements will be even if all of the items are even or two are odd and one is even. There are three even numbers and three odd numbers in the original set. Thus there is only one subset that has three even numbers. We can draw a subset of two odd numbers in $\binom{3}{2} = 3$ ways. We can draw a single even number in $\binom{3}{1} = 3$ ways. This gives

$$3 \times 3 = 9\,,$$

subsets with two odd numbers and one even number. The total number of sets is then $1 + 9 = 10$.

Problem 16

Let the radius of the circle be r. The points that are closer to the center of the circle are ones that are less than $\frac{r}{2}$ from the center i.e. "inside" the circle of radius $\frac{r}{2}$. This smaller circle has an area of

$$A_I = \pi \left(\frac{r}{2}\right)^2 = \frac{\pi r^2}{4}\,.$$

The area of the original circle is πr^2 and so the area of the region where the points are closer to the boundary is

$$\pi r^2 - \frac{\pi r^2}{4} = \frac{3\pi r^2}{4}.$$

This means that the probability a random point is closer to the center is then

$$\frac{A_I}{\pi r^2} = \frac{1}{4}.$$

Problem 17

Pages $1-9$ require nine digits leaving $600-9=591$ digits remaining.

Pages $10-99$ are $99-10+1=90$ pages and require $2(90)=180$ digits leaving $591-180=411$ remaining.

Pages $100-999$ are $999-100+1=900$ pages and require $3(900)=2700$ digits which is more than we have remaining. This means that each of the 411 remaining digits will produce

$$\frac{411}{3} = 137,$$

pages. Thus the total number of pages is

$$9+90+137=236.$$

Note that the answer in the back of the book seems to be for a slightly different problem.

Problem 19

These two cubic polynomials can be represented as

$$y_1(x) = x^3 + Ax^2 + Bx + C$$
$$y_2(x) = x^3 + ax^2 + bx + c.$$

They will intersect if there exists an x such that $y_1(x) = y_2(x)$ or $y_1(x) - y_2(x) = 0$. The left-hand-side of this last expression is a quadratic expression in x and can thus have at most two real values for x that satisfy it.

Problem 20

If we draw these two curves the first is concave down and the second is concave up. The points where they intersect will be when

- $x < a$ and $x < c$ or
- $x > b$ and $x > c$.

The first condition is the intersection of the pair of lines

$$
\begin{aligned}
y &= +(x - a) + b \\
y &= -x + c + d \,.
\end{aligned}
$$

Taking $(x, y) = (2, 5)$ in the above and simplifying we get

$$a - b = -3 \tag{621}$$

$$c + d = 7 \,. \tag{622}$$

The second condition is the intersection of the pair of lines

$$
\begin{aligned}
y &= -(x - a) + b \\
y &= x - c + d \,.
\end{aligned}
$$

Taking $(x, y) = (8, 3)$ in the above and simplifying we get

$$a + b = 11 \tag{623}$$

$$c - d = 5 \,. \tag{624}$$

If we add Equations 621 and 623 we get $a = 4$. If we add Equations 622 and 624 we get $c = 6$. Together these give $a + c = 10$.

Problem 21

Lets try to "pick" the false statement.

- If I is false, the three remaining statements are consistent.
- If II is false, the three remaining statements are consistent.
- If III is false, the three remaining statements are not consistent as I and II cannot both be true.
- If IV is false, the three remaining statements are not consistent as I and II cannot both be true.

Thus we see that III and IV must both be true. This is answer (E).

Problem 22

As $3^2 + 4^2 = 5^2$ this triangle is a right triangle. When we circumscribe a circle about this right triangle the diameter of the circle must be the triangles hypotenuse and is thus the diameter is of length five. This means that the radius of the circle is $r = \frac{5}{2}$. From the problem statement then C is the area of $\frac{1}{2}$ of the circle so

$$C = \frac{1}{2}\pi\left(\frac{5}{2}\right)^2 = \frac{25\pi}{16}\,.$$

Next notice that the other one-half of the circle is made up of areas of size A, B and the area of the right triangle itself. This means that

$$C = A + B + \frac{1}{2}(3)(4) = A + B + 6\,.$$

Problem 23

Lets write down the given requirements. We have

$$\begin{aligned} 7 + a + b + 1 = K \quad &\text{or} \quad a + b + 8 = K && (625)\\ 3 + e + f + 10 = K \quad &\text{or} \quad e + f + 13 = K && (626)\\ 7 + c + 3 = K \quad &\text{or} \quad c + 10 = K && (627)\\ 1 + d + 10 = K \quad &\text{or} \quad d + 11 = K\,. && (628) \end{aligned}$$

As we know that each of a, b, c, d, e, and f are larger than or equal to two we know that from Equation 627 that

$$K \geq 10 + 2 = 12\,.$$

Also as we know that each of a, b, c, d, e, and f are smaller than or equal to nine from Equation 627 we have that

$$K \leq 9 + 10 = 19\,.$$

These only eliminate one choice. If we subtract Equation 628 from Equation 627 we get

$$c - d - 1 = 0 \quad \text{or} \quad c - d = 1\,.$$

From the choices given we then must have

$$(c, d) \in \{(9, 8)\,, (6, 5)\,, (5, 4)\}\,. \qquad (629)$$

If we take $(c, d) = (9, 8)$ then from the above we find $K = 19$ and Equations 625 and 626 become

$$\begin{aligned} a + b &= 11\\ e + f &= 6\,. \end{aligned}$$

This will hold true if $(a, b, e, f) = (5, 6, 2, 4)$. Other choices for (c, d) found in Equation 629 don't give consistent solutions.

Problem 24

Let $OA = OB = r$ the radius of the circle. The area of the triangle is

$$A_T = \frac{1}{2}r^2 \,.$$

Now by "right triangles" $AB = \sqrt{2}r$ so that the radius of the circle with AB as its diameter is then given by $\frac{\sqrt{2}}{2}r = \frac{r}{\sqrt{2}}$. The area of the lune is given by half the area of a circle with radius $\frac{r}{\sqrt{2}}$ minus the area in the circle and in sector AOB but not in the right triangle $\triangle AOB$ or

$$A_L = \frac{1}{2}\left(\pi\frac{r^2}{2}\right) - \left(\frac{1}{4}\pi r^2 - A_T\right) = \frac{r^2}{2} \,,$$

when we simplify. The ratio we seek is then

$$\frac{A_L}{A_T} = 1 \,.$$

Problem 25

The sum of the interior angles of an n-sided figure is $S = 180(n-2)$. For a "regular" figure (with n sides) each interior angle must then be

$$\theta = \frac{180(n-2)}{n} \,.$$

For $n = 6$ (hexagon) and $n = 5$ (pentagon) these are

$$\theta_h = 120^\circ = \angle CBG$$
$$\theta_p = 108^\circ = \angle ABG \,.$$

This means that

$$\angle CBA = 360^\circ - \angle CBG - \angle ABG = 132^\circ \,.$$

Now as $BC = BA$ from properties of isosceles triangles we have

$$\angle BAC = \angle BCA = \frac{180^\circ - \angle CBA}{2} = 24^\circ \,.$$

The 2000 AMC 12 Examination (AHSME 51)

Problem 1

If we factor 2001 into its factor we have $2001 = 3 \cdot 667$. Assigning this to $I \cdot M \cdot O$ we would then have

$$I + M + O = 1 + 3 + 667 = 671\,.$$

Problem 2

This is

$$2000(2000)^{2000} = 2000^{2001}\,.$$

Problem 3

If N_i is the number of jellybeans at the end of the ith day. Then we are told that

$$N_i = (1 - 0.2)N_{i-1} = 0.8N_{i-1} = \frac{4}{5}N_{i-1}\,,$$

for $i \geq 1$. This means that at the end of the second day we have

$$N_2 = \frac{4}{5}N_1 = \left(\frac{4}{5}\right)^2 N_0 = 32\,.$$

Here N_0 is the number of jellybeans Jenny started with. Solving the above for N_0 gives $N_0 = 50$.

Problem 4

From the given Fibonacci numbers in this small sample we see that in the units position we have "observed" the digits

$$1\,, 2\,, 3\,, 5\,, 8\,.$$

Computing some more Fibonacci numbers gives

$$34\,, 55\,, 89\,, 144\,, 233\,, 377\,, 610\,.$$

Thus we now "observe" the new ones digits

$$4\,, 9\,, 7\,, 0\,.$$

At this point we have "seen" all digits but 6.

Problem 5

As $x < 2$ we have that $|x-2| = -(x-2) = 2-x$. Thus we have that $|x-2| = p$ is

$$2 - x = p \quad \text{so} \quad x = 2 - p\,.$$

This means that $x - p = 2 - p - p = 2 - 2p$.

Problem 6

The primes between 4 and 18 are

$$\{5\,, 7\,, 11\,, 13\,, 17\}\,.$$

There are $\left(\begin{array}{c} 5 \\ 2 \end{array}\right) = 10$ pairs from this set. One way to proceed is to table all of these numbers and compute the needed expressions and then see which one is a solution. We can do this with the R programming language as

```
primes = c(5, 7, 11, 13, 17)
a_times_b = outer(primes, primes, "*")
a_plus_b = outer(primes, primes, "+")
res = a_times_b - a_plus_b
print(sort(res[upper.tri(res)]))
```

This gives

```
 [1]  23  39  47  59  63  71  95 119 159 191
```

We see that 119 is a solution.

Problem 7

If we have that $\log_b(729) = n$ then $b^n = 729$. Factoring 729 we get that $729 = 3^6$ so we are looking for the number of integers n such that

$$b^n = 3^6\,.$$

From this we must have n be an integer factor of 6 and thus $n \in \{1, 2, 3, 6\}$. For these values of n we have that b is given by

$$3^6\,, 3^3\,, 3^2\,, 3\,.$$

Thus there are four solutions.

Problem 8

Looking at figure 3 we see that it has $2(3)+1$ boxes in the central vertical column. We have $2(2)+1$ boxes in the column to the right of this central vertical column, $2(1)+1$ boxes in the column to the right-right of this central vertical column and finally a single box to the right-right-right of this central vertical column. There are also the same number of vertical columns to the left as to the right. Thus it looks like the number of boxes the figure f has is given by

$$N_f = (2f+1) + 2\sum_{k=0}^{f-1}(2k+1) \,.$$

Here we have added the number of boxes in the central column and then the number of boxes in the columns to the right/left of this central column. We can evaluate this sum to get

$$N_f = 2f^2 + 2f + 1 \,.$$

This formula works for $f \in \{0, 1, 2, 3\}$. Taking $f = 100$ we get

$$N_{100} = 20201 \,.$$

Problem 9

From the problem statement the sum of these numbers (which is 400) must be divisible by five (which it is). If we "remove" one of these numbers we must get a number that is divisible by four. Thus we consider

$$\begin{aligned} 400 - 71 &= 329 \\ 400 - 76 &= 324 \\ 400 - 80 &= 320 \\ 400 - 82 &= 318 \\ 400 - 91 &= 309 \,. \end{aligned}$$

Only 324 and 320 are divisible by four. Thus the fifth number entered x_5 must be either $x_5 = 76$ or $x_5 = 80$.

Lets assume that $x_5 = 76$. Then the sum of the first four numbers is $400 - 76 = 324$. We then have four choices for what the fourth number x_4 could be. We find

$$\begin{aligned} 324 - 71 &= 253 \\ 324 - 80 &= 244 \\ 324 - 82 &= 242 \\ 324 - 91 &= 233 \,. \end{aligned}$$

These numbers would need to be divisible by three. Unfortunately none are. Thus it cannot be true that $x_5 = 76$ and we must conclude that $x_5 = 80$.

At this point we have answered the question asked but we can determine other things about this problem. In the case when $x_5 = 80$ the sum of the first four numbers is $400 - 80 = 320$. We now have four choices for what the fourth number x_4 could be. We find

$$
\begin{aligned}
320 - 71 &= 249 \\
320 - 76 &= 244 \\
320 - 82 &= 238 \\
320 - 91 &= 229 \, .
\end{aligned}
$$

Only 249 is divisible by three. This means that $x_4 = 71$.

We can follow the logic above to find that $x_3 = 91$ and x_1 and x_2 can then be taken from $\{76, 82\}$.

Problem 10

If we follow the given transformations we have

$$Q = (1, 2, -3) \, ,$$

after reflection. Then

$$R = (1, -2, 3) \, ,$$

after reflection. Then

$$S = (1, -2 + 5, 3) = (1, 3, 3) \, ,$$

after translation.

Problem 11

From $ab = a - b$ we will solve for b to get $b = \frac{a}{1+a}$. If we put that into the second expression we get

$$\frac{a}{\frac{a}{1+a}} + \frac{a}{a(1+a)} - a\left(\frac{a}{1+a}\right) = 2 \, ,$$

when we simplify.

Problem 12

Call the expression we seek to maximize J i.e.

$$J(A, M, C) \equiv AMC + AM + MC + CA \, .$$

Note that J is "equivalent" under permutations of (A, M, C). This means that if J takes an value at (A^*, M^*, C^*) then

$$J(A^*, M^*, C^*) = J(M^*, C^*, A^*) = J(C^*, A^*, M^*) \,,$$

i.e. J takes the same value at three different points. Now J will be largest if A, M, and C are as large as possible and by the above we want them to increase "together". Given the constraint $A + M + C = 12$ this means that $A = M = C = x$ or

$$3x = 12 \quad \text{so} \quad x = 4 \,.$$

The maximum is then

$$J(4, 4, 4) = 112 \,,$$

when we evaluate.

Problem 13

Let c_i and m_i be the amount of coffee and milk (in ounces) respectively for each of the N family members. Then we must have

$$c_i + m_i = 8 \,, \tag{630}$$

for each i. Let $i = 1$ be Angela's "index". Then in the problem statement we are told that

$$\frac{c_1}{\sum_{i=1}^N c_i} = \frac{1}{6} \,, \tag{631}$$

and

$$\frac{m_1}{\sum_{i=1}^N m_i} = \frac{1}{4} \,. \tag{632}$$

From Equation 631 we can solve for c_1 to get

$$c_1 = \frac{1}{5} \sum_{i=2}^N c_i \,.$$

From Equation 632 we can solve for m_1 to get

$$m_1 = \frac{1}{3} \sum_{i=2}^N m_i \,.$$

If I sum Equation 630 for $i = 2, 3, \ldots, N$ I get

$$\sum_{i=2}^N c_i + \sum_{i=2}^N m_i = 8(N - 1) \,. \tag{633}$$

Also using the above expressions for c_1 and m_1 in $c_1 + m_1 = 8$ we get

$$\frac{1}{5}\sum_{i=2}^{N} c_i + \frac{1}{3}\sum_{i=2}^{N} m_i = 8\,. \tag{634}$$

The above are two equations for the two unknowns $\sum_{i=2}^{N} c_i$ and $\sum_{i=2}^{N} m_i$. Solving for each gives

$$\sum_{i=2}^{N} c_i = 20(N-4)$$

$$\sum_{i=2}^{N} m_i = 15(6-N)\,.$$

As $c_i > 0$ from the first of these we have that $20(N-4) > 0$ so $N > 4$. As $m_i > 0$ from the second of these we have that $15(6-N) > 0$ so $N < 6$. As N must be an integer we have that $N = 5$ is the only valid choice.

Problem 14

If we order these numbers from least to greatest (with x at the "end") we get

$$2, 2, 2, 4, 5, 10, x\,.$$

The mode of these numbers is two.

The mean of these numbers is

$$\frac{2+2+2+4+5+10+x}{7} = \frac{25+x}{7}\,.$$

As there are seven numbers the median is the number in the fourth position. Depending on the value of x the number in that position can be different things.

- If $x \geq 4$ then the median is four.
- If $2 \leq x < 4$ then the median is x.
- If $x < 2$ then the median is two.

We will consider each case in tern.

Case 1: Lets assume that $x \geq 4$ then the increasing order of the mean, median, and mode is

$$2, 4, \frac{25+x}{7}\,.$$

If these are in arithmetic sequence then they have a common difference or

$$\frac{25+x}{7} - 4 = 2 \quad \text{so} \quad x = 17\,.$$

Case 2: Next lets assume that $2 \le x < 4$. Then the increasing order of the mean, median, and mode is

$$2, x, \frac{25+x}{7}\,.$$

Again if these are in arithmetic sequence then they have a common difference or

$$\frac{25+x}{7} - x = x - 2 \quad \text{so} \quad x = 3\,.$$

Case 3: Next lets assume that $x < 2$. Then the increasing order of the mean, median, and mode is

$$2, 2, \frac{25+x}{7}\,.$$

This could not be a non-constant arithmetic progression since the first two terms are equal.

Thus the sum of the possible x's is

$$17 + 3 = 20\,.$$

Problem 15

If we take $v = \frac{x}{3}$ then $x = 3v$ and what we are given becomes

$$f(v) = 9v^2 + 3v + 1\,.$$

From this we find that

$$f(3z) = 81z^2 + 9z + 1\,,$$

so $f(3z) = 7$ is

$$81z^2 + 9z - 6 = 0\,.$$

Dividing by 81 gives

$$z^2 + \frac{1}{9}z - \frac{2}{27} = 0\,.$$

From Vieta's formulas the sum of the roots is then $-\frac{1}{9}$.

Problem 16

Assuming "matrix" indexing (i, j) where i counts the number of rows from top to bottom (starting at one) and j counts the number of columns from left to right (starting at one) then under the first checkerboard numbering we have each "cell" taking the value

$$17(i-1) + j\,,$$

for the "slow index" $1 \leq i \leq 13$ and the "fast index" $1 \leq j \leq 17$.

Under the second checkerboard numbering we have each "cell" taking the value

$$13(j-1)+i\,,$$

for the "slow index" $1 \leq j \leq 17$ and the "fast index" $1 \leq i \leq 13$.

Notice that in each case i and j are constrained over the same domain.

The problem asks us which cells (under each numbering) have the same value. This means that

$$17(i-1)+j = 13(j-1)+i\,.$$

We can simplify this as

$$4i = 1+3j\,,$$

or

$$i = \frac{1+3j}{4}\,. \tag{635}$$

As i must be an integer this means that $1+3j$ must be divisible by four or

$$(1+3j) \equiv 0 \pmod 4\,,$$

or

$$3j \equiv -1 \pmod 4\,,$$

or adding four to both sides

$$3j \equiv 3 \pmod 4\,,$$

or

$$j \equiv 1 \pmod 4\,.$$

This means that j takes the values

$$j \in \{1, 5, 9, 13, 17\}\,.$$

Using Equation 635 we get that i is then given by

$$i \in \{1, 4, 7, 10, 13\}\,.$$

Using either of the formulas above the values in these cells are given by

$$\{1, 56, 111, 166, 221\}\,.$$

The sum of these numbers is 555.

Problem 17

From the given right triangle $\triangle OAB$ we have

$$\tan(\theta) = \frac{AB}{AO} = \frac{AB}{1} = AB\,.$$

Using the angle bisector theorem on the angle $\angle OBA$ we have

$$\frac{OB}{OC} = \frac{AB}{AC} = \frac{AB}{1-OC} = \frac{\tan(\theta)}{1-OC}\,. \tag{636}$$

Now we determine the length OB in terms of θ. We have

$$AO = BO\cos(\theta)\,,$$

but $AO = 1$ so $BO = \frac{1}{\cos(\theta)}$. Using Equation 636 we get

$$\frac{1}{OC\cos(\theta)} = \frac{\tan(\theta)}{1-OC}\,.$$

Solving this for OC gives

$$OC = \frac{1}{1+\sin(\theta)}\,.$$

Problem 18

If day 300 in year N is a Tuesday then every day $300 + 7n$ (for $n \in \mathbb{Z}$) is also a Tuesday. As

$$300 \equiv 6 \bmod 7\,,$$

This means that day six of year N is also a Tuesday. Working backwards this means that in year N we have that

- Day five is a Monday
- Day four is a Sunday
- Day three is a Saturday
- Day two is a Friday
- Day one is a Thursday

Thus day 365 of year $N-1$ is a Wednesday. We now need to shift "backwards" in multiples of seven towards the day 100. As

$$365 - 100 = 265 \equiv 6 \bmod 7\,,$$

we know that day $100 + 6 = 106$ is also a Wednesday. This means that day $100 - 7 = 99$ is also a Wednesday so that day 100 is a Thursday.

Problem 19

As

$$15^2 = 225 \quad \text{and}$$
$$13^2 + 14^2 = 365 > 15^2 \,,$$

the angle opposite AC i.e $\angle B$ is larger than $90°$ and the triangle is obtuse.

As D is the midpoint of BC we have that

$$CD = DB = \frac{14}{2} = 7 \,.$$

As E is on the angle bisector of $\angle BAC$ by the angle bisector theorem we have

$$\frac{AC}{CE} = \frac{AB}{BE} \quad 15CE = \frac{13}{CB - CE} = \frac{13}{14 - CE} \,.$$

We can solve that for CE and find $CE = \frac{15}{2}$. Now using this ED then has the length

$$ED = \frac{15}{2} - 7 = \frac{1}{2} \,.$$

Now as all three sides of this triangle are given the semi perimeter is

$$s = \frac{1}{2}(13 + 14 + 15) = 21 \,,$$

Thus Heron's formula for the area of $\triangle ABC$ is

$$\sqrt{s(s-a)(s-b)(s-c)} = \sqrt{21(8)(7)(6)} = 84 \,.$$

As D is the midpoint of one side of the triangle the area of the triangles $\triangle CDE$ and $\triangle BDA$ are $\frac{1}{2}$ of the area of $\triangle ABC$ or $\frac{84}{2} = 42$.

Drop a perpendicular from A and intersecting the extension of the side BC at a point A'. Then we an write the area of $\triangle BDA$ as

$$[BDA] = 42 = \frac{1}{2}(DB)AA' = \frac{7}{2}AA' \,.$$

Solving for AA' gives $AA' = 12$. Using this we have that the area of the triangle of interest $\triangle ADE$ given by

$$[ADE] = \frac{1}{2}(DE)AA' = \frac{1}{2}\left(\frac{1}{2}\right) 12 = 3 \,.$$

Problem 20

Method 1: If we sum these three equations we get

$$x + \frac{1}{y} + y + \frac{1}{z} + z + \frac{1}{x} = 5 + \frac{7}{3} = \frac{22}{3}\,,$$

or

$$x + y + z + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{22}{3}\,. \tag{637}$$

If we take the product of these three equations we get

$$\left(x + \frac{1}{y}\right)\left(y + \frac{1}{z}\right)\left(z + \frac{1}{x}\right) = 4(1)\left(\frac{7}{3}\right) = \frac{28}{3}\,.$$

On expanding the products on the left-hand-side and simplifying we get

$$xyz + x + z + \frac{1}{y} + y + \frac{1}{z} + \frac{1}{x} + \frac{1}{xyz} = \frac{28}{3}\,.$$

Using Equation 637 to simplify the left-hand-side of this we find

$$xyz + \frac{22}{3} + \frac{1}{xyz} = \frac{28}{3}\,,$$

or

$$xyz + \frac{1}{xyz} = 2\,.$$

If we multiply by xyz we can write this as

$$(xyz)^2 - 2(xyz) + 1 = 0\,,$$

or

$$(xyz - 1)^2 = 0\,,$$

Thus $xyz = 1$.

Method 2: From the first and second equation we have that $x = 4 - \frac{1}{y}$ and $y = 1 - \frac{1}{z} = \frac{z-1}{z}$. This means that

$$x = 4 - \frac{z}{z-1} = \frac{3z-4}{z-1}\,.$$

From the third equation we have $z = \frac{7}{3} - \frac{1}{x} = \frac{7x-3}{3x}$ so using this in the above we get

$$x = \frac{3\left(\frac{7x-3}{3x}\right) - 4}{\frac{7x-3}{3x} - 1} = \frac{9(x-1)}{4x-3}\,.$$

We can solve the above for x and find $x = \frac{3}{2}$. From this we can determine $y = \frac{2}{5}$ and $z = \frac{5}{3}$. Thus

$$xyz = \left(\frac{3}{2}\right)\left(\frac{2}{5}\right)\left(\frac{5}{3}\right) = 1\,.$$

Problem 21

Lets draw a x-y Cartesian coordinate system with the square with side length s drawn in the first quadrant. Then introduce the right triangle $\triangle BAC$ with $A = (0, 0)$, B on the x-axis and C on the y-axis. As the square has a side of length s the let point B be a distance x from the vertical face of the square so $B = (s + x, 0)$. In the same way let C be y from the horizontal face of the square so $C = (0, s + y)$. Let the "top most" triangle be denoted I so that according to the problem statement its area is

$$A_I = ms^2 \, .$$

Of course $A_I = \frac{1}{2}ys$ so that

$$m = \frac{A_I}{s^2} = \frac{y}{2s} \, . \tag{638}$$

Let the "right most" triangle be denoted as III and we want to know the value of

$$\frac{A_{III}}{s^2} = \frac{\frac{1}{2}xs}{s^2} = \frac{x}{2s} \, . \tag{639}$$

Evaluating the area of the right triangle $\triangle BAC$ in two different ways we have

$$\frac{1}{2}(x + s)(y + s) = s^2 + \frac{1}{2}sy + \frac{1}{2}sx \, .$$

If we expand the left-hand-side and simplify we get

$$xy = s^2 \, .$$

Dividing both sides of this by s^2 we get

$$\frac{x}{s} \cdot \frac{y}{s} = 1 \quad \text{so} \quad \frac{y}{s} = \frac{s}{x} \, .$$

Using this in Equation 638 we get

$$m = \frac{s}{2x} \quad \text{so} \quad \frac{x}{s} = \frac{1}{2m} \, .$$

Using this in Equation 639 we get

$$\frac{A_{III}}{s^2} = \frac{1}{4m} \, .$$

Problem 22

From the graph it looks like $P(-1) \approx 4$ and the two real zeros are close to $1\frac{2}{3} = \frac{5}{3}$ and $3\frac{2}{3} = \frac{11}{3}$. The product of all of the zeros of $P(x)$ will be the value of d which is also the value of $P(0) \approx 5\frac{1}{3} = \frac{16}{3}$ (using the graph). The product of the non-real zeros of P will be the product of all zeros divided by the product of the real zeros or

$$\frac{\frac{16}{3}}{\frac{5}{3} \cdot \frac{11}{3}} = \frac{48}{55} < 1 \, .$$

The sum of the coefficients of P will be $P(1) - 1^4 = P(1) - 1 \approx 4 - 1 = 3$. Finally, the sum of the real zeros of P is given by

$$\frac{5}{3} + \frac{11}{3} = \frac{16}{3}.$$

The smallest of these is the product of the non-real zeros.

Problem 23

Let p_i be the ith number "picked" where $1 \leq p_i \leq 46$ for $1 \leq i \leq 6$. Then we are told that for both Professor Gamble and the winning ticket we have

$$\sum_{i=1}^{6} \log_{10}(p_i) = N,$$

for N an integer. We can write this as

$$\log_{10}\left(\prod_{i=1}^{6} p_i\right) = N,$$

or

$$\prod_{i=1}^{6} p_i = 10^N = 2^N \cdot 5^N. \tag{640}$$

This means that the numbers p_i must only have prime factors of two and five. Thus they look like the numbers

$$p_i = 2^{n_i} \cdot 5^{m_i}.$$

With this expression Equation 640 becomes

$$2^{\sum_{i=1}^{6} n_i} \cdot 5^{\sum_{i=1}^{6} m_i} = 2^N \cdot 5^N,$$

and thus we have

$$\sum_{i=1}^{6} n_i = N = \sum_{i=1}^{6} m_i,$$

or

$$\sum_{i=1}^{6} (n_i - m_i) = 0. \tag{641}$$

Since we are told that $1 \leq p_i \leq 46$ these p_i are "products" of numbers like

$$2^{n_i} \in \{1, 2, 4, 8, 16, 32\} \quad \text{and} \quad 5^{m_i} \in \{1, 5, 25\}.$$

Thus

$$n_i \in \{0, 1, 2, 3, 4, 5\}$$
$$m_i \in \{0, 1, 2\}.$$

The actual products that satisfy $1 \le p_i \le 46$ are

$$\begin{aligned} p_i &\in \{1,2,4,8,16,32\} \cup \{5,10,20,40\} \cup \{25\} \\ &\in \{1,2,4,5,8,10,16,20,25,32,40\} \\ &\in \{2^0\cdot 5^0, 2^1\cdot 5^0, 2^2\cdot 5^0, 2^0\cdot 5^1, 2^3\cdot 5^0, 2^1\cdot 5^1, 2^4\cdot 5^0, 2^2\cdot 5^1, 2^0\cdot 5^2, 2^5\cdot 5^0, 2^3\cdot 5^1\}\,. \end{aligned}$$

Now to have Equation 641 be satisfied when we draw six p_i from the above set we need to have the sum of $n_i - m_i$ be zero. If we compute the value of $n_i - m_i$ for each of the above p_i we get

$$\begin{aligned} n_i - m_i &\in \{0,1,2,-1,3,0,4,1,-2,5,2\} \\ &\in \{-2,-1,0,0,1,1,2,2,3,4,5\}\,. \end{aligned}$$

We now ask how many ways are there to draw six numbers (without replacement) from the above set such that the sum of the numbers is zero.

If we don't draw any negative numbers the sum of $n_i - m_i$ will not be zero. If we draw only the single negative number of -1 then the smallest we can make the sum is

$$-1+0+0+1+1+2 = 3 \neq 0\,.$$

If we draw only the single negative number of -2 then the smallest we can make the sum is

$$-2+0+0+1+1+2 = 2 \neq 0\,.$$

Thus we must draw both the -1 and the -2. Notice that there are four sums that start with $\{-2,-1,0,0\}$ and then select a single one and a single two that will sum to zero. Thus there are four total numbers of the given form and Professor Gamble has a $\frac{1}{4}$ chance of winning.

Problem 24

By connecting the three points together we form the triangle $\triangle ABC$ which can be shown to be an equilateral triangle. Drop a vertical from C intersecting the segment AB at the point C'. By symmetry we have $\angle CC'A = \angle CC'B = 90°$. Also by symmetry the center of the circle must be on the segment CC'. Call the center of the circle the point O.

As we are told that $\widehat{BC} = 12$ if we let R be the length of AB we must have

$$\frac{60}{360}(2\pi R) = 12 \quad \text{so} \quad R = \frac{36}{\pi}\,.$$

Let r be the radius of the internal circle we seek the circumference of. When we draw the segment from A to O we introduce the right triangle $\triangle AC'O$ that has sides of length

$$\begin{aligned} AC' &= \frac{R}{2} = \frac{18}{\pi} \\ C'O &= r \\ AO &= R - r = \frac{36}{\pi} - r\,. \end{aligned}$$

Using the Pythagorean theorem we then have

$$\left(\frac{36}{\pi} - r\right)^2 = r^2 + \left(\frac{36}{2\pi}\right)^2$$

If we expand and simplify we find that

$$r = \frac{27}{2\pi}\,.$$

Thus the circumference desired is $2\pi r = 27$.

The 2000 AMC 10 Examination

Problem 1

This is the same as Problem 1 worked on Page 895.

Problem 2

This is the same as Problem 2 worked on Page 895.

Problem 3

This is the same as Problem 3 worked on Page 895.

Problem 4

Let the total fee be F. Then we are told that

$$F = f_0 + hT\,,$$

for a fixed cost f_0, a cost per unit time h, and an amount of time T. In December we are told that

$$12.48 = f_0 + hT\,,$$

while in January we have

$$17.54 = f_0 + h(2T)\,.$$

If we solve for hT in the first expression and put this in the second expression we get

$$17.54 = f_0 + 2(12.48 - f_0)\,.$$

In the above if we solved for f_0 we find $f_0 = 7.42$.

Problem 5

By the midpoint theorem in triangles, for the segment AB fixed as we move P since M and N are midpoints the length of MN is equal to $\frac{1}{2}AB$ and is thus fixed and MN is parallel to AB.

As P is always the same distance from AB with a base on AB and a height from P to AB (which does not change) the area of triangle $\triangle PAB$ does not change.

In the trapezoid $ABNM$ as we have discussed that the length of MN, AB, and the distance between these two are fixed the area of this trapezoid is constant.

The only thing that changes as we move P is the perimeter of the triangle $\triangle PAB$.

Problem 6

This is the same as Problem 4 worked on Page 895.

Problem 7

As the angle $\angle ADC$ is trisected each angle there is $30°$. Thus $\angle BDC = 30°$. Using this in the right triangle $\triangle BCD$ we have

$$\tan(30°) = \frac{BC}{DC} \quad \text{so} \quad DC = \frac{BC}{\tan(30°)} = \frac{1}{\frac{1}{\sqrt{3}}} = \sqrt{3}\,.$$

The Pythagorean theorem in that triangle gives

$$BD = \sqrt{1+3} = 2\,.$$

In a similar way using the right triangle $\triangle DAP$ we have

$$\tan(30°) = \frac{AP}{AD} \quad \text{so} \quad AP = AD\tan(30°) = \frac{1}{\sqrt{3}}\,.$$

This means that

$$PB = AB - AP = \sqrt{3} - \frac{1}{\sqrt{3}}\,.$$

The Pythagorean theorem in that right triangle $\triangle DAP$ gives

$$DP = \sqrt{1^2 + \frac{1}{3}} = \frac{2}{\sqrt{3}}\,.$$

Using these parts we find the perimeter of $\triangle BDP$ given by

$$BD + DP + PB = 2 + \frac{2}{\sqrt{3}} + \sqrt{3} - \frac{1}{\sqrt{3}} = 2 + \frac{4\sqrt{3}}{3}\,,$$

when we simplify.

Problem 8

Let F be the number of freshmen and S the number of sophomores. From the problem statement we are told that

$$\frac{2}{5}F = \frac{4}{5}S \quad \text{or} \quad F = 2S\,,$$

which is choice (D).

Problem 9

This is the same as Problem 5 worked on Page 896.

Problem 10

By the triangle inequality we must have

$$\begin{aligned} 4 + 6 &> x \\ 4 + x &> 6 \\ 6 + x &> 4\,. \end{aligned}$$

These simplify to $2 < x < 10$. The same expression must hold for y also so that

$$2 < y < 10\,.$$

We can write the above as $-10 < -y < -2$ which if we add to the inequality in x gives

$$-8 < x - y < 8 \quad \text{or} \quad |x - y| < 8\,.$$

Thus eight is not possible.

Problem 11

If we start with all numbers between 4 and 18 we have

$$4\,,5\,,6\,,7\,,8\,,9\,,10\,,11\,,12\,,13\,,14\,,15\,,16\,,17\,,18\,.$$

If we drop composite numbers we get

$$5\,,7\,,11\,,13\,,17\,.$$

We can now draw two different prime numbers from this set we can compute $p_1p_2 - (p_1 + p_2)$. This can be done by hand but its also easily done in `R`

```
ps = c(5, 7, 11, 13, 17)
p1_times_p2 = outer(ps, ps, FUN='*')
p1_plus_p2 = outer(ps, ps, FUN='+')
p1_times_p2 - p1_plus_p2
```

This gives

```
     [,1] [,2] [,3] [,4] [,5]
[1,]   15   23   39   47   63
[2,]   23   35   59   71   95
[3,]   39   59   99  119  159
[4,]   47   71  119  143  191
[5,]   63   95  159  191  255
```

We see that 191 is a possibility.

Problem 12

This is the same as Problem 8 worked on Page 897.

Problem 13

As we have five yellow pegs (and there are five rows total) we must place one of them in each row. This is the "pigeonhole principle" in action. Starting at the top and placing these pegs we see that to avoid duplicating a color in a column we must place these pegs "on the diagonal". The same argument applied to the red pegs means they must be placed on the "diagonal". Following this logic for all colors gives that there is only one configuration of the desired type.

Problem 14

This is the same as Problem 9 worked on Page 897.

Problem 15

This is the same as Problem 11 worked on Page 898.

Problem 16

Method 1: One way to solve this problem is find the Cartesian x-y equation of the line that goes through AB and the one that goes though CD. Then point E is then the intersection of these two lines. The distance AE is then given by the distance between points formula. For example, if we let a vertical line through A be the y-axis and a horizontal line though DB be an x-axis of a Cartesian coordinate system then we can assign x-y coordinate to the points

$$\begin{aligned} A &= (0,3) \\ B &= (6,0) \\ C &= (4,2) \\ D &= (2,0)\,. \end{aligned}$$

The details of this solution method are left for the reader.

Method 2: Extend the segment DC to intersect the horizontal though A at a point F. Then as AF and DB are parallel to each other the two triangles $\triangle AEF$ and $\triangle BED$ similar and thus we have

$$\frac{AF}{AE} = \frac{DB}{EB} \quad \text{or} \quad \frac{5}{AE} = \frac{4}{AB - AE}\,.$$

Now AB is the hypotenuse of a right triangle with legs of length three and six and thus $AB = \sqrt{3^2 + 6^2} = 3\sqrt{5}$. Putting this in the above and solving for AE gives $AE = \frac{5\sqrt{5}}{3}$.

Problem 17

Denote the "state" of systems as (p, n, q) which means that we have p pennies, n nickles, and q quarters. Under the three transformations given we have

$$\begin{aligned} (p,n,q) &\to (p, n+5, q-1) \\ (p,n,q) &\to (p+5, n-1, q) \\ (p,n,q) &\to (p-1, n, q+5)\,. \end{aligned}$$

Under a state of (p, n, q) the "value" (in pennies) is

$$V = p + 5n + 25q\,.$$

Now under the first two transformations the value does not change. Under the third transformation the initial value and the "next" values are

$$\begin{aligned} V &= p + 5n + 25q \\ V' &= p - 1 + 5n + 25(q+5)\,. \end{aligned}$$

This gives a value increase of

$$\Delta = V' - V = -1 + 25(5) = 124\,.$$

As we start with a value of $V_0 = 1$ the only possible values under these transformations are given by

$$1 + 124n\,,$$

for $n \geq 0$. We can let n be integers and see what values we can get. In R we can do that with

```
ns = seq(1, 10)
124 * ns + 1
```

which gives

```
 [1]  125  249  373  497  621  745  869  993 1117 1241
```

Notice that \$7.45 is one of these numbers (when $n = 6$). Another way to say the same thing as above is that the possible values we can get must be a number x such that $x \equiv 1 \pmod{124}$. We can take the given numbers and see if any of them have this property. We can do this in R with

```
choices = c(363, 513, 630, 745, 907)
choices %% 124
```

which gives

```
[1] 115  17  10   1  39
```

This gives the same amount of \$7.45.

Problem 18

If we sketch the region that Charlyn "can see" we see that it is a "square" with a side length of $5 + 2 = 7$ but with rounded corners *minus* an internal square with a side length of $5 - 2 = 3$. Breaking the outer region into a rectangle "tall" rectangle, two "tall" rectangles (of size 1×5) and four quarters of a circle with radius one. The outer region then has an area of

$$5 \times 7 + 2(1 \times 5) + 4\left(\frac{1}{4}\pi(1^2)\right) = 45 + \pi\,.$$

The internal region has an area of $3^2 = 9$ so the region Charlyn can see is

$$(45 + \pi) - 9 = 36 + \pi \approx 39\,,$$

to the nearest kilometer squared.

Problem 19

This is the same as Problem 21 worked on Page 906.

Problem 20

This is similar to Problem 12 on Page 898.

Call the expression we seek to maximize J i.e.

$$J(A, M, C) \equiv AMC + AM + MC + CA\,.$$

Note that J is "equivalent" under permutations of (A, M, C). This means that if J takes an value at (A^*, M^*, C^*) then

$$J(A^*, M^*, C^*) = J(M^*, C^*, A^*) = J(C^*, A^*, M^*)\,,$$

i.e. J takes the same value at three different points. Now J will be largest if A, M, and C are as large as possible and by the above we want them to increase "together". Given the constraint $A + M + C = 10$ this means that $A = M = C = x$ or

$$3x = 10 \quad \text{so} \quad x = \frac{10}{3} = 3\,\frac{1}{3}\,.$$

This is not an integer solution and is thus not a valid solution for A, M, and C.

We expect the solutions for A, M and C to be "close" to $3\,\frac{1}{3}$. If we take each to be equal to three then $A + M + C = 9 < 10$. If we take two of them equal to three and the other equal to four we get the correct sum/constraint. Notice that both the expressions

$$AMC \quad \text{and} \quad AM + MC + CA\,,$$

are invariant under the transformation where we pick one of A, M, or C to be four and the other two variables to be three.

The maximum we seek is then

$$3^2 \cdot 4 + 3 \cdot 4 + 3 \cdot 4 + 3^2 = 69\,.$$

Problem 21

From the statements in terms of Venn diagrams if we draw the set of ferocious creatures (FC) then the set of alligators (A) must be inside of this set. The set of creepy crawlers (CC) must then include some alligators and thus must overlap with ferocious creatures and alligators. These alligators that are in the creepy crawlers set are also ferocious creatures and thus there are some ferocious creatures are creepy crawlers. Depending on if the set of alligators is a proper subset of the set of creepy crawlers (or not) I or III might or might not be true.

Problem 22

This is the same as Problem 13 worked on Page 899.

Problem 23

This is the same as Problem 14 worked on Page 900.

Problem 24

This is the same as Problem 15 worked on Page 901.

Problem 25

This is the same as Problem 18 worked on Page 903.

The 2000 AHSME Anniversary Examination

I worked these "review" problems to see if my problem solving ability had gotten better between now and when I first worked them. In many cases I solved the problem in the same way I had done earlier. Rather than present the "same" solution I'll just link to the earlier solution of the problem (there seemed no reason to just duplicate text). If my solution method was different than before I include the new solution here.

Problem 1950-10

This is worked on Page 12.

Problem 1951-48

This is worked on Page 39.

Problem 1952-44

This is worked on Page 61.

Problem 1953-50

This is worked on Page 82.

Problem 1954-38

This is worked on Page 95.

Problem 1955-33

This is worked on Page 114.

Problem 1956-39

This is worked on Page 136.

Problem 1957-26

This is worked on Page 148.

Problem 1958-45

This is worked on Page 177.

Problem 1959-22

This is a problem I worked slightly differently the second time than I did when I worked it the first time. Using the points labeled in Figure 1 recall that as the segment MN is on a *midline* of the trapezoid it passes though the midpoints E (of AD) and F (of BC).

Now as EN is a a midline in the triangle $\triangle ADB$ we have that

$$EN = \frac{1}{2}AB = \frac{97}{2}\,.$$

As MF is a a midline in the triangle $\triangle CAB$ we have that

$$MF = \frac{1}{2}AB = \frac{97}{2}\,.$$

Subtracting the length MN from each we get

$$EM = NF = \frac{97}{2} - 3 = \frac{91}{2}\,.$$

Now as EM is a midline in the triangle $\triangle ADC$ we have that

$$\frac{1}{2}DC = EM = \frac{91}{2} \quad \text{so} \quad DC = 91\,.$$

Problem 1960-19

This is worked on Page 205.

Problem 1961-5

This is worked on Page 217.

Problem 1962-27

This is worked on Page 236.

Problem 1963-37

This is worked on Page 258.

Problem 1964-15

This is worked on Page 263.

Problem 1965-29

So I had a few different insights on working this problem a second time. The first is that it feels like a "set problem" and thus set notation/arguments are probably in order. Towards that direction we can start with a Venn diagram where there is a "circle" for mathematics, english, and history. I drew the mathematics circle in the second quadrant, the english circle in the first quadrant, and the history circle below both of these others. I let x denote the number of students that are taking mathematics only (which is also equal to the number of students that are taking mathematics and english only). Thus in these specific regions of overlap in the Venn diagram I would draw and x. The problem statement then gives us numbers for every section of the Venn diagram. For example if z is the number of students taking all three classes than the number of students taking english and history is $5z$.

We can sum the count information in each region of the Venn diagram to get the total number of students or

$$2x + 6 + 6z = 28 \quad \text{or} \quad x + 3z = 11\,.$$

As we are told that $z = 2n$ for $n \geq 1$ we have

$$x + 6n = 11\,.$$

Only $n = 1$ works in the above and gives $x = 5$ which is the desired answer.

Problem 1966-39

This is worked on Page 300.

Problem 1967-31

This is worked on Page 313 but we can show that $\sqrt{D}$ is always odd in an easier way. Using the fact that $\sqrt{D} = a^2 + a + 1$ if a were even it would look like $a = 2n$ for $n \geq 1$ which gives

$$\sqrt{D} = 4n^2 + 2n + 1\,,$$

which is odd. If a were odd then $a = 2n + 1$ for $n \geq 0$ and for $\sqrt{D}$ we find

$$\sqrt{D} = (4n^2 + 4n + 1) + 2n + 1 + 1 = (4n^2 + 6n + 2) + 1\,,$$

which is also an odd number.

Problem 1968-32

This is worked on Page 332.

Problem 1969-29

This is worked on Page 346.

Problem 1970-25

This is worked on Page 358.

Problem 1971-31

One way to solve this problem is given on Page 378.

Here is a second method. We let the center of the circle be $O = (0, 0)$ the center of an x-y Cartesian coordinate system. Then $A = (-2, 0)$, $D = (+2, 0)$, and the equation of the circle is

$$x^2 + y^2 = 4\,. \tag{642}$$

From the point A we want to find another point B such that the distance $AB = 1$. This means that while B is on the equation of the larger circle it is also on the circle centered at A with radius of one or

$$(x + 2)^2 + y^2 = 1\,. \tag{643}$$

Solving these two equations and taking the solution where $y > 0$ we find the point B

$$B = \left(-\frac{7}{4}, \frac{\sqrt{15}}{4}\right)\,.$$

Now the point C is one unit away from B and on the circle given by Equation 642. The fact that it is one unit away means that the point C is on the equation

$$\left(x + \frac{7}{4}\right)^2 + \left(y - \frac{\sqrt{15}}{4}\right)^2 = 1\,. \tag{644}$$

The point C that is on the intersection of Equation 642 and 644 is

$$C = \left(-\frac{17}{16}, \frac{7\sqrt{15}}{16}\right)\,.$$

Now that we have the location of point C the distance desired is given by

$$CD^2 = \left(2 + \frac{17}{16}\right)^2 + \frac{7^2 \cdot 15}{16^2} = \frac{49 \cdot 64}{16^2}\,,$$

so $CD = \frac{7}{2}$.

Problem 1972-35

This is worked on Page 398.

Problem 1973-31

This is worked on Page 417.

On my second working of this problem I did many of the same steps as on the first working but with a couple of simplifications. One observation we will use below is that the units digit of E^2 must be the number T.

Now as discussed in the previous solution as TTT factors at $T \cdot 3 \cdot 37$ one of YE or ME must be $1 \times 37 = 37$ or $2 \times 37 = 74$ (other multiples of 37 are not two digit numbers). If we assume that $YE = 74$ we have that $E = 4$ so that $E^2 = 16$ and we must have $T = 6$. This means that $TTT = 666$ and ME must equal

$$ME = \frac{666}{74} = 9\,,$$

which is not a two digit number and thus is not correct. If instead we assume that $YE = 37$ then $E = 7$ so that $E^2 = 49$ and $T = 9$. This means that $TTT = 999$ and ME must be

$$ME = \frac{999}{37} = 27\,.$$

Thus we have found that $Y = 3$, $E = 7$, $T = 9$, and $M = 2$ so that

$$E + M + T + Y = 21\,.$$

This is more similar to the solution presented in the back of the book.

Problem 1974-20

This is worked on Page 429.

Problem 1976-30

This is worked on Page 458.

a	b	c	sign(abc)	E
+	+	+	+	3+1 = 4
-	+	+	-	1-1 = 0
+	-	+	-	0
+	+	-	-	0
+	-	-	+	0
-	+	-	+	0
-	-	+	+	0
-	-	-	-	-4

Table 19: The signs for the numbers a, b, c, the product abc, and the value of E.

Problem 1977-8

Now $\frac{a}{|a|}$ will be ± 1 if a is nonzero. Call the expression given in the problem E. In Table 19 we enumerate all possible signs for a, b, and c and then the corresponding value of E. From that expression we see that the only choices for E are from the set $\{-4, 0, +4\}$.

Problem 1978-22

This is worked on Page 484.

Problem 1979-26

This is worked on Page 500. Many of the steps I took on the second solving of this problem were similar to my first attempt. One difference was that rather than solving the resulting difference equation one could hypothesis a solution to Equation 295 and then solve for any unknown coefficients. For example after taking $y = 1$ we are left with Equation 295. We might attempt to find a solution to this equation by assuming that $f(x)$ takes the form

$$f(x) = Ax + B\,.$$

When this is put into Equation 295 one gets

$$Ax + B + 1 = Ax + A + B - x - 1\,,$$

or

$$1 = A - x - 1\,,$$

which has no solution. As another attempt we might consider solutions to this equation that take the form

$$f(x) = Ax^2 + Bx + C\,.$$

When this is put into Equation 295 one gets

$$A(x+1)^2 + B(x+1) + C = Ax^2 + Bx + C + x + 2\,.$$

This simplifies to

$$2Ax + A + B = y + 2\,,$$

so $2A = 1$ or $A = \frac{1}{2}$ and $A + B = 2$ so $B = \frac{3}{2}$. This means that the solution must take the form

$$f(x) = \frac{1}{2}x^2 + \frac{3}{2}x + C\,.$$

Using the fact that $f(1) = 1$ in the above we find $C = -1$ so that

$$f(x) = \frac{1}{2}x^2 + \frac{3}{2}x - 1\,.$$

The equation $f(n) = n$ is then equivalent to

$$n^2 + 3n - 2 = 2n\,,$$

which has the solutions $n = -2$ and $n = 1$.

Problem 1980-22

This is worked on Page 513.

Problem 1981-24

This is worked on Page 526.

Problem 1982-16

This is worked on Page 535.

Problem 1983-26

We are asked to bound $p = P(A \cap B)$ above and below using $P(A)$ and $P(B)$. An upper bound is given by

$$P(A \cap B) \leq \min(P(A), P(B)) = \frac{2}{3}\,. \tag{645}$$

We also have

$$P(A \cap B) = P(A) + P(B) - P(A \cup B) = \frac{3}{4} + \frac{2}{3} - P(A \cup B) = \frac{17}{12} - P(A \cup B)\,.$$

If we make $P(A \cup B)$ as large as possible we will make $P(A \cap B)$ as small as possible so taking $P(A \cup B) = 1$ we have

$$P(A \cap B) \geq \frac{17}{12} - 1 = \frac{5}{12}.$$

This means that the region for p in $\left[\frac{5}{12}, \frac{2}{3}\right]$.

Problem 1984-11

This is worked on Page 567.

Problem 1985-24

This is worked on Page 599.

Problem 1986-14

This is worked on Page 619.

Problem 1987-12

This is worked on Page 636.

Problem 1988-6

This is a rectangle.

Problem 1989-23

This is worked on Page 730.

As another way to solve this problem we can count the number of steps to get to various points on the x-axis. To get to various points I find

$$\begin{aligned}
&(1,0) : 1 \\
&(2,0) : 1+2+1+2\times 2 = 8 \\
&(3,0) : 8+1 = 9 \\
&(4,0) : 9+2(3)+1+2(4) = 24 \\
&(5,0) : 24+1 = 25\,,
\end{aligned}$$

steps. From the few locations documented above it looks like to get to locations with *odd* x values say the point $(2n+1,0)$ will require

$$(2n+1)^2\,,$$

steps. We then might ask what is the largest n value such that

$$(2n+1)^2 \approx 1989\,.$$

Using iteration we find that

$$\begin{aligned}
43^2 &= 1849 \\
45^2 &= 2025\,.
\end{aligned}$$

The second value is closer to 1989 than the first we can imagine "starting" at the location $(45,0)$ and stepping "backwards" $2025-1989 = 36$ times. One step backwards brings us to $(44,0)$ and we need to step backwards 35 more times. This involves moving upwards 35 units to place us as $(44,35)$.

Problem 1990-14

This is worked on Page 742.

Problem 1991-28

This is worked on Page 770.

Another slightly different way to solve this problem is the following. If b and w are the number of black and white marbles in the urn at a given time each of the given transformations we take the "state" of $\begin{bmatrix} b \\ w \end{bmatrix}$ and transform it into another state. For the first transformation we have

$$T_1 \begin{bmatrix} b \\ w \end{bmatrix} = \begin{bmatrix} (b-3)+1 \\ w \end{bmatrix} = \begin{bmatrix} b-2 \\ w \end{bmatrix}\,,$$

and we must have at least three black marbles to apply this transformation. For the second transformation we have

$$T_2 \begin{bmatrix} b \\ w \end{bmatrix} = \begin{bmatrix} (b-2)+1 \\ (w-1)+1 \end{bmatrix} = \begin{bmatrix} b-1 \\ w \end{bmatrix}\,,$$

and we must have at least two black marbles and one white marble to apply this transformation. For the third transformation we have

$$T_3 \begin{bmatrix} b \\ w \end{bmatrix} = \begin{bmatrix} b-1 \\ (w-2)+2 \end{bmatrix} = \begin{bmatrix} b-1 \\ w \end{bmatrix},$$

and we must have at least one black marble and two white marbles to apply this transformation. Finally for the fourth transformation we have

$$T_4 \begin{bmatrix} b \\ w \end{bmatrix} = \begin{bmatrix} b+1 \\ (w-3)+1 \end{bmatrix} = \begin{bmatrix} b+1 \\ w-2 \end{bmatrix},$$

and we must have at least three white marbles to apply this transformation.

If we look at the net result of each of the above we see that the number of white marbles does not change or decreases by two. This means that the number of white marbles at any time in this process must be even. This means that choices (D) and (E) cannot be true.

Now we can reduce the number of white marbles by two (leaving the number of black marbles unchanged) by using the composition of $T_4 \cdot T_2$ or $T_4 \cdot T_3$. For example $T_4 \cdot T_2$ gives

$$T_4 \cdot T_2 \begin{bmatrix} b \\ w \end{bmatrix} = T_4 \begin{bmatrix} b-1 \\ w \end{bmatrix} = \begin{bmatrix} b \\ w-2 \end{bmatrix}.$$

To apply T_2 we must have at least two black and one white marble and to apply T_4 we must have at least three white marbles. This means that we *cannot* apply the above on states where $w < 3$ and as w must be even this means that we cannot reduce w below two and thus $w \geq 2$.

This means that choices (A) and (C) are not possible so the answer must be (B).

Problem 1992-14

This is worked on Page 777.

Problem 1993-22

This is worked on Page 800.

Problem 1994-6

This is worked on Page 808.

Problem 1995-30

This is worked on Page 830.

Problem 1996-27

This is worked on Page 845.

Problem 1998-22

This is worked on Page 868.

Problem 1999-18

This is worked on Page 878.

The Contest Problem Book VII: Additional Problems

Dinner Bill Splitting

We seek to find (s, t) such that

$$2(100t + s) - (100s + t) = \pm 1 \,.$$

Here the expression $100t + s$ is "$\frac{1}{2}$ the bill" and the expression $100s + t$ is "the bill" both measured in cents. The above is equivalent to the expression

$$|199t - 98s| = 1 \,. \tag{646}$$

Now as discussed in the text the solution to the problem: minimize g in

$$|199t - 98s| = g \,,$$

is $g = \text{GCD}(199, 98)$ and t and s can be found from the Euclidean algorithm. The steps of this algorithm for the numbers given here are

$$199 = 2 \cdot 98 + 3 \tag{647}$$
$$98 = 32 \cdot 3 + 2 \tag{648}$$
$$3 = 1 \cdot 2 + 1 \,. \tag{649}$$

q	$p+r=14-q$	Possible (p,r) with p odd and $p\neq r$
1	13	$\{(1,12),(11,2),(3,10),(9,4),(5,8),(7,6)\}$
3	11	$\{(1,10),(9,2),(3,8),(7,4),(5,6)\}$
5	9	$\{(1,8),(7,2),(3,6),(5,4)\}$
7	7	$\{(1,6),(5,2),(3,4)\}$
9	5	$\{(1,4),(3,2)\}$
11	3	$\{(1,2)\}$

Table 20: Values for q and possible (p,r) values. This table is constructed after Q learns that p must be odd. Before that information, for each (p,r) in the right-hand column we can have the pair (r,p). Its only when $q\in\{7,9,11\}$ that Q can conclude that all p, q, and r are different.

Now we start with Equation 649 written as

$$1=1\cdot 3-1\cdot 2\,.$$

Then we replace the two in that equation with the remainder from Equation 648 to get

$$1=1\cdot 3-1\cdot(98-32\cdot 3)=33\cdot 3-1\cdot 98\,.$$

Then we replace the three in that equation with the remainder from Equation 647 to get

$$1=33\cdot(199-2\cdot 98)-1\cdot 98=33\cdot 199-67\cdot 98\,.$$

Comparing this to Equation 646 we see that $t=33$ and $s=67$. Thus $\frac{1}{2}$ of the bill was \$33.67. The full original bill was \$67.33. We can check that

$$2(33.67)-67.33=0.01\,.$$

Thirty Digits

Let our number be denoted N. Since there are a total of 30 digits in N in order that *no* digit repeat at least four times would mean that each digit must appear exactly three times. This means that the sum of the digits in N is given by

$$3(0+1+2+\cdots 8+9)=3\left(\frac{9(10)}{2}\right)=135\,.$$

Note that this number is divisible by nine and thus our original number must be divisible by nine. As this means that 3^2 needs to be a factor of N. From the given prime factorization of N we see that this cannot be true and thus at least one digit of N must repeat four or more times.

The Whispered Number Problem

Version A: After P's statement we note that if p were an odd number then $q+r=14-p$ is odd and q and r cannot be equal or else they wouldn't be integers. Thus we conclude that p must be an odd number.

p	q	r
1	7	6
5	7	2
3	7	4
1	9	4
3	9	2
1	11	2

Table 21: Values of p, q, and r for The Whispered Number Problem Version a.

From Q's statement for P and R to have different numbers q must be odd. If we take $q \in \{1, 3, 5, 7, 9, 11\}$, compute $p + r = 14 - q$, and consider the possible (p, r) pairs that sum to $14 - 1$ we get Table 20. Note that for cases where q is *small* we can construct cases where either p or r equals q and for those values Q could not make the statement that he did. This is not true if $q \in \{7, 9, 11\}$. If we table these valid (p, q, r) numbers we get Table 21. The only row in that table that gives a unique solution is $(p, q, r) = (1, 7, 6)$ which has a product of 42.

Version B: As before, the first statement tells us that p must be an odd number.

Now before information from P is given we could imagine Q making a table with his q value, the value of $14 - q = p + r$, and possible (p, r) that sum to $14 - q$. Once Q realizes that p must be odd many of the possible (p, r) are removed and Q can look at the possible (p, q, r) tuples and concludes that all p, q, and r are different. Since we don't know the value of q we must do this for a range of q values. A table of this form is given in Table 22.

Note that if q is *odd* then $14 - q$ will be odd and only decompose into the sum of two odd (and non equal) numbers. This means that for some odd q values Q would *already* (before P statement) know that the three numbers p, q, and r were different. The values of q for which this is true are $q \in \{7, 9, 11\}$. In the "Any problem" column of Table 22 I denote these rows as "EB" for "excluded before" P makes his statement.

Thus in looking at Table 22 only some rows are possible. In the "Any problem" column of that table I denote rows that would not be consistent with the statement that all three variables are not equal. This only gives

$$q \in \{2, 6, 10\},$$

as possible valid choices. If we now consider the possible different r values (knowing that r must be odd) and count the number of valid Q tuples in Table 22 we get Table 23. Notice in that case that if $(p, q, r) = (1, 2, 11)$ then R would have known that tuple (without the statement from Q) after the statement from P. Thus $(p, q, r) = (3, 2, 9)$ and we have $pqr = 54$.

Possible q	$14-q=p+r$	Possible (p,r) with p odd	Any problems
1	13	$\{(1,12),(3,10),(5,8),(7,6),(9,4),(11,2)\}$	$p=q=1$
2	12	$\{(1,11),(3,9),(5,7),(7,5),(9,3),(11,1)\}$	
3	11	$\{(1,10),(3,8),(5,6),(7,4),(9,2)\}$	$p=q=3$
4	10	$\{(1,9),(3,7),(5,5),(7,3),(9,1)\}$	$p=r=5$
5	9	$\{(1,8),(3,6),(5,4),(7,2)\}$	$q=p=5$
6	8	$\{(1,7),(3,5),(5,3),(7,1)\}$	
7	7	$\{(1,6),(3,4),(5,2)\}$	EB
8	6	$\{(1,5),(3,3)\}$	$p=r=3$
9	5	$\{(1,4),(3,2)\}$	EB
10	4	$\{(1,3),(3,1)\}$	
11	3	$\{(1,2)\}$	EB
12	2	$\{(1,1)\}$	$p=r=1$

Table 22: Values of p, q, and r for The Whispered Number Problem Version b.

Possible r value	Number of valid Q tuples	Only (p,q,r) Tuple
1	3	NA
3	3	NA
5	2	NA
7	2	NA
9	1	(3, 2, 9)
11	1	(1, 2, 11)

Table 23: For various r values how many valid Q tuples are there with this value of r.

The Staircase Problem

Piles of Stones

Flipping Pennies

Back to Front

Face Painting

High Slopes

An Odd End

Prime Leaps

Multiple Quotients

Unsquare Party

Double or Add Seven

From (a) and (b) we see that powers of two are in S. The ones less than 2004 are

$$\{2, 4, 8, 16, 32, 64, 128, 256, 512, 1024\} .$$

From (c) some odd numbers are in S. For various "starts" in the above set condition (c) imply

$$\begin{aligned} n &\equiv 2 \pmod 7 \\ n &\equiv 4 \pmod 7 \\ n &\equiv 1 \pmod 7 , \end{aligned}$$

and the above pattern seems to repeat from that point onward. This means that the numbers

$$\begin{aligned} n &\equiv 0 \pmod 7 \\ n &\equiv 3 \pmod 7 \\ n &\equiv 5 \pmod 7 \\ n &\equiv 6 \pmod 7 , \end{aligned}$$

are *not* in S. These are numbers of the form

$$\{7k\,, 7k+3\,, 7k+5\,, 7k+6\} ,$$

for some k. The first number of this form that is *larger* than 2004 are

```
[1] 2009 2005 2007 2008
```

These can be calculated with the following simple `R` code

```
7*(floor(c(2004/7, (2004-3)/7, (2004-5)/7, (2004-6)/7))+1) + c(0, 3, 5, 6)
```

The smallest of these is 2005. **Note:** this does not match the answer in the back of the book which I think is in error. If anyone sees anything wrong with my arguments above please contact me.

Double or Subtract Twelve

Counting Transitive Relations

The 2001 AMC 10 Examination

Problem 1

The numbers are written in increasing order and as there are nine of them the median is the fifth element. Thus we are told that $n + 6 = 10$ so that $n = 4$.

The mean of these numbers is given by

$$\frac{9n + (3 + 4 + 5 + 6 + 8 + 10 + 12 + 15)}{9} = n + 7 = 11\,.$$

Problem 2

This is the given equation

$$x = 2 + \frac{1}{x}(-x)\,.$$

The right-hand-side simplifies and we get $x = 1$ which is in interval (C).

Problem 3

Let the two numbers be a and b. Then we are told that $S = a + b$. If we perform the given transforms we want to evaluate

$$2(a + 3) + 2(b + 3) = 2(a + b) + 6 + 6 = 2S + 12\,.$$

Problem 4

A circle can be drawn "cutting" off all three corners of the triangle. This would give six intersections.

Problem 5

I count six.

Problem 6

Let the two digit number N be written as $10a + b$ where $1 \leq a \leq 9$ and $0 \leq b \leq 9$. Here b is the units digit. Then we are told that

$$P(N) = ab$$
$$S(N) = a + b\,.$$

Then as we were told $P(N) + S(N) = N$ or

$$ab + a + b = 10a + b\,.$$

This simplifies to

$$a(b - 9) = 0\,.$$

We can't have $a = 0$ so we must have $b = 9$.

Problem 7

If x is the unknown number then the given relationship can be described as

$$x \times 10^4 = \frac{4}{x}\,.$$

Solving this for x gives $x = \frac{2}{10^2} = \frac{2}{100} = 0.02$.

Problem 8

If we denote the days from today with the variable n (then $n = 0$ is today, $n = 1$ is tomorrow etc) then from the problem statement we have that

- D works when $n \equiv 0 \pmod 3$.
- W works when $n \equiv 0 \pmod 4$.
- B works when $n \equiv 0 \pmod 6$.
- C works when $n \equiv 0 \pmod 7$.

The next day when all are working will be the least common multiple of these four numbers $\{3, 4, 6, 7\}$. Factoring them we have

$$\{3\,, 2^2\,, 2 \cdot 3\,, 7\}$$

and so the least common multiple is $2^2 \times 3^1 \times 5^0 \times 7^1 = 84$. Thus they are all working together again 84 days from today.

Problem 9

Let I be Kristin's annual income. Then the given relationship can be written as

$$28000\left(\frac{p}{100}\right)+\left(\frac{p+2}{100}\right)(I-28000)=\left(\frac{p+0.25}{100}\right)I\,.$$

We can expand and simplify this. In doing so all terms with p "drop out" and we can solve for I. We find $I = 32000$.

Problem 10

These equations are

$$xy = 24 \tag{650}$$
$$xz = 48 \tag{651}$$
$$yz = 72\,. \tag{652}$$

Using Equation 650 we have

$$x=\frac{24}{y}\,.$$

Solving for y in Equation 652 and putting that into the above we get

$$x=\frac{24}{72}z=\frac{z}{3}\,.$$

Solving for z in Equation 651 and putting that into the above we get

$$x=\frac{1}{3}\left(\frac{48}{x}\right)=\frac{16}{x}\,.$$

Solving this for x gives $x = \pm 4$ where the only positive solution is $x = 4$. Using this in Equation 650 we have $y = 6$. Using Equation 651 we have $z = 12$. With these we have

$$x+y+z=4+6+12=22\,.$$

Problem 11

Using the diagram we can compute the number of squares in the rings $r = 1$ and $r = 2$ by subtracting the area of a smaller square from a larger square as

$$N_1 = 3\times 3-1\times 1=9-1=8$$
$$N_2 = 5\times 5-3\times 3=25-9=16\,.$$

Let S_r be the squares side length of ring $r+1$ for $r \geq 0$. Then

$$
\begin{aligned}
S_1 &= 3 \\
S_2 &= S_1 + 2 = 5 \\
&\vdots \\
S_r &= S_{r-1} + 2\,.
\end{aligned}
$$

Thus solving the above difference equation looks like $S_r = 2r+1$. Then we have

$$N_r = S_r^2 - S_{r-1}^2 = (2r+1)^2 - (2r-1)^2 = 8r\,,$$

when we simplify. Lets check a few of these using this formula

$$
\begin{aligned}
N_1 &= 8 \\
N_2 &= 16 \\
&\vdots \\
N_{100} &= 800\,.
\end{aligned}
$$

Problem 12

We are told that $n \equiv (m-1)m(m+1)$ for $m \geq 2$. From this we see that

- If m is even then two is a factor of n.
- If m is odd then four is a factor of n.
- With three consecutive integers at least one of them has three as a factor.

We are also told that seven is a factor of n. These also mean that factors of n are

$$2 \cdot 3 = 6\,, 2 \cdot 7 = 14\,, 3 \cdot 7 = 21\,, 2 \cdot 3 \cdot 7 = 42\,,$$

are all factors of n. From the choices given n is not divisible by $2 \cdot 2 \cdot 7 = 28$.

To find a number n such that n is not divisible by 28 lets try $m = 6$. Then $n = 5 \cdot 6 \cdot 7 = 210$ which is not divisible by 28.

Problem 13

From the given problem the D, E, F digits can be

$$
\begin{aligned}
(D, E, F) &= (8, 6, 4) \\
(D, E, F) &= (6, 4, 2) \\
(D, E, F) &= (4, 2, 0)\,.
\end{aligned}
$$

The G, H, I, and J digits can be

$$(G, H, I, J) = (9, 7, 5, 3)$$
$$(G, H, I, J) = (7, 5, 3, 1).$$

If we build up the number $ABC - DEF - GHIJ$ using the choices based on the above we see that the digits (A, B, C) can be

$$ABC - 864 - 9753 \Rightarrow (A, B, C) = (2, 1, 0)$$
$$ABC - 864 - 7531 \Rightarrow (A, B, C) = (9, 2, 0)$$
$$ABC - 642 - 9753 \Rightarrow (A, B, C) = (8, 1, 0)$$
$$ABC - 642 - 7531 \Rightarrow (A, B, C) = (9, 8, 0)$$
$$ABC - 420 - 9753 \Rightarrow (A, B, C) = (8, 6, 1)$$
$$ABC - 420 - 7531 \Rightarrow (A, B, C) = (9, 8, 6).$$

From these choices to have $A + B + C = 9$ we must have $(A, B, C) = (8, 1, 0)$.

Problem 14

Let w be the number of whole price tickets, p the number of partial price tickets, and F the full price ticket value. Then from the problem statement we have that

$$w + p = 140 \tag{653}$$
$$wF + p\left(\frac{F}{2}\right) = 2001. \tag{654}$$

From Equation 653 we conclude that $0 \le w \le 140$ and $0 \le p \le 140$ and that $p = 140 - w$. Putting this latter expression into Equation 654 to get

$$wF + (140 - w)\frac{F}{2} = 2001.$$

We can manipulate the above into

$$(w + 140)F = 4002.$$

As everything is an integer this means that $w + 140$ must be a factor of 4002. We can factor 4002 as $4002 = 2 \cdot 3 \cdot 23 \cdot 29$. Using $0 \le w \le 140$ we have

$$140 \le w + 140 \le 280.$$

Now

$$2 \cdot 3 \cdot 23 = 138 < 140$$
$$2 \cdot 3 \cdot 29 = 174 > 140.$$

Using this second expression we have $w + 140 = 174$ so $w = 34$ and

$$F = \frac{4002}{w + 140} = 23.$$

The money raised by the full price ticket sales is $wF = 34(23) = 782$.

Problem 15

Lets draw two horizontal lines (representing the curbs) and place the "bottom" line on the x-axis of an x-y Cartesian plane. Let the "top" horizontal line through $H = (x, y) = (0, 40)$.

Let one side of the crosswalk go from $A = (0, 0)$ "diagonally" from A to $B = (b, 40)$ and the other side of the crosswalk go from $D = (15, 0)$ to $C = (c, 40)$. We are told that the distance from A to B is 50. Draw a perpendicular from D to the segment AB and denote that intersection as the point E. We want to know the distance DE. From the length of two sides of the right triangle $\triangle AHB$ we have $HB = 30$ (so $b = 30$). From the parallel lines HBC and AD we have

$$\angle HBA = \angle BAD \,.$$

Using this we know that $\triangle HBA \sim \triangle EAD$ so

$$\frac{AD}{ED} = \frac{AB}{AH} \quad \text{or} \quad \frac{15}{ED} = \frac{50}{40} \,.$$

This means that $ED = \frac{4}{5}(15) = 12$.

Problem 16

Let these three numbers be x, y, and z such that $x < y < z$. Then we are told that

$$\frac{1}{3}(x + y + z) - 10 = x$$
$$\frac{1}{3}(x + y + z) + 15 = z$$
$$\text{median}(x, y, z) = y = 5 \,.$$

Lets put this value of y into the two equations above and we get

$$2x - z = -25$$
$$x - 2z = -50 \,.$$

Solving these two equations gives $x = 0$ and $z = 25$ and the three numbers are $(x, y, z) = (0, 5, 25)$. The sum of these is 30.

Problem 17

The circumference of the base circle in the cone would be cut from a length of

$$\frac{252}{360}(2\pi(10)) = 14\pi \,.$$

This means that the radius of the base circle must be given by

$$2\pi r = 14\pi \quad \text{so} \quad r = 7 \,.$$

The arc length of the code is the same as the radius of the large circle. Thus the cone of interest is (C).

Problem 18

The total area of this figure is $9 \times 9 = 81$. Each smaller pentagon is composed of one small square and $\frac{1}{4}$ of a square "above" the base square for an area of

$$1 + \frac{1}{4} = \frac{5}{4}.$$

If we count the number of pentagons I find $4 \times 9 = 36$ and thus the total area of the pentagons is

$$36 \times \left(\frac{5}{4}\right) = 45.$$

As a percentage of the total this is

$$\frac{45}{9^2} = \frac{5}{9}.$$

As $\frac{1}{9} = 0.111111$ and so $\frac{5}{9} = 0.555556$ which is 55.5556%.

Problem 19

Let g, c, and p be the number of glazed, chocolate, and powdered donuts respectively then we want to know the number of solutions to

$$g + c + p = 4,$$

where $g \geq 0$, $c \geq 0$, and $p \geq 0$. We are trying to count the number of nonnegative integer solutions to this equation. In [4] this is discussed where it is found that this number is

$$\left(\begin{array}{c} 4+3-1 \\ 3-1 \end{array}\right) = \left(\begin{array}{c} 6 \\ 2 \end{array}\right) = 15.$$

Problem 20

We are told that square has a side length of $s = 2000$. In the isosceles right triangles cut from the corners of the square I let the hypotenuse be denoted by h and the lengths of the legs be denoted l. Then as the octagon is regular h is the length of each of the octagons sides and since the side of the square is made up of two legs and one side we must have

$$2l + h = s = 2000.$$

From the fact that the corner triangles are right isosceles triangles we have

$$2l^2 = h^2 \quad \text{so} \quad l = \frac{h}{\sqrt{2}}.$$

Putting that in the first of the above equations gives

$$\frac{2h}{\sqrt{2}} + h = 2000 \quad \text{so} \quad h = 2000(\sqrt{2} - 1).$$

Problem 21

Let the radius of the right circular cylinder be denoted r and the radius of the right circular cone be denote $R = \frac{10}{2} = 5$. Draw an x-y Cartesian coordinate axis with the y-axis though the axis of the cone. Then the vertex of the cone is located at $(0, 12)$ and a point on the base will be at $(R, 0) = (5, 0)$. The slant height of the cone must be on a line that goes through the two points $(5, 0)$ and $(0, 12)$. This line is given by

$$y = -\frac{12}{5}(x - 5)\,.$$

Evaluating this line at the point $x = r$ will produce a point on the right circular cylinder which has a height of $h = 2r$ (its height is equal to its diameter). Thus

$$-\frac{12}{5}(r - 5) = 2r\,.$$

Solving this for r we find $r = \frac{30}{11}$.

Problem 22

If we let C be the common sum then from the variables given in this magic square we have

$$\begin{aligned} v + 24 + w &= C \\ 18 + x + y &= C \\ 25 + z + 21 &= C \\ v + 18 + 25 &= C \\ 24 + x + z &= C \\ w + y + 21 &= C \\ v + x + 21 &= C \\ w + x + 25 &= C\,. \end{aligned}$$

If we organize and simplify these some we can write them as

$$v + w + 24 = C \tag{655}$$
$$x + y + 18 = C \tag{656}$$
$$z + 46 = C \tag{657}$$
$$v + 43 = C \tag{658}$$
$$x + z + 24 = C \tag{659}$$
$$w + y + 21 = C \tag{660}$$
$$v + x + 21 = C \tag{661}$$
$$w + x + 25 = C\,. \tag{662}$$

Now from Equations 655 and 658 we get

$$v + w + 24 = v + 43 \quad \text{so} \quad v = 19\,.$$

Now from Equations 657 and 659 we get

$$z + 46 = x + z + 24 \quad \text{so} \quad x = 22\,.$$

Using the known values for v and z we can write the above set of equations as

$$\begin{aligned} v + 43 &= C \\ 40 + y &= C \\ 46 + z &= C \\ v + 43 &= C \\ 46 + z &= C \\ 40 + y &= C \\ v + 43 &= C \\ 25 + 22 + 19 &= C\,. \end{aligned}$$

Some of these equations are redundant (i.e. duplicated). This last equation tells us that $C = 66$ and then allows us to use the previous equations to compute

$$\begin{aligned} v &= 66 - 43 = 23 \\ y &= 66 - 40 = 26 \\ z &= 66 - 46 = 20\,. \end{aligned}$$

From these we see that $y + z = 46$.

Problem 23

As this is a "small" probability problem a first attempt should always be made by enumeration using an **outcome tree**. Specifically for this problem if we let (R, W) be the number of red/white chips and let the branches of an outcome tree denote the color of the chip drawn we have the following

```
Start_3_2/
|-- R_2_2
|   |-- R_1_2
|   |   |-- W_0_2
|   |   --- W_1_1
|   |       |-- R_0_1
|   |       --- W_1_0
|   --- W_2_1
|       |-- R_1_1
|       |   |-- R_0_1
|       |   --- W_1_0
|       --- W_2_0
--- W_3_1
    |-- R_2_1
    |   |-- R_1_1
    |   |   |-- R_0_1
    |   |   --- R_1_0
    |   --- W_2_0
    --- W_3_0
```

This diagram indicates we start at a state $(R, W) = (3, 2)$ and then can transition by drawing either a red R or a white W chip. After each transition we have the number of red and white chips shown as the two following numbers. As there are 10 outcomes (leaf nodes with a zero) and there are six of these that have the last chip drawn as white we have a probability of

$$\frac{6}{10} = \frac{3}{5}.$$

Problem 24

I drew this trapezoid with the base DC along the x-axis of an x-y Cartesian coordinate system and the above base AD "above" DC. This would place $D = (0, 0)$ and $C = (d, 0)$ where d is the distance DC. Since we are told that AB and CD are perpendicular to AD and $AD = 7$ we have $A = (0, 7)$ and I took $B = (b, 7)$ where b is the distance AB.

Dropping a perpendicular from B to the segment DC (call that point B') we have the right triangle $\triangle BB'C$. Let $B'C = x$ and we have

$$x^2 + 7^2 = BC^2 = (b + d)^2.$$

As $CD = d = b + x$ we have $x = d - b$. Putting that in the above gives

$$(d - b)^2 + 7^2 = (b + d)^2.$$

Expanding this and simplifying we can solve for bd (everything else cancels) and find

$$bd = \frac{49}{4} = 12.25.$$

The 2001 AMC 12 Examination

Problem 1

If our two numbers are a and b the first statement we are given is that

$$a + b = S\,.$$

We are then told to compute

$$2(a+3) + 2(b+3) = 2(a+b) + 12 = 2S + 12\,.$$

Problem 2

WWX: Working here.

The 2002 AMC 10A Examination

Problem 1

WWX: DP

Problem 2

For this we find

$$(2, 12, 9) = \frac{2}{12} + \frac{12}{9} + \frac{9}{2} = \frac{1}{6} + \frac{4}{3} + \frac{9}{2} = \frac{1}{6} + \frac{27}{6} + \frac{8}{6} = \frac{36}{6} = 6\,.$$

Problem 3

WWX: working here

Problem 4

WWX: working here

Problem 5

WWX: working here

Problem 6

Let x be Cindy's number. Then what she did was

$$\frac{x-9}{3} = 43 \quad \text{so} \quad x = 138\,.$$

What she should have done was

$$\frac{x-3}{9} = \frac{135}{9} = 15\,.$$

Problem 7

WWX: DP

Problem 8

WWX: DP

Problem 9

Write these equations as (there are other ways to "see" this)

$$\begin{aligned}(1000+1)C-(2000+2)A &= 4000+4\\(1000+1)B+(3000+3)A &= 5000+5\,.\end{aligned}$$

Then divide both equations by $1000+1$ to get

$$\begin{aligned}C-2A &= 4\\B+3A &= 5\,.\end{aligned}$$

If we add these two equations together we get

$$A+B+C=9\,,$$

so the average of these three numbers is $\frac{9}{3}=3$.

Problem 10

Factor to write this as

$$(2x+3)(x-4+x-6)=(2x+3)(2x-10)=0\,.$$

The the roots solve $2x+3=0$ or $2x-10=0$ so $x=-\frac{3}{2}$ and $x=5$. The sum of these roots is then $\frac{7}{2}$.

Problem 11

WWX: DP

Problem 12

Let the amount of time needed to travel and get to work "on time" be T (in hours). Let the distance traveled be D (in miles). Then from what we are told we have

$$\frac{D}{40} = T + \frac{3}{60} \quad (663)$$
$$\frac{D}{60} = T - \frac{3}{60}\,. \quad (664)$$

We want to know V where V is such that $\frac{D}{V} = T$ i.e. we arrive at work "on time". This is $V = \frac{D}{T}$. Solving Equation 663 for T and putting that into Equation 664 we get $D = 12$ (miles). Putting that value into Equation 664 we get $T = \frac{1}{4}$ (an hour). This means that $V = \frac{D}{T} = \frac{12}{1/4} = 48$ (miles per hour).

Problem 13

Notice that the given sides are all a multiple of five of a $(3, 4, 5)$ right triangle. This means that our triangle is right and two of the altitudes are the lengths of the legs or of length 15 and 20. The third altitude is the one to the hypotenuse. Lets denote that length as h. As the full area of this triangle is given by

$$\frac{1}{2}(15)(20) = 150\,.$$

If we compute this area using the altitude h we get

$$150 = \frac{1}{2}h(25) \quad \text{so} \quad h = 12\,.$$

Thus the smallest altitude is $h = 12$.

Problem 14

Let the two roots of this quadratic be p and q. Without loss of generality let $p < q$. Then by Vieta's formula

`https://en.wikipedia.org/wiki/Vieta's_formulas`

we have $pq = k$ and $p + q = 63$. If p and q are odd numbers then $p + q$ will be even. Thus the only possible choices for p and q to sum to an odd number is to have one of them p be even. Thus $p = 2$ and $q = 61$. Thus there is one possible value for k of $k = 2(61) = 122$.

Problem 15

To form two digit prime numbers using these digits we cannot have the units digit be a two, a four, or a six (else the number is even and not prime), or a five (else the number is disable by five). Thus we have the four "template" prime numbers

$$2X\,, 4Y\,, 6Z\,, 5W\,,$$

where X, Y, Z, and W are digits drawn from $\{1, 3, 7, 9\}$. We could now assign these remaining digits into the spots for X, Y, Z, and W to form four prime numbers but we don't need to. As we are asked for the sum of the four primes. We would have

$$2X + 4Y + 6Z + 5W = 20 + 40 + 60 + 50 + (X + Y + Z + W) = 170 + (1 + 3 + 7 + 9) = 190\,.$$

Problem 16

Define S as $S = a + b + c + d$. Then if we solve for a, b, c, and d in the given equations we have

$$\begin{aligned} a &= S + 4 \\ b &= S + 3 \\ c &= S + 2 \\ d &= S + 1\,. \end{aligned}$$

If we add these together we get

$$S = 4S + 4 + 3 + 2 + 1 = 4S + 10\,.$$

Solving for S we find $S = -\frac{10}{3}$.

Problem 17

After the first pour occurs we have

- 2 ounces coffee and 0 ounces cream in the first cup and
- 2 ounces coffee and 4 ounces cream with in the second cup.

For the second cup this is $\frac{2}{6}$ coffee and $\frac{4}{6}$ cream. After the transfer of three ounces ($\frac{4+2}{2} = 3$) from the second cup to the first cup we will have

$$\begin{aligned} 2 + 3\left(\frac{2}{6}\right) &= 3 \quad \text{ounces coffee} \\ 0 + 3\left(\frac{4}{6}\right) &= 2 \quad \text{ounces cream} \end{aligned}$$

The fraction of the liquid in the first cup that is now cream is

$$\frac{2}{2+3} = \frac{2}{5} \,.$$

Problem 18

We want the "smallest" sides of the die to face outwards. This means that in the corners we will place the one, two, and three dots outwards, on an edge we will place the one and two dots forward and on a face we will place the one dot forward. If we start at a top corner and place a die with a three "up" and a two "forward" we will leave a one on the blank face. We can position die in similar ways on each corner of the "top" face. We would then place a one "upwards" in the center location. This gives a sum of $4(3) + 4(2) + 1 = 21$. By symmetry we can position die in the same way on the "bottom" of the cube. With these placed we can place die such that two faces have all ones (by symmetry again) and one face has six twos with vertical strip of ones (and another face the same by symmetry again). This gives a total of

$$2(21) + 2(9) + 2(6(2) + 3(1)) = 90 \,.$$

Problem 19

Draw our regular hexagon with one corner at the origin of an x-y Cartesian coordinate system. Denote that corner as O. Label the vertexes of the regular hexagon from O as A, B, C, D, and E walking counterclockwise. From O with a rope of length two Spot can reach the semicircle of radius two that is "to the left" of the segment OE. Call the area of that region A_I. He can also reach a section of a circle of radius two from O and "below" the segment OA. Call the area of that region A_{II}. He can also reach sections of a circle of radius one that are

- "to the left of" the segment ED (call this area A_{III})
- "above" the segment OA and "below" the segment AB (call this area A_{IV})

Note that $A_{III} = A_{IV}$. We will now compute the different areas above. We find

$$A_I = \frac{1}{2}\left(\pi 2^2\right) = 2\pi \,.$$

Now as the interior angle of a regular hexagon is given by $\frac{180(n-2)}{n} = 120^\circ$ when $n = 6$ we have that the sector cut by A_{II} has an area of

$$A_{II} = \left(\frac{60}{360}\right) \pi 2^2 = \frac{2\pi}{3} \,,$$

since in this sector the radius is $r = 2$.

In the same way the areas A_{III} and A_{IV} are given by

$$A_{III} = A_{IV} = \left(\frac{60}{360}\right)\pi 1^2 = \frac{\pi}{6}\,,$$

since in these two sectors the radius is $r = 2$. The area we want is then given by

$$A_I + A_{II} + A_{III} + A_{IV} = 3\pi\,,$$

when we simplify.

Problem 20

In the triangle $\triangle GAD$ the segment HC is parallel to AG and thus divides the triangle into two similar triangles such that

$$\frac{GA}{HC} = \frac{AD}{CD} = 3\,. \tag{665}$$

In a similar way we have $\triangle AFG \sim \triangle EFJ$ so that

$$\frac{GA}{JE} = \frac{AF}{EF} = \frac{5}{1} = 5\,. \tag{666}$$

If we take Equation 666 and divide by Equation 665 we get

$$\frac{GA}{JE} \times \frac{HC}{GA} = \frac{HC}{JE} = \frac{5}{3}\,.$$

Problem 22

Recalling that

$$\begin{aligned}
1^2 &= 1\\
2^2 &= 4\\
3^2 &= 9\\
4^2 &= 16\\
5^2 &= 25\\
6^2 &= 36\\
7^2 &= 49\\
8^2 &= 64\\
9^2 &= 81\\
10^2 &= 100\,.
\end{aligned}$$

We note that in the first pass we will drop 10 numbers and on renumbering will have the numbers $1 - 90$. We can answer this problem by performing these operations until we end with a single tile. We have

- On iteration 2, we drop nine tiles and renumber to get tiles numbered $1 - 81$.
- On iteration 3, we drop nine tiles and renumber to get tiles numbered $1 - 72$.
- On iteration 4, we drop eight tiles and renumber to get tiles numbered $1 - 64$.
- On iteration 5, we drop eight tiles and renumber to get tiles numbered $1 - 56$.
- On iteration 6, we drop seven tiles and renumber to get tiles numbered $1 - 49$.
- On iteration 7, we drop seven tiles and renumber to get tiles numbered $1 - 42$.
- On iteration 8, we drop six tiles and renumber to get tiles numbered $1 - 36$.
- On iteration 9, we drop six tiles and renumber to get tiles numbered $1 - 30$.
- On iteration 10, we drop five tiles and renumber to get tiles numbered $1 - 25$.
- On iteration 11, we drop five tiles and renumber to get tiles numbered $1 - 20$.
- On iteration 12, we drop four tiles and renumber to get tiles numbered $1 - 16$.
- On iteration 13, we drop four tiles and renumber to get tiles numbered $1 - 12$.
- On iteration 14, we drop three tiles and renumber to get tiles numbered $1 - 9$.
- On iteration 15, we drop three tiles and renumber to get tiles numbered $1 - 6$.
- On iteration 16, we drop two tiles and renumber to get tiles numbered $1 - 4$.
- On iteration 17, we drop two tiles and renumber to get tiles numbered $1 - 2$.
- On iteration 18, we drop one tile and renumber to get a single tile numbered 1.

Thus it will take 18 iterations.

Problem 23

The perimeter of triangle $\triangle BEC$ is

$$2(10) + 12 = 32\,.$$

Now we are told the perimeter of triangle $\triangle AED$ is twice this or

$$AE + ED + 2AB + 12 = 2(32) = 64\,.$$

As $AE = ED$ this is

$$2AE + 2AB + 12 = 64 \quad AE + AB = 26\,. \tag{667}$$

	$T_2 = 1$	$T_2 = 2$	$T_2 = 3$	$T_2 = 4$	$T_2 = 5$
$T_1 = 1$	X	3	4	5	6
$T_1 = 2$		X	5	6	7
$T_1 = 3$			X	7	8
$T_1 = 4$				X	9
$T_1 = 5$					X

Table 24: Possible sums for the two draws T_1 and T_2 from the set $\{1, 2, 3, 4, 5\}$.

We would like to determine AE in terms of AB. Lets drop a perpendicular from E that intersects the segment AD at a point denoted E'. In the right triangle $\triangle EE'B$ one leg has a length of

$$BE' = \frac{12}{2} = 6\,,$$

and the hypotenuse has a length of 10. Notice that the right triangle $\triangle BE'E$ is a "three-four-five" right triangle and so the altitude has a length of 8.

Now in right triangle $\triangle EE'A$ the Pythagorean theorem gives

$$AE^2 = (AB + 6)^2 + 8^2\,,$$

which we can simplify to

$$AE = \sqrt{AB^2 + 12AB + 100}\,.$$

Lets put this into Equation 667 as $AE = 26 - AB$ or $AE^2 = (26 - AB)^2$ to get

$$AB^2 + 12AB + 100 = 26^2 - 52AB + AB^2\,.$$

Solving this for AB gives $AB = 9$.

Problem 24

Let the first of Tina's two numbers be denoted by T_1 and the second by T_2. Let Sergio's number be denoted as S. Then by conditioning on S we can compute the probability of what we want as

$$P(S > T_1 + T_2) = P(T_1 + T_2 < S) = \sum_{i=1}^{10} P(T_1 + T_2 < i)P(S = i) = \frac{1}{10}\sum_{i=1}^{10} P(T_1 + T_2 < i)\,.$$

Lets compute $P(T_1 + T_2 < i)$ for each of the i possible values Sergio can draw. Using Table 24 we compute the sum of T_1 and T_2 for various possible values for T_1 and T_2.

Once we have this we can compute the probability of $P(T_1 + T_1 < i)$ for various values for i. This is done in Table 25.

i	$P(T_1 + T_2 < i)$
1	0
2	0
3	0
4	1/10
5	2/10
6	4/10
7	6/10
8	8/10
9	9/10
10	10/10

Table 25: The value of $P(T_1 + T_2 < i)$ for various values of i.

Once we have this we see that the probability we are interested in is given by

$$\begin{aligned} P(S > T_1 + T_2) &= \frac{1}{10}\left(0 + 0 + 0 + \frac{1}{10} + \frac{2}{10} + \frac{4}{10} + \frac{6}{10} + \frac{8}{10} + \frac{9}{10} + \frac{10}{10}\right) \\ &= \frac{1}{10}\left(\frac{40}{10}\right) = \frac{2}{5}\,. \end{aligned}$$

Problem 25

Drop a vertical from D towards the segment AB and intersecting AB at the point D'. Drop a vertical from C towards the segment AB and intersecting AB at the point C'. Then let $DD' = CC' = h$ (the height of the trapezoid), let $AD' = a$ and $BC' = b$. Then from the lengths of AB and CD we have that

$$a + b = 52 - 39 = 13\,. \tag{668}$$

From the Pythagorean theorem in the right triangles $\triangle DD'A$ and $\triangle CC'B$ we have

$$h^2 = 5^2 - a^2 = 12^2 - b^2\,.$$

Solving Equation 668 for a and putting it into the above gives one equation for b. Solving this we get $b = \frac{144}{13}$. This means that

$$h^2 = 12^2 - b^2 = \frac{(5 \cdot 12)^2}{13^2}\,,$$

when we simplify and thus $h = \frac{5 \cdot 12}{13}$. Using this the area of this trapezoid is given by

$$\frac{1}{2}\left(\frac{5 \cdot 12}{13}\right)(39 + 52) = 210\,,$$

when we simplify.

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Problem 1

This would be

$$\frac{6^{2001} \cdot 3^2}{6^{2002}} = \frac{3^2}{6} = \frac{3}{2}\,.$$

Problem 2

We have

$$(2, 4, 6) = \frac{2 \cdot 4 \cdot 6}{2 + 4 + 6} = \frac{2 \cdot 4 \cdot 6}{12} = 4\,.$$

Problem 3

To evaluate the average we first need to evaluate the sum

$$S = 9 + 99 + 999 + 9999 + 99999 + 999999 + 9999999 + 99999999 + 999999999\,,$$

or

$$S = 9(1 + 11 + 111 + 1111 + 11111 + 111111 + 1111111 + 11111111 + 111111111)\,.$$

To evaluate this sum lets start with the last two terms in parenthesis or

$$11111111 + 111111111 = 122222222\,.$$

To this term we add the previous term to get

$$1111111 + 122222222 = 123333333\,.$$

To this term we add the previous term to get

$$111111 + 123333333 = 123444444\,.$$

The pattern we are building up looks clear. After adding all terms we would find

$$S = 9(123456789)\,.$$

The average is then this number divided by nine or 123456789. This has every digit but zero.

Problem 4

We can write this expression as

$$(3x - 2)(1) + 1 = 3x - 1\,.$$

When $x = 4$ this is eleven.

Problem 5

The diameter of the outer circle is given by $3+3+2+2=10$ so has a radius of $R=5$. The area of the shaded region is then given by

$$\pi \cdot 5^2 - \pi \cdot 3^2 - \pi \cdot 2^2 = 12\pi \,.$$

Problem 6

Factoring this expression we have the given number equal to $(n-2)(n-1)$. Thus for n large enough this will not be a prime number (as it factors into two parts). If $n \in \{1,2\}$ this number is zero (which is not prime). If $n=3$ this number is two which is prime. If $n=4$ this number is six which is not prime. Thus there is *one* prime of this form.

Problem 7

Let this expression equal N which we are told is an integer. Thus we have

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{n} = N \,.$$

As we are asked about n lets solve for it. Using the above we find

$$n = \frac{42}{42N - 41} \,, \tag{669}$$

or

$$(42N - 41)n = 42 = 2 \cdot 3 \cdot 7 \,.$$

As n is also an integer and so is $42N-41$ the left-hand-side is an integer factorization of 42. This means that $42N-41$ must be a member of the following set

$$\{1, 2, 3, 7, 6, 14, 21, 42\}$$

Setting $42N-41$ equal to each of these and solving for N only when $42N-41=1$ do we get an integer solution for N of $N=1$. Then using that in Equation 669 we find $n = \frac{42}{1} = 42$ which is not greater than 84.

Problem 8

Now four full weeks will be $4 \times 7 = 28$ days leaving $31 - 28 = 3$ other (consecutive) days. As we know that we have five Mondays in July one of these "extra" days must be a Monday. We will also assume that a full week starts on a Monday and ends on a Sunday. As one of the extra days is a Monday the sequence of extra weekdays that we might have includes

- Monday, Tuesday, Wednesday
- Sunday, Monday, Tuesday
- Saturday, Sunday, Monday

We will consider each case individually.

- In the first case above the month of August would then have the additional weekdays following of Thursday, Friday, Saturday, and Sunday. This means that August would have $31 - 4 = 27$ additional days or three full weeks and $27 - 21 = 6$ additional days of Monday, Tuesday, Wednesday, Thursday, Friday and Saturday. Thus in this case there are five Thursday's, Friday's, and Saturday's.
- In the second case above the month of August would have the additional weekdays following of Wednesday, Thursday, Friday, Saturday, and Sunday. This means that August would have $31 - 5 = 26$ additional days or three full weeks and $26 - 21 = 5$ additional days of Monday, Tuesday, Wednesday, Thursday, and Friday. Thus in this case there are five Wednesday's, Thursday's and Friday's.
- In the third case above the month of August would have the additional weekdays following of Tuesday, Wednesday, Thursday, Friday, Saturday, and Sunday. This means that August would have $31-6 = 25$ additional days or three full weeks and $25-21 = 4$ additional days of Monday, Tuesday, Wednesday, and Thursday. Thus in this case there are five Tuesday's, Wednesday's, and Thursday's.

The "common" fifth day in all three cases is Thursday and thus August must have five Thursdays.

Problem 9

Note that

- There are $4! = 24$ words that start with an A for a total of 24 words
- There are $4! = 24$ words that start with an M for a total of $24 + 24 = 48$ words
- There are $4! = 24$ words that start with an O for a total of $48 + 24 = 72$ words
- There are $4! = 24$ words that start with an S for a total of $72 + 24 = 96$ words
- There are $3! = 6$ words that start with UA for a total of $96 + 6 = 102$ words
- There are $3! = 6$ words that start with UM for a total of $102 + 6 = 108$ words
- There are $3! = 6$ words that start with UO for a total of $108 + 6 = 114$ words

The word USAMO will be the first word after all the words that start with UO and thus will be at position $114 + 1 = 115$.

Problem 10

Let the two roots of this quadratic be p and q. Then by Vieta's formula

https://en.wikipedia.org/wiki/Vieta's_formulas

we would have

$$\begin{aligned} p + q &= -a \\ pq &= b \,, \end{aligned}$$

or since $p = a$ and $q = b$ we have

$$a + b = -a \tag{670}$$
$$ab = b \,. \tag{671}$$

Equation 671 gives us that $b(a - 1) = 0$ so $b = 0$ or $a = 1$. As we are told that $b \neq 0$ we must have $a = 1$. Using this in Equation 670 gives $1 + b = -1$ so $b = -2$ and we have $(a, b) = (1, -2)$.

Problem 11

Let the three integers be n, $n + 1$, and $n + 2$. Then we are told that

$$n(n + 1)(n + 2) = 8(n + n + 1 + n + 2) \,,$$

or

$$n(n + 1)(n + 2) = 24(n + 1) \,,$$

when we simplify a bit. As we are told that $n > 0$ we have $n \neq -1$ then we can write the above as

$$n(n + 2) = 24 \quad \text{or} \quad n^2 + 2n - 24 = 0 \,.$$

This later expression factors as $(n + 6)(n - 4) = 0$. The only valid solution is $n = 4$ where we find

$$n^2 + (n + 1)^2 + (n + 2)^2 = 4^2 + 5^2 + 6^2 = 77 \,.$$

Problem 12

From the form of this equation there might be "easy" values for k that we could check to see if "anything special" happens in those cases. The "easy" values of k where "something special" might happen would include $k \in \{1, 2, 6\}$. For $k = 1$ notice that this expression is

$$\frac{x-1}{x-2} = \frac{x-1}{x-6}\,.$$

This will have no solution. Taking $k = 6$ gives another expression that has no solution. If we "cross multiply" we get

$$(x-1)(x-6) = (x-k)(x-2)\,,$$

or expanding and simplifying gives

$$(k-5)x = 2(k-3)\,.$$

This will have no solution if $k = 5$.

Problem 13

To have this be true for all values of y we want to "solve for x". Writing this expression with x "all on one side" gives

$$(8y+2)x = 12y+3\,,$$

or

$$2(4y+1)x = 3(4y+1)\,.$$

If $4y + 1 \neq 0$ we can divide by it to get $2x = 3$ so $x = \frac{3}{2}$. If we set $x = \frac{3}{2}$ in the original expression we see that it simplifies to $0 = 0$ (an identity) for any value of y.

Problem 14

Call this number n. Note that we can write n as

$$n = 5^{2(64)} \cdot 8^{2(25)} = (5^{64} \cdot 8^{25})^2 = (5^{64} \cdot 2^{75})^2 = (10^{64} \cdot 2^{11})^2\,.$$

Now $2^{11} = 2048$ and multiplying this by 10^{64} adds 64 zero digits to the end of 2048. Thus N is 2048 followed by 64 zeros and the sum of the digits in N is 14.

Problem 15

If A and B are both odd (which most primes are) then $A - B$ and $A + B$ will both be even and not prime unless one of them is two. The smaller of these two numbers would be $A - B$ so we assume that $A - B = 2$ (so that $A = B + 2$) and our four numbers are then

$$B+2\,, B\,, 2\,, 2B+2\,.$$

This cannot be correct as $2B+2$ has two as a factor and is not prime.

This means that one of A or B is the number two. If $A=2$ then $A-B=2-B$ will be negative for B an odd prime. Thus $B=2$ and our four numbers are

$$A\,,2\,,A-2\,,A+2\,.$$

Now A is a number between $A-2$ and $A+2$ so the correct ordering of these numbers is

$$2\,,A-2\,,A\,,A+2\,.$$

For these to be *consecutive* primes (a stipulation given in the solution but not in the problem statement) then we need $A-2=3$ so $A=5$ and our numbers are

$$2\,,3\,,5\,,7\,.$$

Thus our sum is 17 which is prime.

Problem 16

If we are to have

$$\frac{n}{20-n}=m^2\,,$$

then as $m^2 \geq 0$ if $n \geq 0$ we must have $20-n>0$ so $n<20$ or $n \leq 19$. If $n<0$ then we must have $20-n<0$ or $n>20$ which is a contradiction. Thus for our fraction to be positive it looks like n must be in the set

$$n \in \{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19\}\,.$$

We can just compute our fraction for each of these n and look to see if the given expression is a perfect square. Doing this we find that only for the four numbers $n \in \{0,10,16,18\}$ is this true.

Problem 17

I drew this octagon with the point A on the negative x-axis, the point E on the positive x-axis, the point G on the negative y-axis, and the point C on the positive y-axis. Then when we place the other points D, F, H, and B they will be at $\frac{360}{8}=45°$ from the x (or y) axis.

Call the distance from the origin to any two points on the octagon R. Note that two adjacent "radii" and one side of the octagon will form an isosceles triangle with vertex angle $45°$, two equal sides of length R, and a base of length two. Using the law of cosigns in that triangle we have

$$2^2=R^2+R^2-2R^2\cos(45°)\,.$$

We can solve this for R^2 and find $R^2 = \frac{2\sqrt{2}}{\sqrt{2}-1}$.

Now the coordinates of the triangle $\triangle AGD$ we are interested in are given by

$$\begin{aligned} A &= (-R, 0) \\ G &= (0, -R) \\ D &= (R\cos(45), R\sin(45)) = \left(\frac{R}{\sqrt{2}}, \frac{R}{\sqrt{2}}\right). \end{aligned}$$

To evaluate the area of this triangle we will use the fact that it is equal to $\frac{1}{2}||\mathbf{DA} \times \mathbf{DG}||$. We have

$$\begin{aligned} \mathbf{DA} &= \left(-R - \frac{R}{\sqrt{2}}, 0 - \frac{R}{\sqrt{2}}\right) \\ \mathbf{DG} &= \left(-\frac{R}{\sqrt{2}}, -R - \frac{R}{\sqrt{2}}\right), \end{aligned}$$

and thus find

$$\begin{aligned} \mathbf{DA} \times \mathbf{DG} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -R - \frac{R}{\sqrt{2}} & -\frac{R}{\sqrt{2}} & 0 \\ -\frac{R}{\sqrt{2}} & -R - \frac{R}{\sqrt{2}} & 0 \end{vmatrix} = \mathbf{k} \begin{vmatrix} -R - \frac{R}{\sqrt{2}} & -\frac{R}{\sqrt{2}} \\ -\frac{R}{\sqrt{2}} & -R - \frac{R}{\sqrt{2}} \end{vmatrix} \\ &= \mathbf{k}\left(\left(R + \frac{R}{\sqrt{2}}\right)^2 - \frac{R^2}{2}\right) = \mathbf{k}R^2(1 + \sqrt{2}). \end{aligned}$$

Using what we know about R^2 we find the area to be

$$\frac{1}{2}||\mathbf{DA} \times \mathbf{DG}|| = \frac{1 + \sqrt{2}}{2}\left(\frac{2\sqrt{2}}{\sqrt{2} - 1}\right) = 4 + 3\sqrt{2},$$

when we simplify.

Problem 18

If we draw two distinct circles we see that they can intersect in at most two points. Drawing a third circle we can get *four* more intersections for a total of $2+4 = 6$. With this drawing we can imagine a third circle cutting and creating *six* more intersections for a total of $6+6 = 12$ intersection points.

Problem 19

An arithmetic sequence takes the form $a_n = a_1 + b(n-1)$ for $n \geq 1$. Note that in this form we have $a_2 - a_1 = b$. For this form of $\{a_n\}$ we have

$$\begin{aligned}\sum_{i=1}^{100} a_i &= \sum_{i=1}^{100}(a_1 + b(i-1)) = 100a_1 + b\sum_{i=0}^{99} i \\ &= 100a_1 + b\left(\frac{100(99)}{2}\right) = 100a_1 + 50(99)b = 100\,. \quad (672)\end{aligned}$$

In the same way we have

$$\begin{aligned}\sum_{i=101}^{200} a_i &= \sum_{i=101}^{200}(a_1 + b(i-1)) = 100a_1 + b\sum_{i=100}^{199} i \\ &= 100a_1 + b\sum_{i=0}^{99}(i+100) = 100a_1 + b\left(100^2 + \sum_{i=1}^{99} i\right) \\ &= 100a_1 + b100^2 + \frac{99(100)}{2}b = 200\,. \quad (673)\end{aligned}$$

Using Equation 672 to replace $100a_1 + 50(99)b$ as 100 in Equation 673 that equation becomes

$$100 + 100^2 b = 200 \quad \text{so} \quad b = \frac{1}{100} = 0.01\,.$$

Problem 20

Write this system as

$$\begin{aligned} a - 7b &= 4 - 8c \\ 8a + 4b &= 7 + c\,. \end{aligned}$$

Solve this for a and b gives

$$\begin{aligned} a &= \frac{1}{12}(13 - 5c) \\ b &= \frac{1}{12}(-5 + 13c)\,. \end{aligned}$$

Note that $a^2 - b^2 = (a+b)(a-b)$ and from the above we have that

$$\begin{aligned} a + b &= \frac{8 + 8c}{12} = \frac{2}{3}(1 + c) \\ a - b &= \frac{18 - 18c}{12} = \frac{3}{2}(1 - c)\,. \end{aligned}$$

This means that $(a+b)(a-b) = 1 - c^2$. Thus the expression we want to evaluate is $a^2 - b^2 + c^2 = 1$.

Problem 21

Let the areas of Andy's, Beth's, and Carlos' lawns be A, B, and C respectively. Then we are told that

$$A = 2B = 3C\,.$$

Writing everything in terms of A we get

$$B = \frac{A}{2} \quad \text{and} \quad C = \frac{A}{3}\,.$$

Let the "mowing capacity" of Andy's, Beth's, and Carlos' "cutting machines" be a, b, and c respectively. Then we are told that

$$c = \frac{b}{2} = \frac{a}{3}\,.$$

Writing everything in terms of a we get

$$b = \frac{2}{3}a \quad \text{and} \quad c = \frac{a}{3}\,.$$

We want to compare $\frac{A}{a}$, $\frac{B}{b}$, and $\frac{C}{c}$. Expressing everything in terms of A and a these are $\frac{A}{a}$ and

$$\frac{B}{b} = \frac{A}{2} \times \frac{3}{2a} = \frac{3}{4}\frac{A}{a}$$
$$\frac{C}{c} = \frac{A}{3} \times \frac{3}{a} = \frac{A}{a}\,.$$

The smallest of these three numbers is $\frac{3}{4}\frac{A}{a}$ which is Beth.

Problem 22

Let O be the origin of an x-y Cartesian coordinate plane Then let the point $X = (x, 0)$ be on the x-axis and the point $Y = (0, y)$ be on the y-axis. Then from what we are told the points M and N are given by

$$M = \left(\frac{x}{2}, 0\right)$$
$$N = \left(0, \frac{y}{2}\right)\,.$$

Using the fact that $XN = 19$ in the right triangle $\triangle NOX$ we have

$$19^2 = x^2 + \frac{y^2}{4}\,. \tag{674}$$

Using the fact that $YM = 22$ in the right triangle $\triangle YOM$ we have

$$22^2 = \frac{x^2}{4} + y^2 \,. \tag{675}$$

If we solve this equation for y^2 and put it into Equation 674 we get a single equation for x. Solving this we get $x = 16$. Using this in any of the above gives $y = 2\sqrt{105}$.

As we desire to compute XY note that using the right triangle $\triangle XOY$ we have

$$XY^2 = x^2 + y^2 \,,$$

thus it might be easier to solve for x^2 and y^2 (in the above equations) rather than x and y (without the square roots). In either case we should find

$$XY^2 = 16^2 + 4(105) = 676 \,.$$

This means that $XY = 26$.

Problem 23

Let $m = 1$ and we get

$$a_{n+1} = a_1 + a_n + n = n + 1 + a_n \,,$$

or

$$a_{n+1} - a_n = n + 1 \,.$$

If we sum both sides of this from $n = 1$ to $n = N$ we get

$$a_{N+1} - a_1 = \sum_{k=1}^{N} (k+1) = \frac{N(N+1)}{2} + N \,.$$

This means that a_N is given by

$$a_N = 1 + \frac{(N-1)N}{2} + N - 1 = \frac{N(N+1)}{2}$$

From this if $N = 12$ we find $a_{12} = 78$.

Problem 24

Draw this wheel and let the center of the wheel be denoted as O. Next drop a perpendicular from its center vertically downwards intersecting a horizontal line though the bottom of the wheel at O'. Now position the rider at a point R on the wheel and 10 feet from the bottom level of the wheel. From R draw a horizontal line intersecting this vertical segment OO' at a point P. Note that as $PO' = 10$ and $OO' = 20$ we have $OP = 10$. If we draw a line from the rider to the center of the wheel we form a right triangle $\triangle OPR$ where $OR = 20$. If we let $\theta = \angle O'OR$ note that

$$\theta = \cos^{-1}\left(\frac{OP}{OR}\right) = \cos^{-1}\left(\frac{10}{20}\right) = \frac{\pi}{3} = 60^\circ \,.$$

This is

$$\frac{60}{360} = \frac{1}{6} \,,$$

of a rotation of the wheel. As it take one minute for a full rotation it should take $\frac{1}{6}$ of a minute to reach point R or

$$\frac{60}{6} = 10 \,,$$

seconds.

Problem 25

Let n be the number of elements in the original list, S their sum, and m the original mean. Then we know that

$$m = \frac{S}{n} \,, \tag{676}$$

and we are told that

$$\frac{S+15}{n+1} = m + 2 \tag{677}$$

$$\frac{S+15+1}{n+2} = m + 1 \,. \tag{678}$$

If we subtract Equation 678 from Equation 677 we get

$$\frac{S+15}{n+1} - \frac{S+16}{n+2} = 1 \,.$$

If we multiply this by $(n+1)(n+2)$ we get

$$(S+15)(n+2) - (S+16)(n+1) = (n+1)(n+2) \,,$$

or

$$(n+2)S - (n+1)S = (n+1)(n+2) + 16(n+1) - 15(n+2) \,,$$

or

$$S = (n+1)(n+2) + n - 14 = n^2 + 3n + 2 + n - 14 = n^2 + 4n - 12 \,. \tag{679}$$

Now lets use Equation 677 and Equation 676 to solve for S in terms of n. We start with

$$\frac{S+15}{n+1} = \frac{S}{n} + 2 \,,$$

If we multiply by $n(n+1)$ this is

$$n(S+15) = S(n+1) + 2n(n+1) \,,$$

or solving for S we get

$$S = -2n^2 + 13n \,.$$

Setting this equal to Equation 679 and solving for n we get $n \in \{-1, 4\}$. As $n > 0$ we have $n = 4$.

The 2002 AMC 12A Examination

Problem 1

The 2002 AMC 12B Examination

Problem 1

The 2003 AMC 10A Examination

Problem 1

This would be

$$\sum_{k=1}^{2003}(2k) - \sum_{k=1}^{2003}(2k+1) = \sum_{k=1}^{2003} 1 = 2003\,.$$

Problem 2

Socks cost $S = 4$ and T-shirts cost $T = 4+5 = 9$. Each member needs $2(S+T) = 2(13) = 26$ dollars worth of stuff. If n are the number of members of the team we are told that $26n = 2366$ so $n = 91$.

Problem 3

The volume of the original shape is $V_0 = 15 \times 10 \times 8$ while the volume of one removed corner is $V_{\text{corner}} = 3^3$. As there are eight corners we have the fraction removed is

$$\frac{8 \times 3^3}{15 \times 10 \times 8} = 0.18\,,$$

when we simplify. This is 18%.

Problem 4

The average velocity is total distance divided by total time so

$$\frac{2(1)}{\frac{30+10}{60}} = 3\,,$$

in units of kilometers-per-hour.

Problem 5

Write this as $2x^2 + 3x - 5 = (2x+5)(x-1) = 0$. Thus the two roots are $d = 1$ and $e = -\frac{5}{2}$. From this we see that $(d-1)(e-1) = 0$.

Problem 6

Answer (C) is not true as $x\heartsuit 0 = |x - 0| = |x| \neq x$ unless $x \geq 0$.

Problem 7

Let the three integer sides be a, b, and c such that $a \geq b \geq c$ and that

$$a + b + c = 7\,. \tag{680}$$

From this as we have

$$7 = a + b + c \geq 3c \quad \text{so} \quad c \leq \frac{7}{3} < 3\,.$$

Thus we have found that $c \leq 2$. We also know that by the triangle inequality that

$$a - b < c\,. \tag{681}$$

To count the number of triangles with the desired conditions we will take $c \in \{1, 2\}$ and see how many triangles with that value of c exist.

If $c = 1$ then Equation 680 gives

$$a + b = 6\,,$$

and Equation 681 gives

$$a - b < 1\,.$$

The integer solutions to the above we are looking for are when $a - b = 0$ which give $a = b = 3$ and only one triangle with this value of c.

If $c = 2$ then Equation 680 gives

$$a + b = 5\,,$$

and Equation 681 gives

$$a - b < 2\,.$$

The integer solutions to the above we are looking for are when $a - b \in \{0, 1\}$. This gives two systems of equations to solve. The only integer solutions are when $a - b = 1$ and we get $a = 3$ and $b = 2$ and only one triangle with this value of c.

Adding up all of these there are $1 + 1 = 2$ triangles of the given form.

Problem 8

When we factor 60 we have $60 = 2^2 \cdot 3 \cdot 5$. Thus the factors of 60 are

$$2^{n_2} \cdot 3^{n_3} \cdot 5^{n_5}\,,$$

for

$$\begin{aligned}0 &\le n_2 \le 2\\0 &\le n_3 \le 1\\0 &\le n_5 \le 1\,.\end{aligned}$$

This gives a total of $3 \times 2 \times 2 = 12$ factors. We can count the number of these factors that are less than seven. These are the factors $\{1, 2, 3, 4, 5, 6\}$ thus the probability we seek is

$$\frac{6}{12} = \frac{1}{2}\,.$$

Problem 9

As

$$\sqrt[3]{x\sqrt{x}} = \sqrt[3]{x^{3/2}} = x^{1/2}\,,$$

we can apply this same procedure three times to get that the given expression is equal to $x^{1/2} = \sqrt{x}$.

Problem 10

If we look at the given configuration it looks like numbers 4, 5, 6, 7, 8, and 9 can be folded into a cube with one face missing. This is six choices.

Problem 11

We are told that

```
  AMC10
+ AMC12
-------
 123422
```

This means that $C + C$ must end in a four. This means that $C = 2$ or $C = 7$. If $C = 2$ then $M + M$ would need to end in a three which is not possible. Thus $C = 7$ and our addition (with a carry) looks like the following

```
   1
  AM710
+ AM712
-------
 123422
```

This now means that $M + M + 1$ must end in a three so $M + M$ must end in a two. Thus $M = 1$ or $M = 6$. If $M = 1$ then $A + A$ would need to equal 12 so $A = 6$. If $M = 6$ then $A + A + 1 = 12$ which is not possible.

Thus we have found that

$$(A, M, C) = (6, 1, 7) .$$

The sum of these three numbers is 14.

Problem 12

To start we draw this rectangle in the x-y coordinate plane. This rectangle has an area of four. Next the region with $x < y$ forms a right triangle with vertices $(0, 0)$, $(1, 1)$ and $(0, 1)$ and has an area of $\frac{1}{2}$. The probability we seek is then

$$\frac{1/2}{4} = \frac{1}{8} .$$

Problem 13

Let the three numbers be a, b, and c. Then we are told that

$$\begin{aligned} a + b + c &= 20 \\ a &= 4(b + c) \\ b &= 7c . \end{aligned}$$

From the second we have $b + c = \frac{a}{4}$. Putting that in the first we get

$$a + \frac{a}{4} = 20 \quad \text{so} \quad a = 16 .$$

This means that $b + c = 4$. Putting $b = 7c$ in that we see that $8c = 4$ so $c = \frac{1}{2}$. Then $b = \frac{7}{2}$. We then compute

$$abc = 16 \cdot \frac{7}{2} \cdot \frac{1}{2} = 28 .$$

Problem 14

Single digit primes are $\{2, 3, 5, 7\}$. To maximize our number $n = de(10d + e)$ we want to take d and e as large as possible. As de is symmetric in the variables (d, e) while $10d + e$ is not we will want to take $d > e$. Based on this we might try $d = 7$ and $e = 5$. In this case we have $10d + e = 75$ which is not prime and thus $(d, e) = (7, 5)$ is not a valid choice. Next we try $d = 7$ and $e = 3$. In that case we have $10d + e = 73$ which is prime. In this case we have

$$n = 7 \cdot 3 \cdot 73 = 1533 ,$$

and the sum of these digits is 12.

Problem 15

Let A be the event that our number is divisible by two. Thus $P(A) = \frac{50}{100} = \frac{1}{2}$. Let B be the event that our number is divisible by three. Then as there are $\lfloor \frac{100}{3} \rfloor = 33$ numbers of this form we have $P(B) = \frac{33}{100}$ so $P(B^c) = \frac{67}{100}$. We want to evaluate $P(A \cap B^c)$ which we can write as

$$P(A \cap B^c) = P(B^c|A)P(A) = (1 - P(B|A))P(A) \,.$$

Now the probability $P(B|A)$ is the probability a number is divisible by three given it is divisible by two. From the 50 numbers that are divisible by two we expect $\lfloor \frac{50}{3} \rfloor = 16$ of them to be divisible by three. Thus

$$P(B|A) = \frac{16}{50} \,,$$

and the above becomes

$$P(A \cap B^c) = \left(1 - \frac{16}{50}\right) \frac{1}{2} = \frac{17}{50} \,,$$

when we simplify.

Problem 16

Note that

$$\begin{aligned}13^2 &= 169 \\ 13^3 &= 2197 \\ 13^4 &= 28561 \,.\end{aligned}$$

This means that 13^{4p} ends in a one for all $p \in \mathbb{N}$. As

$$2003 = 500(4) + 3 \,,$$

the last digit of 13^{2003} we have the same last digit as 13^3 which from the above is seven.

Problem 17

Let the triangle have a side length of s and the circle have a radius of r. Then by symmetry from the center of the circle we can draw radii to each of the corners of the triangle that create three isosceles triangles with a vertex angle of $\frac{360}{3} = 120°$. Then in ones of these isosceles triangles using the law of cosigns to "link" the base of the isosceles triangle to the two radii that form its legs we have

$$s^2 = r^2 + r^2 - 2r^2 \cos(120°) = 2r^2 - 2r^2 \left(-\frac{1}{2}\right) = 3r^2 \,.$$

This means that $s = \sqrt{3}r$. As we are also told that

$$3s = \pi r^2 \,,$$

we can solve for s and r. We find $s = \frac{9}{\pi}$ and $r = \frac{3\sqrt{3}}{\pi}$.

Problem 18

Write this as

$$\frac{2003}{2004}x^2 + x + 1 = 0\,,$$

or

$$x^2 + \frac{2004}{2003}x + \frac{2004}{2003} = 0\,.$$

Lets denote $\frac{2004}{2003}$ by r. Now using Vieta's formula

`https://en.wikipedia.org/wiki/Vieta's_formulas`

if x_1 and x_2 are the two roots of the above quadratic then we have

$$\begin{aligned} x_1 x_2 &= r \\ x_1 + x_2 &= -r\,, \end{aligned}$$

or

$$x_1 + x_2 = -x_1 x_2\,.$$

If we divide this by $x_1 x_2$ we get

$$\frac{1}{x_1} + \frac{1}{x_2} = -1\,.$$

Problem 19

Let the "top" horizontal line shown in the figure intersect the larger circle of radius $R = \frac{2}{2} = 1$ at the point A (on the left) and B on the right. Let the center of the larger circle be denoted O. The smaller circle has a radius of $r = \frac{1}{2}$.

Now the area of the lune is the area of the top semicircle *minus* the area of the "cap". Here the "cap" is the area in the top semicircle and also above the horizontal segment AB of length one. This means that

$$A_{\text{lune}} = \frac{1}{2}\pi\left(\frac{1}{2}\right)^2 - A_{\text{cap}} = \frac{\pi}{8} - A_{\text{cap}}\,.$$

Now if in the larger circle we draw two radii from the center of the circle O to the points A and B then as $R = 1$ the triangle $\triangle OAB$ is an equilateral triangle with a side s of length one. The area of the cap is the area of the sector of the larger circle from AB minus the equilateral triangle $\triangle OAB$. This means that

$$A_{\text{cap}} = \frac{60}{360}\left(\pi 1^2\right) - \left.\frac{\sqrt{3}}{4}s^2\right|_{s=1} = \frac{\pi}{6} - \frac{\sqrt{3}}{4}\,.$$

Using this we find the area of the lune to be given by

$$A_{\text{lune}} = \frac{\pi}{8} - \frac{\pi}{6} + \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{4} - \frac{\pi}{24}\,.$$

Problem 20

WWX: Here

The 2003 AMC 10B Examination

Problem 1

WWX: DP

WWX: Problems here.

Problem 18

WWX: DP

Problem 19

WWX: DP

Problem 20

Breaking the area we want into the area of the "base" trapezoid and a top triangle we have

$$[AEB] = [AFGB]+[FEG] = \frac{1}{2}AD(AB+FG)+[FEG] = \frac{1}{2}(3)(5+2)+[FEG] = \frac{21}{2}+[FEG]\,.$$

We know the length of the base in $\triangle FEG$ from

$$FG = CD - DF - CG = 5 - 1 - 2 = 3\,,$$

to determine $[FEG]$ we need to determine its "height". Drop a perpendicular from E to the segment FG intersecting FG at a point E'. Then

$$\triangle FE'E \sim \triangle FDA\,,$$

so

$$\frac{EE'}{FE'} = \frac{3}{1}\,. \tag{682}$$

Next

$$\triangle GE'E \sim \triangle GCB\,,$$

so

$$\frac{EE'}{E'G} = \frac{3}{2} \quad \text{or} \quad \frac{EE'}{2 - FE'} = \frac{3}{2}\,. \tag{683}$$

If we divide Equation 682 by 683 we get

$$\frac{2 - FE'}{FE'} = \frac{2}{3}(3) = 2\,,$$

so $FE' = \frac{2}{3}$. Using that we have $E'G = FG - FE' = 2 - \frac{2}{3} = \frac{4}{3}$ and using Equation 682 we have $EE' = 3\left(\frac{2}{3}\right) = 2$. This means that $[EFG] = \frac{1}{2}FG(EE') = \frac{1}{2}(2)(2) = 2$ and thus

$$[AEB] = \frac{21}{2} + 2 = \frac{25}{2}\,.$$

Problem 21

WWX: DP

Problem 22

WWX: DP

Problem 23

WWX: DP

Problem 24

As this is an arithmetic sequence the difference of two consecutive terms must be d the common difference (denoted here as d). This means that

$$x - y - (x + y) = -2y = d \tag{684}$$

$$xy - (x - y) = xy - x + y = d \tag{685}$$

$$\frac{x}{y} - xy = d\,. \tag{686}$$

This is a system of three equations in three unknowns x, y, and d. If we use Equation 684 into Equations 685 and 686 we get

$$xy - x + y = -2y \quad \text{so} \quad xy - x = -3y\,, \tag{687}$$

and

$$\frac{x}{y} - xy = -2y \quad \text{so} \quad \frac{x}{y^2} - x = -2\,. \tag{688}$$

From Equation 687 we have

$$x = -\frac{3y}{y-1} = \frac{3y}{1-y}\,. \tag{689}$$

If we put this into Equation 688 we have

$$\left(\frac{1}{y^2} - 1\right)\left(\frac{3y}{1-y}\right) = -2\,.$$

Solving this for y gives $y = -\frac{3}{5}$. Using this in Equation 689 gives $x = -\frac{9}{8}$. Then from Equation 684 we get $d = \frac{6}{5}$. We can also compute the fourth term to be

$$\frac{x}{y} = \frac{15}{8}\,.$$

The fifth term of this sequence would then be

$$\frac{x}{y} + d = \frac{15}{8} + \frac{6}{5} = \frac{123}{40}\,.$$

Problem 25

WWX: DP

The 2003 AMC 12A Examination

Problem 1

WWX: DPs

The 2003 AMC 12B Examination

Problem 1

The 2004 AMC 10A Examination

Problem 1

This would be $\frac{1500}{6} = 250$.

Problem 2

Note that we have

$$\begin{aligned}
\P(1,2,3) &= \frac{1}{2-3} = -1 \\
\P(2,3,1) &= \frac{2}{3-1} = 1 \\
\P(3,1,2) &= \frac{3}{1-2} = -3\,,
\end{aligned}$$

thus we are looking for

$$\P(-1,1,-3) = \frac{-1}{1-(-3)} = -\frac{1}{4}\,.$$

Problem 3

Alicia's \$20 pay is 2000 cents and the taxes on this are

$$2000\left(\frac{1.45}{100}\right) = 2\times 10^3\left(\frac{145}{10^4}\right) = \frac{2(145)}{10} = 29\,.$$

Problem 4

Note that if $x > 2$ then both $x-1$ and $x-2$ are positive and this expression is then

$$x - 1 = x - 2\,,$$

which has no solutions. If $1 < x < 2$ then this is

$$x - 1 = -(x-2)\,,$$

which has the solution $x = \frac{3}{2}$. If $x < 1$ then this is

$$-(x-1) = -(x-2)\,,$$

which also has no solution. Thus the only solution is $x = \frac{3}{2}$.

Problem 5

There are $\binom{9}{3} = 84$ possible combinations of three points. To line on the same straight line there are three horizontal, three vertical, and two diagonal lines. This gives a probability of

$$\frac{8}{84} = \frac{2}{21}\,.$$

Problem 6

Let I_i be an indicator variable indicating whether or not the ith daughter of Bertha (for $1 \leq i \leq 6$) has six daughters herself. Then as the total number of daughters and granddaughters is 30 we have

$$30 = 6 + \sum_{i=1}^{6} 6 I_i\,.$$

Solving this for $\sum_{i=1} 6I_i$ we get

$$\sum_{i=1}^{6} I_i = 4\,,$$

meaning that Bertha has four daughters that have six daughters so $4 \times 6 = 24$ granddaughters have no children as do $6 - 4 = 2$ of Bertha's daughters. This gives a total of

$$24 + 2 = 26\,.$$

Problem 7

We have

- In the first stack $5 \times 8 = 40$ oranges
- In the second stack $4 \times 7 = 28$ oranges
- In the third stack $3 \times 6 = 18$ oranges
- In the fourth stack $2 \times 5 = 10$ oranges
- In the fifth stack $1 \times 4 = 4$ oranges

This gives

$$40 + 28 + 18 + 10 + 4 = 100\,,$$

total oranges.

Problem 8

At the end of round zero (the start) we are in the "state"

$$(A, B, C) = (15, 14, 13) .$$

At the end of round one we are in the "state"

$$(A, B, C) = (12, 15, 14) .$$

At the end of round two we are in the "state"

$$(A, B, C) = (13, 12, 15) .$$

At the end of round three we are in the "state"

$$(A, B, C) = (14, 13, 12) .$$

Thus in three rounds every player has the number of tokens reduced by one. Notice that we cannot just play 15 groups of three rounds however because some player will end up with zero tokens before all the rounds are finished.

Now if we play 12 groups of three rounds we will end in the state of

$$(A, B, C) = (15 - 12, 14 - 12, 13 - 12) = (3, 2, 1) .$$

One more round takes this state to

$$(A, B, C) = (0, 3, 2) ,$$

and the game ends for a total of $3(12) + 1 = 37$ rounds.

Problem 9

Some areas we can compute directly. We have

$$[\triangle EAB] = \frac{1}{2}(4)(8) = 16$$
$$[\triangle ABC] = \frac{1}{2}(4)(6) = 12 .$$

From these we can compute the area of interest as

$$[\triangle EAB] - [\triangle ABC] = 16 - 12 = 4 .$$

Problem 10

Let H_A and H_B be the number of heads A and B obtain when they flip their coins. Then we want to compute $P\{H_A = H_B\}$. To compute this we can condition on H_A as

$$\begin{aligned}
P\{H_A = H_B\} &= P\{H_B = 0|H_A = 0\}P\{H_A = 0\} + P\{H_B = 1|H_A = 1\}P\{H_A = 1\} \\
&+ P\{H_B = 2|H_A = 2\}P\{H_A = 2\} + P\{H_B = 3|H_A = 3\}P\{H_A = 3\} \\
&= \binom{3}{0}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^3\binom{4}{0}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^4 + \binom{3}{1}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^2\binom{4}{1}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^3 \\
&+ \binom{3}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^1\binom{4}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 + \binom{3}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^0\binom{4}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^1 \\
&= \binom{3}{0}\binom{4}{0}\left(\frac{1}{2}\right)^7 + \binom{3}{1}\binom{4}{1}\left(\frac{1}{2}\right)^7 + \binom{3}{2}\binom{4}{2}\left(\frac{1}{2}\right)^7 + \binom{3}{3}\binom{4}{3}\left(\frac{1}{2}\right)^7 \\
&= \frac{1 + 3(4) + 3\left(\frac{4\cdot 3}{2}\right) + 4}{2^7} = \frac{35}{128}.
\end{aligned}$$

Problem 11

The volume V of each jar is given by

$$V = \pi r^2 h = \pi\left(\frac{d}{2}\right)^2 h = \frac{1}{4}\pi d^2 h\,,$$

where d is the diameter and h is the height. If the diameter changes to $1.25d = \frac{5}{4}d$ then the new volume V' is

$$V' = \frac{\pi}{4}\left(\frac{5}{4}d\right)^2 h' = \frac{\pi}{4}d^2\left(\frac{25}{16}h'\right).$$

If we want $V' = V$ then we need to have

$$\frac{25}{16}h' = h \quad \text{so} \quad h' = \frac{16}{25}h = \left(1 - \frac{9}{25}\right)h = (1 - 0.36)h\,.$$

This means that the height is reduced by 36%.

Problem 12

We must choose one of the three kinds of meat patties. For the number of choices available for the condiments note that we could have any "subset" of condiments from the set of condiments with eight elements. The number of subsets from a set with eight elements is 2^8. Thus the total number of possible hamburgers is

$$3 \cdot 2^8 = 768\,.$$

Problem 13

WWX: DP

Problem 14

Let p, n, d, and q be the number of pennies, nickles, dimes, and quarters Paula holds. Then we are told that

$$\frac{p + 5n + 10d + 25q}{p + n + d + q} = 20\,, \tag{690}$$

and

$$\frac{p + 5n + 10d + 25(q + 1)}{p + n + d + q + 1} = 21\,. \tag{691}$$

From Equation 690 we get

$$p + 5n + 10d + 25q = 20(p + n + d + q)\,. \tag{692}$$

From Equation 691 we get

$$p + 5n + 10d + 25q + 25 = 21(p + n + d + q) + 21\,. \tag{693}$$

If we use the left-hand-side of Equation 692 into the left-hand-side of Equation 693 we get

$$20(p + n + d + q) + 25 = 21(p + n + d + q) + 21\,,$$

or

$$p + n + d + q = 4\,. \tag{694}$$

From this we conclude that each of the variables p, n, d, and q above must be less than or equal to four. Using this result in Equation 692 we get

$$p + 5n + 10d + 25q = 20(4) = 80\,.$$

As $5n + 10d + 25q = 5(n + 2d + 5q)$ this expression ends in a five or a zero and as $0 \le p \le 4$ we have that to have the above be true we must have $p = 0$ and thus we have shown that

$$n + 2d + 5q = 16\,. \tag{695}$$

The system we seek a solution to is then

$$\begin{aligned} n + d + q &= 4 \\ n + 2d + 5q &= 16\,. \end{aligned}$$

or

$$\begin{aligned} n + q &= 4 - d \\ n + 5q &= 16 - 2d\,. \end{aligned}$$

I wrote the above in that form since we are asked about the value of d and in that form there is (hopefully) only one value of d that will give integer solutions for n and q. Subtracting the first equation from the second gives

$$5q - q = 4q = 12 - d \quad \text{so} \quad q = 3 - \frac{d}{4}\,.$$

Using this we have $n = 4 - d - q = 1 - \frac{3}{4}d$. The only value for $0 \le d \le 4$ where n and q are positive integers is $d = 0$ where we find $q = 3$ and $n = 1$. As a check using the values we have found in Equation 690 we get

$$\frac{0 + 5 + 0 + 25(3)}{0 + 1 + 0 + 3} = \frac{80}{4} = 20\,,$$

and using these values we have found in Equation 691 we get

$$\frac{0 + 5 + 0 + 25(4)}{0 + 1 + 0 + 4} = \frac{105}{5} = 21\,.$$

Problem 15

Write this as

$$\frac{x + y}{x} = 1 + \frac{y}{x}\,,$$

and lets seek to bound $\frac{y}{x}$. Now as $-4 \le x \le -2$ we have

$$-\frac{1}{2} \le \frac{1}{x} \le -\frac{1}{4}\,.$$

As $2 \le y \le 4$ we have that

$$-\frac{4}{2} \le \frac{y}{x} \le -\frac{2}{4} \quad \text{or} \quad -2 \le \frac{y}{x} \le -\frac{1}{2}\,.$$

Adding one we see that

$$-1 \le 1 + \frac{y}{x} \le \frac{1}{2}\,.$$

Thus the largest value of the expression is $\frac{1}{2}$.

Problem 16

Problem 17

Problem 18

Problem 19

Problem 20

Problem 21

Problem 22

In the given figure let O be the center of the circle on the segment AB so that $AO = OB = 1$. In addition we will denote $DE = l$ so $AE = 2 - l$. Next in the given figure draw the segment OC. Then in the right triangle $\triangle CBO$ we have

$$OC^2 = OB^2 + BC^2 = 1^2 + 2^2 = 5\,.$$

Let T be the point of tangency of the segment EC and the circle. Then in the right triangle $\triangle OTC$ we have

$$OC^2 = OT^2 + TC^2 \quad \text{so} \quad 5 = 1^2 + TC^2 \quad \text{so} \quad TC = 2\,.$$

Now in the right triangle $\triangle OTE$ we have

$$OE^2 = ET^2 + OT^2\,,$$

so

$$OE^2 = ET^2 + 1\,. \tag{696}$$

In the right triangle $\triangle OAE$ we have

$$OA^2 + AE^2 = OE^2\,,$$

so

$$1^2 + AE^2 = OE^2\,. \tag{697}$$

If we set these two equations for OE^2 equal we get

$$ET = AE = 2 - l\,.$$

Now in the right triangle $\triangle EDC$ we have

$$ED^2 + DC^2 = EC^2\,,$$

or

$$l^2 + 2^2 = (2 - l + 2)^2\,.$$

Solving this for l gives $l = \frac{3}{2}$. Using this we find that

$$EC = 2 - l + 2 = 4 - l = 4 - \frac{3}{2} = \frac{5}{2}\,.$$

Problem 23

Let the centers of A, B, and D be denoted by the points E, H and F respectively. Let G be the point of external tangency between the circles B and C. Consider the triangles $\triangle EGH$ and $\triangle FGH$.

By the symmetry of the figure the $\angle EGH$ is $90°$ and the two triangles above are right triangles. Let $x = FG$ and the radius of the circle B be y. Then as the radius of circle A is one we have

$$EH = 1 + y\,,$$

and

$$EG = 1 + x\,.$$

In the right triangle $\triangle EGH$ using the Pythagorean theorem we have

$$(x+1)^2 + y^2 = (1+y)^2\,,$$

which simplifies to

$$x^2 + 2x = 2y\,. \tag{698}$$

As the radius of the larger circle D is two and the radius of circle B is y the radial segment from F to H must be of length $2 - y$. Then the Pythagorean theorem in the right triangle $\triangle FGH$ gives

$$x^2 + y^2 = (2-y)^2\,,$$

which simplifies to

$$x^2 = 4 - 4y\,. \tag{699}$$

Solving these two equations gives $y = \frac{8}{9}$.

Problem 24

From the problem statement we will evaluate a few values for a_n. We have

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 1a_1 = 1 \\ a_4 &= 2a_2 = 2 \\ a_8 &= 4a_4 = 8\,. \end{aligned}$$

Based on this we will take $n = 2^m$ in the relationship $a_{2n} = na_n$ we get

$$a_{2^{m+1}} = 2^m a_{2^m}\,.$$

To simplify this lets take $b_m = a_{2^m}$ and find that

$$b_{m+1} = 2^m b_m\,.$$

We want to know $a_{2^{100}} = b_{100}$. To evaluate this lets "work backwards". We find

$$\begin{aligned} b_{100} &= 2^{99} b_{99} = 2^{99} 2^{98} b_{98} = 2^{99} 2^{98} 2^{97} b_{97} = 2^{99} 2^{98} 2^{97} \cdots 2^2 2^1 b_1 \\ &= 2^{\sum_{k=0}^{99} k} b_1 = 2^{\frac{99(100)}{2}} b_1 = 2^{4950} b_1 \,. \end{aligned}$$

Now $b_1 = a_2 = 1$ and so $a^{2^{200}} = 2^{4950}$.

Problem 25

The three spheres will have centers that are on a horizontal plane that is parallel and elevated one unit above the "base" plane they are resting on. Their centers are also the corners of an equilateral triangle with a side of length $1 + 1 = 2$. The center of the larger sphere with radius of two will have its center above the "center of mass" of the three smaller spheres and thus above the intersection of the medians of the equilateral triangle (i.e. its centroid). Let this centroid have a point location of D.

Let one of the centers of one of the "base" spheres be denoted A, the center of the "top" sphere be denoted E. First we note that the the distance from A to E must be $1 + 2 = 3$. Next we note that the distance AD can be given by

$$AD \cos(30^\circ) = 1 \quad \text{so} \quad AD = \frac{2}{\sqrt{3}} \,.$$

Now the triangle $\triangle EDA$ is a right triangle and so we have

$$ED = \sqrt{AE^2 - AD^2} = \sqrt{9 - \frac{4}{3}} = \sqrt{\frac{23}{3}} = \frac{\sqrt{69}}{3} \,.$$

Taken together these mean that the distance from the plane to the top of the topmost sphere is

$$1 + \frac{\sqrt{69}}{3} + 2 = 3 + \frac{\sqrt{69}}{3} \,.$$

The 2004 AMC 10B Examination

Problem 1

Rows 12 through 22 are $22 - 12 + 1 = 11$ rows. Thus there are $11 \times 33 = 363$ seats.

Problem 2

There are 10 numbers of the form 7? i.e. $70\,, 71\,, 72 \ldots 79$. There are nine numbers of the form ?7 i.e. $17\,, 27\,, 37 \ldots 97$. The number 77 is common to both sets. Thus there are

$$10 + 9 - 1 = 18\,,$$

numbers of the given form.

Problem 3

Let J be the number of fee throws made by Jenny on the first practice. Then she made

$$J\,,2J\,,4J\,,8J\,,16J\,,$$

at each of her practices. We are told that $16J = 48$ or $J = 3$.

Problem 4

Let P_i be the product of the faces of the die when face i is "down". Then we compute

$$\begin{aligned}
P_1 &= 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 2^4 \cdot 3^2 \cdot 5 \\
P_2 &= 1 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 2^3 \cdot 3^2 \cdot 5 \\
P_3 &= 1 \cdot 2 \cdot 4 \cdot 5 \cdot 6 = 2^4 \cdot 3 \cdot 5 \\
P_4 &= 1 \cdot 2 \cdot 3 \cdot 5 \cdot 6 = 2^2 \cdot 3^2 \cdot 5 \\
P_5 &= 1 \cdot 2 \cdot 3 \cdot 4 \cdot 6 = 2^4 \cdot 3^2 \\
P_6 &= 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 2^3 \cdot 3 \cdot 5\,.
\end{aligned}$$

We seek the greatest common denominator d of these six numbers. Thus we have

$$d = 2^{\min(4,3,4,2,4,3)} \cdot 3^{\min(2,2,1,2,2,1)} \cdot 5^{\min(1,1,1,1,0,1)} = 2^2 \cdot 3^1 = 12\,.$$

Problem 5

To make this expression as large as possible we would want to subtract nothing so $d = 0$ and we want to maximize the expression $c \cdot a^b$. If we let $c = 1$ then we have the two numbers

$$\begin{aligned}
2^3 &= 8 \\
3^2 &= 9\,.
\end{aligned}$$

If we let $c = 2$ then we have the two numbers

$$\begin{aligned}
2 \cdot 1^3 &= 2 \\
2 \cdot 3^1 &= 6\,.
\end{aligned}$$

If we let $c = 3$ then we have the two numbers

$$\begin{aligned}
3 \cdot 1^2 &= 3 \\
3 \cdot 2^1 &= 6\,.
\end{aligned}$$

The largest of all of these numbers is nine.

Problem 6

Notice that (C) gives

$$99! \cdot 100! = (99!)^2 \cdot 100 = (99!)^2 \cdot 10^2 \,,$$

which is a perfect square. All of the other numbers given don't factor into perfect squares in this way.

Problem 7

If we start with d US dollars then after exchanging we have $\frac{7}{10}d$ Canadian dollars. Then spending 60 Canadian dollars she had d Canadian dollars left. This means that

$$\frac{7}{10}d - 60 = d \,.$$

Solving the above for d I find $d = 140$. The sum of the digits is five.

Problem 8

From the problem statement the "airport" is the right angle of a triangle with sides 10 and 8 representing the distance between the airport and Minneapolis and St Paul respectively. The distance between Minneapolis and St Paul is then given by the Pythagorean theorem where

$$\sqrt{10^2 + 8^2} = \sqrt{164} \,.$$

As $12^2 = 144$, $13^2 = 169$, and $14^2 = 196$ the closest distance is 13 miles.

Problem 9

Draw a square with a side length of ten and a circle centered at its lower left vertex. Then adding the area of the circle and the square "double counts" the area of the quarter of the circle in the square. Thus the area we seek is

$$\pi(10^2) + 10^2 - \frac{1}{4}\pi(10^2) = 75\pi + 100 \,,$$

when we simplify.

Problem 10

Let C_n be the number of can at level n. We are told that $C_1 = 1$ and $C_{n+1} = C_n + 2$. Solving this last difference equation we have $C_n = 2n + D$ for a constant D. Then when $n = 1$ we

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$
$m=1$	N	N	N	N	N	N	N	N
$m=2$	N	N	Y	Y	Y	Y	Y	Y
$m=3$	N	Y	Y	Y	Y	Y	Y	Y
$m=4$	N	Y	Y	Y	Y	Y	Y	Y
$m=5$	N	Y	Y	Y	Y	Y	Y	Y
$m=6$	N	Y	Y	Y	Y	Y	Y	Y
$m=7$	N	Y	Y	Y	Y	Y	Y	Y
$m=8$	N	Y	Y	Y	Y	Y	Y	Y

Table 26: Possible values for n and m when two eight sided die are rolled.

get

$$2 + D = 1 \quad \text{so} \quad D = -1 \,.$$

Thus we have that $C_n = 2n - 1$. Let T_n be the total number of cans in the entire stack. Then we have

$$T_n = \sum_{k=1}^{n} C_k = \sum_{k=1}^{n} (2k - 1) = n^2 \,,$$

when we simplify. If we are told that $T_n = 100$ that means that $n = 10$.

Problem 11

Let m be the number on the first die and n the number on the second die. Then the event we are interested is when

$$nm > n + m \,.$$

In Table 26 I explicitly enumerate all of the choices for m and n and denote whether the above condition is true (with a Y) or false (with a N). From that table we see that this event *does not* happen 16 times so the probability that it does happen is given by

$$1 - \frac{16}{64} = \frac{3}{4} \,.$$

Problem 12

From the drawing given we see that the area of the annulus is given by

$$\pi b^2 - \pi c^2 \,.$$

Using the right triangle $\triangle OZX$ we can write this as

$$\pi(c^2 + a^2) - \pi c^2 = \pi a^2 \,.$$

Problem 13

WWX: working here

Problem 14

Let R and B be the initial number of red and blue marbles (respectively) and we are told that $B > R$. Let r be the number of red marbles that are first added to make $\frac{1}{3}$ of the marbles in the bag blue. This means that

$$\frac{1}{3}(R + r + B) = B\,, \tag{700}$$

and we have $R + r + B$ marbles in the bag. Let Y be the number of yellow marbles added to make

$$\frac{1}{5}(R + r + B + Y) = B\,, \tag{701}$$

and we have $R+r+B+Y$ marbles in the bag. If we then double the number of blue marbles the number of marbles in the bag is $N = R + r + 2B + Y$ and we want to determine the fraction $\frac{2B}{N}$.

Now from Equation 701 we have

$$R + r + B + Y = 5B\,.$$

If we add B to both sides we get

$$R + r + 2B + Y = 6B\,.$$

This means that

$$\frac{2B}{N} = \frac{2B}{R + r + 2B + Y} = \frac{2B}{6B} = \frac{1}{3}\,.$$

Problem 15

Let n and d be the number of nickels and dimes that Patty has initially. Then we are told that

$$20 = n + d\,, \tag{702}$$

and the value V_0 of these coins is

$$5n + 10d = V_0\,. \tag{703}$$

If her nickels were dimes and her dimes were nickels then we are told that

$$5d + 10n = V_0 + 70\,. \tag{704}$$

If we put Equation 703 for V_0 into the above (and simplify) we get

$$14 = -d + n\,.$$

Using Equation 702 we can solve for n and d and find $d = 3$ and $n = 17$. This means that

$$V_0 = 5(17) + 10(3) = 115\,,$$

cents.

Problem 16

Joining the centers of the three circles will give an equilateral triangle with a side of length $1 + 1 = 2$. Denoted these centers as the points A, B, and C. By symmetry the center of the larger "outer" circle will be located at the centroid of this equilateral triangle. Denote this centroid point as E. Dropping a perpendicular from C to its base AB and though E. Let this perpendicular intersect AB at a point C'.

Then in the right triangle $\triangle AC'C$ (as $AC' = 1$) the Pythagorean theorem gives

$$CC' = \sqrt{2^2 - 1^2} = \sqrt{3}\,.$$

Next in the right triangle $\triangle AC'E$ as $\angle EAC' = \frac{60^\circ}{2} = 30^\circ$ we have

$$\tan(30^\circ) = \frac{EC'}{AC'} = \frac{EC'}{1} \quad \text{so} \quad EC' = \frac{1}{\sqrt{3}}\,.$$

Then using

$$EC = CC' - EC' = \sqrt{3} - \frac{1}{\sqrt{3}} = \frac{2}{\sqrt{3}}\,.$$

This means that the radius of the larger circle is given by

$$1 + EC = 1 + \frac{2}{\sqrt{3}} = \frac{3 + 2\sqrt{3}}{3}\,.$$

Problem 17

Let the two digits in Jack's age be a and b so that $1 \le a \le 9$ and $1 \le b \le 9$ with his age J given by

$$J = 10a + b\,.$$

Then we are told that Bill's age B is

$$B = 10b + a\,.$$

In five years we are told that

$$J + 5 = 2(B + 5) \quad \text{or} \quad J = 2B + 5\,.$$

In terms of a and b this is

$$10a + b = 20b + 2a + 5 \quad \text{or} \quad a = \frac{5 + 19b}{8}\,.$$

Integer solutions to this can be found by taking $b \in \{1, 2, \ldots, 8, 9\}$ and computing a using the above. Doing this in the following `R` code we have

```
bs = seq(1, 9)
as = (5 + 19*bs)/8
print(data.frame(a=as, b=bs))
```

This gives

```
       a b
1  3.000 1
2  5.375 2
3  7.750 3
4 10.125 4
5 12.500 5
6 14.875 6
7 17.250 7
8 19.625 8
9 22.000 9
```

The only "digit" integer solution is $(a, b) = (3, 1)$. This means that $J = 31$ and $B = 13$ so $J - B = 18$.

Problem 18

WWX: working here

Problem 19

WWX: working here

Problem 20

WWX: working here

Problem 21

WWX: working here

Problem 22

WWX: working here

Problem 23

WWX: working here

Problem 24

WWX: working here

Problem 25

WWX: working here

The 2004 AMC 12A Examination

Problem 1

WWX: Working here.

The 2004 AMC 12B Examination

Problem 1

WWX: Working here.

The 2005 AMC 10A Examination

Problem 1

Let m be Mike's bill. Then $0.1m = 2$ so $m = 20$. Let j be Joe's bill. Then $0.2j = 2$ so $j = 10$. This means that

$$m - j = 10\,.$$

Problem 2

We first compute $1 \star 2 = \frac{3}{1-2} = -3$. Then

$$(1 \star 2) \star 3 = (-3) \star 3 = \frac{0}{-3-3} = 0\,.$$

Problem 3

Solving $2x+7 = 3$ gives $x = -2$. Putting that into the second equation gives $-2b-10 = -2$. Solving that for b gives $b = -4$.

Problem 4

Let the width of our rectangle be w and the length l is then $l = 2w$. As the diagonal is of length x we have

$$x^2 = 4w^2 + w^2 \quad \text{so} \quad w = \frac{x}{\sqrt{5}}\,,$$

Thus the area is given by $w(2w) = \frac{2}{5}x^2$.

Problem 5

Dave by himself can buy the windows he needs by buying four (getting one free) and then buying two more to get a total of seven windows. This is a total cost of

$$C_{\text{Dave}} = 6(100) = 600\,.$$

Doug by himself can buy the windows he needs by buying four (getting one free) and then buying three more to get a total of eight windows. This is a total cost of

$$C_{\text{Doug}} = 7(100) = 700\,.$$

The total cost in this way is $600 + 700 = 1300$.

Dave and Doug together can buy the $7 + 8 = 15$ windows they need if they

- buying four (getting one free) and
- buying four (getting one free) and
- buying four (getting one free)

The total cost under this method is then $12(100) = 1200$. Thus the savings is $1300 - 1200 = 100$.

Problem 6

From the mean of the first 20 numbers the sum of the first 20 numbers must be

$$20(30) = 600\,.$$

From the mean of the second 30 numbers the sum of the second 30 numbers must be

$$30(20) = 600\,.$$

The sum of all the 50 numbers is then $600 + 600 = 1200$. The means of all 50 numbers is then $\frac{1200}{50} = 24$.

Problem 7

Let Josh start at the location $x = 0$ (moving rightwards) and Mike start at $x = 13$ (moving leftwards). Then as a function of time Josh's position is given by

$$x_{\text{Josh}}(t) = v_J t\,,$$

and Mike's position is given by

$$x_{\text{Mike}}(t) = 13 - v_M(t - d)\,,$$

where d is the "delay" in Mike's start. Let the meet at a time $t = T$. Then we know that

$$x_{\text{Josh}}(T) = x_{\text{Mike}}(T)\,,$$

or with $v_J = \frac{4}{5}v_M$ this is

$$\frac{4}{5}v_M T = 13 - v_M(T - d)\,.$$

We are also told that when they meet $T = 2(T - d)$ so $d = \frac{T}{2}$. Using this in the above we get

$$\frac{4}{5} v_M T = 13 - \frac{1}{2} v_M T \,.$$

Solving this for $v_M T$ we get

$$v_M T = 10 \,.$$

Now the number of miles that Mike rode when the meet is given by

$$13 - x_{\text{Mike}}(T) = 13 - \left(13 - \frac{1}{2} v_M T\right) = \frac{1}{2} v_M T = 5 \,.$$

Problem 8

WWX: DP

Problem 9

WWX: DP

Problem 10

This is the equation

$$4x^2 + (a+8)x + 9 = 0 \,.$$

To have only one root means that the discriminant is zero or

$$(a+8)^2 - 4(4)(9) = 0 \quad \text{or} \quad a + 8 = \pm 12 \,.$$

This means that $a \in \{-20, 4\}$. The sum of these is then -16.

Problem 11

WWX: DP

Problem 12

WWX: DP

Problem 13

There are $\left(\begin{array}{c}3\\2\end{array}\right) = 3$ inequalities we can form from the given statement. From

$$n^{100} > 2^{200}\,,$$

we get

$$n > 2^2 = 4\,.$$

From

$$(130n)^{50} > n^{100}\,,$$

we get

$$130n > n^2 \quad \text{or} \quad n(n-130) < 0 \quad \text{or} \quad 0 < n < 130\,.$$

Finally from

$$(130n)^{50} > 2^{200}\,,$$

we get

$$130n > 2^4 = 16 \quad \text{so} \quad n > \frac{8}{65}\,.$$

This last inequality is satisfied if $n > 4$ (a condition above) is satisfied. Thus we have

$$4 < n < 130 \quad \text{or} \quad 5 \le n \le 129\,.$$

This is $129 - 5 + 1 = 125$ numbers.

Problem 14

If our number is $\overline{ABC}$ with A, B, and C digits and $A \ge 1$. Then we are told that $B = \frac{1}{2}(A + C)$ so $A + C = 2B$. This means that $A + C$ must be an even number so

- if A is odd then C must be odd
- if A is even then C must be even

In the range $1 \le A \le 9$ we have five odd numbers and four even numbers. If we pick an odd or even number for A then in the range $0 \le C \le 9$ we have five odd numbers and five even numbers. This means that we have

$$5 \times 5 + 4 \times 5 = 45\,,$$

numbers of this form.

Problem 15

Call this number N. We can write N as

$$N = (3 \cdot 2 \cdot 1)(5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)(7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1) = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7\,.$$

Cube divisors must take the form $2^{3p}3^{3q}5^{3r}7^{3s}$. Thus from the above we can have $0 \le p \le 2$, $0 \le q \le 1$, $r = 0$ and $s = 0$. This gives a total of $3 \times 2 = 6$ divisors of the requested form.

Problem 16

Let our two digit number be $n = (ab) = 10a + b$ with a and b digits such that $1 \le a \le 9$ and $0 \le b \le 9$. Then for n we have

$$10a + b - (a + b) = 9a\,.$$

To have a units digit of six when $1 \le a \le 9$ means that $a = 4$ so $9a = 36$. Then our number $n = (4b)$ and there are ten such numbers.

Problem 17

These five numbers have $\binom{5}{2} = \frac{5 \cdot 4}{2} = 10$ pairs. The sums of these pairs are

$$\begin{aligned}
3 + 5 &= 8\\
3 + 6 &= 9\\
3 + 7 &= 10\\
3 + 9 &= 12\\
5 + 6 &= 11\\
5 + 7 &= 12\\
5 + 9 &= 14\\
6 + 7 &= 13\\
6 + 9 &= 15\\
7 + 9 &= 16\,.
\end{aligned}$$

Looking at these numbers we ask what five element arithmetic sequences can we form from them. It looks like we we might be able to have common differences h of

$$\begin{aligned}
9 - 8 &= 1\\
10 - 8 &= 2\\
12 - 8 &= 4\\
12 - 9 &= 3\,.
\end{aligned}$$

The possible sequences with $h = 1$ are

$$S_1 = \{8, 9, 10, 11, 12\}$$
$$S_2 = \{9, 10, 11, 12, 13\}$$
$$S_3 = \{10, 11, 12, 13, 14\}$$
$$S_4 = \{11, 12, 13, 14, 15\}$$
$$S_5 = \{12, 13, 14, 15, 16\}.$$

The possible sequences with $h = 2$ *might* be

$$S_6 = \{8, 10, 12, 14, 16\}$$
$$S_7 = \{9, 11, 13, 15, 17\}.$$

Note that S_7 cannot exist as we can't form the sum of 17 from our number pairs. Other common differences $h > 2$ are thus not possible.

If we start with the sequence S_1 we see that the only way to form an 8 we need to sum 3 and 5 and to form 9 we need to sum 3 and 6. To form a 10 we need to sum 3 and 7 and thus this we have "used" the number 3 more than two times and this sequence is not possible given the constraints of the problem.

If we consider with the sequence S_2 we see that the only way to form an 9 we need to sum 3 and 6, to form 10 we need to sum 3 and 7, to form a 11 we need to sum 5 and 6. To form a 12 we must sum a 5 and 7, and for form a 13 we sum a 6 and 7. Thus 7 is used three times and this sequence is not possible.

If we consider with the sequence S_3 we see that the only way to form an 10 we need to sum 3 and 7, to form 11 we need to sum 5 and 6, to form a 12 we can sum either a 3 and 9 or a 5 and 7. To form a 13 we must sum a 6 and 7, and to form a 14 we must sum a 5 and 9. In the second case we would have used the 5 three times. In the first case we have the solution

3
5 6
7 9

This satisfies the conditions of the problem and has a middle term of 12.

If we consider with the sequence S_4 it will have three of one of the base numbers and thus does not give a valid sequence for this problem.

If we consider with the sequence S_5 it will have three 9s.

Finally if we consider the sequence S_6 it will have three of one of the base numbers and thus does not give a valid sequence for this problem.

Problem 18

All probability problems should be first attempted by drawing a tree of possible outcomes. From each outcome we can then draw two branches the top one indicating that A won that game and the bottom one indicating that B won that game. We can do this until one of the teams wins the series. Now for the first game either A or B wins each with a probability of $\frac{1}{2}$. For the second game we are told that B wins. All subsequent games have A or B winning each with a probability of $\frac{1}{2}$.

If A wins the first game these are the games

$$ABAA\,, ABABA\,, ABABB\,, ABBAA\,, ABBAB\,, ABBB$$

Given that we know that B wins the second game these have probabilities of $\frac{1}{2^i}$ for i the number of games played (minus the second game where B always wins) and thus

$$\frac{1}{8}\,, \frac{1}{16}\,, \frac{1}{16}\,, \frac{1}{16}\,, \frac{1}{16}\,, \frac{1}{8}\,.$$

Only the first, second, and fourth sequences in the above have A winning the series.

If B wins the first game these are the games

$$BBAAA\,, BBAAB\,, BBAB\,, BBB\,.$$

Again given that we know that B wins the second game these have probabilities of $\frac{1}{2^i}$ for i the number of games played (minus the second game where B always wins) and thus

$$\frac{1}{16}\,, \frac{1}{16}\,, \frac{1}{8}\,, \frac{1}{4}\,.$$

Only the first of these has A winning the series.

From the above outcomes if we add up the probability that A wins the series we get

$$\frac{1}{8} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} = \frac{5}{16}\,.$$

From the above outcomes the probability that B wins the first game (and A wins the series) is $\frac{1}{16}$. Thus the probability of this event (given A wins the series) is

$$\frac{\frac{1}{16}}{\frac{5}{16}} = \frac{1}{5}\,.$$

Problem 19

WWX: DP

Problem 20

WWX: DP

Problem 21

WWX: DP

Problem 22

WWX: DP

Problem 23

WWX: DP

Problem 24

WWX: DP

Problem 25

WWX: DP

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Problem 1

Problem 2

Problem 3

Problem 4

Problem 5

Problem 6

Problem 7

Problem 8

Problem 9

Problem 10

Problem 11

Problem 12

Problem 13

Problem 14

Problem 15

Problem 16

Problem 17

Take the logarithm (base e) of each equation to solve for each variable. We have

$$a = \frac{\ln(5)}{\ln(4)}$$

Then the product is given by

$$abcd = \frac{\ln(8)}{\ln(4)} \,.$$

Using the "change of base" formula

$$\log_b(a) = \frac{\log_c(a)}{\log_c(b)} \,, \tag{705}$$

in the numerator and denominator of the expression for $abcd$ (with a new base $c = 2$) we can write the product above as

$$abcd = \frac{\frac{\log_2(8)}{\log_2(e)}}{\frac{\log_2(4)}{\log_2(e)}} = \frac{3}{2} \,.$$

Problem 18

Problem 19

Problem 20

Problem 21

Problem 22

Problem 23

Problem 24

Problem 25

The 2005 AMC 12A Examination

Problem 1

The 2005 AMC 12B Examination

Problem 1

The 2006 AMC 10A Examination

Problem 1

Let s be the number of sandwiches and t the number of sodas then this purchase will cost

$$3s + 2t\,,$$

when $(s, t) = (5, 8)$. We find this number to be 31.

Problem 2

Note that $h \otimes h = h^3 - h$ so that

$$h \otimes (h \otimes h) = h^3 - (h^3 - h) = h\,.$$

Problem 3

Let m be Mary's age and a be Alice's age. Then we are told that $\frac{m}{a} = \frac{3}{5}$ so

$$m = \frac{3}{5}a\,.$$

If $a = 30$ then $m = 18$.

Problem 4

The largest sum we can get from the digits in the minutes is $5 + 9 = 14$. To this we would need to add the largest digit sum from the numbers

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\,.$$

Which is nine. Thus the largest total digit sum is $14 + 9 = 23$.

Problem 5

An \$8 dollar pizza cut into 8 pieces has a cost per slice of $\frac{8}{8} = 1$ dollar. As Dave ate $\frac{1}{2}$ of the pizza plus one slice of cheese he ate $4 + 1 = 5$ pieces. Since only Dave wants anchovy's he should have to pay the extra \$2 and thus Dave should pay $5 + 2 = 7$ dollars. Doug ate $8 - 5 = 3$ slices and thus should pay \$3. The difference between these two is \$4.

Problem 6

We are told that $x \neq 0$ so we can divide by x^7 to get

$$7^{14}x^7 = 14^7 \quad \text{or} \quad x^7 = \frac{14^7}{(7^2)^7} = \left(\frac{14}{7^2}\right)^7$$

Thus we have that

$$x = \frac{14}{7^2} = \frac{2}{7}\,.$$

Problem 7

To help determine lengths of the segments involved let the point of intersection between the vertical line and the segment CD be E and the point of intersection between the vertical line and AB be F.

Now let $DE = FB = h$ be the length of one of the horizontal segments in AB and CD. Let the vertical distance from either AB or CD to get to the horizontal segment of length y be denoted v. Then starting at A and "walking" counter clockwise around the left-most hexagon we have the lengths

$$18 - h\,, v\,, y\,, v\,, h\,, 8\,.$$

In the same way starting at F and 'walking" counter clockwise around the right-most hexagon we have the lengths

$$h\,, 8\,, 18 - h\,, v\,, y\,, v\,.$$

Now to form the square suggested we will take the right-most hexagon and shift it "up" and "leftwards" to place it on top of the left-most hexagon.

Then the vertical edge length in this square is $8 + v$ and the horizontal edge length is $18 - h$. Walking across the square "in the middle" we see that $y = h$ and that the side length can be represented as $2h$. Setting this equal to $18 - h$ we can solve for h to get $h = 6$. We can also determine that $v = 4$.

Problem 8

Putting these two points into the equation of the curve gives

$$3 = 4 + 2b + c$$
$$3 = 16 + 4b + c\,,$$

or

$$\begin{aligned} 2b + c &= -1 \\ 4b + c &= -13 \,. \end{aligned}$$

From the first equation we have $2b = -1 - c$. If we put that into the second equation we can solve for c to find $c = 11$.

Problem 9

Consider starting the sum at k and summing a total of n terms to get 15. This sum is

$$S \equiv \sum_{i=k}^{k+(n-1)} i = \sum_{i=0}^{n-1} (i+k) = \sum_{i=0}^{n-1} i + \sum_{i=0}^{n-1} k = \frac{(n-1)n}{2} + kn = \frac{n(n+2k-1)}{2} \,.$$

Lets see for which positive integer k and $n \geq 2$ we can get $S = 15$.

If $n = 2$ then the above is

$$2k + 1 = 15 \quad \text{so} \quad k = 7 \,,$$

and the terms are $7\,, 8$.

If $n = 3$ then the above is

$$\frac{3}{2}(3 + 2k - 1) = 15 \quad \text{so} \quad k = 4 \,,$$

and the terms are $4\,, 5\,, 6$.

It seems like given n we are finding k. Solving the above for k gives

$$2k - 1 = \frac{30}{n} - n \,.$$

Then for k to be an integer means that n is a factor of 30. Thus we can skip to $n = 5$ and find that $k = 1$ and the terms are $1\,, 2\,, 3\,, 4\,, 5$.

Thus there are three such sequences of the given form.

Problem 10

Write this requirement as

$$\sqrt{120 - \sqrt{x}} = n \,,$$

with n an integer. As the function $f(v) = \sqrt{v} \geq 0$ for all $v \geq 0$ we have that $n \geq 0$ as a lower bound for n. If we square this and solve for $\sqrt{x}$ we get

$$120 - n^2 = \sqrt{x} \,. \tag{706}$$

Again using the fact that $\sqrt{v} \geq 0$ we have that

$$120 - n^2 \geq 0 \quad \text{so} \quad n^2 \leq 120 \quad \text{so} \quad n \leq 10\,,$$

for an upper bound on n. Finally note that for each value of n in the domain $0 \leq n \leq 10$ we can solve Equation 706 for x (by squaring). Thus there are 11 integer solutions for x.

Problem 11

If we expand the left-hand-side and cancel the $x^2 + y^2$ terms from both sides we get

$$2xy = 0\,.$$

This is solved by $x = 0$ or $y = 0$ which are the equations for two lines.

Problem 12

The area of the left-most region would be that of a semicircle of radius eight or $\frac{1}{2}\pi(8^2) = 32\pi$.

The area of the second region would be that of a semicircle of radius eight *plus* $\frac{1}{4}$ the area of a circle of radius four or

$$\frac{1}{4}(\pi(4^2)) = 4\pi\,.$$

The second figure has a larger area by 4π.

Problem 13

Let W be the amount won. Then the expected win W can be computed as

$$E(W) = E(W|\text{odd})P(\text{odd}) + E(W|\text{even})P(\text{even}) = \frac{1}{2}E(W|\text{even})\,.$$

Now let M be the event that the second roll matches the number on the first roll (and M' that it does not) then we compute

$$\begin{aligned} E(W|\text{even}) &= E(W|\text{even}, M)P(M) + E(W|\text{even}, M')P(M') \\ &= E(W|\text{even}, M)P(M) = \frac{W}{6}\,. \end{aligned}$$

Thus we have shown that

$$E(W) = \frac{W}{12}\,.$$

If we have to pay \$5 to play this game it will be fair when $E(W) - 5 = 0$ or $W = 60$.

Problem 14

We are told that the outside diameter of the rings are given by

$$d_i = 20 - (i-1) = 21 - i \quad \text{for} \quad 1 \le i \le I\,,$$

where I is the last ring. We are also told that $d_I = 3$. Using the above we find that $I = 18$.

Now starting at the topmost ring the total distance D to the bottom most ring can be computed from

$$D = (d_1 - 2) + (d_2 - 2) + (d_3 - 2) + \cdots\,.$$

In the above we start at the top of ring i then move to the bottom of ring i (for a distance of d_i) and then back upwards two centimeters corresponding to the thicknesses of rings i and $i+1$. The only time we *don't* have to subtract this two is on the last ring I. Thus the full sum representing the total distance is

$$D = (d_1 - 2) + (d_2 - 2) + (d_3 - 2) + \cdots + (d_{17} - 2) + d_{18}\,.$$

We can write this as

$$D = \sum_{i=1}^{18} (d_i - 2) + 2\,.$$

Evaluating this as

$$D = \sum_{i=1}^{18} (21 - i - 2) + 2 = \sum_{i=1}^{18} (19 - i) + 2 = \sum_{j=1}^{18} j + 2 = \frac{18(19)}{2} + 2 = 173\,.$$

Problem 15

From the problem statement we are told that the velocities of the two runners are given by $v_O = 250$ and $v_K = 300$ (in meters per minute) for Odell and Kershaw respectively. The radii of the track these two runners run on is also given by $r_O = 50$ and $r_K = 60$ (in meters). We are told that both runners run for a total of thirty minutes. Now the angular location if each runs for t minutes is given by

$$\theta_O = -\frac{v_O t}{2\pi r_O}(2\pi) = -\frac{v_O t}{r_O}$$
$$\theta_K = +\frac{v_K t}{2\pi r_K}(2\pi) = \frac{v_K t}{r_K}\,.$$

Here we have taken the linear distance that each has run vt and divided it by the circumference of the track $2\pi r$ to determine the number of rotations. We then multiply this by 2π to get the angular location of Odell and Kershaw for each time t. They will pass each other when

$$\theta_K - \theta_O = 2\pi n\,,$$

for integers n. This is equivalent to

$$\frac{v_K t}{r_K} + \frac{v_O t}{r_O} = 2\pi n \,.$$

Solving for t we get

$$t = \frac{2\pi n}{\frac{v_K}{r_K} + \frac{v_O}{r_O}} \,.$$

For the numbers given we have $\frac{v_K}{r_K} + \frac{v_O}{r_O} = \frac{300}{60} + \frac{250}{50} = 10$. Thus the above is

$$t = \frac{2\pi n}{10} = \frac{\pi n}{5} \,.$$

We are told that $0 \le t \le 30$ so that

$$0 \le \frac{\pi n}{5} \le 30 \quad \text{or} \quad 0 \le n \le \frac{150}{\pi} \,.$$

Now $\frac{150}{\pi} = 47.74648$ so the two runners cross 47 times.

Problem 16

Drop a perpendicular from A to the segment BC. Let that point of intersection be denoted a O. Make the point O the origin of an $x-y$ Cartesian coordinate system such that $O = (0, 0)$ and with the segment OA lies along the y-axis. Next denote the center of the larger "bottom" circle as $O_B = (0, 2)$ and the center of the smaller "top" circle as $O_T = (0, 5)$. From O_T, O_B, and O draw lines that will perpendicularly intersect the segment AC at the points P, Q, and R respectively. Thus segments $O_T P$ and $O_B Q$ are drawn to their circles tangents.

To start solving this problem we note that we have three similar right triangles namely

$$\triangle APO_T \sim \triangle AQO_B \sim \triangle ARO \,. \tag{707}$$

From these similar triangles we have that $\angle AO_T P = \angle AO_B Q = \angle AOR$. Denote this common angle as θ. As

$$\angle AOC = 90^\circ = \angle AOR + \angle ROC = \theta + \angle ROC \,,$$

we have $\angle ROC = 90^\circ - \theta$. Looking at the value of $90^\circ - \theta$ in the right triangle $\triangle APO_T$ we see that

$$\angle ROC = \angle O_T AP \,.$$

This means that we have another similar right triangle relationship

$$\triangle ARO \sim \triangle ORC \,. \tag{708}$$

Using $\triangle APO_T \sim \triangle AQO_B$ from Equation 707 we have

$$\frac{AO_T}{O_T P} = \frac{AO_B}{O_B Q} \quad \text{or} \quad \frac{AO_T}{1} = \frac{AO_T + 3}{2} \,.$$

Solving this for AO_T we find $AO_T = 3$. Thus we now know that

$$AO = 3 + 1 + 2 + 2 = 8\,,$$

for the height of the triangle $\triangle ABC$.

Using $\triangle APO_T \sim \triangle ARO$ from Equation 707 we have

$$\frac{OR}{AO} = \frac{O_T P}{AO_T} \quad \text{or} \quad \frac{OR}{8} = \frac{1}{3}\,.$$

Solving this for OR gives $OR = \frac{8}{3}$.

Now using Equation 708 we have

$$\frac{OC}{OR} = \frac{AO}{AR} \quad \text{or} \quad \frac{OC}{\frac{8}{3}} = \frac{8}{\sqrt{8^2 - OR^2}}\,,$$

but we know $OR = \frac{8}{3}$ so we can solve the above for OC and find $OC = 2\sqrt{2}$. This is one-half the base of the triangle $\triangle ABC$.

Thus the area of triangle $\triangle ABC$ is given by "one-half of the base times the height" which in this case is given by

$$\frac{1}{2}bh = \frac{1}{2}(4\sqrt{2})8 = 16\sqrt{2}\,.$$

Problem 17

As $AC = 2$ and B and C trisect the segment AD we have $AB = BC = CD = 1$.

Lets next place this rectangle in an x-y Cartesian coordinate system where $A = (0,0)$, $D = (3,0)$, $E = (3,2)$, and $H = (0,2)$. From the dimensions given and deduced we have that $B = (1,0)$, $C = (2,0)$, $G = (1,2)$, and $F = (2,2)$.

We now ask what are the x-y locations of the points X, Y, Z, and W. To find these we will look for intersections of various lines. We have

- Line AF is given by
$$y = \left(\frac{2-0}{2-0}\right) x = x\,.$$
- Line BE is given by
$$y - 0 = \left(\frac{2-0}{3-1}\right)(x-1) = x - 1\,.$$
- Line HC is given by
$$y - 2 = \left(\frac{0-2}{2-0}\right)(x-0) = -x\,,$$
or $y = 2 - x$.

- Line GD is given by

$$y - 2 = \left(\frac{0-2}{3-1}\right)(x-1) = -(x-1)\,,$$

  or $y = 3 - x$.

To find the location of the points above we look for the intersection of two specific lines. The point X is given by the intersection of lines AF and HC which gives $X = (1,1)$. The point Z is given by the intersection of lines BE and GD which gives $Z = (2,1)$. The point Y is given by the intersection of lines BE and HC which gives $Y = \left(\frac{3}{2}, \frac{1}{2}\right)$. Finally, by symmetry (or line intersections) we can conclude that $W = \left(\frac{3}{2}, 2 - \frac{1}{2}\right) = \left(\frac{3}{2}, \frac{3}{2}\right)$.

We can now compute the area we seek from

$$[XYZW] = [ADW] - [ACX] - [BDZ] + [BCY]\,.$$

Computing each of these gives

$$[XYZW] = \frac{1}{2}(3)\left(\frac{3}{2}\right) - \frac{1}{2}(2)(1) - \frac{1}{2}(2)(1) + \frac{1}{2}(1)\left(\frac{1}{2}\right) = \frac{1}{2}\,,$$

when we simplify.

Problem 18

Consider the two letters as "one unit". Then this unit has $26 \times 26 = 26^2$ possible values for its value. If we place four digits down we can do this in 10^4 ways. We can then place the two letter unit in one of the five locations "around" the four digits i.e. before the first digit, after the first digit, after the second digit, etc. This gives

$$10^4 \times 26^2 \times 5\,,$$

license plates.

Problem 19

Let the three angles be $\theta_0 - h$, θ_0, and $\theta_0 + h$. Then the sum of these angles must be 180 so

$$3\theta_0 = 180 \quad \text{so} \quad \theta_0 = 60\,.$$

This means that the three angles are

$$60 - h\,, 60\,, 60 + h\,.$$

Now we must have $60 - h > 0$ so $h < 60$. We cannot have $h = 0$ or else all three angles are equal and not distinct. We can't have $h = 60$ for then one angle is zero. Thus the smallest we can have for h is $h = 1$ and the largest is $h = 59$ for 59 triangles.

Problem 20

Now a number when divided by five will have a remainder that is one of

$$0\,,1\,,2\,,3\,,4\,.$$

Note that if I draw two numbers that have the *same* remainder (when divided by five) the difference between these two numbers will have remainder of zero and will thus be divisible by five. If I draw six numbers I must draw two numbers that have the same remainder (this is the "Pigeonhole Principle") and thus six numbers will always have a pair with a difference that is divisible by five. Thus the probability of this happening is one.

Problem 21

There are $9 \times 10 \times 10 \times 10 = 9000$ four digit numbers. Numbers *without* the digits two or three will have the first digit drawn from

$$\{1\,,4\,,5\,,6\,,7\,,8\,,9\}\,,$$

or $9 - 2 = 7$ and the other three digits drawn from the above set plus $\{0\}$ or eight numbers. Thus there are

$$7 \cdot 8^3 = 3584\,,$$

four digit numbers without a two or a three. The number of numbers with at least one two or three is then

$$9000 - 3584 = 5416\,.$$

Problem 22

Let $p = 300$ and $g = 210$ then we are asked to find

$$\min_{x,y}|xp - yg|\,,$$

over integer x and y since that would be the smallest amount we could "transfer" between the two farmers when they exchange pigs and goats. It can be shown that the solution to this problem is given by the greatest common divisor of p and g often denoted as GCD(p, g). One way to find this value is using the Euclidean algorithm which involves repeated integer divisions. The steps of this algorithm for the numbers given here are

$$300 = 1 \cdot 210 + 90 \tag{709}$$

$$210 = 2 \cdot 90 + 30\,. \tag{710}$$

Now we start with Equation 710 written as

$$30 = 1 \cdot 210 - 2 \cdot 90\,.$$

Then we replace the 90 in that equation with the remainder from Equation 709 to get

$$30 = 1 \cdot 210 - 2 \cdot (300 - 1 \cdot 210) = 3 \cdot 210 - 2 \cdot 300\,.$$

This means that 30 can be obtained by giving three goats and receiving two pigs.

Problem 23

Draw the segments AC and DB. Then because of the similarity between the right triangles $\triangle ACE \sim \triangle BDE$ we have

$$\frac{AC}{AE} = \frac{BD}{BE},$$

or using given lengths

$$\frac{3}{5} = \frac{8}{BE}.$$

Thus $BE = \frac{40}{3}$. In the right triangle $\triangle CAE$ we have

$$CE = \sqrt{5^2 - 3^2} = 4.$$

In the right triangle $\triangle EDB$ we have

$$ED = \sqrt{\left(\frac{40}{3}\right)^2 - 8^2} = \frac{2^5}{3}.$$

This means that

$$CD = CE + ED = 4 + \frac{2^5}{3} = \frac{44}{3}.$$

Problem 24

Imagine the cube sitting on the top of a table. Note that four of the vertices of the octahedron are located on the vertical faces of the cube and two vertices are located on the top and bottom of the cube. Notice that the four corners on the vertical faces are themselves corners of a square. Lets call this square B for the "base" of the octahedron (its a "base" in the fact that the full octahedron is completed by placing pyramids on the top and bottom of the "base"). If the original cube has a side length of s then these four corners are located $\frac{s}{2}$ from the vertical edges of the cube. This means that the side of B is given by

$$\sqrt{\left(\frac{s}{2}\right)^2 + \left(\frac{s}{2}\right)^2} = \frac{s}{\sqrt{2}}.$$

Notice that the "height" of each pyramid on top of this base B is $\frac{s}{2}$.

Now recall that the volume of a pyramid is $\frac{1}{3}A_B h$ where A_B is the area of the "base" B and h is the height. Thus in this case this is

$$\frac{1}{3}\left(\frac{s}{\sqrt{2}}\right)^2 \frac{s}{2} = \frac{s^3}{12}.$$

The total volume of the octahedron is twice this or $\frac{s^3}{6}$. If $s = 1$ then this is $\frac{1}{6}$.

Problem 25

Consider a cube with side length s placed in an x-y-z Cartesian coordinate axis where we specify the eight vertices as the points $A = (0, 0, 0)$, $B = (0, s, 0)$, $C = (s, s, 0)$, $D = (s, 0, 0)$, $E = (0, 0, s)$, $F = (0, s, s)$, $G = (s, s, s)$, and $H = (s, 0, s)$. By symmetry we can assume the bug starts at the vertex H and on the first step goes to vertex E. At vertex E there is a $\frac{2}{3}$ chance we will select a vertex that we have not already visited. Again by symmetry (meaning we could rotate whatever path was actually taken on the cube by the bug onto the described path) we can assume that from E the bug steps to vertex F.

At this point to avoid vertices already visited the bug must either step to G or B. Each happens with a $\frac{1}{3}$ probability. Stepping to G would mean that the bug has visited all vertices on the plane $z = s$ and thus must step "downwards" to start visiting vertexes on the plane $z = 0$. Stepping to B means that the bug must then walk in a "circle" on $z = 0$ such that it can finally step upwards into vertex G to finish its visit of all vertices. From $F \to G$ or $F \to B$ are then two different paths each with probabilities of success. We will compute the probability the bug successfully visits each vertex under each choice.

In stepping from F to G the bug must then step "downwards" and then sequentially step through all vertices on the plane $z = 0$. This would happen with probability

$$\frac{1}{3} \times \frac{2}{3} \times \frac{1}{3} \times \frac{1}{3} = \frac{2}{3^4}.$$

In stepping from F to B the bug first steps "downwards" and then sequentially step through all vertices on the plane $z = 0$ ending at C and then step back upwards to visit G. This would happen with probability

$$\frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} = \frac{1}{3^4}.$$

Remembering the original probability of $\frac{2}{3}$ in stepping from E to F and the probability of $\frac{1}{3}$ in stepping from F to either G or B, the total probability that the bug visits every vertex is then

$$\frac{2}{3} \times \frac{1}{3} \left(\frac{2}{3^4} + \frac{1}{3^4} \right) = \frac{2}{3^5} = \frac{2}{243}.$$

The 2006 AMC 10B Examination

Problem 1

Call this expression E. Then we can write E as

$$E = \sum_{k=0}^{1002} (-1)^{2k+1} + \sum_{k=1}^{1003} (-1)^{2k} = \sum_{k=0}^{1002} (-1) + \sum_{k=1}^{1003} 1 = -1(1003) + 1003 = 0.$$

Problem 2

We have $4 \spadesuit 5 = 9 \cdot (-1) = -9$ and thus

$$3 \spadesuit (4 \spadesuit 5) = 3 \spadesuit (-9) = (3 - 9) \cdot (3 + 9) = -72\,.$$

Problem 3

Let c and p be the points scored by the Cougars and Panthers respectively. Then we are told that

$$c + p = 34 \tag{711}$$
$$c - p = 14\,. \tag{712}$$

If we sum these two equations we get $2c = 48$ or $c = 24$. If we put this value into Equation 712 we get $p = 10$.

Problem 4

This would be

$$\frac{\pi\left(\frac{3}{2}\right)^2 - \pi\left(\frac{1}{2}\right)^2}{\pi\left(\frac{1}{2}\right)^2} = 9 - 1 = 8\,.$$

Problem 5

Note that the area of the bounding square must be larger than the sum of the areas of the two rectangles or

$$2(3) + 3(4) = 18\,.$$

I start by laying the 3×4 rectangle (R_1) at the corner of a Cartesian coordinate plane with the side of length four along the x-axis and the side of length three along the y-axis.

First I could place the 2×3 rectangle (R_2) "to the right" of R_1 and sharing the common side of length three. This would take up $4 + 2 = 6$ units along the x-axis and the bounding square would have area $6 \times 6 = 36$.

Second I could place the 2×3 rectangle (R_2) "above" of R_1 with the side of length three adjacent to the side of length four in R_1. Then this takes up $3 + 2 = 5$ vertical units along the y-axis and the bounding square would have area $5 \times 5 = 25$.

The smaller of these two is 25.

Problem 6

The radius of each circle must be $\frac{1}{2}\left(\frac{2}{\pi}\right) = \frac{1}{\pi}$. Then the perimeter we seek is the sum of the perimeters of four semicircles or two circles (with this radius) or

$$2\left(2\pi\left(\frac{1}{\pi}\right)\right) = 4\,.$$

Problem 7

WWX: Here.

The 2006 AMC 12A Examination

Problem 1

The 2006 AMC 12B Examination

Problem 1

The 2007 AMC 10A Examination

Problem 1

WWX: Working here.

Problem 2

From the definitions given we have

$$\frac{6@2}{6\#2} = \frac{6(2) - 4}{8 - 6(4)} = -\frac{8}{16} = -\frac{1}{2} \,.$$

Problem 3

WWX: Working here.

Problem 4

Two consecutive odd numbers can be written as $2n+1$ and $2n+3$. For the given relationship we must have

$$2n + 3 = 3(2n + 1) \,.$$

Solving this for n gives $n = 0$ so the two numbers are one and three. Their sum is then four.

Problem 5

WWX: Working here.

Problem 6

WWX: Working here.

Problem 7

WWX: Working here.

Problem 8

WWX: Working here.

Problem 9

Write the first equation as

$$3^a = 81^{b+2} = (9^2)^{b+2} = (3^4)^{b+2} = 3^{4b+8}\,,$$

thus we must have $a = 4b + 8$. Write the second equation as

$$5^{a-3} = 125^b = 5^{3b}\,,$$

thus we must have $a - 3 = 3b$. Using what we know about a from before we get

$$(4b + 8) - 3 = 3b\,.$$

Solving we get $b = -5$ and $a = -12$. Thus $ab = 60$.

Problem 10

WWX: From here down.

Problem 23

As the circles are symmetric the points E and F are directly above the points A and B respectively. This means that $AE \perp EF$ and $EF \perp BF$. As the triangle $\triangle ACO$ is a right triangle we have

$$OC = \sqrt{AO^2 - AC^2} = \sqrt{(2\sqrt{2})^2 - 2^2} = 2\,.$$

Now the area of the desired region is the area of rectangle $AEFB$ *minus* the area of the two right triangles $\triangle ACO$ and $\triangle ODB$ and *minus* the area of the "sectors" AEC and BFD. From the known side lengths we have the area of the two triangles given by

$$[ACO] = [ODB] = \frac{1}{2}(2)(2) = 2\,.$$

We now seek to determine the area of the two sectors. Note that

$$\tan(\angle CAO) = \frac{OC}{AC} = \frac{2}{2} = 1\,,$$

thus $\angle CAO = \frac{\pi}{4}$. Then

$$\angle EAC = \angle EAO - \angle CAO = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}\,.$$

The area of each sector is then

$$\frac{\frac{\pi}{4}}{2\pi}(\pi(2^2)) = \frac{\pi}{2}\,.$$

Using the above "subtraction formula" the area of the desired region is given by

$$2(2(2\sqrt{2})) - 2(2) - 2\left(\frac{\pi}{2}\right) = 8\sqrt{2} - 4 - \pi\,.$$

Problem 24

If we square the given expression we get

$$4^2 = a^2 + 2 + a^{-2} \quad \text{so} \quad a^2 + a^{-2} = 16 - 2 = 14\,.$$

Squaring again gives

$$14^2 = a^4 + 2 + a^{-4} \quad \text{so} \quad a^4 + a^{-4} = 14^2 - 2 = 194\,.$$

Problem 25

Consider n_3 a value of "n" that has three digits. Now if n_3 has three digits then at most each can be nine so we have

$$S(n_3) \le 9 + 9 + 9 = 3(9) = 27\,.$$

This means that $S(n_3)$ has at most two digits and from the above result by looking at all numbers less than 27 we can conclude that $S(S(n_3)) \le S(19) = 10$. This means that

$$n_3 + S(n_3) + S(S(n_3)) \le 999 + 27 + 10 = 1036 < 2007\,,$$

Thus there are no three digit numbers that solve this equation.

Clearly $S(n)$ and $S(S(n))$ are both greater than zero. Thus if $n \ge 2007$ the left-hand-side is larger than the right-hand-side and there can be no solution. Thus $n < 2007$ to have a solution.

If $n < 2007$ we have that

$$S(n) \le S(1999) = 28\,,$$

and if $n \le 28$ then $S(n) \le 10$. Then using

$$n = 2007 - S(n) - S(S(n)) \ge 2007 - 28 - 10 = 1969\,,$$

and the range of n where solutions must lie is $1969 \le n \le 2006$. We could check each of these values of n one at a time (there are only 38 of them). This can be done with the simple `R` code (or by hand)

```
S = function(n){
  digit_sum = 0
  while ( n!=0 ) {
    digit_sum = digit_sum + (n %% 10)
    n = ( n - (n %% 10) )/10
  }
  digit_sum
}

ns = seq(1969, 2006)
S_n = sapply(ns, S)
S_S_n = sapply(S_n, S)
ns + S_n + S_S_n
```

Running the above gives

```
 [1] 2001 1995 1998 2001 1995 1998 2001 2004 2007 2010 2013 2007 2010 2004 2007
[16] 2010 2013 2016 2019 2022 2025 2019 2013 2016 2019 2022 2025 2028 2031 2034
[31] 2037 2004 2007 2010 2013 2016 2019 2022
```

Counting the number of these that equal 2007 we find four.

The 2007 AMC 10B Examination

Problem 1

Each bedroom has two walls with areas $12 \times 8 = 96$ and $10 \times 8 = 80$ square-feet. This gives an area of

$$2(96 + 80) = 352 .$$

Removing the area of the doorways and windows gives $352 - 60 = 292$. As we have three bedrooms the total area that must be painted is $3(292) = 876$.

Problem 2

From the definition we have

$$3 \star 5 = 8(5) = 40$$
$$5 \star 3 = 8(3) = 24 ,$$

so $(3 \star 5) - (5 \star 3) = 40 - 24 = 16$.

Problem 3

The average gas mileage would be

$$\frac{2(120)}{\frac{120}{30} + \frac{120}{20}} = 24\,,$$

miles-per-gallon.

Problem 4

We have $\angle AOC = 360 - 140 - 120 = 100$. The arch length $\widehat{AC} = \angle AOC = 100$ and

$$\angle ABC = \frac{1}{2}\widehat{AC} = 50\,.$$

Problem 5

Denoting this information as "set membership" we have

$$\begin{aligned} A &\subset B \\ C &\subset B \\ D &\subset A \\ C &\subset D\,. \end{aligned}$$

Putting the third of these into the first gives

$$\begin{aligned} &D \subset A \subset B \\ &C \subset B \\ &C \subset D\,. \end{aligned}$$

Putting the third of these into the first gives

$$\begin{aligned} &C \subset D \subset A \subset B \\ &C \subset B\,. \end{aligned}$$

Now the second condition is a consequence of the first condition. Reading the answers with this information we see that D is correct.

Problem 6

If c, i, and b are the number of questions correct, incorrect, and blank respectively then we are told that the score is given by

$$6c + 0i + 1.5b = 6c + 1.5b\,.$$

We know that Sarah will leave the last three unanswered so $b = 3$ so her score is

$$6c + 4.5\,.$$

To have this larger than 100 means that

$$c \geq \frac{100 - 4.5}{6} = 15.9167\,.$$

Thus Sarah must answer at least 16 problems correctly.

Problem 7

From Equation 5 the sum of the interior angles of this pentagon must be $180°(n-2) = 540°$ when $n = 5$.

Assume the common side length is l. Now if we draw this pentagon in an x-y Cartesian coordinate plane then from the given angles at A and B we can place $A = (0, l)$, $B = (0, 0)$, $C = (l, 0)$, and $E = (l, l)$. Now to have $ED = CD = l$ the point D will need to be on the perpendicular bisector of AB which means that its y coordinate is located at $\frac{l}{2}$. Dropping a perpendicular from D to the x-axis (called D') we form the right triangle $\triangle CD'D$ and note that

$$\sin(\angle DCD') = \frac{l/2}{l} = \frac{1}{2} \quad \text{so} \quad \angle DCD' = 30°\,.$$

This means that $\angle DCB = 180° - \angle DCD' = 150°$. By symmetry this is the same as the angle $\angle E$.

Problem 8

From the constrains that $0 \leq a < b < c \leq 9$ and that b must be a natural number we have that if

- $a = 0$ then c must be even so $c \in \{2, 4, 6, 8\}$.
- $a = 1$ then c must be odd so $c \in \{3, 5, 7, 9\}$.
- $a = 2$ then c must be $c \in \{4, 6, 8\}$.
- $a = 3$ then c must be $c \in \{5, 7, 9\}$.
- $a = 4$ then c must be $c \in \{6, 8\}$.
- $a = 5$ then c must be $c \in \{7, 9\}$.
- $a = 6$ then c must be $c \in \{8\}$.
- $a = 7$ then c must be $c \in \{9\}$.

Counting these up we have

$$4 + 4 + 3 + 3 + 2 + 2 + 1 + 1 = 20\,,$$

numbers of this form.

Problem 9

If we count the number of letter "s" we find in the string we find twelve. Then we would move

- Replace the first "s" with the letter 1 to its right
- Replace the second "s" with the letter $1 + 2 = 3$ to its right
- Replace the third "s" with the letter $1 + 2 + 3 = 6$ to its right

Then we replace the kth "s" with the letter

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2}\,,$$

to its right. For the 12th "s" this number is $\frac{12(13)}{2} = 78$. As $78 \equiv 0 \pmod{26}$ the replacement letter has "wrapped around" and is back at "s". Thus this "s" is replaced with another one.

Problem 10

If we draw the segment BC and imagine the point A "above" (or "below") the segment then to have an area of one means that this triangles height must satisfy

$$\frac{1}{2}(BC)h = 1 \quad \text{so} \quad h = \frac{2}{BC}\,.$$

If we draw two parallel lines "above" and "below" the segment BC by an amount h than any point A on these two parallel lines will form a triangle with height h and thus an area of one.

Problem 11

Draw the triangle $\triangle ABC$ in its circle with the segment BC "horizontal" and A "above" BC. Then by symmetry the segment from A "towards" BC will pass though the origin of the circle O and be a perpendicular bisector of BC. Call its intersection with BC the point A'. Then

$$BA' = A'C = \frac{BC}{2} = \frac{2}{2} = 1\,.$$

Let the distance OA' be h and draw the radii (of length r) from O to A and C. Then in the right triangle $\triangle OA'C$ the Pythagorean theorem gives

$$h^2 + 1^2 = r^2\,. \tag{713}$$

Next in the right triangle $\triangle AA'C$ the Pythagorean theorem gives

$$(r+h)^2 + 1^2 = 3^2\,,$$

or simplifying and square rooting

$$r + h = 2\sqrt{2}\,. \tag{714}$$

Solving this for h and putting it into Equation 713 we can solve for r. We find $r = \frac{9}{4\sqrt{2}}$. Thus the area of this circle is

$$\pi r^2 = \frac{81\pi}{32}\,.$$

Problem 12

Let C_i be the age of Tom's children and let T be Tom's age. Then we are told that

$$T = C_1 + C_2 + C_3\,, \tag{715}$$

and

$$T - N = 2(C_1 - N + C_2 - N + C_3 - N)\,.$$

This second equation can be written as

$$T - N = 2(C_1 + C_2 + C_3) - 2(3N)\,.$$

Using Equation 715 to replace $C_1 + C_2 + C_3$ this becomes

$$T - N = 2T - 6N\,,$$

or $5N = T$ so that $\frac{T}{N} = 5$.

Problem 13

Draw this region in the x-y Cartesian coordinate plane. By inspection the two circle intersect at $(0,0)$ and $(2,2)$. The region of overlap is a inside a square with a length of two. This square has an area of $A_{\text{square}} = 2^2 = 4$. Viewed from each circle this square is composed of and area that is $\frac{1}{4}$ the area of a circle (with radius two) and an area that is outside this circle. The area of the sector is then

$$A_{\text{sector}} = \frac{1}{4}(\pi(2^2)) = \pi\,.$$

Now the area of the square can be written as

$$A_{\text{square}} = 2A_{\text{sector}} - A_{\text{intersection}}\,.$$

As we are looking for $A_{\text{intersection}}$ and we know the other two expressions we can solve for it to find

$$A_{\text{intersection}} = 2\pi - 4\,.$$

Problem 14

Let N be the total number of students (boys plus girls). Then we are told that

$$0.4N = G\,,$$

where G is the number of girls. After two girls leave and two boys arrive the total number of students N is the same and we are told that

$$0.3N = G - 2\,.$$

Solving the first equation for N in terms of G and putting that into the above gives a single equation for G. Solving that we find $G = 8$.

Problem 15

Writing each angle in terms of $\angle A$ and then summing all angles gives that

$$A + \frac{A}{2} + \frac{A}{3} + \frac{A}{4} = 360\,.$$

This means that $A = \frac{864}{5} = \frac{860}{5} + \frac{4}{5} = 172.8$. To the nearest whole number this is 173.

Problem 16

Let N be the number of students in the junior/senior class. Let N_J and N_S be the number of juniors and seniors respectively. Then we are told that

$$N_J = 0.1N$$
$$N_S = 0.9N\,.$$

Let j_i be the ith junior score and s_i be the ith seniors score on the test. Then from the average of all students we have that

$$84 = \frac{1}{N}\left(\sum_{i=1}^{N_J} j_i + \sum_{i=1}^{N_S} s_i\right)\,.$$

Since all of the juniors got the same score $j_i \equiv j$ for all i and the average score of the seniors was 83 we have that

$$83 = \frac{1}{N_S} \sum_{i=1}^{N_S} s_i \, .$$

Using these two in the expression for the total class average gives

$$84 = \frac{1}{N} \left(N_J j + 83 N_S \right) \, .$$

Writing N_J and N_S in terms of N we can solve the above for j to find $j = 93$.

Problem 17

Let the side of this triangle be denoted by s. Consider the perpendicular from P to the side AB (intersecting at Q) and assume this has a length of h. If this perpendicular splits the side into lengths x and $s - x$ note that the area of the triangle $\triangle PAB$ can be computed by summing the two right triangles $\triangle AQP$ and $\triangle BQP$ as

$$\frac{1}{2} h x + \frac{1}{2} h (s - x) = \frac{1}{2} h s \, ,$$

and thus x does not need to be known. Thus we can evaluate the total area of this equilateral triangle by summing the area of the three triangles $\triangle APB$, $\triangle BPC$, and $\triangle APC$ as

$$\frac{1}{2}(1)s + \frac{1}{2}(2)s + \frac{1}{2}(3)s = 3s \, .$$

The area of an equilateral triangle is given by $\frac{\sqrt{3}}{4} s^2$. Setting this equal to the above and solving for s we get $s = 4\sqrt{3}$.

Problem 18

Connect the centers of the larger "outside" circles together. Then from all of the tangents involved these segments form a square with side lengths of $2r$. Call the center of the upper right circle O and the center of the smaller center circle o. Draw the segment Oo which will be of length $1 + r$. That segment forms the hypotenuse of a right triangle with one leg of length r. The other leg goes though the center of the square and must have a length

$$\sqrt{(1 + r)^2 - r^2} = \sqrt{1 + 2r} \, .$$

Two of these will be equal to the side length of the square or

$$2\sqrt{1 + 2r} = 2r \, .$$

Squaring and solving for r gives $r = 1 + \sqrt{2}$ (taking the positive root).

Problem 19

WWX: Working from here down.

Problem 20

WWX: Working from here down.

Problem 21

Let the square have a side of length s so $s = XY = YZ = ZW = WX$. Then from the given lengths of the larger right triangle $\triangle ABC$ we have

$$\begin{aligned} AW + WB &= 3 \quad (716) \\ BZ + ZC &= 4 \\ AX + s + YC &= 5 \,. \end{aligned}$$

Next notice that in this diagram many angles are equal. We have

$$\angle BAC = \angle BWZ = \angle CZY \,,$$

and

$$\angle AWX = \angle BZW = \angle ZCY \,.$$

This means that many right triangles are similar. For example we see that $\triangle AXW \sim \triangle ABC$ so we can conclude that

$$\frac{AW}{s} = \frac{5}{4} \,.$$

We also have $\triangle WBZ \sim \triangle ABC$ so we can conclude that

$$\frac{WB}{s} = \frac{3}{5} \,.$$

We also have $\triangle CYZ \sim \triangle CBA$ but we won't use that relationship. Using the above two expressions (for AW and WB) in Equation 716 we have

$$\frac{5}{4}s + \frac{3}{5}s = 3 \,.$$

Solving we get $s = \frac{60}{37}$.

Problem 22

WWX: Working from here down.

Problem 23

WWX: Working from here down.

Problem 24

WWX: Working from here down.

Problem 25

WWX: Working from here down.

The 2007 AMC 12A Examination

Problem 1

Problem 2

The 2007 AMC 12B Examination

Problem 1

The 2010 AMC 8 Examination

Problem 1

WWX: Working here.

The 2010 AMC 10A Examination

Problem 1

WWX: Working here.

The 2010 AMC 10B Examination

Problem 1

We have

$$100(100-3)-(100^2-3)=100^2-300-100^2+3=-297\,.$$

Problem 2

WWX: Here

The 2010 AMC 12A Examination

Problem 1

Write this expression as

$$(20-2010+201)+(2010-201+20)=40\,.$$

Problem 2

From the problem statement the number of passengers taken on trip i or N_i is given by

$$\begin{aligned}
N_1 &= 100 \\
N_2 &= 100 - 1 = 100 - (2 - 1) \\
N_3 &= 100 - 2 = 100 - (3 - 1) \\
&\vdots \\
N_i &= 100 - (i - 1)\,.
\end{aligned}$$

There are a total of six trips. The total number of passengers P taken is then

$$\begin{aligned}
P &= \sum_{i=1}^{6} N_i = \sum_{i=1}^{6}(100 - (i - 1)) = \sum_{i=1}^{6}(100 - (i - 1)) = \sum_{i=1}^{6}(101 - i) \\
&= 101(6) - \sum_{i=1}^{6} i = 606 - \frac{6(7)}{2} = 585\,.
\end{aligned}$$

Problem 3

The first statement means that the shaded region is $\frac{1}{2}$ the total of $[ABCD]$ or

$$HG = \frac{1}{2}AB\,.$$

The second statement means that

$$\frac{AD \cdot HG}{EH \cdot HG} = 0.2 = \frac{1}{5}\,,$$

or $\frac{AD}{EH} = \frac{1}{5}$. We want to evaluate

$$\frac{AB}{AD} = \frac{2HG}{AD} = \frac{2HG}{\frac{EH}{5}} = 10\frac{HG}{EH} = 10\,,$$

as $EFGH$ is a square and so $HG = EH$.

Problem 4

This would be the function $-x^{-1}$.

Problem 5

After 50 shorts let Chelsea's score be C_{50}. After 50 more shots where n of them are bulleyes Chelsea's will add at least the additional

$$10n + 4(50 - n)\,,$$

points to her score. Thus her score at the end will be

$$C_{100} \geq C_{50} + 10n + 4(50 - n)\,.$$

At shot 50 the "next highest" competitor had a score H_{50} where $C_{50} - H_{50} = 50$. With n bulleyes to be guaranteed victor means that even if the next highest opponent gets all bulleyes so that $H_{100} = H_{50} + 50(10)$ the value of C_{100} will still be larger or

$$C_{100} > H_{50} + 500\,.$$

The smallest value of n where this will happen is the smallest value of n where

$$C_{50} + 10n + 4(50 - n) \geq H_{50} + 500\,,$$

or subtracting H_{50} from both sides gives

$$50 + 10n + 4(50 - n) \geq 500\,.$$

Solving this for n we get $n \geq \frac{125}{3} = 41.6667$. As n is an integer we need to take $n \geq 42$ so the smallest number is 42.

Problem 6

Let our x be given by pmp where p and m are digits such that $1 \leq p \leq 9$ and $0 \leq m \leq 9$. We are also told that $x + 32$ is a four digit palindrome so $x + 32 \geq 1001$ or $x \geq 969$. This means that $p = 9$ and $6 \leq m \leq 9$. If we consider each of these four numbers we see that when

$$\begin{aligned}
m = 6 &\quad \text{so} \quad x = 969 \quad \text{and} \quad x + 32 = 1001 \\
m = 7 &\quad \text{so} \quad x = 979 \quad \text{and} \quad x + 32 = 1011 \\
m = 8 &\quad \text{so} \quad x = 989 \quad \text{and} \quad x + 32 = 1021 \\
m = 9 &\quad \text{so} \quad x = 999 \quad \text{and} \quad x + 32 = 1031\,,
\end{aligned}$$

only the first is a palindrome. Thus $x = 969$ and so the sum of the digits is $9 + 6 + 9 = 24$.

Problem 7

The "full" height H and volume V are given by

$$\begin{aligned}
H &= 40 \\
V &= \frac{4}{3}\pi R^3 = 100000\,,
\end{aligned}$$

while the "model" volume v is given by

$$v = 0.1 = \frac{4}{3}\pi r^3 \,.$$

Taking the ratio of the two volume expressions we have

$$\frac{V}{v} = \left(\frac{R}{r}\right)^3 = 10^6 \quad \text{so} \quad \frac{r}{R} = 10^{-2} \,.$$

This means that the model height h is given by

$$h = H\left(\frac{r}{R}\right) = 40 \times 10^{-2} = 0.4 \,.$$

Problem 8

The volume of the original cube is $V_0 = 3^3 = 27$. Removing one "column" will take $2^2 \cdot 3 = 12$ units of volume. Removing all three "columns" will over remove the center cube of side length 2 two additional times. Thus the volume remaining is

$$27 - 3(12) + 2(2^3) = 7 \,.$$

Problem 9

WWX: DP

Problem 10

For the arithmetic sequence of numbers p, 9, $3p - q$, and $3p + q$ we have a common difference d given by several differences

$$\begin{aligned} d &= 9 - p & (717) \\ &= 3p - q - 9 & (718) \\ &= 3p + q - (3p - q) = 2q \,. & (719) \end{aligned}$$

If we set equal Equations 718 and 719 we get

$$3p - q - 9 = 2q \quad \text{or} \quad q = p - 3 \,.$$

If we put that expression into Equation 719 and set this equal to Equation 717 we get

$$2p = 6 = 9 - p \quad \text{or} \quad p = 5 \,.$$

This means that $q = 5 - 3 = 2$ and the common difference is $d = 9 - p = 4$. The general term of this sequence is then given by

$$a_n = 5 + 4(n - 1) \quad \text{for} \quad n \geq 1 \,.$$

We find

$$a_{2010} = 5 + 4(2009) = 8041 \,.$$

Problem 11

We want to solve

$$7^{x+7} = 8^x\,,$$

or

$$7^7 \cdot 7^x = 8^x\,,$$

or

$$7^7 = \left(\frac{8}{7}\right)^x\,.$$

Then "taking" the $\log_{\frac{8}{7}}(x)$ of both sides gives

$$x = \log_{\frac{8}{7}}(7^7)\,.$$

From this we see that $b = \frac{8}{7}$.

Problem 12

WWX: DP

Problem 13

Write the second expression as $y = \frac{k}{x}$ and put this into the first equation to get

$$x^2 + \frac{k^2}{x^2} = k^2\,.$$

We can write this as

$$x^4 - k^2x^2 + k^2 = 0\,.$$

Solving for x^2 using the quadratic equation gives

$$x^2 = \frac{k^2 \pm \sqrt{k^4 - 4k^2}}{2} = \frac{k^2 \pm |k|\sqrt{k^2 - 4}}{2}\,.$$

Now to have no intersections means that there are no real solutions and we must have $k^2 - 4 < 0$. If $k = 0$ this inequality is true but the equations are degenerate in that case and we have $(x, y) = (0, 0)$ (only) so the two curves do intersect. If $k = \pm 1$ the inequality is true and if $|k| \geq 2$ it is not. Thus there are $k \in \{-1, +1\}$ or two values of k.

Problem 14

Let the "base" of the triangle be AC with the vertex B "above" AC. Then from B draw the angle bisector to AC intersecting AC at D. Using the "angle bisector theorem" we have that

$$\frac{AB}{AD} = \frac{BC}{DC} \quad \text{or} \quad \frac{AB}{3} = \frac{BC}{8},$$

using the information from the problem.

The triangles perimeter P is given by

$$P = AB + BC + CA = AB + BC + 11 .$$

Using the above relationship we have

$$P = \frac{3}{8}BC + BC + 11 = \frac{11}{8}BC + 11 .$$

The smallest value for P would be if $BC = 0$ but in that case the triangle is degenerate. We need to determine a lower bound on BC. From the triangle inequality we have that

$$AB + BC > AC = 11 .$$

In the above we can replace AB with $\frac{3}{8}BC$ to get

$$\frac{11}{8}BC > 11 \quad \text{or} \quad BC > 8 .$$

To have the side AB integer we need to take $BC = 2 \times 8 = 16$ (so that $AB = 6$) and then find that $P = 11(2) + 11 = 33$.

The 2010 AMC 12B Examination

Problem 1

Let M be the number of minutes Makayla spends in meetings. Then we are told that

$$M = 45 + 2(45) = 135 .$$

Her work day is $9(60) = 360$ minutes. The percent she spends in meetings is then

$$\frac{135}{360} = \frac{1}{4},$$

which is 25%.

Problem 2

Breaking this up into two rectangles I find its area to be

$$A = 2(6) + 2(5) = 22\,.$$

Problem 3

WWX: DP

Problem 4

WWX: DP

Problem 5

WWX: DP

Problem 6

WWX: DP

Problem 7

WWX: DP

Problem 8

WWX: DP

Problem 9

WWX: DP

Problem 10

From the problem statement we are told that

$$\frac{1 + 2 + 3 + \cdots + 98 + 99 + x}{100} = 100x \,,$$

or

$$\sum_{k=1}^{98} k + x = 100^2 x \,.$$

Using Equation 22 the above is

$$\frac{99(98)}{2} = (100^2 - 1)x = (100 - 1)(100 + 1)x = 101(99)x \,,$$

or solving for x we get

$$x = \frac{50}{101} \,.$$

Problem 11

WWX: DP

Problem 12

Using Equation 705 with $c = 2$ for each term the given expression becomes

$$\frac{1}{2}\left(\frac{\log_2(x)}{\log_2(\sqrt{2})}\right) + \frac{\log_2(x)}{\log_2(2)} + 2\left(\frac{\log_2(x)}{\log_2(4)}\right) + 3\left(\frac{\log_2(x)}{\log_2(8)}\right) + 4\left(\frac{\log_2(x)}{\log_2(16)}\right) = 40 \,.$$

Using the fact that $\log_2(2^p) = p$ the above becomes

$$5\log_2(x) = 40 \quad \text{or} \quad \log_2(x) = 8 \quad \text{so} \quad x = 2^8 = 256 \,.$$

Problem 13

WWX: DP

Problem 14

WWX: DP

Problem 15

WWX: DP

Problem 16

WWX: DP

Problem 17

WWX: DP

Problem 18

WWX: DP

Problem 19

WWX: DP

Problem 20

As we are told this is a geometric sequence we have that its common ratio must satisfy

$$r = \frac{a_2}{a_1} = \frac{\cos(x)}{\sin(x)} = \frac{a_3}{a_2} = \frac{\tan(x)}{\cos(x)} = \frac{\sin(x)}{\cos^2(x)}\,,$$

or

$$\frac{\cos(x)}{\sin(x)} = \frac{\sin(x)}{\cos^2(x)} \quad \text{or} \quad \cos^3(x) = \sin^2(x)\,. \tag{720}$$

We can write this as

$$\cos(x)(1 - \sin^2(x)) = \sin^2(x)\,,$$

or

$$\cos(x) = \sin^2(x)(1 + \cos(x))\,,$$

or

$$\frac{\cos(x)}{\sin^2(x)} = 1 + \cos(x)\,. \tag{721}$$

Now a_4 is given by

$$a_4 = ra_3 = \frac{\sin(x)}{\cos(x)} \cdot \frac{\cos(x)}{\sin(x)} = 1\,.$$

Now a_5 is given by

$$a_5 = ra_4 = \frac{\cos(x)}{\sin(x)}\,.$$

Now a_6 is given by

$$a_6 = \frac{\cos^2(x)}{\sin^2(x)}\,.$$

Now a_7 is given by

$$a_7 = \frac{\cos^3(x)}{\sin^3(x)}\,.$$

Now a_8 is given by

$$a_6 = \frac{\cos^3(x)}{\sin^2(x)} \cdot \frac{\cos(x)}{\sin^2(x)}\,.$$

Now using Equation 720 we have $\cos^3(x) = \sin^2(x)$ so

$$a_8 = \frac{\cos(x)}{\sin^2(x)}\,.$$

Now using Equation 721 this is $1 + \cos(x)$.

Problem 21

WWX: DP

The 2013 AMC 8 Examination

Problem 1

As $4 \times 6 = 24$ and $5 \times 6 = 30$ she needs one more car.

Problem 2

Let x be the regular price of a 1/2 pound of fish. Then $0.5x = 3$ so $x = 6$. The regular price of a full pound is then $2x = 12$.

Problem 3

In this sum there are $\frac{1000}{2} = 500$ “pairs” each of which adds together to give one. Thus

$$\begin{aligned} S &= 4((-1+2) + (-3+4) + (-5+6) + \cdots + (-999+1000)) \\ &= 4\sum_{k=1}^{500} 1 = 4(500) = 2000\,. \end{aligned}$$

Problem 4

Let B be the cost of the total bill. Then the per person cost is $\frac{B}{8}$. From what we are told about paying the bill without Judi we have that

$$7\left(\frac{B}{8} + 2.5\right) = B\,.$$

Solving this gives $B = 140$.

Problem 5

The average weight is

$$A = \frac{1}{5}(5 + 5 + 6 + 8 + 106) = 26\,.$$

The median is $M = 6$. Thus the average is larger by 20.

Problem 6

The unknown box in the row above the bottom must have a value $\frac{600}{30} = 20$. The unknown box in the first row must have a value of $\frac{20}{5} = 4$.

Problem 7

The rate of cars-per-second is given by

$$\frac{6}{10} = \frac{3}{5}\,.$$

The train too $120 + 45 = 165$ seconds to pass so the number of cars should be

$$165\left(\frac{3}{5}\right) = 99\,,$$

cars. This is closest to 100 cars.

Problem 8

If we enumerate all possible three toss outcomes we find we will be given one of

$$\{HHH\,, THH\,, HTH\,, HHT\,, HTT\,, THT\,, TTH\,, TTT\}\,.$$

Each element in this event set has a probability of $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$ and three of them have at least two consecutive heads giving a total probability of $\frac{3}{8}$.

Problem 9

Let J_k be the jump amount on the kth jump. Then from the problem statement we are told that

$$\begin{aligned} J_1 &= 1 \\ J_2 &= 2 \\ J_3 &= 2^2 = 4 \\ J_4 &= 2^3 = 8 \\ &\vdots \\ J_n &= 2^{n-1}\,. \end{aligned}$$

We want to know when $J_n > 1000$. As $2^9 = 512$ and $2^{10} = 1024$ we see that when $n - 1 = 10$ or $n = 11$ is when we jump more than one kilometer.

Problem 10

If we have the prime factorization of a and b as

$$a = p_1^{a_1} p_2^{a_2} p_3^{a_3} \cdots p_n^{a_n}$$
$$b = p_1^{b_1} p_2^{b_2} p_3^{b_3} \cdots p_n^{b_n} \,,$$

then the least common multiple and the greatest common factor can be computed as

$$\text{LCM}(a,b) = p_1^{\max(a_1,b_1)} p_2^{\max(a_2,b_2)} p_3^{\max(a_3,b_3)} \cdots p_n^{\max(a_n,b_n)}$$
$$\text{GCF}(a,b) = p_1^{\min(a_1,b_1)} p_2^{\min(a_2,b_2)} p_3^{\min(a_3,b_3)} \cdots p_n^{\min(a_n,b_n)} \,.$$

For the numbers given we have

$$a = 180 = 2^2 \cdot 3^2 \cdot 5$$
$$b = 594 = 2 \cdot 3^3 \cdot 11 \,.$$

Thus we have

$$\text{LCM}(a,b) = 2^2 \cdot 3^3 \cdot 5^1 \cdot 11$$
$$\text{GCF}(a,b) = 2^1 \cdot 3^2 \,.$$

Then the ratio we seek is given by

$$\frac{2^2 \cdot 3^3 \cdot 5^1 \cdot 11}{2^1 \cdot 3^2} = 2 \cdot 3 \cdot 5 \cdot 11 = 330 \,.$$

Problem 11

The time using the treadmill on each day is given by

$$T_{\text{Monday}} = \frac{2}{5}$$
$$T_{\text{Wednesday}} = \frac{2}{3}$$
$$T_{\text{Friday}} = \frac{2}{4} \,,$$

in units of hours. The total of all of this time is to be compared to $\frac{6}{4}$. We find

$$\frac{2}{5} + \frac{2}{3} + \frac{2}{4} - \frac{6}{4} = \frac{1}{15} \,,$$

hours or $\frac{60}{15} = 4$ minutes.

Problem 12

Let F the the price of the "fair special". Let S be the price of one pair of sandals (we are told that $S = 50$). Then we are told that

$$F = S + 0.6S + 0.5S = 2.1S\,.$$

The normal cost of three pairs of sandals would be $3S$. The savings is then $3S - 2.1S = 0.9S$. The percentage this is of $3S$ is then

$$\frac{0.9S}{3S} = 0.3\,,$$

or 30%.

Problem 13

Clara's total should have been

$$C = s_1 + s_2 + \cdots + s_k + \cdots + s_n\,,$$

where her correct kth score has a decimal representation of $s_k = 10t_k + u_k$. If instead she added $10u_k + t_k$ she would have found the total

$$I = s_1 + s_2 + \cdots + (10u_k + t_k) + \cdots + s_n\,,$$

thus

$$C - I = (10t_k + u_k) - (10u_k + t_k) = 10(t_k - u_k) + u_k - t_k = (t_k - u_k(10 - 1) = 9(t_k - u_k)\,.$$

Thus the difference must be divisible by nine. Only the number in (A) has that property.

Problem 14

Let E be the event that the two beans agree. Then we have

$$\begin{aligned} P(E) &= P(\text{Abe shows green and Bea shows green}) + P(\text{Abe shows red and Bea shows red}) \\ &= P(\text{Abe shows green})P(\text{Bea shows green}) + P(\text{Abe shows red})P(\text{Bea shows red}) \\ &= \frac{1}{2}\left(\frac{1}{4}\right) + \frac{1}{2}\left(\frac{2}{4}\right) = \frac{3}{8}\,. \end{aligned}$$

Problem 15

These equations are

$$\begin{aligned} &3^p = 90 - 81 = 9 \quad \text{so} \quad p = 2 \\ &2^r = 32 \quad \text{so} \quad r = 5 \\ &6^s = 1421 - 125 = 1296 = 6^4 \quad \text{so} \quad s = 4\,. \end{aligned}$$

These mean that $prs = 2 \times 5 \times 4 = 40$.

Problem 16

Let x_i be the number of students in grade $i \in \{6, 7, 8\}$. Then we are told that

$$\frac{x_8}{x_6} = \frac{5}{3} \quad \text{and} \quad \frac{x_8}{x_7} = \frac{8}{5}\,.$$

This means that

$$x_6 = \frac{3}{5} x_8 \quad \text{and} \quad x_7 = \frac{5}{8} x_8\,.$$

The total number of students N is

$$N = x_6 + x_7 + x_8 = \frac{3}{5} x_8 + \frac{5}{8} x_8 + x_8 = \frac{89}{40} x_8\,.$$

To have N be as small as possible and an integer means that $x_8 = 40$ (so that $N = 89$). This means that $x_6 = 3(8) = 24$ and $x_7 = 25$.

Problem 17

Let n be the smallest integer. We are told that

$$n + (n + 1) + (n + 2) + (n + 3) + (n + 4) + (n + 5) = 2013\,,$$

or

$$6n + 15 = 2013\,.$$

Solving for n we find $n = 333$. The largest integer is then $n + 5 = 338$.

Problem 18

For this problem we need the volume of this structure. For the "front" faces we see that each is $10 \times 5 \times 1 = 50$ (and there are two faces this size). Each side face is then of volume $(12 - 1 - 1) \times 5 \times 1 = 50$ (again there are two). Adding all of these we get $4(50) = 200$. We now need to add the volume of the floor. Note that the volume of the floor would be $12 \times 10 = 120$ is it were "attached" to the bottom of the "frame". As I think the frame is to be considered "inside" the "frame" (not not just below it) its volume is given by

$$(12 - 2) \times (10 - 2) \times 1 = 10 \times 8 = 80\,.$$

Adding that gives $200 + 80 = 280$.

Problem 19

Let the letters B, C, and H stand for the scores of the given girls. From the problem statement we have that Cassie knows the values of C and H, Bridget knows the values of B and H, and Hannah knows the value of H. From Cassie's statement we can conclude that $H < C$. From Bridget's statement we can conclude that $B < H$. Thus combining these two gives $B < H < C$.

Problem 20

The center of the circle will be at the center of the longer side of the rectangle (and of length $\frac{2}{2} = 1$). Thus the radius of the semicircle is then the hypotenuse of a isosceles right triangle with legs of length one or $r = \sqrt{2}$. Thus the area of the semicircle is then

$$\frac{1}{2}\pi r^2 = \pi \, .$$

Problem 21

In going from home to the northeast corner of the park in three ways (Samantha must pick one of the three vertical paths). She can then go from the northeast corner of the park to her school in six ways. You can verify that by starting at the school and counting "backwards" and adding up the number of ways one can go along each leg. The total paths are then $3 \times 6 = 18$ ways.

Problem 22

All rows have $(32 + 1)(60) = 1980$ toothpicks. All columns have $(60 + 1)(32) = 1952$ toothpicks. The total number of toothpicks are then $1980 + 1952 = 3932$.

Problem 23

From the information about the area of the semicircle on AB we have that

$$\frac{1}{2}\pi \left(\frac{AB}{2}\right)^2 = 8\pi \, .$$

Solving for AB gives $AB = 8$. From the information about the arc of the semicircle on AC we have that

$$\frac{1}{2}\left(2\pi \left(\frac{AC}{2}\right)\right) = \frac{17}{2}\pi \, .$$

Solving for AC gives $AC = 17$. Then the Pythagorean theorem gives

$$AC^2 - AB^2 = BC^2 \quad \text{or} \quad 17^2 - 8^2 = BC^2\,.$$

This gives $BC = 15$. Thus the radius of the semicircle on BC is $\frac{BC}{2} = 7.5$.

Problem 24

Let the side of each square be denoted by s. Let the intersection of the segment AJ with the segment DC at a point x units to the left of the point C at a point we will denote P so $CP = x$. Finally let then angle $\angle BAD = \angle IJG = \theta$. Then as we move from A to J along AJ we move *down* $2s$ units and horizontally $AB + CI = s + \frac{s}{2} = \frac{3s}{4}$ units. Thus

$$\tan(\theta) = \frac{2s}{\frac{3s}{2}} = \frac{4}{3}\,.$$

This means that if we move from point J vertically by s units we will move horizontally by Δx units where

$$\tan(\theta) = \frac{4}{3} = \frac{s}{\Delta x} \quad \text{so} \quad \Delta x = \frac{3}{4}s\,.$$

This means that the point P is $\frac{3}{4}s$ to the left of J so that

$$x = CP = PI - CI = \frac{3}{4}s - \frac{s}{2} = \frac{s}{4}\,.$$

This means that the area of the shaded region (denoted by R) is

$$R = \frac{1}{2}s(s+x) + \frac{1}{2}s\left(x + \frac{s}{2}\right) = \frac{s}{2}\left(s + 2x + \frac{s}{2}\right)\,.$$

with $x = \frac{s}{4}$ this becomes $R = s^2$. Thus the desired ratio is

$$\frac{s^2}{3s^2} = \frac{1}{3}\,.$$

Problem 25

Note that traveling $\frac{1}{2}$ of a circumference of a circle of radius R one would travel

$$\frac{2\pi R}{2} = \pi R\,.$$

To measure how much the *center* of the ball traveled during the first arc we recognize that the center will travel on a smaller radius of

$$R_1 - \frac{4}{2} = R_1 - 2 = 98\,.$$

For the second semicircular path the center of the ball travels on a radius of

$$R_2 + 2 = 62\,.$$

For the third semicircular path the center of the ball travels on a radius of

$$R_3 - 2 = 78\,.$$

These mean that the total path traveled by the center of the ball will be

$$\pi(98) + \pi(62) + \pi(78) = 238\pi\,.$$

The 2014 AMC 8 Examination

Problem 1

For Harry and Terry's calculations we find

$$H = 8 - (2 + 5) = 8 - 7 = 1$$
$$T = 8 - 2 + 5 = 11\,.$$

Thus $H - T = 1 - 11 = -10$.

Problem 2

The largest number of coins would be seven five cent coins. The smallest number of coins would consist of one 25 cent coin and one 10 cent coin for a total of two coins. This difference is $7 - 2 = 5$.

Problem 3

The number of pages in the book must be

$$P = 36 \times 3 + 44 \times 3 + 10 = 250\,.$$

Problem 4

Let these two prime numbers be p and q. Then we know that $p + q = 85$. Now all primes larger than two are odd and the sum of two odd numbers is even. Thus one of p or q must be two and the other prime must be $85 - 2 = 83$. Their product is $2 \times 83 = 166$.

Problem 5

With \$20 dollars Margie can buy five gallons of gas and can thus drive $5 \times 32 = 160$ miles.

Problem 6

Let the total area be S. Then we have

$$S = 2 \times (1 + 4 + 9 + 16 + 25 + 36) = 2 \times (10 + 20 + 61) = 182\,.$$

Problem 7

Let G be the number of girls and B be the number of boys. Then we are told that

$$G - B = 4$$
$$G + B = 28\,.$$

Solving this system we get $B = 12$ and $G = 16$ so $G : B = 16 : 12 = 4 : 3$.

Problem 8

Let d be the amount each member payed. Then we know that

$$11d = 1A2\,,$$

for some digit A. As the right-hand-side is even the number d must also be even and so ends in a two. Let the number d have two digits say as $d = B2$ with B a "digit" i.e. $1 \leq B \leq 9$. Lets compute the product of the numbers $B2$ and 11. We have

$$B2 \times 11 = B2 \times (10 + 1) = B20 + B2\,.$$

Note that this will end in a two as it must. Setting this equal to $1A2$ from the first digit we see that $B = 1$ so the number $d = B2 = 12$ and $11 \times d = 132$ so that $A = 3$.

Problem 9

As $BD = DC$ we have $\angle DCB = \angle DBC = 70$. Using this we have that

$$\angle BDC = 180 - 2(70) = 40\,,$$

and

$$\angle ADB = 180 - \angle BDC = 180 - 40 = 140\,.$$

Problem 10

The first AMC was given in 1985, the second in 1986, etc. The seventh AMC would be held on year

$$1985 + (7 - 1) = 1991\,.$$

This means that Samantha was born in $1991 - 12 = 1979$.

Problem 11

Drawing a Cartesian coordinate grid we can place Jack at $(0,0)$ and Jill at $(3,2)$. The intersection to avoid is located at $(1,2)$. Starting at Jack's location we can count the number of ways to get to each location in this grid. For example there is only one way to get to each of the locations $(i,0)$ for $i \in \{1,2,3\}$ and one way to get to each of the locations $(0,j)$ for $j \in \{1,2\}$. Then at any internal location the number of ways to get to a given location is the sum of the number of ways to get to the location *below* and *left* of the given location.

Following this rule the number $N_{i,j}$ of ways to get to the following location are computed to be

$$
\begin{aligned}
N_{0,0} &= 0 \\
N_{1,0} &= 1 \\
N_{2,0} &= 1 \\
N_{3,0} &= 1 \\
N_{0,1} &= 1 \\
N_{0,2} &= 1 \\
N_{1,2} &= 1 \\
N_{2,0} &= 1 \\
N_{2,1} &= 1 \\
N_{2,2} &= N_{2,1} + N_{1,2} = 2 \\
N_{3,1} &= N_{3,0} + N_{2,1} = 2 \\
N_{3,2} &= N_{3,1} + N_{2,2} = 4\,.
\end{aligned}
$$

The value of $N_{3,2}$ is the desired answer.

Problem 12

There is a $\frac{1}{3}$ chance of matching the first celebrity correctly randomly. Then if that is done there is a $\frac{1}{2}$ chance of matching the second celebrity correctly. This gives a total probability of $\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$.

Problem 13

Note (A) and (B) are possible with the examples $(n,m) = (2,2)$ and $(n,m) = (1,1)$. Note (C) is possible if we take $(n,m) = (1,3)$ so that $n+m = 4$ is even and $n^2 + m^2 = 1 + 9 = 10$ is even. We have that (D) is impossible since if $n+m$ is odd then one of n or m must be odd and the other even. Without loss of generality lets have n odd and m even. Then n^2 will be odd and m^2 will be even so $n^2 + m^2$ will then be odd.

Problem 14

The two equivalent areas we have $[ABCD] = 30 = \frac{1}{2}DC \cdot CE = \frac{1}{2}(5)CE$. Thus $CE = 12$. Then we have that

$$DE = \sqrt{CD^2 + CE^2} = \sqrt{25 + 144} = \sqrt{169} = 13\,.$$

Problem 15

Each arc is $\frac{360}{12} = 30$. Based on the total arcs cut off by x and y we have

$$\angle x = \frac{1}{2}(2(30)) = 30$$
$$\angle y = \frac{1}{2}(4(30)) = 60\,.$$

This means that $x + y = 90$.

Problem 16

Let n be the number of "Middle School Eight" teams (we know $n = 8$). Then the number of games played is

$$2\binom{n}{2} + 4n = n^2 - n + 4n = n^2 + 3n\,.$$

Taking $n = 8$ and evaluating the above gives 88.

Problem 17

George normally takes $T = \frac{1}{3}$ hours to get to school. On this day, the time it took him to talk the first $\frac{1}{2}$ hour is

$$\frac{1/2}{2} = \frac{1}{4}\,,$$

of an hour. This means that to reach school on time he has to travel the other $\frac{1}{2}$ of a mile in

$$\frac{1}{3} - \frac{1}{4} = \frac{1}{12}\,,$$

hours. His velocity must then be

$$\frac{\frac{1}{2}}{\frac{1}{12}} = 6\,,$$

miles-per-hour.

Problem 18

If we let $p = \frac{1}{2}$ the probability that a couple has a boy then each of the given probabilities can be computed using the binomial distribution. We have (once we simplify)

$$
\begin{aligned}
P_A = P_B = p^4 &= \frac{1}{16} \\
P_C = \left(\begin{array}{c} 4 \\ 2 \end{array} \right) p^2 (1-p)^{4-2} &= \frac{3}{8} \\
P_D = \left(\begin{array}{c} 4 \\ 3 \end{array} \right) p^3 (1-p)^{4-3} + \left(\begin{array}{c} 4 \\ 1 \end{array} \right) p^1 (1-p)^{4-1} &= \frac{1}{2} \, .
\end{aligned}
$$

From these we see that (D) is the most likely.

Problem 19

Counting the location and number of small cubes needed to create the larger "outer" cube we see that there are 26 cubes on the "faces" of the large cube with a single remaining small cube located at the center of the large cube. To minimize the surface area colored white means that we want to place one white cube at the center and then the remaining five white cubes at the center location of each larger face (where only one white face of the small cube will be showing). There are six total faces of the large cube but we have only five white cubes to distribute. This means that the displayed area of white will be five while the total surface area is $6(3^2) = 54$. The fraction of surface area that is white is then $\frac{5}{54}$.

Problem 20

The three circles cut out areas equal to $\frac{1}{4}$ of their total area. Thus the area asked for can be computed as

$$
3(5) - \frac{1}{4}(\pi(3^2)) - \frac{1}{4}(\pi(2^2)) - \frac{1}{4}(\pi(1^2)) = 15 - \frac{7}{2}\pi \approx 15 - \frac{7}{2}\left(\frac{22}{7}\right) = 15 - 11 = 4 \, .
$$

Problem 21

As our number is divisible by three the sum of its digits must be divisible by three. For the first number this means that

$$
7 + 4 + A + 5 + 2 + B + 1 \equiv 0 \pmod{3} \, ,
$$

or

$$
19 + A + B \equiv 0 \pmod{3} \, .
$$

As $19 \equiv 1 \pmod 3$ the above is equivalent to

$$A + B \equiv 2 \pmod 3 . \tag{722}$$

For the second number, this same reasoning means that

$$3 + 2 + 6 + A + B + 4 + C \equiv 0 \pmod 3 ,$$

or

$$15 + A + B + C \equiv 0 \pmod 3 .$$

As $15 \equiv 0 \pmod 3$ this means that

$$A + B + C \equiv 0 \pmod 3 .$$

Using Equation 722 in the above we see that $C \equiv 1 \pmod 3$. Thus $C \in \{1, 4, 7\}$. The only valid choice given is $C = 1$.

Problem 22

Let n and m be the two digits of our number such that our number N can be written as $N = 10n + m$. Then we are told that

$$nm + n + m = 10n + m ,$$

or

$$n(m - 9) = 0 .$$

If $n = 0$ we don't have a two digit number thus $m = 9$.

Problem 23

Let A, B, and C stand for the numbers on Ashley's, Bethany's, and Caitlin's uniforms respectfully. Then from the statements given we have that $A + C$ is "earlier" in the month, $A + B$ is "later" in the month, and $B + C$ is "today". This means that

$$A + C < B + C < A + B \leq 31 .$$

The first inequality gives that $A < B$. The second inequality gives that $C < A$ and thus the ordering of the uniform numbers is

$$C < A < B , \tag{723}$$

and each is a two digit prime. This means that they come from the set

$$\{11, 13, 17, 19, 23, 27, 29, 31\} .$$

Each number cannot be too large or else the sums $A + C$, $A + B$, or $B + C$ will be larger than 31 and not represent a day of the month. Since C is one numbers above we know that $C \geq 11$. Since we must have $A + C < 31$ (equivalently $A + C \leq 30$) this means that $A \leq (30 - 11) = 19$. In the same way to have $B + C < 31$ means that $B \leq 19$. This means that A, B, and C must come from the set

$$\{11, 13, 17, 19\}.$$

Using the ordering from Equation 723 we *might* have $B = 19$. If that were true then $A \in \{13, 17\}$ but either of those two choices violates the condition that $A + B \leq 31$. Thus $B \neq 19$. This means that $B = 17$, $A = 13$, and $C = 11$.

Problem 25

The radius of each circle is $r = \frac{40}{2} = 20$ feet. Robert needs to ride along the circumference of N_C circles where

$$N_C = \frac{5280}{2r} = \frac{5280}{40} = 132\,,$$

to move one mile horizontally. The distance traveled along each semi-circular path is

$$\frac{1}{2}\pi(2r) = 20\pi\,,$$

feet. Thus the total distance he covers along the semi-circles is $20\pi N_C = 2640\pi$ feet. This is

$$\frac{2640\pi}{5280} = \frac{\pi}{2}\,,$$

miles.

Lecture Notes on
Mathematical Olympiad Courses: Vol. 1

Lecture 1: Operations on Rational Numbers

Example 2 Notes

Part (iv): To show this we use

$$\frac{75}{13} < \frac{78}{13} = 6$$
$$\frac{37}{13} < \frac{39}{13} = 3\,.$$

Part (v): To show this we use

$$\left(1 - \left(\frac{6}{7}\right)^7\right)\left(9 + \frac{246}{247} - 0.666\right) < 1(9+1) = 10\,.$$

Example 5 Notes

To show this we use

$$\frac{100 \times (83^2 - 83 \cdot 17 + 17^2)}{83 \cdot 66 + 17^2)} = \frac{100 \times (83(83-17) + 17^2)}{83 \cdot 66 + 17^2)} = 100\,.$$

Example 7 Notes

Note that this expression is

$$\frac{x^2}{(x-1)^2 + (x+1)^2 - 2} = \frac{x^2}{x^2 - 2x + 1 + x^2 + 2x + 1 - 2} = \frac{x^2}{2x^2} = \frac{1}{2}\,,$$

when $x = 20092008$.

Example 10 Notes

Comparing $(a-b)^2 = a^2 - 2ab + b^2$ and $(a+b)^2 = a^2 + 2ab + b^2$ are the same as comparing $-ab$ and ab. As $ab < 0$ the first is positive and the second is negative.

Example 11 Notes

I will work this problem in parts. Note that when $-1 < a < 0$ we have $a < a^3 < 0$ and $-a^3 > 0$ so that we have

$$a < a^3 < 0 < -a^3 \, .$$

Now $a^4 > 0$ and $a^4 < -a^3$ so we have

$$a < a^3 < 0 < a^4 < -a^3 \, .$$

Now $-a^4 < 0$ and $|a^4| < |a^3|$ so we have

$$a < a^3 < -a^4 < 0 < a^4 < -a^3 \, .$$

Now $\frac{1}{a} < 0$ and $\left|\frac{1}{a}\right| > |a|$ so we have

$$\frac{1}{a} < a < a^3 < -a^4 < 0 < a^4 < -a^3 < -\frac{1}{a} \, ,$$

as the final ordering.

Testing Question A.1

We can write this sum S as

$$S = -1 + (-1)^2 + (-1)^3 + (-1)^4 + \cdots + (-1)^{100} + (-1)^{101} = \sum_{k=1}^{101} (-1)^k \, .$$

This sum has 50 terms with even powers 51 terms with odd powers to give a sum of $S = 50 + 51(-1) = -1$.

Testing Question A.2

To evaluate this expression we have

$$\begin{aligned} E &= 2008 \times 20092009 - 2009 \times 20082008 \\ &= 2008 \times 2009(1 + 10^4) - (2008 + 1) \times 2008(1 + 10^4) \\ &= 2008(1 + 10^4) \left[2009 - (2008 + 1)\right] = 0 \, . \end{aligned}$$

Testing Question A.3

We start with $x_0 = 2009$ and then following the steps outlined in the problem our next number is given by

$$x_1 = 2009 - \frac{1}{2}(2009) = 2009 \left(1 - \frac{1}{2}\right) = x_0 \left(1 - \frac{1}{2}\right) \, .$$

Next we would have

$$x_2 = x_1 - \frac{1}{3}x_1 = x_1\left(1 - \frac{1}{3}\right) = 2009\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right).$$

Next we would have

$$x_3 = x_2 - \frac{1}{4}x_2 == 2009\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{4}\right).$$

In general the pattern is

$$x_{2008} = 2009\prod_{k=2}^{2009}\left(1 - \frac{1}{k}\right) = 2009\prod_{k=2}^{2009}\frac{k-1}{k} = 2009\left[\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdots\frac{2007}{2008}\cdot\frac{2008}{2009}\right] = 1\,.$$

Testing Question A.4

This sum can be written using partial fractions as

$$\begin{aligned}
\sum_{k=2}^{6}\frac{1}{(2k+1)(2(k+1)+1)} &= \sum_{k=2}^{6}\frac{1}{(2k+1)(2k+3)} \\
&= \sum_{k=2}^{6}\left[\frac{1}{2(2k+1)} - \frac{1}{2(2k+3)}\right] \\
&= \frac{1}{2}\sum_{k=2}^{6}\frac{1}{2k+1} - \frac{1}{2}\sum_{k=2}^{6}\frac{1}{2k+3} \\
&= \frac{1}{2}\sum_{k=2}^{6}\frac{1}{2k+1} - \frac{1}{2}\sum_{k=3}^{7}\frac{1}{2k+1} = \frac{1}{2}\left[\frac{1}{5} - \frac{1}{15}\right].
\end{aligned}$$

Testing Question A.5

After experimenting with a couple different ways of writing the fractions in this sum S it looks like the best method is to write it as

$$S = \frac{1}{2\cdot 5} + \frac{1}{5\cdot 8} + \frac{1}{8\cdot 11} + \frac{1}{11\cdot 14} + \frac{1}{14\cdot 17}.$$

This shows that we can write our sum S as

$$S = \sum_{n\in\{2,5,8,11,14\}}\frac{1}{n(n+3)} = \sum_{k=1}^{5}\frac{1}{(3k-1)(3k+2)}.$$

Next using partial fractions we can write

$$\frac{1}{(3k-1)(3k+2)} = \frac{1}{3(3k-1)} - \frac{1}{3(3k+2)} = \frac{1}{3(3k-1)} - \frac{1}{3(3(k+1)-1)}.$$

This means that our sum can be evaluated as

$$\begin{aligned}\sum_{k=1}^{5}\frac{1}{(3k-1)(3k+2)} &= \sum_{k=1}^{5}\frac{1}{3(3k-1)}-\sum_{k=1}^{5}\frac{1}{3(3(k+1)-1)}\\ &= \sum_{k=1}^{5}\frac{1}{3(3k-1)}-\sum_{k=2}^{6}\frac{1}{3(3k-1)}\\ &= \frac{1}{3(3-1)}-\frac{1}{3(18-1)}=\frac{1}{6}-\frac{1}{51}\,.\end{aligned}$$

Testing Question A.6

Let $x=\sum_{k=3}^{2009}\frac{1}{k}$ and $y=\sum_{k=2}^{2008}\frac{1}{k}$ then the expression we want to evaluate is given by

$$\begin{aligned}E &= x(1+y)-(1+x)y = x+xy-y-xy = x-y\\ &= \sum_{k=3}^{2009}\frac{1}{k}-\sum_{k=3}^{2008}\frac{1}{k}=\frac{1}{2009}-\frac{1}{2}\,.\end{aligned}$$

Testing Question A.7

Recalling that

$$1+2+3+\cdots+(n-1)+n=\frac{n}{2}(n+1)\,,$$

then this sum can be written

$$\begin{aligned}\sum_{k=2}^{51}\frac{1}{\frac{k}{2}(k+1)} &= \sum_{k=2}^{51}\frac{2}{k(k+1)}=2\sum_{k=2}^{51}\left(\frac{1}{k}-\frac{1}{k+1}\right)\\ &= 2\sum_{k=2}^{51}\frac{1}{k}-2\sum_{k=3}^{52}\frac{1}{k}=2\left[\frac{1}{2}-\frac{1}{52}\right]\,.\end{aligned}$$

Testing Question A.8

We should write this sum S as

$$S\equiv 1+\left(\frac{1}{2}+\frac{2}{2}+\frac{1}{2}\right)+\left(\frac{1}{3}+\frac{2}{3}+\frac{3}{3}+\frac{2}{3}+\frac{1}{3}\right)+\cdots\,,$$

where we see that inside each grouped expression the denominator stays the same but the numerator increases and then decreases. Given that pattern we can write and evaluate this sum as

$$\begin{aligned} S &= \sum_{m=1}^{n} \left(\sum_{k=1}^{m-1} \frac{k}{m} + 1 + \sum_{k=m-1}^{1} \frac{k}{m} \right) = \sum_{m=1}^{n} \left(1 + 2 \sum_{k=1}^{m-1} \frac{k}{m} \right) \\ &= n + 2 \sum_{m=1}^{n} \frac{1}{m} \sum_{k=1}^{m-1} k = n + 2 \sum_{m=1}^{n} \frac{1}{m} \left(\frac{m(m-1)}{2} \right) \\ &= n + \sum_{m=1}^{n} (m-1) = n + \left(\sum_{m=1}^{n} m \right) - n \\ &= \frac{n(n+1)}{2} \,. \end{aligned}$$

We can check this for $n \in \{1, 2, 3\}$ and verify that it is correct.

Testing Question A.9

We can write and evaluate this sum S as

$$\begin{aligned} S \equiv \sum_{n=1}^{2009} n^2 (-1)^{n+1} &= \sum_{n=1,3,5,\cdots}^{2009} n^2 (-1)^{n+1} + \sum_{n=2,4,6\cdots}^{2008} n^2 (-1)^{n+1} \\ &= \sum_{k=0}^{1004} (2k+1)^2 + \sum_{k=1}^{1004} (2k)^2 = 1 + \sum_{k=1}^{1004} [(2k+1)^2 - (2k)^2] \\ &= 1 + \sum_{k=1}^{1004} (2k+1-2k)(2k+1+2k) = 1 + \sum_{k=1}^{1004} (4k+1) \\ &= 1 + 1004 + 4 \sum_{k=1}^{1004} k = 1005 + 4 \left(\frac{1004(1005)}{2} \right) \\ &= 2019045 \,. \end{aligned}$$

Testing Question A.10

Note that we can write this sum as

$$(20 - 9) + (200 - 8) + (2000 - 7) + (20000 - 6) + \cdots \,.$$

The general term of the above sum looks to be

$$2 \cdot 10^n - (10 - n) \,,$$

for $1 \le n \le 9$. Thus we want to evaluate

$$S = \sum_{n=1}^{9} [2 \cdot 10^n - (10 - n)] \,.$$

Note that

$$\sum_{n=1}^{9}(10-n)=\sum_{n=1}^{9}n=\frac{1}{2}9(10)=45\,.$$

Thus the sum above becomes

$$S=2\sum_{n=1}^{9}10^n-45=2\left(\frac{1-10^{10}}{1-10}-1\right)-45=2222222175\,.$$

Testing Question B.1

We can evaluate this sum as follows

$$\begin{aligned}
\sum_{k=3,5,7\ldots}\frac{k^2+1}{k^2-1} &= \sum_{n=1}^{49}\frac{(2n+1)^2+1}{(2n+1)^2-1}=\sum_{n=1}^{49}\frac{4n^2+4n+2}{(2n+1-1)(2n+1+1)} \\
&= \sum_{n=1}^{49}\frac{4n^2+4n+2}{2n(2n+2)}=\sum_{n=1}^{49}\frac{4n^2+4n+2}{4n^2+4n} \\
&= \sum_{n=1}^{49}1+\sum_{n=1}^{49}\frac{1}{2n^2+2n}=49+\frac{1}{2}\sum_{n=1}^{49}\frac{1}{n(n+1)} \\
&= 49+\frac{1}{2}\sum_{n=1}^{49}(\frac{1}{n}-\frac{1}{n+1})=49+\frac{1}{2}\left[\sum_{n=1}^{49}\frac{1}{n}-\sum_{n=2}^{50}\frac{1}{n}\right] \\
&= 49+\frac{1}{2}\left[1-\frac{1}{50}\right]\,.
\end{aligned}$$

Testing Question B.2

We can write this sum S as

$$\begin{aligned}
S&=1-\sum_{n=2}^{100}\frac{n}{(1+2+3+\cdots+(n-2)+(n-1))(1+2+3+\cdots+(n-1)+n)} \\
&=1-\sum_{n=2}^{100}\frac{n}{\left(\frac{(n-1)n}{2}\right)\left(\frac{n(n+1)}{2}\right)} \\
&=1-4\sum_{n=2}^{100}\frac{1}{(n-1)n(n+1)}\,.
\end{aligned}$$

Now to continue to evaluate it we need the partial fraction expansion of $\frac{1}{(n-1)n(n+1)}$ which is given by

$$\frac{1}{(n-1)n(n+1)}=-\frac{1}{n}+\frac{1}{2(n+1)}+\frac{1}{2(n-1)}=\frac{1}{2(n-1)}-\frac{1}{2n}-\frac{1}{2n}+\frac{1}{2(n+1)}\,. \quad (724)$$

Which is an expression is in a form that we can sum easily. Using this the sum we need to evaluate is

$$\begin{aligned}\sum_{n=2}^{100}\frac{1}{(n-1)n(n+1)} &= \frac{1}{2}\sum_{n=2}^{100}\left(\frac{1}{n-1}-\frac{1}{n}\right)-\frac{1}{2}\sum_{n=2}^{100}\left(\frac{1}{n}-\frac{1}{n+1}\right)\\ &= \frac{1}{2}\sum_{n=1}^{99}\left(\frac{1}{n}-\frac{1}{n+1}\right)-\frac{1}{2}\sum_{n=2}^{100}\left(\frac{1}{n}-\frac{1}{n+1}\right)\\ &= \frac{1}{2}\left(1-\frac{1}{2}\right)-\frac{1}{2}\left(\frac{1}{100}-\frac{1}{101}\right).\end{aligned}$$

Using this our total sum is given by

$$S=\frac{1}{5050}.$$

Testing Question B.3

This is the sum

$$S=\sum_{n=2}^{101}\frac{1}{(n-1)n(n+1)}.$$

Now using partial fraction expansion given by Equation 724 we can evaluate this. This is exactly the same sum evaluated in the previous question but with an upper limit of 101 (instead of 100) and thus increased from the previous expression by $\frac{1}{100(101)(102)}$. Using the previous result and adding this one more term we find this sum to be

$$\frac{2575}{10302}.$$

Testing Question B.4

This is the sum

$$S=\sum_{n=1}^{50}\frac{n}{1+n^2+n^4}.$$

Note that we can write the denominator as

$$1+n^2+n^4=(n^2+1)^2-n^2=(n^2+1-n)(n^2+1+n)=(n^2-n+1)(n^2+n+1).$$

Note that if we take $n\to n+1$ in the first factor of the above we get

$$n^2-n+1\to n^2+n+1.$$

Based on this we will use partial fractions to write

$$\frac{n}{1+n^2+n^4}=\frac{An+B}{n^2-n+1}+\frac{Cn+D}{n^2+n+1}.$$

Solving we get

$$\begin{aligned}
A &= 0 \\
B &= \frac{1}{2} \\
C &= 0 \\
D &= -\frac{1}{2} \,.
\end{aligned}$$

Thus we have

$$\frac{n}{1+n^2+n^4} = \frac{1}{2}\left(\frac{1}{n^2-n+1}\right) - \frac{1}{2}\left(\frac{1}{n^2+n+1}\right) \,.$$

Using this we have

$$\begin{aligned}
S &= \frac{1}{2}\sum_{n=1}^{50}\frac{1}{n^2-n+1} - \frac{1}{2}\sum_{n=1}^{50}\frac{1}{n^2+n+1} \\
&= \frac{1}{2}\sum_{n=0}^{49}\frac{1}{(n+1)^2-(n+1)+1} - \frac{1}{2}\sum_{n=1}^{50}\frac{1}{n^2+n+1} \\
&= \frac{1}{2}\sum_{n=0}^{49}\frac{1}{n^2+n+1} - \frac{1}{2}\sum_{n=1}^{50}\frac{1}{n^2+n+1} \\
&= \frac{1}{2}\left[1 - \frac{1}{50^2+50+1}\right] = \frac{1275}{2551} \,,
\end{aligned}$$

when we simplify.

Testing Question B.5

I think there is a typo in this question. I think that the terms of the sum should be

$$\frac{n^2}{n^2-10n+50} \,,$$

for $1 \leq n \leq 8$. To solve this problem we would first notice that

$$\frac{n^2}{n^2-10n+50} + \frac{(10-n)^2}{(10-n)^2-10(10-n)+50} = 2 \,.$$

Truthfully I'm not sure how one would would come to this conclusion and if anyone has any idea please contact me. Once we have noted this however the solution is straightforward. Summing both sides from $n = 1$ to $n = 8$ gives

$$\sum_{n=1}^{8}\frac{n^2}{n^2-10n+50} + \sum_{n=1}^{8}\frac{(10-n)^2}{(10-n)^2-10(10-n)+50} = 2(8) = 16 \,.$$

In the second sum above let $m = 10 - n$ and it can then be written as

$$\sum_{m=9}^{2}\frac{m^2}{m^2-10m+50} \,,$$

which is the same as the first sum but starting at $n = 2$ and ending at $n = 9$. Using this result we can write the above as

$$\frac{1}{1-10+50} + 2\sum_{n=2}^{8}\frac{n^2}{n^2-10n+50} + \frac{9^2}{9^2-90+50} = 16\,.$$

Solving for the sum above we get

$$\sum_{n=2}^{8}\frac{n^2}{n^2-10n+50} = 7\,,$$

when we simplify. Thus the sum we want

$$\sum_{n=1}^{8}\frac{n^2}{n^2-10n+50} = 7 + \frac{1}{1-10+50} = \frac{288}{41}\,.$$

If we add one more term (corresponding to $n = 9$) to this we would get

$$\sum_{n=1}^{8}\frac{n^2}{n^2-10n+50} = 9\,.$$

Lecture 2: Monomials and Polynomials

Testing Question A.1

B is not monomial.

Testing Question A.2

It cannot be greater than four. It could be less than four if the highest power terms cancel each other.

Testing Question A.3

Let $P(x)$ be the polynomial that Adam started with. Then we are told that

$$P(x) - (2x^2 + x + 1) = 5x^2 - 2x + 4\,.$$

Solving for $P(x)$ we find

$$P(x) = 7x^2 - x + 5\,.$$

Then the correct expression should be

$$P(x) + (2x^2 + x + 1) = 9x^2 + 6\,.$$

Testing Question A.4

From what we are told we have

$$0.75x^b y^c - 0.5x^{m-1}y^{2n-1} = 1.25ax^n y^m\,.$$

From the exponent of x, the exponent of y, and the leading coefficient this means that

$$b = m - 1 = n \tag{725}$$

$$c = 2n - 1 = m \tag{726}$$

$$0.25 = 1.25a\,. \tag{727}$$

From the last equation we have $a = 0.2$ and Equations 725 and 726 are two equations for the two variables m and n. Solving them we find $m = 3$ and $n = 2$. Thus we have shown that

$$a = 0.2\,, \quad b = n = 2\,, \quad c = m = 3\,,$$

so

$$abc = 1.2\,.$$

Testing Question A.5

Note this product would be

$$x^5\left(x+\frac{1}{x}\right)\left(1+\frac{2}{x}+\frac{3}{x^2}\right)=x^2(x^2+1)(x^2+2x+3)\,,$$

and has a degree of six.

Testing Question A.6

For this question try to find the simplest thing that might work. To do that note that

$$2^8+2^{10}+2^n=(2^4)^2+2(2^4)(2^5)+(2^5)^2-2^{10}+2^n=(2^4+2^5)^2-2^{10}+2^n\,.$$

Thus if we take $n=10$ the above is a perfect square.

Testing Question A.7

From the given expression we note that

$$3x^2=1-x\,,$$

Thus using the above we have replaced higher powers of x with lower powers of x. Putting that into the expression we want to evaluate E gives

$$E=2x(1-x)-x^2-3x+2010=-3x^2-x+2010\,.$$

Using what we know for $3x^2$ again in the above we get

$$E=-(1-x)-x+2010=2009\,.$$

Testing Question A.8

If we subtract two forms of x that are known to be equal we have

$$\frac{a}{b+c}-\frac{b}{a+c}=\frac{(a-b)(a+b+c)}{(a+c)(b+c)}=0\,.$$

This means that $a=b$ or $a+b+c=0$. Doing the same thing for two additional forms for x we have

$$\frac{b}{a+c}-\frac{c}{a+b}=\frac{(b-c)(a+b+c)}{(a+c)(a+b)}=0\,.$$

Thus we have $b = c$ or $a+b+c = 0$. Doing the same thing for the final two additional forms for x we have

$$\frac{a}{b+c} - \frac{c}{a+b} = \frac{(a-c)(a+b+c)}{(b+c)(a+b)} = 0\,.$$

Thus we have $a = c$ or $a+b+c = 0$.

Now if $a = b$ then we see that x is given by

$$x = \frac{a}{a+c} = \frac{c}{2a}\,.$$

We can write the above as

$$c^2 + ac - 2a^2 = 0\,,$$

so using the quadratic equation solving for c in terms of a gives $c = -2a$ or $c = a$.

If $c = a$ then we have $a = b = c$ and $x = \frac{1}{2}$. If $c = -2a$ then we have $a+b+c = 0$ and $x = \frac{a}{a-2a} = -1$. These give two solutions for x.

Testing Question A.9

Write the given expression we want to evaluate as

$$\frac{2(x+2xy-y)}{x-y-2x}\,.$$

Multiply by $\frac{1}{xy}$ on the "top and bottom" to write this as

$$\frac{2\left(\frac{1}{y}+2-\frac{1}{x}\right)}{\left(\frac{1}{y}-\frac{1}{x}-2\right)}\,.$$

Then from the expression we are given the above is

$$\frac{2(2-4)}{(-4-2)} = \frac{2}{3}\,.$$

Testing Question B.1

We will denote the length of the side of each square by its "name" in lower case. Thus the length of the side of the square D will be denoted as d. Next if we pick a spot on the vertical edge of the rectangle and walk rightward we see that the horizontal edge length of this rectangle can be written as the sum of a number of different square edges. I find

$$d+i = c+e+i = c+f+h = b+1+f+h = b+g+h\,. \tag{728}$$

We will call the above the "horizontal equations". In the same way if we select a location on the horizontal edge and walking upward we see that the vertical edge can be written as the sum of a number of different square edges. I find

$$d + c + b = d + c + 1 + g = d + e + f + g = i + h = i + f + g \,. \tag{729}$$

We will call the above the "vertical equations". To solve this problem We will then select certain simple equations from the above two groups and replace all of one variable with another eventually working to a single equation we can solve. Examples of this will make it clear.

One of the equations in Equations 728 is

$$b + 1 + f + h = b + g + h \quad \text{so} \quad g = f + 1 \,.$$

Thus in Equations 728 and 729 we will replace all $g \to f + 1$. Doing this gives

$$d + i = c + e + i = c + f + h = b + 1 + f + h \,, \tag{730}$$

and

$$d + c + b = d + c + f + 2 = d + e + 2f + 1 = i + h = i + 2f + 1 \,. \tag{731}$$

In Equation 731 we have that $d + c + b = d + c + 2f$ so we see that $f = b - 2$. Thus in Equations 730 and 731 we will replace all $f \to b - 2$. Doing this gives

$$d + i = c + e + i = c + b + h - 2 = 2b + h - 1 \,, \tag{732}$$

and

$$d + c + b = d + e + 2b - 3 = i + h = i + 2b - 3 \,. \tag{733}$$

In Equation 733 we have that $i + 2b - 3 = i + h$ so we see that $h = 2b - 3$. Thus in Equations 732 and 733 we will replace all $h \to 2b - 3$. Doing this gives

$$d + i = c + e + i = c + 3b - 5 = 4b - 4 \,, \tag{734}$$

and

$$d + c + b = d + e + 2b - 3 = i + 2b - 3 \,. \tag{735}$$

In Equation 734 we have that $c + 3b - 5 = 4b - 4$ so we see that $c = b + 1$. Thus in Equations 734 and 735 we will replace all $c \to b + 1$. Doing this gives

$$d + i = b + e + i + 1 = 4b - 4 \,, \tag{736}$$

and

$$d + 2b + 1 = d + e + 2b - 3 = i + 2b - 3 \,. \tag{737}$$

In Equation 737 we have that $d + 2b + 1 = d + e + 2b - 3$ so we see that $e = 4$. Using that in Equations 736 and 737 gives

$$d + i = b + i + 5 = 4b - 4 \,, \tag{738}$$

and

$$d + 2b + 1 = i + 2b - 3\,. \tag{739}$$

In Equation 738 we have that $d + i = b + i + 5$ so we see that $d = b + 5$. Replacing $d \to b + 5$ in Equations 738 and 739 gives

$$b + 5 + i = 4b - 4\,, \tag{740}$$

and

$$i - 3 = b + 6\,. \tag{741}$$

Based on Equation 741 we will replace all i with $i \to b + 9$ in all other equations. In this case that is only Equation 740 which then means that $b = 9$. Working backwards with what we have above we can determine the value of all variables. We see that

$$\begin{aligned} b &= 9 \\ i &= 18 \\ d &= 14 \\ c &= 10 \\ h &= 15 \\ f &= 7 \\ g &= 8 \\ e &= 4\,. \end{aligned}$$

Using these, one side of the large rectangle is $d + i = 14 + 18 = 32$ and another side has a length of $d + c + b = 14 + 10 + 9 = 33$. Thus the area of the rectangle is then 32×33.

Testing Question B.2

We have

$$\begin{aligned} P(7) &= a7^7 + b7^3 + c7 - 5 \\ &= -a(-7)^7 - b(-7)^3 - c(-7) - 5 \\ &= -(a(-7)^7 + b(-7)^3 + c(-7) - 5 + 5) - 5 \\ &= -(P(-7) + 5) - 5 = -(7 + 5) - 5 = -17\,. \end{aligned}$$

Testing Question B.3

Multiply both sides by $a + b + c$ to get

$$\frac{a+b+c}{a} + \frac{a+b+c}{b} + \frac{a+b+c}{c} = 1\,,$$

or

$$1 + \frac{b}{a} + \frac{c}{a} + \frac{a}{b} + 1 + \frac{c}{b} + \frac{a}{c} + \frac{b}{c} + 1 = 1\,,$$

or

$$\left(\frac{b}{a} + \frac{1}{\frac{a}{b}}\right) + \left(\frac{c}{a} + \frac{1}{\frac{c}{a}}\right) + \left(\frac{c}{b} + \frac{1}{\frac{c}{b}}\right) = -2\,.$$

Now the left-hand-side of the above can be written as

$$f\left(\frac{b}{a}\right) + f\left(\frac{c}{a}\right) + f\left(\frac{c}{b}\right)\,,$$

for $f(x)$ defined as

$$f(x) \equiv x + \frac{1}{x}\,.$$

By plotting, we can see that its domain is all x except $x \neq 0$. Now when $x > 0$ we have

$$x + \frac{1}{x} \geq 2\,,$$

with the minimum value of $f = 2$ when $x = 1$. Now when $x < 0$ we have

$$x + \frac{1}{x} \leq -2\,.$$

with the maximum value of $f = -2$ when $x = -1$.

With this information we are ready to solve our problem. We know that $f\left(\frac{b}{a}\right)$, $f\left(\frac{c}{a}\right)$, $f\left(\frac{c}{b}\right)$ can't all be of the same sign for if they were all positive then they would all be larger than or equal to two and the left-hand-side would be larger than six. The same argument means that they can't all be negative. Thus at least one of these expressions is positive and one is negative. With out loss of generality lets assume that

$$f\left(\frac{b}{a}\right) < 0\,.$$

Then from the definition of f this means that

$$\frac{b}{a} + \frac{a}{b} < 0 \quad \text{or} \quad \frac{a}{b} < -\frac{b}{a}\,.$$

Now if a and b are both positive the above inequality cannot be made true. If a and b are both negative the above inequality cannot be made true. It is only if a and b are of different sign (one positive and one negative) that it can be true (say for $a = -1$ and $b = 2$).

Testing Question B.4

From the given equations by replacing the variables y and z in tern we can write x as

$$x = \frac{a}{y} = \frac{az}{c} = \frac{ab}{cx} \quad \text{so} \quad x^2 = \frac{ab}{c}\,.$$

By replacing the variables x and z in tern we can write y as

$$y = \frac{a}{x} = \frac{az}{b} = \frac{ac}{by} \quad \text{so} \quad y^2 = \frac{ac}{b}\,.$$

By replacing the variables y and x in tern we can write z as

$$z = \frac{c}{y} = \frac{cx}{a} = \frac{cb}{az} \quad \text{so} \quad z^2 = \frac{cb}{a} \, .$$

Using these three expressions we find that

$$x^2 + y^2 + z^2 = \frac{ab}{c} + \frac{ac}{b} + \frac{cb}{a} = \frac{a^2b^2 + a^2c^2 + c^2b^2}{abc} \, .$$

Testing Question B.5

Using the fact that

$$\sum_{k=0}^{4} a^k = \frac{1 - a^5}{1 - a} \, ,$$

when we set this equal to zero we see that $a^5 = 1$. From that we have that

$$a^{2000} = (a^5)^{400} = 1 \, ,$$

and

$$a^{2010} = (a^5)^{402} = 1 \, .$$

Thus the expression we want to evaluate is then $1 + 1 + 1 = 3$.

Testing Question B.6

Note that the expansion in a_k is a polynomial expansion of the expression $(x^2 - x - 1)^n$ but that the thing we wan to evaluate is the sum of only the even terms of this expansion. Towards that end let the polynomial expansion be written as

$$(x^2 - x - 1)^n = \sum_{k=0}^{2n} a_k x^k \, .$$

Then replacing $x \to -x$ in that we have

$$(x^2 + x - 1)^n = \sum_{k=0}^{2n} a_k (-1)^k x^k \, .$$

Note that if we add these two expressions we get

$$(x^2 - x - 1)^n + (x^2 + x - 1)^n = \sum_{k=0}^{2n} a_k [1 + (-1)^k] x^k = 2 \sum_{k=0}^{n} a_{2k} x^{2k} \, .$$

Taking $x = 1$ in the above then gives

$$\sum_{k=0}^{n} a_{2k} = \frac{1}{2}((1 - 1 - 1)^n + (1 + 1 - 1)^n) = \left\{ \begin{array}{ll} 1 & n \text{ even} \\ 0 & n \text{ odd} \end{array} \right. \, .$$

Lecture 3: Linear Equations of a Single Variable

Testing Question A.1

Putting these two values of x into the given expressions only D gives the value of zero.

Testing Question A.2

If $k = 0$ there are no solutions so we must have $k \neq 0$. In that case we have

$$x = \frac{12}{k} \,,$$

which for x to be an integer means that k must be a divisor of 12. Thus k could be one of $\{1, 2, 3, 4, 6, 12\}$.

Testing Question A.3

If $x = 1$ the left-hand-side is

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} = 1 + \frac{5}{6} = \frac{11}{6} > \frac{13}{12} \,.$$

If $x = 2$ the left-hand-side is

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{5}{6} + \frac{1}{4} = \frac{13}{12} \,,$$

so we have one integer solution. As the left-hand-side is a decreasing function of x for $x \geq 3$ the left-hand-side will be smaller than its value when $x = 2$ and thus not equal to $\frac{13}{12}$. Thus there is only one solution.

Testing Question A.4

Put $x = 4$ into the given expression to get

$$3a - 4 = 2 + 3 \quad \text{or} \quad a = 3 \,.$$

Then $(-a)^2 - 2a = 9 - 2(3) = 3$.

Testing Question A.5

Lets multiply the given expression by $mn \neq 0$ to get

$$n(x - n) - m(x - m) = m^2 \,,$$

or

$$(n-m)x = n^2 + m^2 - m^2 = n^2\,.$$

If $n-m=0$ there are no solutions unless $n=0$ then there are an infinite number of solutions. As in this problem we are told that $n \neq 0$ (since $nm \neq 0$) we conclude that if $n-m=0$ there are no solutions. If $n-m \neq 0$ then the solution is

$$x = \frac{n^2}{n-m}\,.$$

Testing Question A.6

If $a+b=0$ then all x are solutions (this condition is equivalent to if $a=-b$). If $a+b \neq 0$ we can divide by it to get $4ax = a+b$. If $a=0$ there are no solutions (unless $b=0$). If $a \neq 0$ the solution is $x = \frac{a+b}{4a}$.

Testing Question A.7

If we put $x=-2$ into the given expression we get

$$-\frac{2}{3}m = -10+4 = -6 \quad \text{so} \quad m = 9\,.$$

In that case $m^2 - 11m + 17 = -1$ so this raised to the power of 2007 is -1.

Testing Question A.8

Write this equation as $(m^2-m)x = m-1$ or $m(m-1)x = m-1$. If $m=1$ then all x are solutions. If $m \neq 1$ then we get $mx=1$. From this we see that if $m=0$ there are no solutions and otherwise the single solution is $x = \frac{1}{m}$.

Testing Question A.9

Write this equation as

$$k(k-2)x = k(k-5)\,.$$

If $k=0$ than any x value is a solution. If $k \neq 0$ then we have

$$(k-2)x = k-5\,.$$

Now if $k=2$ then there are no solutions so if $k \neq 2$ we have

$$x = \frac{k-5}{k-2}\,.$$

As x is positive so

$$\frac{k-5}{k-2} > 0\,.$$

The left-hand-side of this expression changes sign at the points as k crosses $k = 2$ and $k = 5$. Over the range of possible k values we find that when $0 < k < 2$ the ratio is positive, when $2 < k < 5$ the ratio is negative and when $k > 5$ the ratio is positive again.

Testing Question A.10

Write this equation as

$$(2a-3)x = a-3\,.$$

If $a = \frac{3}{2}$ then the left-hand-side is always zero but the right-hand-side is not. In that case there is no solution to this equation.

Testing Question B.1

If we simplify the first equation we get

$$x = -\frac{2}{5}a\,.$$

If we simplify the second equation we get

$$2x + 2a = 1\,.$$

If we put this first expression into the second expression we get $a = \frac{5}{6}$ which then means that $x = -\frac{1}{3}$.

Testing Question B.2

Write the given equation as

$$\frac{a}{ab+a+1} + \frac{b}{bc+b+1} + \frac{c}{ca+c+1} = \frac{1}{2x}\,.$$

Then using the fact that $c = \frac{1}{ab}$ to write everything in terms of a and b we have

$$\frac{a}{ab+a+1} + \frac{b}{\frac{1}{a}+b+1} + \frac{\frac{1}{ab}}{\frac{1}{b}+\frac{1}{ab}+1} = \frac{1}{2x}\,.$$

In the third fraction on the left-hand-side multiply by b "on the top and bottom" to get

$$\frac{a}{ab+a+1} + \frac{b}{\frac{1}{a}+b+1} + \frac{\frac{1}{a}}{1+b+\frac{1}{a}} = \frac{1}{2x}\,.$$

Adding the last two fractions on the left-hand-side together we get

$$\frac{a}{ab+a+1} + \frac{b+\frac{1}{a}}{\frac{1}{a}+b+1} = \frac{1}{2x}\,.$$

In the first fraction on the left-hand-side multiply by $\frac{1}{a}$ "on the top and bottom" to get

$$\frac{1}{b+1+\frac{1}{a}} + \frac{b+\frac{1}{a}}{\frac{1}{a}+b+1} = \frac{1}{2x}\,.$$

Adding the first two fractions we get

$$\frac{1+b+\frac{1}{a}}{b+1+\frac{1}{a}} = \frac{1}{2x}\,,$$

or

$$1 = \frac{1}{2x} \quad \text{so} \quad x = \frac{1}{2}\,.$$

Testing Question B.3

If we solve the given equation for x we find

$$x = \frac{12}{5}(m+123)\,. \tag{742}$$

For x to be a positive integer we must have $x \geq 1$ so using the above this means that

$$m \geq \frac{5}{12} - 123 = -122.5833\,.$$

For m to be a positive integer the smallest value that satisfies the above is $m = 1$ but in Equation 742 we see that x will not be an integer. For x to be an integer we need $m + 123$ to be divisible by five so the number $m + 123$ needs to end in a zero or a five. We can make that number end in a five if we take $m = 2$. Then we find $x = 300$.

Testing Question B.4

Starting with the given equation simplify it and solve for x to find $x = 2$. Then we want an equation of the form

$$ax - \frac{1}{2} = 0\,,$$

that has $x = 2$ as a solution. Putting that value of x into this and solving for a we find $a = \frac{1}{4}$ and the full equation is

$$\frac{1}{4}x - \frac{1}{2} = 0\,.$$

Testing Question B.5

By starting with $a_1 = 1$ we see that

$$a_2 = \frac{1}{2}$$
$$a_3 = \frac{1}{3}$$
$$a_4 = \frac{1}{4},$$

and it looks like the pattern is

$$a_n = \frac{1}{n}.$$

We can prove this by induction for if the above holds then from the formula for a_{n+1} we have

$$a_{n+1} = \frac{1}{1 + \frac{1}{\frac{1}{n}}} = \frac{1}{n+1}.$$

This means that the desired sum S is given by

$$S = \sum_{i=1}^{2008} \frac{1}{i(i+1)}.$$

Using partial fractions we have that

$$\frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1},$$

and thus the sum we want to evaluate is given by

$$S = \sum_{i=1}^{2008} \frac{1}{i} - \sum_{i=1}^{2008} \frac{1}{i+1} = \sum_{i=1}^{2008} \frac{1}{i} - \sum_{i=2}^{2009} \frac{1}{i} = 1 - \frac{1}{2009} = \frac{2008}{2009}.$$

Lecture 4: Systems of Simultaneous Linear Equations

Testing Question A.1

Putting these values of x and y into the equations given we get

$$2a + b = 7$$
$$2b + c = 5\,.$$

Solving for b in the first equation and putting that into the second equation gives

$$4a - c = 9\,,$$

when we simplify.

Testing Question A.2

Using

$$3x - y = 5$$
$$2x + 3y = -4\,,$$

from the first and second sets respectively gives $x = 1$ and $y = -2$. Using these two values in $2x + y - z = 0$ gives $z = 0$. Using these three values in both sets of given equations gives

$$4a - 10b = -22$$
$$a + 2b = 8$$
$$1 - 2 + 5 = c\,.$$

Solving these we find $(a, b, c) = (2, 3, 4)$.

Testing Question A.3

From the first equation we have

$$kx - y = -\frac{1}{3} \quad \text{or} \quad y = kx + \frac{1}{3}\,.$$

If we put that into the second equation we get

$$3kx + 1 = 1 - 6x\,,$$

or

$$(3k + 6)x = 0 \Rightarrow (k + 2)x = 0\,.$$

Now if $k = -2$ there are an infinite number of solutions for x (and y), if $k \neq -2$ there is one solution $x = 0$ and $y = \frac{1}{3}$.

Testing Question A.4

Starting with the first equation we can write it as

$$(b-2)a = 2b\,.$$

Now if $b \neq 2$ then we get

$$a = \frac{2b}{b-2}\,.$$

If we put this into the second of the given equations we get

$$\frac{\left(\frac{2b}{b-2}\right)c}{\left(\frac{2b}{b-2}\right)+c} = 5\,.$$

This can be manipulated into

$$-3bc = 10(b-c)\,. \tag{743}$$

Now if we take the third of the given equations written in the form $bc = 4(b+c)$ by taking the ratio of these two we get

$$-3 = \frac{10(b-c)}{4(b+c)}\,.$$

We can manipulate this into $11b^2 = 40b$. From this we see that $b = 0$ or $b = \frac{40}{11}$. If $b = 0$ then $c = 0$ and $a = 0$. That set of numbers cannot be a solution as it will not satisfy the original equations. If $b = \frac{40}{11}$ then using Equation 743 we get $c = -40$. Finally using $ac = 5(a+c)$ we get $a = \frac{40}{9}$. Thus the solution is

$$(a, b, c) = \left(\frac{40}{9}, \frac{40}{11}, -40\right)\,.$$

Testing Question A.5

If we add these equations together we get

$$-x - y - z = -9 \quad \text{or} \quad x + y + z = 9\,.$$

From the first of the equations given in the question we have $x = y + z + 5$. If we put that in the equation just derived we get

$$(y+z) = 2\,.$$

If we put the expression $y + z$ in the first equation given in the question we have

$$x - 2 = 5 \quad \text{or} \quad x = 7\,.$$

Taking $x = 7$ in the three equations given in the question we have

$$-y - z = -2 \tag{744}$$

$$y - z = 8 \tag{745}$$

$$z - y = -15 + 7 = -8\,. \tag{746}$$

Adding Equations 744 and 745 we derive $z = -3$. Putting this value of z into Equation 744 we derive $y = 5$.

Testing Question A.6

Add all equations together we get

$$x + y + z + u + v = 15\,. \tag{747}$$

Next if we add sequential pairs of the given equations we get

$$\begin{aligned} x + u &= 3 && (748)\\ y + v &= 5 && (749)\\ z + x &= 7 && (750)\\ u + y &= 9 && (751)\\ v + z &= 6\,. && (752) \end{aligned}$$

Now using Equation 747 with Equations 748 and 749 we get

$$3 + 5 + z = 15 \quad \text{or} \quad z = 7\,.$$

Using that in Equation 750 gives $x = 0$. Then Equation 748 gives $u = 3$, Equation 751 gives $y = 6$, and Equation 752 gives $v = -1$. Thus we have found the solution

$$(x, y, z, u, v) = (0, 6, 7, 3, -1)\,.$$

Testing Question A.7

Write the first equation as

$$\frac{1}{x} = -\frac{2}{y} - \frac{3}{z}\,, \tag{753}$$

and the second equation as

$$\frac{1}{x} = \frac{6}{y} + \frac{5}{z}\,.$$

If we set these two expressions for $\frac{1}{x}$ equal we get

$$-\frac{2}{y} - \frac{3}{z} = \frac{6}{y} + \frac{5}{z}\,.$$

We can write that expression as

$$-\frac{8}{y} - \frac{8}{z} = 0 \Rightarrow -\frac{1}{y} = \frac{1}{z} \Rightarrow y = -z\,.$$

If we put that into Equation 753 we get

$$\frac{1}{x} = -\frac{2}{y} + \frac{3}{y} = \frac{1}{y}\,.$$

so we see that $x = y$. Thus we have learned that $x = y = -z$ and so

$$\frac{x}{y} + \frac{y}{z} + \frac{z}{x} = 1 - 1 - 1 = -1\,.$$

Testing Question A.8

The system we want to solve is

$$\begin{aligned} mx + 2y &= 10 \\ 3x - 2y &= 0 \,. \end{aligned}$$

If we add these two equations together we get $(3+m)x = 10$ and thus

$$x = \frac{10}{3+m} \,.$$

Using the second equation above this means that

$$2y = 3x = \frac{30}{3+m} \Rightarrow y = \frac{15}{3+m} \,.$$

For x and y to be integers the number $3+m$ must divide by ten and fifteen. The numbers that divide ten are $\{1, 2, 5, 10\}$ and the numbers that divide fifteen are $\{1, 3, 5, 15\}$. Only the numbers one and five divide both ten and fifteen. This means we have two potential solutions.

One solution is if $3+m = 1$ so that $m = -2$ so that $x = 10$ and $y = 15$ and the original system then takes the form

$$\begin{aligned} -2x + 2y &= 10 \\ 3x - 2y &= 0 \,. \end{aligned}$$

This means that $m^2 = 4$.

Another solution is if $3+m = 5$ so that $m = 2$ so that $x = 2$ and $y = 3$ and the original system in this case takes the form

$$\begin{aligned} 2x + 2y &= 10 \\ 3x - 2y &= 0 \,. \end{aligned}$$

In this case $m^2 = 4$ also.

Testing Question A.9

Let t be the sum of any of the rows, any of the columns, or any diagonal of the given grid. Then from the given letters in the grid we have

$$a+b+6 = t \quad (754)$$
$$c+d+e = t \quad (755)$$
$$f+7+2 = t \quad \text{or} \quad f+9=t \quad (756)$$
$$a+c+f = t \quad (757)$$
$$b+d+7 = t \quad (758)$$
$$6+e+2 = t \quad \text{or} \quad e+8=t \quad (759)$$
$$a+d+2 = t \quad (760)$$
$$f+d+6 = t\,. \quad (761)$$

If we equate Equation 756 to Equation 759 we get

$$f+9=e+8 \quad \text{or} \quad e=f+1\,.$$

Using this we will replace all e's with f's in Equation 755 to get

$$c+d+f+1=t\,,$$

for a full system given by

$$a+b = t-6 \quad (762)$$
$$c+d+f = t-1 \quad (763)$$
$$f = t-9 \quad (764)$$
$$a+c+f = t \quad (765)$$
$$b+d = t-7 \quad (766)$$
$$a+d = t-2 \quad (767)$$
$$f+d = t-6\,. \quad (768)$$

In this system we next replace all t's with f's using Equation 764 or $t=f+9$ to get

$$a+b=f+3 \Rightarrow a+b-f=3 \quad (769)$$
$$c+d+f=f+8 \Rightarrow c+d=8 \quad (770)$$
$$a+c+f=f+9 \Rightarrow a+c=9 \quad (771)$$
$$b+d=f+2 \Rightarrow b+d-f=2 \quad (772)$$
$$a+d=f+7 \Rightarrow a+d-f=7 \quad (773)$$
$$f+d=f+3 \Rightarrow d=3\,. \quad (774)$$

Using the last equation we can put $d=3$ into everything. In Equation 770 we will get $c=5$. In equation 771 we get $a=4$. As we now know a and d using Equation 767 we find $t=4+3+2=9$. Then using this value in Equation 759 we get $e=1$. Using this value in

Equation 756 we get $f = 0$. Using what we know in Equation 754 we get $b = 9 - 6 - 4 = -1$. With all of this our square looks like

$$\left|\begin{array}{|c|c|c|} \hline 4 & -1 & 6 \\ \hline 5 & 3 & 1 \\ \hline 0 & 7 & 2 \\ \hline \end{array}\right| .$$

Note the book gets different numbers than what I get here but the numbers quoted here satisfy the required sums in the square.

Testing Question A.10

Our equations are

$$\begin{aligned} x + y + z + u &= 10 \quad (775) \\ 2x + y + 4z + 3u &= 29 \quad (776) \\ 3x + 2y + z + 4u &= 27 \quad (777) \\ 4x + 3y + z + 2u &= 22 . \quad (778) \end{aligned}$$

If we add all of these together we get

$$10x + 7y + 7z + 10u = 88 ,$$

or

$$10(x + u) + 7(y + z) = 88 .$$

Now from Equation 775 we have

$$1(x + u) + 1(y + z) = 10 .$$

Motivated by these two expressions let $a \equiv x + u$ and $b \equiv y + z$ and we end with the system

$$\begin{aligned} a + b &= 10 \\ 10a + 7b &= 88 . \end{aligned}$$

Solving this we get $a = 6$ and $b = 4$. Then from the definitions of a and b this means that

$$x + u = 6 \Rightarrow u = 6 - x ,$$

and

$$y + z = 4 \Rightarrow z = 4 - y .$$

Putting these into Equation 776 and simplifying we get

$$-x - 3y = -5 . \quad (779)$$

Putting these into Equation 777 and simplifying we get

$$-x + y = -1 . \quad (780)$$

These give two equations in terms of x and y. Solving we get $x = 2$ and $y = 1$. From these we then have

$$u = 4 \quad \text{and} \quad z = 3 .$$

Testing Question B.1

To start we write the two given equations as

$$x + \frac{m}{3}y = \frac{7}{3}$$
$$x + \frac{n}{2}y = 2\,.$$

Next put x from the first equation into second equation to get

$$-\frac{m}{3}y - \frac{7}{3} + \frac{n}{2}y = 2\,,$$

which we can eventually write as

$$(3n - 2m)y = 28\,.$$

This will not have any solutions if $3n - 2m = 0$ or $3n = 2m$. This last equation means that m is divisible by three and n is divisible by two. As we are told that m and n are integers in the domain $[-10, +10]$ this means that the possible choices for m are from the set

$$m \in \{-9, -6, -3, 0, 3, 6, 9\}\,,$$

and the possible choices for n are from the set

$$n \in \{-10, -8, -6, -4, -2, 0, 2, 4, 6, 8, 10\}\,.$$

These results then tell us that

$$2m \in \{-18, -12, -6, 0, 6, 12, 18\}\,,$$

and

$$3n \in \{-30, -24, -18, -12, -6, 0, 6, 12, 18, 24, 30\}\,.$$

If we match pairs of (m, n) where $3n = 2m$ we see that for m taken from the set

$$m \in \{-9, -6, -3, 0, 3, 6, 9\}\,,$$

and then paired with n taken from the set

$$n \in \{-6, -4, -2, 0, 2, 4, 6\}\,,$$

we will have the needed solutions to this problem.

Testing Question B.2

We start by writing these three equations as

$$\frac{x+y+z}{x(y+z)} = \frac{1}{2}$$
$$\frac{x+y+z}{y(x+z)} = \frac{1}{3}$$
$$\frac{x+y+z}{z(x+y)} = \frac{1}{4}\,,$$

or

$$x + y + z = \frac{1}{2}x(y+z) = \frac{1}{3}y(x+z) = \frac{1}{4}z(x+y)\,.$$

Lets call the common left-hand-side of each expression t. Then we can write the above as

$$2t = xy + xz \tag{781}$$

$$3t = yz + xy \tag{782}$$

$$4t = xz + yz\,. \tag{783}$$

If we add these three equations together we get

$$xy + xz + yz = \frac{9}{2}t\,. \tag{784}$$

As Equations 781, 782, and 783 have left-hand-sides that are multiples of t lets "subtract" these to get other expressions for t. If we take Equation 782 minus Equation 781 we get

$$t = yz - xz\,. \tag{785}$$

If we take Equation 783 minus Equation 782 we get

$$t = xz - xy\,. \tag{786}$$

Finally, if we take Equation 783 minus Equation 781 we get

$$2t = yz - xy\,. \tag{787}$$

Now using Equation 785 and 786 we get

$$yz - xz = xz - xy \quad \text{or} \quad xy + yz = 2xz\,.$$

If we put that into Equation 784 we get

$$3xz = \frac{9}{2}t \quad \text{or} \quad xz = \frac{3}{2}t\,.$$

Using this value for xz in Equation 786 and 785 we get $xy = \frac{t}{2}$ and $yz = \frac{5}{2}t$. At this point to summarize then we have shown that

$$xy = \frac{1}{2}t \tag{788}$$

$$xz = \frac{3}{2}t \tag{789}$$

$$yz = \frac{5}{2}t\,. \tag{790}$$

Taking the ratio of Equation 788 to 788 I get

$$\frac{y}{z} = \frac{1/2}{3/2} = \frac{1}{3} \quad \text{or} \quad y = \frac{z}{3}\,.$$

Taking the ratio of Equation 788 to 790 I get

$$\frac{x}{z} = \frac{1/2}{5/2} = \frac{1}{5} \quad \text{or} \quad x = \frac{z}{5}\,.$$

Putting these two expressions into Equation 788 we have

$$2\left(\frac{z}{3}\right)\left(\frac{z}{5}\right) = x + y + z = \frac{z}{5} + \frac{z}{3} + z\,,$$

which is a single equation in z. Simplifying this we get $z = 0$ or $z = \frac{23}{2}$. If $z = 0$ then $x = y = 0$. If $z = \frac{23}{2}$ then $x = \frac{z}{5} = \frac{23}{10}$ and $y = \frac{z}{3} = \frac{23}{6}$.

Testing Question B.3

To start we bring the terms $2x^2$, $2y^2$ and $2z^2$ over to the left-hand-side to get

$$\begin{aligned} x(y+z+x) &= 60 \quad (791)\\ y(z+x+y) &= 75 \quad (792)\\ z(x+y+z) &= 90 \quad (793) \end{aligned}$$

If we take Equation 791 divide it by Equation 792 we get

$$\frac{x}{y} = \frac{60}{75} = \frac{4}{5}\,. \quad (794)$$

If we take Equation 791 divide it by Equation 793 we get

$$\frac{x}{z} = \frac{60}{90} = \frac{2}{3}\,. \quad (795)$$

Finally if we take Equation 792 and divide it by Equation 793 we get

$$\frac{y}{z} = \frac{25}{90} = \frac{5}{6}\,. \quad (796)$$

Thus from these we conclude that

$$\begin{aligned} x &= \frac{4}{5}y\\ x &= \frac{2}{3}z\\ y &= \frac{5}{6}z\,. \end{aligned}$$

This means that $y = \frac{5}{4}x$ and $z = \frac{3}{2}x$. If we put these into Equation 791 we get one equation in terms of x of

$$x\left(\frac{5}{4}x + \frac{3}{2}x + x\right) = 60\,,$$

or

$$x^2\left(\frac{5}{4} + \frac{6}{4} + \frac{4}{4}\right) = 60\,.$$

Solving for x gives $x^2 = 16$ so $x = \pm 4$. Thus $y = \pm 5$ and $z = \pm 6$. We can check that $(x, y, z) = \pm(4, 5, 6)$ are solutions to the original equations.

Testing Question B.4

We are given the system

$$\begin{aligned} x + 2y &= a + 6 \quad (797)\\ 2x - y &= 25 - 2a\,. \quad (798) \end{aligned}$$

If we multiply Equation 798 by two and add that to Equation 797 we get

$$5x = 56 - 3a \quad \text{so} \quad x = \frac{56 - 3a}{5} \,. \tag{799}$$

Putting this into Equation 798 and solving for y we get

$$y = \frac{-13 + 4a}{5} \,. \tag{800}$$

Now for x to be a positive we must have

$$56 - 3a > 0 \quad \text{or} \quad a < \frac{56}{3} \,,$$

and for y to be positive we must have

$$-13 + 4a > 0 \quad \text{or} \quad a > \frac{13}{4} = 3 \,.$$

This means that a is bounded as

$$\frac{13}{4} < a < \frac{56}{3} \,. \tag{801}$$

Next for x and y to be integers means that $56 - 3a$ and $-13 + 4a$ must both be divisible by five. This means that

$$56 - 3a = 5n$$
$$-13 + 4a = 5m \,,$$

for integers $n > 0$ and $m > 0$. From these two expressions we can solve for a in each to get

$$a = \frac{56 - 5n}{3} \tag{802}$$
$$a = \frac{5m + 13}{4} \,. \tag{803}$$

To have a given by Equation 802 satisfy Equation 801 we need

$$\frac{13}{4} < \frac{56 - 5n}{3} < \frac{56}{3} \,.$$

In terms of n this is

$$0 < n < 9.25 \,.$$

Thus the possible integer values for n are $n \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. To have a given by Equation 803 satisfy Equation 801 we need to have

$$\frac{13}{4} < \frac{5m + 13}{4} < \frac{56}{3} \,.$$

In terms of m this is

$$0 < m < 12.33 \,.$$

Thus the possible integer values for m are $m \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$.

Setting the two expressions for a given by Equation 802 and 803 equal to each other we can get one expression relating n and m given by

$$m = \frac{37 - 4n}{3} \,.$$

If we put the possible values for n into this expressions we get

```
[1] 11.00  9.67  8.33  7.00  5.67  4.33  3.00  1.67  0.33
```

The values above that are permissible for m are three, seven, and eleven and correspond to the pairs

$$(n, m) \in \{(7, 3), (4, 7), (1, 11)\} ,$$

and correspond to the values of a given by 7, 12, and 17. From these a values we can compute x and y and find

$$x \in \{7, 4, 1\} \quad \text{with} \quad y \in \{3, 7, 11\} .$$

Note: I solved this question under the constraints that $x > 0$ and $y > 0$ and not $x \geq 1$ and $y \geq 1$ thus I obtained a few more solutions than the book did.

Testing Question B.5

If we add all five equations together we get

$$6(x + y + z + u + v) = 96 ,$$

so that $x + y + z + u + v = 16$. Next write the first equation as

$$x + (x + y + z + u + v) = 16 ,$$

and put in what we know for the sum above. This gives $x = 0$. Next write the second equation as

$$y + (x + y + z + u + v) = 17 \quad \text{so} \quad y = 1 .$$

The other equations can be solved in the same way. Doing this we find $z = 19 - 16 = 3$, $u = 5$, and $v = 7$.

Lecture 5: Multiplication Formula

Testing Question A.1

Write the given expression like

$$(a^2 + 8a) + (b^2 - 14b) + 65 = 0\,.$$

Complete the square of the two quadratic terms as

$$(a^2 + 8a + 16 - 16) + (b^2 - 14b + 49 - 49) + 65 = 0\,,$$

which becomes

$$(a+4)^2 + (b-7)^2 = 0\,.$$

This means that $a = -4$ and $b = 7$. Using these we have that

$$a^2 + ab + b^2 = 16 - 28 + 49 = 37\,.$$

Testing Question A.2

One way to work this problem is write all variables in terms of a single other and simplify. To do this we note that from what we are given we have $a = 2 + b$ and $c = b - 4$. If we call the given expression we seek E we have

$$\begin{aligned}
E &= a^2 + b^2 + c^2 - ab - bc - ca \\
&= (2+b)^2 + b^2 + (b-4)^2 - (2+b)b - b(b-4) - (2+b)(b-4) \\
&= 4 + 4b + b^2 + b^2 + b^2 - 8b + 16 - 2b - b^2 - b^2 + 4b - (2b - 8 + b^2 - 4b) \\
&= 20 - 2b + b^2 - (b^2 - 2b - 8) = 20 - 2b + 2b + 8 = 28\,.
\end{aligned}$$

Another way to work this problem is to first note that by adding $a - b = 2$ and $b - c = 4$ by together we get $a - c = b$. Next we note that

$$\begin{aligned}
a^2 + b^2 - ab &= (a-b)^2 + ab \\
b^2 + c^2 - bc &= (b-c)^2 + bc \\
a^2 + c^2 - ac &= (a-c)^2 + ac\,.
\end{aligned}$$

If we add these together we get

$$2a^2 + 2b^2 + 2c^2 - ab - bc - ac = (a-b)^2 + (b-c)^2 + (a-c)^2 + ab + bc + ac\,,$$

or

$$2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ac = (a-b)^2 + (b-c)^2 + (a-c)^2\,.$$

The left-hand-side is $2E$ so we have

$$\begin{aligned}
E &= a^2 + b^2 + c^2 - ab - bc - ac = \frac{1}{2}\left[(a-b)^2 + (b-c)^2 + (a-c)^2\right] \\
&= \frac{1}{2}\left[4 + 16 + 36\right] = 28\,.
\end{aligned}$$

Testing Question A.3

Call this expression E. Then we have

$$\begin{aligned} E &= (a^2 + b^2)(c^2 + d^2) \\ &= a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 \\ &= (ac)^2 + (bd)^2 + (ad)^2 + (bc)^2 \\ &= (ac + bd)^2 - 2abcd + (ad - bc)^2 + 2abcd \\ &= (ac + bd)^2 + (ad - bc)^2 \,, \end{aligned}$$

which is the form we want.

Testing Question A.4

Expand the right-hand-side to get

$$14a^2 + 14b^2 + 14c^2 = a^2 + 4b^2 + 9c^2 + 4ab + 6ac + 12bc \,,$$

or

$$13a^2 + 10b^2 + 5c^2 - 4ab - 6ac - 12bc = 0 \,.$$

Now we will try to "complete the square" in the left-hand-side and see what information that might provide. Note that in doing this as we have terms $-4ab$ and $-6ac$ we might have to split the term $13a^2$ into a "part" for b^2 and a "part" for c^2. While that may seem complicated observing what we do might make things clearer. Towards that end we will write the above as

$$(A_1a^2 - 4ab + B_1b^2) + (A_2a^2 - 6ac + C_1c^2) + (B_2b^2 - 12bc + C_2c^2) = 0 \,. \tag{804}$$

The splitting of a^2, b^2, and c^2 mean that

$$\begin{aligned} A_1 + A_2 &= 13 \\ B_1 + B_2 &= 10 \\ C_1 + C_2 &= 5 \,. \end{aligned}$$

Now we expect (hope) that each of the term groupings above will factor. This means that each of the numbers above should actually be a perfect square. This means that

$$\begin{aligned} A_1 &= a_1^2 \\ A_2 &= a_2^2 \,, \end{aligned}$$

and so on. Thus we have

$$a_1^2 + a_2^2 = 13 \tag{805}$$

$$b_1^2 + b_2^2 = 10 \tag{806}$$

$$c_1^2 + c_2^2 = 5 \,. \tag{807}$$

Assuming all of the above are integers then from Equation 805 we have that

$$(a_1, a_2) = (2, 3) \quad \text{or} \quad (a_1, a_2) = (3, 2) \,.$$

From Equation 806 we have that

$$(b_1, b_2) = (1, 3) \quad \text{or} \quad (b_1, b_2) = (3, 1) \,.$$

From Equation 807 we have that

$$(c_1, c_2) = (1, 2) \quad \text{or} \quad (c_1, c_2) = (2, 1) \,.$$

If we then look at the first term we would factor in Equation 804 we see that to get the middle term of $-4ab$ we must have

$$-2a_1 b_1 = -4 \quad \text{or} \quad a_1 b_1 = 2 \,.$$

The only choices that do this are

$$(a_1, a_2) = (2, 3) \quad \text{and} \quad (b_1, b_2) = (1, 3) \,.$$

If we then look at the third term we would factor in Equation 804 we see that to get the middle term of $-12bc$ we must have

$$-2b_2 c_2 = -12 \quad \text{or} \quad b_2 c_2 = 6 \,.$$

The only choice that do this (given $b_2 = 3$) is

$$(c_1, c_2) = (1, 2) \,.$$

With these choices, we can write Equation 804 as

$$(4a^2 - 4ab + b^2) + (9a^2 - 6ac + c^2) + (9b^2 - 12bc + 4c^2) = 0 \,.$$

This we can factor! We find

$$(2a - b)^2 + (3a - c)^2 + (3b - 2c)^2 = 0 \,.$$

This means that we have

$$\begin{aligned} 2a &= b \\ 3a &= c \,, \end{aligned}$$

Thus

$$a : b : c = a : 2a : 3a = 1 : 2 : 3 \,.$$

Testing Question A.5

Note that we can write

$$x^2 + 3x + 1 = x^2 + 2x + 1 + x = (x+1)^2 + x\,.$$

Using this our expression is

$$\frac{x}{(x+1)^2 + x} = a\,,$$

or

$$\frac{1}{\frac{(x+1)^2}{x} + 1} = a\,.$$

If we "flip" this we get

$$\begin{aligned} \frac{(x+1)^2}{x} + 1 &= \frac{1}{a} \quad \text{or} \\ \frac{x^2+2x+1}{x} &= \frac{1}{a} - 1 \quad \text{or} \\ x + 2 + \frac{1}{x} &= \frac{1}{a} - 1\,, \end{aligned}$$

or

$$x + \frac{1}{x} = \frac{1}{a} - 3\,. \tag{808}$$

Next using the same factoring the expression we want can be written as

$$\begin{aligned} \frac{x^2}{x^4 + 3x^2 + 1} &= \frac{x^2}{(x^2+1)^2 + x^2} = \frac{1}{\frac{1}{x^2}(x^2+1)^2 + 1} \\ &= \frac{1}{\frac{x^4+2x^2+1}{x^2} + 1} = \frac{1}{x^2 + 2 + \frac{1}{x^2} + 1} \\ &= \frac{1}{\left(x + \frac{1}{x}\right)^2 + 1}\,. \end{aligned}$$

But we know how to expression $x + \frac{1}{x}$ in terms of a from Equation 808. Using that we get

$$\frac{1}{\left(\frac{1}{a} - 3\right)^2 + 1}\,,$$

for the value of the desired expression.

Testing Question A.6

Note that if we can get an expression for

$$x^3 + \frac{1}{x^3}\,,$$

in terms of a then we can square this to get the desired expression. If we recall that

$$X^3 + Y^3 = (X + Y)(X^2 - XY + Y^2)\,.$$

Thus if we let $X = x$ and $Y = \frac{1}{x}$ we get

$$x^3 + \frac{1}{x^3} = a\left(x^2 - 1 + \frac{1}{x^2}\right)\,.$$

This means that we need to be able to evaluate $x^2 + \frac{1}{x^2}$. If we square the given expression we get

$$x^2 + 2 + \frac{1}{x^2} = a^2 \quad \text{so} \quad x^2 + \frac{1}{x^2} = a^2 - 2\,.$$

If we use this in the above we have

$$x^3 + \frac{1}{x^3} = a(a^2 - 3)\,.$$

If we then square this we get

$$x^6 + 2 + \frac{1}{x^6} = a^2(a^2 - 3)^2\,,$$

thus

$$x^6 + \frac{1}{x^6} = a^2(a^2 - 3)^2 - 2 = a^6 - 6a^4 + 9a^2 - 2\,.$$

Testing Question A.7

Starting with

$$a^4 + b^4 + c^4 + d^4 = 4abcd\,,$$

we will write the given expression as

$$(a^2 - b^2)^2 + 2a^2b^2 + (c^2 - d^2)^2 + 2c^2d^2 = 4abcd\,,$$

or

$$(a^2 - b^2)^2 + (c^2 - d^2)^2 + 2(a^2b^2 - 2abcd + c^2d^2) = 0\,,$$

or

$$(a^2 - b^2)^2 + (c^2 - d^2)^2 + 2(ab - cd)^2 = 0\,.$$

This means that $a^2 = b^2$, $c^2 = d^2$, and $ab = cd$. If we square this last expression we get

$$a^2b^2 = c^2d^2\,.$$

Using the fact that $a^2 = b^2$ and $c^2 = d^2$ this means that

$$b^4 = c^4 = d^4\,.$$

If we take square roots of the above (assuming everything is real so that $b^2 > 0$) we get

$$b^2 = c^2 = d^2\,.$$

Thus we have shown that

$$a^2 = b^2 = c^2 = d^2\,.$$

Testing Question A.8

From what we are given we have

$$a + b = -(c + d) .$$

If we cube each side we get

$$a^3 + 3a^2b + 3ab^2 + b^3 = -(c^3 + 3c^2d + 3cd^2 + d^3) ,$$

or

$$\begin{aligned} a^3 + b^3 + c^3 + d^3 &= -3a^2b - 3ab^2 - 3c^2d - 3cd^2 \\ &= -3(a^2b + ab^2 + c^2d + cd^2) \\ &= -3(ab(a + b) + cd(c + d)) . \end{aligned}$$

Using the fact $a + b = -(c + d)$ and $c + d = -(a + b)$ we get can write the above as

$$a^3 + b^3 + c^3 + d^3 = +3(ab(c + d) + cd(a + b)) = 3(abc + abd + acd + bcd) .$$

Testing Question A.9

Warning: I think there is a typo in this problem. To get the solution that the book presents at the end of the book we need to have the third expression be

$$a + b + c = 6 ,$$

rather than

$$a + b + c = 2 .$$

We will assume the first of these two in what follows.

To start this problem lets define

$$\begin{aligned} u &= a - 2 \\ v &= b - 2 \\ w &= c - 2 . \end{aligned}$$

Then we can write

$$a + b + c = 6 \Rightarrow u + v + w = 0 .$$

The first expression we are given can be written in terms of u, v, and w as

$$u^3 + v^3 + w^3 = 0 .$$

Then using the identity

$$u^3 + v^3 + w^3 - 3uvw = (u + v + w)(u^2 + v^2 + w^2 - uv - vw - wu) ,$$

we have

$$-3uvw = 0 .$$

Thus we must have one of $u = 0$, $v = 0$, and $w = 0$.

Testing Question A.10

Lets consider the expression E defined as

$$E = (a+b+c)^3 - a^3 - b^3 - c^3 .$$

We can manipulate this as

$$\begin{aligned}
E &= ((a+b+c)^3 - a^3) - (b^3 + c^3) \\
&= ((a+b+c) - a)[(a+b+c)^2 + (a+b+c)a + a^2] - (b+c)(b^2 - bc + c^2) \\
&= (b+c)[(a+b+c)^2 - b^2 + (a+b+c)a + bc + a^2 - c^2] \\
&= (b+c)(3a^2 + 3ab + 3ac + 3bc) \\
&= 3(b+c)(a^2 + ab + ac + bc) \\
&= 3(b+c)(a(a+b) + c(a+b)) \\
&= 3(b+c)(a+b)(a+c) .
\end{aligned}$$

As $E = 0$ from the above we see that one of $b = -c$, $a = -b$, $a = -c$ must be true.

Under each of these conditions we see that

$$a^{2n+1} + b^{2n+1} + c^{2n+1} = (a+b+c)^{2n+1} .$$

For example if $b = -c$ then the above is

$$a^{2n+1} - c^{2n+1} + c^{2n+1} = (a-c+c)^{2n+1} ,$$

which is true.

Testing Question B.1

We write M in a "special" way to try and separate out various parts to factor. We have

$$\begin{aligned}
M &= 3x^2 - 8xy + 9y^2 - 4x + 6y + 13 \\
&= \left(X_1x^2 - 8xy + Y_1y^2\right) + \left(X_2x^2 - 4x + N_1\right) + \left(Y_2y^2 + 6y + N_2\right) .
\end{aligned}$$

For this "expansion" to be valid we need to enforce that

$$\begin{aligned}
X_1 + X_2 &= 3 \\
Y_1 + Y_2 &= 9 \\
N_1 + N_2 &= 13 .
\end{aligned}$$

Now one way for the right-hand-side to factor into "squares" we would need to have X_1, X_2, Y_1, Y_2, N_1, and N_2 be perfect squares. We might start by trying $Y_2 = 1$. Then in that case to factor the last expression above or $Y_2y^2 + 6y + N_2 = y^2 + 6y + N_2$ we could take $N_2 = 9$ and get

$$(y+3)^2 .$$

If we do that then $N_1 = 13 - N_2 = 4$ and $Y_1 = 9 - Y_2 = 8$.

Next to factor the second to last expression above or

$$X_2 x^2 - 4x + N_1 = X_2 x^2 - 4x + 4\,,$$

we could take $X_2 = 1$ and get

$$(x-2)^2\,.$$

This means that $X_1 = 3 - X_2 = 3 - 1 = 2$ and the left-most expression above looks like

$$\begin{aligned} X_1 x^2 - 8xy + Y_1 y^2 &= 2x^2 - 8xy + 8y^2 \\ &= 2\left(x^2 - 4xy + 4y^2\right) \\ &= 2\left(x - 2y\right)^2\,. \end{aligned}$$

Thus using all of the above we have shown that

$$M = 2(x-2y)^2 + (x-2)^2 + (y+3)^2\,.$$

So $M \geq 0$. Now $M \neq 0$ for if it was zero we would need $x = 2$, $y = -3$ and $x - 2y = 0$ which is an inconsistent set of equations. Thus we conclude that $M > 0$.

Testing Question B.2

Method 1 (incomplete): We start with the given expressions

$$a + b = c + d \tag{809}$$

$$a^2 + b^2 = c^2 + d^2\,. \tag{810}$$

Then using the identities

$$\begin{aligned} a^2 + b^2 &= (a+b)^2 - 2ab \\ c^2 + d^2 &= (c+d)^2 - 2cd\,, \end{aligned}$$

in Equation 810 we get

$$(a+b)^2 - 2ab = (c+d)^2 - 2cd\,.$$

But from Equation 809 we know that $a + b = c + d$ so the above becomes

$$-2ab = -2cd \quad \text{so} \quad ab = cd\,.$$

Thus we have shown that the products are equal. Now consider $a^3 + b^3$. We find

$$\begin{aligned} a^3 + b^3 &= (a+b)(a^2 - ab + b^2) \\ &= (c+d)(a^2 + b^2 - ab) \\ &= (c+d)(c^2 + d^2 - cd) \\ &= c^3 + d^3\,. \end{aligned}$$

Next consider $a^4 + b^4$. We find

$$\begin{aligned} a^4 + b^4 &= (a^2 + b^2)^2 - 2a^2b^2 \\ &= (c^2 + d^2)^2 - 2c^2d^2 \\ &= c^4 + d^4 \,. \end{aligned}$$

Finally consider $a^5 + b^5$. We find

$$\begin{aligned} a^5 + b^5 &= (a+b)(a^4 - a^3b + a^2b^2 - ab^3 + b^4) \\ &= (c+d)(c^4 + d^4 - a^3b + a^2b^2 - ab^3) \\ &= (c+d)\left[c^4 + d^4 + ab(-a^2 + ab - b^2)\right] \\ &= (c+d)\left[c^4 + d^4 + cd(-c^2 - d^2 + cd)\right] \\ &= c^5 + d^5 \,. \end{aligned}$$

Thus we have shown that $a^n + b^n = c^n + d^n$ for $n \le 5$ and using similar steps as above these calculations could be continued up to a specified (but fixed) n. This setup looks like it is primed for a induction proof but I was unable to develop one. If anyone sees how to do this please contact me.

Method 2: From the given expression we have that

$$\begin{aligned} a - c &= d - b \\ a^2 - c^2 &= d^2 - b^2 \,. \end{aligned}$$

By factoring the second expression we get $(a+c)(a-c) = (d+b)(d-b)$. Using the first equation above in this to replace $a - c$ we get

$$(a+c)(d-b) = (d+b)(d-b) \,.$$

One way the above can be true is if $d - b = 0$. In that case then $d = b$ and from Equation 809 we have that $a = c$. These together give that

$$a^n + b^n = c^n + d^n \,. \tag{811}$$

for all n. If $d - b \neq 0$ then we can divide by it on both sides to get

$$a + c = d + b \,.$$

If we subtract this from Equation 809 we get

$$b - c = c - b = -(b - c) \,,$$

or

$$b = c \,.$$

Using that in Equation 809 gives $a = d$ and we again get Equation 811.

Testing Question B.3

Lets define E as the difference between these two expressions. Then we have

$$\begin{aligned} E &= (a^2+b^2+c^2)^2 - 2(a^4+b^4+c^4) \\ &= a^4+b^4+c^4+2a^2b^2+2a^2c^2+2b^2c^2-2a^4-2b^4-2c^4 \\ &= -a^4-b^4-c^4+2a^2b^2+2a^2c^2+2b^2c^2 \,. \end{aligned}$$

Lets write this as

$$E = a^2(-a^2+2b^2+2c^2)+b^2(-b^2+2c^2)-c^4 \,.$$

Then as $a=-b-c$ we have

$$a^2 = b^2+2bc+c^2 = (b+c)^2 \,.$$

Thus we can write E as

$$\begin{aligned} E &= a^2(-b^2-2bc-c^2+2b^2+2c^2)+b^2(-b^2+2c^2)-c^4 \\ &= a^2(b^2+c^2-2bc)+b^2(-b^2+2c^2)-c^4 \\ &= a^2(b-c)^2-b^4+2b^2c^2-c^4 \\ &= (b+c)^2(b-c)^2-(b^2-c^2)^2 = 0 \,, \end{aligned}$$

as we wanted to show.

Testing Question B.4

We start with $a+b=1$. Then using

$$a^2+b^2 = (a+b)^2-2ab \,.$$

we get

$$2 = 1-2ab \Rightarrow 1 = -2ab \Rightarrow ab = -\frac{1}{2} \,.$$

Thus we now know the product of ab. Next we have

$$\begin{aligned} a^3+b^3 &= (a+b)(a^2-ab+b^2) \\ &= 1\left(2-\left(-\frac{1}{2}\right)\right) = 2+\frac{1}{2} = \frac{5}{2} \,. \end{aligned}$$

Next we have

$$\begin{aligned} a^4+b^4 &= (a^2+b^2)^2-2a^2b^2 \\ &= 2^2-2(ab)^2 = 4-2\left(-\frac{1}{2}\right)^2 = 4-\frac{1}{2} = \frac{7}{2} \,. \end{aligned}$$

Next we have

$$
\begin{aligned}
a^5+b^5 &= (a+b)(a^4-a^3b+a^2b^2-ab^3+b^4)\\
&= 1\left(\frac{7}{2}+ab(-a^2+ab-b^2)\right)\\
&= \left(\frac{7}{2}-\frac{1}{2}\left(-2-\frac{1}{2}\right)\right)=\frac{7}{2}+\frac{1}{2}\left(\frac{5}{2}\right)=\frac{19}{4}\,.
\end{aligned}
$$

Next we have

$$
\begin{aligned}
a^6+b^6 &= (a^3+b^3)^2-2a^3b^3\\
&= \left(\frac{5}{2}\right)^2-2\left(-\frac{1}{2}\right)^3\\
&= \frac{25}{4}+\frac{2}{8}=\frac{25}{4}+\frac{1}{4}=\frac{26}{4}=\frac{13}{2}\,.
\end{aligned}
$$

Finally we have

$$
\begin{aligned}
a^7+b^7 &= (a+b)\left(a^6-a^5b+a^4b^2-a^3b^3+a^2b^4-ab^5+b^6\right)\\
&= 1\left(\frac{13}{2}+ab(-a^4+a^3b-a^2b^2+ab^3-b^4)\right)\\
&= \frac{13}{2}+\left(-\frac{1}{2}\right)\left(-\frac{7}{2}+ab(a^2-ab+b^2)\right)\\
&= \frac{13}{2}+\left(-\frac{1}{2}\right)\left(-\frac{7}{2}-\frac{1}{2}\left(2+\frac{1}{2}\right)\right)\\
&= \frac{13}{2}+\left(-\frac{1}{2}\right)\left(-\frac{19}{4}\right)=\frac{13\cdot 4}{8}+\frac{19}{8}=\frac{71}{8}\,.
\end{aligned}
$$

Testing Question B.5

Note that

$$
\begin{aligned}
(a+b)^3 &= a^3+3a^2b+3ab^2+b^3\\
&= a^3+b^3+3ab(a+b)\,.
\end{aligned}
$$

If we let $v \equiv a+b$ then the above can be written as

$$
\begin{aligned}
v^3 &= a^3+b^3+3ab-3ab+3abv\\
&= 1+3ab(v-1)\,.
\end{aligned}
$$

Note that $v=1$ is a solution to the above equation. We can write this expression as

$$
v^3-3abv+3ab-1=0\,.
$$

We would like to solve this for v. As $v=1$ is a root we should be able to "factor" $v-1$ "out" of the polynomial above. Performing polynomial long division we find

$$
v^3-3abv+3ab-1=(v-1)(v^2+v+1-3ab)=0\,.
$$

One solution to the above is $v = 1$ or $a + b = 1$.

Another solution is given by $v^2 + v + 1 - 3ab = 0$. This means that

$$a^2 + 2ab + b^2 + a + b + 1 - 3ab = 0\,,$$

or

$$a^2 - ab + b^2 + a + b + 1 = 0\,.$$

Let's try to factor this. Based on the "middle" terms we might look for a representation like

$$(A_1 a^2 - ab + B_1 b^2) + (A_2 a^2 + a + N_1) + (B_2 b^2 + b + N_2) = 0\,.$$

In order for this to be true we need to have

$$\begin{aligned} A_1 + A_2 &= 1 & (812)\\ B_1 + B_2 &= 1 & (813)\\ N_1 + N_2 &= 1\,. & (814) \end{aligned}$$

In addition, if we are lucky enough to be able to factor this as

$$(\sqrt{A_1}a - \sqrt{B_1}b)^2 + (\sqrt{A_2}a + \sqrt{N_1})^2 + (\sqrt{B_2}b + \sqrt{N_2})^2 = 0\,.$$

Then we must also have

$$\begin{aligned} 2\sqrt{A_1 B_1} &= 1 & (815)\\ 2\sqrt{A_2 N_1} &= 1 & (816)\\ 2\sqrt{B_2 N_2} &= 1\,. & (817) \end{aligned}$$

To satisfy these lets take

$$A_1 = \frac{1}{2}\,, B_1 = \frac{1}{2}\,, N_1 = \frac{1}{2}\,.$$

Then we must also have $A_2 = \frac{1}{2}$, $B_2 = \frac{1}{2}$ and $N_2 = \frac{1}{2}$. Thus the above becomes

$$\left(\frac{1}{2}a^2 - ab + \frac{1}{2}b^2\right) + \left(\frac{1}{2}a^2 + a + \frac{1}{2}\right) + \left(\frac{1}{2}b^2 + b + \frac{1}{2}\right) = 0\,.$$

We can factor this then as

$$\left(\frac{a}{\sqrt{2}} - \frac{b}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}a + \frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}b + \frac{1}{\sqrt{2}}\right)^2 = 0\,,$$

or

$$\frac{1}{\sqrt{2}}(a-b)^2 + \frac{1}{\sqrt{2}}(a+1)^2 + \frac{1}{\sqrt{2}}(b+1)^2 = 0\,.$$

This means that $a = b$, $a = -1$, and $b = -1$ so that $a + b = -2$.

Lecture 6: Some Methods of Factorization

Testing Question A.1

Part (i): Call this expression E. Then we have

$$\begin{aligned} E &= x^9 + 7x^6y^3 + 7x^3y^6 + y^9 \\ &= x^6(x^3 + 7y^3) + y^6(7x^3 + y^3) \\ &= x^6(7x^3 + 7y^3 - 6x^3) + y^6(7x^3 - 7y^3 - 6y^3) \\ &= (7x^3 + 7y^3)(x^6 + y^6) - 6x^9 - 6y^9 \\ &= 7(x^3 + y^3)(x^6 + y^6) - 6(x^9 + y^9)\,. \end{aligned}$$

Next use the identity

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)\,,$$

on the factor $x^9 + y^9$ as

$$x^9 + y^9 = (x^3 + y^3)(x^6 - x^3y^3 + y^6)\,.$$

Then we can write E as

$$\begin{aligned} E &= (x^3 + y^3)\left[7(x^6 + y^6) - 6(x^6 - x^3y^3 + y^6)\right] \\ &= (x^3 + y^3)(x^6 + y^6 + 6x^3y^3)\,. \end{aligned}$$

Part (ii): To start with we note that

$$(2x + y + 3z)^2 = 4x^2 + y^2 + 9z^2 + 4xy + 12xz + 6yz\,.$$

Thus using this, if we call the given expression E we can write E as

$$\begin{aligned} E &= (2x + y + 3z)^2 - 4xy - 12xz - 6yz - 6yz + 12xz - 4xy \\ &= (2x + y + 3z)^2 - 8xy - 12yz \\ &= (2x + 3z + y)^2 - 4y(2x + 3z) \\ &= (2x + 3z)^2 + 2y(2x + 3z) + y^2 - 4y(2x + 3z) \\ &= (2x + 3z)^2 - 2y(2x + 3z) + y^2 \\ &= (2x + 3z - y)^2\,. \end{aligned}$$

Part (iii): Call this expression E. Then we can write E as

$$\begin{aligned} E &= (x+1)(x+3)(x-1)(x+5) + 16 \\ &= (x^2 + 4x + 3)(x^2 + 4x - 5) + 16 \\ &= (x^2 + 4x)^2 - 2(x^2 + 4x) - 15 + 16 \\ &= (x^2 + 4x)^2 - 2(x^2 + 4x) + 1 \\ &= (x^2 + 4x - 1)^2\,. \end{aligned}$$

Part (iv): Call this expression E. Then we have

$$\begin{aligned} E &= (2x^2 - 4x + 1)^2 - 7(2x^2 - 4x) + 3 \\ &= (2x^2 - 4x + 1)^2 - 7(2x^2 - 4x + 1) + 7 + 3 \\ &= (2x^2 - 4x + 1)^2 - 7(2x^2 - 4x + 1) + 10\,. \end{aligned}$$

Now let $v = 2x^2 - 4x + 1$ then the above is

$$E = v^2 - 7v + 10 = (v - 2)(v - 5)\,,$$

so

$$E = (2x^2 - 4x - 1)(2x^2 - 4x - 4)\,.$$

Part (v): Note that in this expression if $x = 2$ we have

$$8 - 12 + 2a + 4 - 2a = 0\,.$$

This means that $x-2$ is a factor of this expression. To find this factor let b and c be unknown (for now) and write our expression E as

$$\begin{aligned} E &= (x-2)(x^2 + bx + c) \\ &= x^3 + bx^2 + cx - 2x^2 - 2bx - 2c \\ &= x^3 + (b-2)x^2 + (c-2b)x - 2c\,. \end{aligned}$$

We then set this equal to what we were originally given or

$$x^3 - 3x^2 + (a+2)x - 2a\,.$$

This gives

$$b - 2 = -3 \Rightarrow b = -1\,,$$

and

$$c - 2b = a + 2 \Rightarrow c = 2b + a - 2 = a\,.$$

Thus the factoring we have found is

$$E = (x-2)(x^2 - x + a)\,.$$

Part (vi): Call this expression E. Then we have

$$\begin{aligned} E &= x^{10}(x+1) + x^8(x+1) + x^6(x+1) + x^4(x+1) + x^2(x+1) + (x+1) \\ &= (x+1)(x^{10} + x^8 + x^6 + x^4 + x^2 + 1) \\ &= (x+1)(x^8(x^2+1) + x^4(x^2+1) + x^2 + 1) \\ &= (x+1)(x^2+1)(x^8 + x^4 + 1)\,. \end{aligned}$$

As another way to factor this note that from the expression for E we have

$$(x-1)E = x^{12} - 1\,.$$

This means that we have

$$E = \frac{x^{12} - 1}{x - 1}\,.$$

The numerator of this we can factor to get

$$\begin{aligned} E &= \frac{(x^6-1)(x^6+1)}{x-1} \\ &= \frac{(x^3-1)(x^3+1)x^6+1)}{x-1} \\ &= \frac{(x-1)(x^2+x+1)(x^3+1)(x^6+1)}{x-1} \\ &= (x^2+x+1)(x^3+1)(x^6+1) \\ &= (x^2+x+1)(x+1)(x^2-x+1)(x^6+1)\,. \end{aligned}$$

Testing Question A.2

Part (i): Call this expression E. Then we have

$$\begin{aligned} E &= (x^2-(a^2+b^2))^2-(a^2+b^2)^2+(a^2-b^2)^2 \\ &= (x^2-(a^2+b^2))^2+(a^2-b^2)^2-(a^2+b^2)^2 \\ &= (x^2-(a^2+b^2))^2+[a^2-b^2-(a^2+b^2)][a^2-b^2+(a^2+b^2)] \\ &= (x^2-(a^2+b^2))^2-4a^2b^2 \\ &= [x^2-(a^2+b^2)-2ab][x^2-(a^2+b^2)+2ab] \\ &= [x^2-a^2-2ab-b^2][x^2-(a^2-2ab+b^2)] \\ &= [x^2-(a+b)^2][x^2-(a-b)^2]\,. \end{aligned}$$

Part (ii): Call this expression E. Then we have

$$\begin{aligned} E &= (ab+1)(ab+a+b+1)+ab \\ &= (ab+1)(ab+1+a+b)+ab \\ &= (ab+1)(ab+b+1)+a(ab+1)+ab \\ &= (ab+1)(ab+b+1)+a[ab+b+1] \\ &= (ab+b+1)(ab+a+1)\,. \end{aligned}$$

Testing Question A.3

Note that

$$\begin{aligned} 81 &= (3^2)^2 = 3^4 \\ 27 &= 3\cdot 9 = 3^3\,, \end{aligned}$$

thus we can write our expression E as

$$\begin{aligned} E &= (3^4)^6-3^2\cdot(3^3)^7-(3^2)^{11} \\ &= 3^{24}-3^{23}-3^{22} \\ &= 3^{22}(3^2-3-1) \\ &= 3^{22}(9-3-1) \\ &= 3^{22}\cdot(5)\,. \end{aligned}$$

Next note that $45 = 3^2\cdot 5$ which is a factor of E.

Testing Question A.4

Call this expression E. Then we have

$$\begin{aligned} E &= 3\underbrace{(11\cdots 11)}_{\text{2n digits}} - 6\underbrace{(11\cdots 11)}_{\text{n digits}} \\ &= 3[\underbrace{11\cdots 11}_{\text{2n digits}} - 2\underbrace{(11\cdots 11)}_{\text{n digits}}]\,. \end{aligned}$$

Now if $n = 1$ this is

$$11 - 2(1) = 9\,.$$

If $n = 2$ this is

$$1111 - 2(11) = 1089\,.$$

If $n = 3$ this is

$$E_3 = 111111 - 2(111) = 110889\,.$$

These observations motivate us to write E_3 as

$$E_3 = 111111 - 111 - 111 = 111000 - 111 = 111 \cdot 10^3 - 111\,.$$

This then motivates what to do in the general case. In the general case we have

$$\begin{aligned} E &= \underbrace{11\cdots 11}_{\text{2n digits}} - 2\cdot\underbrace{11\cdots 11}_{\text{n digits}} \\ &= \underbrace{11\cdots 11}_{\text{2n digits}} - \underbrace{11\cdots 11}_{\text{n digits}} - \underbrace{11\cdots 11}_{\text{n digits}} \\ &= \underbrace{11\cdots 11}_{\text{n digits}}\underbrace{00\cdots 00}_{\text{n digits}} - \underbrace{11\cdots 11}_{\text{n digits}} \\ &= \underbrace{11\cdots 11}_{\text{n digits}}\cdot 10^n - \underbrace{11\cdots 11}_{\text{n digits}} \\ &= \underbrace{11\cdots 11}_{\text{n digits}}(10^n - 1)\,. \end{aligned}$$

Based on this for we next note that

$$10^n - 1 = \underbrace{99\cdots 99}_{\text{n digits}} = 9\cdot\underbrace{11\cdots 11}_{\text{n digits}}\,.$$

Using this for E we can write

$$E = 9\cdot(\underbrace{11\cdots 11}_{\text{n digits}})^2 = 3^2\cdot(\underbrace{11\cdots 11}_{\text{n digits}})^2\,,$$

which is a perfect square. If we take $n = 1$, $n = 2$, and $n = 3$ in the above we see that the derived formula matches the special cases above. Note that there seems to be a typo in the solution to this problem in the back of the book (it does not match the above for $n = 1$).

Testing Question A.5

Part (i): Call this expression E. Then we have

$$E = (x^2 + x - 1)^2 + (x^2 + x - 1) - 2\,.$$

If we let $v = x^2 + x - 1$ then we see that

$$E = v^2 + v - 2\,,$$

which we can factor as

$$(v + 2)(v - 1)\,.$$

This means that our expression E factors as

$$\begin{aligned} E &= (x^2 + x - 1 + 2)(x^2 + x - 1 - 1) \\ &= (x^2 + x + 1)(x^2 + x - 2)\,. \end{aligned}$$

Part (ii): Call this expression E. Then we have

$$\begin{aligned} E &= (x - y)^3 + (y - x - 2)^3 + 8 \\ &= (x - y)^3 + (y - x - 2)^3 + 2^3\,. \end{aligned}$$

From this form it is helpful to recall that

$$a^3 + b^3 + c^3 = 3abc + (a + b + c)(a^2 + b^2 + c^2 - ab - ac - bc)\,.$$

We can use this with our problem by

$$\begin{aligned} a &= x - y \\ b &= y - x - 2 \\ c &= 2\,. \end{aligned}$$

In the case we have here we have $a + b + c = 0$ so the above becomes

$$a^3 + b^3 + c^3 = 3abc\,.$$

Thus E takes the form

$$E = 6(x - y)(y - x - 2)\,.$$

Part (iii): Call this expression E and let $v \equiv 6x + 5$. Then we have

$$\begin{aligned} E &= v^2 \left(\frac{v}{2} - \frac{1}{2}\right)\left(\frac{v+1}{6}\right) - 6 = \frac{v^2(v-1)(v+1)}{2 \cdot 6} - 6 \\ &= \frac{1}{12}(v^2(v-1)(v+1) - 72) = \frac{1}{12}(v^2(v^2 - 1) - 72) \\ &= \frac{1}{12}(v^4 - v^2 - 72) = \frac{1}{12}(v^2 - 9)(v^2 + 8)\,. \end{aligned}$$

Using the definition of v this is

$$\begin{aligned}
E &= \frac{1}{12}((6x+5)^2-9)((6x+5)^2+8) \\
&= \frac{1}{12}(6x+5-3)(6x+5+3)(36x^2+60x+25+8) \\
&= \frac{1}{12}(6x+2)(6x+8)(36x^2+60x+33) \\
&= \frac{2^2\cdot 3}{12}(3x+1)(3x+4)(12x^2+20x+11) \\
&= (3x+1)(3x+4)(12x^2+20x+11)\,.
\end{aligned}$$

Part (iv): Call this expression E and let

$$v \equiv x^2+5x+6 = (x+2)(x+3)\,.$$

Then we have

$$\begin{aligned}
E &= v(v+x)-2x^2 = v^2+xv-2x^2 = (v+2x)(v-x) \\
&= (x^2+5x+6+2x)(x^2+5x+6-x) \\
&= (x^2+7x+6)(x^2+4x+6) \\
&= (x+6)(x+1)(x^2+4x+6)\,.
\end{aligned}$$

Part (v): Based on the form of this expression lets define

$$\begin{aligned}
a &= x^2-2x \\
b &= x^2-4x+2 \\
c &= -2(x^2-3x+1)
\end{aligned}$$

Then our expression E is the sum $a^3+b^3+c^3$. To *factor* this recall that we can also write this sum as

$$a^3+b^3+c^3 = 3abc+(a+b+c)(a^2+b^2+c^2-ab-ac-bc)\,,$$

in the case considered here we have

$$a+b+c = -6x+2+6x-2 = 0\,,$$

thus we have

$$\begin{aligned}
E &= a^3+b^3+c^3 = 3abc = 3(x^2-2x)(x^2-4x+2)(-2(x^3-3x+1)) \\
&= -6x(x-2)(x^2-4x+2)(x^2-3x+1)\,.
\end{aligned}$$

Part (vi): Call this expression E, then by expanding and then grouping terms we have

$$\begin{aligned}
E &= a^3+b^3+c^3+(a+b)(b+c)(c+a)-2abc \\
&= a^3+b^3+c^3+(a+b)(bc+ab+c^2+ac)-2abc \\
&= a^3+b^3+c^3+abc+a^2b+ac^2+a^2c+b^2c+ab^2+bc^2+abc-2abc \\
&= a^3+b^3+c^3+a^2b+ac^2+a^2c+b^2c+ab^2+bc^2 \\
&= a^3+b^3+c^3+a^2(b+c)+b^2(a+c)+c^2(a+b) \\
&= a^3+a^2(b+c)+b^3+b^2(a+c)+c^3+c^2(a+b) \\
&= a^2(a+b+c)+b^2(a+b+c)+c^2(a+b+c) \\
&= (a+b+c)(a^2+b^2+c^2)\,.
\end{aligned}$$

Testing Question A.6

Part (i): I believe there is a typo in the book's solution to this problem in that the coefficient of the xy term in the expression they start with is the negative of what it seems to be in the problem statement. What follows is the solution to the expression given in the problem statement.

We are starting with

$$E \equiv x^2 + xy - 2y^2 + 8x + ay - 9\,. \tag{818}$$

Given that we can write

$$x^2 + xy - 2y^2 = (x + 2y)(x - y)\,,$$

lets assume/hope that we can write E as

$$E = (x + 2y + A)(x - y + B)\,.$$

If we "expand" this we find

$$E = (x + 2y)(x - y) + B(x + 2y) + A(x - y) + AB\,.$$

Setting this equal to the right-hand-side of Equation 818 and canceling the common terms from $(x + 2y)(x - y)$ we get

$$Bx + 2By + Ax - Ay + AB = 8x + ay - 9\,.$$

Grouping terms by x and y on the left-hand-side of this expression we get

$$(A + B)x + (2B - A)y + AB = 8x + ay - 9\,.$$

Equating the coefficients of x and y in the above we get

$$A + B = 8 \tag{819}$$

$$2B - A = a \tag{820}$$

$$AB = -9\,. \tag{821}$$

Using Equations 819 and 821 we have

$$A - \frac{9}{A} = 8\,,$$

or

$$A^2 - 8A - 9 = 0\,,$$

or

$$(A - 9)(A + 1) = 0\,.$$

This means that $A = 9$ or $A = -1$. Using these in Equation 821 we get $B = -1$ or $B = 9$.

Using Equation 820 on these pairs for $a(A, B)$ we find

$$a(9, -1) = 2(-1) - 9 = -11\,,$$

and

$$a(-1,9) = 2(9) - (-1) = 19\,.$$

Part (ii): When they say "linear polynomials" I think they mean an expression of the form

$$(x^2 + Ax + B)(x^2 + Cx + D)\,.$$

Lets call this expression E and set it equal to the form above. We would have

$$\begin{aligned}
E &\equiv x^4 - x^3 + 4x^2 + 3x + 5 \\
&= (x^2 + Ax + B)(x^2 + Cx + D) \\
&= x^4 + Cx^3 + Dx^2 + Ax^3 + ACx^2 + ADx + Bx^2 + BCx + BD \\
&= x^4 + (C + A)x^3 + (D + AC + B)x^2 + (AD + BC)x + BD\,.
\end{aligned}$$

If we equate coefficients of powers of x this means that

$$\begin{aligned}
A + C &= -1 \\
D + AC + B &= 4 \\
AD + BC &= 3 \\
BD &= 5\,.
\end{aligned}$$

We could try to solve these in general but lets instead look if we can just find one (integer) solution. If we consider the last equation one solution is to take $B = 1$ and then $D = 5$. With these the other equations are

$$\begin{aligned}
A + C &= -1 \\
5 + AC + 1 &= 4 \Rightarrow AC = -2 \\
5A + C &= 3\,.
\end{aligned}$$

Solving the first and the last equations we see that $A = 1$ and $C = -2$. The second equation then gives $C = -2$. All together the solution we found is given by

$$A = 1\,, B = 1\,, C = -2\,, \quad \text{and} \quad D = 5\,.$$

Using these in the expression for E above we have

$$E = (x^2 + x + 1)(x^2 - 2x + 5)\,.$$

Testing Question A.7

We are told that

$$y^4 - 6y^3 + my^2 + ny + 36 = (y^2 + 3y + 6)(y^2 + Ay + B)\,,$$

for some A and B. If we expand the right-hand-side of the above we get

$$\begin{aligned}
\text{RHS} &= y^4 + Ay^3 + By^2 + 3y^3 + 3Ay^2 + 3By + 6y^2 + 6Ay + 6B \\
&= y^4 + (A + 3)y^3 + (B + 3A + 6)y^2 + (3B + 6A)y + 6B\,.
\end{aligned}$$

Equating the coefficient of y^3 this means that

$$A + 3 = -6 \Rightarrow A = -9\,.$$

Equating the constant term we have that

$$6B = 36 \Rightarrow B = 6\,.$$

Equating the coefficients of y^2 and y we have that

$$\begin{aligned} m &= B + 3A + 6 = 6 - 27 + 6 = 12 - 27 = -15 \\ n &= 3B + 6A = 3 \cdot 6 + 6 \cdot (-9) = 18 - 54 = -36\,. \end{aligned}$$

Testing Question B.1

Lets call this expression E and let $v = x^2 + 6x + 1$ then we have

$$\begin{aligned} E &= 2v^2 + 5(x^2+1)v + 2(x^2+1)^2 \\ &= (2v + (x^2+1))(v + 2(x^2+1)) \\ &= (2(x^2+6x+1) + x^2 + 1)(x^2 + 6x + 1 + 2x^2 + 2) \\ &= (2x^2 + 12x + 2 + x^2 + 1)(3x^2 + 6x + 3) \\ &= 3(x^2 + 2x + 1)(3x^2 + 12x + 3) \\ &= 9(x+1)^2(x^2 + 4x + 1)\,. \end{aligned}$$

Testing Question B.2

From the problem statement We must have

$$x^4 + ax^2 + b = (x^2 + 2x + 5)(x^2 + Ax + B)\,.$$

If we expand the right-hand-side we get

$$\begin{aligned} x^4 + ax^2 + b &= x^4 + Ax^3 + Bx^2 + 2x^3 + 2Ax^2 + 2Bx + 5x^2 + 5Ax + 5B \\ &= x^4 + (A+2)x^3 + (B + 2A + 5)x^2 + (2B + 5A)x + 5B\,. \end{aligned}$$

Now the coefficients of x^3 and x must vanish so

$$A + 2 = 0 \quad \text{so} \quad A = -2\,,$$

and

$$2B + 5A = 0 \quad \text{so} \quad 2B = 10 \quad \text{so} \quad B = 5\,.$$

From the coefficient for x^2 and the constant term we have

$$\begin{aligned} a &= B + 2A + 5 = 5 - 4 + 5 = 6 \\ b &= 5B = 25\,. \end{aligned}$$

This means that $a + b = 31$.

Testing Question B.3

Lets call this expression E. Then by expanding to simplify we have

$$\begin{aligned}
E &= (ab+cd)(a^2-b^2+c^2-d^2)+(ac+bd)(a^2-d^2+b^2-c^2) \\
&= a^3b-ab^3+abc^2-abd^2+a^2cd-b^2cd+c^3d-cd^3 \\
&+ a^3c-acd^2+ab^2c-ac^3+a^2bd-bd^3+b^3d-bc^2d \\
&= a^3(b+c)+b^3(-a+d)+c^3(d-a)+d^3(-c-b) \\
&+ a^2(cd+bd)+b^2(-cd+ac)+c^2(ab-bd)+d^2(-ab-ac) \\
&= a^3(b+c)+b^3(-a+d)+c^3(d-a)+d^3(-c-b) \\
&+ da^2(c+b)+b^2c(-d+a)+bc^2(a-d)+ad^2(-b-c) \\
&= a^2(c+b)(a+d)+b^2(a-d)(c-b)+c^2(b-c)(a-d)+d^2(a+d)(-b-c) \\
&= [a^2(c+b)-d^2(c+b)](a+d)+[b^2(a-d)-c^2(a-d)](c-b) \\
&= (a^2-d^2)(b+c)(a+d)+(b^2-c^2)(a-d)(c-b) \\
&= (a-d)(a+d)(b+c)(a+d)+(b-c)(b+c)(a-d)(c-b) \\
&= (a-d)(b+c)[(a+d)^2-(c-b)^2] \\
&= (a-d)(b+c)[a+d-(c-b)][a+d+c-b] \\
&= (a-d)(b+c)(a+b-c+d)(a-b+c+d)\,.
\end{aligned}$$

Testing Question B.4

Recall that we have

$$a^3+b^3=(a+b)(a^2-ab+b^2)\,.$$

Thus we can write the expression $E \equiv (ay+bx)^3+(ax+by)^3$ as

$$\begin{aligned}
E &= (ay+bx+ax+by)[(ay+bx)^2-(ay+bx)(ax+by)+(ax+by)^2] \\
&= (a(x+y)+b(x+y))[a^2y^2+2abxy+b^2x^2-a^2xy-aby^2-abx^2-b^2xy+a^2x^2+2abxy+b^2y^2] \\
&= (a+b)(x+y)[(a^2-ab+b^2)y^2+(2ab-a^2-b^2+2ab)xy+(b^2-ab+a^2)x^2] \\
&= (x+y)[(a^3+b^3)y^2+(a^3+b^3)x^2+(a+b)(4ab-a^2-b^2)xy] \\
&= (a^3+b^3)(x+y)(x^2+y^2)+(a+b)(x+y)xy(4ab-a^2b^2)\,.
\end{aligned}$$

From this we next add the term $-(a^3+b^3)(x^3+y^3)$ to get

$$(x^3+b^3)[(x+y)(x^2+y^2)-(x^3+y^3)]+(a+b)(x+y)xy(4ab-a^2-b^2)\,.$$

Expanding the second factor in the first term we have this equal to

$$(x^3+b^3)[x^3+xy^2+yx^2+y^3-x^3-y^3]+(a+b)(x+y)xy(4ab-a^2-b^2)\,,$$

or

$$(a^3+b^3)xy(y+x)+(a+b)(x+y)xy(4ab-a^2-b^2)\,,$$

or factoring common factors

$$xy(x+y)(a+b)[a^2-ab+b^2+4ab-a^2-b^2]=3abxy(a+b)(x+y)\,.$$

Testing Question B.5

By factoring, we can write these three numbers as

$$ab(a-b)(a+b)(a^2+b^2)\,,$$

$$bc(b-c)(b+c)(b^2+c^2)\,,$$

and

$$ac(c-a)(c+a)(c^2+a^2)\,.$$

Note that each of these is the product of five integers.

To solve this question we will argue that for any configuration of a, b, and c that are even/odd one of the numbers above will have the product of three even numbers and thus be divisible by eight.

For example, if a, b, and c are all odd numbers than the numbers $(a-b)$, $(a+b)$, and (a^2+b^2) are even numbers so first number has three even factors and is thus divisible by eight.

If we have two odd numbers say a and b then $a-b$, $a+b$, and a^2+b^2 are even and the first number has three even factors and is thus divisible by eight.

If we have one odd number and two even numbers let a be the odd number. Then the second number above is the product of at least three even numbers and is divisible eight.

If we have no odd numbers then many of the product above are divisible by eight.

Lecture 7: Absolute Value and Its Applications

Testing Question A.1

If $x > 0$ then $x + |x| = 2x$ so
$$\frac{|x + |x||}{x} = \frac{|2x|}{x} = 2\,.$$
If $x < 0$ then $x + |x| = 0$ so the expression given is zero. If $x = 0$ the expression is undefined.

Testing Question A.2

The given condition on x simplifies to the statement that $x \leq \frac{7}{11}$. To determine the extremes of the function $f(x) = |x - 1| - |x + 3|$ we consider the locations where the absolute values change their definitions. We have

- If $x < -3$ then $f(x) = -(x - 1) + (x + 3) = 4$.
- If $-3 < x < 1$ then $f(x) = -(x - 1) - (x + 3) = -2x - 2$.
- If $x > 1$ then $f(x) = x - 1 - (x + 3) = -4$.

This is a slanted line between two constant values (positive four and negative four). When $x \leq \frac{7}{11}$ the largest value of this function is the value of four and the smallest value of this function is the expression $-2x - 2$ evaluated at $\frac{7}{11}$.

Testing Question A.3

In the expression
$$|1 - x| = 1 + |x|\,,$$
the absolute values "change" across the points $x = 0$ and $x = 1$. If $x \leq 0$ this expression is
$$1 - x = 1 - x\,,$$
which is an identity and is thus true for all x. If $0 < x < 1$ this expression is
$$1 - x = 1 + x\,,$$
which is satisfied for $x = 0$ only. If $x \geq 1$ this expression is
$$-(1 - x) = 1 + x\,,$$
which has no solution. Thus there are only solutions to the given expression when $x \leq 0$. Then from the expression given we see that $|1 - x|$ is the left-hand-side and must equal the right-hand-side which is $1 + |x|$. When $x \leq 0$ this is $1 - x$.

Testing Question A.4

Note that the "sign" of the absolute values in this expression will change at the points $x = -1$, $x = 2$, $x = 3$. Using that we can evaluate it. For example, if $x < -1$ then it is

$$-(x + 1) - (x - 2) - (x - 3) = -3x + 4 .$$

At $x = -1$ this takes the value seven. If $-1 < x < 2$ then this expression is

$$x + 1 - (x - 2) - (x - 3) = 6 - x .$$

At $x = 2$ this takes the value four. If $2 < x < 3$ then this expression is

$$x + 1 + x - 2 - (x - 3) = x + 2 .$$

At $x = 3$ this is the value five. As our expression is piecewise linear for all x the minimum must occur at $x \in \{-1, 2, 3\}$. The smallest value of this expression is when $x = 2$ where it takes the value of four.

Testing Question A.5

If $x < 0$ then $|x| = -x$ and our expression is

$$\frac{|-x - 2x|}{3} = \frac{|-3x|}{3} = |x| = -x .$$

Testing Question A.6

As $x \to \pm\infty$ this expression tends to infinity and is piecewise linear in between. At the points $x \in \{a, b, c, d\}$ one line segment ends and another one begins. Thus the minimum value of this expression will be at one of these points. For any value of x call this expression $v(x)$.

Now if $x = a$ this expression is

$$v(a) = |a - b| + |a - c| + |a - d| = b - a + c - a + d - a = b + c + d - 3a .$$

Now if $x = b$ this expression is

$$v(b) = |b - a| + |b - c| + |b - d| = b - a + c - b + d - b = d + c - a - b .$$

Now if $x = c$ this expression is

$$v(c) = |c - a| + |c - b| + |c - d| = c - a + c - b + d - c = c + d - a - b .$$

Now if $x = d$ this expression is

$$v(d) = |d - a| + |d - b| + |d - c| = 3d - a - b - c .$$

Note that $v(b) = v(c)$. The smallest of these three numbers is the minimum of the entire expression over all x. We ask ourselves is $v(a) < v(b)$ or

$$b + c + d - 3a < d + c - a - b \,.$$

We can simplify the above to

$$b < a \,,$$

which is *not* true. Thus we have learned that instead we must have $v(b) > v(a)$.

Next we ask ourselves is $v(b) < v(d)$ or

$$d + c - a - b < 3d - a - b - c \,.$$

We can simplify the above to

$$c < d \,,$$

which *is* true. Thus we have learned that instead we must have $v(b) < v(d)$. In all cases when $a < b < c < d$ the smaller of the three numbers above is $d + c - a - b$

Testing Question A.7

If $a + b > 0$ then this expression is

$$a + b = a - b \quad \text{or} \quad b = -b \quad \text{or} \quad b = 0 \,.$$

In this case our expression is $|a| = a$ so $a > 0$. In this case we have $ab = 0$.

If $a + b < 0$ then this expression is

$$-(a + b) = a - b \quad \text{or} \quad -a = a \quad \text{or} \quad a = 0 \,.$$

In this case our expression is $|b| = -b$ so $b < 0$. In this case we have $ab = 0$.

Testing Question A.8

If a and b are integers then $|a - b| \in \{0, 1, 2, \dots\}$ and for any x and y $|x - y|^{19} \geq 0$. Thus for the sum of these two terms to be equal to one can only happen if

$$|a - b|^{19} = 1 \quad \text{and} \quad |c - a|^{19} = 0 \quad \text{or} \tag{822}$$

$$|a - b|^{19} = 0 \quad \text{and} \quad |c - a|^{19} = 1 \,. \tag{823}$$

If Equation 822 is true then $a = c$ and $a - b = \pm 1$ so $a = b \pm 1$ and in summary we have

$$a = c = b \pm 1 \,.$$

This means that the value we want to evaluate is given by

$$|c - a| + |a - b| + |b - c| = 0 + 1 + 1 = 2 \,.$$

If Equation 823 is true then $a = b$ and $c - a = \pm 1$ so $a = c \pm 1$ and in summary we have

$$a = b = c \pm 1\,.$$

This means that the value we want to evaluate is given by

$$|c - a| + |a - b| + |b - c| = 1 + 0 + 1 = 2\,.$$

Testing Question A.9

Note that the first expression in absolute values can be written as

$$\begin{aligned} 2a^3 - 3a^2 - 2a + 1 &= a(2a^2 - 3a - 2) + 1 \\ &= a(2a + 1)(a - 2) + 1\,. \end{aligned}$$

For $a = 2009$ everything in the above is positive so the expression given by the left-hand-side is positive and we have

$$|2a^3 - 3a^2 - 2a + 1| = 2a^3 - 3a^2 - 2a + 1\,.$$

The second expression in absolute values can be written as

$$\begin{aligned} 2a^3 - 3a^2 - 3a - 2009 &= a(2a^2 - 3a - 3) - 2009 \\ &= a(2a^2 - 3a - 2 - 1) - 2009 \\ &= a(2a + 1)(a - 2) - a - 2009 \\ &= a(2a + 1)(a + 2) - (a + 2009)\,. \end{aligned}$$

Now $a + 2009 = 2a$ for the given value of a we are considering so

$$a(2a + 1)(a + 2) - (a + 2009) = a(2a + 1)(a + 2) - 2a = a[(2a + 1)(a + 2) - 2] > 0\,.$$

Thus we have

$$|2a^3 - 3a^2 - 3a - 2009| = 2a^3 - 3a^2 - 3a - 2009\,.$$

The expression we want to evaluate is given by

$$2a^3 - 3a^2 - 2a + 1 - (2a^3 - 3a^2 - 3a - 2009) = a + 2010 = 4019\,.$$

Testing Question A.10

From the given expression we know that

$$|x - a| - b = \pm 3 \quad \text{so} \quad |x - a| = b \pm 3\,.$$

If we consider the positive term on the "three" we have

$$x - a = \pm(b + 3) \quad \text{so} \quad x = a \pm (b + 3)\,.$$

This has two solutions for x given by

$$\{a + b + 3, a - b - 3\} . \tag{824}$$

If we consider the negative term on the "three" we have

$$x - a = \pm(b - 3) \quad \text{so} \quad x = a \pm (b - 3) .$$

This has two solutions for x given by

$$\{a + b - 3, a - b + 3\} . \tag{825}$$

From these it looks like there are four distinct solutions which would be the numbers

$$\{b + 3 , -b - 3 , b - 3 , -b + 3\} ,$$

with a added to each. As we are told that there are only three solutions to this expression then two of these numbers must be the same. Now if $|b| > 3$ or $|b| < 3$ then these are four distinct numbers and the hypothesis of the question are not satisfied. The only way we get three distinct numbers from the above is if $|b| = 3$. If $b = \pm 3$ then the above set becomes

$$\{6 , -6 , 0\} .$$

Thus $b = \pm 3$.

Testing Question B.1

As $|x_i| < 1$ we have that

$$\sum_{i=1}^{n} |x_i| < n .$$

Thus from the expression given we have

$$49 + \left| \sum_{i=1}^{n} x_i \right| = \sum_{i=1}^{n} |x_i| < n .$$

As $|\sum_{i=1}^{n} x_i| \geq 0$ and the above we see that

$$n > 49 + \left| \sum_{i=1}^{n} x_i \right| \geq 49 .$$

From the above we know that $n > 49$ so we might try and see if we can find $n = 50$ values of x_i that satisfy the given expression.

As $n = 50$ is an even number if we make the x_i each have different signs (and the same magnitude) then we can make $\sum_{i=1}^{n} x_i = 0$. If we make them all have the same magnitude (say x where $x > 0$) then we need to have

$$\sum_{i=1}^{n} |x_i| = xn = 49 ,$$

so $x = \frac{49}{n} = 0.98$ when $n = 50$. Thus we can satisfy the given equation if we take

$$x_i = \begin{cases} \pm 0.98 & i \text{ even} \\ \mp 0.98 & i \text{ odd} \end{cases} .$$

Testing Question B.2

As this expression tends to infinity as $x \to \pm\infty$ the minimum must be in the interior of the domain. Also note that this expression is piecewise linear on the interior of the domain thus the minimum value must be at a point where one linear segment stops and another one begins i.e. at one of the points $\{a_1, a_2, \ldots, a_n\}$. Lets denote this expression as

$$E(x) = \sum_{k=1}^{n} |x - a_k| \,,$$

and use the notation $E_k \equiv E(x_k)$. To study this problem lets evaluate expression in the question at the value a_p for $1 \le p \le n$. Based on the ordering of the a_ks this can be written as

$$\begin{aligned} E_p &= \sum_{k=1}^{p} |a_p - a_k| + \sum_{k=p+1}^{n} |a_p - a_k| \\ &= \sum_{k=1}^{p} (a_p - a_k) + \sum_{k=p+1}^{n} (a_k - a_p) \,. \end{aligned} \tag{826}$$

Lets compare this to E_{p+1} which is

$$E_{p+1} = \sum_{k=1}^{p+1} (a_{p+1} - a_k) + \sum_{k=p+2}^{n} (a_k - a_{p+1}) \,.$$

Using these we compute the difference between them or

$$\begin{aligned} E_{p+1} - E_p &= \sum_{k=1}^{p+1} (a_{p+1} - a_k) - \sum_{k=1}^{p} (a_p - a_k) + \sum_{k=p+2}^{n} (a_k - a_{p+1}) - \sum_{k=p+1}^{n} (a_k - a_p) \\ &= \sum_{k=1}^{p} (a_{p+1} - a_k) - \sum_{k=1}^{p-1} (a_p - a_k) + \sum_{k=p+2}^{n} (a_k - a_{p+1}) - \sum_{k=p+2}^{n} (a_k - a_p) - (a_{p+1} - a_p) \\ &= (a_{p+1} - a_p) + \sum_{k=1}^{p-1} (a_{p+1} - a_p) + \sum_{k=p+2}^{n} (a_p - a_{p+1}) - (a_{p+1} - a_p) \\ &= \sum_{k=1}^{p} (a_{p+1} - a_p) - \sum_{k=p+1}^{n} (a_{p+1} - a_p) \\ &= p(a_{p+1} - a_p) - (n - p - 1 + 1)(a_{p+1} - a_p) \\ &= (a_{p+1} - a_p)(2p - n) \,. \end{aligned} \tag{827}$$

Now as $a_{p+1} - a_p > 0$ for all p we see that the sign of the above difference only depends on the sign of $2p - n$. If $p < \frac{n}{2}$ then $E_{p+1} - E_p < 0$ and if $p > \frac{n}{2}$ then $E_{p+1} - E_p > 0$. This means that when $p < \frac{n}{2}$ we have $E_{p+1} < E_p$ so E_p is *decreasing* in that range and then once $p > \frac{n}{2}$ that $E_{p+1} > E_p$ so that E_p is then *increasing* in that range. The smallest value of E_p is then when p is as close as it can be to $\frac{n}{2}$.

If n is even then the smallest value will be at $E_{\frac{n}{2}}$. If n is odd then we need to find the smaller of

$$E_{\lfloor \frac{n}{2} \rfloor} \quad \text{or} \quad E_{\lceil \frac{n}{2} \rceil} \,.$$

Using Equation 827 we see that if $p = \lfloor \frac{n}{2} \rfloor$ then

$$E_{p+1} - E_p = (a_{p+1} - a_p)\left(2\lfloor \frac{n}{2} \rfloor - n\right) < (a_{p+1} - a_p)\left(2\left(\frac{n}{2}\right) - n\right) = 0 \,.$$

Thus

$$E_{\lfloor \frac{n}{2} \rfloor + 1} > E_{\lfloor \frac{n}{2} \rfloor} \,.$$

As

$$\lceil \frac{n}{2} \rceil = \lfloor \frac{n}{2} \rfloor + 1 \,,$$

we have just shown that

$$E_{\lceil \frac{n}{2} \rceil} > E_{\lfloor \frac{n}{2} \rfloor} \,.$$

Thus when n is odd the smallest value of $E(x)$ is

$$E_{\lfloor \frac{n}{2} \rfloor} \,.$$

Testing Question B.3

The sign of the absolute values "change" when $x = \frac{4}{5}$ and $x = \frac{1}{3}$. Thus if $x < \frac{1}{3}$ this expression is

$$2x + 4 - 5x + (1 - 3x) + 4 = -6x + 8 \,,$$

which is not a constant. If $\frac{1}{3} < x < \frac{4}{5}$ this expression is

$$2x + 4 - 5x - (1 - 3x) + 4 = 7 \,,$$

which is a constant.

Testing Question B.4

Using $a + b + c = 0$ we have that $b + c = -a$, $a + c = -b$, and $a + b = -c$. Thus x can be written

$$x = \frac{|a|}{-a} + \frac{|b|}{-b} - \frac{|c|}{-c} = -\frac{|a|}{a} - \frac{|b|}{b} + \frac{|c|}{c} \,. \tag{828}$$

From the definition of $|v|$ note that

$$|v| = \begin{cases} v & v > 0 \\ -v & v < 0 \end{cases} \,,$$

so

$$\frac{|v|}{v} = \begin{cases} 1 & v > 0 \\ -1 & v < 0 \end{cases} = \text{sign}(v) \,.$$

Thus we can write x as

$$x = -\text{sign}(a) - \text{sign}(b) + \text{sign}(c)\,. \tag{829}$$

In order for $a + b + c = 0$ we cannot have all of the be of the same sign. Thus at least one of the numbers must be of a different sign than the other three. Enumerating the possible signs that a, b, and c and take (and then computing the value of x based on Equation 828) we have

$$\begin{aligned}
a > 0\,, b > 0\,, c < 0 \quad &\text{so} \quad x = -1 - 1 - 1 = -3 \\
a > 0\,, b < 0\,, c > 0 \quad &\text{so} \quad x = -1 + 1 + 1 = +1 \\
a < 0\,, b > 0\,, c > 0 \quad &\text{so} \quad x = +1 - 1 + 1 = +1 \\
a < 0\,, b < 0\,, c > 0 \quad &\text{so} \quad x = +1 + 1 + 1 = +3 \\
a < 0\,, b > 0\,, c < 0 \quad &\text{so} \quad x = +1 - 1 - 1 = -1 \\
a > 0\,, b < 0\,, c < 0 \quad &\text{so} \quad x = -1 + 1 - 1 = -1\,.
\end{aligned}$$

Given these values for x we could then evaluate the given expression.

Testing Question B.5

By running the code `LNOMOC_Vol_1_Lecture_7_B_5.py` for several values of n that it looks like this value should be equal to n^2. Here $n = 100$ so this value would be 100^2.

Lecture 8: Linear Equations with Absolute Values

Testing Question A.1

This is equivalent to

$$5x - 4 = 3 + 2x \quad \text{or} \quad 5x - 4 = -(3 + 2x)\,.$$

Solving each we get

$$x = \frac{7}{3} \quad \text{and} \quad x = \frac{1}{7}\,.$$

Testing Question A.2

Notice that the absolute values on the left-hand-side change their sign when $a = -\frac{7}{2}$ or $a = \frac{1}{2}$.

Now if $a < -\frac{7}{2}$ our expression is

$$-(2a + 7) - (2a - 1) = 8\,.$$

Solving this we get $a = -\frac{7}{2}$ which is not an integer.

Now if $-\frac{7}{2} < a < \frac{1}{2}$ then our expression is

$$2a + 7 - (2a - 1) = 8\,.$$

This simplifies to an identity. The integer values of a in the domain above are $a \in \{-3, -2, -1, 0\}$.

Now if $a > \frac{1}{2}$ then our expression is

$$2a + 7 + (2a - 1) = 8\,.$$

This simplifies to $a = \frac{1}{2}$ which is not an integer.

Testing Question A.3

From the given expression we have that

$$x - |2x + 1| = \pm 3\,.$$

Now if $2x + 1 \geq 0$ (or $x \geq -\frac{1}{2}$) then the above expression is

$$x - (2x + 1) = \pm 3\,.$$

This has two solutions $x = -4$ and $x = 2$. Only the solution $x = 2$ is larger than $-\frac{1}{2}$.

If $2x + 1 < 0$ (or $x < -\frac{1}{2}$) then the above expression is

$$x + (2x + 1) = \pm 3 \, .$$

This has two solutions $x = -\frac{4}{3}$ and $x = \frac{2}{3}$. Only the solution $x = -\frac{4}{3}$ is less than $-\frac{1}{2}$.

Thus this expression has two solutions.

Testing Question A.4

In the case where $x < 0$ our equation is

$$-x = ax + 1 \, ,$$

which has a solution ($a \neq -1$) given by

$$x = -\frac{1}{1 + a} \, .$$

This will be negative (and give a single solution) if

$$-\frac{1}{1 + a} < 0 \quad \text{or} \quad a > -1 \, .$$

In the case when $x \geq 0$ our equation is

$$x = ax + 1 \, .$$

If $a \neq 1$ then a solution is given by

$$x = \frac{1}{1 - a} \, .$$

This will be a positive solution if $a < 1$. To *not* have a positive solution we must take $a \geq 1$.

Combining these two conditions ($a > -1$ and $a \geq 1$) we must have $a \geq 1$.

Testing Question A.5

As the right-hand-side is equal to an absolute value we know that $a \geq 0$. We can write our given expression as

$$|x - 2| - 1 = \pm a \, ,$$

or

$$|x - 2| = 1 \pm a \, .$$

We must have $1 \pm a \geq 0$ so $1 - a \geq 0$ so we have learned that $a \leq 1$. From the above we have that

$$x = 2 \pm (1 \pm a) \, .$$

For general a the above would give *four* solutions (two for the first $\pm$ and for each of these we get two from the second $\pm$). If $a = 1$ we see that

$$|x - 2| \in \{0, 2\} .$$

If $|x - 2| = 0$ then $x = 2$. If $|x - 2| = 2$ then $x - 2 = \pm 2$ so $x \in \{0, 4\}$ and we have three total solutions for x.

Testing Question A.6

When $x < 0$ our equation is

$$-\frac{ax}{2008} - x - 2008 = 0 .$$

Solving for x we get

$$x = -\frac{2008^2}{a + 2008} .$$

This will be negative solution if $a > -2008$.

When $x > 0$ our equation is

$$\frac{ax}{2008} - x - 2008 = 0 .$$

Solving for x we get

$$x = \frac{2008^2}{a - 2008} .$$

This will be positive solution if $a > 2008$.

To only have one negative solution (and no positive solutions) we must have $a > -2008$ and $a \leq 2008$. Thus the valid range for a is

$$-2008 < a \leq 2008 .$$

Testing Question A.7

For **(ii)** to have only one solution means that

$$|4x - 5| = -3n = 0 ,$$

so that $n = 0$.

For **(i)** to have no solutions means that

$$|3x - 4| = -2m < 0 ,$$

so that $m > 0$.

For **(iii)** to have two solutions means that

$$|5x - 6| = -4k > 0\,,$$

so that $k < 0$.

Taking these together we get that $k < n < m$.

Testing Question A.8

From the first equation if $x \geq y$ then we get

$$x - y = x + y - 2 \quad \text{or} \quad y = 1\,.$$

Thus we have that $x \geq 1$ and putting $y = 1$ in the second equation gives

$$|x + 1| = x + 2\,,$$

or

$$x + 1 = x + 2 \quad \text{or} \quad x + 1 = -(x + 2)\,.$$

The first of these has no solution and the second has the solution $x = -\frac{3}{2}$ which violates the condition that $x \geq 1$. Thus there is no solution when $x \geq y$.

If $x < y$ then the first equation is

$$-x + y = x + y - 2 \quad \text{or} \quad x = 1\,.$$

Then to have $x < y$ means that $1 < y$ and the second equation gives

$$|1 + y| = 3\,,$$

or

$$1 + y = -3 \quad \text{or} \quad 1 + y = 3\,.$$

The first of these has the solution $y = -4$ and the second has the solution $y = 2$. Only the second solution satisfies $1 < y$ so the only solution is $(x, y) = (1, 2)$.

Testing Question A.9

Let $v = |x|$ then

$$v^2 + v - 6 = 0 \quad \text{or} \quad (v + 3)(v - 2) = 0\,.$$

This means that $v = -3$ or $v = 2$. We know that $v \geq 0$ so the only solution is $v = |x| = 2$. Thus $x = \pm 2$ and the sum is zero.

Testing Question A.10

From the second equation we have $y = 6 - 2x$. Putting this in the first equation we get

$$x + 3(6 - 2x) + |3x - 6 + 2x| = 19\,,$$

or

$$-5x + |5x - 6| = 1\,.$$

If $5x - 6 \geq 0$ (or $x \geq \frac{6}{5}$) this is $-6 = 1$ which means there is no solution for this range of x.

If $5x - 6 < 0$ (or $x < \frac{6}{5}$) this is

$$-10x = -5 \quad \text{or} \quad x = \frac{1}{2}\,.$$

For this value of x we have $y = 6 - 1 = 5$. Thus the only solution is $(x, y) = \left(\frac{1}{2}, 5\right)$.

Testing Question B.1

The sign of the expression in the absolute value in the first equation changes when

$$x - 2y = 0\,.$$

It is positive when

$$x - 2y \geq 0 \quad \text{or} \quad y \leq \frac{1}{2}x\,.$$

The expression $y = \frac{1}{2}x$ is a line above and below which $|x - 2y|$ has different expressions. Thus we will break the x-y plane up into regions where x, y, and $x - 2y$ are of known signs. If you draw the line $y = \frac{1}{2}x$ in the x-y plane I've numbered these regions $I\,, II\,, III\,, IV\,, V\,, VI$ counterclockwise.

Now region I is

$$x \geq 0\,, \quad y \geq\,, \quad y \geq \frac{1}{2}x\,,$$

and the equations become

$$\begin{aligned} -(x - 2y) &= 1 \\ x + y &= 2\,. \end{aligned}$$

This has the solution $(x, y) = (1, 1)$. This solution falls in this region and so is a valid solution to the absolute value system.

Next, region II is

$$x < 0\,, \quad y \geq\,, \quad y \geq \frac{1}{2}x\,,$$

and the equations become

$$\begin{aligned} -(x-2y) &= 1 \\ -x+y &= 2\,. \end{aligned}$$

This has the solution $(x,y)=(-3,-1)$. This solution is outside of this region and is not a solution to the absolute value system.

Next, region III is

$$x<0\,,\quad y<0\,,\quad y\geq \frac{1}{2}x\,,$$

and the equations become

$$\begin{aligned} -(x-2y) &= 1 \\ -x-y &= 2\,. \end{aligned}$$

This has the solution $(x,y)=\left(-\frac{5}{3},-\frac{1}{3}\right)$. This solution is in the given region and is a solution to the absolute value system.

Next, region IV is

$$x<0\,,\quad y<0\,,\quad y< \frac{1}{2}x\,,$$

and the equations become

$$\begin{aligned} x-2y &= 1 \\ -x-y &= 2\,. \end{aligned}$$

This has the solution $(x,y)=(-1,-1)$. This solution is in the given region and is a solution to the absolute value system.

Next, region V is

$$x\geq 0\,,\quad y<0\,,\quad y< \frac{1}{2}x\,,$$

and the equations become

$$\begin{aligned} x-2y &= 1 \\ x-y &= 2\,. \end{aligned}$$

This has the solution $(x,y)=(3,1)$. This solution is not in the given region and is not a solution to the absolute value system.

Finally, region VI is

$$x\geq 0\,,\quad y\geq 0\,,\quad y< \frac{1}{2}x\,,$$

and the equations become

$$\begin{aligned} x-2y &= 1 \\ x+y &= 2\,. \end{aligned}$$

This has the solution $(x, y) = \left(\frac{5}{3}, \frac{1}{3}\right)$. This solution is in the given region and is thus a solution to the absolute value system.

In summary then the solutions to the absolute value system are

$$(1, 1), \quad \left(-\frac{5}{3}, -\frac{1}{3}\right), \quad (-1, -1), \quad \left(\frac{5}{3}, \frac{1}{3}\right).$$

Testing Question B.2

The first absolute value is easy to remove and we must have

$$|||x - 1| - 1| - 1| = 1.$$

Removing one more gives

$$||x - 1| - 1| - 1 = \pm 1,$$

or

$$||x - 1| - 1| = 1 \pm 1.$$

The two possible values for the right-hand-side are $\{0, 2\}$ and we have two equations

$$||x - 1| - 1| = 0 \quad (830)$$

$$||x - 1| - 1| = 2. \quad (831)$$

Equation 830 is easier to solve and we have

$$|x - 1| = 1.$$

This means that

$$x - 1 = \pm 1 \quad \text{or} \quad x \in \{0, 2\}.$$

Equation 831 becomes

$$|x - 1| - 1 = \pm 2.$$

This would mean that

$$|x - 1| \in \{-1, 3\}.$$

As the left-hand-side must be positive we have $|x - 1| = 3$ only so that

$$x - 1 = \pm 3.$$

This has the two solutions $x \in \{-2, 4\}$. Thus the full set of solutions is

$$x \in \{-2, 0, 2, 4\}.$$

We can verify that these solutions are valid with the simple `R` code

```
eval_fn = function(x){
    v = x
    for( ii in 1:4 ){
      v = abs(v-1)
    }
    v
}
print(sapply(c(-2, 0, 2, 4), eval_fn))
```

which gives a vector of four zeros.

Testing Question B.3

We square both sides of this expression to get

$$a^2 - 2(a+b)|a| + (a+b)^2 < a^2 - 2a|a+b| + (a+b)^2 \, .$$

If we cancel common terms we get

$$-(a+b)|a| < -a|a+b| \, ,$$

or

$$(a+b)|a| > a|a+b| \, .$$

From this we see that $a \neq 0$ and $a+b \neq 0$ or else the above inequality would be an equality. Dividing by $|a|$ we get

$$a+b > \frac{a}{|a|}|a+b| \, .$$

If $a > 0$ then this would be

$$a+b > |a+b| \, ,$$

which is not possible for any sign (positive or negative) for $a+b$. Thus we must have $a < 0$ and the above becomes

$$a+b > -|a+b| \, .$$

If $a+b > 0$ then this is true while if $a+b < 0$ it is not. Thus $a+b > 0$ so $b > -a$. As $a < 0$ this means that $b > 0$. In summary then, the conditions are $a < 0$ and $b > 0$.

Testing Question B.4

Part (i): If we assume that $2 < 52a$ we can look for solutions x where x might be $x < 2$, $2 < x < 52a$, or $x > 52a$. In the first case where we assume that $x < 2$ we get every x is a solution if and only if $52a = 2$ which is a contradiction to the assumption that $2 < 52a$. In a similar way we also have no solutions if we assume that $x > 52a$. If we assume that $2 < x < 52a$ the given equation is

$$\frac{1}{x-2} = -\frac{1}{x-52a} \, ,$$

or

$$x - 52a = -x + 2\,.$$

Solving for x we get

$$x = \frac{1}{2}(52a + 2) = 26a + 1\,.$$

Notice that this is the midpoint between the two locations $x = 2$ and $x = 52a$. This is the only solution to this equation.

Part (ii): If a is the square of an odd prime then $a = p^2$ where p is an odd prime. This means that

$$x = 26p^2 + 1\,.$$

If $p = 3$ this is 235 which is composite (as it is divisible by five). I'm not sure how to show this in the general odd prime case where $p = 2k + 1$ for $k \geq 1$. Using that expansion gives

$$x = 26(2k + 1)^2 + 1 = 104k^2 + 104k + 27\,,$$

which I can't prove factors. If anyone sees anything I've missed please contact me.

Lecture 9: Sides and Angles of a Triangle

Example 5 Notes

From the given diagram and the property that isosceles triangles have equal angles from their base to their legs we have that

$$\angle BCD = 180 - 2(2\beta) = 180 - 4\beta\,.$$

Next note that as $\angle ACB = \beta$ we have

$$\begin{aligned}\angle DCE &= 180 - \angle ACB - \angle BCD \\ &= 180 - \beta - (180 - 4\beta) = 3\beta\,.\end{aligned}$$

This process can be continued in each triangle until we get to the angle α.

Example 6 Notes

Now in $\triangle BGC$ we have

$$\angle BGC + \angle GBC + \angle GCB = 180\,.$$

Next we will break the angles $\angle GBC$ and $\angle GCB$ into parts as

$$\begin{aligned}\angle GBC &= \angle GBD + \angle DBC \\ \angle GCB &= \angle GCD + \angle DCB\,,\end{aligned}$$

to get

$$\angle BGC + \angle GBD + \angle DBC + \angle GCD + \angle DCB = 180\,.$$

To use this note that

$$\angle DBC + \angle DCB = 180 - 150 = 30\,, \tag{832}$$

and that

$$\angle BGC = 100\,,$$

which means that

$$100 + \angle GBD + \angle GCD + 30 = 180\,,$$

or

$$\angle GBD + \angle GCD = 50\,.$$

From the fact that the problem gives us angle bisectors have that

$$\angle ABD + \angle ACD = 2\angle GBD + 2\angle GCD = 2(50) = 100\,.$$

Thus for $\angle A$ we have

$$\begin{aligned}\angle A &= 180 - \angle ABC - \angle ACB \\ &= 180 - (\angle ABD + \angle DBC) - (\angle ACD + \angle DCB) \\ &= 180 - (\angle ABD + \angle ACD) - (\angle DBC + \angle DCB) \\ &= 180 - 100 - 30 = 50\,.\end{aligned}$$

In the above we have used Equation 832 to evaluate $\angle DBC + \angle DCB$.

Example 7 Notes

Introduce the angles as suggested in the problem. Now note that $\beta = 2\gamma$ from the fact that opposite angles are congruent and BE bisects the external angle at B. Next we have that

$$\beta = \alpha + \delta \,,$$

from the exterior angle theorem in the triangle $\triangle ABC$ at vertex B. Next

$$\delta = \alpha + \gamma \,,$$

from the exterior angle theorem in the triangle $\triangle BCE$ at vertex C. Finally from triangle $\triangle AEB$ as the external angle at B is $\beta - \gamma$ we have that

$$\begin{aligned} \beta - \gamma &= \angle BEA + \angle BAC \\ &= \alpha + \alpha = 2\alpha \,. \end{aligned}$$

This means that

$$\beta = \gamma + 2\alpha \,.$$

As we know that $\beta = 2\gamma$ when we use this in the above we get

$$\gamma = 2\alpha \,.$$

Next summing all of the angles in the triangle $\triangle ADB$ we get

$$\angle DAB + 2\beta = 180 \,,$$

or

$$\frac{1}{2}(180 - \alpha) + 2\beta = 180 \,.$$

This is equivalent to

$$-\frac{\alpha}{2} + 2\beta = 90 \,.$$

Let now write this expression in terms of the variable γ by using the facts that $\alpha = \frac{\gamma}{2}$ and $\beta = 2\gamma$ to get

$$-\frac{\gamma}{4} + 4\gamma = 90 \,.$$

Solving this we get $\gamma = 24$. Using this we have that

$$\angle A = \alpha = \frac{\gamma}{2} = 12 \,.$$

Testing Question A.1

The sum of all the interior angles in a convex n-sided polygon is $180(n-2)$. Thus we need to have

$$180(n-2) < 2007 \,,$$

so

$$n - 2 < \frac{2007}{180} = 11.15 \, .$$

If n is an integer $n-2$ will also be one and so we must have

$$n - 2 \le 11 \, ,$$

so $n \le 13$. The maximum value for n is $n = 13$.

Testing Question A.2

We draw triangle $\triangle ABC$ with AC on the x-axis and B "above" the segment AC. Introduce the "base" angles of several isosceles triangles as

$$\begin{aligned} \alpha &= \angle QAP \\ \beta &= \angle B \\ \gamma &= \angle C \, . \end{aligned}$$

Then because the various equal segments produce a number of isosceles triangles we have

$$\begin{aligned} \angle BPQ &= \beta \\ \angle AQP &= \alpha \\ \angle CPA &= \gamma \, . \end{aligned}$$

From the fact that supplemental angles sum to 180 degrees we have that

$$\angle BQP = 180 - \alpha \, ,$$

From the fact that the sum of the angles in a triangle sum to 180 degrees we have that

$$\begin{aligned} \angle QPA &= 180 - 2\alpha \\ \angle PAC &= 180 - 2\gamma \, . \end{aligned}$$

Now as $AB = BC$ we have

$$\angle BAC = \angle BCA \, ,$$

or

$$\alpha + (180 - 2\gamma) = \gamma \, ,$$

or

$$\alpha + 180 = 3\gamma \, . \tag{833}$$

To get a second relationship relating α, β, and γ note that $\angle BQP = 180 - 2\beta$. We can also evaluate $\angle BQP$ by recognizing that it is the external angle of the vertex Q in triangle $\triangle QPA$ and thus we can also write it as

$$\angle BQP = \alpha + \angle QPA = \alpha + (180 - \beta - \gamma) \, .$$

Using both of these expressions we have

$$\angle BQP = 180 - 2\beta = \alpha + 180 - \beta - \gamma \, ,$$

which is equivalent to

$$\gamma = \alpha + \beta\,. \tag{834}$$

We would like to get one more equation involving the three unknowns α, β, and γ. To do that we will use the fact that the sum of the angles in $\triangle BQP$ is 180 degrees so

$$2\beta + (180 - \alpha) = 180\,,$$

so

$$\alpha = 2\beta\,. \tag{835}$$

From Equation 834 this means that $\gamma = 3\beta$. Using these two expressions in Equation 833 we get

$$2\beta + 180 = 9\beta\,.$$

Solving this we get $\beta = \frac{180}{7} = 25\,\frac{5}{7}$.

Testing Question A.3

We draw this triangle with the right angle C at the origin of an x-y Cartesian coordinate system, the segment AC along the y-axis, the segment CB along the x-axis, and finally the points E and F on the hypotenuse AB in the order A, F, E, and B. Note that in placing the points E and F we know that there is *not* a "gap" between E and F because if you draw the triangle that way (with points in the order A, E, F, and then B) and using the equal lengths given in this question we would have

$$AB = AE + EF + FB = AC + CB + EF\,,$$

so that as $EF > 0$ we see that

$$AB > AC + CB\,,$$

which violates the triangle inequality.

Lets draw segments to introduce angles that are the base angles for isosceles triangles. If we draw the segments CF and CE then as $AC = AE$ we introduce α as

$$\angle ACE = \angle AEC = \alpha\,.$$

As $BC = BF$ we introduce β as

$$\angle BCF = \angle BFC = \beta\,.$$

Now in triangle $\triangle FCE$ we have

$$\angle FCE = 180 - \alpha - \beta\,.$$

As $\angle ACB = 90°$ we get

$$90 = \alpha + \beta - \angle FCE = \alpha + \beta - (180 - \alpha - \beta)\,.$$

This simplifies to

$$\alpha + \beta = 135\,. \tag{836}$$

This means that

$$\angle FCE = 180 - (\alpha + \beta) = 180 - 135 = 45\,.$$

Testing Question A.4

Let the three integer sides be a, b, and c such that $a \geq b \geq c$ and that

$$a + b + c = 17 \,. \tag{837}$$

From this as we have

$$17 = a + b + c \geq 3c \quad \text{so} \quad c \leq \frac{17}{3} < 6 \,.$$

Thus we have found that $c \leq 5$. We also know that by the triangle inequality that

$$a - b < c \,. \tag{838}$$

To count the number of triangles with the desired conditions we will take $c \in \{1, 2, 3, 4, 5\}$ and see how many triangles with that value of c exist.

If $c = 1$ then Equation 837 gives

$$a + b = 16 \,,$$

and Equation 838 gives

$$a - b < 1 \,.$$

The integer solutions to the above we are looking for are when $a - b = 0$ which give $a = b = 8$ and only one triangle with this value of c.

If $c = 2$ then Equation 837 gives

$$a + b = 15 \,,$$

and Equation 838 gives

$$a - b < 2 \,.$$

The integer solutions to the above we are looking for are when $a - b \in \{0, 1\}$. This gives two systems of equations to solve. The only integer solutions are when $a - b = 1$ and we get $a = 8$ and $b = 7$ and only one triangle with this value of c.

If $c = 3$ then Equation 837 gives

$$a + b = 14 \,,$$

and Equation 838 gives

$$a - b < 3 \,.$$

The integer solutions to the above we are looking for are when $a - b \in \{0, 1, 2\}$. This gives three systems of equations to solve. The only integer solutions are when $a - b = 0$ (where $a = b = 7$) and when $a - b = 2$ (where $a = 8$ and $b = 6$). Thus there are two valid triangle when $c = 3$.

If $c = 4$ then Equation 837 gives

$$a + b = 13 \,,$$

and Equation 838 gives

$$a - b < 4 \,.$$

The integer solutions to the above we are looking for are when $a-b \in \{0, 1, 2, 3\}$. This gives four systems of equations to solve. The only integer solutions are when $a - b = 1$ (where $a = 7$ and $b = 6$) and when $a - b = 3$ (where $a = 8$ and $b = 5$). Thus there are two valid triangle when $c = 4$.

If $c = 5$ then Equation 837 gives

$$a + b = 12\,,$$

and Equation 838 gives

$$a - b < 5\,.$$

The integer solutions to the above we are looking for are when $a - b \in \{0, 1, 2, 3, 4\}$. This gives five systems of equations to solve. One can show that the only integer solutions are when $a - b =\in \{0, 2, 4\}$ with roots $(a, b) = (6, 6)$, $(a, b) = (7, 5)$, and $(a, b) = (8, 4)$. Thus there are three valid triangle when $c = 5$.

Adding up all of these there are

$$1 + 1 + 2 + 2 + 3 = 9\,,$$

triangles of the given form.

Testing Question A.5

We are told that $a < b < c$ with $b = 2$ so that

$$a < 2 < c\,.$$

For a to be a positive integer means that $a = 1$ is the only solution to the above inequality and we have

$$1 < 2 < c\,.$$

By the triangle inequality we must have

$$c < 1 + 2 = 3\,.$$

To be an integer this means that $c \in \{1, 2\}$ but neither of these will satisfy $1 < 2 < c$ and thus there are no integer solutions of this requested form.

Testing Question A.6

Let m be the length of the hypotenuse and n the length of the unknown side. Then we must have

$$m^2 = n^2 + 21^2\,,$$

or

$$m^2 - n^2 = 3^2 \cdot 7^2\,,$$

or

$$(m - n)(m + n) = 3^2 \cdot 7^2 .$$

Lets see for which values of m and n will this be satisfied.

- If $m - n = 1$ then $m + n = 3^2 \cdot 7^2 = 441$.
- If $m - n = 3$ then $m + n = 3 \cdot 7^2 = 147$.
- If $m - n = 3^2 = 9$ then $m + n = 7^2 = 49$.
- If $m - n = 3^2 = 3 \cdot 7 = 21$ then $m + n = 3 \cdot 7 = 21$.

We can stop there since valid solutions must have $m - n < m + n$ (not all of the ones listed above will be valid). We can solve all of these systems for the value of m and n and then look at the solution that gives the smallest perimeter. Another way is note that the perimeter can be written as

$$21 + m + n ,$$

and thus the smallest value for this will have $m+n$ the smallest. From the above the smallest happens when $m + n = 49$ which gives a minimum perimeter of $21 + 49 = 70$.

Testing Question A.7

We draw this figure with the segment AD on the x-axis with points from left-to-right in the order A, B, then D. The point E is above D and to the right of it so that $\angle ADE = 140°$. Let

$$\alpha = \angle EAD .$$

Then as $AB = BC$ we have $\angle ACB = \alpha$ and $\angle ABC = 180 - 2\alpha$. Next note that

$$\angle CBD = 180 - \angle ABC = 180 - (180 - 2\alpha) = 2\alpha .$$

Then as $BC = CD$ we have $\angle BDC = \angle CBD = 2\alpha$ and

$$\angle BCD = 180 - 2(2\alpha) = 180 - 4\alpha .$$

Next as

$$\angle ACB + \angle BCD + \angle DCE = 180 ,$$

we have

$$\angle DCE = 180 - \alpha - (180 - 4\alpha) = 3\alpha .$$

Then as $CD = DE$ we have

$$\angle DEC = \angle DCE = 3\alpha ,$$

and

$$\angle CDE = 180 - 2(3\alpha) = 180 - 6\alpha .$$

Now are are told the angle measure of

$$\angle ADE = \angle ADC + \angle CDE = 2\alpha + (180 - 6\alpha) = 180 - 4\alpha .$$

Setting this equal to $140°$ we solve for α and find $\alpha = 10°$.

Testing Question A.8

Draw this triangle with its base BC along the x-axis and the point A above the segment BC. Lets draw the segments FD, FE, and DE. As $AB = AC$ we have

$$\angle ABC = \angle ACB = \frac{180 - \angle BAC}{2} = \frac{180 - 80}{2} = 50\,.$$

As $BD = BF$ we get

$$\angle BDF = \angle BFD = \frac{180 - \angle FBD}{2} = \frac{180 - 50}{2} = 65\,.$$

In the same way

$$\angle EDC = \angle DEC = 65\,.$$

Thus we have

$$\angle EDF = 180 - \angle FDB - \angle EDC = 180 - 2(65) = 50\,.$$

Testing Question A.9

Let the sides of the triangle be $b-1$, b, and $b+1$ for some integer b. Then we are told that

$$3b \leq 100\,,$$

so

$$b \leq 33\frac{1}{3}\,.$$

As b is a positive integer we have learned that $b \leq 33$. The triangle inequality used as

$$(b-1) + b > b + 1 \quad \text{so} \quad b > 2\,.$$

Thus we have learned that $b \geq 3$. As the triangle must be acute means that the angle opposite the largest side is less than $\frac{\pi}{2}$. If θ is this angle this means that

$$\cos(\theta) < 1\,.$$

Using the law of cosigns we have that

$$(b+1)^2 = b^2 + (b-1)^2 - 2b(b-1)\cos(\theta)\,,$$

so as $\cos(\theta) < 1$ this means that

$$(b+1)^2 < b^2 + (b-1)^2\,.$$

Expanding and simplifying this can be written as

$$b(b-4) > 0\,.$$

This means that $b > 4$. Thus we have learned that $b \geq 5$.

Combining what we know we have that $5 \leq b \leq 33$ which is $33 - 5 + 1 = 29$ triangles.

Testing Question A.10

I found this figure hard to draw just from the text description. Quickly looking at the solutions in the back gave a diagram that I could reason with. Refer to that diagram in what follows.

To start we will define the angles $\theta \equiv \angle ACB$ and $\phi \equiv \angle BDA$. Then as $AB = AC$ we know that $\angle ABC = \angle ACB = \theta$. As $AB = BD$ we know that $\angle BAD = \angle BDA = \phi$.

Using the right triangle $\triangle AED$ we have $\angle EAD = \frac{\pi}{2} - \phi$. Then using triangle $\triangle ABC$ we have

$$\angle BAC = \pi - 2\angle BCA = \pi - 2\theta \, .$$

Next we can write

$$\angle BAC + \angle CAD = \angle BAD = \angle BDA = \phi \, ,$$

as

$$\pi - 2\theta + \frac{\pi}{2} - \phi = \phi \, ,$$

or

$$\frac{3\pi}{2} = 2\phi + 2\theta \, .$$

This means that

$$\angle C + \angle D = \theta + \phi = \frac{3\pi}{4} = 135^\circ \, .$$

Testing Question B.1

Draw the triangle with the AC side along an x-axis and the vertex B above. Draw the angle bisector of $\angle A$ as the segment AE. Draw the segment BD intersecting AE at H.

To solve this problem draw the segment CH. Then let the areas of the following triangles be denoted as

$$\begin{aligned} S_0 &\equiv \text{Area}(\triangle BHE) \\ S_1 &\equiv \text{Area}(\triangle BHA) \\ S_2 &\equiv \text{Area}(\triangle AHD) \\ S_3 &\equiv \text{Area}(\triangle DHC) \\ S_4 &\equiv \text{Area}(\triangle CHE) \, . \end{aligned}$$

Lets assume that $S_0 = 1$. Then from

$$\frac{AH}{HE} = \frac{3}{1} \quad \text{so} \quad S_1 = 3S_0 = 3 \, .$$

From

$$\frac{BH}{HD} = \frac{5}{3} \, ,$$

we have

$$HD = \frac{3}{5} BH \,,$$

so

$$S_2 = \frac{3}{5} S_1 = \frac{9}{5} \,.$$

We now want to compute S_3 and S_4. From $\triangle AEC$ we have

$$\frac{S_2 + S_3}{S_4} = \frac{AH}{HE} = 3 \,.$$

From $\triangle BDC$ we have

$$\frac{S_3}{S_0 + S_4} = \frac{HD}{BH} = \frac{3}{5} \,.$$

Using what we know about $S_0 = 1$ and $S_2 = \frac{9}{5}$ we get

$$\frac{9}{5} + S_3 = 3S_4 \,, \tag{839}$$

and

$$S_3 = \frac{3}{5}(1 + S_4) \,. \tag{840}$$

Solving these two we find $S_3 = \frac{6}{5}$ and $S_4 = 1$. These then mean that

$$S_1 + S_0 = 4$$
$$S_2 + S_3 + S_4 = \frac{9}{5} + \frac{6}{5} + 1 = 4 \,.$$

Since this means that the areas of the triangles $\triangle AEB$ and $\triangle AEC$ are equal and since they share a common side we must have $CE = BE$ and $AB = AC$. The later expression means that the triangle $\triangle BAC$ is isosceles. This means that

$$\angle C = \angle B = \frac{1}{2}(180 - \angle A) = \frac{1}{2}(180 - 70) = 55 \,.$$

Testing Question B.2

I found this figure hard to draw just from the text description. Quickly looking at the solutions in the back gave a diagram that I could reason with.

From the diagram let $\angle B = \beta$ and $\angle ACD = \gamma$. As the sums of the angles in a triangle must equal 180 degrees we have

$$\angle ACB = 180 - \beta - 96 = 84 - \beta \,.$$

Next from the exterior angle theorem we have

$$\gamma = \beta + 96 \,.$$

Using the three angles in $\triangle BA_1C$ we have

$$\begin{aligned}\angle A_1 &= 180 - \frac{\beta}{2} - \left(84 - \beta + \frac{\gamma}{2}\right) \\ &= 96 + \frac{1}{2}(\beta - \gamma) = 96 + \frac{1}{2}(-96) = \frac{96}{2} = 48\,.\end{aligned}$$

Using the three angles in $\triangle BA_2C$ we have

$$\angle A_2 = 180 - \frac{\beta}{2^2} - \left(84 - \beta + \left(1 - \frac{1}{2^2}\right)\gamma\right)\,.$$

From the above it looks like the general pattern is

$$\begin{aligned}\angle A_n &= 180 - \frac{\beta}{2^n} - \left(84 - \beta + \left(1 - \frac{1}{2^n}\right)\gamma\right) \\ &= 96 - \frac{\beta}{2^n} + \beta - \left(1 - \frac{1}{2^n}\right)\gamma \\ &= 96 + \left(1 - \frac{1}{2^n}\right)\beta + \left(1 - \frac{1}{2^n}\right)\gamma \\ &= 96 + \left(1 - \frac{1}{2^n}\right)(\beta - \gamma) \\ &= 96 + \left(1 - \frac{1}{2^n}\right)(-96) = \frac{96}{2^n}\,.\end{aligned}$$

Using this we find that $\angle A_5 = \frac{96}{2^5} = 3$.

Testing Question B.3

Draw this figure with BC on the x-axis and the vertex A "above" this segment. Draw the smaller triangle $\triangle DEF$ inside this larger one. Then as the triangle $\triangle DEF$ has all equal sides each angle is $60°$. Define

$$\begin{aligned}\angle ADF &= \alpha \\ \angle CEF &= \beta \\ \angle DEB &= \gamma\,.\end{aligned}$$

Then using supplementary angles we have

$$\begin{aligned}\angle BDE &= \pi - \frac{\pi}{3} - \alpha \\ \angle FEC &= \pi - \frac{\pi}{3} - \gamma \\ \angle AFD &= \pi - \frac{\pi}{3} - \beta\,.\end{aligned}$$

As all three angles in a triangle must sum to 180 degrees we have

$$\angle A = \pi - \angle ADF - \angle AFD = \pi - \alpha - \left(\pi - \frac{\pi}{3} - \beta\right) = -\alpha + \beta + \frac{\pi}{3}$$
$$\angle B = \pi - \angle BDE - \angle DEB = \pi - \left(\pi - \frac{\pi}{3} - \alpha\right) - \gamma = \alpha - \gamma + \frac{\pi}{3}$$
$$\angle C = \pi - \angle FEC - \angle EFC = \pi - \left(\pi - \frac{\pi}{3} - \gamma\right) - \beta = \gamma - \beta + \frac{\pi}{3}.$$

As $AB = AC$ we know that $\angle B = \angle C$ or

$$\frac{\pi}{3} + \alpha - \gamma = \frac{\pi}{3} + \gamma - \beta\,,$$

or

$$\gamma = \frac{1}{2}(\alpha + \beta)\,,$$

which is the expression we wanted to derive.

Testing Question B.4

For this diagram we will draw the triangle $\triangle ACB$ with vertex C at the origin of the x-y plane, vertex A on the y-axis above C, and vertex B on the x-axis. Then vertex E is between C and B and vertex D is on the hypotenuse connecting A and B.

Using similar triangles we have that $\triangle ACB \sim \triangle DEB$ so that

$$\frac{AC}{DE} = \frac{CB}{EB} = \frac{AB}{DB}\,.$$

As there are a lot of variables in the above equations lets try to simplify the above by writing it using only variables in the larger triangle. As $DB = \frac{1}{2}$ we have $\frac{AB}{DB} = 2AB$. Next as we are told that $DE = 1 - BC$ and $EB = AC$ the above can be written as

$$\frac{AC}{1 - BC} = \frac{BC}{AC} = 2AB\,. \tag{841}$$

Using the Pythagorean theorem in the right triangle $\triangle ACB$ we have

$$AC^2 + BC^2 = AB^2\,. \tag{842}$$

From Equation 841 we get

$$AC^2 = BC(1 - BC)\,,$$

which if we put this into Equation 842 we get

$$BC(1 - BC) + BC^2 = AB^2\,,$$

or

$$AB^2 = BC\,.$$

Using this in Equation 841 (specifically $\frac{BC}{AC} = 2AB$) we get

$$\frac{AB^2}{AC} = 2AB\,,$$

or

$$AB = 2AC\,,$$

so that

$$\frac{AC}{AB} = \frac{1}{2} = \sin(\angle B)\,.$$

This means that $\angle B = 30°$.

Lecture 10: Pythagoras' Theorem and Its Applications

Example 5 Notes

Drop AD perpendicular to BC as shown in the diagram in the text. Then using the Pythagorean theorem in the "left" right triangle we have

$$AB^2 = (BM + MD)^2 + AD^2 = BM^2 + 2BM \cdot MD + MD^2 + AD^2 \,. \tag{843}$$

Using the Pythagorean theorem in the triangle $\triangle MDA$ gives

$$MD^2 + AD^2 = AM^2 \,. \tag{844}$$

Using this we have

$$AB^2 = BM^2 + AM^2 + 2BM \cdot MD \,.$$

Also another application of the Pythagorean theorem (this time in the "right" right triangle) gives

$$AC^2 = CD^2 + AD^2 = (CM - DM)^2 + AD^2 = CM^2 - 2CM \cdot DM + DM^2 + AD^2 \,.$$

Using Equation 844 this becomes

$$AC^2 = CM^2 + AM^2 - 2CM \cdot DM \tag{845}$$

If we add Equations 843 and Equations 845 then we get

$$AB^2 + AC^2 = 2BM^2 + 2AM^2 \,,$$

as $BM = CM$.

Testing Question A.1

Method 1: Let BC be along the x-axis with B to the left of C and A above BC. Connect A to D with a segment and also drop a vertical from A to BC to a point E. Then we have

$$\begin{aligned} BD^2 + CD^2 &= (BE - DE)^2 + (DE + CE)^2 \\ &= BE^2 - 2BE \cdot DE + DE^2 + DE^2 + 2DE \cdot CE + CE^2 \,. \end{aligned}$$

Now

$$DE^2 = AD^2 - AE^2 \,,$$

so the above is

$$BD^2 + CD^2 = 2(AD^2 - AE^2) + BE^2 - 2BE \cdot DE + 2DE \cdot CE + CE^2 \,.$$

Now as AE is perpendicular to BC and $\angle ABC = \angle ACB = 45°$. This means that $\angle BAE = \angle CAE = 45°$. This means that $\triangle ABE$ and $\triangle ACE$ are isosceles right triangles. This means that

$$BE = AE = EC \,.$$

Using this we have

$$-2BE \cdot DE + 2DE \cdot CE = 0\,.$$

So back to the above expression we have

$$BD^2 + CD^2 = 2AD^2 - 2AE^2 + BE^2 + CE^2 = 2AD^2\,,$$

as we were to show.

Method 2: Draw the right triangle $\triangle CAB$ with the right angle at A in the x-y Cartesian coordinate plane with the point $A = (0,0)$, $B = (L,0)$, and $C = (0,L)$. A point (x,y) on the line between the points BC is given by the equation

$$y = L + \left(\frac{0-L}{L-0}\right)(x-0) = L - x\,.$$

Let the point D be on this line so that $D = (x, L-x)$. Then the Euclidean distance formula gives

$$\begin{aligned} AD^2 &= x^2 + (L-x)^2 \\ BD^2 &= (x-L)^2 + (L-x)^2 = 2(L-x)^2 \\ CD^2 &= (0-x)^2 + (L-L+x)^2 = 2x^2\,. \end{aligned}$$

Thus from these we see that

$$BD^2 + CD^2 = 2AD^2\,,$$

as we were to show.

Testing Question A.2

Let a be a leg length along the x-axis and b a leg length along the y-axis in a right triangle. Then we are told that

$$a + b + \sqrt{a^2+b^2} = 30\,, \tag{846}$$

and

$$\frac{1}{2}ab = 30 \Rightarrow ab = 60\,. \tag{847}$$

If we square the first equation we get

$$(a+b)^2 + 2(a+b)\sqrt{a^2+b^2} + (a^2+b^2) = 900\,,$$

or

$$2a^2 + 2b^2 + 2(a+b)\sqrt{a^2+b^2} = 900 - 2ab\,,$$

or using what we know about ab this is

$$2a^2 + 2b^2 + 2(a+b)\sqrt{a^2+b^2} = 900 - 120 = 780\,.$$

Dividing this by two we get

$$a^2 + b^2 + (a+b)\sqrt{a^2+b^2} = 390\,.$$

If we write this in terms of $c = \sqrt{a^2 + b^2}$ we have

$$c^2 + (30 - c)c = 390 \, .$$

We can solve this for c where we find $c = 13$. Then using that in Equation 846 we have $a + b = 17$. Next using Equation 847 as $b = \frac{60}{a}$ into that gives

$$a + \frac{60}{a} = 17 \, ,$$

or

$$a^2 - 17a + 60 = 0 \, ,$$

or

$$(a - 5)(a - 12) = 0 \, .$$

This means that the solutions for a are $a \in \{5, 12\}$. Using this in $a + b = 17$ means that $b \in \{12, 5\}$. This means that the sides of the triangle are 5, 12, and 13.

Testing Question A.3

Let the the segment AB be along the x-axis and C "above" the segment AB. As $\triangle ACB$ is a right triangle we have

$$BC^2 = AB^2 - AC^2 = 225 - 81 = 144 \quad \text{so} \quad BC = 12 \, .$$

As $BD : DC = 5 : 3$ we have

$$\begin{aligned} BD &= \frac{5}{8}(12) = \frac{15}{2} \\ DC &= \frac{3}{8}(12) = \frac{9}{2} \, . \end{aligned}$$

Now using the right triangle $\triangle ACD$ we have

$$AD^2 = AC^2 + CD^2 = 81 + \frac{81}{4} = \frac{405}{4} \quad \text{so} \quad AD = \frac{9}{2}\sqrt{5} \, .$$

From the point D drop a vertical to the segment AB intersecting at E. Then as $\angle CAD = \angle DAB$ we have

$$\triangle ACD \sim \triangle AED \, .$$

This means that

$$\frac{ED}{DC} = \frac{AD}{AD} = 1 \Rightarrow ED = DC = \frac{9}{2} \, .$$

Note in solving this problem we didn't actually need the length AD.

Testing Question A.4

Let the the segment AB be along the x-axis and C "above" the segment AB. From the problem statement D is on AB and E is on CB. As D is on the perpendicular bisector of AB we have

$$AD = DB = \frac{1}{2} AB = 3\sqrt{5}\,.$$

As $\angle ACB = 90°$ we have

$$AB = \sqrt{BC^2 + AC^2} = \sqrt{144 + 36} = \sqrt{180} = 6\sqrt{5}\,.$$

As $\triangle BDE \sim \triangle BCA$ we have

$$\frac{BE}{DB} = \frac{AB}{CB}\,,$$

or

$$\frac{BE}{3\sqrt{5}} = \frac{6\sqrt{5}}{12} \quad \text{so} \quad BE = \frac{15}{2}\,.$$

Using that length we have that

$$CE = BC - BE = 12 - \frac{15}{2} = \frac{9}{2}\,.$$

Testing Question A.5

I found this figure hard to draw just from the text description. Quickly looking at the solutions in the back gave a diagram that I could reason with.

The Pythagorean theorem applied to the diagonal of the rectangle $ABCD$ gives

$$AC^2 = AB^2 + BC^2 = BD^2\,.$$

Using the Pythagorean theorem again in the right triangles $\triangle DEC$ and $\triangle CEB$ gives

$$DE^2 + 25 = \frac{9}{16} BD^2 + 25 = CD^2\,,$$

and

$$EB^2 + 25 = \frac{1}{16} BD^2 + 25 = BC^2\,.$$

If we next add these two equations together we get

$$\frac{10}{16} BD^2 + 50 = CD^2 + BC^2\,.$$

In this note that the right-hand-side is BD^2 again thus

$$50 = \frac{6}{16} BD^2 \Rightarrow BD = \frac{20}{\sqrt{3}}\,,$$

which is also the length AC.

Testing Question A.6

Draw the right triangle $\triangle ACB$ with AC on the y-axis and CB on the x-axis and D the midpoint of AC. Then the Pythagorean theorem in the triangle $\triangle ACB$ is

$$\begin{aligned} AB^2 &= AC^2 + BC^2 \\ &= (AD + DC)^2 + BC^2 \,, \end{aligned}$$

but since $AD = DC$ we can let $AD + DC = 2DC$ to get

$$\begin{aligned} AB^2 &= 4DC^2 + BC^2 \\ &= 4(DC^2 + BC^2) - 3BC^2 \,. \end{aligned}$$

Now the Pythagorean theorem in triangle $\triangle DCB$ is

$$DC^2 + BC^2 = BD^2 \,.$$

Using this in the above we get

$$AB^2 + 3BC^2 = 4BD^2 \,,$$

as we were to show.

Testing Question A.7

Draw the right triangle $\triangle ACB$ with AC on the y-axis and CB on the x-axis. The points D and E are placed on BC and AC respectively.

Now consider the left-hand-side of the given expression. We have

$$AD^2 + BE^2 = AC^2 + CD^2 + EC^2 + CB^2$$

But in $\triangle ACB$ we have

$$AC^2 + CB^2 = AB^2 \,,$$

and in $\triangle ECD$ we have

$$CD^2 + EC^2 = BDDE^2 \,.$$

Using these in the above we get

$$AD^2 + BE^2 = AB^2 + DE^2 \,.$$

Testing Question A.8

Let the triangle $\triangle ABC$ have BC along the x-axis with A "above" that segment. From A drop a vertical to BC intersecting at D.

Recall that m_i is defined as

$$m_i = AP_i^2 + BP_iP_iC\,.$$

Now we can write BP_i and P_iC as

$$BP_i = BD - DP_i\,,$$

and

$$P_iC = DC + DP_i\,.$$

As triangle $\triangle ABC$ is isosceles we know that $BD = DC$ thus we can write m_i as

$$\begin{aligned} m_i &= AP_i^2 + (BD - DP_i)(BD + DP_i) \\ &= AP_i^2 + BD^2 + BD \cdot DP_i - BD \cdot DP_i - DP_i^2 \\ &= BD^2 + AP_i^2 - DP_i^2 \end{aligned}$$

But by the Pythagorean theorem in $\triangle ADP_i$ we have

$$AP_i^2 = DP_i^2 + AD^2\,,$$

so using this m_i becomes

$$m_i = BD^2 + AD^2\,,$$

which is independent of i. The Pythagorean theorem in $\triangle ADB$ gives that $BD^2 + AD^2 = AB^2 = 4$ thus we have

$$\sum_{i=1}^{100} m_i = 400\,.$$

Testing Question A.9

I found this figure hard to draw just from the text description. Quickly looking at the solutions in the back gave a diagram that I could reason with.

By symmetry we have $FD = DG$, $AD = DB$, and $AG = BF$. As ED is perpendicular to FG and equidistant between F and G we have that $EF = EG$. This means that

$$\begin{aligned} EF^2 &= EG^2 \\ &= AE^2 + AG^2\,, \end{aligned}$$

using the right triangle $\triangle EAG$. As $AG = BF$ we get

$$EF^2 = AE^2 + BF^2\,,$$

as we were to show.

Testing Question A.10

I found this figure hard to draw just from the text description. Quickly looking at the solutions in the back gave a diagram that I could reason with.

Rotate $\triangle BPA$ in a counterclockwise direction around the point B by $60°$. Let M be the image of P after this rotation. Then $\angle MBP = 60$, $MB = BP$, and $MC = PA = 2$.

Draw the segment MP. With that segment drawn the triangle $\triangle MBP$ is isosceles with a vertex angle $\angle MBP = 60$ so

$$\angle BMP = \angle BPM = 60\,,$$

also. This means that $\triangle MBP$ is an equilateral triangle and $MP = BP = 2\sqrt{3}$. As we are told the length $PC = 4$ we know all of the sides of the triangle $\triangle PMC$. We know that

$$\begin{aligned} MP^2 &= 12 \\ MC^2 &= 4 \\ CP^2 &= 16\,. \end{aligned}$$

Note that $CP^2 = MP^2 + MC^2$ and thus $\triangle PMC$ is a right triangle with $\angle PMC = 90°$. Note also that one of the sides of this triangle is $\frac{1}{2}$ the length of the hypotenuse so

$$\sin(\angle CPM) = \frac{CM}{CP} = \frac{1}{2} \quad \text{so} \quad \angle CPM = 30°\,.$$

Now

$$\angle BPC = \angle BPM + \angle MPC = 60 + 30 = 90\,.$$

This means that in the right triangle $\triangle BPC$ we have

$$BC^2 = BP^2 + PC^2 = 12 + 4^2 = 12 + 16 = 28\,,$$

so $BC = \sqrt{28} = 2\sqrt{7}$.

Testing Question B.1

From the center of the circle draw a perpendicular towards the chord AB intersecting AB at O'. As $OA = OB$ this point O' is on the perpendicular bisector of AB so that

$$AO' = O'B = \frac{63 + 33}{2} = 48\,.$$

Then using the right triangle $\triangle AO'O$ we have that

$$OO' = \sqrt{AO^2 - AO'^2} = \sqrt{52^2 - 48^2} = 20\,.$$

Next we have that $O'M = AM - AO' = 63 - 48 = 15$. Using the right triangle $\triangle MO'O$ we have

$$OM^2 = O'O^2 + O'M^2 = 20^2 + 15^2 \quad \text{so} \quad OM = 25\,.$$

Testing Question B.2

I drew my rectangle $ABCD$ with AB along the x-axis and AD along the y-axis so that walking counterclockwise around the rectangle starting at $A = (0, 0)$ we have the points B, C, and then D. Let the point P be inside the rectangle such that P is x units to the right of A and y units to the left of B. P is also p units above the horizontal x-axis and q units below the point C.

Now given that we know the distances AP, BP, and CP using right triangles formed by dropping vertical and horizontal perpendiculars to AB and BC we have that

$$x^2 + p^2 = 3^2 \quad (848)$$
$$y^2 + p^2 = 4^2 \quad (849)$$
$$y^2 + q^2 = 5^2 \,. \quad (850)$$

To answer this question we want to know the value of $\sqrt{x^2 + q^2}$. Lets use these three equations to evaluate

$$\begin{aligned} x^2 + q^2 &= (9 - p^2) + q^2 \quad \text{using Equation 848} \\ &= 9 - (16 - y^2) + q^2 \quad \text{using Equation 849} \\ &= -5 + y^2 + q^2 \\ &= -5 + 25 \quad \text{using Equation 850} \\ &= 18 \,. \end{aligned}$$

Thus $PD = \sqrt{18} = 3\sqrt{2}$.

Testing Question B.3

Draw our right triangle with one leg along the positive x axis of length m and another leg along the positive y axis of length $n = km$ with k a natural number. The length of the hypotenuse is then given by

$$\sqrt{k^2m^2 + m^2} = m\sqrt{k^2 + 1} \,.$$

Now as m is an integer in the above expression for the hypotenuse we see that to have integer sides we need $\sqrt{k^2 + 1}$ to be an integer. This means that we must have

$$\sqrt{k^2 + 1} = p \quad \text{or} \quad k^2 + 1 = p^2 \,,$$

for some integer p. This then means that

$$p^2 - k^2 = (p - k)(p + k) = 1 \,. \quad (851)$$

Now if p and k are integers then $p-k$ and $p+k$ are also and thus we are looking for a integer factorization of one. One way that can happen is if

$$\begin{aligned} p + k &= 1 \\ p - k &= 1 \,. \end{aligned}$$

If we solve the above system we get $p = 1$ and $k = 0$. The fact that $k = 0$ means that there is no triangles of the required form.

Another way that can happen is if

$$\begin{aligned} p + k &= -1 \\ p - k &= -1 \,. \end{aligned}$$

If we solve the above system we get $p = -\frac{1}{2}$ and $k = \frac{3}{2}$. As these are not integers we again have a contradiction to the assumptions and again conclude that no triangles of the required form exist.

Testing Question B.4

I drew my rectangle $ABCD$ with AB along the x-axis and AD along the y-axis so that walking counterclockwise around the rectangle starting at $A = (0, 0)$ we have the points B, C, and then D. Let the point E be on AB and F on AD with the segments CF and CE trisecting the right angle $\angle DCB$. Let θ be the angles

$$\theta = \angle DCF = \angle FCE = \angle ECB = \frac{1}{3}\left(\frac{\pi}{2}\right) = \frac{\pi}{6} \,.$$

Now in the right triangle $\triangle CBE$ we have

$$\tan(\theta) = \frac{EB}{CB} \,,$$

or since we know the value of θ and the length BE we have

$$\frac{1}{\sqrt{3}} = \frac{6}{CB} \Rightarrow CB = 6\sqrt{3} \,.$$

Using this we have

$$DF = AD - AF = CB - AF = 6\sqrt{3} - 2 \,.$$

Now in the right triangle $\triangle CDF$ we have

$$\tan(\theta) = \frac{1}{\sqrt{3}} = \frac{DF}{DC} \,.$$

This means that

$$DC = DF\sqrt{3} = 6 \cdot 3 - 2\sqrt{3} = 18 - 2\sqrt{3} \,.$$

We now know the length of all sides of the rectangle so its area is given by

$$\begin{aligned} CB \cdot DC &= 6\sqrt{3} \cdot (18 - 2\sqrt{3}) = 6 \cdot 2\sqrt{3}(9 - \sqrt{3}) \\ &= 12(9\sqrt{3} - 3) = 36(3\sqrt{3} - 1) \,. \end{aligned}$$

Lets try to approximate this we have

$$3\sqrt{3} \gtrapprox 3 \cdot 1.7 = 3 + 2.1 = 5.1 \,.$$

This means that the area is approximately $36 \cdot (4.1) = 147.6$.

Testing Question B.5

I was only able to prove one direction of the given statement. If we draw the given quadrilateral and then the diagonals AC and BD and assume that they are perpendicular at their intersection (denoted O) then using the Pythagorean theorem several times (by including the point O) we have

$$\begin{aligned} AB^2 + CD^2 &= (AO^2 + OB^2) + (CO^2 + OD^2) \\ &= (AO^2 + OD^2) + (OB^2 + CO^2) \\ &= AD^2 + BC^2 \,. \end{aligned}$$

I was not sure how to show the other direction.

Lecture 11: Congruence of Triangles

Testing Question A.1

Draw CB on the x-axis and A "above" that segment. As we are told two of the angles in our triangle the third is given by

$$\angle ABC = 180 - 60 - 75 = 45\,.$$

In the right triangle $\triangle ADC$ we have

$$\angle CAD = 90 - \angle ACD = 90 - 60 = 30\,.$$

Then as $\angle CAB = 75$ we have

$$\angle DAB = 75 - \angle CAD = 75 - 30 = 45\,.$$

Lets draw the segment CF which will pass through H and be perpendicular to AB. Then using the right triangle AFH as $\angle HAF = 45$ we have

$$\angle AHF = 90 - \angle HAF = 90 - 45 = 45\,.$$

Thus we have

$$\angle CHD = \angle AHF = 45\,.$$

Testing Question A.2

Draw the line AB along the x-axis of a Cartesian x-y plane and the point C "above" the segment AB. Place the point D on the "inside" of the triangle $\triangle ABC$ and the point P such that the given conditions in the problem are true. In doing this I drew the point P "below" the segment AB.

To solve this problem we first draw the segment PD. Note that $\triangle DBP \cong \triangle DBC$ by "Side-Angle-Side". This means that

$$\angle BPD = \angle BCD\,.$$

Note that $\triangle BDC \cong \triangle ADC$ by "Side-Side-Side". This means that $\angle BCD = \angle ACD$.

Now $\angle BCD + \angle ACD = 60^\circ$ and from the above those two angles are equal. Thus each is 30° so $\angle BPD = 30^\circ$.

Testing Question A.3

I found this figure hard to draw just from the text description. Quickly looking at the solutions in the back gave a diagram that I could reason with.

In the problem we are told that $AB = 1$ and

$$AP + PQ + QA = 2\,. \tag{852}$$

Let $a \equiv AP$ and $b \equiv AQ$ and extend AD to a point P' such that DP' is of length $1-a$. Note that by "Side-Angle-Side" the right triangle $\triangle P'DC \cong \triangle PBC$.

Now

$$QP' = QD + DP' = 1 - b + 1 - a = 2 - (a+b)\,,$$

But by using Equation 852 $a + b = 2 - PQ$ so that we have

$$QP' = PQ\,.$$

Using the above arguments note that in the triangles $\triangle CPQ$ and $\triangle CQP'$ we have

- $CP = CP'$ by using the right triangles $\triangle PBC$ and $\triangle P'DC$ (discussed above).
- $PQ = QP'$ (proved above).
- $QC = QC$ (reflexivity).

Thus by "Side-Side-Side" $\triangle CPQ \cong \triangle CP'Q$. Because of this we have $\angle PCQ = \angle QCP'$.

Now define the three angles in $\angle BCD$ as

$$\begin{aligned} \angle BCP &= \alpha \\ \angle PCQ &= \beta \\ \angle QCD &= \gamma\,. \end{aligned}$$

Thus as $\angle BCD$ is a right angle we have

$$\alpha + \beta + \gamma = 90^\circ\,. \tag{853}$$

But using what we proved above

$$\begin{aligned} \angle QCP' = \beta &= \angle QCD + \angle DCP \\ &= \gamma + \alpha\,. \end{aligned}$$

If we put this into Equation 853 we get

$$\alpha + \gamma + (\alpha + \gamma) = 90^\circ \Rightarrow \alpha + \gamma = 45^\circ.$$

If we put this into Equation 853 we get $\beta = \angle PCQ = 45^\circ$ also.

Testing Question A.4

I found this figure hard to draw just from the text description. Quickly looking at the solutions in the back gave a diagram that I could reason with.

Extend from AD leftwards until it would intersect with the extension of EC to a point denoted N. Now NAD and BC are parallel so $\angle ANE = \angle ECB$.

Note that $\angle NAE = \angle EBC = 90°$ and $AE = EB$ (due to the fact that E is the midpoint of the side AB).

Taken together this means that the right triangles $\triangle NAE \cong \triangle CBE$ and we have that

$$NA = BC = AD\,.$$

Now if $\angle NMD = 90°$ then the segment AM is the median of the hypotenuse DN in the right triangle $\triangle NMD$ and so by the theorem that "the median to the hypotenuse of a right triangle is half the hypotenuse" we would have

$$AM = \frac{1}{2}DN = AD\,.$$

This would prove what we are interested in if we can argue that $\angle NMD = 90$. To show this note that

$$\angle ANE = \angle ECB = \angle MDC = \tan^{-1}\left(\frac{1}{2}\right)\,.$$

Thus

$$\angle NDM = 90 - \angle MDC = 90 - \angle ANE\,.$$

Thus in triangle $\triangle DNM$ we have

$$\angle NMD = 180 - (\angle NDM + \angle ANE) = 180 - 90 = 90\,.$$

Testing Question A.5

I found this figure hard to draw just from the text description. Quickly looking at the solutions in the back gave a diagram that I could reason with. Refer to that diagram in what follows.

Extend AE to intersect the extension of BC at F. Then $DE = EC$, $\angle DEA = \angle CEF$, and $\angle ADC = \angle ECF$ so that we have $\triangle EDA \cong \triangle ECF$. Using that we can conclude that $AE = EF$.

Now note that BE is the median to the hypotenuse in the right triangle $\triangle ABF$ so

$$BE = AE = EF\,.$$

Let $\alpha = \angle CEB$. Then in $\triangle BEC$ as we are told that $BC = CE$ so we have $\angle EBC = \alpha$.

Now in triangle $\triangle BEF$ as $BE = EF$ we have $\angle EBC = \angle BFE = \alpha$.

As $AD||BF$ so $\angle DAF = \angle BFA = \alpha$.

As $AE = BE$ in $\triangle AEB$ we have $\angle EAB = \angle EBA$. Let that angle be β. Then as we know $\angle ABC = 90$ we have

$$\angle ABC = \angle ABE + \angle EBC = 90\,,$$

which is the statement that $\beta + \alpha = 90$.

Using the isosceles triangle $\triangle AEB$ (where $AE = BE$) we get

$$\angle AEB = 180 - 2\angle EAB = 180 - 2\beta = 180 - 2(90 - \alpha) = 2\alpha\,.$$

Thus

$$\angle AEC = \angle AEB + \angle BEC = 2\alpha + \alpha = 3\alpha = 3\angle DAE\,,$$

as we were to show.

Testing Question A.7

I found this figure hard to draw just from the text description. Quickly looking at the solutions in the back gave a diagram that I could reason with. Refer to that diagram in what follows.

Extend the segments AE and BC until they intersect at a point denoted as F. In the triangle $\triangle AED$

$$\angle FAC = 90 - \angle ADE = 90 - \angle CDB = \angle DBC\,.$$

As we have shown the three angles to be equal we have $\triangle FAC \sim \triangle DBC$ or

$$\frac{AC}{BC} = \frac{AF}{BD}\,.$$

As $AC = BC$ in the above we get $AF = BD$. We are told that $BD = 2AE$ so we have that $AF = 2AE$. Now as $AF = AE + EF$ we have we see that $EF = AE$.

Now considering the two right triangles $\triangle AEB$ and $\triangle FEB$ as $FE = EA$ and $EB = EB$ (reflexivity) we have that $\triangle AEB \cong \triangle FEB$, so $\angle EBA = \angle EBF$ (which is what we wanted to prove).

Testing Question A.8

The book has one solution to this problem. Another very simple solution seems to be the following. We draw the square with $A = (0, 8)$, $B = (0, 0)$, $C = (8, 0)$ and $D = (8, 8)$. Then in the right triangle $\triangle ADP$ using the Pythagorean theorem we have

$$DP = \sqrt{AP^2 - AD^2} = \sqrt{10^2 - 8^2} = 6\,.$$

This means that $PC = DC - DP = 8 - 6 = 2$.

Testing Question A.9

I found this figure hard to draw just from the text description. Quickly looking at the solutions in the back gave a diagram that I could reason with. Refer to that diagram in what follows.

Draw the segments AC and AD introducing the right triangles $\triangle ABC$ and $\triangle AED$. Extend the segment CB to a points P such that $BP = DE$ so that once drawn we have

$$CP = CB + BP = CB + DE = 1\,,$$

using the assumptions in the problem.

Now since the segments $AB = AE$ and $BP = ED$ by using "Side-Angle-Side" (since the angle in-between is $90°$) we have that $\triangle ABP \cong \triangle AED$. This means that $AP = AD$.

Now in the triangles $\triangle APC$ and $\triangle ADC$ we have that $CP = 1 = CD$, $AP = AD$, and $AC = AC$ (by reflexivity) so that by "Side-Side-Side" we have that $\triangle APC \cong \triangle ADC$.

We will now decompose the area we want into two areas we can calculate. We have

$$[ABCDE] = [ABC] + [ACD] + [ADE] = [APC] + [APC] = 2[APC]\,.$$

Now we can compute this last area as

$$[APC] = \frac{1}{2}(AB)(PB + BC) = \frac{1}{2}(1)(1) = \frac{1}{2}\,.$$

Therefore we find

$$[ABCDE] = 2\left(\frac{1}{2}\right) = 1\,.$$

Testing Question A.10

I found this figure hard to draw just from the text description. Quickly looking at the solutions in the back gave a diagram that I could reason with. Refer to that diagram in what follows.

Rotate $\triangle ADC$ counterclockwise around the point A. Then the "new" AC will lie along the "old" AB and the point D is mapped to the point D' which is "outside" of the triangle $\triangle ACB$.

Now $\triangle AD'B \cong \triangle ADC$ since all we did was rotate the triangle $\triangle ACD$. This means that $AD' = AD$ and $D'B = DC$. Note that $\angle D'AD = 60°$ since that's the amount by which we rotated the segment AC.

If we draw the segment DD' then as $\angle DAD = 60°$ and $AD' = AD$ we see that the triangle $\triangle D'AD$ is isosceles with a vertex angle of 60°. This means that it is an equilateral triangle and all angles are 60°.

Now as $\angle AD'B = \angle ADC = 150°$ and $\angle AD'D = 60°$ we have that $\angle DD'B = 150 - 60 = 90°$ thus the triangle $\triangle BD'D$ is a right triangle made of the segments $BD' = DC$ (from the rotation), $D'D = AD$ (by the equilateral triangle $\triangle D'AD$), and $DB = DB$ (by reflexivity).

Testing Question B.1

I found this figure hard to draw just from the text description. Quickly looking at the solutions in the back gave a diagram that I could reason with. Refer to that diagram in what follows.

Drop perpendiculars from A, C and E onto the line l connecting B and D. Call the points where these perpendiculars intersect this line the points A_1, C_1, and E_1 respectively.

From the right triangles introduced we see that $\triangle AA_1B \cong \triangle CC_1B$ and $\triangle CC_1D \cong \triangle DE_1E$ and thus we have

$$A_1B = CC_1 = DE_1 \,.$$

Now the perpendicular dropped from M (the midpoint of AE) to the line l will go through the midpoint of A_1E_1.

But as $A_1B = DE_1$ the midpoint of the segment A_1E_1 is the midpoint of the segment BD. Thus the location of M is on the perpendicular bisector of BD. Thus the point M is located on the perpendicular bisector of BD and as such its horizontal location within the segment BD is independent from C.

Note that the distance of M from the line l is given by

$$\frac{1}{2}(AA_1 + EE_1) = \frac{1}{2}(BC_1 + C_1D) = \frac{1}{2}BD \,,$$

which is a length that is also independent of C. Thus M is independent of C.

Testing Question B.2

I found this figure hard to draw just from the text description. Quickly looking at the solutions in the back gave a diagram that I could reason with. Refer to that diagram in what follows.

Drop a perpendicular from F and call the intersection with the segment AB the point H. Then by using the common side AF and the angles on either end of it we have

$$\triangle ACF \cong \triangle AHF \,,$$

from "Angle-Side-Angle". Thus we have that $CF = FH$.

Note that

$$\angle ACD = 90 - \angle A = \angle B \,.$$

Using that we have that

$$\begin{aligned} \angle FEC = 180 - \angle AEC = 180 - \left(180 - \frac{1}{2}\angle A - \angle ACD\right) \\ = \angle ACD + \frac{1}{2}\angle A = \angle B + \frac{1}{2}\angle A = \angle CFE \,. \end{aligned}$$

This means that the triangle $\triangle FCE$ has two equal angles and is thus isosceles so we now have that

$$CF = CE = FH \,.$$

Now as $CED \parallel FH$ we have $\angle BFH = \angle BCD$ and since $CE = FH$ using "Angle-Side-Angle" we have

$$\triangle ECG \cong \triangle HFB \,,$$

and thus conclude that $CG = FB$. Using that as FG is common to both of these segments CG and FB we can conclude that $CF = GB$ (as we were to show).

Testing Question B.3

I drew the segment AC along an Cartesian coordinate axes with A at the origin and C "to the right" of A. The point B was "above" AC.

We are told for this question that $AC = 2AB$ and $\angle A = 2\angle C$. Let D be the intersection of the angle bisector of $\angle BAC$ with D the point where this angle bisector intersects BC. From D drop a perpendicular to AC and denote its intersection with AC as the point E.

Now as $\triangle DCA$ and $\triangle DEC$ are right triangles with a common side DE and $\angle DAE = \angle DCE$ by "Angle-Side-Angle" we have

$$\triangle ADE \cong \triangle CDE \,,$$

thus $AE = EC$ and we see that $AB = \frac{1}{2}AC = AE$.

Now as $AB = AE$, $\angle BAD = \angle DAE$, and the segment AD is common between the two triangles we have $\triangle ADE \cong \triangle ADB$. Using this we can conclude that $\angle ABD = \angle AED = 90°$.

Testing Question B.4

I found this figure hard to draw just from the text description. Quickly looking at the solutions in the back gave a diagram that I could reason with. Refer to that diagram in what follows.

Draw the segment BD. Then as $\angle A = 60^\circ$ and $AB = AD$ the triangle $\triangle ABD$ is equilateral so $\angle ABD = \angle ADB = 60^\circ$ and all sides have the same length.

Extend the segment BC to a point E such that $CE = CD$ and draw the segment DE. Then as

$$\angle ECD = 180 - \angle DCB = 180 - 120 = 60\,,$$

and $CE = CD$ we have another equilateral triangle $\triangle CDE$.

Now $BD = AD$ and $CD = ED$ and that the angle "between" these segments

$$\angle ADC = 60 + \angle BDC = \angle BDE\,,$$

is equal so from "Side-Angle-Side" we have that

$$\triangle ADC \cong \triangle BDE\,,$$

so $BE = AC$. Using that and from how BE is constructed we have

$$AC = BE = BC + CE = BC + CD\,.$$

Testing Question B.5

I found this figure hard to draw just from the text description. Quickly looking at the solutions in the back gave a diagram that I could reason with. Refer to that diagram in what follows.

As $\angle B = \angle C = 80$ we have that $AB = AC$.

As $\angle BPC = 30$ we have that $\angle CPA = 180 - 30 = 150$.

Draw the segment BQ such that $\angle CBQ = 20$ and $BQ = AB$. Draw the segment AQ. Then as $\angle ABQ = 80 - 20 = 60$ and $AB = BQ$ we have that $\triangle ABQ$ is equilateral.

Draw the segment CQ then

$$\angle CAQ = \angle BAQ - \angle PAC = 60 - \angle PAC\,,$$

and since

$$\angle PAC = 180 - \angle ABC - \angle BCA = 180 - 80 - 80 = 20\,,$$

we have that $\angle CAQ = 60 - 20 = 40$.

In triangle $\triangle APC$ we have

$$\angle PCA = 180 - \angle CPA - \angle CAP = 180 - 150 - 20 = 10\,.$$

In triangle $\triangle ABC$ we have

$$\angle ACB = 80 = \angle BCP + \angle PCA \quad \text{so} \quad \angle BCP = 80 - 10 = 70\,.$$

Since $AQ = AB = AC$ in the $\triangle ACQ$ is isosceles and we get

$$\angle ACQ = \angle AQC = \frac{1}{2}(180 - \angle CAQ) = \frac{1}{2}(180 - 40) = 70\,.$$

As $\angle AQB = 60$ and we must have $\angle ACQ = \angle AQC$ we get $\angle BQC = 10$.

From all of the smaller angles we have computed we have

$$\angle BCQ = \angle BCP + \angle PCA + \angle ACQ = 70 + 10 + 70 = 150\,.$$

Then from all of these angles and the fact that $AC = BQ$ we have that

$$\triangle QBC \cong \triangle CAP\,,$$

by "Angle-Side-Angle" (since the two triangles have equal angles and $AC = BQ$). Thus we can conclude that $BC = PA$ by the fact that corresponding sides of congruent triangles are congruent.

Lecture 12: Applications of Midpoint Theorems

Testing Question A.1

Quickly looking at the solutions in the back gave a diagram that I could reason with. Refer to that diagram in what follows.

Draw the diagonal BD and let P be the midpoint of that segment. Draw the segments EP and PF. Then in triangle $\triangle ADB$ the midpoint theorem gives

$$EP = \frac{1}{2}AB\,.$$

In triangle $\triangle DBC$ the midpoint theorem gives

$$PF = \frac{1}{2}CD\,.$$

The triangle inequality in the triangle $\triangle EPF$ gives

$$EF < EP + PF\,.$$

Using the above two expressions this becomes

$$EF < \frac{1}{2}(AB + DC)\,.$$

Testing Question A.2

If we extend AD and BC until the meet at a point E. Then as $CD \parallel AB$ and $CD = \frac{1}{2}AB$ by the midpoint theorem we would then have $DE = AD$ and $CE = BC$.

In the triangle $\triangle EDB$ as C and N are the midpoints of EB and DB respectively we have $CN = \frac{1}{2}ED = \frac{1}{2}AD$ and $CN \parallel DE$.

In the triangle $\triangle ECA$ as D and M are the midpoints of EA and CA respectively we have $DM = \frac{1}{2}CE = \frac{1}{2}CB$ and $DM \parallel EC$.

Let F be the midpoint of the segment AB. Then as M and F are the midpoints of AC and AB respectively by the midpoint theorem in triangle $\triangle CAB$ we have $MF \parallel CB$ and $MF = \frac{1}{2}CB$. This means the full segments $DMF \parallel ECB$. In the same way as as N and F are the midpoints of BD and AB respectively by the midpoint theorem in triangle $\triangle DBA$ we have $NF \parallel DA$ and $NF = \frac{1}{2}AD$. This means the full segments $CNF \parallel EDA$.

Notice that $CN = \frac{1}{2}AD = NF$ and $DM = \frac{1}{2}CB = MF$. Thus M and N bisect the segments DF and CF respectively. Then by the midpoint theorem in the triangle $\triangle DFC$ we have that $MN \parallel CD$ and $MN = \frac{1}{2}CD$.

Using what we know from the above we have

$$l_2 = DM + MN + NC + CD = \frac{1}{2}CB + \frac{1}{2}CD + \frac{1}{2}AD + \frac{1}{2}AB$$
$$= \frac{1}{2}(CB + CD + AD + AB) = \frac{1}{2}l_1 \,.$$

Thus $l_1 = 2l_2$ and $n = 2$.

Testing Question A.3

Method 1: We can solve this with the angle bisector theorem. Let the square have a side of length s. Then the diagonal has a length of $\sqrt{2}s$ and the distance $AO = OC$ is half of this length. In triangle $\triangle ACB$ the angle bisector theorem states

$$\frac{AC}{CF} = \frac{AB}{BF} \,,$$

or

$$\frac{\sqrt{2}s}{CF} = \frac{s}{s - CF} \,.$$

Solving this for CF gives

$$CF = \left(\frac{\sqrt{2}}{\sqrt{2}+1} \right) s \,.$$

In triangle $\triangle AOB$ the angle bisector theorem states

$$\frac{AO}{OF} = \frac{AB}{EB} \,,$$

or

$$\frac{\frac{\sqrt{2}}{2}s}{OF} = \frac{s}{\frac{\sqrt{2}}{2}s - OE} \,.$$

Solving this for OE gives

$$OE = \left(\frac{\sqrt{2}}{\sqrt{2}+1} \right) \frac{s}{2} \,,$$

which is $\frac{1}{2}$ of the expression for CF given above.

Method 2: Extend from the point C and parallel to DB a segment. Let that segment intersect the extension of AF at the point G. Then by the midpoint theorem we have $OE = \frac{1}{2}CG$.

Now by construction we have

$$\angle BAF = \angle FAC = \frac{1}{2}(45) = 22.5^\circ \,.$$

As $\angle ABC = 90°$ we have $\angle AFB = 90 - 22.5 = 67.5 = \angle CFG$.

Note that $\angle ACB = 45°$ and since $CG \parallel OB$ and $OB \perp AC$ we have $\angle ACG = 90°$. This means that

$$\angle FCG = \angle ACG - \angle ACB = 90 - 45 = 45° .$$

Finally we can compute the last angle in the triangle $\triangle FCG$. We have

$$\angle FGC = 180 - \angle CFG - \angle FCG = 180 - 67.5 - 45 = 67.5 = \angle CFG .$$

This means that the triangle $\triangle FCG$ is isosceles and $CF = CG = 2OE$.

Testing Question A.4

Quickly looking at the solutions in the back gave a diagram that I could reason with. Refer to that diagram in what follows.

Let F be the midpoint of AC. Draw the segment EF. By the midpoint theorem we have $EF \parallel AB$ and $EF = \frac{1}{2}AB = DE$.

Draw the segment DF. As $DE = EF$ we have $\angle FDE = \angle DFE$. As triangle $\triangle ADC$ is a right triangle and DF connects the right angle D to the midpoint of the hypothesis we have that $DF = AF = FC$. This means that

$$\angle FDC = \angle FCD = \angle DFE .$$

As $EF \parallel AB$ we have $\angle B = \angle FEC$ but $\angle FEC$ is an exterior angle in the triangle $\triangle DEF$ so

$$\angle B = \angle FEC = \angle EDF + \angle EFD = 2\angle C .$$

Testing Question A.6

Quickly looking at the solutions in the back gave a diagram that I could reason with. Refer to that diagram in what follows.

Let the intersection of AD and BE be denoted O. Draw from D downwards and parallel to BE a segment that intersects AC at a point F. Then in triangle $\triangle BEC$ by the midpoint theorem we have $EF = FC$ and $DF = \frac{1}{2}BE = 2$.

Note that $\triangle BOD \cong \triangle BOA$ (by Angle-Side-Angle with the common side BO). This means that $AB = BD$ and $AO = OD$. Thus BE bisects the segment AD and so $AO = OD = \frac{1}{2}AD = 2$.

Now in triangle $\triangle ADF$ the segment OE is parallel to DF and passes though a midpoint. This means that $OE = \frac{1}{2}DF = \frac{1}{2}(2) = 1$ and $AE = EF$. From the length of OE we have $BO = BE - OE = 4 - 1 = 3$.

Now using the right triangle $\triangle AOB$ we have

$$AB = \sqrt{AO^2 + OB^2} = \sqrt{4+9} = \sqrt{13}\,.$$

As $BC = BD + DC = 2AB = 2\sqrt{13}$.

Now using the right triangle $\triangle AOE$ we have

$$AE = \sqrt{AO^2 + OE^2} = \sqrt{4+1} = \sqrt{5}\,.$$

As $AC = AE + EF + FC = 3AE = 3\sqrt{5}$.

Testing Question B.1

Quickly looking at the solutions in the back gave a diagram that I could reason with. Refer to that diagram in what follows.

Draw the segment AC and denote its midpoint at the point P. Draw the segments PE and PF. By the midpoint theorem we have $PE \parallel CG$ and $PF \parallel DH$.

As $PF \parallel DH$ we have $\angle AHE = \angle PFE$. As $PE \parallel CG$ we have $\angle EGB = \angle PEF$.

Again using the midpoint theorem on the segments PF and PE we have

$$PF = \frac{1}{2}AD > \frac{1}{2}BC = PE\,.$$

But these are two sides of the triangle $\triangle PFE$ so we have

$$\angle PEF > \angle PFE\,.$$

From the equivalence of the angles derived above this leads to $\angle EGB > \angle AHE$ which is what we desired to show.

Testing Question B.4

Quickly looking at the solutions in the back gave a diagram that I could reason with. Refer to that diagram in what follows.

From the point E draw a line "vertical" and parallel to both BC and AD. Then we have $\angle FEA = \angle EAD = \theta$ and $\angle FEB = \angle EBC = \phi$ where I have defined both θ and ϕ. From all of the angles in the triangle $\triangle ABE$ we have

$$2\phi + 2\theta = 180 \quad \text{so} \quad \phi + \theta = 90\,.$$

This means that $\angle BEA = \theta + \phi = 90°$ i.e. a right angle. As $\angle FBE = \angle FEB$ we have that $\triangle BFE$ is an isosceles triangle and so $BF = FE$. As $\angle FAE = \angle AEF$ we have that $\triangle EFA$ is also an isosceles triangle and so $EF = FA$. Thus

$$BF = FE = FA\,,$$

and F is the midpoint of AB. Because of this and the fact that FE is parallel to AD and BC it bisects the segment CD at the point E. This means that EF is the midline of the trapezium $ABCD$ and we have

$$EF = \frac{1}{2}(BC + AD)\,.$$

Using the fact that $EF = \frac{1}{2}AB$ in the above gives

$$\frac{1}{2}AB = \frac{1}{2}(BC + AD)\,,$$

which is equivalent to what we desired to show.

Lecture 13: Similarity of Triangles

Testing Question A.1

I found this figure hard to draw just from the text description. Quickly looking at the solutions in the back gave a diagram that I could reason with. Refer to that diagram in what follows.

As EF is parallel to AB we have that $\triangle EFC \sim \triangle ABC$. Thus $\frac{EF}{AB} = \frac{CF}{BC}$ or using the given lengths

$$\frac{EF}{20} = \frac{CF}{100}\,. \tag{854}$$

As EF is parallel to CD we have that $\triangle EFB \sim \triangle DCB$. Thus $\frac{EF}{CD} = \frac{BF}{BC}$ or using the given lengths

$$\frac{EF}{80} = \frac{BF}{100}\,. \tag{855}$$

Now $BF = BC - CF = 100 - CF$. Using that in Equation 855 gives

$$\frac{EF}{80} = \frac{100 - CF}{100} = 1 - \frac{CF}{100}\,. \tag{856}$$

Equation 854 gives $\frac{CF}{100}$ in terms of EF or

$$\frac{EF}{80} = 1 - \frac{EF}{20}\,.$$

Solving this we find $EF = 16$.

Testing Question A.2

I found this figure hard to draw just from the text description. Quickly looking at the solutions in the back gave a diagram that I could reason with. Refer to that diagram in what follows.

From the statement that $\triangle PAB \sim \triangle PCA$ we have

$$\frac{BP}{AP} = \frac{AP}{CP} = \frac{AB}{AC}\,,$$

or

$$\frac{CP + BC}{AP} = \frac{AP}{CP} = \frac{8}{6}\,,$$

or

$$\frac{CP + 7}{AP} = \frac{AP}{CP} = \frac{4}{3}\,.$$

From the second of these relations we have $AP = \frac{4}{3}CP$. If we put this into the first of these relationships we get

$$\frac{CP + 7}{\frac{4}{3}CP} = \frac{4}{3}\,.$$

Solving this for CP gives $CP = 9$.

Testing Question A.3

I found this figure hard to draw just from the text description. Quickly looking at the solutions in the back gave a diagram that I could reason with. Refer to that diagram in what follows.

Let the segment BE intersect AD at a point denoted O. Now D is a median of BC so $BD = CD = \frac{a}{2}$. As $\triangle ABC$ is an isosceles right triangle we have $AC = \sqrt{2}a$. Using the right triangle $\triangle ABD$ we have

$$AD = \sqrt{AB^2 + BD^2} = \sqrt{a^2 + \frac{a^2}{4}} = \frac{\sqrt{5}}{2}a\,.$$

Let $\angle ABE = x$ and $\angle EBC = y$ then $x + y = 90$. Also

$$\begin{aligned}
\angle ADB &= x \\
\angle BAD &= y \\
\angle DAC &= 45 - \angle BAD = 45 - y \\
\angle AEB &= 90 - \angle DAC = 45 + y\,.
\end{aligned}$$

As $\angle BCA = 45$ and EF is perpendicular to BC we have $\angle FEC = 45$ also. Then

$$\angle BEF = 180 - \angle AEB - \angle FEC = 180 - (45 + y) - 45 = 90 - y = x\,.$$

Using all of these angles we have several similar triangles. We have

$$\triangle AOB \sim \triangle BOD\,, \tag{857}$$

and

$$\triangle AOB \sim \triangle BFE\,. \tag{858}$$

Now from Equation 857 we get

$$\frac{AB}{BD} = \frac{BO}{DO} = \frac{AO}{BO}\,,$$

or as $\frac{AB}{BD} = \frac{a}{a/2} = 2$ this is

$$2 = \frac{BO}{DO} = \frac{AO}{BO}\,, \tag{859}$$

Now as $\triangle AOB$ is a right triangle we have that $AB^2 = BO^2 + AO^2$ or $AO = \sqrt{a^2 - BO^2}$. Putting that into Equation 859 gives

$$2 = \frac{\sqrt{a^2 - BO^2}}{BO} \quad \text{so} \quad BO = \frac{a}{\sqrt{5}}\,,$$

and then

$$AO = \sqrt{a^2 - BO^2} = \frac{2a}{\sqrt{5}}\,.$$

From Equation 859 we also get

$$DO = \frac{BO}{2} = \frac{a}{2\sqrt{5}}\,.$$

Using Equation 858 we get

$$\frac{AB}{BE} = \frac{BO}{EF} = \frac{AO}{BF}\,,$$

or using what we know this can be written as

$$\frac{a}{BE} = \frac{\frac{a}{\sqrt{5}}}{EF} = \frac{\frac{2a}{\sqrt{5}}}{BF}\,. \tag{860}$$

Now $\triangle EFC$ is an isosceles right triangle so $EF = CF$.

The segment BC can be written in two parts

$$BC = a = BF + CF = BF + EF\,. \tag{861}$$

From Equation 860 we get

$$BF = \frac{2a}{\sqrt{5}} \cdot \frac{\sqrt{5}}{a} \cdot EF = 2EF\,.$$

Putting that into Equation 861 gives

$$a = BF + EF = 3EF \quad \text{so} \quad EF = \frac{a}{3}\,.$$

Testing Question A.4

Let $AC = BC = a$. Then as both $\angle CAB = \angle CBA = \angle MCN = 45°$ we have

$$\triangle CMN \sim \triangle ANC\,, \tag{862}$$

and

$$\triangle CMN \sim \triangle BMC\,. \tag{863}$$

From Equation 862 we have

$$\frac{x}{CN} = \frac{CN}{m+x} = \frac{CM}{a}\,,$$

thus $CN^2 = \frac{m+x}{x}$ so $CN = \frac{m+x}{a}CM$. These give

$$CN = \frac{\sqrt{m+x}}{\sqrt{x}} \quad \text{and} \quad CM = \frac{a}{\sqrt{x(m+x)}}\,. \tag{864}$$

From Equation 863 we have

$$\frac{x}{CM} = \frac{CN}{a} = \frac{CM}{n+x}\,,$$

thus $CM^2 = \frac{n+x}{x}$ so $CN = \frac{ax}{CM}$. These give

$$CM = \frac{\sqrt{n+x}}{\sqrt{x}} \quad \text{and} \quad CN = \frac{a}{\sqrt{x(n+x)}}\,. \tag{865}$$

Setting the two expressions for CN in Equations 864 and 865 equal to each other we have

$$\frac{\sqrt{m+x}}{\sqrt{x}} = \frac{a}{\sqrt{x(n+x)}}$$

or

$$\sqrt{(m+x)(n+x)} = a\,,$$

so

$$a^2 = (m+x)(n+x)\,. \tag{866}$$

Now as $m+x+n$ is the length of the hypotenuse of the right triangle $\triangle ACB$ we have

$$2a^2 = (m+x+n)^2\,.$$

Using this in Equation 866 we get

$$2(m+x)(n+x) = (m+x+n)^2\,.$$

Expanding we get

$$2(mn + mxnx + x^2) = m^2 + x^2 + n^2 + 2mx + 2mn + 2xn\,,$$

or canceling common terms we get

$$x^2 = m^2 + n^2\,.$$

Thus the triangle formed is a right triangle.

Testing Question A.5

I found this figure hard to draw just from the text description. Quickly looking at the solutions in the back gave a diagram that I could reason with. Refer to that diagram in what follows.

As $AC = 3EC$ we have that $AE = 2EC$.

Now draw a segment from D and parallel to the segment BGE such that it intersects the segment AC ad the point F. Now $\triangle AGE \sim \triangle ADF$ so

$$\frac{AG}{AD} = \frac{GE}{DF} = \frac{AE}{AF}\,. \tag{867}$$

Now $AD = AG + GD$ and $AF = 2EC + EF$ so using the above we get

$$\frac{AG}{AG + GD} = \frac{2EC}{2EC + EF}\,. \tag{868}$$

We now ask what fraction of EC is EF. Note that in triangle $\triangle BEC$ the segment DF is parallel the segment BGE and passes though the midpoint D of BC. Thus by the midpoint theorem we have $EF = FC$ so $EF = FC = \frac{1}{2}EC$. Thus Equation 868 becomes

$$\frac{AG}{AG + GD} = \frac{2EC}{2EC + \frac{1}{2}EC} = \frac{4}{4+1} = \frac{4}{5}\,.$$

This means that $5AG = 4AG + 4GD$ so $\frac{AG}{GD} = \frac{4}{1}$ so $AG : GD = 4 : 1$.

Testing Question A.6

I found this figure hard to draw just from the text description. Quickly looking at the solutions in the back gave a diagram that I could reason with. Refer to that diagram in what follows.

As $AE = \frac{1}{3}AD$ we have

$$ED = 2AE\,. \tag{869}$$

Now draw from D and parallel to FEC a segment that will intersect AB at a point D'. Then

$$\triangle AD'D \sim \triangle AFE\,,$$

which means that

$$\frac{AF}{AD'} = \frac{FE}{D'D} = \frac{AE}{AD}\,. \tag{870}$$

From Equation 869 we know that $\frac{AE}{AD} = \frac{1}{3}$. In the triangle $\triangle BFC$ the segment DD' is parallel to CF and intersects the midpoint of BC at the point D. Thus by the midpoint theorem we have

$$FD' = BD' = \frac{1}{2}BF\,.$$

Then using $\frac{AE}{AD} = \frac{1}{3}$ in Equation 870 we have $\frac{AF}{AD'} = \frac{1}{3}$ so $AD' = 3AF$. Also $AD' = AF + FD' = AF + \frac{1}{2}BF$. Thus

$$3AF = AF + \frac{1}{2}BF \quad \text{so} \quad BF = 4AF = 4(1.2) = 4.8\,,$$

centimeters. Next

$$AB = AF + BF = 5AF = 6.0\,,$$

centimeters.

Testing Question A.7

I found this figure hard to draw just from the text description. Quickly looking at the solutions in the back gave a diagram that I could reason with. Refer to that diagram in what follows.

From the statement that $AP : PB = 2 : 1$ we have $\frac{AP}{PB} = \frac{2}{1}$ or $AP = 2PB$. Then since

$$AB = AP + PB = 3PB = 4\,,$$

we have $PB = \frac{4}{3}$ and $AP = \frac{8}{3}$.

Now let $\angle ADP = x$ and $\angle PDC = y$ so $x + y = 90^\circ$. Then from the diagram we have $\angle DPA = y$ and $\angle CDE = x$ so that $\triangle DAP \sim \triangle CED$ so

$$\frac{AD}{CE} = \frac{AP}{DE} = \frac{DP}{DC}\,,$$

or

$$\frac{2}{CE} = \frac{8/3}{DE} = \frac{\sqrt{2^2 + \left(\frac{8}{3}\right)^2}}{4}\,.$$

This gives

$$CE = \frac{8}{\sqrt{2^2 + \left(\frac{8}{3}\right)^2}} = \frac{12}{5}\,,$$

when we simplify.

Testing Question A.10

I found this figure hard to draw just from the text description. Quickly looking at the solutions in the back gave a diagram that I could reason with. Refer to that diagram in what follows.

Let $\angle CBA = \beta$ and draw the angle bisector from A to the segment BC intersecting at the point D. Then as this segment is the angle bisector and we are told that $\angle CAB = 2\angle CBA = 2\beta$ we have

$$\angle CDA = \angle DAB = \beta\,.$$

As the angles in a triangle must sum to 180° in $\triangle ADB$ we have $\angle ADB = 180^\circ - \angle DAB - \angle DBA = 180^\circ - \beta - \beta = 180^\circ - 2\beta$. Then by supplementary angles we have

$$\angle CDA = 180^\circ - \angle ADB = 180^\circ - (180^\circ - 2\beta) = 2\beta\,.$$

Then from the sum of the angles in the triangle $\triangle ACD$ we have

$$\angle ACD = 180^\circ - \angle CAD - \angle CDA = 180^\circ - \beta - 2\beta = 180^\circ - 3\beta\,.$$

We now have all of the angles denoted in terms of β.

If we note that $\angle DAB = \angle DBA = \beta$ we see that the triangle $\triangle ADB$ is isosceles and so

$$AD = BD\,. \tag{871}$$

From the angles given note that we can conclude that $\triangle DAC \sim \triangle ABC$ and so

$$\frac{AD}{AB} = \frac{CD}{AC} = \frac{AC}{BC}\,. \tag{872}$$

We also have

$$BC = CD + DB\,. \tag{873}$$

We will now use Equation 871 and 873 into Equation 872 to write everything in terms of the original triangle sides and the segment BD. Then Equation 872 becomes

$$\frac{BD}{AB} = \frac{BC - BD}{AC} = \frac{AC}{BC}\,.$$

By equating the second and third expression above we get

$$AC^2 = BC^2 - BC \cdot BD\,. \tag{874}$$

By equating the first and third expression above we get

$$BC \cdot BD = AC \cdot AB\,,$$

which if we use in Equation 874 gives

$$AC^2 = BC^2 - AC \cdot AB\,,$$

the desired expression.

Lecture 14: Areas of Triangles and Applications of Area

Testing Question A.1

Draw the triangle $\triangle ABC$ with AB on the x-axis of an x-y coordinate plane and the point C "above" the segment AB. Then extend the segment AB to a point B', the segment BC to a point C', and the segment CA to a point A' in the given proportions. Lets decompose the desired area into triangular regions as

$$[A'B'C'] = [ABC] + [BB'C'] + [C'A'C] + [AA'B'] \,.$$

Next we will relate the three triangles that are not $\triangle ABC$ to the area of $\triangle ABC$.

As $AB' = 2AB$ we see that $\triangle BB'C'$ has base BB' equal to that of ABC's base AB. Next as $CC' = 2BC$ the triangle $\triangle BB'C'$ has a "height" three times as large since $BC' = 3BC$. This means that $[BB'C'] = 3[ABC] = 3$.

Next as $CC' = 2BC$ we see that $\triangle C'A'C$ has a base (the segment CC') two times as long as BC (the base of the triangle $\triangle ABC$). Also as $AA' = 3AC$ (so that $A'C = 4AC$) a "height" four times as long as that in $\triangle ABC$. This means that $[C'A'C] = 2 \cdot 4[ABC] = 8$.

Finally, as triangle $\triangle AA'B'$ has a base $AA' = 3AC$ and a "height" (the segment AB') that is two times as large as "height" of AB in the triangle $\triangle ABC$ as $AB' = 2AB$. This means that $[AA'B] = 6[ABC] = 6$.

All of these together mean that

$$[A'B'C'] = 1 + 3 + 8 + 6 = 18 \,.$$

Testing Question A.2

I found this figure hard to draw just from the text description. Quickly looking at the solutions in the back gave a diagram that I could reason with. Refer to that diagram in what follows.

Let O be the intersection of the diagonals. As $AO = OD$ we have

$$\begin{aligned} (PE + PF)AO &= PE \cdot AO + PF \cdot AO \\ &= PE \cdot OD + PF \cdot AO \\ &= 2[POD] + 2[APO] = 2\left([POD] + [APO]\right) \\ &= 2[AOD] \\ &= 2\left(\frac{1}{4}[ABCD]\right) = \frac{1}{2}[ABCD] = \frac{1}{2}(5 \cdot 12) = 30 \,. \end{aligned}$$

Now AO is one-half the length of the diagonal or

$$AO = \frac{1}{2}\sqrt{5^2 + 12^2} = \frac{13}{2}\,.$$

This means that

$$PE + PF = \frac{30}{AO} = \frac{30}{\frac{13}{2}} = \frac{60}{13} > 4\,.$$

Testing Question A.3

Quickly looking at the solutions in the back gave a diagram that I could reason with. Refer to that diagram in what follows.

Lets connect the point P to the points A, B, and C forming the segments PA, PB, and PC. Then with s as the length of the side of the equilateral triangle $\triangle ABC$ we have that

$$\begin{aligned}\text{Area}(\triangle ABC) &= \text{Area}(\triangle ABP) + \text{Area}(\triangle APC) - \text{Area}(\triangle CBP)\\ &= \frac{1}{2}(AB)h_3 + \frac{1}{2}(AC)h_1 - \frac{1}{2}(CB)h_2\,.\end{aligned}$$

Using the fact that $AB = AC = CB = s$ we have

$$\text{Area}(\triangle ABC) = \frac{1}{2}s(h_3 + h_1 - h_2) = 3s\,.$$

Now all equilateral triangles with a side of s have an area equal to $\frac{\sqrt{3}}{4}s^2$. Setting this equal to $3s$ and solving for s gives

$$s = 4\sqrt{3}\,.$$

Using this the area of the equilateral triangle is

$$\text{Area}(\triangle ABC) = 3s = 12\sqrt{3}\,.$$

Testing Question A.4

The area of triangle $\triangle ABC$ (denoted A) can be written in three ways

$$A = \frac{1}{2}h_a a = \frac{1}{2}h_b b = \frac{1}{2}h_c c\,. \tag{875}$$

Using the above we have

$$\begin{aligned}a &= \frac{2A}{h_a}\\ b &= \frac{2A}{h_b}\\ c &= \frac{2A}{h_c}\,.\end{aligned}$$

If we put these into $2b = a + c$ we get

$$\frac{4A}{h_b} = \frac{2A}{h_a} + \frac{2A}{h_c} \, .$$

If we multiply this by $\frac{1}{2A}$ we get the desired relationship.

Testing Question A.5

Quickly looking at the solutions in the back gave a diagram that I could reason with. Refer to that diagram in what follows.

Now as $BD = 2CD$ and $[GCD] = 4$ the triangle $\triangle GBD$ will have the same height as triangle $\triangle GCD$ but a base twice as large as triangle $\triangle GCD$. Thus

$$[GBD] = 2[GCD] = 8 \, .$$

Now in $\triangle BEC$ the subtriangles $\triangle BGC$ and $\triangle EGC$ have their "bases" on BE and have the same height say h. Thus we have

$$\text{Area } \triangle EGC = EG \times h$$
$$\text{Area } \triangle BGC = BG \times h \, .$$

Dividing these two expressions gives

$$\frac{EG}{BG} = \frac{\text{Area } \triangle EGC}{\text{Area } \triangle BGC} = \frac{S_1}{S_3 + S_4} = \frac{3}{12} = \frac{1}{4} \, . \tag{876}$$

This has allowed us to get an expression relating EG and BG. Let

$$S_4 = \text{Area of } \triangle BGF$$
$$S_5 = \text{Area of } \triangle AGF$$
$$S_6 = \text{Area of } \triangle AGE \, .$$

Then using the same logic for the subtriangles $\triangle BGA$ and $\triangle EGA$ as above we have

$$\frac{EG}{BG} = \frac{S_6}{S_4 + S_5} \, . \tag{877}$$

We can use the same logic as above along the segments AD and CF i.e. equating parts of each segment's lengths to sums of component areas. For example, for the triangles "to the left" of AD we have

$$\frac{AG}{GD} = \frac{S_4 + S_5}{S_3} = \frac{S_4 + S_5}{8} \, . \tag{878}$$

For the triangles "to the right" of AD we have

$$\frac{AG}{GD} = \frac{S_6 + S_1}{S_2} = \frac{S_6 + 3}{4} \, . \tag{879}$$

Along the segment FC for the triangles "above" FC we have

$$\frac{FG}{GC} = \frac{S_5}{S_6 + S_1} = \frac{S_5}{S_6 + 3} \,. \tag{880}$$

For the triangles "below" of FC we have

$$\frac{FG}{GC} = \frac{S_4}{S_3 + S_2} = \frac{S_4}{12} \,. \tag{881}$$

Now equating $\frac{EG}{GD}$ in Equations 876 and 877 gives

$$\frac{S_6}{S_4 + S_5} = \frac{1}{4} \,. \tag{882}$$

Equating $\frac{AG}{GD}$ in Equations 878 and 879 gives

$$\frac{S_4 + S_5}{8} = \frac{S_6 + 3}{4} \,. \tag{883}$$

Equating $\frac{FG}{GC}$ in Equations 880 and 881 gives

$$\frac{S_5}{S_6 + 3} = \frac{S_4}{12} \,. \tag{884}$$

These give three equations for the three unknowns S_4, S_5, and S_6. The first two equations are

$$\begin{aligned} S_4 + S_5 - 4S_6 &= 0 \\ S_4 + S_5 - 2S_6 &= 6 \,. \end{aligned}$$

Using the first of these in the second gives $2S_6 = 6$ or $S_6 = 3$. Using that value in Equations 882 and 884 gives

$$\begin{aligned} S_4 + S_5 &= 12 \\ 2S_5 &= S_4 \,. \end{aligned}$$

Together these give $S_4 = 8$ and $S_5 = 4$.

Using all of this we have that the total area of the triangle $\triangle ABC$ is given by

$$S_1 + S_2 + S_3 + S_4 + S_5 + S_6 = 3 + 4 + 8 + 8 + 4 + 3 = 30 \,.$$

Testing Question A.6

Quickly looking at the solutions in the back gave a diagram that I could reason with. Refer to that diagram in what follows.

We will start by using the angle bisector theorem three times (for the angles A, B, and C).

For the angle A the angle bisector theorem gives

$$\frac{BD}{AB} = \frac{CD}{AC},$$

or

$$\frac{BD}{c} = \frac{CD}{b},$$

or as $CD = BC - BD = a - BD$ this is

$$\frac{BD}{c} = \frac{a - BD}{b}.$$

We can solve fore BD in the above to get

$$BD = \frac{ac}{b + c}. \tag{885}$$

Then using this we have

$$CD = a - BD = \frac{ab}{b + c}. \tag{886}$$

These two equations given expressions for the "parts" of BC that depend on the length of the sides of the full triangle.

For the angle B the angle bisector theorem gives

$$\frac{CE}{a} = \frac{AE}{c},$$

or

$$\frac{CE}{a} = \frac{b - CE}{c}.$$

Solving this for CE and then $AE = AC - CE = b - CE$ gives

$$CE = \frac{ab}{a + c} \tag{887}$$

$$AE = b - CE = \frac{cb}{a + c}. \tag{888}$$

For the angle C the angle bisector theorem gives

$$\frac{AF}{b} = \frac{BF}{a},$$

or

$$\frac{AF}{b} = \frac{c - AF}{a}.$$

Solving this for AF and then $BF = AB - AF = c - AF$ gives

$$AF = \frac{bc}{a + b} \tag{889}$$

$$BF = c - AF = \frac{ac}{a + b}. \tag{890}$$

Now we will evaluate the ratio of the three "corner" triangles to the area of the full triangle. We find

$$\begin{aligned}
\frac{[AEF]}{[ABC]} &= \frac{AF \times AE}{AB \times AC} = \frac{\left(\frac{bc}{a+b}\right)\left(\frac{cb}{a+c}\right)}{cb} = \frac{bc}{(a+b)(a+c)} \\
\frac{[BFD]}{[ABC]} &= \frac{BF \times BD}{AB \times BC} = \frac{\left(\frac{ac}{a+b}\right)\left(\frac{ac}{b+c}\right)}{ca} = \frac{ac}{(a+b)(b+c)} \\
\frac{[CED]}{[ABC]} &= \frac{CE \times CD}{CA \times CB} = \frac{\left(\frac{ab}{a+c}\right)\left(\frac{ab}{b+c}\right)}{ba} = \frac{ab}{(a+c)(b+c)} .
\end{aligned}$$

As

$$[ABC] = [EDF] + [AEF] + [BFD] + [CED] ,$$

and using the above we have

$$\begin{aligned}
\frac{[EDF]}{[ABC]} &= 1 - \frac{[AEF]}{[ABC]} - \frac{[BFD]}{[ABC]} - \frac{[CED]}{[ABC]} \\
&= 1 - \frac{bc}{(a+b)(a+c)} - \frac{ac}{(a+b)(b+c)} - \frac{ab}{(a+c)(b+c)} \\
&= \frac{2abc}{(a+b)(a+c)(b+c)} ,
\end{aligned}$$

when we simplify.

Testing Question A.8

Consider a general triangle $\triangle ABC$ with the side opposite the angle A having length BC, the side opposite the angle B having length AC, and the side opposite the angle C having length AB. Let P be a point in the interior of this triangle.

Let t_A, t_B, and t_C be the perpendicular distance from P to the sides BC, AC, and AB respectively. Now the total area of the triangle can be written using these variables as

$$\text{Area} = \frac{1}{2} t_C AB + \frac{1}{2} t_A BC + \frac{1}{2} t_B AC . \tag{891}$$

Let h_A, h_B, and h_C be the heights from A, B, and C to the sides opposite the given vertex. Then the area A can be written using these variables as

$$\text{Area} = \frac{1}{2} h_C AB = \frac{1}{2} h_A BC = \frac{1}{2} h_B AC . \tag{892}$$

Solving the above for AB, BC, and AC in terms of A and putting those expressions into Equation 891 we get

$$1 = \frac{t_C}{h_C} + \frac{t_A}{h_A} + \frac{t_B}{h_B} . \tag{893}$$

For the given triangle and notation here we note that the perpendicular bisector and the height to the side opposite the angle A

$$\frac{t_A}{h_A} = \frac{[CPF]}{[CAF]} = \frac{d}{a+d}\,,$$

in the same way we have

$$\frac{t_B}{h_B} = \frac{d}{d+b}\,,$$

and

$$\frac{t_C}{h_C} = \frac{d}{d+c}\,.$$

Using these in Equation 893 gives

$$1 = \frac{d}{d+a} + \frac{d}{d+b} + \frac{d}{d+c}\,.$$

If we multiply this by $\frac{(d+a)(d+b)(d+c)}{d}$ we get

$$\begin{aligned}\frac{(d+a)(d+b)(d+c)}{d} &= (d+b)(d+c) + (d+a)(d+c) + (d+a)(d+b)\\ &= d^2 + (b+c)d + bc + d^2 + (a+c)d + ac + d^2 + (a+b)d + ab\\ &= 3d^2 + 2(a+b+c)d + ab + ac + bc\,.\end{aligned}$$

Multiplying by d on both sides and expanding the left-hand-side gives

$$d^3 + (a+b+c)d^2 + (ab+ac+bc)d + abc = 3d^3 + 2(a+b+c)d^2 + (ab+ac+bc)d\,,$$

or simplifying we get

$$abc = 2d^3 + (a+b+c)d^2\,.$$

Using what we are told we have that $abc = 2(3^3) + 43(3^2) = 441$.

Testing Question A.9

Quickly looking at the solutions in the back gave a diagram that I could reason with. Refer to that diagram in what follows.

Using the Pythagorean theorem we have that

$$BE = \sqrt{AB^2 + AE^2} = \sqrt{1 + \frac{1}{4}} = \frac{\sqrt{5}}{2}$$

Introduce a point D drawn from C and perpendicular to AC. Then as $\angle BED = 90$ we have that

$$\angle BEA + \angle DEC = 90\,.$$

From this we can conclude that

$$\angle DEC = \angle ABE$$
$$\angle EDC = \angle AEB\,.$$

This means that the two right triangle $\triangle BAE$ and $\triangle ECD$ are similar. This means that

$$\frac{CE}{CD} = \frac{AB}{AE} = \frac{1}{\frac{1}{2}} = 2\,.$$

But $CE = \frac{1}{2}$ so we have that $CD = \frac{1}{4}$. In addition, we have

$$\frac{AB}{BE} = \frac{1}{\frac{\sqrt{5}}{2}} = \frac{EC}{ED} = \frac{1/2}{ED}\,,$$

so

$$ED = \frac{1}{\sqrt{5}}\,. \tag{894}$$

As we have an isosceles right triangle in $\triangle BAC$ we have

$$\angle ECF = \angle ACB = 45\,,$$

and since $\angle ACD = 90$ this means that $\angle ACB = 45 = \angle BCD$ and the segment FC is the angle bisector of $\angle ECD$ in the triangle $\triangle ECD$. By the angle bisector theorem in that triangle we have that

$$\frac{CD}{FD} = \frac{EC}{EF}\,,$$

or since we know $CD = \frac{1}{4}$ and $EC = \frac{1}{2}$ this is

$$\frac{(1/4)}{FD} = \frac{(1/2)}{EF} \quad \text{so} \quad FD = \frac{1}{2}EF\,.$$

Using this in Equation 894 written as

$$ED = \frac{1}{\sqrt{5}} = EF + FD\,,$$

we have

$$\frac{1}{\sqrt{5}} = EF + \frac{EF}{2}\,.$$

This gives $EF = \frac{2}{3\sqrt{5}}$ and so

$$FD = ED - EF = \frac{1}{3\sqrt{5}}\,.$$

Now that we know the lengths of EF and FD since these two triangles share a common base we can write

$$\frac{[EFC]}{[ECD]} = \frac{EF}{ED} = \frac{\frac{2}{3\sqrt{5}}}{\frac{1}{\sqrt{5}}} = \frac{2}{3}\,.$$

We can evaluate $[ECD]$ using the formula for the area of a right triangle

$$[ECD] = \frac{1}{2}\left(\frac{1}{4}\right)\left(\frac{1}{2}\right) = \frac{1}{16}\,.$$

Using this in the above we find

$$[EFC] = \frac{1}{16} \times \frac{2}{3} = \frac{1}{24}\,.$$

Testing Question B.1

Quickly looking at the solutions in the back gave a diagram that I could reason with. Refer to that diagram in what follows. In addition, I labeled the intersection of the segments BF and AE as Q and the intersection of the segments CF with DE as P. I also denoted the areas of the internal polygons (six triangles and a single quadrilateral) as S_i for $1 \le i \le 7$ so that

$$\begin{aligned} S_1 &= [ECP] \\ S_2 &= [CPD] \\ S_3 &= [PFD] \\ S_4 &= [PFQE] \\ S_5 &= [BQE] \\ S_6 &= [BQA] \\ S_7 &= [AQF] . \end{aligned}$$

We can evaluate some of the areas in this question terms of the above notation. We have

$$[EDA] + [FBC] = S_7 + S_4 + S_3 + S_5 + S_4 + S_1 ,$$

and

$$[ABCD] = S_1 + S_2 + S_3 + S_4 + S_5 + S_6 + S_7 .$$

If we subtract these two we find

$$[ABCD] - ([EDA] + [FBC]) = S_2 + S_6 - S_4 .$$

Thus we will have $[ABCD] = [EDA] + [FBC]$ if $S_4 = S_2 + S_6$. Let h_1, h_2, and h_3 be the heights of the triangles $\triangle ABE$, $\triangle BFC$, and $\triangle EDC$ respectively. From the fact that E and F are midpoints we have that

$$h_2 = \frac{1}{2}(h_1 + h_3) .$$

We will use this in evaluating $[BFC]$ we have

$$\begin{aligned} [BFC] &= \frac{1}{2} h_2 BC \\ &= \frac{1}{4}(h_1 + h_3) BC = \frac{1}{4} h_1 BC + \frac{1}{4} h_3 BC . \end{aligned}$$

Now $BC = 2BE = 2EC$ so the above equals

$$\frac{1}{2} h_1 BE + \frac{1}{2} h_3 EC = (S_6 + S_5) + (S_1 + S_2) .$$

Noting that $[BFC] = S_5 + S_4 + S_1$ when we set these two expressions equal we get

$$S_4 = S_2 + S_6 ,$$

as we desired to show.

Testing Question B.2

Quickly looking at the solutions in the back gave a diagram that I could reason with. Refer to that diagram in what follows.

Extend AG to a point P such that $AG = GP$. Let AP intersect BC at a point D. Now D is the midpoint of BC (as it is a median of the triangle) and $GD = \frac{1}{3}AG$ (due to the medians of a triangle splitting the segments in ratios of $2:1$). Thus $DP = GD$. Now the quadrilateral $BPCG$ have diagonals that are bisected. This means that this quadrilateral is a parallelogram so we know that $BP = GC = 2$ and $PC = BG = \sqrt{2}$.

Now note that

$$BP^2 + BG^2 = 4 + 4 \cdot 2 = 12$$
$$GP^2 = 4 \cdot 3 = 12\,,$$

which means that the triangle $\triangle GBP$ is a right triangle and $\angle GBP = 90°$ and in fact this parallelogram is actually a rectangle. This means that

$$[GBPC] = 2\sqrt{2} \times 2 = 4\sqrt{2}\,.$$

Thus

$$[GBC] = \frac{1}{2}[GBPC] = 2\sqrt{2}\,,$$

and

$$[ABC] = 3[GBC] = 6\sqrt{2}\,.$$

Testing Question B.3 (Ceva's Theorem)

Quickly looking at the solutions in the back gave a diagram that I could reason with. Refer to that diagram in what follows.

Consider the ratio $r \equiv \frac{BD}{DC}$. Then as the numerator and denominator are "bases" of the triangles $\triangle ABD$ and $\triangle ACD$ we have

$$r = \frac{[ABD]}{[ACD]}\,.$$

The segments BD and DC are also "bases" of the triangles $\triangle PBD$ and $\triangle PCD$ respectively so we also have

$$r = \frac{[PBD]}{[PCD]}\,.$$

Recognizing that each of these triangles is a part of the two larger triangles $\triangle ABD$ and $\triangle ACD$ we also have

$$r = \frac{[ABD] - [APB]}{[ACD] - [APC]}\,.$$

We can write the above as

$$([ACD] - [APC])r = [ABD] - [APB]\,.$$

If we divide both sides by $[ACD]$ we get

$$\left(1 - \frac{[APC]}{[ACD]}\right) r = r - \frac{[APB]}{[ACD]}\,.$$

This simplifies to give

$$r = \frac{BD}{DC} = \frac{[APB]}{[APC]}\,. \tag{895}$$

This expresses $\frac{BD}{DC}$ as the ratio of the triangles "above" but not adjacent to the segments BD and DC. In the same way as above we could derive

$$\frac{AE}{EC} = \frac{[BPA]}{[BPC]}$$
$$\frac{AF}{FB} = \frac{[APC]}{[BPC]}\,.$$

Using these expression we can write

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = \frac{[APB]}{[APC]} \cdot \frac{[BPC]}{[BPA]} \cdot \frac{[APC]}{[BPC]} = 1\,,$$

when we simplify.

Testing Question B.4

Quickly looking at the solutions in the back gave a diagram that I could reason with. Refer to that diagram in what follows.

If we consider AO and OA' "bases" of two triangles that have a vertex at point C then as the height to C from AA' is the same for both triangles the ratio of them is equal to the ratio of the two triangles areas or

$$\frac{AO}{OA'} = \frac{[COA]}{[COA']}\,.$$

If we consider AO and OA' "bases" of two triangles that have a vertex at point B then as the height to B from AA' is the same for both triangles the ratio of them is equal to the ratio of the two triangles areas or

$$\frac{AO}{OA'} = \frac{[BOA]}{[BOA']}\,.$$

Thus we have shown that

$$\frac{AO}{OA'} = \frac{[COA]}{[COA']} = \frac{[BOA]}{[BOA']}\,. \tag{896}$$

We can do the same thing for the ratio $\frac{BO}{OB'}$ where with a "vertex" at point A we get

$$\frac{BO}{OB'} = \frac{[AOB]}{[AOB']},$$

and with a vertex at the point C we get

$$\frac{BO}{OB'} = \frac{[COB]}{[COB']}.$$

Thus we have shown that

$$\frac{BO}{OB'} = \frac{[AOB]}{[AOB']} = \frac{[COB]}{[COB']}. \quad (897)$$

We can do the same thing for the ratio $\frac{CO}{OC'}$ where with a "vertex" at point A we get

$$\frac{CO}{OC'} = \frac{[AOC]}{[AOC']},$$

and with a vertex at the point C we get

$$\frac{CO}{OC'} = \frac{[COC]}{[COC']}.$$

Thus we have shown that

$$\frac{CO}{OC'} = \frac{[AOC]}{[AOC']} = \frac{[COC]}{[COC']}. \quad (898)$$

We will now prove the following lemma. If

$$\frac{a}{b} = \frac{c}{d}, \quad (899)$$

then

$$\frac{a \pm c}{b \pm d} = \frac{a}{b} = \frac{c}{d}. \quad (900)$$

We can prove this by "cross-multiplying" and using Equation 899. Applying this lemma to Equation 896 we get

$$\frac{AO}{OA'} = \frac{[AOC] + [AOB]}{[A'OC] + [A'OB]} = \frac{[AOC] + [AOB]}{[BOC]},$$

when we recognize the simpler form for the sum $[A'OC] + [A'OB]$. Applying this lemma to Equation 897 we get

$$\frac{BO}{OB'} = \frac{[AOB] + [COB]}{[AOB'] + [COB']} = \frac{[AOB] + [COB]}{[AOC]}.$$

Applying this lemma to Equation 898 we get

$$\frac{CO}{OC'} = \frac{[AOC] + [BOC]}{[AOC'] + [BOC']} = \frac{[AOC] + [BOC]}{[AOB]}.$$

If we introduce $x = [BOC]$, $y = [COA]$, and $z = [AOB]$ then using the above what we want to compute is given by

$$\begin{aligned}
\frac{AO}{OA'} \cdot \frac{BO}{OB'} \cdot \frac{CO}{OC'} &= \left(\frac{y+z}{x}\right)\left(\frac{z+x}{y}\right)\left(\frac{y+x}{z}\right) \\
&= 2 + \frac{y+z}{x} + \frac{x+y}{z} + \frac{z+x}{y} \\
&= 2 + \frac{AO}{OA'} + \frac{CO}{OC'} + \frac{BO}{OB'} = 2 + 92 = 94 \,,
\end{aligned}$$

where we have used the information given in the problem.

Testing Question B.5

Warning: I was able to make progress on this problem but was unable to full finish it. If anyone sees anyways to continue from where I left off please contact me.

Note: Please refer to the diagram given in the problem in what follows.

From D draw a line parallel to AC that will intersect the segment PB at a point L. Then as $\angle CAP = \angle ADL$ and $\angle EPA = \angle DPL$ we have that $\triangle APE \sim \triangle DPL$. As $AP = PD$ these two triangles are actually congruent so $\triangle APE \cong \triangle DPL$ and $PL = PE = 3$. This means that

$$BL = 9 - PL = 9 - 3 = 6 \,.$$

Note that $EL = 3 + PL = 3 + 3 = 6 = LB$ and LD is parallel to $AEC = EC$ thus D must be the midpoint of CB (in triangle $\triangle BEC$ the segment LD is parallel to the segment EC and passes through the midpoint of one side EB).

In the same way, parallel to AB from D draw a segment. Let this segment intersect CF at the point K. As DK is parallel to the base of triangle $\triangle CFB$ and goes through a midpoint of the side BC we have that K is a midpoint of the side CF. This means that

$$KF = CK = \frac{1}{2} CF = 10 \,.$$

Since $\angle KPD = \angle FPA$, $AP = PD = 6$, and $\angle KDP = \angle FAP$ by angle-side-angle we have $\triangle KPD \cong \triangle FPA$. Thus $PK = PF$ and as $PK + PF = KF = 10$ we have that

$$PK = PF = 5 \,.$$

In $\triangle PCB$ the segment KD connects two midpoints and thus its length is one-half of the base PB or

$$KD = \frac{1}{2} PB = \frac{1}{2}(3 + 6) = \frac{9}{2} \,.$$

In $\triangle FCB$ the segment KD also connects two midpoints and thus its length is one-half of the base FB in this case so

$$FB = 2KD = 9 \,.$$

Finally as $\triangle KPD \cong \triangle FPA$ we have $AF = KD = \frac{9}{2}$.

We have now derived the distances from P to all of the points A, B, C, D, E, F and the two distances AF and FB. Lets denote the areas of the six internal triangles as

$$
\begin{aligned}
S_1 &= [FPB] \\
S_2 &= [PBD] \\
S_3 &= [PDC] \\
S_4 &= [PCE] \\
S_5 &= [EPA] \\
S_6 &= [AFP] \,.
\end{aligned}
$$

Then using the ratios of the segments "out" of P as bases of triangles (as in the previous question) we have

$$
\begin{aligned}
\frac{AP}{PD} &= \frac{6}{6} = 1 = \frac{S_4 + S_5}{S_3} = \frac{S_6 + S_1}{S_2} \\
\frac{EP}{PB} &= \frac{3}{9} = \frac{S_4}{S_3 + S_2} = \frac{S_5}{S_1 + S_6} \\
\frac{CP}{PF} &= \frac{15}{5} = 3 = \frac{S_4 + S_5}{S_6} = \frac{S_2 + S_3}{S_1} \,.
\end{aligned}
$$

The same thing for the ratio $AF : FB$ gives

$$
\frac{AF}{FB} = \frac{1}{2} = \frac{S_4 + S_5 + S_6}{S_1 + S_2 + S_3} \,. \tag{901}
$$

These give a linear system of equations for S_i for $1 \le i \le 6$. If we could solve this we could sum S_i to get the answer we seek. Unfortunately this system seems to be indeterminate and I was unable to solve for S_i.

As a final comment, to solve this problem we really don't need to know S_i for each i only $\sum_{i=1}^{6} S_i$. Using Equation 901 we see that

$$
\sum_{i=1}^{3} S_i = 2 \sum_{i=4}^{6} S_i \,,
$$

so

$$
\sum_{i=1}^{6} S_i = 3 \sum_{i=4}^{6} S_i \,.
$$

From the above we have $S_4 + S_5 = 3S_6$ so we get

$$
\sum_{i=1}^{6} S_i = 3(3S_6 + S_6) = 12S_6 \,.
$$

Thus if we could evaluate S_6 we would be done.

If anyone sees a better approach or a place where the above can be extended (or an error) please contact me.

Lecture 15: Division of Polynomials

Testing Question A.1

Using long division we have

$$\begin{array}{r|l} & 3x^2 \quad + 6x \quad + 7 \\ \hline x-2 & 3x^3 \qquad\quad - 5x \quad + 6 \\ & -3x^3 + 6x^2 \\ \hline & \qquad 6x^2 \quad - 5x \\ & \qquad -6x^2 + 12x \\ \hline & \qquad\qquad 7x \quad + 6 \\ & \qquad\qquad -7x + 14 \\ \hline & \qquad\qquad\qquad 20 \end{array}$$

This means that the quotient is $q(x) = 3x^2 + 6x + 7$ and the remainder is $r(x) = 20$.

Using synthetic division produces a table like the following

$$\begin{array}{c|cccc} & 3 & 0 & -5 & 6 \\ 2 & & 6 & 12 & 14 \\ \hline & 3 & 6 & 7 & 20 \end{array}$$

This means that we can write

$$3x^3 - 5x + 6 = (3x^2 + 6x + 7)(x - 2) + 20\,.$$

Testing Question A.2

Motivated by the simple fact that $2x - 1 = 2\left(x - \frac{1}{2}\right)$ we will use synthetic division with respect to $\frac{1}{2}$ to get

$$\begin{array}{c|ccccc} & -6 & 0 & -7 & 8 & 9 \\ \frac{1}{2} & & -3 & -\frac{3}{2} & -\frac{17}{4} & \frac{15}{8} \\ \hline & -6 & -3 & -\frac{17}{2} & \frac{15}{4} & \frac{87}{8} \end{array}$$

This means that we have shown that

$$-6x^4 - 7x^2 + 8x + 9 = \left(x - \frac{1}{2}\right)\left(-6x^3 - 3x^2 - \frac{17}{2}x + \frac{15}{4}\right) + \frac{87}{8}\,.$$

If we "pull a factor of two from the second factor" we get

$$-6x^4 - 7x^2 + 8x + 9 = (2x - 1)\left(-3x^3 - \frac{3}{2}x^2 - \frac{17}{4}x + \frac{15}{8}\right) + \frac{87}{8}\,.$$

The quotient $q(x)$ is the second factor above (the one after the $2x - 1$) and the remainder r is $r = \frac{87}{8}$.

Testing Question A.3

For this question we will use synthetic division with respect to -3. As a *partial* result for this we get

	1	3	8	$-k$	11
-3		-3	0	-24	
	1	0	8	$(-24+-1k)$	

At this point the algebraic result from `polyhornerscheme` was wrong and I decided to complete this part of the question "by hand". To finish this synthetic division we next multiply $-24-k$ by -3 to get

$$72+3k\,.$$

We then add this to 11 to get

$$83+3k\,.$$

As we are told that $x+3$ must be a factor of the original polynomial we know that this final expression must be zero so that $k=-\frac{83}{3}$.

Testing Question A.4

Method 1: Synthetic division of $f(x)$ with respect to -1 gives

$$\begin{array}{r|ccccc} -1 & 1 & 0 & -a & -b & 2 \\ & & -1 & 1 & -1+a & 1-a+b \\ \hline & 1 & -1 & 1-a & -1+a-b & 3-a+b \end{array}$$

Synthetic division of $f(x)$ with respect to -2 gives

$$\begin{array}{r|ccccc} -2 & 1 & 0 & -a & -b & 2 \\ & & 2 & 4 & 8-2a & 16-4a+2b \\ \hline & 1 & -2 & 4-a & -8+2a-b & 18-4a+2b \end{array}$$

As we are told that both $x+1$ and $x+2$ are factors of $f(x)$ we must have

$$3-a+b=0$$
$$18-4a+2b=0\,.$$

Solving these we get $a=6$ and $b=3$.

Method 2: From what we are told we know that $f(-1)=f(-2)=0$. These are the expressions

$$f(-1)=1-a+b+2=0$$
$$f(-2)=16-4a+2b+2=0\,.$$

These are equivalent to the equations above and have the same solution.

Testing Question A.5

From what we are told we know we can write

$$f(x) = (x-1)q_1(x) + 1\,. \tag{902}$$

Now lets divide $q_1(x)$ by $x-2$ to write it as

$$q_1(x) = (x-2)q_2(x) + k\,,$$

for some k. If we put that expression into Equation 902 we get

$$f(x) = (x-1)(x-2)q_2(x) + k(x-1) + 1\,.$$

We also know that $f(2) = 2$ so this means that

$$k(2-1) + 1 = 2 \quad \text{so} \quad k = 1\,,$$

and we have shown that

$$f(x) = (x-1)(x-2)q_2(x) + (x-1) + 1 = (x-1)(x-2)q_2(x) + x\,.$$

Now lets divide $q_2(x)$ by $x-3$ to write it as

$$q_2(x) = (x-3)q_3(x) + k\,,$$

for some *other* k. This means that we would have

$$f(x) = (x-1)(x-2)(x-3)q_3(x) + k(x-1)(x-2) + x\,.$$

We know that $f(3) = 3$ so in the above that means that $k(3-1)(3-2) + 3 = 3$ so $k = 0$. This means that we have shown we can write $f(x)$ as

$$f(x) = (x-1)(x-2)(x-3)q_3(x) + x\,.$$

This means that the remainder is x when we divide $f(x)$ by $(x-1)(x-2)(x-3)$.

Testing Question A.6

We are told that

$$f(x) = x^5 - 5qx + 4r = (x-2)^2 Q(x)\,.$$

This means that

$$f(2) = 32 - 10q + 4r = 0\,, \tag{903}$$

and that because

$$f'(x) = 2(x-2)Q(x) + (x-2)^2 Q(x)\,,$$

we would have $f'(2) = 0$ also. Because we can write $f'(x) = 5x^4 - 5q$ this means that

$$f'(2) = 5 \cdot 2^4 - 5q = 0\,. \tag{904}$$

We can solve the above for q and find $q = 2^4 = 16$. Using that in Equation 903 gives

$$32 - 10(16) + 4r = 0 \quad \text{so} \quad r = 32\,.$$

Testing Question A.7

From the fact that $f(x)$ is a polynomial of degree three and the information about the remainder when we divide by $x^2 - 1$ we have that

$$f(x) = (2x - 5) + (Ax + B)(x^2 - 1) . \tag{905}$$

If we expand this and group we can write $f(x)$ as

$$f(x) = Ax^3 + Bx^2 + (2 - A)x + (-B - 5) .$$

We now divide this form by $x^2 - 4$. I'll skip the long division steps (as they are hard to typeset) and only note that when we do this we find

$$f(x) = [(2 + 3A)x + (3B - 5)] + (Ax + B)(x^2 - 4) .$$

The question tells us the remainder when we divide $f(x)$ by $x^2 - 4$ is $-3x + 4$ which means that

$$2 + 3A = -3 \quad \text{so} \quad A = -\frac{5}{3} ,$$

and

$$3B - 5 = 4 \quad \text{so} \quad B = 3 .$$

From these and Equation 905 we have that

$$f(x) = (2x - 5) + \left(-\frac{5}{3}x + 3\right)(x^2 - 1) = -\frac{5}{3}x^3 + 3x^2 + \frac{11}{3}x - 8 .$$

when we expand and simplify.

Testing Question A.8

From the fact that this polynomial has integer coefficients lets start by look for rational roots. All rational roots of this polynomial will be of the form $\frac{q}{p}$ where q is a factor of the *constant* term i.e. eight and p is a factor of the coefficient of x^3 i.e. one. Possible values for q include

$$q \in \{\pm 1, \pm 2, \pm 4, \pm 8\} .$$

If we start with $x = -1$ we see that

$$-1 + 7 - 14 + 8 = 0 ,$$

Thus $x + 1$ is a factor. Synthetic division by -1 gives

$$\begin{array}{r|rrrr} & 1 & 7 & 14 & 8 \\ -1 & & -1 & -6 & -8 \\ \hline & 1 & 6 & 8 & 0 \end{array}$$

This means that

$$x^3 + 7x^2 + 14x + 8 = (x + 1)(x^2 + 6x + 8) .$$

The second factor is simple to factor and we have

$$x^3 + 7x^2 + 14x + 8 = (x + 1)(x + 2)(x + 4) .$$

Testing Question A.9

Note that this expression is symmetric in x and y. This means that we expect to be able to write it in terms of the "components" $x+y$ and xy. Motivated by this fact we might write

$$\begin{aligned} f(x) &= x^4 + y^4 + (x+y)^4 \\ &= (x^2+y^2)^2 - 2x^2y^2 + (x+y)^4 \\ &= [(x+y)^2 - 2xy]^2 - 2x^2y^2 + (x+y)^4 \\ &= (x+y)^4 - 4xy(x+y)^2 + 4x^2y^2 - 2x^2y^2 + (x+y)^4 \\ &= 2(x+y)^4 - 4xy(x+y)^2 + 2x^2y^2 = 2[(x+y)^4 - 2xy(x+y)^2 + x^2y^2] \\ &= 2[(x+y)^2 - xy]^2 \\ &= 2[x^2+2xy+y^2-xy]^2 = 2[x^2+xy+y^2]^2\,. \end{aligned}$$

Testing Question A.10

Lets call this expression $E(x,y,z)$. Then notice that

$$E(y,z,x) = yz(y^2-z^2) + zx(z^2-x^2) + xy(x^2-y^2)\,,$$

which is the same as $E(x,y,z)$. Next notice that

$$E(z,x,y) = zx(z^2-x^2) + xy(x^2-y^2) + yz(y^2-z^2)\,,$$

which is also the same as $E(x,y,z)$. This means that $E(x,y,z)$ is a cyclical polynomial. Now for cyclical polynomials if we can find one cyclical factor we can get all other factors by "cycling the variables". Consider taking $x=y$ in $E(x,y,z)$. We would have

$$E(y,y,z) = y^2(0) + yz(y^2-z^2) + yz(z^2-y^2) = 0\,,$$

Thus $x-y$ is a factor of $E(x,y,z)$. By "cycling the variables" this means that $y-z$ and $z-x$ are also factors of E. To prove/verify this lets write E as a polynomial in the variable x as

$$\begin{aligned} E &= yx^3 - y^3x + z^3x - zx^3 + yz(y^2-z^2) \\ &= (y-z)x^3 + (-y^3+z^3)x + yz(y^2-z^2)\,. \end{aligned}$$

From this by using

$$z^3 - y^3 = (z-y)(z^2+zy+y^2) \quad \text{and} \quad z^2-y^2 = (z-y)(z+y)\,,$$

we immediately see that $z-y$ is a factor of E which we could take out and use to simplify E further. If we didn't see that, we could use synthetic division to evaluate the x polynomial form of E at $x=y$ as

$$\begin{array}{r|cccc} y & y-z & 0 & -y^3+z^3 & yz(y^2-z^2) \\ & & y^2-yz & y^3-y^2z & yz^3-y^3z \\ \hline & y-z & y^2-yz & z^3-y^2z & 0 \end{array}$$

This means that we can write E as

$$E = (x - y)[(y - z)x^2 + y(y - z)x + z(z^2 - y^2)] \,.$$

We still must have the two other factors listed above. Again we see the factor $z - y$ is there. But assuming we didn't want to use that factor right now we also know that $E(x, y, z)$ must also vanish when $x = z$ so we can do synthetic division on the second factor (the factor in brackets) in E above as

$$\begin{array}{c|ccc} z & y - z & y(y - z) & z(z^2 - y^2) \\ & & yz - z^2 & y^2 z - z^3 \\ \hline & y - z & y^2 - z^2 & 0 \end{array}$$

This means that we can write E as

$$E = (x - y)(x - z)[(y - z)x + (y^2 - z^2)] \,.$$

We sill need to "remove" the third factor $y - z$ but that is easy to do given the above form and we have

$$E = (x - y)(x - z)(y - z)[x + y + z] \,.$$

Testing Question B.1

If $f(x)$ is a common factor of $g(x)$ and $h(x)$ then it is a common factor of

$$\begin{aligned} h(x) - 3g(x) &= 3x^4 - 9x^3 + 2x^2 + 3x - 1 - 3x^4 + 9x^3 - 6x^2 + 9x - 3 \\ &= -4x^2 + 12x - 4 = -4(x^2 - 3x + 1) \,. \end{aligned}$$

The only way $f(x) = x^2 + ax + b$ is a factor of the above is if $a = -3$ and $b = 1$.

Testing Question B.2

Consider the function $f(y) = y^m - 1$. Note that $y - 1$ must be a factor of $f(y)$ as $f(1) = 1^m - 1 = 0$. This means that we can write $f(y) = (y - 1)g(y)$ for some $g(y)$. If we take $y = x^3$ this means that

$$x^{3m} - 1 = (x^3 - 1)q(x^3) = (x - 1)(x^2 + x + 1)q(x^3) \,. \tag{906}$$

This means that $x^2 + x + 1$ is a factor of $x^{3m} - 1$.

Next consider

$$x^{3n+1} - x = x(x^{3n} - 1) = x(x - 1)(x^2 + x + 1)\tilde{q}(x^3) \,.$$

This means that $x^2 + x + 1$ is a factor of $x^{3n+1} - x$.

Next consider

$$x^{3p+2} - x^2 = x^2(x^{3p} - 1) = x^2(x - 1)(x^2 + x + 1)\hat{q}(x^3) \,.$$

This means that $x^2 + x + 1$ is a factor of $x^{3p+2} - x^2$.

If we take the expression given we can write it as

$$x^{3m} + x^{3n+1} + x^{3p+2} = (x^{3m} - 1) + (x^{3n+1} - x) + (x^{3p+2} - x^2) + (1 + x + x^2)\,.$$

Using the three facts above we see that the right-hand-side is divisible by $x^2 + x + 1$ and thus the left-hand-side must also be dividable by $x^2 + x + 1$.

Testing Question B.3

We are told that we can write

$$f(x) = (x - a)m(x) + a\,.$$

Lets now divide $m(x)$ by $x - b$ to write it as

$$m(x) = (x - b)n(x) + C_1\,,$$

for some constant C_1. With these formulas we see that $f(b)$ (which must equal b) is given by

$$f(b) = (b - a)m(b) + a = (b - a)C_1 + a\,.$$

For this to equal b means that $C_1 = 1$. This means that we have shown that $f(x)$ takes the form

$$f(x) = (x - a)(x - b)n(x) + (x - a) + a = (x - a)(x - b)n(x) + x\,.$$

Lets divide $n(x)$ by $x - c$ as

$$n(x) = (x - c)q(x) + C_2\,,$$

for some constant C_2. With these formulas we see that $f(c)$ (which must equal c) is given by

$$f(c) = (c - a)(c - b)C_2 + c\,.$$

For this to equal c means that $C_2 = 0$. This means that we have shown that $f(x)$ takes the form

$$f(x) = (x - a)(x - b)(x - c)q(x) + x\,.$$

From this form the remainder when we divide $f(x)$ by $(x - a)(x - b)(x - c)$ is x.

Testing Question B.4

Call this expression $E(x, y, z)$. Then I claim that E is cyclical. This means that

$$E(y, z, x) = E(z, x, y) = E(x, y, z)\,.$$

This means that if I can find a single factor of E I can find others by cyclically permuting its variables. Lets evaluate $x = z$ in E. We have

$$E(z, y, z) = (y^2 - z^2)(1 + zy)(1 + z^2) + 0 + (z^2 - y^2)(1 + z^2)(1 + zy) = 0\,.$$

This means that $x - z$ is a factor of E. Another then must be $y - x$ and another must be $z - y$. Using this information we will first factor out $x - z$ and see what expression that gives us. Note that as the second term in E i.e.

$$(z^2 - x^2)(1 + yz)(1 + yx)\,,$$

already has $x - z$ as a factor the other two terms in E must have $x - z$ as a factor (but one that is harder to "see" directly). Call this part E_{13} for the first and third terms of E. Then we will write E_{13} as a polynomial in x as

$$\begin{aligned}
E_{13} &= (y^2 - z^2)(1 + (y + z)x + yzx^2) + (x^2 - y^2)(1 + zx)(1 + zy) \\
&= (y^2 - z^2)(yzx^2 + (y + z)x + 1) + (x^2 + zx^3 - y^2 - y^2zx)(1 + zy) \\
&= (y^2 - z^2)(yzx^2 + (y + z)x + 1) + (zx^3 + x^2 - y^2zx - y^2)(1 + zy) \\
&= (1 + zy)zx^3 + [(y^2 - z^2)yz + (1 + zy)]x^2 + [(y^2 - z^2)(y + z) - y^2z(1 + zy)]x - z^2 - zy^3\,.
\end{aligned}$$

Lets use synthetic division with respect to z on the above. We have

z	$(1 + zy)z$	$y^3z - yz^3 + 1 + zy$	$y^3 - yz^2 + zy^2 - z^3 - y^2z - z^2y^3$	$-z^2 - zy^3$
		$z^2 + z^3y$	$y^3z^2 + z^3 + z^2y + z$	$zy^3 + z^2$
	$z + z^2y$	$y^3z + z^2 + zy + 1$	$y^3 + zy^2 - y^2z + z$	0

This means that we can write E as

$$E = (x - z)[(z + z^2y)x^2 + (y^3z + z^2 + zy + 1)x + y^3 + z - (z + x)(1 + yz)(1 + yx)]\,.$$

Lets simplify the second factor (called F) above. I find (with perhaps too many details)

$$\begin{aligned}
F &= (z + z^2y)x^2 + (y^3z + z^2 + zy + 1)x + y^3 + z - (1 + yz)[z + yzx + x + yx^2] \\
&= (z + z^2y)x^2 + (y^3z + z^2 + zy + 1)x + y^3 + z - (1 + yz)[yx^2 + (1 + yz)x + z] \\
&= [z + z^2y - y - y^2z]x^2 + [y^3z + z^2 + zy + 1 - (1 + yz)^2]x + y^3 + z - z - yz^2 \\
&= [(z - y) + zy(z - y)]x^2 + [y^3z + z^2 + zy - 2yz - y^2z^2]x + y(y^2 - z^2) \\
&= (z - y)(1 + zy)x^2 + [y^3z + z^2 - yz - y^2z^2]x + y(y^2 - z^2) \\
&= (z - y)(1 + zy)x^2 + [y^2z(y - z) + z(z - y)]x + y(y^2 - z^2) \\
&= (z - y)(1 + zy)x^2 + (z - y)[z - zy^2]x + y(y^2 - z^2)\,.
\end{aligned}$$

From this we see the factor $z - y$ and we can write E as

$$E = (x - z)(z - y)[(1 + zy)x^2 + z(1 - y^2)x - y(y + z)]\,.$$

We still expect a factor $y - x$ of E. When we view the term in braces above as a polynomial in x we can find this factor by performing synthetic division with respect to y on the above. We have

y	$1 + zy$	$z(1 - y^2)$	$-y^2 - yz$
		$y + zy^2$	$y^2 + zy$
	$1 + zy$	$y + z$	0

This means that we have

$$(1 + zy)x^2 + z(1 - y^2)x - y(y + z) = (x - y)[(1 + zy)x + y + z]\,,$$

and finally for E the factorization

$$E = -(x - z)(z - y)(y - x)(xyz + x + y + z)\,.$$

Testing Question B.5

In general, polynomial division of $f(x)$ by $g(x)$ gives

$$f(x) = g(x)q(x) + r(x)\,,$$

for a polynomial quotient $q(x)$ and remainder $r(x)$. The degree of $r(x)$ is one less than the degree of $g(x)$. For this question we are told that

$$q(x) = r(x) = h(x)\,,$$

so

$$\begin{aligned} f(x) = x^3 + 2x^2 + 3x + 2 &= g(x)h(x) + h(x) \\ &= (g(x) + 1)h(x)\,. \end{aligned} \tag{907}$$

If $h(x)$ is not constant then by comparing the highest polynomial powers on the left and right hand sides we could have

- $g(x)$ could be quadratic with $h(x)$ linear or
- $g(x)$ could be linear with $h(x)$ quadratic

To have the degree of $r(x) = h(x)$ be one less than the degree of $g(x)$ we need to take the first of the two choices above. This means that

$$\begin{aligned} g(x) &= x^2 + Ax + B \\ h(x) &= x + C\,, \end{aligned}$$

for constants A, B, and C. If we put these in the above we see that we need to have

$$\begin{aligned} x^3 + 2x^2 + 3x + 2 &= (x^2 + Ax + B + 1)(x + C) \\ &= x^3(A + C)x^2 + (B + 1 + AC)x + (B + 1)C\,. \end{aligned}$$

Equating coefficients this means that

$$\begin{aligned} A + C &= 2 \\ B + 1 + AC &= 3 \\ (B + 1)C &= 2\,. \end{aligned}$$

From the last equation we have $B + 1 = \frac{2}{C}$ which if we put in the above we get

$$\begin{aligned} A + C &= 2 \\ \frac{2}{C} + AC &= 3\,. \end{aligned}$$

From the first equation we have $A = 2 - C$ when in the second equation gives

$$\frac{2}{C} + (2 - C)C = 3\,.$$

We can write this as

$$C^3 - 2C^2 + 3C - 2 = 0\,.$$

Looking for integer solutions we would need to have $C \in \{\pm 1, \pm 2\}$. Testing each of these in the left-hand-side of the above we find that only $C = 1$ is a solution. In that case $A = 2 - C = 1$ and

$$B + 1 = \frac{2}{C} = 2 \quad \text{so} \quad B = 1\,.$$

This means that we have shown that

$$x^3 + 2x^2 + 3x + 2 = (x^2 + x + 1)(x + 1) + (x + 1)\,.$$

Lecture Notes on Mathematical Olympiad Courses: Vol. 2

Lecture 16: Quadratic Surd Expressions and Their Operations

Notes on Example 3

In this example we first note that $7^4 = 2401$ ends in a one. Now recall that if a number ends in an one then all integer powers of that number will also end in a one. Thus

$$(7^4)^{500},$$

has a ones digit of one. This means that units digit of the full product

$$(7^4)^{500} \cdot 7^3,$$

can be determined from the units digit of the second factor i.e. $7^3 = 343$. Thus the units digit of the full product will be three.

Testing Question A.1

Call this expression E. Now if $x < 2$ then $x - 2 < 0$ and $-x > -2$ so $3 - x > 1 > 0$. Using these we can "take the square roots of the squares" as

$$\begin{aligned} E &= \left|\sqrt{(x-2)^2} + \sqrt{(3-x)^2}\right| = \left|\sqrt{(2-x)^2} + \sqrt{(3-x)^2}\right| \\ &= |(2-x) + (3-x)| = |5 - 2x| . \end{aligned}$$

Now $2x < 4$ so $-2x > -4$ so $5 - 2x > 1$ and the argument of the absolute value above is positive. This means that and we have

$$E = 5 - 2x .$$

Testing Question A.2

Call this expression E. Now we have the given expression equal to

$$\begin{aligned} E &= \frac{1+\sqrt{2}+\sqrt{3}}{1-\sqrt{2}+\sqrt{3}}\left(\frac{1-\sqrt{2}-\sqrt{3}}{1-\sqrt{2}-\sqrt{3}}\right) \\ &= \frac{(1+\sqrt{2}+\sqrt{3})(1-\sqrt{2}-\sqrt{3})}{(1-\sqrt{2})^2-3} = \frac{1-(\sqrt{2}+\sqrt{3})^2}{1-2\sqrt{2}+2-3} = \frac{1-(\sqrt{2}+\sqrt{3})^2}{-2\sqrt{2}} \\ &= -\left(\frac{1-(2+2\sqrt{6}+3)}{2\sqrt{2}}\right) = -\left(\frac{-4-2\sqrt{6}}{2\sqrt{2}}\right) = \frac{2+\sqrt{6}}{\sqrt{2}} \\ &= \frac{2\sqrt{2}+\sqrt{12}}{2} = \sqrt{2}+\sqrt{3}\,. \end{aligned}$$

Testing Question A.3

Lets call this expression E. We start by factoring and then pulling out "common factors" as

$$\begin{aligned} E &= \frac{(x-3)(x-1)+(x+1)\sqrt{x^2-9}}{(x+3)(x+1)+(x-1)\sqrt{x^2-9}} \\ &= \frac{\sqrt{(x-3)^2}(x-1)+(x+1)\sqrt{(x-3)(x+3)}}{\sqrt{(x+3)^2}(x+1)+(x-1)\sqrt{(x-3)(x+3)}} \\ &= \frac{\sqrt{x-3}}{\sqrt{x+3}}\left[\frac{\sqrt{x-3}(x-1)+(x+1)\sqrt{x+3}}{\sqrt{x+3}(x+1)+(x-1)\sqrt{x-3}}\right]. \end{aligned}$$

Note that this second factor in brackets is one so the above is equal to

$$E = \frac{\sqrt{x-3}}{\sqrt{x+3}} = \frac{\sqrt{x^2-9}}{x+3}\,.$$

Testing Question A.4

Lets call this expression E then we can write E as

$$\begin{aligned} E &= \frac{2+3\sqrt{3}+\sqrt{5}}{(2+\sqrt{3})(2\sqrt{3}+\sqrt{5})} = \frac{2+\sqrt{3}+2\sqrt{3}+\sqrt{5}}{(2+\sqrt{3})(2\sqrt{3}\sqrt{5})} \\ &= \frac{1}{2\sqrt{3}+\sqrt{5}} + \frac{1}{2+\sqrt{3}} \\ &= \frac{2\sqrt{3}-\sqrt{5}}{4\cdot 3-5} + \frac{2-\sqrt{3}}{4-3} = \frac{1}{7}(2\sqrt{3}-\sqrt{5})+2-\sqrt{3}\,. \end{aligned}$$

Testing Question A.5

Lets call this expression E then we can write E as

$$\begin{aligned} E &= \left(\sqrt{5}+\sqrt{6}+\sqrt{7}\right)\left(\sqrt{5}+\sqrt{6}-\sqrt{7}\right)\left(\sqrt{5}-\sqrt{6}+\sqrt{7}\right)\left(-\sqrt{5}+\sqrt{6}+\sqrt{7}\right) \\ &= \left(\sqrt{5}+\sqrt{6}+\sqrt{7}\right)\left(\sqrt{5}+\sqrt{6}-\sqrt{7}\right)\left(\sqrt{7}+\sqrt{5}-\sqrt{6}\right)\left(\sqrt{7}-\sqrt{5}+\sqrt{6}\right) \\ &= \left(\left(\sqrt{5}+\sqrt{6}\right)^2-7\right)\left(7-\left(\sqrt{5}-\sqrt{6}\right)^2\right) \\ &= \left(5+2\sqrt{30}+6-7\right)\left(7-\left(5-2\sqrt{30}+6\right)\right) = \left(4+2\sqrt{30}\right)\left(-4+2\sqrt{30}\right) \\ &= 4\cdot 30-16 = 104\,. \end{aligned}$$

Testing Question A.6

Write a as

$$a = \sqrt{6}-2 = \sqrt{6}-\sqrt{2}\cdot\sqrt{2} = \sqrt{2}\left(\sqrt{3}-\sqrt{2}\right)\,.$$

Write b as

$$b = 2\sqrt{2}-\sqrt{6} = -\sqrt{6}+2\sqrt{2} = \sqrt{2}\left(-\sqrt{3}+2\right)\,.$$

With these expressions we have that

$$\frac{a}{b} = \frac{\sqrt{3}-\sqrt{2}}{2-\sqrt{3}}\,.$$

If we "remove" the square root in the denominator we can write this as

$$\begin{aligned} \frac{a}{b} &= \frac{\sqrt{3}-\sqrt{2}}{2-\sqrt{3}}\left(\frac{2+\sqrt{3}}{2+\sqrt{3}}\right) = \frac{2\sqrt{3}-2\sqrt{2}+3-\sqrt{6}}{4-3} \\ &= 3+2\sqrt{3}-2\sqrt{2}-\sqrt{6} \\ &= \sqrt{3}\cdot\sqrt{3}+2\sqrt{3}-2\sqrt{2}-\sqrt{2\cdot 3} \\ &= \sqrt{3}\left(\sqrt{3}+2\right)-\sqrt{2}\left(2+\sqrt{3}\right) \\ &= \left(\sqrt{3}+2\right)\left(\sqrt{3}-2\right) = 3-4 = -1\,. \end{aligned}$$

This means that $a = -b$. If we note that $a > 0$ so the above means that $b < 0$. Thus we the order is $a > b$.

Testing Question A.7

We have

$$\begin{aligned} a &= \sqrt{27}-\sqrt{26} \\ b &= \sqrt{28}-\sqrt{27} \\ c &= \sqrt{29}-\sqrt{28}\,. \end{aligned}$$

To help see what we should do we note that

$$\left(\sqrt{27} - \sqrt{26}\right)\left(\sqrt{27} + \sqrt{26}\right) = 26 - 26 = 1\,.$$

The same pattern holds for b and c. This means that we have

$$\begin{aligned} a &= \frac{1}{\sqrt{27} + \sqrt{26}} \\ b &= \frac{1}{\sqrt{28} + \sqrt{27}} \\ c &= \frac{1}{\sqrt{29} + \sqrt{28}}\,. \end{aligned}$$

Since we know that

$$\sqrt{26} + \sqrt{27} < \sqrt{27} + \sqrt{28} < \sqrt{28} + \sqrt{29}\,,$$

we have that

$$\frac{1}{\sqrt{28} + \sqrt{29}} < \frac{1}{\sqrt{27} + \sqrt{28}} < \frac{1}{\sqrt{26} + \sqrt{27}}\,,$$

which means that $c < b < a$.

Testing Question A.8

We start with

$$\frac{3}{1 + \sqrt{3}} < x < \frac{3}{\sqrt{5} - \sqrt{3}}\,. \tag{908}$$

Now using the fact that

$$1 + \sqrt{3} > 1 + 2 = 3\,,$$

thus we see that

$$\frac{3}{1 + \sqrt{3}} > \frac{3}{1 + 2} = 1\,.$$

Thus the left-hand-side of the above is larger than one and thus the smallest x can be is $x = 2$. This means that $x \geq 2$. We now need to evaluate a bound on the right-hand-side to see how large x can become. Note that we can write the right-hand-side above as

$$\frac{3\left(\sqrt{5} + \sqrt{3}\right)}{5 - 3} = \frac{3}{2}\left(\sqrt{5} + \sqrt{3}\right)\,.$$

We can bound the right-hand-side above using the Bernoulli inequality 951 as follows

$$\begin{aligned} \sqrt{5} + \sqrt{3} &= \sqrt{4 + 1} + \sqrt{4 - 1} \\ &= 2\left(\sqrt{1 + \frac{1}{4}} + \sqrt{1 - \frac{1}{4}}\right) \\ &< 2\left(1 + \frac{1}{8} + 1 - \frac{1}{8}\right) = 4\,. \end{aligned}$$

This means that

$$x < \frac{3}{2}(4) = 6\,.$$

Thus it looks like the we can have

$$x \in \{2, 3, 4, 5\}\,.$$

Another way to work the second half of this problem is to realize that when we ask for which x values do we have

$$x < \frac{3}{2}(\sqrt{5} + \sqrt{3})\,.$$

We can answer that problem by "squaring until there are no more square roots". Thus the above is equal to

$$\frac{4x^2}{9} < 5 + 3 + 2\sqrt{15}\,,$$

or

$$\frac{4x^2}{9} < 8 + 2\sqrt{15}\,,$$

or

$$\frac{2x^2}{9} < 4 + \sqrt{15}\,,$$

or

$$2x^2 < 36 + 9\sqrt{15}\,,$$

or

$$(2x^2 - 36)^2 < 81 \cdot 15 = 1215\,.$$

We can then evaluate the above for various integer x values and stop when we get the first x value that does not satisfy this inequality. This method gives the same x values as before.

Testing Question A.9

Lets call this expression E so that

$$E = \frac{1}{1 - \sqrt[4]{5}} + \frac{1}{1 + \sqrt[4]{5}} + \frac{2}{1 + \sqrt{5}}\,.$$

If we rationalize the denominator of the first two fractions by multiplying by "forms of one" with denominators $1 + \sqrt[4]{5}$ and $1 - \sqrt[4]{5}$ respectively we can write E as

$$E = \frac{1 + \sqrt[4]{5}}{1 - \sqrt{5}} + \frac{1 - \sqrt[4]{5}}{1 - \sqrt{5}} + \frac{2}{1 + \sqrt{5}} = \frac{2}{1 - \sqrt{5}} + \frac{2}{1 + \sqrt{5}}\,.$$

If we rationalize the denominators of these two fractions by multiplying by "forms of one" with denominators $1 + \sqrt{5}$ and $1 - \sqrt{5}$ respectively we can write E as

$$E = \frac{2\left(1 + \sqrt{5}\right) + 2\left(1 + \sqrt{5}\right)}{(1 - 5)} = \frac{4}{-4} = -1\,.$$

Testing Question A.10

We are told that $c > b > c > d > 0$ and that

$$U = \sqrt{ab} + \sqrt{cd}$$
$$V = \sqrt{ac} + \sqrt{bd}$$
$$W = \sqrt{ad} + \sqrt{bc}\,.$$

From definition of U replace b is the first square root with c. Then since $b > c$ this means that

$$U > \sqrt{ac} + \sqrt{cd}\,.$$

From the definition of V we can replace $\sqrt{ac}$ in the above to get

$$U > (V - \sqrt{bd}) + \sqrt{cd}\,.$$

This means that

$$U - V > \sqrt{cd} - \sqrt{bd} = \sqrt{d}\left(\sqrt{c} - \sqrt{d}\right) > 0\,.$$

Thus we have shown that $U > V$.

Next consider the difference $V - W$. We find

$$\begin{aligned} V - W &= \sqrt{a}\left(\sqrt{c} - \sqrt{d}\right) + \sqrt{b}\left(\sqrt{d} - \sqrt{c}\right) \\ &= \sqrt{a}\left(\sqrt{c} - \sqrt{d}\right) - \sqrt{b}\left(\sqrt{c} - \sqrt{d}\right) \\ &= \left(\sqrt{a} - \sqrt{b}\right)\left(\sqrt{c} - \sqrt{d}\right) > 0\,. \end{aligned}$$

This then means that $V > W$.

Combining these two results we have that $U > V > W$.

Testing Question B.1

To have both of the square roots have positive arguments means that

$$|a| - 3 \geq 0 \Rightarrow |a| \geq 3\,,$$

and

$$3 - |a| \geq 0 \Rightarrow 3 \geq |a|\,.$$

Taken together this means that $|a| = 3$ or $a = \pm 3$. If $a = 3$ the denominator of second fraction is zero thus $a = -3$. This means that x is given by

$$x = \left(-\frac{2(-3)}{4-3}\right)^{1993} = 6^{1993}\,.$$

To determine the units digit of this product note that $6^1 = 6$, $6^2 = 36$, and $6^3 = 216$. Thus all powers of six will end in a six. Thus the last digit of x is six.

Testing Question B.2

Lets call this expression E then we have

$$
\begin{aligned}
E &= \sqrt[3]{3}\left(\sqrt[3]{\frac{4}{9}} - \sqrt[3]{\frac{2}{9}} + \sqrt[3]{\frac{1}{9}}\right)^{-1} \\
&= \sqrt[3]{3}\left(\frac{\sqrt[3]{4} - \sqrt[3]{2} + \sqrt[3]{1}}{\sqrt[3]{9}}\right)^{-1} \\
&= \sqrt[3]{3}\left(\frac{\sqrt[3]{9}}{\sqrt[3]{4} - \sqrt[3]{2} + \sqrt[3]{1}}\right) = \frac{3}{\sqrt[3]{4} - \sqrt[3]{2} + 1}\,.
\end{aligned}
$$

We would like to multiply by a "form of one" that will rationalize the denominator. Towards that end lets multiply it by something like $\sqrt[3]{2} + A$ where we will seek to determine what A is that will make the results the simplest. We have

$$
\left(\sqrt[3]{4} - \sqrt[3]{2} + 1\right)\left(\sqrt[3]{2} + A\right) = \sqrt[3]{8} - \sqrt[3]{4} + \sqrt[3]{2} + A\sqrt[3]{4} - A\sqrt[3]{2} + A\,.
$$

If we take $A = 1$ in the above we get

$$
\sqrt[3]{8} - \sqrt[3]{4} + \sqrt[3]{2} + \sqrt[3]{4} - \sqrt[3]{2} + 1 = \sqrt[3]{8} + 1 = 2 + 1 = 3\,.
$$

Thus lets multiply the numerator and denominator of E by

$$
\frac{\sqrt[3]{2} + 1}{\sqrt[3]{2} + 1}\,,
$$

to get

$$
E = \frac{3\left(\sqrt[2]{3} + 1\right)}{3} = 1 + \sqrt[3]{2}\,.
$$

Testing Question B.3

Lets call this expression E then we can write E as

$$
E = \sqrt{\frac{n(n+1)(n+2)(n+3)+1}{4}}\,,
$$

with $n = 1998$. Now take the products of the factors n with $n+3$ and also $n+1$ with $n+2$ to write the above as

$$
\begin{aligned}
E &= \sqrt{\frac{(n^2+3n)(n^2+3n+2)+1}{4}} \\
&= \frac{\sqrt{(n^2+3n)^2 + 2(n^2+3n)+1}}{2} = \frac{\sqrt{(n^2+3n+1)^2}}{2} \\
&= \frac{n^2+3n+1}{2}\,.
\end{aligned}
$$

Now when $n = 1988$ this is 1999000.

Testing Question B.4

In this question we take $a = \sqrt[3]{4} + \sqrt[3]{2} + 1$ where we note that from Question B2 we might consider the product

$$\begin{aligned}(\sqrt[3]{4} + \sqrt[3]{2} + 1)(\sqrt[3]{2} - 1) &= \sqrt[3]{8} + \sqrt[3]{4} + \sqrt[3]{2} - \sqrt[3]{4} - \sqrt[3]{2} - 1 \\ &= \sqrt[3]{8} - 1 = 2 - 1 = 1\,.\end{aligned}$$

Thus we have that

$$a = \frac{1}{\sqrt[3]{2} - 1}\,.$$

From this we find

$$a^2 = \frac{1}{\sqrt[3]{4} - 2\sqrt[3]{2} + 1} = \frac{1}{2^{\frac{2}{3}} - 2 \cdot 2^{\frac{1}{3}} + 1} = \frac{1}{2^{\frac{2}{3}} - 2^{\frac{4}{3}} + 1}\,,$$

and

$$\begin{aligned}a^3 = \frac{1}{(2^{\frac{2}{3}} - 2^{\frac{4}{3}} + 1)}\frac{1}{(2^{\frac{1}{3}} - 1)} &= \frac{1}{2 - 2^{\frac{5}{3}} + 2^{\frac{1}{3}} - 2^{\frac{2}{3}} + 2^{\frac{4}{3}} - 1} \\ &= \frac{1}{1 - 2 \cdot 2^{\frac{2}{3}} + 2^{\frac{1}{3}} - 2^{\frac{2}{3}} + 2 \cdot 2^{\frac{1}{3}}} \\ &= \frac{1}{1 - 3 \cdot 2^{\frac{2}{3}} + 3 \cdot 2^{\frac{1}{3}}}\,.\end{aligned}$$

Call the expression we want to evaluate E. Using the above E is then given by

$$\begin{aligned}E &= \frac{3}{a} + \frac{3}{a^2} + \frac{1}{a^3} \\ &= (2^{\frac{1}{3}} - 1) + 3(2^{\frac{2}{3}} - 2 \cdot 2^{\frac{1}{3}} + 1) + 1 - 3 \cdot 2^{\frac{2}{3}} + 3 \cdot 2^{\frac{1}{3}} \\ &= (3 - 6 + 3)2^{\frac{1}{3}} - 3 + 3 + 1 + (3 - 3)2^{\frac{2}{3}} = 1\,.\end{aligned}$$

Testing Question B.5

Define A and B as $A = \sqrt{13} + \sqrt{11}$ and $B = \sqrt{13} + \sqrt{11}$. Then notice that

$$\begin{aligned}M &= A^6 \\ A + B &= 2\sqrt{13} \\ A \cdot B &= 13 - 11 = 2\,.\end{aligned}$$

We can get even powers of A (or B) using the above expressions. In that direction note that

$$A^2 + B^2 = (A + B)^2 - 2AB = \left(2\sqrt{13}\right)^2 - 2 \cdot 2 = 4 - 13 - 4 = 48\,.$$

As $M = A^6$ we need higher powers. To get that we consider

$$\begin{aligned} A^6 + B^6 = (A^2)^3 + (B^2)^3 &= (A^2 + B^2)(A^4 - A^2 B^2 + B^4) \\ &= (A^2 + B^2)\left((A^2)^2 + (B^2)^2 - (AB)^2\right) \\ &= (A^2 + B^2)\left((A^2 + B^2)^2 - 2A^2B^2 - (AB)^2\right) \\ &= (A^2 + B^2)\left((A^2 + B^2)^2 - 3(AB)^2\right) . \end{aligned}$$

Notice that we know all of the factors in the above and we can evaluate this as

$$A^6 + B^6 = 48 \cdot (48^2 - 3 \cdot 2^2) = 110016 \, ,$$

an integer. Now given this, one of the things we want can be written as

$$A^6 = \left[A^6 + B^6\right] - B^6 = 110016 - B^6 \, .$$

Now we can bound B as

$$0 < B < \sqrt{16} - \sqrt{9} = 4 - 3 = 1 \, ,$$

thus

$$B^6 < 1 \, .$$

As $M = A^6$ is an integer minus a number less than one (i.e. B^6) the decimal part of M is $P = 1 - B^6$. This is because in computing M we are subtracting a number less than one from an integer. Given this we have

$$M(1 - P) = \left(\left[A^6 + B^6\right] - B^6\right) B^6 = (AB)^6 = 2^6 = 64 \, .$$

Lecture 17: Compound Quadratic Surd Form $\sqrt{a \pm \sqrt{b}}$

Testing Question A.1

For this we have

$$\sqrt{12 - 4\sqrt{5}} = \sqrt{2(6 - 2\sqrt{5})} = \sqrt{2(\sqrt{5} - \sqrt{1})^2} = \sqrt{2}(\sqrt{5} - 1)\,.$$

Testing Question A.2

Lets call this expression E. Then we have

$$\begin{aligned}
E &= \sqrt{2 + \sqrt{3}} + \sqrt{2 - \sqrt{3}} \\
&= \sqrt{\frac{1}{2}(4 + 2\sqrt{3})} + \sqrt{\frac{1}{2}(4 - 2\sqrt{3})} \\
&= \frac{1}{\sqrt{2}}\sqrt{(\sqrt{3} + \sqrt{1})^2} + \frac{1}{\sqrt{2}}\sqrt{(\sqrt{3} - \sqrt{1})^2} \\
&= \frac{1}{\sqrt{2}}(\sqrt{3} + 1) + \frac{1}{\sqrt{2}}(\sqrt{3} - 1) = \frac{2\sqrt{3}}{\sqrt{2}} = \sqrt{6}\,.
\end{aligned}$$

Testing Question A.3

Lets call this expression E. Then we have

$$E = \sqrt{14 + 6\sqrt{5}} - \sqrt{14 - 6\sqrt{5}}\,.$$

Now in order for the argument of the first square root to be of the form $(a + \sqrt{5}b)^2$ would mean that a and b would have to satisfy

$$a^2 + 5b^2 = 14\,.$$

One way this can be true is if $a = 3$ and $b = 1$. This means that

$$\begin{aligned}
E &= \sqrt{(3 + \sqrt{5})^2} - \sqrt{(3 - \sqrt{5})^2} \\
&= 3 + \sqrt{5} - 3 + \sqrt{5} = 2\sqrt{5}\,.
\end{aligned}$$

Testing Question A.4

Lets call this expression E. Then we have

$$\begin{aligned} E &= \sqrt{8+\sqrt{63}} - \sqrt{8-\sqrt{63}} \\ &= \sqrt{8+3\sqrt{7}} - \sqrt{8-3\sqrt{7}}\,. \end{aligned}$$

Method 1: If we square E we get

$$\begin{aligned} E^2 &= 8+3\sqrt{7}+8-3\sqrt{3}-2\sqrt{(8+3\sqrt{3})(8-3\sqrt{7})} \\ &= 16-2\sqrt{64-9\cdot 7} = 16-2 = 14\,. \end{aligned}$$

Thus $E = \sqrt{14}$.

Method 2: Another way to work this is to write E as

$$\begin{aligned} E &= \sqrt{8+\sqrt{63}} - \sqrt{8-\sqrt{63}} \\ &= \sqrt{\frac{1}{2}(16+2\sqrt{63})} - \sqrt{\frac{1}{2}(16-2\sqrt{63})} \\ &= \sqrt{\frac{1}{2}(\sqrt{9}+\sqrt{7})^2} - \sqrt{\frac{1}{2}(\sqrt{9}-\sqrt{7})^2} \\ &= \frac{\sqrt{9}+\sqrt{7}}{\sqrt{2}} - \frac{\sqrt{9}-\sqrt{7}}{\sqrt{2}} = \frac{2\sqrt{7}}{\sqrt{2}} = \sqrt{14}\,. \end{aligned}$$

Testing Question A.5

Lets call this expression E. Then we have

$$E = \sqrt{4+\sqrt{7}} + \sqrt{4-\sqrt{17}}\,.$$

We can write E as

$$\begin{aligned} E &= \sqrt{\frac{1}{2}(8+2\sqrt{7})} + \sqrt{\frac{1}{2}(8-2\sqrt{7})} \\ &= \sqrt{\frac{1}{2}(\sqrt{7}+1)^2} + \sqrt{\frac{1}{2}(\sqrt{7}-1)^2} \\ &= \frac{1}{\sqrt{2}}(\sqrt{7}+1) + \frac{1}{\sqrt{2}}(\sqrt{7}-1) \\ &= \frac{2\sqrt{7}}{\sqrt{2}} = \sqrt{2}\sqrt{7} = \sqrt{14}\,. \end{aligned}$$

Testing Question A.6

Lets call this expression E. Then we have

$$E = \sqrt{7 - \sqrt{15} - \sqrt{16 - 2\sqrt{15}}}\,.$$

Now in order for the argument of the last square root to be of the form $(a + \sqrt{15}b)^2$ would mean that a and b would have to satisfy

$$a^2 + 15b^2 = 16\,.$$

One way this can be true is if $a = 1$ and $b = 1$. This means that we can write E as

$$\begin{aligned} E &= \sqrt{7 - \sqrt{15} - \sqrt{(\sqrt{15} - 1)^2}} \\ &= \sqrt{7 - \sqrt{15} - (\sqrt{15} - 1)} = \sqrt{8 - 2\sqrt{15}} \\ &= \sqrt{(\sqrt{5} - \sqrt{3})^2} = \sqrt{5} - \sqrt{3}\,. \end{aligned}$$

Testing Question A.7

Let $a = \sqrt{3\sqrt{5} - \sqrt{2}}$ and $b = \sqrt{3\sqrt{2} - \sqrt{5}}$ then our system is

$$\begin{aligned} x + y &= a \\ x - y &= b\,. \end{aligned}$$

If we add and then subtract these two equations we get the solutions

$$\begin{aligned} x &= \frac{a + b}{2} \\ y &= \frac{a - b}{2}\,. \end{aligned}$$

This would mean that $xy = \frac{a^2 - b^2}{4}$.

Now from the definition of a and b we have that

$$\begin{aligned} a^2 - b^2 &= 3\sqrt{5} - \sqrt{2} - (3\sqrt{2} - \sqrt{5}) \\ &= 4\sqrt{5} - 4\sqrt{2} = 4(\sqrt{5} - \sqrt{2})\,. \end{aligned}$$

Thus $xy = \sqrt{5} - \sqrt{2}$.

Testing Question A.8

Lets call this expression E then we can write E as

$$\begin{aligned} E &= \sqrt{8 + 2(4 + 2\sqrt{7} + 2\sqrt{5} + \sqrt{35})} \\ &= \sqrt{16 + 4\sqrt{5} + 4\sqrt{7} + 2\sqrt{35}}\,. \end{aligned}$$

Based on this form we might hope that we can write E in the following form

$$E = a + b\sqrt{5} + c\sqrt{7}\,.$$

From the expression for E given in this question and the form we hope it takes we have

$$\begin{aligned} E^2 &= 16 + 4\sqrt{5} + 4\sqrt{7} + 2\sqrt{35} \\ &= a^2 + 5b^2 + 7c^2 + 2ab\sqrt{5} + 2ac\sqrt{7} + 2cb\sqrt{35}\,. \end{aligned}$$

This will be true if

$$a^2 + 5b^2 + 7c^2 = 16 \qquad (909)$$

$$2ab = 4 \qquad (910)$$

$$2ac = 4 \qquad (911)$$

$$2cb = 2\,. \qquad (912)$$

Based on Equation 909 lets "try" $a = 2$, $b = 1$, and $c = 1$. From these we see that the other three equations are also true. This means that

$$E = 2 + \sqrt{5} + \sqrt{7}\,.$$

Testing Question A.9

Denote this expression by E. Then we have

$$\begin{aligned} E &= \sqrt{a + 3 + 4\sqrt{a-1}} + \sqrt{a + 3 - 4\sqrt{a-1}} \\ &= \sqrt{(\sqrt{a-1} + 2)^2} + \sqrt{(\sqrt{a-1} - 2)^2} \\ &= \sqrt{a-1} + 2 + \sqrt{(\sqrt{a-1} - 2)^2}\,. \end{aligned}$$

We can perform the last step above because $\sqrt{a-1} + 2 \geq 2 > 0$ for all $a \geq 1$. Now if

$$\sqrt{a-1} - 2 > 0 \quad \text{or} \quad a \geq 5\,,$$

then E is equal to

$$E = \sqrt{a-1} + 2 + \sqrt{a-1} - 2 = 2\sqrt{a-1}\,.$$

If

$$\sqrt{a-1} - 2 < 0\,,$$

or $\sqrt{a-1} < 2$ or as this expression must be positive we have

$$0 \leq \sqrt{a-1} < 2\,,$$

or $1 \leq a < 5$ then E is equal to

$$E = \sqrt{a-1} + 2 + 2 - \sqrt{a-1} = 4\,.$$

Testing Question A.10

We can write A as

$$\begin{aligned} A &= \frac{\sqrt{\sqrt{3}+1} - \sqrt{\sqrt{3}-1}}{\sqrt{\sqrt{3}+1} + \sqrt{\sqrt{3}-1}} \times \frac{\sqrt{\sqrt{3}+1} - \sqrt{\sqrt{3}-1}}{\sqrt{\sqrt{3}+1} - \sqrt{\sqrt{3}-1}} \\ &= \frac{(\sqrt{3}+1) - 2\sqrt{(\sqrt{3}+1)(\sqrt{3}-1)} + (\sqrt{3}-1)}{\sqrt{3}+1-(\sqrt{3}-1)} \\ &= \frac{2\sqrt{3} - 2\sqrt{3-1}}{2} = \sqrt{3} - \sqrt{2}\,. \end{aligned}$$

Now from the expression given we have

$$\begin{aligned} x &= \frac{\sqrt{6}-\sqrt{30}}{\sqrt{2}-\sqrt{10}} - 2 = \frac{\sqrt{3}(\sqrt{2}-\sqrt{10})}{\sqrt{2}-\sqrt{10}} - 2 \\ &= \sqrt{3} - 2\,, \end{aligned}$$

which is *not* equal to A.

Testing Question B.1

Lets denote this expression E. Now for $\sqrt{ab}$ to be defined in the real numbers we must have $ab \geq 0$. Thus a and b are the same sign (both positive or both negative). If $a > 0$ (then $b > 0$) so that both of $\sqrt{a}$ and $\sqrt{b}$ are defined and

$$\begin{aligned} (\sqrt{a})^2 &= a \\ (\sqrt{b})^2 &= b\,. \end{aligned}$$

Now for E to be real means the argument of the square root must be positive or

$$\sqrt{ab} \geq \frac{a+b}{2}\,.$$

This is the *opposite* of the arithmetic-mean geometric-mean (AM-GM) inequality and would only be true if $a = b$ giving a contradiction. Thus a cannot be $a > 0$ and thus we conclude that $a < 0$ (so that $b < 0$ also). This means that $\sqrt{-a}$ and $\sqrt{-b}$ are defined. Using that information we can write E as

$$\begin{aligned} E &= \sqrt{2\sqrt{ab} + (-a) + (-b)} = \sqrt{2\sqrt{ab} + (\sqrt{-a})^2 + (\sqrt{-b})^2} \\ &= \sqrt{(\sqrt{-a} + \sqrt{-b})^2} = \sqrt{-a} + \sqrt{-b}\,. \end{aligned}$$

Testing Question B.2

Call this expression E. For E to be a real number we must have $a \geq 1$. We can simplify the argument of the first cube root as

$$\begin{aligned} \frac{(\sqrt{a-1} - \sqrt{a})^5}{\sqrt{a-1} + \sqrt{a}} \times \frac{\sqrt{a-1} - \sqrt{a}}{\sqrt{a-1} - \sqrt{a}} &= \frac{(\sqrt{a-1} - \sqrt{a})^6}{a-1-a} \\ &= -(\sqrt{a-1} - \sqrt{a})^6\,. \end{aligned}$$

In the same way, the argument of the second cube root can be simplified to get

$$(\sqrt{a-1} + \sqrt{a})^6\,.$$

Taking the needed cube roots of these we get

$$\begin{aligned} E &= -(\sqrt{a-1} - \sqrt{a})^2 + (\sqrt{a-1} + \sqrt{a})^2 \\ &= -((a-1) - 2\sqrt{a(a-1)} + a) + ((a-1) - 2\sqrt{a(a-1)} + a) \\ &= 4\sqrt{a(a-1)}\,. \end{aligned}$$

Testing Question B.3

Let E_i (i for "inner") be defined as

$$E_i = 1 + a^2 + \sqrt{1 + a^2 + a^4}\,.$$

Next note that

$$1 + a^2 + a^4 = 1 + 2a^2 + a^4 - a^2 = (1 + a^2)^2 - a^2\,.$$

Thus we have shown that we can write E_i as

$$\begin{aligned} E_i &= 1 + a^2 + \sqrt{(a^2 - a + 1)(a^2 + a + 1)} \\ &= \frac{1}{2}(2 + 2a^2 + 2\sqrt{(a^2 - a + 1)(a^2 + a + 1)}) \\ &= \frac{1}{2}\left((a^2 - a + 1) + 2\sqrt{(a^2 - a + 1)(a^2 + a + 1)} + (a^2 + a + 1)\right) \\ &= \frac{1}{2}\left(\sqrt{a^2 - a + 1} + \sqrt{a^2 + a + 1}\right)^2\,. \end{aligned}$$

This means that the total expression we want to evaluate is given by

$$E = \sqrt{E_i} = \frac{1}{\sqrt{2}}(\sqrt{a^2 - a + 1} + \sqrt{a^2 + a + 1})\,.$$

Testing Question B.4

Lets call this expression E. To solve this question we first try to factor (a multiple) of the argument of the second square root. That is, we ask if we can write

$$2x - 4 + 2\sqrt{2x-5} = (a + b\sqrt{2x-5})^2\,,$$

for some a and b. If we expand the right-hand-side of the above we get

$$a^2 + (2x-5)b^2 + 2ab\sqrt{2x-5}\,.$$

If we take $ab = 1$ to make the above equal we need to have

$$a^2 + (2x-5)b^2 = 2x - 4\,.$$

This will be satisfied if we take $b = 1$ and $a = 1$. This means that

$$\begin{aligned}\sqrt{x - 2 + \sqrt{2x-5}} &= \sqrt{\frac{1}{2}(2x - 4 + 2\sqrt{2x-5})}\\ &= \frac{1}{\sqrt{2}}\sqrt{(1+\sqrt{2x-5})^2} = \frac{1+\sqrt{2x-5}}{\sqrt{2}}\,.\end{aligned}$$

Next we try to factor (another multiple) of the argument of the first square root. That is, we ask if we can write

$$2x + 4 + 6\sqrt{2x-5} = (a + b\sqrt{2x-5})^2\,,$$

for some (perhaps different) a and b. If we expand the right-hand-side of the above we get

$$a^2 + (2x-5)b^2 + 2ab\sqrt{2x-5}\,.$$

If we take $2ab = 6$ or $ab = 3$ to make the above equal we need to have

$$a^2 + (2x-5)b^2 = 2x + 4\,.$$

This will be satisfied if we take $b = 1$ and $a = 3$. This means that

$$\begin{aligned}\sqrt{x + 2 + 3\sqrt{2x-5}} &= \sqrt{\frac{1}{2}(2x + 4 + 6\sqrt{2x-5})}\\ &= \frac{1}{\sqrt{2}}\sqrt{(3+\sqrt{2x-5})^2} = \frac{3+\sqrt{2x-5}}{\sqrt{2}}\,.\end{aligned}$$

These two things taken together mean that we have

$$E = \frac{3+\sqrt{2x-5}}{\sqrt{2}} - \frac{1+\sqrt{2x-5}}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}\,.$$

Testing Question B.5

We start with

$$\sqrt{x} = \sqrt{a} - \frac{1}{\sqrt{a}}\,. \tag{913}$$

If we square both sides of this we get

$$x = a - 2 + \frac{1}{a}\,,$$

or

$$x + 2 = a + \frac{1}{a}\,.$$

If we square this again we get

$$(x+2)^2 = x^2 + 4x + 4 = a^2 + 2 + \frac{1}{a^2}\,,$$

or

$$x^2 + 4x = a^2 - 2 + \frac{1}{a^2} = \left(a - \frac{1}{a}\right)^2\,.$$

This means that

$$\sqrt{x^2+4x} = \left|a - \frac{1}{a}\right|\,. \tag{914}$$

Now from Equation 913 as $\sqrt{x} \ge 0$ we know that

$$\sqrt{a} - \frac{1}{\sqrt{a}} \ge 0\,,$$

which means that

$$\sqrt{a} \ge \frac{1}{\sqrt{a}}\,,$$

or

$$(\sqrt{a})^2 \ge 1\,,$$

or

$$a \ge 1\,.$$

This means that $\frac{1}{a} < 1$ so that $a > \frac{1}{a}$ and Equation 914 becomes

$$\sqrt{x^2+4x} = a - \frac{1}{a}\,.$$

This means that

$$\frac{x+2+\sqrt{x^2+4x}}{x+2-\sqrt{x^2+4x}} = \frac{a+\frac{1}{a}+a-\frac{1}{a}}{a+\frac{1}{a}-\left(a-\frac{1}{a}\right)} = \frac{2a}{\frac{2}{a}} = a^2\,.$$

Testing Question B.6

Lets call this expression E. Next we ask if we can write the argument of the square root as a square as

$$17 - 12\sqrt{2} = (a - b\sqrt{2})^2 = a^2 + 2b^2 - 2ab\sqrt{2}\,.$$

This is true if

$$\begin{aligned} a^2 + 2b^2 &= 17 \\ 2ab &= 12 \quad \text{or} \quad ab = 6\,. \end{aligned}$$

Both of these can be made true if we take $a = 3$ and $b = 2$. Thus we have

$$17 - 12\sqrt{2} = (3 - 2\sqrt{2})^2\,,$$

so that

$$\sqrt{17 - 12\sqrt{2}} = 3 - 2\sqrt{2}\,.$$

This means that we have shown that

$$E = \frac{1}{3 - 2\sqrt{2}} = \frac{3 + 2\sqrt{2}}{9 - 4 \cdot 2} = 3 + 2\sqrt{2}\,. \tag{915}$$

We now need to find the closest integer to the above expression.

Method 1: Lets evaluate $\sqrt{2}$ to an accuracy such that when we compute E above we can determine the closest integer. One way to compute square roots "by hand" is to use Newton's method on an appropriate function. For this problem we consider

$$f(x) = x^2 - 2\,.$$

Then $f(x) = 0$ has solutions $\pm\sqrt{2}$. For this problem Newton's iterations take the form

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{(x_n^2 - 2)}{2x_n} = \frac{x_n}{2} + \frac{1}{x_n}\,.$$

This is simple enough we can iterate by hand. Using `R` and starting at $x_0 = 1$ as

```
my_g = function(x){
  x - 0.5 * x + 1/x
}

print(c(my_g(1), my_g(my_g(1)), my_g(my_g(my_g(1)))))
```

we find

```
[1] 1.500000 1.416667 1.414216
```

for the first three iterates. Thus we conclude that

$$1.4 < \sqrt{2} < 1.41\,,$$

so that

$$2.8 < 2\sqrt{2} < 2.82\,,$$

and

$$5.8 < 3 + 2\sqrt{2} < 5.82\,.$$

Thus the closest integer to E is six.

Method 2: From Equation 915 we have

$$E = 3 + \sqrt{8}\,.$$

Thus

$$5 = 3 + \sqrt{4} < E < 3 + \sqrt{9} = 6\,,$$

so we now know that $5 < E < 6$. The midpoint of this range is the value 5.5. Write 5.5 as

$$5.5 = 3 + 2.5 = 3 + \sqrt{2.5^2} = 3 + \sqrt{6.25} < 3 + \sqrt{8} = E\,.$$

Thus using this, we have the improved bound that

$$5.5 < E < 6\,.$$

Again we see that the nearest integer to E is six.

Lecture 18: Congruence of Integers

Notes on Example 8

Using the comments/results from Example 5 we have that

$$a^n + b^n = (a+b)(a^{n-1} - a^{n-2}b + a^{n-3}b^2 - \cdots + b^{n-1})\,,$$

thus we see that

$$a^n + b^n \equiv 0 \pmod{a+b}\,.$$

If we write the given expression as suggested we see that the *pairs* of terms will all be divisible by $n+2$ except the first (unit term). This proves that the remainder when divided by $n+2$ is one.

Testing Question A.1

We are looking to find the number of values for n such that

$$2007 = qn + 7\,,$$

for some quotient q. This is equivalent to

$$2000 = qn\,.$$

The prime factorization of 2000 gives

$$2^4 \cdot 5^3 = qn\,.$$

This means that n is a number of the form $n = 2^p \cdot 5^q$ for $0 \le p \le 4$ and $0 \le q \le 3$. There are $(4+1) \times (3+1) = 20$ numbers of this form. Some of these values would produce values for n that were *less than* seven and would thus not have seven as a remainder when we divide by n. These numbers happen when

- If $p = 0$ and $q \in \{0, 1\}$
- If $p = 1$ and $q = 1$.
- If $p = 2$ and $q = 0$.

or four numbers. This means that $20 - 4 = 16$ number of the required type exist.

Testing Question A.2

We will use the idea that since this number 123456789 is to the fourth power if I can evaluate $123456789 \equiv b \pmod{m}$ where b is small then I can raise everything to the fourth power to get

$$123456789^4 \equiv b^4 \pmod{m},$$

and hopefully I can evaluate $b^4 \pmod{m}$. Note that if we define $N \equiv 123456789$ we can write

$$N = 1 \cdot 10^8 + 2 \cdot 10^7 + 3 \cdot 10^6 + 4 \cdot 10^5 + 5 \cdot 10^4 + 6 \cdot 10^3 + 7 \cdot 10^2 + 8 \cdot 10 + 9.$$

Now note that

$$10 \equiv 2 \pmod{8},$$

so squaring this we get

$$10^2 \equiv 2^2 \equiv 4 \pmod{8},$$

while cubing it we get

$$10^3 \equiv 2^3 \equiv 8 \equiv 0 \pmod{8}.$$

This means that

$$10^p \equiv 2^p \equiv 0 \pmod{8} \quad \text{when} \quad p \geq 3.$$

Using these facts we can work on the number N above. We have

$$\begin{aligned} N &\equiv 1 \cdot 0 + 2 \cdot 0 + 3 \cdot 0 + 4 \cdot 0 + 5 \cdot 0 + 6 \cdot 0 + 7 \cdot 4 + 0 + 9 \pmod{8} \\ &\equiv 28 + 9 \equiv 4 + 1 \equiv 5 \pmod{8}. \end{aligned}$$

Thus we will have

$$123456789^4 \equiv 5^4 \pmod{8}.$$

One way to evaluate the right-hand-side of the above is to note that

$$5 \equiv (-3) \pmod{8},$$

so

$$5^4 \equiv (-3)^4 \equiv 81 \equiv 1 \pmod{8}.$$

Another way is to note that $5^4 = (5^2)^2 = 25^2$ and $25 \equiv 1 \pmod{8}$ so

$$25^2 \equiv 1^2 \equiv 1 \pmod{8},$$

the same result we got earlier. Thus as $123456789^4 \equiv 1 \pmod{8}$ this number has a remainder of one when divided by eight.

Testing Question A.3

Note that $2222 = 2 \times 1111$ and $5555 = 5 \times 1111$ and that

$$1111 = 1000 + 100 + 10 + 1\,. \tag{916}$$

Then using

$$\begin{aligned} 10 &\equiv 3 \pmod 7 \quad \text{so} \\ 10^2 &\equiv 3^2 \equiv 2 \pmod 7 \quad \text{and} \\ 10^3 &\equiv 3^3 \equiv 27 \equiv 6 \pmod 7\,, \end{aligned}$$

we have

$$\begin{aligned} 1111 &\equiv 6 + 2 + 3 + 1 \pmod 7 \\ &\equiv 12 \pmod 7 \\ &\equiv 5 \pmod 7\,. \end{aligned}$$

Of course $2 \equiv 2 \pmod 7$ and $5 \equiv 5 \pmod 7$ so multiplying the above result for the number 1111 we get

$$\begin{aligned} 2222 &\equiv 10 \equiv 3 \pmod 7 \\ 5555 &\equiv 25 \equiv 4 \pmod 7\,. \end{aligned}$$

Using the above we can now think about the remainders of powers of 2222. With the above we have

$$\begin{aligned} 2222^2 &\equiv 3^2 \equiv 2 \pmod 7 \\ 2222^3 &\equiv 3^3 \equiv 6 \pmod 7 \\ 2222^4 &\equiv 3^4 \equiv 81 \equiv 4 \pmod 7 \\ 2222^5 &\equiv 3^5 \equiv 3 \cdot 4 \equiv 12 \equiv 5 \pmod 7 \\ 2222^6 &\equiv 3^6 \equiv 3 \cdot 5 \equiv 15 \equiv 1 \pmod 7\,, \end{aligned}$$

and the pattern will repeat cyclically from this point onward. To compute 2222^{5555} we now need to ask how many sixes can we pull out of 5555. That is we need to know what is the remainder of 5555 when we divide by six. For six note that the remainder of powers of ten look like

$$\begin{aligned} 10 &\equiv 4 \pmod 6 \quad \text{so} \\ 10^2 &\equiv 16 \equiv 4 \pmod 6 \quad \text{and} \\ 10^3 &\equiv 4^3 \equiv 64 \equiv 4 \pmod 6\,, \end{aligned}$$

which mans that using Equation 916 we have

$$1111 \equiv 4 + 4 + 4 + 1 \equiv 13 \equiv 1 \pmod 6\,.$$

Thus

$$5555 \equiv 5 \cdot 1 \equiv 5 \pmod 6\,.$$

This means that we can write $5555 = 6q + 5$ for some q and thus since

$$2222^{5555} = 2222^{6q} \cdot 2222^5 \,,$$

Thus

$$2222^{5555} \equiv 1 \cdot 5 \equiv 5 \pmod{7} \,. \tag{917}$$

We can now think about the remainders of powers of 5555. Using the above we have

$$5555 \equiv 4 \bmod 7 \,.$$

so that

$$\begin{aligned} 5555^2 &\equiv 16 \equiv 2 \bmod 7 \\ 5555^3 &\equiv 64 \equiv 1 \bmod 7 \,. \end{aligned}$$

and the pattern will repeat cyclically from this point onward. To compute 5555^{2222} we now need to ask how many threes can we pull out of 2222. That is we need to know what is the remainder of 2222 when we divide by three. For three note that the remainder of powers of ten look like

$$\begin{aligned} 10 &\equiv 1 \pmod{3} \quad \text{so that} \\ 10^p &\equiv 1 \pmod{3} \,, \end{aligned}$$

for all $p \geq 1$. Thus using Equation 916 we have

$$1111 \equiv 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 + 1 \equiv 4 \equiv 1 \pmod{3} \,.$$

Thus

$$2222 \equiv 2 \cdot 1 \equiv 2 \pmod{3} \,.$$

Thus we have shown that $2222 = 3q + 2$ so that

$$5555^{2222} = 5555^{3q} \cdot 5555^2 \,.$$

As

$$\begin{aligned} 5555^{3q} &\equiv 1 \pmod{7} \quad \text{and} \\ 5555^2 &\equiv 2 \pmod{7} \,, \end{aligned}$$

we have by taking their product that

$$5555^{2222} \equiv 2 \pmod{7} \,. \tag{918}$$

Using Equation 917 and 918 we see that

$$2222^{5555} + 5555^{2222} \equiv 5 + 2 \equiv 7 \equiv 0 \pmod{7} \,,$$

this shows that the requested number is divisible by seven.

Testing Question A.4

To start we notice that

$$\begin{aligned} 47 &\equiv 3 \pmod{11} \\ 47^2 &\equiv 3^2 \equiv 9 \pmod{11} \\ 47^3 &\equiv 27 \equiv 5 \pmod{11} \\ 47^4 &\equiv 81 \equiv 4 \pmod{11} \\ 47^5 &\equiv 243 \equiv 1 \pmod{11} \quad \text{as} \quad 11 \cdot 22 = 242 \,. \end{aligned}$$

Thus additional powers of 47^5 will all have $(47^5)^d \equiv 1 \pmod{11}$. We now need to know what is the remainder of 37^{27} when divided by five. To determine this we consider

$$\begin{aligned} 37 &\equiv 2 \pmod{5} \\ 37^2 &\equiv 2^2 \equiv 4 \pmod{5} \\ 37^3 &\equiv 8 \equiv 3 \pmod{5} \\ 37^4 &\equiv 16 \equiv 1 \pmod{5} \,. \end{aligned}$$

Thus additional powers of 37^4 will all have $(37^4)^d \equiv 1 \pmod{5}$. We now need to know what is the remainder of 27 when divided by four. As $27 = 4 \cdot 6 + 3$ we have that

$$37^{27} = (37^4)^6 \cdot 37^3 \,.$$

Then using the facts that

$$\begin{aligned} (37^4)^6 &\equiv 1 \pmod{5} \quad \text{and} \\ 37^3 &\equiv 3 \pmod{5} \,, \end{aligned}$$

the product of the two equations above gives

$$37^{27} \equiv 37^3 \equiv 3 \pmod{5} \,,$$

thus $37^{27} = 5d + 3$ for some d.

Taking N to be the number given in the problem we have shown that

$$N = 47^{5d+3} = (47^5)^d \cdot (47^3) \,.$$

As $(47^5)^d \equiv 1 \pmod{11}$ we have that

$$N \equiv 47^3 \equiv 5 \pmod{11} \,,$$

using results found above. Note that I think there is an error in the solution found in the back of the book where they claim $37^{27} \equiv 2 \pmod{5}$ when in fact $37^{27} \equiv 3 \pmod{5}$.

Testing Question A.5

To start we notice that

$$9 \equiv 9 \pmod{11}$$
$$9^2 \equiv 81 \equiv 4 \pmod{11}\,.$$

If we multiply this expression by three we get

$$3 \cdot 9^2 \equiv 3 \cdot 4 \equiv 12 \equiv 1 \pmod{11}\,. \tag{919}$$

But $3 \cdot 9^2 = 3 \cdot (3^2)^2 = 3^5$. Now we are asked to consider

$$9^{1990} = 3^{2(1990)} = 3^{3980} = (3^5)^{796}\,.$$

Now Equation 919 gives $3^5 \equiv 1 \pmod{11}$ and so from the above we have $9^{1990} \equiv 1 \pmod{11}$.

Testing Question A.6

Let x be the three digit remainder when dividing n by 1000 i.e. $n \equiv x \pmod{1000}$.

To start we note that n is the product of odd integers and will thus will itself be an odd number. Now if we write out some of the factors of n we see that it is

$$n = 3 \times 7 \times 11 \times 15 \times 19 \times 23 \times 27 \times 31 \times 35 \times 39 \times 43 \times 47 \times 51 \times 55 \times \cdots \times 1999 \times 2003\,.$$

Thus n has a factor of $15 \times 35 \times 55$ which is proportional to $5^3 = 125$. Multiples of 125 take the form

$$\begin{aligned}
1 \cdot 125 &= 125\\
2 \cdot 125 &= 250\\
3 \cdot 125 &= 375\\
4 \cdot 125 &= 500\\
5 \cdot 125 &= 625\\
6 \cdot 125 &= 750\\
7 \cdot 125 &= 875\\
8 \cdot 125 &= 1000\,,
\end{aligned}$$

and larger multiples will have repeating last three digits. Thus the only possible three final digits for n (taken from the *odd* ones above) are

$$x \in \{125\,, 375\,, 625\,, 875\}\,.$$

As $n \equiv x \pmod{1000}$ we have $1000 \mid (n - x)$ or as $1000 = 2^3 \cdot 5^3$ we have that $2^3 \mid (n - x)$ i.e. $8 \mid (n - x)$. Thus $n \equiv x \pmod{8}$. This means that the remainder of n (when divided by eight) *and* the remainder of the three digit number x (when divided by eight) must be the same.

From the form of n we can actually compute the remainder of n by eight. To do that note that we can write n as

$$n = \prod_{k=0}^{500}(4k+3) = \prod_{k=0;k \text{ even}}^{500}(4k+3) \prod_{k=1;k \text{ odd}}^{499}(4k+3)$$
$$= \prod_{j=0}^{250}(4(2j)+3)\prod_{j=0}^{249}(4(2j+1)+3) = \prod_{j=0}^{250}(8j+3)\prod_{j=0}^{249}(8j+7)\,.$$

Each factor in the first product has a remainder three (when divided by eight) and each factor in the second product has a remainder of seven (when divided by eight). Thus

$$n \equiv 3^{351} \times 7^{250} \pmod 8\,.$$

Thus

$$n \equiv (21)^{250} \cdot 3 \pmod 8\,. \tag{920}$$

Now $21 \equiv 5 \pmod 8$ so $21^2 \equiv 5^2 \equiv 25 \equiv 1 \pmod 8$ thus

$$21^{250} \equiv (21^2)^{125} \equiv 1^{125} \equiv 1 \pmod 8\,.$$

Using this in Equation 920 we have $n \equiv 1 \cdot 3 \equiv 3 \pmod 8$. From this result, and the above we need to have $x \equiv 3 \pmod 8$. From the choices of x above only $x = 875$ has this property.

Testing Question A.7

From the statement $n \equiv 1 \pmod 5$ we have that $n = 5k + 1$ for some k. If we evaluate the remainder of both sides with respect to seven and since we are told that $n \equiv 2 \pmod 7$ we must have that $5k \equiv 1 \pmod 7$. Thus $5k = 7l + 1$ for some l. The smallest l (and thus k) that satisfies this is $l = 2$ (with $k = 3$) so that the smallest n satisfying the first two conditions is

$$n_2 = 5k + 1 = 7l + 2 = 16\,.$$

To satisfy the second two conditions our n must be larger thus

$$n = n_2 + (5 \times 7)m = 16 + 35m\,,$$

for some m. To have $n \equiv 3 \pmod 9$ based on the above (since $16 \equiv 7 \pmod 9$) we need

$$7 + 35m \equiv 3 \pmod 9 \quad \text{or} \quad 35m \equiv -4 \pmod 9 \quad \text{or} \quad 35m = -4 + 9p\,,$$

for some p. Starting with integer values of $m \in \{1, 2, 3, \dots\}$ and solving for p the smallest integer p is one where $p = 16$ so that $m = 4$ thus our n takes the form

$$n = 16 + 35(4 + 9p') = 156 + 315p'\,,$$

for some p'. Finally to have $n \equiv 4 \pmod{11}$ since $156 \equiv 2 \pmod{11}$ we need to have $315p' \equiv 2 \pmod{11}$ thus

$$315p' = 11q + 2\,,$$

for some q. Starting with integer values of $p' \in \{1, 2, 3, \dots\}$ and solving for q the smallest integer q is one where $q = 143$ so that $p' = 5$ thus our n takes the form

$$n = 156 = 315(5) = 1731\,.$$

Testing Question A.8

Part (a): To start we notice that

$$\begin{aligned} 2 &\equiv 2 \pmod{7} \\ 2^2 &\equiv 2^2 \equiv 4 \pmod{7} \\ 2^3 &\equiv 8 \equiv 1 \pmod{7} \quad \text{thus} \\ 2^{3p} &\equiv 1^p \equiv 1 \pmod{7}\,. \end{aligned}$$

Multiplying the last equation above by 2 and 2^2 we get

$$\begin{aligned} 2^{3p+1} &\equiv 2 \pmod{7} \\ 2^{3p+2} &\equiv 2^2 \equiv 4 \pmod{7}\,. \end{aligned}$$

As all positive integers n can be written as $3p$, $3p+1$, or $3p+2$ from the above we see that the only form where $2^n - 1 \equiv 0 \pmod{7}$ is when $n = 3p$ for $p \geq 1$.

Part (b): Using the results from the above we have

$$\begin{aligned} 2^{3p} + 1 &\equiv (1+1) \equiv 2 \pmod{7} \\ 2^{3p+1} + 1 &\equiv (2+1) \equiv 3 \pmod{7} \\ 2^{3p+2} + 1 &\equiv (4+1) \equiv 5 \pmod{7}\,. \end{aligned}$$

None of these are zero indicating that there are no n where $2^n + 1$ is divisible by seven.

Testing Question A.9

Let N be the number given. Then note that we can write N as

$$N = (3 \cdot 7 \cdot 17)^{1999} \cdot 7 \cdot 17^2 = (357)^{1999} \cdot 7 \cdot 17^2\,. \tag{921}$$

The units digit is the value of x where $N \equiv x \pmod{10}$. Note that

$$\begin{aligned} 357 &\equiv 7 \pmod{10} \\ 357^2 &\equiv 7^2 \equiv 9 \pmod{10} \\ 357^3 &\equiv 7^3 \equiv 7 \cdot 9 \equiv 3 \pmod{10}\,, \end{aligned}$$

which is true as seven and ten are relatively prime. Using this same "trick" we see that

$$357^4 \equiv 7 \cdot 3 \equiv 21 \equiv 1 \pmod{10}.$$

Based on Equation 921 we need to know the remainder of 1999 when divided by four

$$1999 = 2000 - 1 = 4(500) - 1 = 4(499) + 4 - 1 = 4(499) + 3 \quad \text{so} \quad 1999 \equiv 3 \pmod{4}.$$

Then from this $357^{1999} = (357^4)^{499} \cdot 357^3$ we have

$$357^{1999} \equiv 1 \cdot 3 \equiv 3 \pmod{10}.$$

This gives "part" of N. To get all the remaining pieces needed for N note that

$$\begin{aligned} 7 &\equiv 7 \pmod{10} \\ 17 &\equiv 7 \pmod{10} \\ 17^2 &\equiv 7^2 \equiv 49 \equiv 9 \pmod{10}. \end{aligned}$$

Thus with these results we have all we need to compute $N \equiv x \pmod{10}$. We find

$$N \equiv 3 \cdot 7 \cdot 9 \equiv 189 \equiv 9 \pmod{10}.$$

Testing Question A.10

Note that

$$2^{10} \equiv 1024 \equiv 24 \pmod{100}.$$

Thus

$$(2^{10})^2 \equiv 24^2 \equiv 576 \equiv 76 \pmod{100},$$

and

$$(2^{10})^4 \equiv 76^2 \equiv 5776 \equiv 76 \pmod{100}.$$

The pattern continues so if we continued squaring we would see that

$$(2^{10})^{2p} = 76 \pmod{100},$$

or

$$2^{20p} \equiv 76 \pmod{100}.$$

For the $N = 2^{999}$ we start with since $999 = 1000 - 1 = 50(20) - 1 = 49(20) + 19$ we see that

$$N = 2^{20(29)} \times 2^{10} \times 2^9 \equiv 76 \times 24 \times 2^9 \pmod{100},.$$

Now

$$76 \times 24 \equiv 1824 \equiv 24 \pmod{100},$$

and

$$2^9 \equiv 2^{-1} \times 2^{10} \equiv 2^{-1} \times 1024 \equiv 512 \equiv 12 \pmod{100}.$$

Thus

$$N \equiv 24 \times 12 \equiv 288 \equiv 88 \pmod{100}.$$

Testing Question B.1

Let $N = 14^{14^{14}}$. Then I compute

$$\begin{aligned}
14 &\equiv 14 \pmod{100} \\
14^2 &\equiv 196 \equiv 96 \pmod{100} \\
14^3 &\equiv 14(96) \equiv 1344 \equiv 44 \pmod{100} \\
14^4 &\equiv 14(44) \equiv 616 \equiv 16 \pmod{100} \\
14^5 &\equiv 14(16) \equiv 224 \equiv 24 \pmod{100} \\
14^6 &\equiv 14(24) \equiv 336 \equiv 36 \pmod{100} \\
14^7 &\equiv 14(36) \equiv 504 \equiv 4 \pmod{100} \\
14^8 &\equiv 14(4) \equiv 56 \pmod{100} \\
14^9 &\equiv 14(56) \equiv 784 \equiv 84 \pmod{100} \\
14^{10} &\equiv 14(84) \equiv 1176 \equiv 76 \pmod{100} \\
14^{11} &\equiv 14(76) \equiv 1064 \equiv 64 \pmod{100} \\
14^{12} &\equiv 14(64) \equiv 896 \equiv 96 \pmod{100} \,,
\end{aligned}$$

and the sequence repeats from this point forwards. What this means is that every twelfth power of 14 will give us a remainder of 96 (when divided by 100). Thus if we can write

$$14^{14} = 12q + r \,,$$

we can simplify the remainder of N (when divided by 100) greatly. Note that

$$14 \equiv 2 \pmod{12} \tag{922}$$
$$14^2 \equiv 2^2 \equiv 4 \pmod{12} \tag{923}$$
$$14^3 \equiv 2^3 \equiv 8 \pmod{12} \tag{924}$$
$$14^4 \equiv 2^4 \equiv 16 \equiv 4 \pmod{12} \,, \tag{925}$$

and the pattern repeats. Now these equations mean that

$$14^{2p} \equiv 4^p \equiv 4 \pmod{12} \,.$$

If we take $p = 7$ we get $14^{14} \equiv 4 \pmod{12}$ so that we have

$$14^{14} = 12q + 4 \,.$$

This means that

$$N = 14^{12q+4} = 14^{12q} \cdot 14^4 \,.$$

Now from the above we have

$$\begin{aligned}
(14^{12})^9 &\equiv 96 \pmod{100} \\
14^4 &\equiv 16 \pmod{100} \,,
\end{aligned}$$

so their product is then

$$N \equiv 96 \cdot 16 \equiv 1536 \equiv 36 \pmod{100} \,.$$

Testing Question B.2

For the numbers given note that

$$\begin{aligned}
257 &\equiv 7 \pmod{50} \\
257^2 &\equiv 7^2 \equiv 49 \pmod{50} \\
257^3 &\equiv 7^3 \equiv 7(49) \equiv 343 \equiv 43 \pmod{50} \\
257^4 &\equiv 7^4 \equiv 7(43) \equiv 301 \equiv 1 \pmod{50}\,.
\end{aligned}$$

From this we have that $257^{4p} \equiv 1^p \equiv 1 \pmod{50}$. Now as $33 = 4 \cdot 8 + 1$ we have

$$257^{33} \equiv (257^4)^8 \cdot 257^1 \equiv 1^8 \cdot 257^1 \equiv 7 \pmod{50}\,.$$

This means that

$$257^{33} + 46 \equiv 7 + 46 \equiv 53 \equiv 3 \pmod{50}\,.$$

Using this we have that

$$\begin{aligned}
(257^{33} + 46)^2 &\equiv 3^2 \equiv 9 \pmod{50} \\
(257^{33} + 46)^3 &\equiv 3(9) \equiv 27 \pmod{50} \\
(257^{33} + 46)^4 &\equiv 3(27) \equiv 81 \equiv 31 \pmod{50} \\
(257^{33} + 46)^5 &\equiv 3(31) \equiv 93 \equiv 43 \pmod{50} \\
(257^{33} + 46)^6 &\equiv 3(43) \equiv 129 \equiv 29 \pmod{50} \\
(257^{33} + 46)^7 &\equiv 3(29) \equiv 87 \equiv 37 \pmod{50} \\
(257^{33} + 46)^8 &\equiv 3(37) \equiv 111 \equiv 11 \pmod{50} \\
(257^{33} + 46)^9 &\equiv 3(11) \equiv 33 \pmod{50} \\
(257^{33} + 46)^{10} &\equiv 3(33) \equiv 99 \equiv -1 \pmod{50}\,.
\end{aligned}$$

This means that

$$(257^{33} + 46)^{20} \equiv ((257^{33} + 46)^{10})^2 \equiv (-1)^2 \equiv 1 \pmod{50}\,,$$

and thus

$$((257^{33} + 46)^{10})^{26} \equiv ((257^{33} + 46)^{10})^{20}((257^{33} + 46)^{10})^6 \equiv 1 \cdot 29 \equiv 29 \pmod{50}\,.$$

Lecture 19: Decimal Representation of Integers

Notes on Example 5

There is a typo in this example in that the number $11\cdots 1122\cdots 22$ is the number $11\cdots 11$ *times* 10^n (so that it is followed by n zeros) *plus* the number $22\cdots 22$. Then using the decimal representation of a number with repeating digits our number is

$$\frac{1}{9}(10^n-1)\times 10^n+\frac{2}{9}(10^n-1)\,,$$

which factors as

$$\frac{1}{9}(10^n-1)(10^n+2)\,.$$

The rest of the example seems correct.

Testing Question A.1

We are told that

$$\overline{abc}=a10^2+b10+c=37n\,, \tag{926}$$

for some n. As $100\leq\overline{abc}<1000$ we have

$$\left\lceil\frac{100}{37}\right\rceil\leq n\leq\left\lfloor\frac{1000}{37}\right\rfloor\,,$$

or

$$3\leq n\leq 27\,.$$

There are $27-3+1=25$ three digit numbers of this type.

One *computational* way to solve this problem is then to compute *all* 25 three digit numbers of the form $37n$ for $n\in\{3,4,\ldots,26,27\}$. For each number computed we can determine a, b, and c by "digit assignment". Then we can form the three digit number $\overline{bca}$ and verify that it is also divisible by 37. For example the second number of this form is $4\times 37=148$ so $a=1$, $b=4$, and $c=8$. Thus $\overline{bca}=481$ which is 13×37 and is thus divisible by 37.

Another way to solve this is to use Equation 926 in the form

$$10b=37n-100a-c\,.$$

If we put this into $\overline{bca}=100b+10c+a$ we get

$$\begin{aligned}\overline{bca}&=100b+10c+a=10(37n-100a-c)+10c+a\\&=370n-10^3a-10c+10c+a=37\cdot 10\cdot n-999a\\&=2\cdot 5\cdot 37n-3^3\cdot 37a=37(10n-27a)\,,\end{aligned}$$

which is obviously divisible by 37.

Testing Question A.2

Let our number be a $n+1$ digit number so that it can be written as $6 \cdot 10^n + x$ where x is a n digit numbers. Then we are told that

$$6 \cdot 10^n + x = 5x \,.$$

Solving for x we get

$$x = \frac{10^n}{4} \,.$$

Now $n \geq 2$ for x to be an integer. Based on that let $n = m + 2$ where $m \geq 0$. Then we have

$$x = \frac{100 \cdot 10^m}{4} = 25 \cdot 10^m \,.$$

This means that our number is

$$6 \cdot 10^{m+2} + 25 \cdot 10^m = 625 \cdot 10^m \,.$$

This is the number 625 with m zeros following it.

Testing Question A.3

Let $x = \overline{abc}$ and $\tilde{x} = \overline{cba}$ with

$$a + b + c = 21 \,. \tag{927}$$

Then we are told that

$$\tilde{x} - x = 495 \,.$$

Using the expressions for x and $\tilde{x}$ in terms of a, b, and c the above is

$$(100c + 10b + a) - (100a + 10b + c) = 99(c - a) = 495 \,.$$

This means that $c - a = 5$. Solving this for c and putting it into Equation 927 we get

$$b = 16 - 2a \,.$$

As $0 \leq b \leq 9$ and using the above we have

$$0 \leq 16 - 2a \leq 9 \quad \text{so} \quad \frac{7}{2} \leq a \leq 8 \,.$$

In addition as $0 \leq c \leq 9$ we have

$$0 \leq a + 5 \leq 9 \quad \text{so} \quad -5 \leq a \leq 4 \,.$$

The only common value for a in these ranges is $a = 4$. In that case $a = 4$, $b = 8$, and $c = 9$ so our number is 489.

Testing Question A.4

Call numbers of this type N_n where N_n has $2n+1$ digits for $n \geq 1$. From the given examples we see that these numbers N_n are of the form

$$N_n = 7 \cdot 10^{2n} + 1 \cdot \sum_{k=1}^{n-1} 10^{k+n} + 2 \cdot 10^n + 8 \cdot \sum_{k=1}^{n-1} 10^k + 9\,.$$

We can evaluate the above sums using $\sum_{k=1}^{n} r^k = \frac{r^{n+1}-r}{r-1}$ where we get

$$\begin{aligned} N_n &= 7 \cdot 10^{2n} + 10^n \left(\frac{10^n - 10}{9}\right) + 2 \cdot 10^n + 8 \left(\frac{10^n - 10}{9}\right) + 9 \\ &= 7 \cdot 10^{2n} + \frac{1}{9} 10^{2n} - \frac{1}{9} 10^{n+1} + 2 \cdot 10^n + \frac{8}{9} 10^n - \frac{8}{9} 10 + 9 \\ &= \frac{1}{9} \left(64 \cdot 10^{2n} + 16 \cdot 10^n + 1\right) = \frac{1}{9} (8 \cdot 10^n + 1)^2 \\ &= \left(\frac{8 \cdot 10^n + 1}{3}\right)^2 . \end{aligned}$$

This will be a perfect square if $8 \cdot 10^n + 1$ is divisible by three.

One way to show that is to note that $8 \cdot 10^n + 1$ is an eight followed by n zeros plus one which has digits that sum to $8 + 1 = 9$ which is divisible by three so the original number is.

Another way to see this is to write this number as

$$\frac{8(10^n - 1) + 8 + 1}{3} = \frac{24(10^n - 1)}{9} + 3 = 24 \left(\frac{10^n - 1}{9}\right) + 3\,.$$

The number $\frac{10^n-1}{9}$ is the sequence of $n - 1$ ones. Thus the full number above is a natural number.

Testing Question A.5

Let the number n have N digits with a last digit of d such that $0 \leq d \leq 9$. From what we are told about n we have that

$$\left(\frac{n - d}{10}\right) + d \cdot 10^{N-1} = 5n\,.$$

If we multiply this by ten we get

$$n - d + d \cdot 10^N = 50n\,.$$

Solving for n we have

$$n = \frac{d}{49}(10^N - 1)\,.$$

To make n be as small as possible means to take d and N as small as possible. For n to be a natural number means that $d(10^N - 1)$ must have 49 as a factor. If we take $d = 7$ then we want the smallest N that has $10^N - 1$ as a factor. This is $N = 6$ so that

$$n = \frac{7(10^6 - 1)}{49} = 142857\,.$$

Note the answer in the back of the book is wrong (its solution does not satisfy the conditions of the question) and the correct statement (to use their logic) is to have

$$5n = d \cdot 10^{m+1} + x\,,$$

where the exponent on the 10 is $m + 1$ and not m.

Testing Question A.6

Let $n = \overline{abcd}$ and we are told that

$$n + a + b + c + d = 2001\,, \tag{928}$$

with $1 \le a \le 9$. Expanding n we have

$$1000a + 100b + 10c + d + a + b + c + d = 2001\,,$$

or

$$1001a + 101b + 11c + 2d = 2001\,. \tag{929}$$

We must have

$$a + b + c + d \le 4 \cdot 9 = 36\,,$$

so using $a + b + c + d = 2001 - n$ we get that

$$2001 - n \le 36 \quad \text{so} \quad n \ge 1965\,.$$

Also as we have $a + b + c + d \ge 1 + 0 + 0 + 0 = 1$ we have

$$2001 - n \ge 1 \quad \text{so} \quad n \le 2000\,.$$

The number $n = 2000$ does not satisfy the conditions of the problem and thus $a = 1$. As $n \ge 1965$ we know that $b = 9$. Using these Equation 928 becomes

$$n + c + d = 1991\,, \tag{930}$$

and Equation 929 becomes

$$11c + 2d = 91\,. \tag{931}$$

As $n \ge 1965$ we know that $c \ge 6$. If we table c for $6 \le c \le 9$ we can compute d using the above and see which values of d are consistent with the above equations. This is done in Table 27. The only consistent choice is $c = 7$ which gives $d = 7$ for a number $n = 1977$.

c	$d = (91 - 11c)/2$
6	12.5
7	7
8	1.5
9	-4

Table 27: Values of c and then d computed using Equation 931.

Testing Question A.8

Our original number n is $n = \overline{abcd}$ and our "adjusted" number is $n' = \overline{cdab}$. We are told that

$$n' - n = 5940\,. \tag{932}$$

Expanding n and n' in terms of their digits we get Equation 932 equivalent to

$$1000c + 100d + 10a + b - (1000a + 100b + 10c + d) = 5940\,,$$

which can be simplified to

$$10(c - a) + (d - b) = 60\,. \tag{933}$$

As a, b, c, and d are digits and thus bounded we can use that fact to bound the expressions $c - a$ and $d - b$. For example we know that $0 \le c \le 9$ and that $1 \le a \le 9$. The later means that $-9 \le -a \le -1$ so adding to the inequality for c we have

$$-9 \le c - a \le 8\,. \tag{934}$$

The same type of considerations for d and b give us

$$-8 \le d - b \le 9\,. \tag{935}$$

Solving Equation 933 for $d - b$ and putting that expression into Equation 935 gives

$$-8 \le 60 - 10(c - a) \le 9\,,$$

or simplifying we get bounds for $c - a$ of

$$5.1 \le c - a \le 6.8\,.$$

As $c - a$ must be an integer we have learned that $c - a = 6$. Using that in Equation 933 we learn that $d = b$.

This means that n takes the form $\overline{adcd}$ where c is the digit $a + 6$. For c to be $c \le 9$ we have that $a \in \{1, 2, 3\}$. To find the smallest number n we start with $a = 1$. In that case then $c = 7$ and our number is $\overline{1d7d}$. As n must be odd we know that $d \in \{1, 3, 5, 7, 9\}$ and we will want this number to have a remainder of eight when divided by nine. Taking $d \in \{1, 3, 5, 7, 9\}$ we find that $d = 9$ has this property and thus $n = 1979$.

d	$2d+1 \pmod{10}$	$2d \pmod{10}$
0	1	0
1	3	2
2	5	4
3	7	6
4	9	8
5	1	0
6	3	2
7	5	4
8	7	6
9	9	8

Table 28: Values of d, the ones digit of $2d+1$, and the number $2d+1 \pmod{10}$ and the ones digit of $2d$.

Testing Question A.9

Let our original number be denoted $x = \overline{abcde}$ with the "adjusted" number $y = \overline{ABCDE}$. Here the digit in y is equal to the same digit in x if that digit is not a five or a two and the replacement as suggested in the problem is done if it is.

We are told that x is odd so that $y = 2(x+1)$ will be an even number. This means that the last digit of y i.e. E will have to be $E \in \{2, 4, 6, 8\}$. If E was any of these numbers (but two) then e would be the same number and x would not be an odd number. Thus we know that $E = 2$ and thus $e = 5$. Thus our numbers look like

$$
\begin{aligned}
x &= \overline{abcd5} \\
y &= \overline{ABCD2}\,.
\end{aligned}
$$

Now the last digit of $x+1$ will be a six which when multiplied by two will be 12 and thus has a carry of one. This means that the tens digit of y (i.e. D) will be the ones digit of the number $2d+1$. Based on different values for d the digit D will take on different values. See the second column in Table 28 where we compute these. Because of how D is obtained from d (it is copied if $d \notin \{2, 5\}$) only two choices in that table give possible values. They are $d = 2$ (where $D = 5$) and $d = 9$ (where $D = 9$).

Case 1: Consider the case where we assume that $d = 2$ (so that $D = 5$). Then our numbers look like

$$
\begin{aligned}
x &= \overline{abc25} \\
y &= \overline{ABC52}\,.
\end{aligned}
$$

Adding one to x and multiplying by two we see that C is the units digit of $2c$ (because there is no carry in this case). Looking at the third column in Table 28 we see that $c = 0$ so that $C = 0$. Continuing to build the number y in this way we see that we would need to have $A = B = 0$ and thus y does not have five digits and we have not found a solution.

Case 2: Consider the case where we assume that $d = 9$ (so that $D = 9$). Then our numbers look like

$$x = \overline{abc95}$$
$$y = \overline{ABC92}\,.$$

Adding one to x and multiplying by two we see that C is the units digit of $2c + 1$. Looking at the second column in Table 28 we see that $c \in \{2, 9\}$. From the arguments made before if $c = 2$ then $A = B = 0$ and the number will not be five digits. This means that $c = 9$ so that $C = 9$ and our numbers look like

$$x = \overline{ab995}$$
$$y = \overline{AB992}\,.$$

These arguments continue one more time to get

$$x = \overline{a9995}$$
$$y = \overline{A9992}\,.$$

At this point we still have $a \in \{2, 9\}$ but if $a = 9$ then we need a carry and the number y has six digits. Thus $a = 2$ (so $A = 5$) and our two numbers are

$$x = 29995$$
$$y = 59992\,.$$

Testing Question A.10

We want to maximize

$$E \equiv \frac{\overline{abc}}{a+b+c} = \frac{100a + 10b + c}{a+b+c}\,.$$

Let

$$a' = \frac{a}{a+b+c}$$
$$b' = \frac{b}{a+b+c}$$
$$c' = \frac{c}{a+b+c}\,,$$

then our expression E becomes

$$E = 100a' + 10b' + c'\,.$$

This is linear in a', b', and c' and will have its optimums at the "corners" of the convex polytope defined by

$$0 \le a' \le 1$$
$$0 \le b' \le 1$$
$$0 \le c' \le 1\,.$$

If we form a table of the value of E at each of the four "corners" of the convex polytope we get Table 29. From that table we see that the maximum is the value 100. Note that this can be achieved with $\overline{abc} = 100$.

a'	b'	c'	$100a' + 10b' + c'$
0	0	0	0
1	0	0	100
0	1	0	10
0	0	1	1

Table 29: Values of a', b', and c' and the objective function $100a' + 10b' + c'$.

Testing Question B.1

We want to find $n = \overline{abc}$ such that

$$\overline{abc} = (a + b + c)^3 .$$

To have n three digits we need

$$100 \leq n \leq 999 \quad \text{or} \quad 100 \leq (a + b + c)^3 \leq 999 .$$

Taking the cube root this means that

$$100^{1/3} \leq a + b + c \leq 999^{1/3} \quad 4.641589 \leq a + b + c \leq 9.996666 .$$

As a, b, and c are integers we can conclude that

$$5 \leq a + b + c \leq 9 .$$

Specifically $a + b + c \in \{5, 6, 7, 8, 9\}$. If we cube each of these numbers we get the numbers

$$\{125, 216, 343, 512, 729\} .$$

Summing the digits in each of these numbers gives

$$\{8, 9, 10, 8, 18\} .$$

The only choice that is consistent is when $a + b + c = 8$ so $n = (a + b + c)^3 = 512$.

Testing Question B.3

Consider the two numbers $n = \overline{ab} = 10a + b$ and $n' = \overline{ba} = 10b + a$ and their ratio

$$\frac{n}{n'} = \frac{10a + b}{10b + a} .$$

As we have a quotient and a remainder we know that $n > n'$ so $10a + b > 10b + a$ or

$$10(a - b) > a - b \quad \text{so} \quad a > b .$$

As the quotient and remainder are the same (call them both q) we have

$$10a + b = q(10b + a) + q .$$

Solving this for b we find

$$b = \frac{(10-q)a - q}{10q - 1} \,. \tag{936}$$

Now as both $10 \leq n \leq 99$ and $10 \leq n' \leq 99$ we have that

$$\frac{10}{99} \leq \frac{n}{n'} \leq \frac{99}{10} \quad \text{so} \quad 0.1010101 \leq \frac{n}{n'} \leq 9.9 \,.$$

As the quotient q must be an integer we know from the above that $1 \leq q \leq 9$.

We can find our numbers n and n' if we take $q \in \{1, 2, \ldots, 8, 9\}$ and then $a \in \{1, 2, \ldots, 8, 9\}$ and use Equation 936 to compute b. If b is an integer then we have found our numbers. We can do this in the simple **R** code

```
for( q in 1:9 ){
    for( a in 1:9 ){
        b = ((10 - q)*a - q)/(10*q-1)
        if( b!=0 && abs(round(b)-b)<1.e-6 ){
            print(sprintf('q= %d; a= %d; b= %d', q, a, b))
        }
    }
}
```

Which gives

```
[1] "q= 2; a= 5; b= 2"
```

This corresponds to $n = 52$ and $n' = 25$.

Lecture 20: Perfect Square Numbers

Notes on Basic Properties of Perfect Square Numbers

Recall from earlier in this chapter that if n is a perfect square then

- When dividing n^2 by two, three, or four the remainder is zero or one.
- When dividing n^2 by eight the remainder is zero, one, or four.

Part (IV): These are proved as follows. Now to determine the possible values of $n^2 \pmod 2$ and $n^2 \pmod 4$ consider that $n^2 = (2m)^2$ or $n^2 = (2m+1)^2$ for some m. This means that

$$\begin{aligned} n^2 &= (2m)^2 = 4m^2 \\ n^2 &= (2m+1)^2 = 4m^2 + 4m + 1\,. \end{aligned}$$

From this we see that

$$\begin{aligned} n^2 &\equiv 0 \pmod 2 \quad \text{or} \\ n^2 &\equiv 1 \pmod 2 \quad \text{and} \\ n^2 &\equiv 0 \pmod 4 \quad \text{or} \\ n^2 &\equiv 1 \pmod 4. \end{aligned}$$

Next to determine the possible values of $n^2 \pmod 3$ consider

$$\begin{aligned} n^2 &= (3m)^2 = 9m^2 \\ n^2 &= (3m-1)^2 = 9m^2 - 6m + 1 \\ n^2 &= (3m+1)^2 = 9m^2 + 6m + 1\,. \end{aligned}$$

In the first case we have that $n^2 \equiv 0 \pmod 3$ or in the second case that $n^2 \equiv 1 \pmod 3$.

Part (V): To determine the possible values of $n^2 \pmod 8$ consider the forms that n can take. We have $n \in \{4m, 4m \pm 1\,, 4m \pm 2\}$ and thus

$$\begin{aligned} n^2 &= (4m)^2 = 16m^2 \\ n^2 &= (4m \pm 1)^2 = 16m^2 \pm 8m + 1 \\ n^2 &= (4m \pm 2)^2 = 16m^2 \pm 16m + 4\,. \end{aligned}$$

From this we see that $n^2 \equiv 0 \pmod 8$, $n^2 \equiv 1 \pmod 8$, or $n^2 \equiv 4 \pmod 8$.

Part (VI): When $m = 10a + b$ for m^2 we have

$$m^2 = 100a^2 + 20ab + b^2\,.$$

Now because

$$100a^2 + 20ab = 10(10a^2 + 2ab)\,,$$

this number must have a zero ones digit and because

$$\frac{100a^2 + 20ab}{10} = 2(5a^2 + ab)\,,$$

and is thus even and thus ends with an even digit. Thus the number $100a^2+20ab$ has an even tens digit. In order for m^2 to be odd the units digit from b^2 must be odd so $b \in \{1,3,5,7,9\}$. For each of these b values b^2 has an even tens digit and thus when b^2 is added to $100a^2+20ab$ we will have a number with an even tens digit.

Part (VII): Now b cannot be an odd digit or else from the above we have an odd perfect square with an even tens digit. If b is even then $b \in \{2,4,6,8\}$ only if $b \in \{4,6\}$ will the tens digit of b^2 be odd and thus have the tens digit of m^2 be odd. Note that in either case where $b \in \{4,6\}$ the units digit of b^2 is six.

Notes on Example 1

Now in this example note that the number $5k+2$ has units digit of two or seven but a perfect square number m^2 can only have units digits of 0, 1, 4, 5, 6, or 9 and thus $5k+2$ cannot be a perfect square number.

The number $5k+3$ has units digit of three or eight but m^2 can only have units digits of 0, 1, 4, 5, 6, or 9 and thus $5k+3$ cannot be a perfect square.

Notes on Example 4

We ask if the number

$$1 + n(n+1)(n+2)(n+3)\,,$$

is a perfect square. Expanding we can write this as

$$n^4 + 6n^3 + 11n^2 + 6n + 1\,.$$

If this is a perfect square for all n it will factor into expressions like

$$(n^2 + An + 1)^2\,,$$

for some integer A. Expanding this we get

$$n^4 + 2An^3 + (2 + A^2)n^2 + 2An + 1\,.$$

This will equal the above if $2A = 6$ and then $2 + A^2 = 11$ which is true and we have the given expression a perfect square.

Next expressing the given number "symmetrically" we consider

$$(m-2)^2 + (m-1)^2 + m^2 + (m+1)^2 + (m+2)^2\,,$$

which we can expand and simplify to

$$5m^2 + 10 = 5(m^2 + 2)\,.$$

For this to be a perfect square $m^2 + 2$ must be divisible by five. This means that the units digit of m^2 would have to be a three or an eight. As the ones digit of a perfect square can only be the digits 0, 1, 4, 5, 6, or 9 this is impossible.

Notes on Example 5

Part (A): To have

$$3n^2 - 3n + 3 = 3(n^2 - n + 1)\,,$$

be a perfect square means that we must have $n^2 - n + 1 = 3m^2$ for some natural number m. We might see if this can be made true for $m = 1$ which means that

$$n^2 - n + 1 = 1 \quad \text{so} \quad n \in \{0, 1\}\,,$$

and thus there exists numbers of this form that are perfect squares.

Testing Question A.1

This sum is

$$3k^2 + 3k - 4 + 7k^2 - 3k + 1 = 10k^2 - 3\,.$$

The number $10k^2$ must end in a zero so the number $10k^2 - 3$ must have a units digit of seven. As perfect squares must have units digit drawn from $\{0, 1, 4, 5, 6, 9\}$ and thus seven is not possible. Thus this sum cannot be a perfect square number.

Testing Question A.2

Expanding this expression we get

$$x^4 - 10x^3 + 5x^2 + 100x - 96 + m\,.$$

If this is a perfect square for all x it must equal an expression of the form

$$(x^2 + Ax + B)^2\,,$$

for some A and B. Expanding the above gives

$$x^4 + 2Ax^3 + (A^2 + 2B)x^2 + 2ABx + B^2\,.$$

This will equal to the above if $2A = -10$ (or $A = -5$) and $A^2 + 2B = 5$ or $B = -10$. This also means that we must have

$$-96 + m = B^2 = 100 \quad \text{so} \quad m = 196\,.$$

Testing Question A.3

If both of these numbers are perfect squares then we can write

$$n + 20 = b^2$$
$$n - 21 = a^2 .$$

Note that $b > a$. Subtracting these two we get

$$41 = b^2 - a^2 = (b - a)(b + a) .$$

Now as 41 is a prime number and that $b - a < b + a$ we have that

$$b - a = 1$$
$$b + a = 41 .$$

Solving this system we get $b = 21$ and $a = 20$. This means that

$$n + 20 = b^2 = 441$$
$$n - 21 = a^2 = 400 .$$

Both of which give $n = 421$.

Testing Question A.4

I think there is a typo in the solution to this question. In that the book gives the 2009 consecutive integers to be

$$x - 1004 , x - 1003 , \cdots x - 1 , x , x + 1 , \cdots , x + 1003 , x + 1004 ,$$

and then claims that $x = 41$. The problem is that this gives the first number $x - 1004 = -963 < 0$ in contrast to the fact that all the integers summed should be positive.

My solution is given here. If our consecutive positive integers are given by $n + i$ for $i \geq 1$ and $n \geq 0$ then we are told that

$$\sum_{i=1}^{2009} (n + i) = m^2 ,$$

for some m. We can evaluate the left-hand-side of the above to first get

$$2009n + \sum_{i=1}^{2009} i = m^2 ,$$

or

$$2009n + \frac{2009(2010)}{2} = m^2 ,$$

or

$$2009(n + 1005) = m^2 .$$

Now as $2009 = 7^2 \cdot 41$ we have

$$7^2 \cdot 41 \cdot (n + 1005) = m^2 .$$

We will have m be as small as possible if $n + 1005$ has a factor of 41 and some other perfect square (say l^2) such that

$$n + 1005 = 41l^2 \quad \text{and} \quad n = 41l^2 - 1005 > 0 .$$

If we try $l \in \{1, 2, 3, 4, 5\}$ the first value of l where the above is true is $l = 5$ and we have

$$n = 41(5^2) - 1005 = 20 .$$

This means that the smallest number we sum is $20 + 1 = 21$ and the largest number we sum is $20 + 2009 = 2029$.

Testing Question A.5

I think there is a typo in the solution to this question. In the problem statement one of the exponents seems to be 10000 while in the solutions it looks to be 1000.

Call this expression E. We have

$$\begin{aligned} E &= 4^{27} + 4^{1000} + 4^x \\ &= 2^{54} + 2^{2000} + 2^{2x} \\ &= 2^{54}(1 + 2^{1946} + 2^{2x-54}) \\ &= 2^{54}(1 + 2 \cdot 2^{1945} + 2^{2x-54}) . \end{aligned}$$

This will be a perfect square if $2x - 54 = 2(1945)$ which means that $x = 1972$. I was not able to reason as to why no larger x exists.

Testing Question A.6

Let E be the given expression. Lets see if we can write E in the form

$$(n^2 + An + 1)^2 .$$

Expanding this we get

$$n^4 + 2An^3 + (A^2 + 2)n^2 + 2An + 1 .$$

If we take $A = 1$ this is

$$n^4 + 2n^3 + 3n^2 + 2n + 1 ,$$

which is *larger* than E. Thus we have shown that

$$E < (n^2 + n + 1)^2 .$$

Based on the above the next smaller perfect square than $(n^2+n+1)^2$ is $(n^2+n)^2$. Expanding this later expression we get

$$(n^2+n)^2 = n^4 + 2n^3 + n^2\,,$$

which is *less* than E. Thus we have shown that

$$(n^2+n)^2 < E < (n^2+n+1)^2\,,$$

showing that E cannot be a perfect square.

Testing Question A.8

We first evaluate the given sum (denoted N). We have

$$\begin{aligned} N &= a10^2 + b10 + c + b10^2 + c10 + a + c10^2 + a10 + b \\ &= (a+b+c)(10^2+10+1) = 111(a+b+c) = 3 \cdot 37 \cdot (a+b+c)\,. \end{aligned}$$

In order for this to be a perfect square we must have $a+b+c \geq 3 \cdot 37 = 111$ but

$$0 < a+b+c \leq 9+9+9 = 27\,,$$

and the above is not possible and no such perfect square exists.

Testing Question A.9

As we must have

$$\begin{aligned} n^2 &\equiv 0 \pmod 8 \quad \text{or} \\ n^2 &\equiv 1 \pmod 8 \quad \text{or} \\ n^2 &\equiv 4 \pmod 8\,, \end{aligned}$$

if we take the "mod" eight of both sides of this expression we get

$$[a^2 \bmod 8] + [b^2 \bmod 8] + 0 = 6\,,$$

which is impossible.

Lecture 21: Pigeonhole Principle

Lecture 22: $[x]$ and $\{x\}$

Lecture 23: Diophantine Equations (I)

Lecture 24: Roots and Discriminant of Quadratic Equation

Notes on Example 9

From the product expression

$$(a^2 - (b+c)^2)(a^2 - (b-c)^2) < 0\,,$$

we must have either

$$a^2 - (b+c)^2 < 0 \quad \text{and} \quad a^2 - (b-c)^2 > 0\,, \tag{937}$$

or

$$a^2 - (b+c)^2 > 0 \quad \text{and} \quad a^2 - (b-c)^2 < 0\,. \tag{938}$$

Now as

$$a^2 - (b+c)^2 < a^2 - (b-c)^2\,,$$

we see that Equation 938 is not possible. From Equation 937 and with a, b, c positive we have

$$a < b + c \quad \text{and} \quad a > |b - c|\,.$$

This last expression means that

$$a > b - c \quad \text{and} \quad a > -b + c\,,$$

or

$$b < a + c \quad \text{and} \quad c < a + b\,.$$

Thus we have derived the triangle inequalities of $a < b + c$, $b < a + c$, and $c < a + b$.

Testing Question A.1

Write the first expression as

$$(2003x)^2 - (2003+1)(2003-1)x - 1 = 0\,,$$

or

$$2003^2 x^2 - (2003^2 - 1)x - 1 = 0\,.$$

In this form its easier to see how it factors and we have

$$(2003^2 x + 1)(x - 1) = 0\,.$$

Thus the two roots are

$$-\frac{1}{2003^2} \quad \text{and} \quad m = 1\,.$$

For the second expression we can factor it as

$$(x-1)(x+2003) = 0\,,$$

Thus the two roots are

$$n = -2003 \quad \text{and} \quad 1\,.$$

Thus we see that

$$m - n = 1 - (-2003) = 2004\,.$$

Testing Question A.2

This expression "changes" when x crosses the points -3 and $+3$. If $x < -3$ then this expression is

$$x^2 - (x+3) - (x-3) - 24 = 0 \quad \text{or} \quad x^2 - 2x - 24 = 0 \quad \text{or} \quad (x-6)(x+4) = 0\,.$$

Thus the two roots are $x = -4$ and $x = 6$. Only the value $x = -4$ is less than -3.

If $-3 < x < 3$ then this expression is

$$x^2 + (x+3) - (x-3) - 24 = 0 \quad \text{or} \quad x^2 - 15 = 0\,.$$

Thus the two roots are $x = \pm\sqrt{15} = \pm 3.87298$. Neither of these two values are in the domain $-3 < x < +3$ and thus there are no solutions for x in this domain.

If $x > 3$ then this expression is

$$x^2 + (x+3) + (x-3) - 24 = 0 \quad \text{or} \quad x^2 + 2x - 24 = 0 \quad \text{or} \quad (x+6)(x-4) = 0\,.$$

Thus the two roots are $x = -6$ and $x = 4$. Only the value $x = 4$ is greater than 3.

Testing Question A.3

If $m = 2$ this expression is linear and has a solution $x = -1$.

Looking to factor this expression we find we can write it as

$$((m-2)x - (2m+1))(x+1) = 0\,.$$

This means that the roots are $x = -1$ and $x = \frac{2m+1}{m-2}$.

Testing Question A.4

Write this expression as $x^2 = 3x - 1$. Then taking $x = a$ and multiplying by a we get

$$a^3 = 3a^2 - a = 3(3a-1) - a = 8a - 3\,.$$

Using these the expression given (called here E) can be written

$$\begin{aligned} E &= \frac{a^2(2(8a-3) - 5(3a-1) + 2a - 8)}{3a - 1 + 1} \\ &= \frac{a(3a-9)}{3} = a(a-3) = a^2 - 3a = 3a - 1 - 3a = -1\,. \end{aligned}$$

Testing Question A.5

Our two equations are

$$1988x^2 + bx + 8891 = 0 \tag{939}$$
$$8891x^2 + bx + 1988 = 0\,. \tag{940}$$

If x is a common root lets subtract these two equations to get

$$-6903x^2 + 6903 = 0 \quad \text{so} \quad x^2 = 1\,.$$

Thus $x = \pm 1$. In Equation 939 if we place $x = 1$ we get

$$1988 + b + 8891 = 0 \quad \text{so} \quad b = -10879\,.$$

In Equation 940 if we place $x = 1$ we also get $b = -10879$.

In Equation 939 if we place $x = -1$ we get

$$1988 - b + 8891 = 0 \quad \text{so} \quad b = 10879\,.$$

In Equation 940 if we place $x = +1$ we also get $b = 10879$.

Thus for b we have $b = \pm 10879$.

Testing Question A.6

If $m^2 - 1 = 0$ then we have a linear equation which will have one real root. These values for m are $m = \pm 1$.

If this equation has at least one real root then we know that $\Delta \geq 0$ or

$$(2(m+2))^2 - 4(m^2 - 1) \geq 0\,.$$

Expanding and simplifying this give $m \geq -\frac{5}{4}$.

As $m = \pm 1$ are in the domain of $m \geq -\frac{5}{4}$ the later is the domain where we have at least one real root for this quadratic equation.

Testing Question A.7

If these two quadratics have a common root and we subtract them we get

$$-kx - 7 + 6x + k + 1 = 0 \quad \text{or} \quad (k-6)x = k - 6\,.$$

From this we see that if $k \neq 6$ then we have $x = 1$ for the common root. Note that if $k = 6$ these two equations are the same and are

$$x^2 - 6x - 7 = (x - 7)(x + 1) = 0\,.$$

This has the two "common" roots of $x = -1$ and $x = 7$.

If $k \neq 6$ then we know a common root is $x = 1$. If we put $x = 1$ in the first expression we get

$$1 - k - 7 = 0 \quad \text{so} \quad k = -6\,.$$

Using that value of k into the two equations gives

$$\begin{aligned} x^2 + 6x - 7 &= (x + 7)(x - 1) = 0 \\ x^2 - 6x + 5 &= (x - 5)(x - 1) = 0\,. \end{aligned}$$

These have the roots $\{-7, 1\}$ and $\{1, 5\}$ respectively.

Testing Question A.8

To have two real roots we must have $\Delta = 0$ or

$$4b^2 - 4(c + a)(c - a) = 0\,,$$

which can be written as

$$a^2 + b^2 = c^2\,.$$

Thus a, b, and c would form the lengths of a right triangle.

Testing Question A.9

To have real roots we must have $\Delta \geq 0$ or

$$4(1 + a)^2 - 4(3a^2 + 4ab + 4b^2 + 2) \geq 0\,.$$

We can write this as

$$4b^2 + 4ab + 2a^2 - 2a + 1 \leq 0\,,$$

or

$$4(b^2 + ab) + 2a^2 - 2a + 1 \leq 0\,,$$

or

$$4\left(b^2 + ab + \left(\frac{a}{2}\right)^2 - \frac{a^2}{4}\right) + 2a^2 - 2a + 1 \leq 0\,,$$

or

$$4\left(b^2 + ab + \frac{a^2}{4}\right) - a^2 + 2a^2 - 2a + 1 \leq 0\,,$$

or

$$4\left(b+\frac{a}{2}\right)^2+a^2-2a+1\leq 0\,,$$

or

$$4\left(b+\frac{a}{2}\right)^2+(a-1)^2\leq 0\,.$$

This means that $a=1$ and $b=-\frac{a}{2}=-\frac{1}{2}$.

Testing Question A.10

Computing the discriminant of this quadratic equation we find

$$\begin{aligned}\Delta &= (a+b+c)^2-4(a^2+b^2+c^2)\\ &= (a+b+c-2\sqrt{a^2+b^2+c^2})(a+b+c+2\sqrt{a^2+b^2+c^2})\,.\end{aligned}$$

Now in the Cauchy-Schwarz inequality $|\mathbf{x}^T\mathbf{y}|\leq ||\mathbf{x}||||\mathbf{y}||$ if we take $\mathbf{x}=(a,b,c)$ and $\mathbf{y}=(1,1,1)$ we get

$$|a+b+c|\leq\sqrt{3}\sqrt{a^2+b^2+c^2}<2\sqrt{a^2+b^2+c^2}\,.$$

This means that

$$\begin{aligned}(a+b+c)-2\sqrt{a^2+b^2+c^2}&<0\\ (a+b+c)+2\sqrt{a^2+b^2+c^2}&>0\,.\end{aligned}$$

This means that Δ which is the product of the two numbers above is negative and there are no real roots to this quadratic.

Testing Question B.1

Fat will win if no matter how Taf assigns the three numbers from Fat to the variables a, b, and c we will have $\Delta>0$ or

$$b^2>4ac\,. \tag{941}$$

I claim that Fat should pick one negative number and two positive numbers. In that case if Taf assigns the negative number to the variable a or c in Equation 941 the inequality will be true and he will loose. Thus Taf must assign the negative number to b. Thus if one of Fat's numbers is -1 he will win if

$$1>4ac\,. \tag{942}$$

By the symmetry of a and c if we take them equal and less than $\frac{1}{2}$ the above will hold true and Fat will win. Thus one (of an infinite many) winning combinations Fat could send is -1, $\frac{1}{3}$, and $\frac{1}{3}$.

Testing Question B.2

If each quadratic equation were to have two equal roots then we need $\Delta = 0$ for each one. That means that we would have

$$\begin{aligned} 4b^2 - 4ac = 0 \quad &\text{or} \quad b^2 = ac \\ 4c^2 - 4ab = 0 \quad &\text{or} \quad c^2 = ab \\ 4a^2 - 4bc = 0 \quad &\text{or} \quad a^2 = bc \,. \end{aligned}$$

If we divide the first equation by the second equation we get

$$\frac{b^2}{c^2} = \frac{c}{b} \quad \text{or} \quad \left(\frac{b}{c}\right)^3 = 1 \,.$$

This means that $\frac{b}{c} = 1$ or $b = c$. If this is true then using the above we have that $a = b = c$. This is in contradiction to the initial assumption that a, b, and c are distinct.

Testing Question B.3

We first note that the common root cannot be $x = 1$ or else $a = 1$ and $b = 1$ for if we take $x = 1$ in the first equation we get

$$a - 1 - a^2 - 2 + a^2 + 2a = a - 3 + 2a = 3a - 3 = 0 \,.$$

If we multiply the first equation by $b - 1$ and the second equation by $a - 1$ we get

$$\begin{aligned} (a-1)(b-1)x^2 - (a^2+2)(b-1)x + (a^2+2a)(b-1) = 0 \\ (a-1)(b-1)x^2 - (a-1)(b^2+2)x + (a-1)(b^2+2b) = 0 \,. \end{aligned}$$

If we subtract these two equations we get

$$\left[-(a^2+2)(b-1) + (a-1)(b^2+2)\right] x + \left[(a^2+2a)(b-1) - (a-1)(b^2+2b)\right] = 0 \,.$$

If we expand the expressions inside the brackets, factor, and simplify we can write the above as

$$(a-b)\left[ab - (a+b) - 2\right](-x+1) = 0 \,.$$

We are told that $a - b \neq 0$ and from the above have concluded that $-x + 1 \neq 0$. Thus we must have

$$ab = 2 + a + b \,. \tag{943}$$

Note that by the symmetry in this equation if $(a, b) = (x, y)$ is a solution then so is $(a, b) = (y, x)$.

For a given value for a by solving Equation 943 for b we have

$$b = \frac{a+2}{a-1} \,. \tag{944}$$

Now as a and b are positive integers from Equation 944 we see that not all integer value for a will give an integer value for b. Now if $a = 2$ then the above gives $b = 4$. Using Equation 944 we see that if a is even then the expression $a+2$ will be even while $a-1$ will be odd and the fractional expression for b will not be an integer. Also if a is odd then the expression $a+2$ will be odd but $a-1$ will be even and the fractional expression for b will not be an integer. Thus the only integer solutions to the above are $(a, b) \in \{(2, 4), (4, 2)\}$.

Note that the expression we are asked to evaluate is symmetric in a and b and thus will give the same value for both the points above. For $(a, b) = (2, 4)$ I find

$$\frac{2^4 + 4^2}{2^{-4} + 4^{-2}} = \frac{16 + 16}{\frac{1}{16} + \frac{1}{16}} = 16^2 = 256\,.$$

Testing Question B.4

Note that as the left-hand-side of this expression is *positive* for all x we must have $mx > 0$ so $m \neq 0$. Otherwise for a solution we must have either

$$m > 0 \quad \text{and} \quad x > 0\,, \tag{945}$$

or

$$m < 0 \quad \text{and} \quad x < 0\,, \tag{946}$$

As the left-hand-side of the original quadratic "changes" at the points $x \in \{-2, -1, +1, +2\}$ we will start by assuming that Equation 945 is true and hypothesize that $0 < x < 1$. In this case the original quadratic becomes

$$-(x^2 - 1) - (x^2 - 4) = -2x^2 + 5\,.$$

Setting this equal to mx we get

$$2x^2 + mx - 5 = 0\,.$$

Solving this for x we get

$$x = \frac{-m \pm \sqrt{m^2 - 4(2)(-5)}}{2(2)} = \frac{-m \pm \sqrt{m^2 + 40}}{4}\,.$$

The negative sign above will give a value for x that is $x < 0$. To have this x in the supposed range $0 < x < 1$ we would need to have

$$\frac{-m + \sqrt{m^2 + 40}}{4} < 1\,,$$

or

$$\sqrt{m^2 + 40} < m + 4\,,$$

or squaring

$$m^2 + 40 < (m + 4)^2 = m^2 + 8m + 16\,,$$

which simplifies to $m > 3$. As this is a positive value for m we have found a solution that satisfies Equation 945.

Next we will assume that Equation 945 is true and hypothesize that $1 < x < 2$. In this case the original quadratic becomes

$$(x^2 - 1) - (x^2 - 4) = 3\,.$$

Setting this equal to mx we get $x = \frac{3}{m}$. To have this x in the supposed range $1 < x < 2$ we would need to have

$$1 < \frac{3}{m} < 2 \quad \text{or} \quad \frac{1}{2} < \frac{m}{3} < 1 \quad \text{or} \quad \frac{3}{2} < m < 3\,.$$

As this is a positive value for m we have found another solution that satisfies Equation 945.

Finally we will assume that Equation 945 is true and hypothesize that $x > 2$. In this case the original quadratic becomes

$$(x^2 - 1) + (x^2 - 4) = 2x^2 - 5\,.$$

Setting this equal to mx we get

$$2x^2 - mx - 5 = 0\,.$$

Solving this for x we get

$$x = \frac{m \pm \sqrt{m^2 - 4(2)(-5)}}{2(2)} = \frac{m \pm \sqrt{m^2 + 40}}{4}\,.$$

The negative sign above will give a value for x that is $x < 0$. To have this x in the supposed range $x > 2$ we would need to have

$$\frac{m + \sqrt{m^2 + 40}}{4} > 2\,.$$

Following the same steps as before simplifies to $m > \frac{3}{2}$. Again as this is a positive value for m we have found a third that satisfies Equation 945.

In summary then for $m > 0$ we have

- If $m > \frac{3}{2}$ then $x = \frac{m+\sqrt{m^2+40}}{4}$ is a positive solution with $x > 2$.
- If $\frac{3}{2} < m < 3$ then $x = \frac{3}{m}$ is a positive solution with $1 < x < 2$.
- If $m > 3$ then $x = \frac{-m+\sqrt{m^2+40}}{4}$ is a positive solution with $0 < x < 1$.

As before as the left-hand-side of the original quadratic "changes" at the points $x \in \{-2, -1, +1, +2\}$ we would now need to start by assuming that Equation 946 is true and hypothesize that for example $-1 < x < 0$. Following all of the same logic as above will gives solutions similar to the ones above. An easier way to get these solutions is to recognize that the original equation is unchanged by the substation $m \to -m$ and $x \to -x$. Thus we can convert the "summary" above for $m > 0$ into a "summary" for $m < 0$ by making the replacement $m \to -m$ and $x \to -x$. In (maybe too much detail) this gives

In summary then for $-m > 0$ we have

- If $-m > \frac{3}{2}$ then $-x = \frac{-m+\sqrt{m^2+40}}{4}$ is a negative solution with $-x > 2$.
- If $\frac{3}{2} < -m < 3$ then $-x = \frac{3}{-m}$ is a negative solution with $1 < -x < 2$.
- If $-m > 3$ then $-x = \frac{m+\sqrt{m^2+40}}{4}$ is a negative solution with $0 < -x < 1$.

or simplifying then

For $m > 0$ we have

- If $m < -\frac{3}{2}$ then $x = \frac{m-\sqrt{m^2+40}}{4}$ is a negative solution with $x < -2$.
- If $-3 < m < -\frac{3}{2}$ then $x = \frac{3}{m}$ is a negative solution with $-2 < x < -1$.
- If $m < -3$ then $x = \frac{-m-\sqrt{m^2+40}}{4}$ is a negative solution with $-1 < x < 0$.

Testing Question B.5

For each equation to have two roots means that $\Delta_1 > 0$ and $\Delta_2 > 0$ or

$$
\begin{aligned}
1^2 - 4q_1 > 0 \\
p^2 - 4q_2 > 0 \,.
\end{aligned}
$$

These are equivalent to the expressions

$$
q_1 < \frac{1}{4} \quad (947)
$$

$$
p^2 > 4q_2 \,. \quad (948)
$$

If both inequalities above are true the both equations have two distinct roots.

Lets assume that Equation 947 is *not* true i.e. $q_1 > \frac{1}{4}$. This means that the first equation does not have two real roots. Using the constraint given in the problem we have that

$$
q_1 = p - q_2 - 1 \,,
$$

so $q_1 > \frac{1}{4}$ means that

$$
p - q_2 - 1 > \frac{1}{4} \quad \text{or} \quad p > q_2 + \frac{5}{4} \,.
$$

Squaring this we get that

$$p^2 > q_2^2 + \frac{5}{2}q_2 + \frac{25}{16}\,.$$

Subtracting $4q_2$ from this we get

$$\begin{aligned}
p^2 - 4q_2 &> q_2^2 - \frac{3}{2}q_2 + \frac{25}{16} \\
&= q_2^2 - \frac{3}{2}q_2 + \frac{9}{25} - \frac{9}{25} + \frac{25}{16} \\
&= \left(q_2 - \frac{3}{2}\right)^2 + 1 > 0\,.
\end{aligned}$$

This means that Equation 948 is true and thus the second equation has two real roots.

Lets now assume that Equation 948 is *not* true. This means that the second equation does not have two real roots. Using the constraint given in the problem we have that

$$q_2 = p - q_1 - 1 > \frac{p^2}{4}\,.$$

This means that

$$4q_1 < -p^2 + 4p - 4\,.$$

The right-hand-side of the above can be written as

$$-p^2 + 4p - 4 = -(p^2 - 4p) - 4 = -(p^2 - 4p + 4) + 4 - 4 = -(p-2)^2 < 0 < 1\,.$$

This means that Equation 947 is true and the first equation has two real roots.

Finally if *both* Equation 947 and 948 are false then $q_1 > \frac{1}{4}$ and $q_2 > \frac{p^2}{4}$ so that

$$q_1 + q_2 + 1 > \frac{1}{4} + \frac{p^2}{4} + 1 = \frac{p^2+5}{4}\,.$$

I claim this can't equal p (as would be required by the constraint) for if it did then p would solve

$$p^2 - 4p + 5 = 0\,,$$

which it can't as the above quadratic has no real roots. Thus this case cannot happen with the given constraint.

Lecture 25: Relation between Roots and Coefficients of Quadratic Equations

Lecture 26: Diophantine Equations (II)

Lecture 27: Linear Inequality and Systems of Linear Inequalities

Lecture 28: Quadratic Inequalities and Fractional Inequalities

Lecture 29: Inequalities with Absolute Values

Lecture 30: Geometric Inequalities

Inequalities: A Mathematical Olympiad Approach

Numerical Inequalities

Exercise 1.5

Part (ii): Write a as $a = b + (a - b)$ and then use the triangle inequality to get

$$|a| \leq |b| + |a - b|\,,$$

or

$$|a - b| \geq |a| - |b|\,.$$

In the same way we can write b as $b = a + (b - a)$ then with the triangle inequality we get

$$|b| \leq |a| + |b - a|\,,$$

or

$$|b| - |a| \leq |b - a|\,.$$

Thus we have shown that

$$\begin{aligned} |a - b| &\geq |a| - |b| \quad \text{and} \\ |a - b| &\geq -(|a| - |b|)\,. \end{aligned}$$

Combining these two we see that

$$|a - b| \geq ||a| - |b||\,.$$

Part (iii): We want to prove that

$$x^2 + xy + y^2 \geq 0\,.$$

To start note that if $x > 0$ and $y > 0$ then $xy > 0$ and every term in the left-hand-side is greater than or equal to zero and the inequality holds. If $x < 0$ and $y < 0$ then the same argument made when both x and y are positive holds in this case. Assume that x and y are not of the same sign, thus $xy < 0$. Then consider

$$(x + y)^2 \geq 0 \quad \text{or} \quad x^2 + xy + y^2 \geq -xy\,.$$

but $-xy \geq 0$ so the left-hand-side is again positive as we were to show.

As a second method to demonstrate this inequality note that we have the identity

$$(x^2 + xy + y^2)(x - y) = x^3 - y^3\,,$$

when we expand and simplify. Now if $x < y$ the right-hand-side is negative and so is the factor $x - y$ which means that $x^2 + xy + y^2$ must be positive. If $x > y$ then the right-hand-side is positive and so is the factor $x - y$. This means that the $x^2 + xy + y^2$ must again be positive.

Part (iv): Note that $(x-y)^2 \geq 0$ so $x^2 - 2xy + y^2 \geq 0$ and

$$x^2 - xy + y^2 \geq xy\,.$$

If both $x > 0$ and $y > 0$ then we have $xy > 0$ so

$$x^2 - xy + y^2 > 0\,.$$

Exercise 1.6

Note that

$$a + b + c - (a+b) - (a+c) - (b+c) + (a+b+c) = 0\,,$$

or

$$a + b + c + (a+b+c) = (a+b) + (a+c) + (b+c)\,.$$

Then using the triangle equality on the left-hand-side we see that

$$|a + b + c + (a+b+c)| \leq |a| + |b| + |c| + |a+b+c|\,.$$

The triangle law on the expression on the right-hand-side gives

$$|(a+b) + (a+c) + (b+c)| \leq |a+b| + |a+c| + |b+c|\,.$$

If we subtract these two expressions (which we know are equal) we get

$$0 \leq |a| + |b| + |c| - |a+b| - |a+c| - |b+c| + |a+b+c|\,.$$

Exercise 1.7

Part (i): Note $b \leq 1$ and $a > 0$ so $ab \leq a$, thus $-a \leq -ab$. Since $b \leq 1$ we can add this inequality to the last one to get

$$b - a \leq 1 - ab\,.$$

If we divide this by $1 - ab$ (assuming its positive for the moment) we get

$$\frac{b-a}{1-ab} \leq 1\,.$$

We now show that $1 - ab > 0$. This is equivalent to $1 > ab$ which we know is true since we assume that $a \leq b \leq 1$.

Part (ii): We assume that $0 \leq a \leq b \leq 1$ and we want to prove

$$0 \leq \frac{a}{1+b} + \frac{b}{1+a} \leq 1\,.$$

As a and b are both positive both $\frac{a}{1+b}$ and $\frac{b}{1+a}$ are both positive so their sum must be positive. This shows the left-side of our inequality. We now try to show that

$$\frac{a}{1+b} + \frac{b}{1+a} \leq 1 \,.$$

Note that the left-hand-side of the above is equivalent to

$$\frac{1}{\frac{1}{a} + \frac{b}{a}} + \frac{1}{\frac{1}{b} + \frac{a}{b}} \,.$$

Now let $x \equiv \frac{a}{b}$ (which by our assumptions in this problem we know that $x \leq 1$) and the above becomes

$$\frac{1}{\frac{1}{a} + \frac{1}{x}} + \frac{1}{\frac{1}{b} + x} \,. \tag{949}$$

Since $a < 1$ we have $\frac{1}{a} > 1$ and the individual terms above can be bounded as

$$\frac{1}{a} + \frac{1}{x} > 1 + \frac{1}{x} \quad \text{or} \quad \frac{1}{\frac{1}{a} + \frac{1}{x}} < \frac{1}{1 + \frac{1}{x}}$$
$$\frac{1}{b} + x > 1 + x \quad \text{or} \quad \frac{1}{\frac{1}{b} + x} < \frac{1}{1 + x} \,.$$

Thus with these Equation 949 becomes

$$\frac{1}{\frac{1}{a} + \frac{1}{x}} + \frac{1}{\frac{1}{b} + x} < \frac{1}{1 + \frac{1}{x}} + \frac{1}{1 + x} = \frac{x}{x+1} + \frac{1}{x+1} = 1 \,,$$

showing the desired result.

Part (iii): One inequality we want to show is

$$0 \leq ab^2 - ba^2 \,.$$

Divide this by $ab > 0$ to get

$$0 \leq b - a \,,$$

which is true by assumption.

For this part using $b < 1$ we have

$$ab^2 - ba^2 < ab^2 - b^2a^2 = b^2(a - a^2) \leq a - a^2 = \frac{1}{4} - \left(\frac{1}{2} - a\right)^2 \leq \frac{1}{4} \,.$$

Where in the second to the last step on the expression $a - a^2$ we have "completed the square".

Exercise 1.8

From the inequality

$$\sqrt{2} < \frac{m + 2n}{m + n} \,,$$

we have

$$\sqrt{2}(m+n) < m+2n\,,$$

or

$$0 < (1-\sqrt{2})m + (2-\sqrt{2})n = (1-\sqrt{2})(m-\sqrt{2}n)\,.$$

As $1-\sqrt{2} < 0$ so if we divide by this number we get

$$m - \sqrt{2}n < 0 \quad \text{or} \quad \frac{m}{n} < \sqrt{2}\,.$$

As each of these steps is reversible we have shown both directions.

Exercise 1.9

We want to show that when $a \geq b$ and $x \geq y$ that

$$ax + by \geq ay + bx\,. \tag{950}$$

We start with that inequalities and move everything to the left-hand-side where we get

$$a(x-y) + b(y-x) \geq 0\,,$$

or

$$(a-b)(x-y) \geq 0\,.$$

This later expression is true by assumption.

Exercise 1.10

Write the expression we want to prove is true as

$$\frac{\sqrt{x^3}}{\sqrt{xy}} + \frac{\sqrt{y^3}}{\sqrt{xy}} \geq \sqrt{x} + \sqrt{y}\,,$$

or

$$\frac{\sqrt{x^3}+\sqrt{y^3}}{\sqrt{xy}} \geq \sqrt{x} + \sqrt{y}\,,$$

or

$$\sqrt{x^3} + \sqrt{y^3} \geq \sqrt{x^2 y} + \sqrt{xy^2}\,,$$

or

$$\sqrt{x^3} - \sqrt{x^2 y} - \sqrt{xy^2} + \sqrt{y^3} \geq 0\,.$$

While the powers on x and y make it somewhat confusing it can be made more clear if we write the above as

$$(x^{1/2})^3 - (x^{1/2})^2 y^{1/2} - (x^{1/2})(y^{1/2})^2 + (y^{1/2})^3 \geq 0\,,$$

or

$$(x^{1/2} - y^{1/2})((x^{1/2})^2 - (y^{1/2})^2) \geq 0\,.$$

This expression is true for if $\sqrt{x} > \sqrt{y}$ then by squaring we have that $x > y$ and both differences above are positive so the product is positive. If on the other hand we have that $\sqrt{x} < \sqrt{y}$ then also by squaring we have that $x < y$ and both differences above are now negative but the product is still positive. In all cases all steps are reversible and we have shown the desired inequality.

Exercise 1.11

We want to prove $E \geq 0$ where we have defined E as

$$E \equiv (a-b)(c-d) + (a-c)(b-d) + (d-a)(b-c)\,.$$

Now since $a+d = b+c$ we have $a-c = b-d$. Lets define x and y such that

$$\begin{aligned} x &= a - b = c - d \\ y &= a - c = b - d\,. \end{aligned}$$

With these we see that $x+y = a-d$ and $x-y = a-b-a+c = -b+c$. From these two expressions we see that we can write $(d-a)(b-c)$ in terms of x and y as

$$(d-a)(b-c) = (-(x+y))(-(x-y)) = (x+y)(x-y)\,.$$

Thus the given expression in terms of x and y is

$$x^2 + y^2 + (x+y)(x-y) = x^2 + y^2 + x^2 - y^2 = 2x^2 \geq 0\,,$$

showing the given expression.

Another perhaps simpler way to work this problem is to expand each term and simplify

$$\begin{aligned} E &= ac - ad - bc + bd + ab - ad - cb + cd + bd - dc - ab + ac \\ &= 2ac - 2ad - 2bc + 2bd = 2a(c-d) - 2b(c-d) \\ &= 2(a-b)(c-d) > 0\,. \end{aligned}$$

This last expression we know is positive because from the given constraint we have that $a - b = c - d$ so that

$$(a-b)(c-d) = (a-b)^2 \geq 0\,.$$

Exercise 1.12

Given that $a < b < c < d$ and the function definition

$$f(a,b,c,d) = (a-b)^2 + (b-c)^2 + (c-d)^2 + (d-a)^2\,.$$

We want to show

$$f(a,c,b,d) > f(a,b,c,d) > f(a,b,d,c)\,.$$

Consider the first inequality. We would like to show or $f(a,c,b,d) > f(a,b,c,d)$. The above expression is equivalent to

$$(a-c)^2 + (c-b)^2 + (b-d)^2 + (a-d)^2 > (a-b)^2 + (b-c)^2 + (c-d)^2 + (a-d)^2\,,$$

or when we cancel common terms on both sides

$$(a-c)^2 + (b-d)^2 > (a-b)^2 + (c-d)^2\,.$$

If we expand each quadratic we get

$$a^2 - 2ac + c^2 + b^2 - 2bd + d^2 > a^2 - 2ab + b^2 + c^2 - 2cd + d^2\,.$$

Again canceling common terms this is

$$-2(ac+bd) > -2(ab+cd)\,,$$

or

$$a(c-b) + d(b-c) < 0\,,$$

or

$$(a-d)(c-b) < 0\,.$$

which is true by the assumptions on a, b, c, and d.

Next we try to prove $f(a,b,c,d) > f(a,b,d,c)$ or

$$(a-b)^2 + (b-c)^2 + (c-d)^2 + (a-d)^2 > (a-b)^2 + (b-d)^2 + (d-c)^2 + (a-c)^2\,.$$

If we cancel common terms and expand the above is equivalent to

$$b^2 - 2bc + c^2 + a^2 - 2ad + d^2 > b^2 - 2bd + d^2 + a^2 - 2ac + c^2\,.$$

Again canceling common terms we get

$$-2(bc+ad) > -2(bd+ac)\,,$$

or

$$bc + ac < bd + ac\,,$$

or

$$b(c-d) + a(d-c) < 0\,,$$

or

$$(d-c)(a-b) < 0\,.$$

This last inequality is known to be true from the assumptions of the problem.

Exercise 1.13

This is a "quadratic surd" expression and we will write the left-hand-side as

$$\left(\frac{2x}{1-\sqrt{1+2x}}\right)^2 .$$

Considering the fraction we are squaring we find

$$\frac{2x}{1-\sqrt{1+2x}}\left(\frac{1+\sqrt{1+2x}}{1+\sqrt{1+2x}}\right) = \frac{2x(1+\sqrt{1+2x})}{1-(1+2x)} = -(1+\sqrt{1+2x}) .$$

Thus the inequality becomes

$$(1+\sqrt{1+2x})^2 < 2x+9 .$$

Expanding this we get

$$1+1+2x+2\sqrt{1+2x} < 2x+9 ,$$

or

$$2\sqrt{1+2x} < 7 ,$$

or

$$0 < 1+2x < \frac{49}{4} \quad \text{or} \quad -\frac{1}{2} < x < \frac{45}{8} .$$

Exercise 1.14

Write this expression as

$$\sqrt{4n^2+n} = \sqrt{4n^2\left(1-\frac{1}{4n^2}\right)} = 2n\sqrt{1-\frac{1}{4n^2}} .$$

Now we use one form of **Bernoulli's inequality** where if $0 \le r \le 1$ and $x \ge -1$ then

$$(1+x)^r \le 1+rx . \tag{951}$$

We can write

$$\sqrt{1-\frac{1}{4n^2}} < 1-\frac{1}{8n^2} ,$$

so that we have

$$\sqrt{4n^2-n} < 2n\left(1-\frac{1}{8n^2}\right) = 2n-\frac{1}{4n} ,$$

Thus the "error" between $\sqrt{4n^2-n}$ and $2n$ is smaller than $\frac{1}{4n}$. For $n \ge 1$ this fractional part is less than $\frac{1}{4}$.

Exercise 1.15

If $a > b$ then $a^3 > b^3$ and $a^2 > b^2$ so

$$(a^3 - b^3)(a^2 - b^2) > 0\,,$$

since both terms in the product are positive. If $a < b$ the taking the cube and the square of this as we just did the above product is the product of two negative numbers and is also positive. Thus the above product is positive in all cases. Expanding the above product gives

$$a^5 + b^5 - a^3b^2 - a^2b^3 > 0\,,$$

or

$$a^5 + b^5 > a^3b^2 + a^2b^3 = a^2b^2(a + b)\,.$$

This means that the first fraction we are considering can be bounded above as

$$\frac{ab}{a^5 + b^5 + ab} < \frac{ab}{a^2b^2(a + b) + ab} = \frac{1}{ab(a + b) + 1}\,.$$

Doing this same "thing" in each of the other fractions we find the sum we are considering is bounded above by

$$\frac{1}{ab(a + b) + 1} + \frac{1}{bc(b + c) + 1} + \frac{1}{ac(a + c) + 1}\,.$$

Using the fact that $ab = \frac{1}{c}$, $bc = \frac{1}{a}$, and $ac = \frac{1}{b}$ in each of the above fractions we get

$$\frac{c}{a + b + c} + \frac{a}{b + c + a} + \frac{b}{a + c + b} = 1\,.$$

The quadratic function $ax^2 + bx + c$

Exercise 1.16

From the quadratic formula a quadratic will have real roots if its discriminant is positive i.e. $\Delta = b^2 - 4ac \geq 0$. Notice that from the given expressions if we let $f(x) = ax^2 + bx + c$ then

$$f(1) = a + b + c \geq 0$$
$$f(-1) = a - b + c \geq 0\,.$$

This means that the function $f(x)$ is above the x-axis at the points $x = \pm 1$. If we complete the square in the quadratic $f(x)$ (as is done in the book) we can write it as

$$f(x) = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a^2}\,.$$

As $a > 0$ the smallest value of $f(x)$ happens when $x_{\text{min}} = -\frac{b}{2a}$ where it takes the value

$$f_{\text{min}} = c - \frac{b^2}{4a^2}\,.$$

From the fact that $\Delta \geq 0$ we can show that $f_{\text{min}} < 0$. Note that x_{min} must be inside the domain $[-1, 1]$ for if not as $f(\pm\infty) = \infty$ the function f would cross the x-axis too many times and have more than two roots. Thus we conclude that $-1 < x_{\text{min}} < 1$.

Finally, as our function $f(x)$ is continuous with $f(-1) > 0$ and $f(x_{\text{min}}) < 0$ there must be a root between these two values. In addition, as our function $f(x)$ is continuous with $f(x_{\text{min}}) < 0$ and $f(+1) > 0$ there must be a root between these two values.

Exercise 1.17

Given that the number "one" is in the left-hand-side of each of these inequalities lets see where a, b, and c fall relative to this number. Thus we start by assuming that all three inequalities are true and ask if $a > 1$. If that is the case then from the inequality

$$c(1 - a) > \frac{1}{4}\,,$$

the left-hand-side would be negative which is a violation of the inequality. Thus we must have $a < 1$. By symmetry of these equations with respect to a, b, and c we know that all of these numbers must be less than one. In the same way we cannot have $a < 0$ and thus the domain of each of these variables is $(0, 1)$.

Again assuming that all three inequalities hold lets take the product of the three of them which we can write as

$$[a(1 - a)][b(1 - b)][c(1 - c)] > \frac{1}{4^3}\,.$$

Now on the domain $0 < x < 1$ the quadratic $x(1 - x) \leq \frac{1}{4}$ so the above cannot hold since the left-hand-side must be the product of three numbers smaller than $\frac{1}{4}$.

A fundamental inequality: arithmetic mean-geometric mean

Exercise 1.18

An equivalent expression is to show that $\frac{1}{2}(1 + x) \geq \sqrt{x}$. Note that the AM-GM inequality with $a = 1$ and $b = x$ is

$$\sqrt{x} \leq \frac{1 + x}{2}\,,$$

and is what we wanted to show.

Exercise 1.19

Note that the AM-GM inequality with $a = x$ and $b = \frac{1}{x}$ is

$$1 \leq \frac{1}{2}\left(x + \frac{1}{x}\right),$$

which is equivalent to what we wanted to show.

Exercise 1.20

Note that

$$(x - y)^2 \geq 0\,,$$

so expanding we get

$$x^2 + y^2 \geq 2xy\,.$$

Another way to prove this is to use AM-GM inequality with $a = x^2$ and $b = y^2$ (with $x \geq 0$ and $y \geq 0$) to get

$$xy \leq \frac{1}{2}(x^2 + y^2)\,,$$

which is equivalent to what we wanted to show.

Exercise 1.21

Expanding the right-hand-side we get

$$2(x^2 + y^2) \geq x^2 + 2xy + y^2\,,$$

Subtracting $x^2 + y^2$ from both sides we get

$$x^2 + y^2 \geq 2xy\,,$$

which is true as this is the AM-GM inequality with $a = x^2$ and $b = y^2$.

Exercise 1.22

Multiply by $x + y$ on both sides to get

$$1 + \frac{y}{x} + \frac{x}{y} + 1 \geq 4\,,$$

or

$$\frac{y}{x} + \frac{x}{y} \geq 2\,.$$

This last expression is the AM-GM inequality with $a = \frac{y}{x}$ and $b = \frac{x}{y}$.

Exercise 1.23

Use the AM-GM inequality with $a = Ax$ and $b = \frac{B}{x}$ to get

$$Ax + \frac{B}{x} \geq 2\sqrt{Ax\left(\frac{B}{x}\right)} = 2\sqrt{AB}\,.$$

Exercise 1.24

Use the AM-GM inequality with $a = \frac{A}{B}$ and $b = \frac{B}{A}$ to get

$$\frac{A}{B} + \frac{B}{A} \geq 2\sqrt{\frac{A}{B} \cdot \frac{B}{A}} = 2\,.$$

Exercise 1.25

Lets introduce E defined as

$$E = \frac{a+b}{2} - \sqrt{ab}\,.$$

Note that we can write E as

$$E = \frac{1}{2}(a - 2\sqrt{ab} + b) = \frac{1}{2}(\sqrt{a} - \sqrt{b})^2\,.$$

As we know that $b \leq a$ we know that $\sqrt{b} \leq \sqrt{a}$ so $\sqrt{a} - \sqrt{b} \geq 0$. Next we write

$$a - b = (\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b})\,.$$

This means that

$$\sqrt{a} - \sqrt{b} = \frac{a-b}{\sqrt{a} + \sqrt{b}}\,.$$

This means that

$$E = \frac{1}{2}\frac{(a-b)^2}{(\sqrt{a} + \sqrt{b})^2}\,.$$

Now using $b \leq a$ we can bound $\sqrt{a} + \sqrt{b}$ as

$$2\sqrt{b} \leq \sqrt{a} + \sqrt{b} \leq 2\sqrt{a}\,.$$

This means that

$$\frac{1}{2}\frac{(a-b)^2}{(2\sqrt{a})^2} \leq E \leq \frac{1}{2}\frac{(a-b)^2}{(2\sqrt{b})^2}\,,$$

or

$$\frac{(a-b)^2}{8a} \leq E \leq \frac{(a-b)^2}{8b}\,.$$

Exercise 1.26

If we use $x + y \geq 2\sqrt{xy}$ three times we have

$$(x+y)(y+z)(z+x) \geq (2\sqrt{xy})(2\sqrt{yz})(2\sqrt{zx}) = 8xyz\,.$$

Exercise 1.27

If we use $xy \leq \frac{1}{2}(x^2 + y^2)$ three times we get

$$xy + yz + zx \leq \frac{1}{2}(x^2+y^2) + \frac{1}{2}(y^2+z^2) + \frac{1}{2}(z^2+x^2) = x^2+y^2+z^2\,. \tag{952}$$

Exercise 1.28

If we use $\sqrt{xy} \leq \frac{1}{2}(x+y)$ three times we get

$$\begin{aligned} x\sqrt{yz} + y\sqrt{zx} + z\sqrt{xy} &\leq x\left(\frac{y+z}{2}\right) + y\left(\frac{z+x}{2}\right) + z\left(\frac{x+y}{2}\right) \\ &= \frac{xy}{2} + \frac{xz}{2} + \frac{yz}{2} + \frac{xy}{2} + \frac{xz}{2} + \frac{zy}{2} \\ &= xy + xz + yz\,. \end{aligned}$$

Exercise 1.29

Use the result above in Equation 952 with $z = 1$ to get

$$x^2 + y^2 + 1 \geq xy + y + x\,.$$

Exercise 1.30

Recall the harmonic-geometric mean inequality (HM-GM)

$$\frac{2}{\frac{1}{x} + \frac{1}{y}} \leq \sqrt{xy}\,, \tag{953}$$

but written as

$$\frac{1}{\sqrt{xy}} \leq \frac{1}{2}\left(\frac{1}{x} + \frac{1}{y}\right)\,. \tag{954}$$

If we use this result three times we get

$$\frac{1}{\sqrt{xy}} + \frac{1}{\sqrt{xz}} + \frac{1}{\sqrt{yz}} \le \frac{1}{2}\left(\frac{1}{x} + \frac{1}{y}\right) + \frac{1}{2}\left(\frac{1}{x} + \frac{1}{z}\right) + \frac{1}{2}\left(\frac{1}{y} + \frac{1}{z}\right)$$
$$= \frac{1}{x} + \frac{1}{y} + \frac{1}{z}\,. \tag{955}$$

Exercise 1.31

If we consider Equation 955 with $x \to x^2$, $y \to y^2$, and $z \to z^2$ we get

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} \ge \frac{1}{xy} + \frac{1}{yz} + \frac{1}{zx}\,.$$

If we multiply both sides by xyz we get

$$\frac{yz}{x} + \frac{xz}{y} + \frac{xy}{z} \ge z + x + y\,.$$

Exercise 1.32

Using the AM-GM inequality written as

$$\sqrt{x^2}\sqrt{y^2+z^2} \le \frac{x^2+y^2+z^2}{2}$$
$$\sqrt{y^2}\sqrt{x^2+z^2} \le \frac{x^2+y^2+z^2}{2}\,,$$

if we add these we get the desired result.

Exercise 1.33

Use the AM-GM by writing this expression as

$$x^4 + y^4 + 4 + 4 = (x^4+4) + (y^4+4)$$
$$\ge 2\sqrt{4x^4} + 2\sqrt{4y^4} = 4(x^2+y^2)$$
$$\ge 4(2\sqrt{x^2y^2}) = 8xy\,.$$

Exercise 1.34

Use the AM-GM inequality as

$$\begin{aligned}(a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right) &\geq (2\sqrt{ab}+2\sqrt{cd})\left(2\sqrt{\frac{1}{ab}}+2\sqrt{\frac{1}{cd}}\right)\\ &= 4(\sqrt{ab}+\sqrt{cd})\left(\sqrt{\frac{1}{ab}}+\sqrt{\frac{1}{cd}}\right)\\ &\geq 4\left(2\sqrt{\sqrt{ab}\sqrt{cd}}\right)\left(2\sqrt{\sqrt{\frac{1}{ab}}\sqrt{\frac{1}{cd}}}\right)\\ &= 16(abcd)^{1/4}\left(\frac{1}{(abcd)^{1/4}}\right) = 16\,.\end{aligned}$$

Exercise 1.35

Use the AM-GM inequality as

$$\begin{aligned}\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a} &\geq 2\sqrt{\frac{a}{b}\cdot\frac{b}{c}}+2\sqrt{\frac{c}{d}\cdot\frac{d}{a}}\\ &= 2\left(\sqrt{\frac{a}{c}}+\sqrt{\frac{c}{a}}\right)\\ &\geq 4\sqrt{\sqrt{\frac{a}{c}}\cdot\sqrt{\frac{c}{a}}} = 4\,.\end{aligned}$$

Exercise 1.36

The AM-GM inequality gives us

$$x_1+x_2+\cdots+x_n \geq n(x_1x_2\cdots x_n)^{\frac{1}{n}}\,,$$

and

$$\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n} \geq \left(\frac{1}{x_1}\cdot\frac{1}{x_2}\cdots\frac{1}{x_n}\right)^{\frac{1}{n}} = \frac{n}{(x_1x_2\cdots x_n)^{\frac{1}{n}}}\,.$$

Taking the product of these two expressions we get

$$(x_1+x_2+\cdots+x_n)\left(\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}\right) \geq n^2\,.$$

Exercise 1.37

The AM-GM inequality directly gives us

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n} \geq \left(\frac{a_1}{b_1} \cdot \frac{a_2}{b_2} \cdots \frac{a_n}{b_n} \right)^{\frac{1}{n}} = n \,,$$

since the product on the right-hand-side simplifies to one as a_i and b_i are permutations of each other.

Exercise 1.38

Note that

$$a^n - 1 = \left(a^{n-1} + a^{n-2} + a^{n-3} + \cdots + a + 1\right)(a-1) \,.$$

Using the AM-GM inequality we have then that

$$a^n - 1 \geq n \left(a^{n-1} a^{n-2} a^{n-3} \cdots a^2 a^1\right)^{\frac{1}{n}} (a-1) \,.$$

Now recalling that

$$1 + 2 + 3 + \cdots + (n-2) + (n-1) = \frac{n(n-1)}{2} \,,$$

we have

$$a^n - 1 \geq n \left(a^{\frac{n-1}{2}}\right)(a-1) = n \left(a^{\frac{n+1}{2}} - a^{\frac{n-1}{2}}\right) \,.$$

Exercise 1.39

We are told that

$$(1+a)(1+b)(1+c) = 8 \,,$$

which is equivalent to

$$\left(\frac{1+a}{2}\right)\left(\frac{1+b}{2}\right)\left(\frac{1+c}{2}\right) = 1 \,.$$

Now the AM-GM inequality tells us that

$$\begin{aligned} \frac{1+a}{2} &\geq \sqrt{a} \\ \frac{1+b}{2} &\geq \sqrt{b} \\ \frac{1+c}{2} &\geq \sqrt{c} \,, \end{aligned}$$

Therefore $1 = \left(\frac{1+a}{2}\right)\left(\frac{1+b}{2}\right)\left(\frac{1+c}{2}\right) \geq \sqrt{abc}$. The relationship $1 \geq \sqrt{abc}$ implies

$$abc \leq 1^2 = 1 \,.$$

Exercise 1.40

The AM-GM inequality tells us that

$$\frac{a^3}{b} + \frac{b^3}{c} + X \geq 3\left(\frac{a^3}{b} \cdot \frac{b^3}{c} \cdot X\right)^{\frac{1}{3}} = 3\left(\frac{a^3 b^2}{c} X\right)^{\frac{1}{3}}.$$

If we take $X = cb$ in the above we get

$$\frac{a^3}{b} + \frac{b^3}{c} + cb \geq 3\left(a^3 b^3\right)^{\frac{1}{3}} = 3ab\,. \tag{956}$$

Now by the same method we can show that

$$\frac{b^3}{c} + \frac{c^3}{a} + ca \geq 3bc\,, \tag{957}$$

and

$$\frac{c^3}{a} + \frac{a^3}{b} + ab \geq 3ca\,. \tag{958}$$

If we add these three equations we get

$$\frac{a^3}{b} + \frac{b^3}{c} + \frac{b^3}{c} + \frac{c^3}{a} + \frac{c^3}{a} + \frac{a^3}{b} + cb + ca + ab \geq 3(ab + bc + ca)\,,$$

or

$$2\left(\frac{a^3}{b} + \frac{b^3}{c} + \frac{c^3}{a}\right) \geq 2(ab + bc + ca)\,.$$

If we divide this by two we get the desired result.

Exercise 1.41

Divide both sides by abc and the given inequality is equivalent to

$$\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b} \geq a + b + c\,.$$

To prove that this inequality is true note that the AM-GM inequality applied on the first two terms would be

$$\frac{ab}{c} + \frac{bc}{a} \geq 2b\,.$$

The AM-GM on the last two terms would give a lower bound of $2c$. Finally the AM-GM inequality applied to the first and third terms give a lower bound of $2a$. This motivates us to write the left-hand-side as

$$\frac{1}{2}\left(\frac{ab}{c} + \frac{ca}{b}\right) + \frac{1}{2}\left(\frac{ab}{c} + \frac{bc}{a}\right) + \frac{1}{2}\left(\frac{bc}{a} + \frac{ca}{b}\right) \geq a + b + c\,,$$

as we desired to show.

Exercise 1.42

Use the AM-GM inequality on first factor as

$$a^2b + b^2c + c^2a \geq 3\sqrt[3]{a^2b \cdot b^2c \cdot c^2a} = 3\sqrt[3]{a^3b^3c^3} = 3abc\,.$$

Use the AM-GM inequality on second factor in the same way as

$$ab^2 + bc^2 + ca^2 \geq 3abc\,.$$

Taken together (by multiplying) we get that the left-hand-side is bounded below by

$$9(abc)^2\,.$$

Exercise 1.43

Using the AM-GM inequality for the expression on the left-hand-side LHS we get

$$\text{LHS} \geq 3\sqrt[3]{\left(\frac{1+ab}{1+a}\right)\left(\frac{1+bc}{1+b}\right)\left(\frac{1+ac}{1+c}\right)}\,.$$

Multiplying the first two factors in the argument of the cube root gives

$$\frac{(1+bc+ab+ab^2c)(1+ac)}{(1+b+a+ab)(1+c)}\,.$$

Multiplying the third factor "in" gives

$$\frac{1+ac+bc+abc^2+a^2b+a^2bc+ab^2c+a^2b^2c^2}{1+c+b+bc+a+ac+ab+abc}\,.$$

Using the fact that $abc = 1$ this simplifies to

$$\frac{2+ac+bc+c+ab+a+b}{2+ac+bc+c+ab+a+b} = 1\,,$$

giving LHS ≥ 3.

Exercise 1.44

Recall that for $a, b > 0$ we have

$$\frac{1}{a} + \frac{1}{b} \geq \frac{4}{a+b+c}\,,$$

which we can prove by multiplying by $a + b$ simplifying and recognizing it as an application of the AM-GM to the sum $\frac{a}{b} + \frac{b}{a}$. To show the first inequality write the left-hand-side as

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right) + \frac{1}{2}\left(\frac{1}{b} + \frac{1}{c}\right) + \frac{1}{2}\left(\frac{1}{a} + \frac{1}{c}\right)\,,$$

and use the above to get

$$\text{LHS} \geq \frac{2}{a+b} + \frac{2}{b+c} + \frac{2}{a+c}\,.$$

The second inequality will be true if

$$\left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right)(2a+2b+2c) \geq 9\,,$$

or

$$\left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right)((a+b)+(b+c)+(c+a)) \geq 9\,.$$

This is true if we make the association that

$$\begin{aligned} x_1 &= a+b \\ x_2 &= b+c \\ x_3 &= c+a\,, \end{aligned}$$

and use Exercise 1.36.

Exercise 1.45

Note that

$$\frac{H_n + n}{n} = \frac{(1+1) + \left(1+\frac{1}{2}\right) + \left(1+\frac{1}{3}\right) + \cdots + \left(1+\frac{1}{n-1}\right) + \left(1+\frac{1}{n}\right)}{n}\,.$$

Using the AM-GM inequality we have

$$\begin{aligned} \frac{H_n + n}{n} &\geq \sqrt[n]{\prod_{k=1}^{n}\left(1+\frac{1}{k}\right)} = \sqrt[n]{\prod_{k=1}^{n}\left(\frac{k+1}{k}\right)} \\ &= \sqrt[n]{\frac{\prod_{k=1}^{n}(k+1)}{\prod_{k=1}^{n} k}} = \sqrt[n]{\frac{\prod_{k=2}^{n+1} k}{n!}} \\ &= \sqrt[n]{\frac{(n+1)!}{n!}} = \sqrt[n]{n+1}\,. \end{aligned}$$

This means that

$$n + H_n \geq n(n+1)^{\frac{1}{n}}\,.$$

Exercise 1.46

Let

$$y_i = \frac{1}{1+x_i} \quad \text{so that} \quad 1 + x_i = \frac{1}{y_i} \quad \text{or} \quad x_i = \frac{1}{y_i} - 1 = \frac{1-y_i}{y_i}\,.$$

So we are told that

$$\sum_{i=1}^{n} y_i = 1\,,$$

or

$$y_i + \sum_{j=1;j\neq i}^{n} y_j = 1\,,$$

or

$$1 - y_i = \sum_{j=1;j\neq i}^{n} y_j\,.$$

Using the AM-GM inequality on the right-hand-side we have

$$1 - y_i = \sum_{j=1;j\neq i}^{n} y_j \geq (n-1)\left(\prod_{j=1;j\neq i}^{n} y_i\right)^{\frac{1}{n-1}}\,.$$

Now consider

$$\begin{aligned}
\prod_i x_i = \prod_i \left(\frac{1-y_i}{y_i}\right) &= \frac{\prod_i (1-y_i)}{\prod_i y_i} = \frac{\prod_i \left(\sum_{j=1;j\neq i}^{n} y_j\right)}{\prod_i y_i} \\
&\geq \frac{\prod_i (n-1)\left(\prod_{j=1;j\neq i}^{n} y_i\right)^{\frac{1}{n-1}}}{\prod_i y_i} \\
&= \frac{(n-1)^n \left[\prod_i \left(\prod_{j=1;j\neq i}^{n} y_i\right)\right]^{\frac{1}{n-1}}}{\prod_i y_i} \\
&= \frac{(n-1)^n \left[(y_2\cdots y_n)(y_1 y_3 \cdots y_n)(y_1 y_2 y_4 \cdots y_n)\cdots(y_1 y_2 \cdots y_{n-1})\right]^{\frac{1}{n-1}}}{\prod_i y_i} \\
&= \frac{(n-1)^n \left(y_1^{n-1} y_2^{n-1} \cdots y_n^{n-1}\right)^{\frac{1}{n-1}}}{\prod_i y_i} = (n-1)^n\,.
\end{aligned}$$

Thus we have shown that

$$\prod_i x_i \geq (n-1)^n\,.$$

Exercise 1.47

Let $a_{n+1} \equiv 1 - a_1 - a_2 - \cdots - a_n$ so that

$$a_1 + a_2 + \cdots + a_n + a_{n+1} = 1\,. \tag{959}$$

Then let

$$a_i = \frac{1}{1+b_i}\,,$$

so that

$$1 + b_i = \frac{1}{a_i}\,,$$

or

$$b_i = \frac{1}{a_i} - 1 = \frac{1 - a_i}{a_i}\,.$$

Putting this into Equation 959 we get

$$\sum_{i=1}^{n+1} \frac{1}{1 + b_i} = 1\,.$$

Now use Exercise 1.46 above to conclude that

$$b_1 \cdot b_2 \cdots b_n \cdot b_{n+1} \geq n^{n+1}\,.$$

In terms of a_i this is

$$\frac{1 - a_1}{a_1} \cdot \frac{1 - a_2}{a_2} \cdots \frac{1 - a_n}{a_n} \cdot \frac{1 - a_{n+1}}{a_{n+1}} \geq n^{n+1}\,,$$

or

$$\frac{1}{n^{n+1}} \geq \frac{a_1 \cdot a_2 \cdots a_n \cdot a_{n+1}}{(1 - a_1)(1 - a_2) \cdots (1 - a_n)(1 - a_{n+1})}\,,$$

which is equivalent to what we are trying to prove.

Exercise 1.48

Note that if

$$\sum_{i=1}^{n} \frac{1}{1 + a_i} = 1\,,$$

then we have

$$\sum_{i=1}^{n} \frac{1 + a_i - a_i}{1 + a_i} = 1\,,$$

or

$$\sum_{i=1}^{n} \left(1 - \frac{a_i}{1 + a_i}\right) = 1\,.$$

or

$$n - \sum_{i=1}^{n} \frac{a_i}{1 + a_i} = 1\,,$$

and finally

$$\sum_{i=1}^{n} \frac{a_i}{1 + a_i} = n - 1\,.$$

We will use this (and the original expression that sums to one) to evaluate

$$
\begin{aligned}
E &\equiv \sum_{i=1}^{n} \sqrt{a_i} - (n-1) \sum_{i=1}^{n} \frac{1}{\sqrt{a_i}} \\
&= \left(\sum_{j=1}^{n} \frac{1}{1+a_j} \right) \sum_{i=1}^{n} \sqrt{a_i} - \left(\sum_{j=1}^{n} \frac{a_j}{1+a_j} \right) \sum_{i=1}^{n} \frac{1}{\sqrt{a_i}} \\
&= \sum_{i,j=1}^{n} \frac{\sqrt{a_i}}{1+a_j} - \sum_{i,j=1}^{n} \frac{a_j}{(1+a_j)\sqrt{a_i}} = \sum_{i,j=1}^{n} \left(\frac{\sqrt{a_i}}{1+a_j} - \frac{a_j}{(1+a_j)\sqrt{a_i}} \right) \\
&= \sum_{i,j=1}^{n} \left(\frac{1}{1+a_j} \right) \left(\frac{a_i}{\sqrt{a_i}} - \frac{a_j}{\sqrt{a_i}} \right) = \sum_{i,j=1}^{n} \frac{a_i - a_j}{(1+a_j)\sqrt{a_i}} \\
&= \sum_{i,j=1;i>j}^{n} \frac{a_i - a_j}{(1+a_j)\sqrt{a_i}} + \sum_{i,j=1;i<j}^{n} \frac{a_i - a_j}{(1+a_j)\sqrt{a_i}} \\
&= \sum_{i,j=1;i>j}^{n} \left(\frac{a_i - a_j}{(1+a_j)\sqrt{a_i}} + \frac{a_j - a_i}{(1+a_i)\sqrt{a_j}} \right) \\
&= \sum_{i,j=1;i>j}^{n} \frac{(a_i - a_j)(1+a_i)\sqrt{a_j} + (a_j - a_i)(1+a_j)\sqrt{a_i}}{(1+a_i)(1+a_j)\sqrt{a_i a_j}} \,.
\end{aligned}
$$

The numerator (denoted by N) in the above fraction is

$$
\begin{aligned}
N &\equiv (a_i - a_j)((1+a_i)\sqrt{a_j} - (1+a_j)\sqrt{a_i}) \\
&= (a_i - a_j)(\sqrt{a_j} - \sqrt{a_i} + a_i\sqrt{a_j} - a_j\sqrt{a_i}) \,.
\end{aligned}
$$

This should factor more but I had a hard time seeing what it factored into. To help with that I let $x \equiv \sqrt{a_i}$ and $y \equiv \sqrt{a_j}$ to write N as

$$
\frac{N}{x^2 - y^2} = y - x + x^2 y - y^2 x = y - x + xy(x-y) = (xy-1)(x-y) \,.
$$

In terms of the original variables this is

$$
N = (a_i - a_j)(\sqrt{a_i a_j} - 1)(\sqrt{a_i} - \sqrt{a_j}) \,,
$$

so that

$$
\begin{aligned}
E &= \sum_{i=1}^{n} \sqrt{a_i} - (n-1) \sum_{i=1}^{n} \frac{1}{\sqrt{a_i}} \\
&= \sum_{i,j=1;i>j}^{n} \frac{(a_i - a_j)(\sqrt{a_i} - \sqrt{a_j})(\sqrt{a_i a_j} - 1)}{(1+a_i)(1+a_j)\sqrt{a_i a_j}} \\
&= \sum_{i,j=1;i>j}^{n} \frac{(\sqrt{a_i} + \sqrt{a_j})(\sqrt{a_i} - \sqrt{a_j})^2(\sqrt{a_i a_j} - 1)}{(1+a_i)(1+a_j)\sqrt{a_i a_j}} \,.
\end{aligned}
$$

Now if $\sqrt{a_i a_j} - 1 > 0$ then we have shown the desired inequality $E > 0$. We can show this in the following way.

From the sum constraint $\sum_{i=1}^n \frac{1}{1+a_i} = 1$ we have

$$1 \geq \frac{1}{1+a_i} + \frac{1}{1+a_j} = \frac{2+a_i+a_j}{1+a_i+a_j+a_ia_j}$$
$$= \frac{1+a_i+a_j+a_ia_j+1-a_ia_j}{1+a_i+a_j+a_ia_j} = 1 + \frac{1-a_ia_j}{1+a_i+a_j+a_ia_j}\,.$$

This means that

$$0 \geq \frac{1-a_ia_j}{1+a_i+a_j+a_ia_j} \quad \text{or} \quad 1-a_ia_j \leq 0\,,$$

or $a_ia_j \geq 1$.

Exercise 1.49

Let $S_a \equiv \sum_i \frac{a_i^2}{a_i+b_i}$ and $S_b \equiv \sum_i \frac{b_i^2}{a_i+b_i}$. Then we have

$$S_a - S_b = \sum_i \frac{a_i^2 - b_i^2}{a_i+b_i} = \sum_i (a_i - b_i) = 0\,.$$

This means that $S_a = S_b$ and we can write

$$S_a = \frac{1}{2}(S_a + S_b) = \frac{1}{2}\sum_i \frac{a_i^2+b_i^2}{a_i+b_i}\,.$$

Note that

$$a_i^2 + b_i^2 \geq \frac{1}{2}(a_i+b_i)^2\,,$$

which can be show true by expanding the right-hand-side and simplifying. This means that

$$S_a \geq \frac{1}{4}\sum_i (a_i+b_i) = \frac{1}{4}\left(\sum_i a_i + \sum_i b_i\right) = \frac{1}{4}\left(\sum_i a_i + \sum_i a_i\right) = \frac{1}{2}\sum_i a_i\,.$$

Exercise 1.50

We start with $(a^2-b^2)(a-b) \geq 0$ or

$$a^3 - ab^2 - a^2b + b^3 \geq 0\,,$$

or

$$a^3 + b^3 \geq ab(a+b)\,.$$

This means that

$$\frac{1}{a^3+b^3+abc} \leq \frac{1}{ab(a+b+c)}\,.$$

In the same way we have for the other terms in the sum on the left-hand-side of the given expression

$$\frac{1}{b^3+c^3+abc} \leq \frac{1}{cb(a+b+c)}$$
$$\frac{1}{c^3+a^3+abc} \leq \frac{1}{ca(a+b+c)}\,.$$

If we add these three inequalities together we get

$$\frac{1}{a^3+b^3+abc} + \frac{1}{b^3+c^3+abc} + \frac{1}{c^3+a^3+abc} \leq \frac{1}{a+b+c}\left(\frac{1}{ab} + \frac{1}{cb} + \frac{1}{ca}\right)\,.$$

Simplifying the right-hand-side gives

$$\frac{1}{a^3+b^3+abc} + \frac{1}{b^3+c^3+abc} + \frac{1}{c^3+a^3+abc} \leq \frac{1}{a+b+c}\left(\frac{c+a+b}{abc}\right) = \frac{1}{abc}\,,$$

the desired expression.

Exercise 1.51

The AM-GM inequality gives

$$abc \leq \left(\frac{a+b+c}{3}\right)^3 = \frac{1}{3^3} = \frac{1}{27}\,.$$

If we consider the expression on the left-hand-side (denoted E) we have

$$\begin{aligned}
E &= \left(\frac{1}{a}+1\right)\left(\frac{1}{b}+\frac{1}{c}+1+\frac{1}{bc}\right) \\
&= \frac{1}{b}+\frac{1}{c}+1+\frac{1}{bc}+\frac{1}{ac}+\frac{1}{a}+\frac{1}{abc} \\
&= 1+\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)+\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ac}\right)+\frac{1}{abc} \\
&\geq 1+3\left(\frac{1}{abc}\right)^{1/3}+3\left(\frac{1}{ab\cdot bc\cdot ac}\right)^{1/3}+27 \\
&\geq 28+3\cdot 3+3\left(\frac{1}{a^2b^2c^2}\right)^{1/3} = 37+3\left(\frac{1}{abc}\right)^{2/3} \\
&\geq 37+3(27)^{2/3} = 37+27 = 64\,.
\end{aligned}$$

Exercise 1.52

Write the left-hand-side (LHS) of this as

$$\begin{aligned}
\text{LHS} &= \left(\frac{1-a}{a}\right)\left(\frac{1-b}{b}\right)\left(\frac{1-c}{c}\right) \\
&= \left(\frac{b+c}{a}\right)\left(\frac{a+c}{b}\right)\left(\frac{a+b}{c}\right)\,.
\end{aligned}$$

Now use the AM-GM inequality in each factor as

$$\begin{aligned}\text{LHS} &\geq \left(\frac{2\sqrt{bc}}{a}\right)\left(\frac{2\sqrt{ac}}{b}\right)\left(\frac{2\sqrt{ab}}{c}\right)\\ &= 8\frac{abc}{abc} = 8\,.\end{aligned}$$

Exercise 1.53

Write the left-hand-side (LHS) of this as

$$\text{LHS} = \frac{a(c+1)+b(a+1)+c(b+1)}{(a+1)(b+1)(c+1)} = \frac{ac+a+ab+b+bc+c}{(a+1)(b+1)(c+1)}\,.$$

Next we note that

$$\begin{aligned}(a+1)(b+1)(c+1) = (a+1)(b+1+bc+c) &= 1+b+c+bc+ab+a+abc+ac\\ &= 1+a+b+c+ab+ac+bc+abc\,.\end{aligned}$$

This means that

$$\begin{aligned}\text{LHS} &= \frac{1+a+b+c+ab+ac+bc+abc-1-abc}{(a+1)(b+1)(c+1)}\\ &= \frac{1+a+b+c+ab+ac+bc+abc-2}{(a+1)(b+1)(c+1)}\\ &= 1-\frac{2}{(a+1)(b+1)(c+1)}\,.\end{aligned}$$

This means that the left-hand-side is larger than $\frac{3}{4}$ if and only if

$$1-\frac{2}{(a+1)(b+1)(c+1)} \geq \frac{3}{4}\,,$$

or (in perhaps too many steps)

$$\frac{1}{4} \geq \frac{2}{(a+1)(b+1)(c+1)}\,,$$

or

$$\frac{1}{8} \geq \frac{1}{(a+1)(b+1)(c+1)}\,,$$

or finally

$$(a+1)(b+1)(c+1) \geq 8\,.$$

We can show that the above is true by using AM-GM three times in the expression on the left-hand-side as

$$(a+1)(b+1)(c+1) \geq (2\sqrt{a})\left(2\sqrt{b}\right)(2\sqrt{c}) = 8\,.$$

Exercise 1.54

We are told that

$$\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} = 1\,,$$

or combining terms on the left-hand-side we get

$$\frac{(1+b)(1+c) + (1+a)(1+c) + (1+a)(1+b)}{(1+a)(1+b)(1+c)} = 1\,,$$

or

$$\frac{1+c+b+bc+1+c+a+ac+1+b+a+ab}{(1+a)(1+b)(1+c)} = 1\,,$$

or

$$\frac{3+2a+2b+2c+ab+ac+bc}{(1+a)(1+b)(1+c)} = 1\,.$$

If we multiply both sides by $(1+a)(1+b)(1+c)$ we get

$$\begin{aligned} 3+2a+2b+2c+ab+ac+bc &= (1+a)(1+b+c+bc) \\ &= 1+b+c+bc+a+ab+ac+abc \\ &= 1+a+b+c+ab+ac+bc+abc\,. \end{aligned}$$

If we cancel common terms this becomes

$$2+a+b+c = abc\,.$$

Now the AM-GM inequality on the left-hand-side states that

$$2+a+b+c \geq 4(2abc)^{\frac{1}{4}}\,.$$

As $2+a+b+c = abc$ this means that

$$abc \geq 4 \cdot 2^{\frac{1}{4}}(abc)^{\frac{1}{4}}\,.$$

As $abc > 0$ we can write the above as

$$(abc)^{\frac{3}{4}} \geq 2^{2+\frac{1}{4}}\,,$$

or

$$abc \geq 2^{\frac{4}{3}\left(\frac{8}{4}+\frac{1}{4}\right)} = 2^{\frac{4}{3}\left(\frac{9}{4}\right)} = 2^3 = 8\,,$$

as we were to show.

Exercise 1.55

Notice that we can write the first term of the left-hand-side as

$$\frac{2ab}{a+b} = \frac{2}{\frac{1}{a}+\frac{1}{b}}\,,$$

which we recognize as the harmonic mean HM. As the harmonic mean is less than the arithmetic mean or

$$\frac{2ab}{a+b} = \frac{2}{\frac{1}{a}+\frac{1}{b}} \le \frac{a+b}{2}\,,$$

we can bound the left-hand-side (LHS) of our given expression as

$$\text{LHS} \le \frac{a+b}{2} + \frac{b+c}{2} + \frac{c+a}{2} = a+b+c\,.$$

Exercise 1.56

Lets use the AM-GM inequality for both sums as

$$\begin{aligned}\left(\sum_{i=1}^{n}\frac{1}{a_ib_i}\right)\left(\sum_{i=1}^{n}(a_i+b_i)^2\right) &\ge n\left(\prod_{i=1}^{n}\frac{1}{a_ib_i}\right)^{\frac{1}{n}} n\left(\prod_{i=1}^{n}(a_i+b_i)^n\right)^{\frac{1}{n}} \\ &= n^2\left(\prod_{i=1}^{n}\frac{(a_i+b_i)^2}{a_ib_i}\right)^{\frac{1}{n}}\,.\end{aligned}$$

We will have proven the given inequality if we can show that

$$(a_i+b_i)^2 \ge 4a_ib_i\,,$$

which can be seen to be true by expanding the left-hand-side and simplifying. Thus we have that

$$\text{LHS} \ge n^2\left(\prod_{i=1}^{n}4\right)^{\frac{1}{n}} = 4n^2\,.$$

Exercise 1.57

Notice that

$$(x+y+z)^2 = x^2+y^2+z^2+2xy+2xz+2yz\,.$$

There are two "groups" in the right-hand-side of the above. The first three terms is the first group and the second three terms is the second group. Notice that we can write the first "group" as

$$x^2+y^2+z^2 = \frac{1}{2}(x^2+y^2) + \frac{1}{2}(x^2+z^2) + \frac{1}{2}(y^2+z^2) \ge xy+xz+yz\,.$$

This means that we have folded the first group in the second group and now have

$$(x+y+z)^2 \ge 3(xy+xz+yz)\,.$$

Using the same trick as above we can write

$$
\begin{aligned}
xy + xz + yz &= \frac{1}{2}(xy + xz) + \frac{1}{2}(xz + yz) + \frac{1}{2}(xy + yz) \\
&\geq \sqrt{x^2yz} + \sqrt{xyz^2} + \sqrt{xy^2z} \\
&= x\sqrt{yz} + y\sqrt{xz} + z\sqrt{xy} \,.
\end{aligned}
$$

Using this in the above we have

$$
(x + y + z)^2 \geq 3(x\sqrt{yz} + y\sqrt{xz} + z\sqrt{xy}) \,.
$$

Exercise 1.58

Use the AM-GM inequality on the first two terms on the left-hand-side as

$$
x^4 + y^4 + z^2 \geq 2\left(\sqrt[2]{x^4y^4}\right) + z^2 = 2x^2y^2 + z^2 \,.
$$

Use the AM-GM inequality on the two terms on the right-hand-side of the above to get

$$
x^4 + y^4 + z^2 \geq 2(2x^2y^2z^2)^{\frac{1}{2}} = 2\sqrt{2}(xyz) = \sqrt{8}(xyz) \,.
$$

Exercise 1.59

To start use the AM-GM inequality as

$$
\frac{x^2}{y-1} + \frac{y^2}{x-1} \geq 2\sqrt{\frac{x^2}{y-1} \cdot \frac{y^2}{x-1}} = \frac{2xy}{\sqrt{xy - (x+y) + 1}} \,.
$$

In the denominator of the above fraction

$$
xy - (x + y) + 1 \,,
$$

if we use $x + y \geq 2\sqrt{xy}$ we get

$$
xy - (x + y) + 1 \leq xy - 2\sqrt{xy} + 1 = \left(\sqrt{xy} - 1\right)^2 \,.
$$

This means that

$$
\sqrt{xy - (x + y) + 1} \leq \sqrt{xy} - 1 \,,
$$

so that

$$
\frac{1}{\sqrt{xy - (x + y) + 1}} \geq \frac{1}{\sqrt{xy} - 1} \,.
$$

Using this in the above we get that

$$
\frac{x^2}{y-1} + \frac{y^2}{x-1} \geq \frac{2\left(\sqrt{xy}\right)^2}{\sqrt{xy} - 1} \,.
$$

Based on the form of the above we define $v \equiv \sqrt{xy}$ which has a domain of $v \geq 1$ (from the domain of x and y). In terms of v the right-hand-side of the above is

$$f(v) \equiv \frac{v^2}{v-1}\,.$$

Lets minimize this function as a function of $v \geq 1$. The extreme values of this function are where the derivative vanishes or where

$$\begin{aligned}\frac{d}{dv}\left(\frac{v^2}{v-1}\right) &= \frac{2v}{v-1} - \frac{v^2}{(v-1)^2} \\ &= \frac{2v(v-1)-v^2}{(v-1)^2} = \frac{v^2-2v}{(v-1)^2} = 0\,.\end{aligned}$$

This will be zero when $v = 0$ or $v = 2$. As $v \geq 1$ the minimum (based on a simple sketch of this function and the observation that $\lim_{v\to 1^+} f(v) = \infty$ and $\lim_{v\to\infty} f(v) = \infty$) of this function is given by

$$\left.\frac{v^2}{v-1}\right|_{v=2} = \frac{4}{1} = 4\,.$$

This means that $f(v) \geq 4$ and thus

$$\frac{x^2}{y-1} + \frac{y^2}{x-1} \geq 8\,.$$

A wonderful inequality: The rearrangement inequality

Exercise 1.60

Order our numbers as $a < b < c$ then we also have $a^2 < b^2 < c^2$. The rearrangement inequality gives when we take our "base" vector to be (a^2, b^2, c^2) and our permutation vector to be (b, c, a) we get

$$a^3 + b^3 + c^3 \geq a^2(b) + b^2(c) + c^2(a)\,,$$

which is the given inequality.

Exercise 1.61

Order our numbers as $a < b < c$ then we also have $a^2 < b^2 < c^2$. The rearrangement inequality gives when we take our "base" vector to be (a^2, b^2, c^2) and our permutation vector to be (b, c, a)

$$a^3 + b^3 + c^3 \geq a^2(b) + b^2(c) + c^2(a) = a^2b + b^2c + c^2a\,. \tag{960}$$

This gives "one-half" of the desired inequality.

Now note that

$$(ab)^3 + (bc)^3 + (ca)^3 = \frac{1}{c^3} + \frac{1}{a^3} + \frac{1}{b^3}\,, \tag{961}$$

when $abc = 1$. Since $a < b < c$ we have

$$\frac{1}{c} < \frac{1}{b} < \frac{1}{a} \quad \text{and} \quad \frac{1}{c^2} < \frac{1}{b^2} < \frac{1}{a^2}\,.$$

Using the rearrangement inequality with our "permutation" vector $\mathbf{a} = \left(\frac{1}{c^2}, \frac{1}{b^2}, \frac{1}{a^2}\right)$ and our "base" vector $\mathbf{b} = \left(\frac{1}{c}, \frac{1}{b}, \frac{1}{a}\right)$ then the rearrangement inequality gives

$$\frac{1}{c^3} + \frac{1}{b^3} + \frac{1}{a^3} \geq \frac{1}{c}a'_1 + \frac{1}{b}a'_2 + \frac{1}{c}a'_3\,,$$

where $\mathbf{a}'$ is any permutation of the vector $\mathbf{a}$.

Now note that when $abc = 1$ we have

$$\begin{aligned}
a^2b &= \frac{1}{(bc)^2}\frac{1}{ac} = \frac{1}{ab^2c^3} = \frac{1}{bc^2} \\
b^2c &= \frac{1}{(ac)^2}\frac{1}{ba} = \frac{1}{a^3bc^2} = \frac{1}{a^2c} \\
c^2a &= \frac{1}{(ba)^2}\frac{1}{bc} = \frac{1}{a^2b^3c} = \frac{1}{ab^2}\,.
\end{aligned}$$

Based on this we let the permutation vector $\mathbf{a}'$ be

$$\left(\frac{1}{a^2}, \frac{1}{c^2}, \frac{1}{b^2}\right),$$

then using the rearrangement inequality (and the above) we have

$$\frac{1}{c^3} + \frac{1}{b^3} + \frac{1}{a^3} \geq \frac{1}{c}\cdot\frac{1}{a^2} + \frac{1}{b}\cdot\frac{1}{c^2} + \frac{1}{c}\cdot\frac{1}{b^2} = b^2c + a^2b + c^2a\,. \tag{962}$$

When we add Equations 960 with Equations 961 and 962 we get the desired inequality.

Exercise 1.63

Let $a < b < c$ so that $\frac{1}{c} < \frac{1}{b} < \frac{1}{a}$. If we take our vectors to be $\mathbf{b} = \mathbf{a} = \left(\frac{1}{c}, \frac{1}{b}, \frac{1}{a}\right)$ then the rearrangement inequality gives with our permutation vector to be $\left(\frac{1}{b}, \frac{1}{a}, \frac{1}{c}\right)$

$$\frac{1}{c^2} + \frac{1}{b^2} + \frac{1}{a^2} \geq \frac{1}{cb} + \frac{1}{ab} + \frac{1}{ac} = \frac{a+c+b}{abc}\,.$$

Exercise 1.64

Warning: I didn't not get the same result as the problem statement even after looking at the "solution" in the back. I'm wondering if there is typo in the problem statement. If anyone sees anything I've done wrong please contact me.

Let our three sides of the triangle be such that $a \le b \le c$. Then the triangle inequality gives

$$\begin{aligned} a < b + c \quad &\text{so} \quad b + c - a > 0 \\ b < a + c \quad &\text{so} \quad a + c - b > 0 \\ c < a + b \quad &\text{so} \quad a + b - c > 0 \,. \end{aligned}$$

From which we see that the denominators in the expression given are all positive numbers.

Next lets add a, b, and c to $a \le b \le c$ to get

$$2a \le \quad a + b \quad \le a + c \tag{963}$$

$$a + b \le \quad 2b \quad \le b + c \tag{964}$$

$$a + c \le \quad b + c \quad \le 2c \,. \tag{965}$$

Using Equation 963 and 965 we get

$$a + b \le a + c \le b + c \,. \tag{966}$$

From the original ordering of a, b, and c we have

$$-c < -b < -a \,,$$

If we add this to Equation 966 we get

$$0 < a + b - c < a + c - b < b + c - a \,,$$

and thus

$$\frac{1}{b + c - a} < \frac{1}{a + c - b} < \frac{1}{a + b - c} \,.$$

Now to use the rearrangement inequality we can let our "base" vector be

$$\mathbf{b} = \left(\frac{1}{b + c - a}, \frac{1}{a + c - b}, \frac{1}{a + b - c} \right) \,,$$

and $\mathbf{a} = (a, b, c)$. Then if our permutation vector is $\mathbf{a}' = (b, c, a)$ the rearrangement inequality gives

$$\text{LHS} = \frac{a}{b + c - a} + \frac{b}{a + c - b} + \frac{c}{a + b - c} \ge \frac{b}{b + c - a} + \frac{c}{a + c - b} + \frac{a}{a + b - c} \,.$$

If our permutation vector is $\mathbf{a}' = (c, a, b)$ the rearrangement inequality gives

$$\text{LHS} \ge \frac{c}{b + c - a} + \frac{a}{a + c - b} + \frac{b}{a + b - c} \,.$$

Of course LHS ≥ 0 so we have

$$\text{LHS} \ge -\frac{a}{b + c - a} - \frac{b}{a + c - b} - \frac{c}{a + b - c} \,.$$

If we add these three inequalities together we get

$$3\text{LHS} \ge \frac{b + c - a}{b + c - a} + \frac{a + c - b}{a + c - b} + \frac{a + b - c}{a + b - c} = 1 + 1 + 1 = 3 \,.$$

This means that

$$\text{LHS} \ge 1 \,.$$

Exercise 1.65

Without loss of generality we can take $a_1 < a_2 < \cdots < a_{n-1} < a_n$. Then with s defined as the sum of all these numbers we have

$$s - a_n < s - a_{n-1} < \cdots < s - a_2 < s - a_1 \,, \tag{967}$$

so

$$\frac{1}{s-a_1} < \frac{1}{s-a_2} < \cdots < \frac{1}{s-a_{n-1}} < \frac{1}{s-a_n} \,. \tag{968}$$

If we take these fractions as $(b_1, b_2, \dots, b_n)$ with $(a_1, a_2, \dots, a_n)$ the rearrangement inequality gives

$$\text{LHS} \equiv \frac{a_1}{s-a_1} + \frac{a_2}{s-a_2} + \cdots + \frac{a_{n-1}}{s-a_{n-1}} + \frac{a_n}{s-a_n} \geq \frac{a'_1}{s-a_1} + \frac{a'_2}{s-a_2} + \cdots + \frac{a'_{n-1}}{s-a_{n-1}} + \frac{a'_n}{s-a_n} \,, \tag{969}$$

for any permutation $(a'_1, a'_2, \dots, a'_n)$ of $(a_1, a_2, \dots, a_n)$. Lets consider the $n-1$ permutations

$$\begin{aligned} (a'_1, a'_2, \dots, a'_n) &= (a_2, a_3, \dots, a_1) \\ (a'_1, a'_2, \dots, a'_n) &= (a_3, a_4, \dots, a_2) \\ (a'_1, a'_2, \dots, a'_n) &= (a_4, a_5, \dots, a_3) \\ &\vdots \\ (a'_1, a'_2, \dots, a'_n) &= (a_n, a_1, \dots, a_{n-1}) \,. \end{aligned}$$

In Equation 969 and then add them together to get

$$(n-1)\text{LHS} \geq \frac{s-a_1}{s-a_1} + \frac{s-a_2}{s-a_2} + \cdots + \frac{s-a_{n-1}}{s-a_{n-1}} + \frac{s-a_n}{s-a_n} = n \,.$$

Solving for LHS we get

$$\text{LHS} \geq \frac{n}{n-1} \,.$$

Exercise 1.66

Using the notation from the previous Exercise if we take $(b_1, b_2, \cdots, b_n)$ from Equation 968 and $(a_1, a_2, \cdots, a_n)$ from Equation 967 then the rearrangement inequality gives

$$\frac{s-a_n}{s-a_1} + \frac{s-a_{n-1}}{s-a_2} + \cdots + \frac{s-a_2}{s-a_{n-1}} + \frac{s-a_1}{s-a_n} \geq n \,,$$

where we have taken $(a'_1, a'_2, \dots, a'_n) = (s-a_1, s-a_2, \cdots, s-a_n)$. We can solve the above for the expression we seek and find

$$\text{LHS} \equiv \frac{s}{s-a_1} + \frac{s}{s-a_2} + \cdots + \frac{s}{s-a_{n-1}} + \frac{s}{s-a_n} \geq n + \frac{a_n}{s-a_1} + \frac{a_{n-1}}{s-a_2} + \cdots + \frac{a_2}{s-a_{n-1}} + \frac{a_1}{s-a_n} \,.$$

From the ordering of the a_i's we have

$$\begin{aligned} \frac{a_n}{s-a_1} + \frac{a_{n-1}}{s-a_2} + \cdots + \frac{a_2}{s-a_{n-1}} + \frac{a_1}{s-a_n} &\geq \frac{a_1}{s-a_1} + \frac{a_2}{s-a_2} + \cdots + \frac{a_{n-1}}{s-a_{n-1}} + \frac{a_n}{s-a_n} \\ &\geq \frac{n}{n-1} \,, \end{aligned}$$

using the result from the previous exercise. This means that

$$\text{LHS} \geq n + \frac{n}{n-1} = n\left(1 + \frac{1}{n-1}\right) = \frac{n^2}{n-1},$$

as we were to show.

Exercise 1.67

Lets "link" the sequence of a's "together" as

$$a_1, a_1, a_2, a_2, \ldots, a_n, a_n.$$

Then lets use the result of Exercise 1.65 so here $s = 2$ and that result is

$$2\frac{a_1}{s-a_1} + 2\frac{a_2}{s-a_2} + \cdots + 2\frac{a_n}{s-a_n} \geq \frac{2n}{2n-1},$$

which is equivalent to the desired expression.

Exercise 1.68

Use Tchebyshev's inequality with $\mathbf{a} = \mathbf{b} = \mathbf{x}$ to get

$$\frac{x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2}{n} \geq \left(\frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}\right)^2.$$

Exercise 1.69

Squaring $a + b + c = 1$ we get

$$a^2 + b^2 + c^2 + 2(ab + ac + bc) = 1. \tag{970}$$

Now from Exercise 1.68 we have

$$\sqrt{\frac{a^2 + b^2 + c^2}{3}} \geq \frac{a + b + c}{3} = \frac{1}{3},$$

or

$$a^2 + b^2 + c^2 \geq \frac{1}{3}.$$

Using this in Equation 970 we get

$$2(ab + ac + bc) \leq 1 - \frac{1}{3} = \frac{2}{3},$$

which is equivalent to the desired expression.

Exercise 1.70

Note: I had to look at the solutions in the back to get definitions for **a** and the hint to use Corollary 1.4.2. Once I had those the rest of the problem was easier to solve.

We first define G as

$$G \equiv \sqrt[n]{x_1 x_2 x_3 \cdots x_n}\,,$$

and then let the vector **a** be

$$\mathbf{a} = \left(\frac{x_1}{G}\,, \frac{x_1 x_2}{G^2}\,, \frac{x_1 x_2 x_3}{G^3}\,, \cdots\,, \frac{x_1 x_2 x_3 \cdots x_n}{G^n} \right)\,.$$

Note that $\frac{x_1 x_2 x_3 \cdots x_n}{G^n} = 1$. Then with $\mathbf{a}'$ being a "circular left shift" of **a** i.e. such that

$$\mathbf{a}' = (a_2\,, a_3\,, \cdots\,, a_n\,, a_1) = \left(\frac{x_1 x_2}{G^2}\,, \frac{x_1 x_2 x_3}{G^3}\,, \cdots\,, \frac{x_1 x_2 x_3 \cdots x_n}{G^n}\,, \frac{x_1}{G} \right)\,,$$

we can use Corollary 1.4.2 to get

$$\frac{a_2}{a_1} + \frac{a_3}{a_2} + \frac{a_4}{a_3} + \cdots + \frac{a_n}{a_{n-1}} + \frac{a_1}{a_n} = \frac{x_2}{G} + \frac{x_3}{G} + \frac{x_4}{G} + \cdots + \frac{x_n}{G} + \frac{x_1}{G}\left(\frac{G^n}{x_1 x_2 \cdots x_n} \right) \geq n\,.$$

This simplifies to

$$G \leq \frac{1}{n}(x_1 + x_2 + x_3 + \cdots + x_n)\,,$$

which is one-half of the desired inequality.

Another arrangement of **a** would be a "circular right shift" of **a** i.e. such that

$$\mathbf{a}' = (a_n\,, a_1\,, \cdots\,, a_{n-2}\,, a_{n-1}) = \left(\frac{x_1 x_2 x_3 \cdots x_n}{G^n}\,, \frac{x_1}{G}\,, \cdots\,, \frac{x_1 x_2 x_3 \cdots x_{n-2}}{G^{n-2}}\,, \frac{x_1 x_2 x_3 \cdots x_{n-1}}{G^{n-1}} \right)\,.$$

With this we can again use Corollary 1.4.2 to get

$$\frac{a_n}{a_1} + \frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_{n-2}}{a_{n-1}} + \frac{a_{n-1}}{a_n} \geq n\,,$$

or

$$\frac{G}{x_1} + \frac{G}{x_2} + \frac{G}{x_3} + \cdots + \frac{G}{x_{n-1}} + \frac{G}{x_n} \geq n\,,$$

or

$$G \geq \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \cdots + \frac{1}{x_n}}\,,$$

which is the second inequality.

Exercise 1.72

Note that we can write

$$\sum_i \frac{a_i}{\sqrt{1-a_i}} = -\sum_i -\frac{a_i}{\sqrt{1-a_i}} = -\sum_i \frac{1-a_i-1}{\sqrt{1-a_i}} = -\sum_i \sqrt{1-a_i} + \sum_i \frac{1}{\sqrt{1-a_i}}\,.$$

Lets write this as averages as

$$\frac{1}{n}\sum_i \frac{a_i}{\sqrt{1-a_i}} = -\frac{1}{n}\sum_i \sqrt{1-a_i} + \frac{1}{n}\sum_i \frac{1}{\sqrt{1-a_i}}\,. \tag{971}$$

The Cauchy-Schwarz inequality on the first term gives

$$\frac{1}{n}\sum_i \sqrt{1-a_i} \le \frac{1}{n}\sqrt{\sum_i (1-a_i)}\sqrt{\sum_i 1} = \frac{1}{n}\sqrt{n-1}\sqrt{n} = \frac{\sqrt{n-1}}{\sqrt{n}}\,.$$

Using the AM-GM inequality on the second term we get

$$\frac{1}{n}\sum_i \frac{1}{\sqrt{1-a_i}} \ge \sqrt[n]{\prod_{i=1}^n \frac{1}{\sqrt{1-a_i}}}\,,$$

Simplifying the right-hand-side of the above we see that we can write

$$\begin{aligned}\text{RHS} &= \prod_{i=1}^n \frac{1}{\sqrt{\sqrt[n]{1-a_i}}} = \sqrt{\prod_{i=1}^n \frac{1}{\sqrt[n]{1-a_i}}}\\ &= \sqrt{\frac{1}{\prod_{i=1}^n \sqrt[n]{1-a_i}}} = \sqrt{\frac{1}{\sqrt[n]{\prod_{i=1}^n (1-a_i)}}}\,.\end{aligned}$$

Now the AM-GM inequality gives

$$\sqrt[n]{\prod_{i=1}^n (1-a_i)} \le \frac{1}{n}\sum_{i=1}^n (1-a_i) = 1 - \frac{1}{n}\sum_{i=1}^n a_i = 1 - \frac{1}{n} = \frac{n-1}{n}\,.$$

Using this we can form a lower bound on RHS as

$$\text{RHS} \ge \sqrt{\frac{1}{\frac{n-1}{n}}} = \sqrt{\frac{n}{n-1}}\,.$$

This means that we have shown that

$$\frac{1}{n}\sum_i \frac{a_i}{\sqrt{1-a_i}} \ge \sqrt{\frac{n}{n-1}} - \frac{\sqrt{n-1}}{\sqrt{n}} = \frac{\sqrt{n}}{\sqrt{n-1}}\left(1 - \frac{n-1}{n}\right) = \frac{1}{\sqrt{n}\sqrt{n-1}}\,.$$

This means that

$$\sum_i \frac{a_i}{\sqrt{1-a_i}} \ge \frac{\sqrt{n}}{\sqrt{n-1}}\,. \tag{972}$$

Using the Cauchy-Schwarz inequality on $\sum_i \sqrt{a_i}$ gives

$$\sum_i \sqrt{a_i} \le \sqrt{\sum_i a_i}\sqrt{\sum_i 1} = \sqrt{1}\sqrt{n} = \sqrt{n}\,.$$

Using this in Equation 972 gives

$$\sum_i \frac{a_i}{\sqrt{1-a_i}} \geq \frac{1}{\sqrt{n-1}} \sum_i \sqrt{a_i}\,,$$

as we were to show.

Exercise 1.73

Part (i): Using the AM-GM inequality we have

$$\sqrt{4a+1} \leq \frac{1}{2}(4a+1+1) = 2a+1\,.$$

Doing this three times gives

$$\sqrt{4a+1} + \sqrt{4b+1} + \sqrt{4c+1} \leq (2a+1) + (2b+1) + (2c+1) = 2(1) + 3 = 5\,.$$

Part (ii): Using the Cauchy-Schwarz inequality on the vectors

$$\begin{aligned}
\mathbf{a} &= (\sqrt{4a+1}\,, \sqrt{4b+1}\,, \sqrt{4c+1}) \\
\mathbf{b} &= (1\,, 1\,, 1)\,,
\end{aligned}$$

gives

$$\begin{aligned}
\sqrt{4a+1} + \sqrt{4b+1} + \sqrt{4c+1} &\leq \sqrt{(4a+1) + (4b+1) + (4c+1)} \times \sqrt{3} \\
&= \sqrt{4(1)+3} \times \sqrt{3} = \sqrt{21}\,.
\end{aligned}$$

Convex functions

Exercise 1.77

Part (i): Consider $f(x) \equiv \left(x + \frac{1}{x}\right)^2$ for $x \in \mathbb{R}^+$. Note that

$$f'(x) = 2\left(x + \frac{1}{x}\right)\left(1 - \frac{1}{x^2}\right) = 2\left(x - \frac{1}{x^3}\right)\,,$$

and

$$f''(x) = 2\left(1 + \frac{3}{x^4}\right) > 0\,,$$

for all $x \in \mathbb{R}^+$. This means that $f(x)$ is convex on $\mathbb{R}^+$. Next recall that if $f(x)$ is convex we have

$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{2}(f(x) + f(y))\,.$$

In words this means that sums of convex functions are bounded below as

$$f(x) + f(y) \geq 2f\left(\frac{x+y}{2}\right).$$

If $x = a$ and $y = b$ and for the $f(x)$ given above this becomes

$$\left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2 \geq 2f\left(\frac{a+b}{2}\right) = 2f\left(\frac{1}{2}\right) = 2\left(\frac{1}{2} + 2\right)^2 = \frac{25}{2}.$$

Part (ii): As $f(x)$ is convex by Jensen's inequality with $t_i = \frac{1}{n} = \frac{1}{3}$ we have

$$f\left(\frac{x_1 + x_2 + x_3}{3}\right) \leq \frac{1}{3}(f(x_1) + f(x_2) + f(x_3)).$$

With $x_1 = a$, $x_2 = b$, and $x_3 = c$ and for the $f(x)$ given above this is

$$\left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2 + \left(c + \frac{1}{c}\right)^2 \geq 3f\left(\frac{a+b+c}{3}\right) = 3f\left(\frac{1}{3}\right) = 3\left(\frac{1}{3} + 3\right)^2 = \frac{100}{3}.$$

Exercise 1.78

Define $f(a, b, c)$ as the left-hand-side of the given expression and notice that $f(a, b, c)$ is equivalent under all permutations of (a, b, c). Computing the first two derivatives of f with respect to a we find

$$\begin{aligned}
\frac{\partial f}{\partial a} &= \frac{1}{b+c+1} - \frac{b}{(c+a+1)^2} - \frac{c}{(a+b+1)^2} - (1-b)(1-c) \\
\frac{\partial^2 f}{\partial a^2} &= \frac{2b}{(c+a+1)^3} + \frac{2c}{(a+b+1)^3} \geq 0,
\end{aligned}$$

for all $0 \leq a \leq 1$. With the above this means that f is convex in each of its arguments and thus f takes its maximum at the endpoints i.e. $(a, b, c) = (0, 0, 0)$ or $(a, b, c) = (1, 1, 1)$. Thus

$$f(a, b, c) \leq \min(f(0, 0, 0), f(1, 1, 1)) = \min(1, 1) = 1.$$

Exercise 1.79

Note that this expression is symmetric in x and y. Now if $x = 0$ this inequality is

$$1 + \frac{1}{\sqrt{1+y^2}} \leq 2,$$

which is true for $0 \leq y \leq 1$. Now based on the idea of "turning products into sums" (here we have the product xy) we will make the substitutions

$$x = e^{-u} \tag{973}$$

$$y = e^{-v}. \tag{974}$$

Now to have $0 \leq x \leq 1$ we need to have u such that

$$0 \leq e^{-u} \leq 1 \,.$$

The inequality $0 \leq e^{-u}$ is satisfied for all real u. For $e^{-u} \leq 1$ to be true we need to have

$$-u \leq \log(1) = 0 \quad \text{so} \quad u \geq 0 \,,$$

and a similar condition on v.

These two transform the given expression in x and y into one in terms of u and v as

$$\frac{1}{\sqrt{1+e^{-2u}}} + \frac{1}{\sqrt{1+e^{-2v}}} \leq \frac{2}{\sqrt{1+e^{-(u+v)}}} \,,$$

or

$$\frac{1}{\sqrt{1+e^{-(u+v)}}} \geq \frac{1}{2}\left(\frac{1}{\sqrt{1+e^{-2u}}} + \frac{1}{\sqrt{1+e^{-2v}}}\right) \,, \tag{975}$$

Based on this if we define

$$f(\xi) = \frac{1}{\sqrt{1+e^{-2\xi}}} \,.$$

then Equation 975 is

$$f\left(\frac{u+v}{2}\right) \geq \frac{1}{2}(f(u) + f(v)) \,,$$

which will be true (and the proof complete) if $f(\xi)$ is concave. For this $f(\xi)$ note that

$$f'(\xi) = -\frac{1}{2}(1+e^{-2\xi})^{-3/2}(-2e^{-2\xi}) = \frac{e^{-2\xi}}{(1+e^{-2\xi})^{3/2}} \,,$$

and

$$\begin{aligned}
f''(\xi) &= \frac{e^{-2\xi}(-2)}{(1+e^{-2\xi})^{3/2}} - \frac{3}{2}\frac{e^{-2\xi}(-2e^{-2\xi})}{(1+e^{-2\xi})^{5/2}} \\
&= -\frac{2e^{-2\xi}}{(1+e^{-2\xi})^{3/2}} + \frac{3e^{-4\xi}}{(1+e^{-2\xi})^{5/2}} \\
&= \frac{-2+e^{-2\xi}}{e^{2\xi}(1+e^{-2\xi})^{5/2}} < 0 \,,
\end{aligned}$$

for all $\xi \geq 0$. Thus $f(\xi)$ *is* concave as we needed to show.

Exercise 1.80

To show that $f(x)$ is concave we can show that $f''(x) \leq 0$ on $x \in [0, \pi]$. We have

$$\begin{aligned}
f(x) &= \sin(x) \\
f'(x) &= \cos(x) \\
f''(x) &= -\sin(x) \,,
\end{aligned}$$

which is less than or equal to zero when $x \in [0, \pi]$ showing $f(x)$ is concave. As $f(x)$ is concave we have that

$$\frac{1}{3}(f(A) + f(B) + f(C)) \leq f\left(\frac{A+B+C}{3}\right),$$

for A, B, C in $[0, \pi]$. If these are three angles in a triangle then $A + B + C = \pi$ and

$$f\left(\frac{A+B+C}{3}\right) = f\left(\frac{\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2},$$

and the above becomes

$$\sin(A) + \sin(B) + \sin(C) \leq \frac{3\sqrt{3}}{2}.$$

Exercise 1.81

Part (i): Following the idea of "turning products into sums" if we take the natural logarithm of both sides this inequality is equivalent to

$$\frac{1}{2}(\ln(\sin(A)) + \ln(\sin(B))) \leq \ln\left(\sin\left(\frac{A+B}{2}\right)\right),$$

which will be true if $f(x) = \ln(\sin(x))$ is concave on $[0, \pi]$. For this $f(x)$ we have

$$\begin{aligned}
f(x) &= \ln(\sin(x)) \\
f'(x) &= \frac{\cos(x)}{\sin(x)} \\
f''(x) &= \frac{-\sin(x)}{\sin(x)} - \frac{\cos^2(x)}{\sin^2(x)} = -1 - \frac{1 - \sin^2(x)}{\sin^2(x)} \\
&= -1 - \frac{1}{\sin^2(x)} + 1 = -\frac{1}{\sin^2(x)} < 0,
\end{aligned}$$

thus $f(x)$ is concave on $[0, \pi]$. This means that

$$f\left(\frac{A+B}{2}\right) \geq \frac{1}{2}f(A) + \frac{1}{2}f(B),$$

and the initial inequality in this section is true.

Part (ii-iii): These are additional expressions of the concavity of $f(x) = \ln(\sin(x))$ for $x \in [0, \pi]$ and are proved as in **Part (i)** above.

Part (iv): For this we use **Part (iii)** above with $A + B + C = \pi$ to get

$$\sin(A)\sin(B)\sin(C) \leq \sin^3\left(\frac{\pi}{3}\right) = \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3\sqrt{3}}{8}.$$

Part (v): For this we use **Part (iii)** above with $\frac{A}{2} + \frac{B}{2} + \frac{C}{2} = \frac{\pi}{2}$ to get

$$\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right) \leq \sin^3\left(\frac{\pi}{6}\right) = \frac{1}{8}\,.$$

Part (vi): For this we start with

$$\cos(x)\cos(y) = \frac{1}{2}(\cos(x+y) + \cos(x-y))\,,$$

as

$$\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right) = \frac{1}{2}\left(\cos\left(\frac{A}{2} + \frac{B}{2}\right) + \cos\left(\frac{A}{2} - \frac{B}{2}\right)\right)\,.$$

Lets multiply this by $\cos\left(\frac{C}{2}\right)$ and apply the above product to sum cosign relationship again as

$$\begin{aligned}\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right) &= \frac{1}{4}\cos\left(\frac{A}{2} + \frac{B}{2} + \frac{C}{2}\right) + \frac{1}{4}\cos\left(\frac{A}{2} - \frac{B}{2} + \frac{C}{2}\right) \\ &- \frac{1}{4}\cos\left(\frac{A}{2} + \frac{B}{2} - \frac{C}{2}\right) + \frac{1}{4}\cos\left(\frac{A}{2} - \frac{B}{2} - \frac{C}{2}\right)\,.\end{aligned}$$

Using $A + B + C = \pi$ to write each "sum" in terms of the other variable. For example in each of the four above we will use

$$\begin{aligned}A + B + C &= \pi \\ A + C &= \pi - B \\ A + B &= \pi - C \\ B + C &= \pi - A\,,\end{aligned}$$

to get

$$\begin{aligned}\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right) &= \frac{1}{4}\cos\left(\frac{\pi}{2}\right) + \frac{1}{4}\cos\left(\frac{\pi - 2B}{2}\right) \\ &- \frac{1}{4}\cos\left(\frac{\pi - 2C}{2}\right) + \frac{1}{4}\cos\left(\frac{A - (\pi - A)}{2}\right) \\ &= \frac{1}{4}\cos\left(\frac{\pi}{2} - B\right) - \frac{1}{4}\cos\left(\frac{\pi}{2} - C\right) + \frac{1}{4}\cos\left(A - \frac{\pi}{2}\right)\,.\end{aligned}$$

Now using $\cos\left(\frac{\pi}{2} - x\right) = \sin(x)$ we get

$$\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right) = \frac{1}{4}(\sin(A) + \sin(B) + \sin(C))\,.$$

Exercise 1.82 (Bernoulli's inequality)

Part (i): This is "easy" to prove using the Taylor's series expansion of $f(x) = (1+x)^n$. For this function we have

$$\begin{aligned}f'(x) &= n(1+x)^{n-1} \\ f''(x) &= n(n-1)(1+x)^{n-2}\,.\end{aligned}$$

Notice that when $n \geq 1$ and $x \geq -1$ we have $f''(x) \geq 0$. Using these the Taylor series of $f(x)$ about the point $x = 0$ is given by

$$
\begin{aligned}
f(x) = (1+x)^n &= f(0) + f'(0)x + \frac{f''(\xi)}{2!}x^2 \quad \text{for } \xi \text{ between 0 and } x \\
&= 1 + nx + \frac{f''(\xi)}{2}x^2 \geq 1 + nx \,.
\end{aligned}
$$

Another way to prove this is to use the GM $\leq$ AM inequality of a set of n specially chosen numbers. Our n numbers are with $n-1$ "ones" and a single $1+nx$. The GM of these is

$$
(1+nx)^{1/n} \,,
$$

while the AM of these is

$$
\frac{n-1+1+nx}{n} = (1+x) \,.
$$

The GM $\leq$ AM is then

$$
(1+nx)^{1/n} \leq 1+x \quad \text{or} \quad 1+nx \leq (1+x)^n \,.
$$

Part (ii): Given $(1+x)^n \geq 1+nx$ we want to prove the GM $\leq$ AM inequality. Let $\mathbf{a}$ be a sequence of positive numbers a_i and define

$$
\sigma_j = \frac{1}{j}\sum_{i=1}^{j} a_i \,.
$$

Then lets use the Bernoulli inequality with $x = \frac{\sigma_j}{\sigma_{j-1}} - 1$ and $n = j$ to get

$$
\left(\frac{\sigma_j}{\sigma_{j-1}}\right)^j \geq 1 + j\left(\frac{\sigma_j}{\sigma_{j-1}} - 1\right) = j\left(\frac{\sigma_j}{\sigma_{j-1}}\right) - (j-1) \,,
$$

or if we multiply by σ_{j-1}^j on both sides we get

$$
\sigma_j^j \geq j\sigma_j\sigma_{j-1}^{j-1} - (j-1)\sigma_{j-1}^j = \sigma_{j-1}^{j-1}(j\sigma_j - (j-1)\sigma_{j-1}) = a_j\sigma_{j-1}^{j-1} \,.
$$

Lets evaluate this at $j = n$ to get

$$
\sigma_n^n \geq a_n\sigma_{n-1}^{n-1} \geq a_n a_{n-1}\sigma_{n-2}^{n-2} \geq \cdots \geq a_n a_{n-1}\cdots a_2\sigma_1^1 = \prod_{i=1}^{n} a_i \,.
$$

Taking the n-th root of both sides we get

$$
\sigma_n = \frac{1}{n}\sum_{i=1}^{n} a_i \geq \left(\prod_{i=1}^{n} a_i\right)^{1/n} \,,
$$

which is the GM $\leq$ AM inequality.

Exercise 1.85

If we define $s \equiv a + b + c$ we can write the first fraction as

$$\frac{a}{(a+b+c-a)^2} = \frac{a}{(s-a)^2} = \frac{a}{s^2\left(1-\frac{a}{s}\right)^2} = \frac{a/s}{s\left(1-\frac{a}{s}\right)^2}.$$

As each of the fractions on the left-hand-side is of the same "form" as this one we can write our desired inequality as

$$\frac{a/s}{s\left(1-\frac{a}{s}\right)^2} + \frac{b/s}{s\left(1-\frac{b}{s}\right)^2} + \frac{c/s}{s\left(1-\frac{c}{s}\right)^2} \geq \frac{9}{4s}.$$

Multiply both sides by s and define $a' = \frac{a}{s}$, $b' = \frac{b}{s}$, $c' = \frac{c}{s}$ to get

$$\frac{a'}{(1-a')^2} + \frac{b'}{(1-b')^2} + \frac{c'}{(1-c')^2} \geq \frac{9}{4},$$

with $a' + b' + c' = 1$. Now based on the functional form of the above we consider

$$\begin{aligned}
f(x) &= \frac{x}{(1-x)^2} \\
f'(x) &= \frac{1}{(1-x)^2} + \frac{2x}{(1-x)^3} = \frac{1+x}{(1-x)^3} \\
f''(x) &= \frac{1}{(1-x)^3} + \frac{3(1+x)}{(1-x)^4} = \frac{4+2x}{(1-x)^4} \geq 0,
\end{aligned}$$

when $x \geq -\frac{4}{2} = -2$. Thus $f(x)$ is convex for $x \geq 0$. This means that

$$\frac{1}{3}(f(a') + f(b') + f(c')) \geq f\left(\frac{a'+b'+c'}{3}\right) = f\left(\frac{1}{3}\right) = \frac{(1/3)}{(2/3)^2} = \frac{3}{4},$$

or

$$f(a') + f(b') + f(c') \geq \frac{9}{4},$$

the expression above.

Exercise 1.86

Part (ii): If we have $abc = 1$ then the inequality we are given is

$$1 + \frac{3}{\frac{1}{bc} + \frac{1}{ac} + \frac{1}{ab}} \geq \frac{6}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}},$$

which is the first inequality in this problem with $a \to \frac{1}{a}$, $b \to \frac{1}{b}$, and $c \to \frac{1}{c}$.

Exercise 1.88

Part (i): Recall Hölder's inequality which states that when $\frac{1}{a}+\frac{1}{b}=1$ that

$$\sum_{i=1}^{n} x_i y_i \le \left(\sum_{i=1}^{n} x_i^a\right)^{1/a} \left(\sum_{i=1}^{n} x_i^b\right)^{1/b} . \quad (976)$$

Now if

$$\frac{1}{a}+\frac{1}{b}=\frac{1}{c} ,$$

and we divide by $\frac{1}{c}$ we get

$$\frac{1}{\frac{a}{c}}+\frac{1}{\frac{b}{c}}=1 ,$$

Then lets write Hölder's inequality with $a \to \frac{a}{c}$ and $b \to \frac{b}{c}$ to get

$$\sum_{i=1}^{n} x_i y_i \le \left(\sum_{i=1}^{n} x_i^{a/c}\right)^{c/a} \left(\sum_{i=1}^{n} x_i^{b/c}\right)^{c/b} .$$

If we take $x_i \to x_i^c$ and $y_i \to y_i^c$ and take the $\frac{1}{c}$ "root" of both sides this is

$$\left(\sum_{i=1}^{n} x_i^c y_i^c\right)^{1/c} \le \left(\sum_{i=1}^{n} x_i^a\right)^{1/a} \left(\sum_{i=1}^{n} x_i^b\right)^{1/b} ,$$

the desired expression.

Part (ii): From the given $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=1$ define C such that

$$\frac{1}{C}=\frac{1}{a}+\frac{1}{b} \quad \text{so} \quad \frac{1}{C}+\frac{1}{c}=1 .$$

Lets use Hölder's inequality with the "pairs" $x_i y_i$ and z_i as

$$\sum_{i=1}^{n} x_i y_i z_i \le \left(\sum_{i=1}^{n} (x_i y_i)^C\right)^{1/C} \left(\sum_{i=1}^{n} z_i^c\right)^{1/c} .$$

Now from **Part (i)** above we can bound $\left(\sum_{i=1}^{n}(x_i y_i)^C\right)^{1/C}$ above by $\left(\sum_{i=1}^{n} x_i^a\right)^{1/a} \left(\sum_{i=1}^{n} y_i^b\right)^{1/b}$ to get the desired result.

A helpful inequality

Exercise 1.92

WWX: working here.

Problem Book in High-School Mathematics

Rational Equations, Inequalities, and Functions of One Variable

Problem 1

This is

$$x + 2 = 3 \quad \text{so} \quad x = 1 \,.$$

Problem 2

WWX: DP

Problem 3

Part (a): WWX: DP

Part (b): WWX: DP

Problem 4

If $a = 0$ then all x are solutions. If $a \neq 0$ then $x = a$ is a solution.

Problem 5

If $a = 2$ then all x are solutions. If $x \neq 2$ then $x = a + 2$.

Problem 6

Write this as

$$(a - 3)(a + 3)x = (a + 3)(a^2 - 3a + 9) \,.$$

If $a = -3$ then all x are solutions. If $a \neq -3$ then we can write the above as

$$(a - 3)x = a^2 - 3a + 9 \,.$$

Here we see that if $a = 3$ then there are no solutions. If $a \neq 3$ the solution is

$$x = \frac{a^2 - 3a + 9}{a - 3}\,.$$

Chinese Mathematics Competitions and Olympiads: 1981-1993

1981/82: Paper I

Section 1: Problem 1

Recall some conditions:

- If P is necessary for Q means that Q cannot be true unless P is true.
- If P sufficient for Q means knowing P is true means knowing that Q true but know $\neg P$ does not mean Q is false.

This means that P is necessary for Q, i.e. if Q is true two triangles are congruent then they have equal areas and two equal sides.

To show that P is not sufficient for Q we can imagine an acute triangle and an obtuse triangle with two equal sides (one of which is the base) and equal heights. These two triangles would have the same area but they are not congruent. Thus P is not sufficient for Q.

Then P is necessary but not sufficient for Q.

Section 1: Problem 2

Now P necessary for Q then this means that Q cannot be true unless P is. Now if we take

$$\theta = \frac{\pi}{2}\,,$$

then Q

$$\sin(\theta) + \cos(\theta) = a\,,$$

becomes

$$\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = a\,,$$

so that $a = \sqrt{2}$. In this case P is

$$a = \sqrt{1 + \sin(\theta)} = \sqrt{1 + 1} = \sqrt{2}\,,$$

which is true.

This simple setting of parameters makes me think that maybe if Q is true then P must be true so that P would be necessary for Q. To try to show that lets square Q to get

$$1 + 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) = a^2 \,.$$

But recall that

$$\sin(\theta) = 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) \,,$$

so Q squared is

$$\sin(\theta) + 1 = a^2 \,.$$

Taking the square root we get

$$\sqrt{1+\sin(\theta)} = a \,,$$

which would be P if $a > 0$ since then $\sqrt{a^2} = a$. This makes me think that P might not be true if $a < 0$ and so in that case P would not be necessary for Q.

Lets find a value for θ so that $a < 0$ in Q. If we take

$$\frac{\theta}{2} = \frac{5\pi}{4} \quad \text{so} \quad \theta = \frac{5\pi}{2} = 2\pi - \frac{\pi}{2} \,.$$

Then Q is

$$-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = a \quad \text{so} \quad a = -\sqrt{2} \,.$$

While for that value of θ P is

$$\sqrt{1-1} = 0 \neq a \,,$$

and thus P is *not* true. This means that P is *not* necessary Q.

We ask now if P is sufficient for Q? This is no because if we take $\theta = \frac{5\pi}{2}$ (as above) P

$$\sqrt{1+\sin(\theta)} = a \,,$$

becomes $0 = a$ while Q is

$$-\sqrt{2} = a \,,$$

which is a contradiction.

Thus P is not necessary or sufficient for Q.

Section 1: Problem 3

We can write T as

$$
\begin{aligned}
T &= \frac{\sin(a)+\tan(a)}{\cos(a)+\cot(a)} = \frac{\sin(a)+\frac{\sin(a)}{\cos(a)}}{\cos(a)+\frac{\cos(a)}{\sin(a)}} \\
&= \frac{\sin(a)}{\cos(a)}\left(\frac{1+\frac{1}{\cos(a)}}{1+\frac{1}{\sin(a)}}\right) \\
&= \frac{\sin(a)}{\cos(a)}\left(\frac{\sin(a)\cos(a)+\sin(a)}{\sin(a)\cos(a)+\cos(a)}\right) \\
&= \frac{\sin(a)^2}{\cos(a)^2}\left(\frac{\cos(a)+1}{\sin(a)+1}\right) .
\end{aligned}
$$

Now when $a \neq \frac{k\pi}{2}$ we have $\cos(a) \neq 0$, $\sin(a) \neq 0$, $\cos(a)+1>0$ and $\sin(a)+1>0$. Thus $T>0$.

Section 1: Problem 4

Lets compute the area of each figure.

Part (a): Place this triangle with the side AB along the x-axis of an x-y Cartesian system. Then the point C is "above" the segment AB. Drop a perpendicular from C towards the segment AB intersecting AB at a point D. Let $CD=h$ and then from the fact that $\angle A = 60^\circ$ we have

$$
\begin{aligned}
CD &= h = \sqrt{2}\sin(60) = \frac{\sqrt{3}}{\sqrt{2}} \\
AD &= \sqrt{2}\cos(60) = \frac{1}{\sqrt{2}} .
\end{aligned}
$$

Now as $\angle B = 45^\circ$ and triangle $\triangle BDC$ is a right triangle we have $BD = CD = h = \frac{\sqrt{3}}{\sqrt{2}}$. The total area of triangle $\triangle ABC$ is then

$$
\begin{aligned}
A_a &= \frac{1}{2}AD\cdot CD + \frac{1}{2}BD\cdot CD = \frac{1}{2}\left(\frac{\sqrt{3}}{\sqrt{2}}\right)(AD+BD) \\
&= \frac{\sqrt{3}}{4}(1+\sqrt{3}) = \frac{3+\sqrt{3}}{4} .
\end{aligned}
$$

Recalling that $\sqrt{3} = 1.73205$ the above becomes $A_a = 1.18301$.

Part (b): Recall that the area of a trapezoid with diagonals of length d_1 and d_2 and an angle between them of θ is given by

$$
\frac{1}{2}d_1 d_2 \sin(\theta) \tag{977}
$$

For this part we then have

$$A_b = \frac{1}{2}\sqrt{2}\sqrt{3}\sin(75^\circ) < \frac{1}{2}\sqrt{2}\sqrt{3} < \frac{1}{2}\sqrt{3}\sqrt{3} = \frac{3}{2} < \pi\,.$$

Part (c): This would be $A_c = \pi = 3.14159$.

Part (d): Let the side of the square be s. Then using the Pythagorean theorem we have that $2s^2 = d^2 = 2.5^2 = \left(\frac{5}{2}\right)^2$. This means that

$$s = \frac{5}{2\sqrt{2}}\,.$$

The area is then $A_d = s^2 = \frac{25}{8} = 3\frac{1}{8} = 3.125$.

Based on these the greatest area is from Part (c).

Section 1: Problem 5

WWX: DP

Section 1: Problem 6

The region M is a triangle and the region N is a region that "straddles" the vertex of this triangle. This region is then the sum of two trapezoids and each has an area given by $\frac{1}{2}h[b_1 + b_2]$. Thus the total area is

$$\frac{1}{2}(1-t)[t+1] + \frac{1}{2}(t+1-1)[1+2-(t+1)]\,.$$

If we simplify this we get the answer (a).

Section 3: Problem 1

WWX: DP

Section 3: Problem 2

WWX: DP

Section 3: Problem 3

WWX: DP

Section 3: Problem 4

WWX: DP

1982/83: Paper I

Section 1: Problem 1

WWX: DP

Section 1: Problem 2

If we "clear fractions" we get

$$p - p\cos(\theta) + p\sin(\theta) = 1\,,$$

or in terms of (x, y) this is

$$\sqrt{x^2 + y^2} - x + y = 1\,,$$

or

$$\sqrt{x^2 + y^2} = 1 + x - y\,.$$

If we square this we get

$$x^2 + y^2 = 1 + x^2 + y^2 + 2x - 2y - 2xy\,,$$

or

$$2xy - 2x + 2y - 1 = 0\,.$$

We can write this as

$$xy - x + y - \frac{1}{2} = 0\,,$$

or

$$(x + 1)(y - 1) + 1 - \frac{1}{2} = 0\,.$$

This is a hyperbola.

Section 1: Problem 3

WWX: DP

Section 1: Problem 4

WWX: DP

Section 1: Problem 5

WWX: DP

Section 1: Problem 6

Write this expression as

$$x^2 = (k-2)x - (k^2+3k+5) \, .$$

Then this means that

$$x_1^2 + x_2^2 = (k-2)(x_1+x_2) - 2(k^2+3k+5) \, .$$

As this expression is a quadratic by factoring we see that the sum of the roots is given by

$$x_1 + x_2 = -(-(k-2)) = k-2 \, ,$$

so that

$$\begin{aligned} x_1^2 + x_2^2 &= (k-2)^2 - 2(k^2+3k+5) = -k^2 - 10k - 6 \\ &= -(k^2+10k) - 6 = -(k^2+10k+25-25) - 6 \\ &= -(k+5)^2 + 19 \, . \end{aligned}$$

Notice that this is an upside down parabola where the largest this value can be is 19 when $k = -5$.

Note that not all values for k will give real roots x_1 and x_2. To have real roots means that the discriminant D is positive or

$$D = (k-2)^2 - 4(k^2+3k+5) = -3k^2 - 16k - 16 > 0 \, .$$

This is equivalent to

$$(3k+4)(k+4) < 0 \, .$$

This will be satisfied when $-4 < k < -\frac{4}{3}$. The value of k that is closest to the location of the maximum $k = -5$ will be the largest this expression can be. Thus $k = -4$ with a value of

$$-1 + 19 = 18 \, ,$$

is the largest value we can have for this expression.

Section 1: Problem 7

WWX: do this problem

1985/86: Paper I

Section 2: Problem 3

WWX: DP

Section 2: Problem 4

Given the condition

$$x \star y = ax + by + cxy\,,$$

and what we are told we have

$$1 \star 2 = a + 2b + 2c = 3 \quad (978)$$

$$2 \star 3 = 2a + 3b + 6c = 4\,. \quad (979)$$

We are looking for a d such that

$$x \star d = ax + bd + cxd = x\,,$$

for all x. This last condition is equivalent to

$$(a - 1 + cd)x + bd = 0\,.$$

The only way this is possible for all x is if

$$a - 1 + cd = 0 \quad (980)$$

$$bd = 0\,. \quad (981)$$

These give us four equations in the four unknowns a, b, c, and d that we need to solve. From Equation 981 we have $b = 0$ or $d = 0$. As $d \neq 0$ we must have $b = 0$. With that our three remaining equations become

$$a + 2c = 3 \quad (982)$$

$$2a + 6c = 4 \quad (983)$$

$$a - 1 + cd = 0\,. \quad (984)$$

Solving Equations 982 and 983 we get $a = 5$ and $c = -1$. Putting these into Equation 984 we get

$$5 - 1 + (-1)d = 0 \quad \text{so} \quad d = 4\,.$$

The 46th Putnam Mathematical Competition

Problem A.1

WWX: DP

Problem A.5

We might see if we can find a pattern for I_m. For $m = 1$ the integrand is $P_1 = \cos(x)$ and we have

$$I_1 = \int_0^{2\pi} \cos(x) dx = \sin(x)|_0^{2\pi} = 0\,.$$

Now lets now consider the expression for I_2. We have

$$P_2 = \cos(x)\cos(2x) = \frac{1}{2}(\cos(x+2x) + \cos(x-2x)) = \frac{1}{2}(\cos(3x) + \cos(x))\,. \tag{985}$$

From this the value of I_2 is

$$I_2 = \frac{1}{2}\left.\frac{\sin(3x)}{3}\right|_0^{2\pi} + \frac{1}{2}\sin(x)|_0^{2\pi} = 0\,.$$

Next consider the integrand in I_3 or P_3 which using Equation 985 is

$$\begin{aligned}
P_3 = \cos(x)\cos(2x)\cos(3x) = P_2\cos(3x) &= \frac{1}{2}(\cos(3x) + \cos(x))\cos(3x) \\
&= \frac{1}{2}\left(\frac{1}{2}(\cos(6x) + \cos(0)) + \frac{1}{2}(\cos(4x) + \cos(2x))\right) \\
&= \frac{1}{4}\cos(6x) + \frac{1}{4} + \frac{1}{4}\cos(4x) + \frac{1}{4}\cos(2x)\,. \qquad (986)
\end{aligned}$$

This expression will have a nonzero integral due to the $\frac{1}{4}$ term.

The pattern we observe at this point is that each P_m can be written like

$$P_m = \sum_{k\in S_m} A_k \cos(kx)\,, \tag{987}$$

where S_n is a set of integers and A_k are nonzero constants. When we multiply a sum like this by a factor $\cos(nx)$ we can use the cosign product formula

$$\cos(kx)\cos(nx) = \frac{1}{2}\left(\cos((k+n)x) + \cos((k-n)x)\right)\,,$$

that will give rise to *another* sum of the form $\sum_{k\in S_{m+1}} A'_k \cos(kx)$ but with a different set of integers S_{m+1}. The value for I_m will be nonzero only when though this process we end with an expansion like the above where $0 \in S_m$.

As a bit of detail we have $P_1 = \cos(x)$ and so $S_1 = \{1\}$. By Equation 985 we see that $S_2 \in \{1, 3\}$. By Equation 986 we see that $S_3 \in \{0, 2, 4, 6\}$. Now in computing S_4 we will take every element in S_3 and add and subtract the number four to get

$$\{4, 6, 8, 10, -4, -2, 0, 2\} ,$$

and since cos is an even function we take the absolute value of the above numbers to get

$$S_4 = \{0, 2, 4, 6, 8, 10\} .$$

At this point we have argued that I_3 and I_4 are nonzero (as $0 \in S_3$ and $0 \in S_4$). We can continue this pattern up to $m = 10$ to answer the given question. While the above calculations are simple enough to do by hand I choose to implement them in `python` to make sure I don't make any mistakes (see the code `1985_Putnam_A5.py`). When we run that code we get

```
P_3 has a zero in the set S_3
P_4 has a zero in the set S_4
P_7 has a zero in the set S_7
P_8 has a zero in the set S_8
```

These are the four integers m where $I_m \neq 0$.

The 60th Putnam Mathematical Competition

Problem A.1

We must have f, g, and h linear or else there will be regions where the expression on the left-hand-side will not be linear. Thus we will take

$$\begin{aligned} f(x) &= f_0 + f_1 x \\ g(x) &= g_0 + g_1 x \\ h(x) &= h_0 + h_1 x \,. \end{aligned}$$

Now if we take $x \to -\infty$ then the left-hand-side of the given expression will trend to

$$|f_1 x| - |g_1 x| + h_1 x = |f_1||x| - |g_1||x| + h_1 x \,.$$

as $x < 0$ this is

$$-|f_1|x + |g_1|x + h_1 x = (-|f_1| + |g_1| + h_1)x \,.$$

As the right-hand-side has no x dependence the coefficient of x must be zero so

$$h_1 = |f_1| - |g_1| \,. \tag{988}$$

Now if we take $x \to +\infty$ then the left-hand-side of the given expression will trend to

$$|f_1 x| - |g_1 x| + h_1 x = |f_1|x - |g_1|x + h_1 x \,.$$

As the right-hand-side has the coefficient of x of -2 we must have

$$|f_1| - |g_1| + h_1 = -2 \,. \tag{989}$$

Using Equation 988 for $|f_1| - |g_1|$ in the above we get $2h_1 = -2$ so $h_1 = -1$. This also gives

$$h_1 = |f_1| - |g_1| = -1 \,. \tag{990}$$

We have shown that our expression can be written as

$$|f_1 x + f_0| - |g_1 x + g_0| - x + h_0 = \begin{cases} -1 & x < -1 \\ 3x + 2 & -1 \le x \le 0 \\ -2x + 2 & x > 0 \end{cases} \,.$$

Moving the $-x$ to the right-hand-side this is equivalent to

$$|f_1 x + f_0| - |g_1 x + g_0| + h_0 = \begin{cases} x - 1 & x < -1 \\ 4x + 2 & -1 \le x \le 0 \\ -x + 2 & x > 0 \end{cases} \,.$$

Taking $x = 0$ in the above we can solve for h_0 and find

$$h_0 = 2 - |f_0| + |g_0| .$$

Putting this into the above and moving the two to the right-hand-side we are left with

$$|f_1 x + f_0| - |f_0| - |g_1 x + g_0| + |g_0| = \begin{cases} x - 3 & x < -1 \\ 4x & -1 \leq x \leq 0 \\ -x & x > 0 \end{cases} . \tag{991}$$

Now in the region $-1 \leq x \leq 0$ comparing the coefficient of x between the left and right sides of the equations means that

$$|f_1| - |g_1| = 4 .$$

With Equation 990 we now have two equations for $|f_1|$ and $|g_1|$. Solving them we find

$$|f_1| = \frac{3}{2}$$
$$|g_1| = \frac{5}{2} .$$

These mean that

$$f_1 = s_f \frac{3}{2}$$
$$g_1 = s_g \frac{5}{2} ,$$

where $s_f = \pm 1$ and $s_g = \pm 1$ are the "signs" of f_1 and g_1 respectively.

Now from the right-hand-side of Equation 991 we see that the only "kinks" can happen at the point $x = -1$ and $x = 0$. From the left-hand-side of that equation the kinks happen when $f_1 x + f_0 = 0$ and $g_1 x + g_0 = 0$ or at

$$-\frac{f_0}{f_1} \quad \text{and} \quad -\frac{g_0}{g_1} .$$

Lets assume that $-\frac{f_0}{f_1} = -1$ and $\frac{g_0}{g_1} = 0$ so that $f_0 = f_1$ and $g_0 = 0$. Using these in the left-hand-side of Equation 991 we get

$$\left| s_f \frac{3}{2} x + s_f \frac{3}{2} \right| - \left| s_f \frac{3}{2} \right| - \left| s_g \frac{5}{2} x \right| = \frac{3}{2}|x+1| - \frac{3}{2} - \frac{5}{2}|x| .$$

We can evaluate this expression in the regions $x < -1$, $-1 < x < 0$ and $x > 0$ and show that it equals the right-hand-side of Equation 991 and we have found a solution.

To see if we have another solution lets now assume that $-\frac{f_0}{f_1} = 0$ and $\frac{g_0}{g_1} = -1$ so that $f_0 = 0$ and $g_0 = f_1$. Using these in the left-hand-side of Equation 991 we get

$$\left| s_f \frac{3}{2} x \right| - 0 - \left| s_g \frac{5}{2} x + s_g \frac{5}{2} \right| + \left| s_g \frac{5}{2} \right| = \frac{3}{2}|x| - \frac{5}{2}|x+1| + \frac{5}{2} .$$

We can evaluate this expression in the regions $x < -1$ and find that its not equal to $x - 3$ as it should be. So we have not found another solution.

Thus is looks like the solutions are

$$\begin{aligned} f(x) &= \pm\frac{3}{2}(x+1) \\ g(x) &= \pm\frac{5}{2}x \\ h(x) &= -x + h_0 = -x + 2 - \frac{3}{2} = -x + \frac{1}{2} . \end{aligned}$$

Problem A.2

WWX: DP

Problem A.3

To determine the form for a_n write the given expression as

$$1 = \left(\sum_{n=0}^{\infty} a_n x^n\right)(1 - 2x - x^2) .$$

Then expanding the terms on the right-hand-side we get

$$1 = \sum_{n=0}^{\infty} a_n x^n - 2\sum_{n=0}^{\infty} a_n x^{n+1} - \sum_{n=0}^{\infty} a_n x^{n+2} ,$$

or

$$1 = \sum_{n=0}^{\infty} a_n x^n - 2\sum_{n=1}^{\infty} a_{n-1} x^n - \sum_{n=2}^{\infty} a_{n-2} x^n .$$

or "releasing" some terms we get

$$\begin{aligned} 1 &= a_0 + a_1 x - 2a_0 x + \sum_{n=2}^{\infty}(a_n - 2a_{n-1} - a_{n-2})x^n \\ &= a_0 + (a_1 - 2a_0)x + \sum_{n=2}^{\infty}(a_n - 2a_{n-1} - a_{n-2})x^n . \end{aligned}$$

If we equate the terms on the right and left of this expression we see that $a_0 = 1$ and $a_1 = 2a_0 = 2$. In addition to make the coefficient of x^n (for $n \geq 2$) zero we must have

$$a_n - 2a_{n-1} - a_{n-2} = 0 .$$

This is a second order difference equation with constant coefficients. Solutions of this type are of the form $a_n = r^n$. If we substitute that expression into the above we get

$$r^n - 2r^{n-1} - r^{n-2} = 0 ,$$

or dividing by r^{n-2} this is the quadratic equation

$$r^2 - 2r - 1 = 0\,.$$

This has solutions given by the quadratic formula or

$$r = \frac{2 \pm \sqrt{4 - 4(-1)}}{2} = 1 \pm \sqrt{2}\,.$$

This means that the general form for a_n is given by

$$a_n = C(1-\sqrt{2})^n + D(1+\sqrt{2})^n\,,$$

for two constants C and D. To evaluate those note that

$$a_0 = C + D = 1\,. \tag{992}$$

and

$$a_1 = C(1-\sqrt{2}) + D(1+\sqrt{2}) = 2\,.$$

If we use Equation 992 in the above we get

$$-\sqrt{2}C + \sqrt{2}D = 1\,.$$

Thus C and D satisfy two linear equations. Solving these we get

$$\begin{aligned} C &= -\left(\frac{1-\sqrt{2}}{2\sqrt{2}}\right) \\ D &= \frac{1+\sqrt{2}}{2\sqrt{2}}\,. \end{aligned}$$

Using these we have that

$$a_n = -\frac{1}{2\sqrt{2}}(1-\sqrt{2})^{n+1} + \frac{1}{2\sqrt{2}}(1+\sqrt{2})^{n+1}\,. \tag{993}$$

We now compute $a_n^2 + a_{n+1}^2$. Using the above formula we find

$$\begin{aligned} a_n^2 + a_{n+1}^2 &= \frac{1}{4\cdot 2}(-(1-\sqrt{2})^{n+1} + (1+\sqrt{2})^{n+1})^2 + \frac{1}{4\cdot 2}(-(1-\sqrt{2})^{n+2} + (1+\sqrt{2})^{n+2})^2 \\ &= \frac{1}{4\cdot 2}\left((1-\sqrt{2})^{2n+2} - 2(1-\sqrt{2})^{n+1}(1+\sqrt{2})^{n+1} + (1+\sqrt{2})^{2n+2}\right) \\ &+ \frac{1}{4\cdot 2}\left((1-\sqrt{2})^{2n+4} - 2(1-\sqrt{2})^{n+2}(1+\sqrt{2})^{n+2} + (1+\sqrt{2})^{2n+4}\right) \\ &= \frac{1}{8}\left((1-\sqrt{2})^{2n+2} - 2(1-2)^{n+1} + (1+\sqrt{2})^{2n+2}\right) \\ &+ \frac{1}{8}\left((1-\sqrt{2})^{2n+4} - 2(1-2)^{n+2} + (1+\sqrt{2})^{2n+4}\right)\,. \end{aligned}$$

In the above we have used the fact that $(1-\sqrt{2})(1+\sqrt{2}) = 1-2$. We now group the first, second, and third terms together taken from each of the two separate expressions above. This gives

$$
\begin{aligned}
8(a_n^2 + a_{n+1}^2) &= (1-\sqrt{2})^{2n+2}\left[(1-\sqrt{2})^2 + 1\right] \\
&- 2((-1)^{n+1} + (-1)^{n+2}) \\
&+ (1+\sqrt{2})^{2n+2}\left[(1+\sqrt{2})^2 + 1\right] \\
&= (1-\sqrt{2})^{2n+2}\left[1 - 2\sqrt{2} + 2 + 1\right] \\
&- 2(-1)^{n+1}(1-1) \\
&+ (1+\sqrt{2})^{2n+2}\left[1 + 2\sqrt{2} + 2 + 1\right] \\
&= (1-\sqrt{2})^{2n+2}\left[4 - 2\sqrt{2}\right] + (1+\sqrt{2})^{2n+2}\left[4 + 2\sqrt{2}\right] .
\end{aligned}
$$

We now factor out a two from the right-hand-side (and cancel it with a two on the left-hand-side) to get

$$4(a_n^2 + a_{n+1}^2) = (1-\sqrt{2})^{2n+2}(2-\sqrt{2}) + (1+\sqrt{2})^{2n+2}(2+\sqrt{2}) .$$

We now factor out a $\sqrt{2}$ from the right-hand-side to get

$$a_n^2 + a_{n+1}^2 = \frac{1}{2\sqrt{2}}(1-\sqrt{2})^{2n+2}(\sqrt{2}-1) + \frac{1}{2\sqrt{2}}(1+\sqrt{2})^{2n+2}(\sqrt{2}+1) ,$$

or

$$a_n^2 + a_{n+1}^2 = -\frac{1}{2\sqrt{2}}(1-\sqrt{2})^{2n+3} + \frac{1}{2\sqrt{2}}(1+\sqrt{2})^{2n+3} ,$$

which we recognize as a_{2n+2}.

Problem A.4

WWX: DP

Problem A.5

WWX: DP

Problem A.6

Write this as

$$a_n = \frac{6a_{n-1}^2}{a_{n-2}} - \frac{8a_{n-1}a_{n-2}}{a_{n-3}} .$$

Then divide both sides by a_{n-1} to get

$$\frac{a_n}{a_{n-1}} = 6\left(\frac{a_{n-1}}{a_{n-2}}\right) - 8\left(\frac{a_{n-2}}{a_{n-3}}\right) .$$

Based on this form we take $r_n \equiv \frac{a_n}{a_{n-1}}$ for $n \geq 2$ and the above becomes

$$r_n = 6r_{n-1} - 8r_{n-2} . \tag{994}$$

The value of r_n for $n \in \{2, 3\}$ are

$$r_2 = \frac{a_2}{a_1} = \frac{2}{1} = 2$$
$$r_3 = \frac{a_3}{a_2} = \frac{24}{2} = 12 .$$

If we take $n = 3$ in Equation 994 we find that

$$r_3 = 6r_2 - 8r_1 .$$

Using what we know for r_3 and r_2 we find that $r_1 = 0$.

The general solution of equations like Equation 994 is to reduce it to characteristic form by assuming a form $r_n = x^n$ and find the value for x. When we do that we get the quadratic equation

$$x^2 = 6x - 8 \quad \text{or} \quad (x-2)(x-4) = 0 .$$

Thus the solution for r_n is

$$r_n = C2^n + D4^n ,$$

for constants C and D. Taking $n = 1$ and using $r_1 = 0$ in the above we see that $C = -2D$ so that

$$r_n = D(2^{n-1} - 1)2^{n+1} .$$

Taking $n = 2$ in the above and using $r_2 = 2$ we see that $D = \frac{1}{4} = 2^{-2}$ and we have

$$r_n = (2^{n-1} - 1)2^{n-1} ,$$

a known function of n for $n \geq 1$.

In summary, at this point for a_n we have shown that

$$a_n = r_n a_{n-1} , \tag{995}$$

for $n \geq 2$ and r_n is a known function of n. If we iterate Equation 995 we get

$$\begin{aligned}
a_2 &= r_2 a_1 \\
a_3 &= r_3 a_2 = r_3 r_2 a_1 \\
a_4 &= r_4 a_3 = r_4 r_3 r_2 a_1 \\
&\vdots \\
a_n &= \left(\prod_{i=2}^{n} r_i\right) a_1 .
\end{aligned}$$

Since $a_1 = 1$ and using the known form for r_i we have

$$
\begin{aligned}
a_n &= \prod_{i=2}^{n} 2^{i-1}(2^{i-1} - 1) \\
&= \prod_{i=1}^{n-1} 2^i(2^i - 1) = \left(\prod_{i=1}^{n-1} 2^i\right)\left(\prod_{i=1}^{n-1}(2^i - 1)\right) \\
&= 2^{\frac{n(n-1)}{2}}\left(\prod_{i=1}^{n-1}(2^i - 1)\right) . \qquad (996)
\end{aligned}
$$

The above give a "functional form" for a_n.

Now if $n = 1$ we have $a_n = 1$ and thus a_n is proportional to n in this case. If $n > 1$ then we perform a prime factorization of n to write it as $n = 2^k m$ for $m \geq 1$ and an odd number. Of course for k in this factorization we have

$$
k \leq n \leq \frac{n(n-1)}{2} .
$$

This means that 2^k will divide the expression $2^{\frac{n(n-1)}{2}}$. Next the question is whether m in the prime factorization of n will divide the product factor in Equation 996. **Warning:** The solution in the back of the book claims that one of the factors $2^i - 1$ will in-fact have m as a divisor. I'm not able to verity that statement. If anyone has a ELI5 (explain it like I'm five) answer for me please contact me.

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