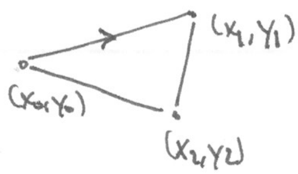
① l_1 def $d = |x_1 - x_0| + |y_1 - y_0|$

D.1 ✓

D.2 ✓

D.3 ✓

D.4 ✓



$\cancel{d}(x_0,y_0)$, $|x_1 - x_0| + |y_1 - y_0| = |x_1 - x_2 + x_2 - x_0| + \cdots$

$\leq \cdots$

D.5 ✓

l_∞ def $d_{l_\infty} = \text{Max}\{|x_0 - x_1|, |y_0 - y_1|\}$.

D.1 ✓

D.2 ✓

D.3 ✓

D.4 ✓

D.5.

$d_{01} = \text{Max}\{|x_0 - x_2 + x_2 - x_1|, |y_0 - y_2 + y_2 - y_1|\}$ $\{d_{02} + d_{21}\}$

$\leq \text{Max}\{|x_0 - x_2| + |x_2 - x_1|, |y_0 - y_2| + |y_2 - y_1|\}$

$\leq \text{Max}\{|x_0 - x_2|, |y_0 - y_2|\} + \text{Max}\{|\cancel{x_2 - x_1}|, |y_2 - y_1|\}$

2

$$d(\vec{r}_0,\vec{r}_1) = \sqrt{(x_0-x_1)^2+(y_0-y_1)^2}$$

$$D \equiv d^2(\vec{r}_0,\vec{r}_1) = (x_0-x_1)^2+(y_0-y_1)^2$$

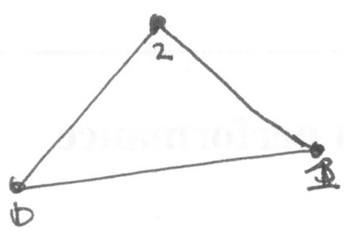
$$= (x_0-x_2+x_2-x_1)^2+(y_0-y_2+y_2-y_1)^2$$

$$(x_0-x_2)^2 + 2(x_0-x_2)(x_2-x_1) + (x_2-x_1)^2$$

$$+ (y_0-y_2)^2 + 2(y_0-y_2)(y_2-y_1) + (y_2-y_1)^2$$

$$(a+b)^2 = a^2 + 2ab + b^2 \overset{?}{\leq} a^2+b^2 \quad \text{iff} \quad a \geq 0, b < 0 \quad \text{or} \quad a<0, b>0$$

Consider



$$D = D(x_2,y_2) = (x_0-x_2)^2 + 2(\ \ )(\ \ ) + (\ \ )^2$$

$$+ (y_0-y_2)^2 + 2(\ \ )(\ \ ) +$$

Find a minim of D

$$\frac{\partial D}{\partial x_2} = 0 \qquad \frac{\partial D}{\partial y_2} = 0 \Rightarrow \text{Solve for } x_2 + y_2$$

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$$D(x_2,y_2) = (x_0-x_2)^2 + 2(x_0-x_2)(x_2-x_1) + (x_2-x_1)^2$$
$$+ (y_0-y_2)^2 + 2(y_0-y_2)(y_2-y_1) + (y_2-y_1)^2$$

$$\frac{\partial D}{\partial x_2} = 2(x_0-x_2) + 2(-1)(x_2-x_1) + 2(x_0-x_2) + 2(x_2-x_1) \overset{\text{set}}{=} 0$$

$$\frac{\partial D}{\partial y_2} = 2(y_0-y_2) + 2(-1)(y_2-y_1) + 2(y_0-y_2) + 2(y_2-y_1) = 0$$

$\Rightarrow$ $\frac{\partial D}{\partial x_2} = $ eq gives

$$x_0 - x_2 - x_2 + x_1 + x_0 - \cancel{x_2} + \cancel{x_2} - x_1 = 0$$

$\frac{\partial D}{\partial y_2} = 0$ eq gives

$$y_0 - y_2 - y_2 + \cancel{y_1} + y_0 - \cancel{y_2} + \cancel{y_2} - y_1 = 0$$

$\Rightarrow$ $2x_0 - 2\bar{x}_2 = 0 \Rightarrow \bar{x}_2 = x_0$ bar means minimum

$2y_0 - 2\bar{y}_2 = 0 \Rightarrow \bar{y}_2 = y_0$

Since B showing that $(\bar{x},\bar{y})$ is a minimum we

have that $D(x_2,y_2) \geq D(\bar{x}_2,\bar{y}_2) = d_{01}$

Actually considering the function of distance from $\bar{r}_0$ to $\bar{r}_1$ + distance from $\bar{r}_1$ to $\bar{r}_2$ + minimizing one should be able to prove this

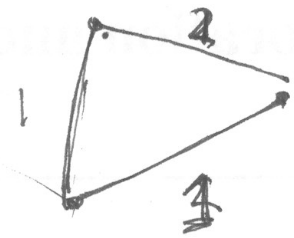
(3) (a) D.1 ✓

D.2 ✓

D.3 ✓

D.4 ✓

D.5 ~~Yes No~~ yes



(b) $S(a,2)$ 10101

* *

# of ways to select 2 elts $\binom{5}{2}$ in $S(a,2)$

$\bar{B}(a,2)$ $\binom{5}{2}+\binom{5}{1}+\binom{5}{0}$

(c) $S(a,5)$ the complement of the input sequence

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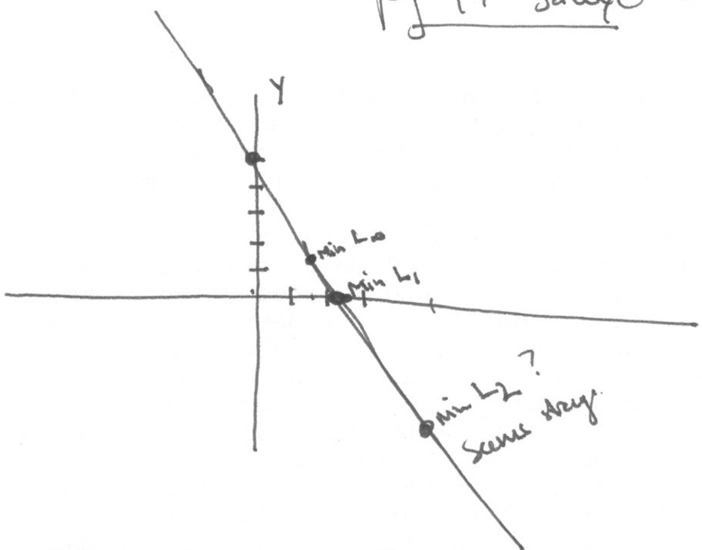
(4)



$2x + y = 5$

x	y
0	5
5/2	0

The distance from the origin to any other point is given by

L_∞:

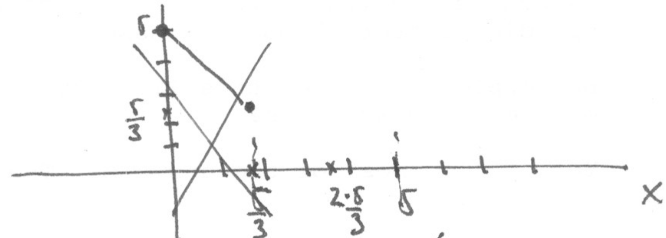
$$d(x,y) = \text{Max}\{|x-0|, |y-0|\}$$

restriction to above line

$$= \text{Max}\{|x|, |5-2x|\}$$

plotting $d(x,y)$ as a fn of x

$\text{Max}\{|x|, |5-2x|\}$



$$|x| = |5-2x|$$

$$\Downarrow$$

$x = -5 + 2x$ or $x = 5 - 2x$

$5 = x$ or $3x = 5$

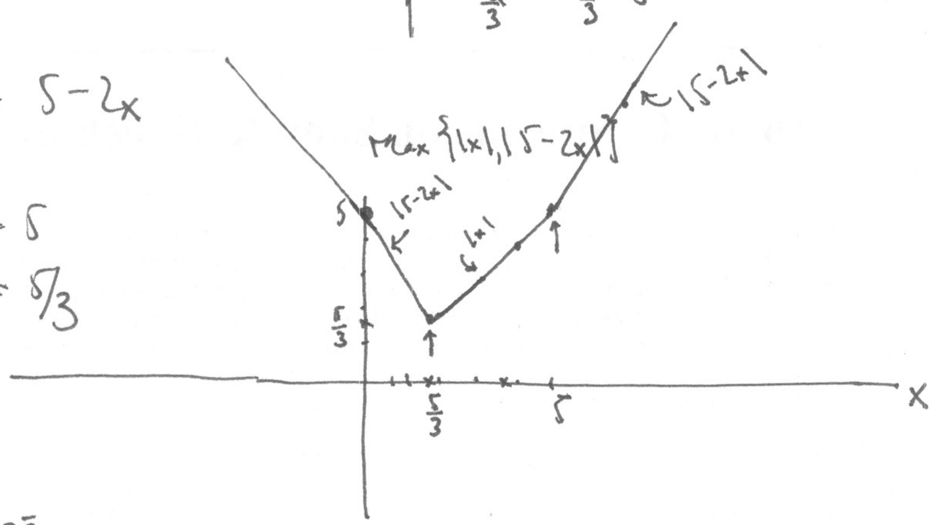
$x = 5/3$

- - - - -

for $x = \frac{5}{6}$

$$\text{Max}\left\{\frac{5}{6}, \left|5 - \frac{5}{3}\right|\right\}$$

$$= \text{Max}\left\{\frac{5}{6}, 2\left(\frac{5}{3}\right)\right\} = \frac{10}{3} = 3.\bar{3}$$



For $x = 4$

$$d = \text{Max}\{4, |5-8|\} = 4$$

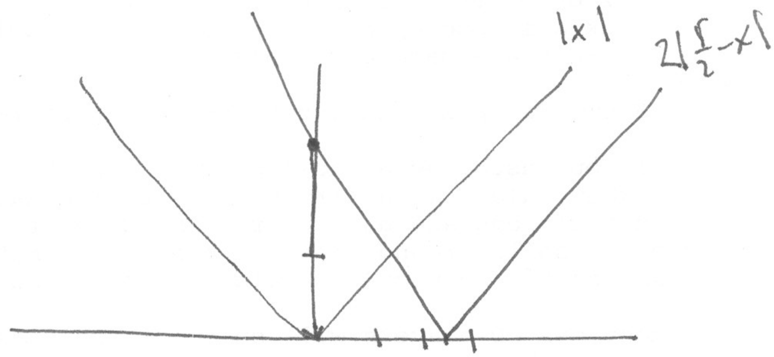
For $x = 6$

$$d = \text{Max}\{6, |5-12|\} = 7$$

The min of d is at $x = \frac{5}{3}$

Then $y = 5 - 2\left(\frac{5}{3}\right) = 5\left(1 - \frac{2}{3}\right) = \frac{5}{3}$

For L_1



$$d(x,y) = |x| + |y| = |x| + |5-2x|$$

$$= |x| + 2\left|\frac{5}{2} - x\right|$$

$$= \begin{cases} x + 2\left|\frac{5}{2} - x\right| & x > 0 \\ -x + 2\left|\frac{5}{2} - x\right| & x < 0 \end{cases}$$

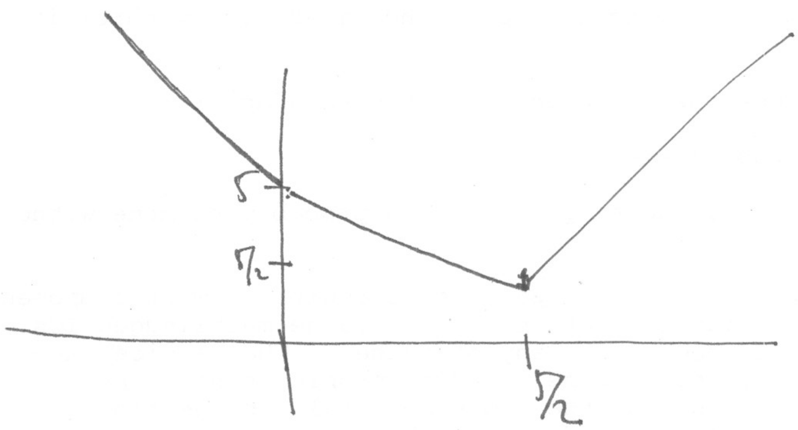
$$= \begin{cases} x + 2\left(\frac{5}{2} - x\right) & x > 0 \ \& \ x < \frac{5}{2} \Rightarrow 0 < x < \frac{5}{2} \\ x + 2\left(x - \frac{5}{2}\right) & x > 0 \ \& \ x > \frac{5}{2} \Rightarrow x > 5/2 \\ -x + 2\left(\frac{5}{2} - x\right) & x < 0 \ \& \ x < \frac{5}{2} \Rightarrow x < 0 \\ -x + 2\left(x - \frac{5}{2}\right) & x < 0 \ \& \ x > 5/2 \end{cases}$$

$$d(x,y) = \begin{cases} -x-2x+5 & x<0 \\ x-2x+5 & 0<x<5/2 \\ 3x-5 & x>5/2 \end{cases}$$

$$d(x,y) = \begin{cases} -3x+5 & x<0 \\ -x+5 & 0<x<5/2 \\ 3x-5 & x>5/2 \end{cases}$$

$3(5/2) - 2(5/2)$

$5/2$



Min $x = 5/2 \quad y = 5 - 2(5/2) = 0$

Now:

L_2:

$$d(x,y)^2 = x^2 + y^2 = x^2 + (5-2x)^2$$

$$\frac{d(d^2)}{dx} = 2x + 2(5-2x) = 0$$

$$x + 5 - 2x = 0 \Rightarrow x = 5 \quad y = -5$$

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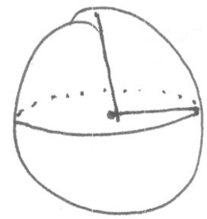
5 (a) $S(N, \frac{\pi R}{2})$ the equator



(b) $B(N, \frac{\pi R}{2})$ the North hemisphere

(c) ~~Clo~~ $\bar{B}(N, \frac{\pi R}{2})$ closed North hem

(d) surface of earth

(e) South pole.

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(1) $Q(x) = ax^2 + bx + c$

V.1 ✓

V.2 ✓ Yes vector space.

V.3 ✓

V.4 ✓

V.5 ✓

V.6 ✓

(2) $Q(0) = 0 \Rightarrow c = 0$

$Q(x) = ax^2 + bx$

Yes. Vector space

(3) $Q(x) = ax^2 + bx$

$Q(1) = a + b = 0 \qquad a = -b$

$Q(x) = ax^2 - \cancel{b}ax = ax(x-1)$

$bx^2 - bx$

$(a+b)x^2 - (a+b)x$ Yes Closed under ~~addition~~ sum

Yes. Vector space

(4) $Q(x) = ax(x-1)$

$Q(2) = 0$ ~~...~~. unless $a = 0$.

~~No quadratic satisfy these 3 conditions~~ only $Q \equiv 0$

Satis this cond.

Yes. Vect. space

(5) $x^2 + bx + c$ No Summing two vectors of

this form does Not yield

V.1 ~~+~~ V.5. No

V.2 ✓ V.6. No

V.3 ✓ V.7. No

V.4 ~~...~~ No

(6) Yes Vector space

(7) Yes

(8) Yes

(9) No. V.1 ~~+~~ . V.[~~3~~ 6,] ~~+~~

(10) No . See above

(11) Yes

(12) ~~No~~ $f(.5)=0$ Yes vector span

~~V.1 No~~ ~~V.5 No.~~

~~V.4 No.~~ ~~V.6 No.~~

(13) No

V.1 No V.5 ... No.

V.4 No V.6 No.

(14) $f \in C^2([0,\pi])$

$$f''+f=0 \Rightarrow f = Ae^{ix}+Be^{-xi}$$

$$\cancel{B} = \cancel{A}$$

$\cancel{f(0) =}$

$$f(0) = A+B=0; \; f(\pi) = Ae^{i\pi}+Be^{-i\pi} = 0$$

$$A=-B \qquad -Be^{i\pi}+Be^{-i\pi}=0$$

$$B(\underbrace{-e^{+i\pi}}_{+1}+\underbrace{e^{-i\pi}}_{-1}) = 0 \quad ✓$$

$$\therefore f(x) = -Be^{ix}+Be^{-ix} = -B(e^{ix}-e^{-ix})$$

$$f(x) = (-B(2i))\frac{e^{ix} - e^{-ix}}{2i} = \tilde{B}\sin(x)$$

~~No.~~ ~~B.1~~ ~~No~~ si

<u>Yes</u>

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Question:

$M0 = 0$ follows from M be Additive for

$$M \cdot 0 = M(v - v) = M(v) + M(-v)$$

$$= Mv - Mv = 0.$$

Question

$$w_{n+1} = U + Mw_n \qquad w_0 = 0.$$

$$w_1 = U \quad , w_2 = U + MU \quad , \quad w_3 = U + MU + M^2U \quad , \ldots$$

in general $\quad w_n = U + MU + M^2U + \cdots + M^{n-1}U.$

If $w_0 \neq 0$. Can this cause the iteration to diverge?

~~$w_1 = w_0$~~ ; ~~$w_2 = U + Mw_0$~~

$$w_1 = U + Mw_0 \; ; \; w_2 = U + M(U + Mw_0) \quad ; \quad w_3 = U + M(M^2w_0 + U + MU)$$

$$= M^2w_0 + U + MU \qquad\qquad = M^3w_0 + U + MU + M^2U$$

$\ldots$ in general $\quad w_n = M^n w_0 + \sum_{l=0}^{n-1} M^k U$.

If the sum converges $M^k U \to 0$ in norm + $\therefore$

$M^n w_0 \to 0$ in norm + also converges

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Ex:

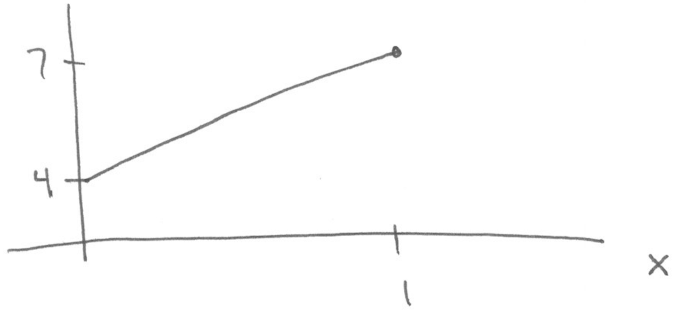
$$g_n(x) = (n^2 x)^n e^{-n^2 x}$$

$$\int_0^1 g_{10}(x)\,dx =$$

The integrals are tending to $+\infty$.

(1) Assume norm is L_∞ norm (max norm).

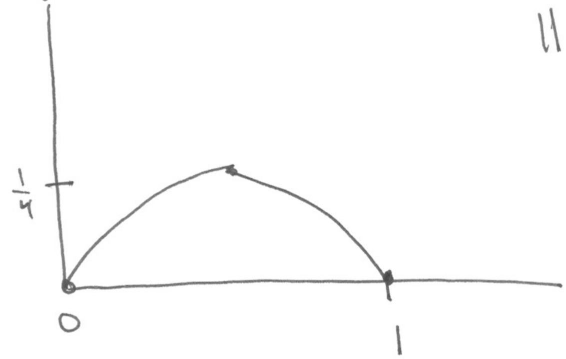
(a)



$\|3x-4\| = 7.$

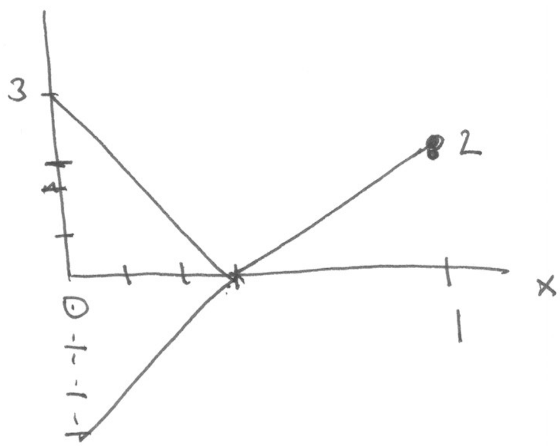
$|x^2-x|$

(b)



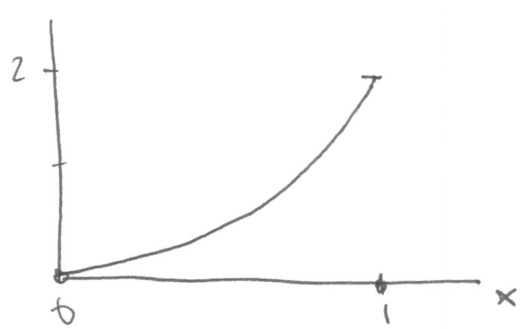
$\|x^2-x\|_\infty = \frac{1}{4}$

(c)



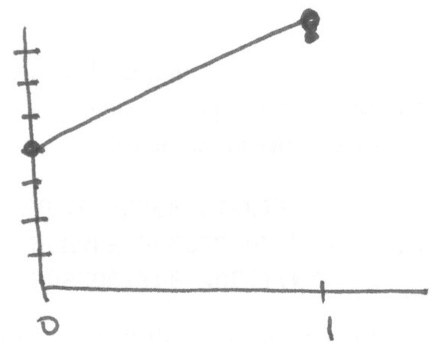
$\|5x-3\|_\infty = 3$

(d) x^2+x



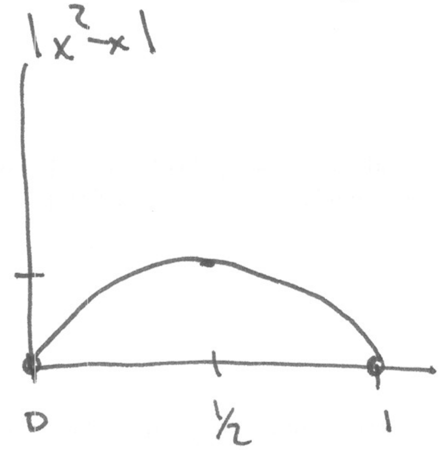
$\|x^2+x\|_\infty = 2.$

① (a)



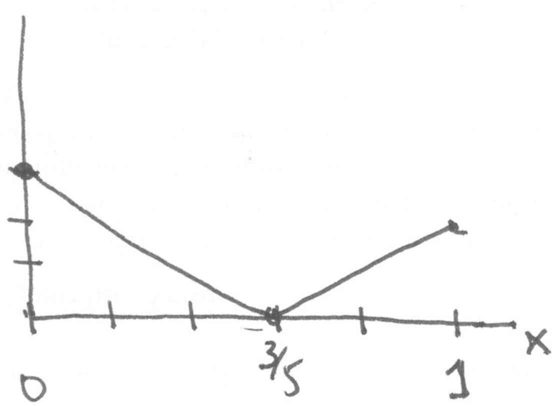
$\|3x+4\|_\infty = 7$

(b)



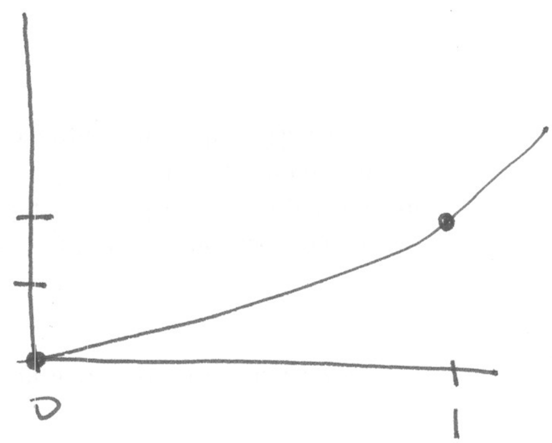
$\|x^2-x\|_\infty = 1/4$

(c)



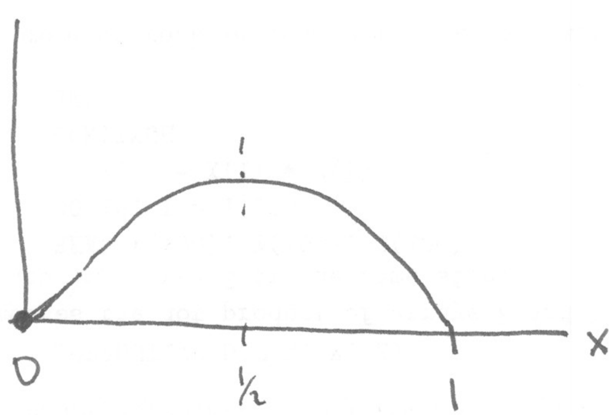
$\|5x-3\|_\infty = 3$

(d)



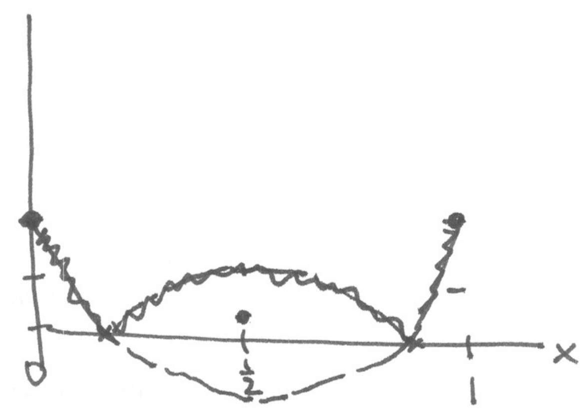
$\|x^2+x\|_\infty = 2$

(e)



$\|\sin \pi x\|_\infty = 1$

(f)



$\frac{1}{4} - \frac{1}{5} = \frac{1}{20} = .05$

$$-x^2+x-.2=0$$

$$-2x+1=0$$
$$x=1/2$$

$$x=\frac{-1\pm\sqrt{1^2-4(-1)(-.2)}}{2(-1)}$$

$$=\frac{-1\pm\sqrt{1-8/10}}{-2}$$

$$=\frac{1\pm\sqrt{1/5}}{2}$$

$$= .276\ ,\ .723$$

$$\|-x^2+x-.2\|_\infty = \max\{.2,\ .2\} = .2$$

(9) $-x^2+x-.1=0$

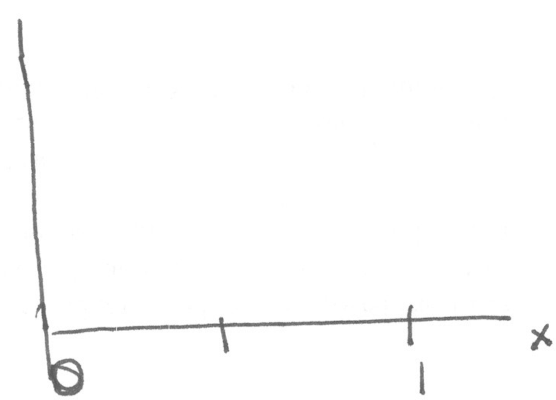
$$x=\frac{-1\pm\sqrt{1^2-4(-1)(-.1)}}{2(-1)} = \frac{-1\pm\sqrt{1-4/10}}{-2} = \frac{1\pm\sqrt{3/5}}{2}$$

$$= 1.127,\ .887$$

$$-2x+1=0 \Rightarrow x=1/2$$

$$(\qquad)\Big|_{1/2} = \frac{1}{4}-\frac{1}{10} = \frac{1}{2}\left(\frac{1}{2}-\frac{1}{5}\right)$$

$$=\frac{1}{2}\left(\frac{5-2}{10}\right) = \frac{3}{20} > \frac{1}{7}$$

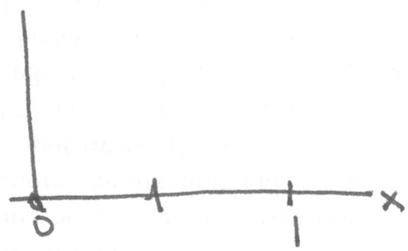


$(\quad)|_0 = -.1$

$(\quad)|_1 = -.1$

$\therefore \quad \| -x^2 + x - .1 \|_\infty = 3/20.$

(h)



$x^{10}(x-1)^{10}$

$\frac{d}{dx} \rightarrow \quad 10(x^2-x)^9(2x-1) = 0$

$(\frac{1}{2})^{10}(\frac{1}{2})^{10} = (\frac{1}{2})^{20}$

$\| \cdots \|_\infty = (\frac{1}{2})^{20}.$

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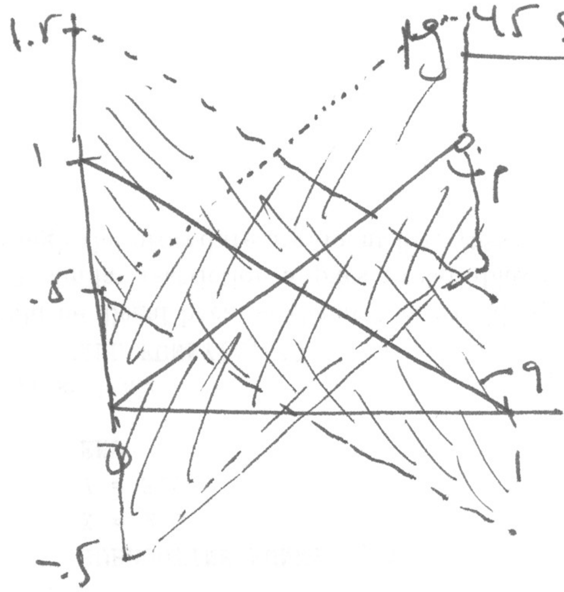
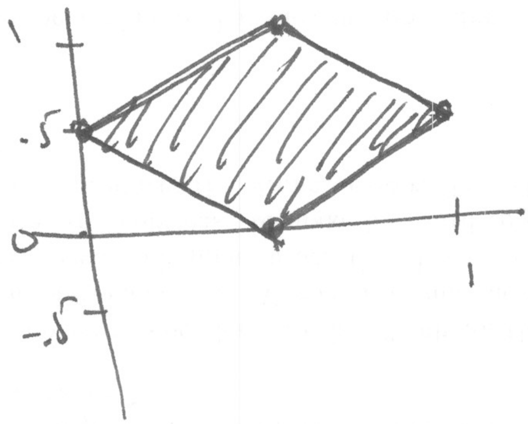
(2)



region the fn must lie



Can any fn . . . No. It would not have a value at $x=0$ or $x=1$.

(3)

$$\|g(x) - f(x)\|_\infty = F(a)$$

$$F(a) = \max_{x\in[0,1]} |x^9(1-x)^{11} - a| \qquad x^9(1-x)^{11}\Big|_{x=1/2} > 0$$

$= \cancel{\ }$

$F(0) = a$ $\quad F'(a) = +1$ $\therefore$ min exists at end pts either zero or 1

$F(1) = a$ $\quad$ since its ~~...~~ ~~...~~ value at $x=0$ is a

$a=0$ gives the smallest.

... How do I analytically minimize a fn like this as a function of a?

One would Assume that the ~~smaller~~ larger # of parameters one has control over the better the accurate fit

Ex:

$$h(x) = \int_0^x k(x,y)\, g(y)\, dy$$

$v_n = T^n v_0 \qquad v_0 \in C[a,b]$

$$v_1(x) = \int_0^x k(x,y)\, v_0(y)\, dy \le M \|v_0\| x$$

$$v_2(x) = \int_0^x k(x,y)\, v_1(y)\, dy \le \int_0^x M^2 \|v_0\| y\, dy = M^2 \|v_0\| \frac{x^2}{2}$$

$\vdots$

$$v_n(x) \le M^n \|v_0\| \frac{x^n}{n!}$$

~~Thus iteration~~ $\cancel{g_{n+1} = v_0 + T g_n}$ ~~has~~ ~~...~~ as ~~formal solution~~

$\cancel{g_n \to v_0}$ No matter how large b is $\exists$ n

$$\lim_{n \to +\infty} v_n(x) = 0 \qquad \forall \quad x \in [0,b]$$

Thus iteration of $g_{n+1} = v_0 + T g_n$ converges...

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① Yes

② Yes

③ Yes

④

$$F\left(a\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + b\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) = F\begin{pmatrix} ax_1 + bx_2 \\ ay_1 + by_2 \end{pmatrix} = \begin{pmatrix} -ay_1 - by_2 \\ ax_1 + bx_2 \end{pmatrix}$$

$$= a\begin{pmatrix} -y_1 \\ x_1 \end{pmatrix} + b\begin{pmatrix} -y_2 \\ x_2 \end{pmatrix}.$$ Yes

⑤ Yes

⑥ No

⑦ No.

⑧ No

⑨ No

⑩ Yes

⑪ No

⑫ $T(af_1 + bf_2) = af_1(x^2) + bf_2(x^2)$

Yes

⑬ Yes

⑭ Yes

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15 $af_1 + bf_2 \to af_1(x+h) + bf_2(x+h)$
$- af_1 - bf_2$.

Yes

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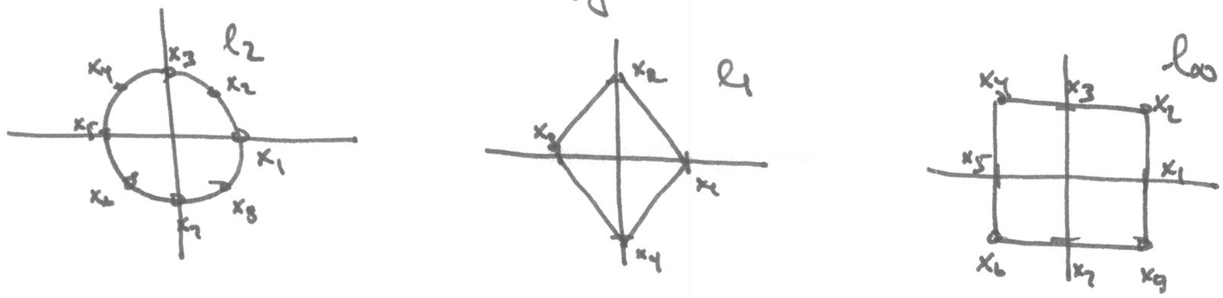


$A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$

For ℓ_2 $S(0,1)$

$\vec{x}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; $x_2 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$x_3 = (0,1)$ $x_4 = \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$x_5 = (-1,0)$ $x_6 = \left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$

$x_7 = (0,-1)$ $x_8 = \left(\frac{+1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$

when mapped $y = Ax$

$y_1 = (1,0)$ $y_2 = \left(2/\sqrt{2}, 2/\sqrt{2}\right)$

$y_3 = (1,2)$ $y_4 = \left(0, 2/\sqrt{2}\right)$

$y_5 = (-1,0)$ $y_6 = \left(0, -2/\sqrt{2}\right)$

$y_7 = (-1,-2)$ $y_8 = \left(0, -2/\sqrt{2}\right)$

For ℓ_1 $S(0,1)$

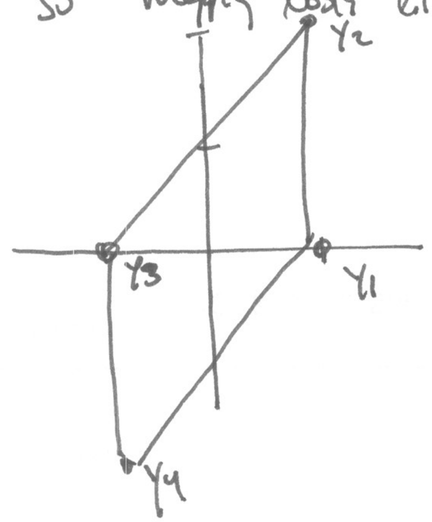
$x_1 = (1,0)$ $x_2 = (0,1)$

$x_3 = (-1,0)$ $x_4 = (0,-1)$

$y_1 = (1,0)$ $y_2 = (1,2)$

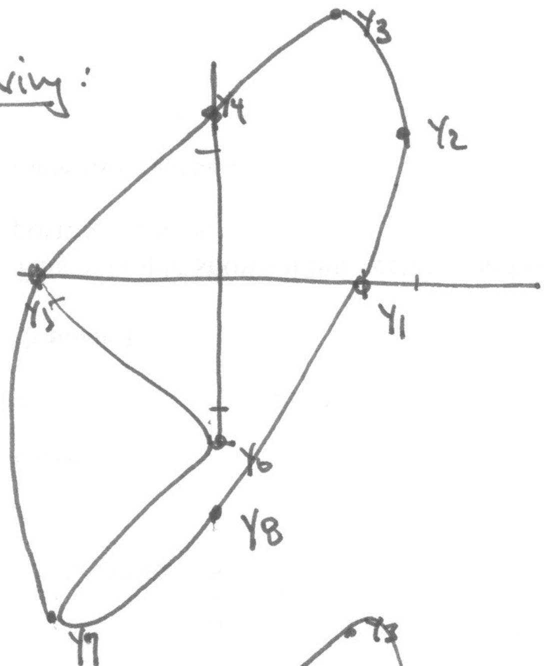
$y_3 = (-1,0)$ $y_4 = (-1,-2)$

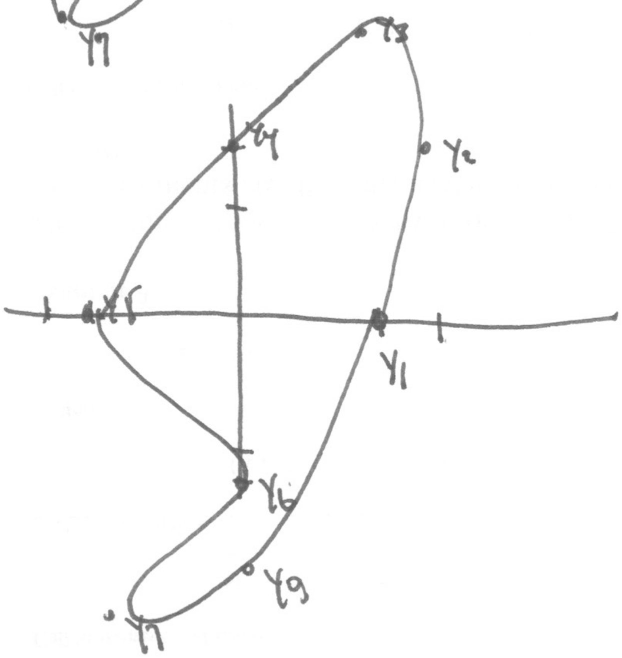
So mapping looks like



The straight lines drawn here are correct since the mapping is linear & preserves the lines.

Giving:





For l_∞ $S(0,1)$:

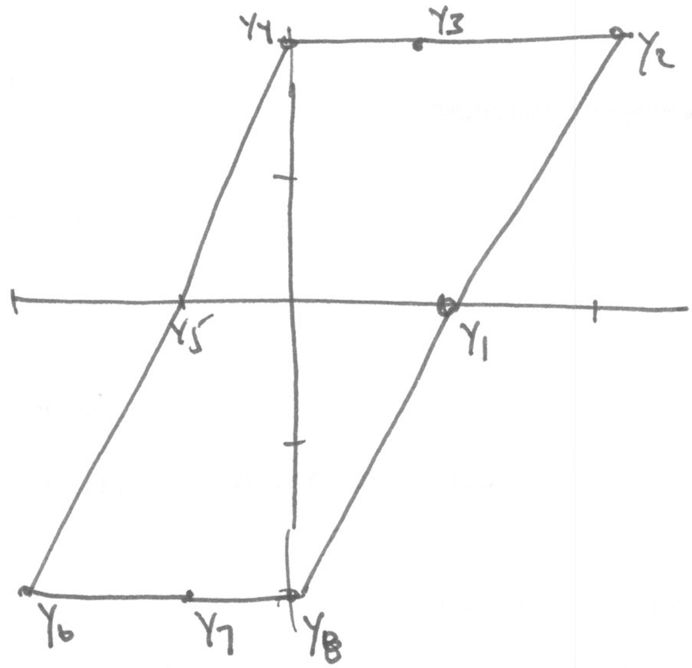
$x_1 = (1,0)$ $x_2 = (1,1)$ $x_3 = (0,1)$ $x_4 = (-1,1)$

$x_5 = (-1,0)$ $x_6 = (-1,-1)$ $x_7 = (0,-1)$ $x_8 = (1,-1)$

$y_1 = (1,0)$ $y_2 = (2,2)$ $y_3 = (1,2)$ $y_4 = (0,2)$

$y_5 = (-1,0)$ $y_6 = (\cancel{-2},-2)$ $y_7 = (-1,-2)$ $y_8 = (0,-2)$

The new plot is



10-3-02

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(1) $w = 2x - 3y + 4z$

point $(+1, -1, +1)$ gives $W = 2+3+4 = 9$

$f: \mathbb{R}^3 \to \mathbb{R}$ $\quad \|f\|_\infty = 9$

(2) $\|f\|_\infty = |a| + |b| + |c|$

(3) $\|f\|_\infty = \sum_1^n |a_r|$

(4) l_∞ $\quad \left\| \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right\|_\infty = \text{Max}\{|w_1|, |w_2|\}$

$$= \underset{\substack{x \in \mathbb{R}^2 \\ \|x\|_\infty = 1}}{\text{Max}} \{ |2x_1 - 3x_2 + 4x_3|, |x_1 + x_2 + x_3| \}$$

$$= \text{Max}\{|2+3+4|, |3|\} = \text{Max}\{9, 3\} = 9.$$

(5) $\|f\|_\infty = 9 = \underset{i}{\text{Max}} \sum_j |a_{ij}|$

(5) Then sum of new w2 row would be

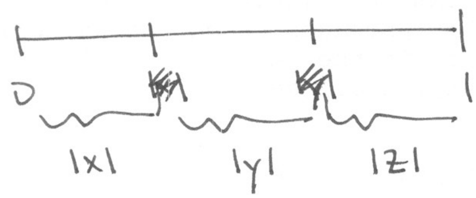
(a) $1 + 2 + 4 = 7 \qquad \|f\|_\infty = 7$

(b) $\sum_j |a_{2j}| = 10 \qquad \|f\|_\infty = 10$ $\quad$ remember ∞ is horizontal

take maximum horizontal row sum

(6) $\max\limits_{1\le r\le m}\{|w_r|\} \le \max\limits_{1\le r\le m}\left\{\sum_{s=1}^{n}|a_{rs}|\right\}$

(7) $|x|+|y|+|z|=1$



put largest component in spot z

so $\|T\|_1 = 4$

(8) $\|T\|_1 = \max\{|a|,|b|,|c|\}$

(9) $\|T\|_1 = \max\limits_{1\le r\le n}\{|a_r|\}$

(10) $\max\{|w_1|+|w_2|\}$

$= \max(|7x_1+2x_2|+|-3x_1+6x_2|\}$

$\le \max(7|x_1|+2|x_2|+3|x_1|+6|x_2|\}$

$\max(10|x_1|+8|x_2|) \le 10$

This is achieved when $\vec{x}=(1,0)$

$\|T\|_1 = 10$

(11) (a) $\|f\|_1 \leq \mathrm{Max}\left(7|x_1| + 2|x_2| + 3|x_1| + 9|x_2|\right)$

$\leq \;\|$ achieved at $x=(0,1)$

(b) $\|f\|_1 \leq \mathrm{Max}\left((7+3)|x_1| + (2+8)|x_2|\right) \leq 10$

achieved at $x = \begin{matrix}1,0\\0,1\end{matrix}$

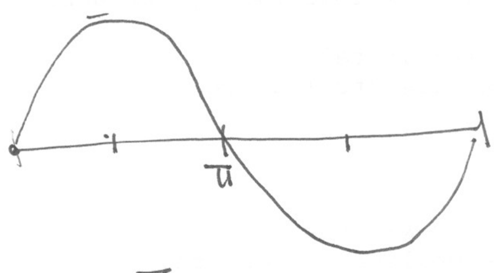
(c) $\|f\|_1 \leq \mathrm{Max}(10, 10)$

(12) $\|f\|_1 = \underset{\leq p \leq}{\mathrm{Max}} \sum_{1\leq r \leq m} |a_{rp}|$

(13) $\mathrm{Max}\, |c| \leq \int_0^1 |3x^2+2x+1|\, dx = x^3 + x^2 + x \Big|_0^1 = 3$

$$x = \frac{-2 \pm \sqrt{4-3(1)}}{2(3)} = \frac{-2\pm 1}{6} = \begin{cases} -3/6 = -\frac{1}{2} \\ -1/6 \end{cases}$$

(14)



$$|c| \leq \int_0^{\pi} (+1)\sin x\, dx + \int_{\pi}^{2\pi} (-1)\sin x = -\cos x \Big|_0^{\pi} + \cos x \Big|_{\pi}^{2\pi} =$$

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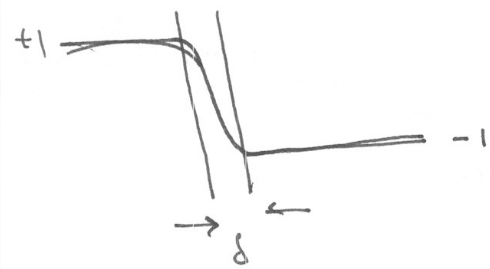
$$|C| \le -[(-1)-1] + [1-(-1)] = 2+2=4.$$

By taking a function like the following



Then supremum = 4.

No fn actually obtains this value

(15) This is the continuous analog to the $\|\cdot\|_\infty$ norm of

$$f:(x_1, x_2, \ldots, x_n) \longrightarrow \sum_{i=1}^{n} a_r x_r$$

Thus

$$\|T\| = \int_a^b |\phi(x)|\, dx.$$

(16) $\cancel{g(x)} = \cancel{\int_0^1} (3y)f(x\;\; y)$

$$\|g\| = \sup_{0 \le x \le 1} \left| \int_0^1 (x^2+2xy+3y^2) f(y)\, dy \right|$$

$$\le \sup_{0 \le x \le 1} \left\{ \int_0^1 |x^2+2xy+3y^2|\, dy \right\}$$

$$\therefore \|T\| = \sup_{0 \le x \le 1} \int_0^1 |x^2+2xy+3y^2|\, dy$$

$$g(x) = \int_0^1 \cancel{(x - 3y)(x \quad y)} f(y)\,dy$$

Since $x^2 + 2xy + 3y^2 \geq 0 \quad \forall \quad x \in [0,1], \; y \in [0,1]$

$f \equiv 1$ is a fn that makes $\|g\|$ a maximum.

$$\|g\| \doteq \sup_{0 \le x \le 1} \left| \int_0^1 (x^2 + 2xy + 3y^2)\,dy \right|$$

$$\sup_{0 \le x \le 1} \left| (x^2 y + xy^2 + y^3 \right|_0^1$$

$$= \sup_{0 \le x \le 1} (x^2 + x + 1) = 3$$

(17) (i) $$|c| \le \int_0^1 |k-y|\,dy$$

$$= \int_0^k (k-y)\,dy + -\int_k^1 (k-y)\,dy$$

$$= ky - \frac{y^2}{2} \Big|_0^k - \left(ky - \frac{y^2}{2}\right|_k^1$$

$$= \cancel{k^2 - \frac{k^2}{2}} + \cancel{k - \frac{1}{2}} - \cancel{k^2 + \frac{k^2}{2}} = \cancel{k - \frac{1}{2}}$$

$$= k^2 - \frac{k^2}{2} - \left(k - \frac{1}{2} - k^2 + \frac{k^2}{2}\right)$$

$$= k^2 - \cancel{\frac{k^2}{2}} - k + \frac{1}{2} + \cancel{k^2} - \cancel{\frac{k^2}{2}} = k^2 - k + \frac{1}{2}$$

$\sup |c| = ?$

$2k - 1 = 0 \Rightarrow k = 1/2$

$\sup |c| = \frac{1}{4} - \frac{1}{2} + \frac{1}{2} = \underline{\underline{\frac{1}{4}}}$

(iii) Based on the above $\|T\| = 1/4$

(18) $$\|T\| = \sup_{x \in [0,1]} \int_0^1 |k(x,y)|\,dy$$

For T given by $g_r = \sum_s k_{rs} f_s$

$$\|T\| = \max_r \left\{ \sum_s |k_{rs}| \right\}$$

① $g(x) = \int_0^x t f(t)\, dt$

Then $\|T\| = ?$

$$\|T\| = \sup_{\substack{0 \le x \le 1 \\ \|f\|_\infty = 1}} \left| \int_0^x t f(t)\, dt \right| \le \sup_{0 \le x \le 1} \int_0^x t\, dt$$

$$= \sup_{0 \le x \le 1} \frac{t^2}{2} \Big|_0^x = \sup_{0 \le x \le 1} \left(\frac{x^2}{2}\right) = \frac{1}{2}.$$

Since $\|T\| \le \frac{1}{2} < 1$ iteration of This operator

converge:

$$f_{n+1} = 1 + \int_0^x t f_n(t)\, dt \qquad f_0 \equiv 0$$

$$f_1 = 1 + 0 = 1 \qquad f_2 = 1 + \int_0^x t\, dt = 1 + \frac{t^2}{2}\Big|_0^x = 1 + \frac{x^2}{2}$$

$$f_3 = 1 + \int_0^x \left(t + \frac{t^3}{2}\right) dt = 1 + \frac{t^2}{2} + \frac{t^4}{4 \cdot 2}$$

$\therefore$ $f_n =$ ~~$f + \frac{t^2}{2} + \frac{t^4}{}$~~

$$f_4 = 1 + \int_0^x t + \frac{t^3}{2} + \frac{t^5}{4 \cdot 2}\, dt$$

$$= 1 + \frac{x^2}{2} + \frac{x^4}{2 \cdot 4} + \frac{x^6}{2 \cdot 4 \cdot 6} + \cdots$$ what is this the series of?

If this iteration does have a limit one might expect it to converge to

{ ~~I~~ ~~So~~ we cannot guarantee convergence although it may happen

Is this correct? The norm of $1 + \int_0^x t f(t)\, dt$ is bounded by $3/2 > 1$?

$$f = 1 + \int_0^x t f\, dt \Rightarrow f(0) = 1$$

$$\frac{d}{dx} \Rightarrow \quad f' = 0 + x f$$

$$\frac{df}{f} = x \Rightarrow \quad \ln f = \frac{x^2}{2} + c_1$$

$$f(x) = c_2 e^{x^2/2} \quad + f(0) = 1 \Rightarrow c_2 = 1$$

$$f(x) = e^{x^2/2}$$

(2) $M = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$

$$M^2 = \begin{pmatrix} ab & 0 \\ 0 & ab \end{pmatrix}$$

$$M^3 = \begin{pmatrix} 0 & a^2b \\ ab^2 & 0 \end{pmatrix}$$

$$M^4 = \begin{pmatrix} a^2b^2 & 0 \\ 0 & a^2b^2 \end{pmatrix} = \begin{pmatrix} a^2b^2 & 0 \\ 0 & a^2b^2 \end{pmatrix}$$

$$M^5 = \begin{pmatrix} 0 & a^3b^2 \\ a^2b^3 & 0 \end{pmatrix}$$

$$\vdots$$

$$M^{2n} = \begin{pmatrix} a^nb^n & 0 \\ 0 & a^nb^n \end{pmatrix} + M^{2n+1} = \begin{pmatrix} 0 & a^{n+1}b^n \\ a^nb^{n+1} & 0 \end{pmatrix}$$

$\therefore$

$$I + M + M^2 + \cdots + M^n + \cdots$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} + \begin{pmatrix} ab & 0 \\ 0 & ab \end{pmatrix} + \begin{pmatrix} 0 & a^2b \\ ab^2 & 0 \end{pmatrix} + \begin{pmatrix} a^2b^2 & 0 \\ 0 & a^2b^2 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & a^3b^2 \\ a^2b^3 & 0 \end{pmatrix} + \cdots + \begin{pmatrix} a^nb^n & 0 \\ 0 & a^nb^n \end{pmatrix} + \begin{pmatrix} 0 & a^{n+1}b^n \\ a^nb^{n+1} & 0 \end{pmatrix} + \cdots$$

$$= \begin{pmatrix} 1 + ab + a^2b^2 + \cdots a^nb^n & a + a^2b + a^3b^2 + \cdots + a^{n+1}b^n \\ b + ab^2 + a^2b^3 + \cdots + a^nb^{n+1} & 1 + ab + a^2b^2 + \cdots + a^nb^n \end{pmatrix}$$

$$= \begin{pmatrix} 1+ab+a^2b^2+\cdots+a^nb^n+\cdots & a(1+ab+a^2b^2+\cdots+a^nb^n+\cdots) \\ b(1+ab+a^2b^2+\cdots+a^nb^n+\cdots) & 1+ab+a^2b^2+\cdots+a^nb^n+\cdots \end{pmatrix}$$

$$= \sum_{k=0}^{\infty}(ab)^k \begin{pmatrix} 1 & a \\ b & 1 \end{pmatrix}$$ to converge we require $|ab|<1$

$$= \frac{1}{1-ab}\begin{pmatrix} 1 & a \\ b & 1 \end{pmatrix}$$

$$(I-M)N \underset{\text{formally}}{=} (I-M)(I+M+M^2+M^3+\cdots+M^n+\cdots)$$

$$= I+\cancel{M}+\cancel{M^2}+\cancel{M^3}+\cdots+\cancel{M^n}+\cdots-\cancel{M}-\cancel{M^2}-\cancel{M^3}-\cancel{M^4}+\cdots = I \checkmark$$

Checking

$(I-M)N =$

$$\begin{pmatrix} 1 & -a \\ -b & 1 \end{pmatrix}\left(\frac{1}{1-ab}\begin{pmatrix} 1 & a \\ b & 1 \end{pmatrix}\right) = \frac{1}{1-ab}\begin{pmatrix} 1-ab & 0 \\ 0 & 1-ab \end{pmatrix} = I \checkmark$$

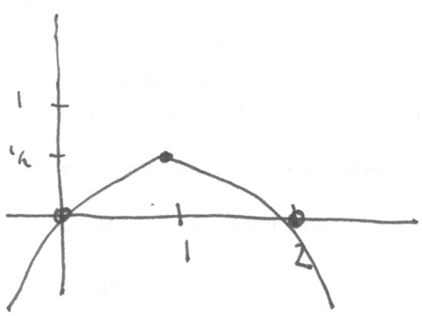
(3) Pg 73 Sawyer 10-05-02 1

$$\|M\| \equiv \sup_{\substack{x \\ f \ni \\ \|f\|=1}} |g(x)| \text{ below}$$

$$|g(x)| \le \left| \int_0^x y f(y)\,dy \right| + \left| \int_x^1 x f(y)\,dy \right|$$

$$\le \int_0^x y\,dy + x \int_x^1 dy = \frac{x^2}{2} + x(1-x)$$

$$= \frac{x^2}{2} + x - \cancel{x^2} = -\frac{x^2}{2} + x = x\left(-\frac{x}{2}+1\right) = \frac{x}{2}(2-x)$$



Max/Min at

$-x+1 = 0$

$x = 1$

$f(1) = \frac{1}{2}(1)$

$\therefore \ |g(x)| \le \frac{1}{2}$

with $f \equiv 1$

$$g(x) = \frac{x^2}{2} + x(1-x) = \frac{x}{2}(2-x)$$

f has max at $x = 1/2$ w/ $x = +1$

$\therefore$ ~~[illegible]~~ $\|M\| = \frac{1}{2}$

Yes $f_{n+1} = 1 + M f_n$ will converge ... I don't think the initial ~~[illegible]~~ presents a problem.

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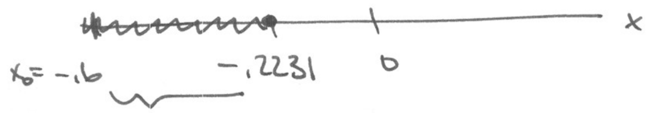
$x = e^x - 1.1$ $\qquad f(x) = e^x - 1.1$

$f'(x) = e^x$



$|f'(x)| = e^x < .8 \quad \forall\, x < -.2231$

$x_0 = -.6 \quad f(x_0) = x_1 = -.55118$

So $|x_\infty - x_0| \le |x_1 - x_0| \sum_{k=0}^{\infty} (.8)^k = \dfrac{|x_1 - x_0|}{1 - .8} = \underline{\underline{.244}}$

$|-.6 + .2231| = .3769$ Thus we stay in the choice $|f'(x)| < .8$ region & the iteration is guaranteed to converge.

~~By the choice~~

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(1) $f(x) = \frac{1}{2}\left(x + \frac{a}{x}\right)$

$f'(x) = \frac{1}{2}\left(1 - \frac{a}{x^2}\right)$ ✓

require $|f'(x)| \le k < 1$

$\frac{1}{2}\left|1 - \frac{a}{x^2}\right| < 1$

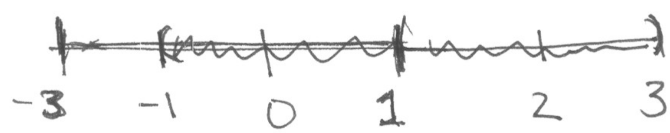
$\left|1 - \frac{a}{x^2}\right| < 2$

✓ $-2 < 1 - \frac{a}{x^2} < 2$

✓ $-3 < -\frac{a}{x^2} < 1$

✓ $-1 < \frac{a}{x^2} < 3$

$\cancel{-1 > \frac{x^2}{a} > \frac{1}{3}}$



$\frac{1}{3} < \frac{x^2}{a} < 1$

$< \frac{x^2}{a}$

here as $a > 0$ & $x^2 > 0$ this can be simplified to

$0 < \frac{a}{x^2} < 3$

$\Rightarrow \frac{1}{3} < \frac{x^2}{a} < \infty$

$\frac{a}{3} < x^2 < \infty \Rightarrow \frac{\sqrt{a}}{\sqrt{3}} < |x| < \infty$

$\Rightarrow x > \frac{\sqrt{a}}{\sqrt{3}}$ or $x < -\frac{\sqrt{a}}{\sqrt{3}}$

We are guaranteed convergence if points inside a region w/ $|f'(x)| < 1$ stay inside this region.

requiring $x > \frac{\sqrt{a}}{\sqrt{3}}$ will be true if $x > \frac{2\sqrt{a}}{\sqrt{3}}$ say.

$x^2 > a/3$

Then $|f'(x)| \leq \frac{1}{2} + \frac{a}{2x^2}$ $\qquad 0 < \frac{1}{x^2} < 3/a$

$\leq \frac{1}{2} + \frac{3}{2} = 2$!! not 1? $\qquad \frac{-3}{a} < \frac{-1}{x^2} < 0$

$|f'(x)| = \left|\frac{1}{2} - \frac{a}{2x^2}\right|$

Why can one not show that $|f'(x)| < 1$? What am I doing wrong?

Take $x > \frac{2\sqrt{a}}{\sqrt{3}}$ $\qquad x^2 < \frac{3}{4a}$

Then $|f'(x)| \leq \frac{1}{2} + \frac{1}{2}\frac{a}{x^2} \leq \frac{1}{2} + \frac{1}{2}a\left(\frac{3}{4a}\right) = \frac{1}{2}\left(1 + \frac{3}{8}\right)$

$= \frac{1}{2}\left(\frac{11}{8}\right) = \frac{11}{16} < 1$

Then starting at a point x_0 the furthest we can walk from x_0 is $|x_0 - x_1|\underbrace{(1 + k + k^2 + \dots)}$

$\sum_{k=0}^{\infty}\left(\frac{11}{16}\right)^k = \frac{1}{1-(11/16)} = \frac{16}{5} > 3$

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(3)

given a positive initial guess the iterations of $f(x)$ always return a positive value, thus the

Is there any kind to use this iteration to converge to a negative #?

(2) $f(x) = \sin x + .5x$ goes to a fixed pt $\Rightarrow \frac{1}{2}x = \sin x$

$f'(x) = \cos x + .5 \qquad |f'(x)| < 1$

$\rightarrow \quad |\cos x + .5| < 1 \qquad \boxed{||a|-|b|| < |a+b|}$

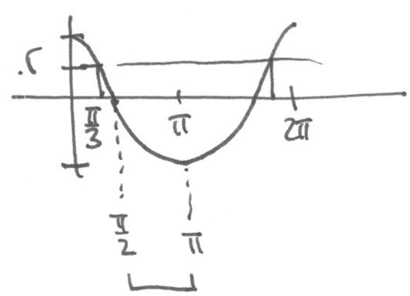
$\Leftrightarrow \quad \cancel{\cos x < .5}$

$\cancel{||\cos x| - .5|} \quad \Leftarrow$

$-1 < \cos x + .5 < 1$

$-1.5 < \cos x < .5$

$\Leftrightarrow \quad -1 < \cos x < .5$



$\Leftrightarrow \quad x \in \left[\frac{\pi}{3}, \frac{5\pi}{3}\right]$

$= [1.047, 5.23]$

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So $|f'(x)| < 1$ when $x \in [\frac{\pi}{2}, \pi] = [1.5707, 3.1415]$

If $x_0 = 2$ $\quad x_1 = f(x_0) = \sin(2) + \frac{1(2)}{2} = 1.03$ ✓

This is ___outside___ of the iteration region.

& $|x_0 - x_1| = .9651$

$|x_2 - x_1| = |f(x_1) - f(x_0)| = \cancel{|f(x_1 - x_0)|}$

$\cancel{\le \|f\| |x_1 - x_0|}$

$= |f'(\xi)(x_1 - x_0)|$ for some ξ between x_1 & x_0.

By the mean value theorem.

$\le \|f'(\xi)\| |x_1 - x_0|$

Sin $|x_3 - x_2|$

$$x_\infty = \lim_{n\to\infty} (x_{n+1} - x_n + x_n - x_{n-1} + x_{n-1} - \cdots)$$

$$x_\infty - x_1 = \lim_{n\to\infty} (x_n - x_1) = \lim_{n\to\infty} (x_n - \underbrace{x_{n-1} + x_{n-1}} - \underbrace{x_{n-2} + x_{n-2}} - \underbrace{x_{n-3} + x_{n-3}} - \cdots x_3 - x_2 + x_2 - x_1$$

$$|x_\infty - x_1| = \left|\lim_{n\to\infty} (\qquad)\right| \le \lim_{n\to\infty} |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \cdots + |x_2 - x_1|$$

$$\le |x_1 - x_0|(1 + k + k^2 + \cdots)$$

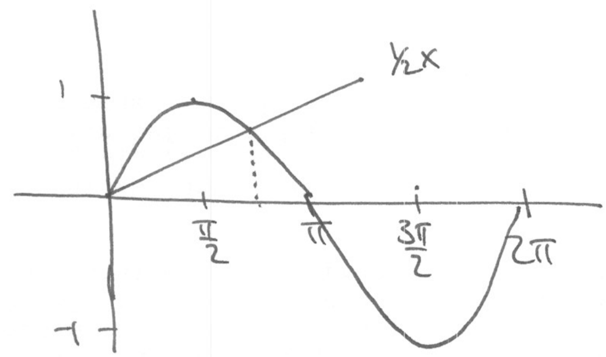
$$\sum_{k=0}^{\infty} \left(\frac{1}{2}\right) = \frac{1}{1 - 1/2} = 2$$

$x_2 = .535$

$x_3 = .277$

$\vdots$

Dammit sin was evaluating in radians!!



$\sin(\frac{\pi}{2}) \leq \sin(2) \nleq$

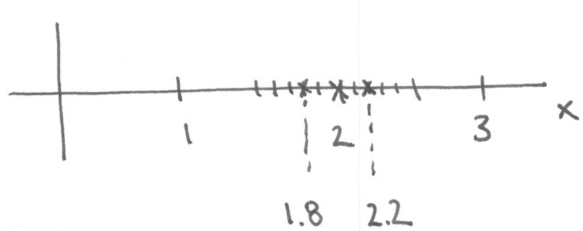
---

w/ correct radian's from pg 4:

If $x_0 = 2$ $\quad x_1 = f(x_0) = 1.909$

$\therefore \; |x_0 - x_1| = .0907 < .1$

So $\quad |x_\infty - x_0| \leq 2|x_0 - x_1| < .2$



Thus all points of this iteration are guaranteed to be w/in the range $(1.8, 2.2) \subset [\frac{\pi}{2}, \pi]$ & thus the iteration converges.

The # of ~~points~~ iteration till convergence would be

$\cancel{|x_1 - x_0|}$ $|x_\infty - x_n| \leq \cancel{x_\infty - x_{\infty-1}}$

$$|x_p - x_n| = |x_p - x_{p-1} + x_{p-1} - x_{p-2} + \cdots + x_{n+1} - x_n|$$

$$\leq |x_p - x_{p-1}| + |x_{p-1}$$

$$K^p |x_0 - x_1| +$$

— —

$$|x_{n+1} - x_n| = |T^{n+1} x_0 - T^n x_0| \leq \|T\|^n |Tx_0 - x_0|$$

$$|x_{n+2} - x_n| = |x_{n+2} - x_{n+1} + x_{n+1} - x_n| = |T^{n+1} x_1 - T^{n+1} x_0 + T^n x_1 - T^n x_0|$$

$$= |T^{n+1}(x_1 - x_0) + T^n (x_1 - x_0)|$$

$$\leq \|T\|^{n+1} |x_1 - x_0| + \|T\|^n |x_1 - x_0|$$

$$= \|T\|^n (1 + \|T\|) |x_1 - x_0|$$

$$\Rightarrow |x_{n+m} - x_n| \leq \|T\|^n \left(\sum_{k=0}^{m-1} \|T\|^k \right) |x_1 - x_0|$$

$m \to \infty$

$$\Rightarrow |x_\infty - x_n| \leq \|T\|^n \left(\sum_{k=0}^{\infty} \|T\|^k \right) |x_1 - x_0| = \frac{\|T\|^n |x_1 - x_0|}{(1 - \|T\|)}$$

$\|T\| < 1$

Thus to achieve an accuracy of 10^{-9} requires

an upper bound of

$|x_\infty - x_n| \leq 10^{-9}$

$\Leftarrow$ $\dfrac{(1/2)^n (1.2)}{(1-1/2)} \leq 10^{-9}$

$\Leftrightarrow \quad n \geq 28.57$

(3) (a) $f(x) = \tan x$

$f'(x) = \sec^2 x$

$f'(4.5) = 22.5 > 1$

(b) $f(x) = \pi + \tan^{-1} x$; ~~f(4.5)~~ =

$f'(x) =$

$y = \pi + \tan^{-1} x$

$\tan(y - \pi) = x$

$\sec^2(y-\pi) \cdot \frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{\sec^2(y-\pi)} = \cos^2(y-\pi)$

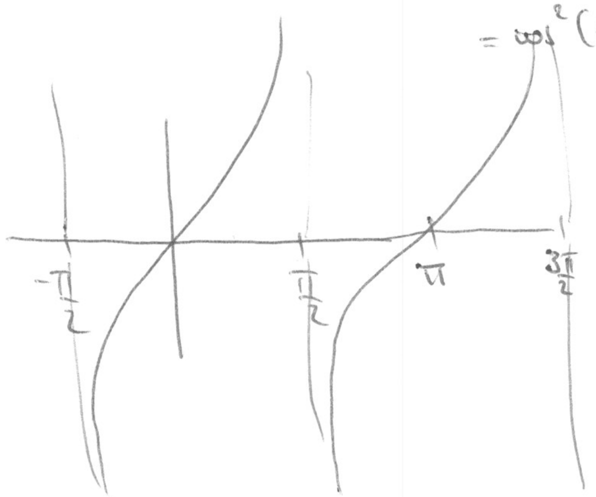
$\frac{dy}{dx} = \cos^2(y-\pi)$

$= \cos^2(\pi + \tan^{-1} x - \pi)$

$= \cos^2(\tan^{-1} x) = 4.7 \cdot 10^{-2} < 1$

$x = \tan x$

$\Leftrightarrow x = \tan^{-1} x$



Use the one given by in pt (b)

in (a) $|f'(x_0)| > 1$ while in (b) $|f'(x)| < 1 \quad \forall x \in \mathbb{R}$

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(1)

(a) $f(x) = \begin{pmatrix} -y \\ x \end{pmatrix}$

$$f'(x) = \begin{pmatrix} \frac{\partial(-y)}{\partial x} & \frac{\partial(-y)}{\partial y} \\ \frac{\partial(x)}{\partial x} & \frac{\partial(x)}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

(b) $f'(x) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

(c) $f'(x) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

(d) $f'(x) = \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix}$

(e) $f'(x) = \begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix}$

(2) (a) $f(x) = x^2 + y^2$

$$f(\vec{x} + \Delta\vec{x}) \cong f(\vec{x}) + f'(\vec{x}) \cdot \Delta\vec{x}$$

$\|$

$\cancel{x^2}$ $(x+\Delta x)^2 + (y+\Delta y)^2 = x^2 + y^2 + f'(\vec{x}) \cdot \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$

To 1st order:

$$* \quad 2x\Delta x + 2y\Delta y = f'(x) \cdot \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$$

$$\Leftrightarrow \quad f'(x) = \begin{pmatrix} 2x \\ 2y \end{pmatrix} \qquad \underline{\underline{f'(x)}} = (2x, 2y).$$

(b) $f'(x) = (-a\sin t, b\cos t)$

1

$F'(f)h \circ (x)$

(a) $[f(x)]^2$

$\Delta F = (f(x)+h(x))^2 - f(x)^2 = f^2 + 2fh + h^2 - f^2 = 2fh + h^2$

$F'(f)h \circ (x) = 2f(x)h(x)$

(b) $(f+h)^n - f^n \equiv \Delta F$

$\sum_{k=0}^{n} \binom{n}{k} f^k h^{n-k} - f^n = \sum_{k=0}^{n-1} \binom{n}{k} f^k h^{n-k}$

$\therefore$ ~~...~~ $= n f^1 h^{n-1} + \cdots + n f^{n-1} h.$

$F'(f)h \cdot (x) = n f^{n-1} h$

(c) $\Delta F = x^3 (f+h)^2 - x^3 f^2$

$= x^3(\cancel{f^2} + 2hf + h^2) - x^3 \cancel{f^2}$

$= x^3(2hf + h^2)$

$F'(f)h \cdot (x) = 2fx^3 h$

(d) $\sin(f(x))$

$$\Delta F = \sin(f+h) - \sin f$$

$$= \sin f \cosh + \cos f \sin h - \sin f$$

$$= \sin f\,\overset{h\to 0 \Rightarrow 0}{(\cos h - 1)} + \cos f \sin h$$

$$\Rightarrow \quad F'(f)h\circ(x) = \cos(f(x))$$

(e) $F(f) = \int_0^x t(f(t))^2\,dt$

$$\Delta F = \int_0^x (t(f+h)^2 - tf^2)\,dt = \int_0^x (t(\cancel{f^2}+2fh+h^2) - t\cancel{f^2})\,dt$$

$$= \int_0^x t(2fh+h^2)\,dt$$

$$F'(f)h\circ(x) = 2\int_0^x tfh\,dt$$

(f) $\Delta F = (f'+h') - f' = h'$

$$F'(f)h\circ(x) = h'(x)$$

(g) $\cancel{f}$ $\qquad ff' = (f^2/2)'$

$$\Delta F = (f+h)(f'+h') - ff'$$

$$= \cancel{ff'} + fh' + hf' + hh' - \cancel{ff'}$$

$$= fh' + hf' + hh'$$

$$F'(f)h\cdot(x) = fh' + hf' = (fh)'$$

(h) $$\Delta F = x\int_0^1 ((f'+h')^2 - f'^2)\,dt = x\int_0^1 (2f'h' + h'^2)\,dt$$

$$F'(f)h\cdot(x) = 2x\int_0^1 f'h'\,dt$$

(i) $$\Delta F = \int_0^x ((f+h)^2 + (f'+h')^2 - f^2 - f'^2)\,dt$$

$$= \int_0^x (\cancel{f^2} + 2fh + h^2 + \cancel{f'^2} + 2f'h' + h'^2 - \cancel{f^2} - \cancel{f'^2})\,dt$$

$$= 2\int_0^x \left(fh + f'h' + \frac{h^2}{2} + \frac{h'^2}{2}\right)dt$$

$$F'(f)h\cdot(x) = 2\int_0^x (fh + f'h')\,dt$$

(j) $\Delta F = \left[\int_0^x (f+h)\,dt\right]^2 - \left[\int_0^x f\,dt\right]^2$ $\left\{ a^2-b^2 = (a-b)(a+b) \right\}$

$$= \left(\int_0^x (f+h)\,dt + \int_0^x f\,dt\right)\left(\int_0^x (f+h)\,dt - \int_0^x f\,dt\right)$$

$$= \left(\int_0^x (2f+h)\,dt\right)\left(\int_0^x h\,dt\right)$$

$$= 2\left(\int_0^x f\,dt\right)\left(\int_0^x h\,dt\right) + \left(\int_0^x h\,dt\right)^2$$

$$F'(f)h\cdot(x) = 2\left(\int_0^x f\,dt\right)\left(\int_0^x h\,dt\right)$$

Ask...

(2) $\Delta F = \cancel{x/3} + \int_0^x (f+h)^2\,dt - \cancel{x/3} - \int_0^x f^2\,dt$

$$= \int_0^x (2fh + h^2)\,dt$$

$$F'(f)h\cdot(x) = 2\int_0^x fh\,dt$$

$$\|S(f)\| \le 2x\|f\|_\infty < 2(1)(1/4) = 1/2.$$

Thus $f_0 \equiv 0$

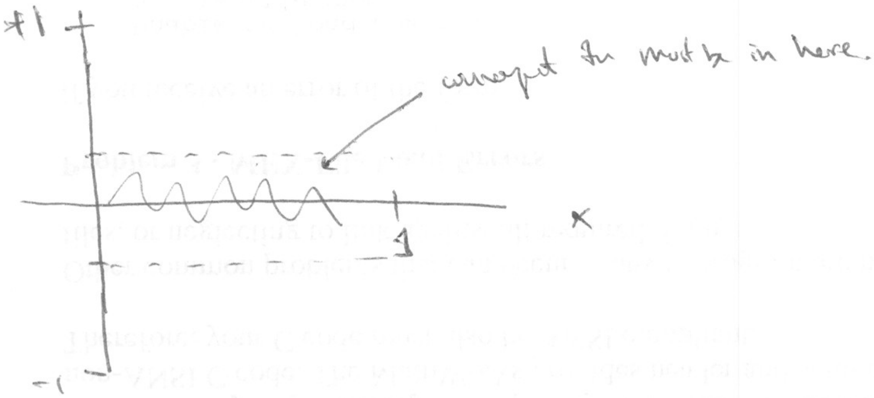
$$f_1 = \frac{x}{8} + \int_0^x 0 \, dt = \frac{x}{8}$$

$$\|f_1 - f_0\|_\infty = 1/8$$

So starting at the point $f_0 \equiv 0$ we can move at most

$$\sum_{k=0}^{\infty} \left(\tfrac{1}{2}\right)^k = 2 \quad \text{from}$$

$$\|x_\infty - x_0\|_\infty \le \|x_1 - x_0\|_\infty \cdot \sum_{k=0}^{\infty} (1/2)^k = 2\|x_1 - x_0\|_\infty = 2(1/8) = 1/4$$



$$e(x) = y'(x) + y(x)^2 - \frac{1}{(1+x)^2}$$

$$y \to y + h$$

$$e(x) \to y' + h' + y^2 + 2yh + h^2 - \frac{1}{(1+x)^2}$$

ef k

$$= \underbrace{y' + y^2 - \frac{1}{(1+x)^2}} + h' + 2yh + h^2$$

$$e + \underbrace{h' + 2yh}_{k(x)} + h^2$$

w/ $y_0 = \frac{1}{x+1}$

$$k(x) = h' + \frac{2h}{x+1} \qquad \text{eq (7)}$$

Solve $F(x) \equiv 0$

$$F(x+h) = F(x) + F'(x)h + O(h^2)$$

$$e(x+h) = e(x) + e'(x)h$$

Now $k(x)$ is the derivative of our error function, evaluated at $y = y_0$. For our newton-rapson iterations we require the inverse of this opperator $\Rightarrow$

$$\therefore \quad h' + \frac{2h}{x+1} \equiv f'(y_0)$$

$$\underbrace{(x+1)^2 h' + 2(x+1)h} = (x+1)^2 f'(y_0)$$

$$\frac{d}{dx}\left((x+1)^2 h\right) = (x+1)^2 f'(y_0)$$

$$k(x) = h'(x) + \frac{2h(x)}{x+1}$$

$$\Rightarrow (x+1)^2 h$$

— — —

$$u\,dv = uv - v\,du$$

$$y_{n+1} = y_n - [F'(y_0)]^{-1} F(y_n)$$

eq $F(y) = e \equiv y' + y^2 - \frac{1}{(1+x)^2}$

$$y_{n+1} = y_n - [F'(y_0)]^{-1}\left[y_n' + y_n^2 - \frac{1}{(1+x)^2} \right]$$

$$\Rightarrow y_{n+1} = y_n - \frac{1}{(x+1)^2}\int_0^x (s+1)^2\left(y_n' + y_n^2 - \frac{1}{(1+s)^2}\right) ds$$

$$= y_n - \frac{1}{(x+1)^2}\left[(s+1)^2 y_n \Big|_0^x - \int_0^x 2(s+1)\,y_n(s)\,ds + \int_0^x (s+1)^2 y_n^2 - x \right]$$

$\{ y_n(0) = 1 \}$

$$y_{n+1} = y_n - \frac{1}{(x+1)^2}\left[y_n(x)(x+1)^2 - 1] - \int_0^x 2(s+1)\,y_n(s)\,ds + \int_0^x (s+1)^2 y_n^2\,ds - x \right]$$

$$= \cancel{y_n} - \cancel{y_n} + \frac{1}{(x+1)^2} + \frac{1}{(x+1)^2}\int_0^x 2(s+1)\,y_n(s)\,ds - \frac{1}{(x+1)^2}\int_0^x (s+1)^2 y_n^2\,ds + \frac{x}{(x+1)^2}$$

$$= (x+1)^{-1} + (x+1)^{-2}\int_0^x 2(s+1)\,y_n(s)\,ds - (x+1)^{-2}\int_0^x (s+1)^2 y_n(s)^2\,ds \qquad \text{eq (10)}$$

$$y_0 = \frac{1}{x+1}$$

$$y_1 = \frac{1}{x+1} + (x+1)^{-2}\int_0^x \cancel{\frac{2(s+1)}{s}}\, 2\,ds - (x+1)^{-2}\int_0^x ds$$

$$= \frac{1}{x+1} + \frac{2x}{(x+1)^2} - \frac{x}{(x+1)^2} = \frac{1}{x+1} + \frac{x}{(x+1)^2}$$

$$y_2 = \frac{1}{x+1} + (x+1)^{-2}\left[\int_0^x 2\left[1+\frac{s}{s+1}\right]ds - \int_0^x (s+1)^2\left(\frac{1}{(s+1)}+\frac{s}{(s+1)^2}\right)^2 ds\right]$$

$$= \frac{1}{x+1} + (x+1)^{-2}\left[2x + 2\int_0^x \frac{s+1-1}{s+1} \qquad \right]$$

eq (16)

$$y_{n+1} = (x+1)^{-1} + (x+1)^{-2}\int_0^x \left(2(s+1)\, y_n(s) - (s+1)^2 y_n(s)^2\right) ds \equiv S(y)$$

$$S(y+h) - S(y) = (x+1)^{-2}\int_0^x \left(2(s+1)(\cancel{y}+h) - (s+1)^2(y+h)^2 - 2(s+1)\cancel{y}\right.$$

$$\left. + (s+1)^2 y^2\right) ds$$

$$= (x+1)^{-2}\int_0^x \left(2(s+1)h(s) - (s+1)^2\left[\cancel{y^2} + 2yh + h^2 - \cancel{y^2}\right]\right) ds$$

$$= (x+1)^{-2}\int_0^x \left(2(s+1)h(s) - 2(s+1)^2\left(yh + \overset{\approx 0}{\frac{h^2}{2}}\right)\right) ds$$

$$= (x+1)^{-2}\int_0^x \left[2(s+1) - 2(s+1)^2 y\right] h \, ds \equiv S'(y)h\circ(x)$$

Want $\|S'(y)\| \equiv \max_{\|h\|=1} \|S'(y)h\circ(x)\|_\infty$

$$\|S'(y)\| = \sup\left\{ \quad \cdots \quad \right\}$$

$$y_0(s) = \frac{1}{1+s} \qquad \|y - y_0\| < t \Rightarrow \left\|y - \frac{1}{s+1}\right\| < t$$

Require $\|\phi'(t)\| \geq \|S'(y)\|$

$$\| \quad \frac{2t(a+1)}{3}$$

let $\phi'(t) = \frac{2t(a+1)}{3}$ w/ $\phi(0) = \frac{a}{(a+1)^2}$

$$\phi(t) = \frac{t^2}{3}(a+1) + \frac{a}{(a+1)^2}$$

$$\|y_2 - Y\| \leq T - t_2$$

$$(x+1)^2 y' + (x+1)^2 y^2 = 1$$

$$\|$$

$$\frac{d}{dx}\left((x+1)^2 y\right) - 2(x+1)y +$$

① $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} xy + .07 \\ x^2+y^2-.41 \end{pmatrix} = \begin{pmatrix} xy + 7\cdot10^{-2} \\ x^2+y^2-41\cdot10^{-2} \end{pmatrix}$

In these exercises ~~we~~ we are looking for the zero of this mapping. Then using the modified constant slope newton-raphson method for finding the zero gives the following iteration

$$\vec{V}_{n+1} = \vec{V}_n - (f'(\vec{V}_n))^{-1}(f(\vec{V}_n)) \cong \vec{V}_{n+1} = \vec{V}_n - (f'(\vec{V}_0))^{-1}(f(\vec{V}_n))$$

Now $f'\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y & x \\ 2x & 2y \end{pmatrix}$

$$f'\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} -6\cdot10^{-1} & 1\cdot10^{-1} \\ 2\cdot10^{-1} & -12\cdot10^{-1} \end{pmatrix} = 10^{-1}\begin{pmatrix} -6 & 1 \\ 2 & -12 \end{pmatrix}$$

$$\left(f'\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}\right)^{-1} = 10\,\frac{1}{(6(12)-2)}\begin{pmatrix} -12 & -1 \\ -2 & -6 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} -12 & -1 \\ -2 & -6 \end{pmatrix} = -\frac{1}{7}\begin{pmatrix} 12 & 1 \\ 2 & 6 \end{pmatrix}$$

Thus

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} + \frac{1}{7}\begin{pmatrix} 12 & 1 \\ 2 & 6 \end{pmatrix}\begin{pmatrix} x_n y_n + 7\cdot10^{-2} \\ x_n^2 + y_n^2 - 41\cdot10^{-2} \end{pmatrix}$$

$$= \begin{pmatrix} x_n \\ y_n \end{pmatrix} + \frac{1}{7}\begin{pmatrix} 12x_n y_n + 84\cdot10^{-2} + 1x_n^2 + 1y_n^2 - 41\cdot10^{-2} \\ 2x_n y_n + 14\cdot10^{-2} + 6x_n^2 + 6y_n^2 - 246\cdot10^{-2} \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} x_n \\ y_n \end{pmatrix} + \frac{1}{7}\begin{pmatrix} x_n^2 + y_n^2 + 12x_n y_n + 43\cdot10^{-2} \\ 6x_n^2 + 6y_n^2 + 2x_n y_n - 232\cdot10^{-2} \end{pmatrix}}_{S\begin{pmatrix} x_n \\ y_n \end{pmatrix}}$$

Then $S'\binom{x}{y} = I + \frac{1}{7}\begin{pmatrix} 2x + 12y & 2y + 12x \\ 12x + 2y & 12y + 2x \end{pmatrix}$

So

$$S'\binom{1\cdot 10^{-1} + X}{-6\cdot 10^{-1} + Y} = I + \frac{1}{7}\begin{pmatrix} 2\cdot 10^{-1} + 2X + -72\cdot 10^{-1} + 12Y & -12\cdot 10^{-1} + 2Y + 12\cdot 10^{-1} + 12X \\ 12\cdot 10^{-1} + 12X - 12\cdot 10^{-1} + 2Y & -72\cdot 10^{-1} + 12Y + 2\cdot 10^{-1} + 2X \end{pmatrix}$$

$$= I + \frac{1}{7}\begin{pmatrix} -70\cdot 10^{-1} + 2X + 12Y & 0 + 2Y + 12X \\ 12X + 2Y & -70\cdot 10^{-1} + 12Y + 2X \end{pmatrix}$$

$$= \frac{1}{7}\begin{pmatrix} 2X + 12Y & 2Y + 12X \\ 12X + 2Y & 12Y + 2X \end{pmatrix}$$

Assuming $\left\|\binom{X}{Y}\right\|_\infty \le t$

Then $\left\|S'\binom{10^{-1} + X}{-6\cdot 10^{-1} + Y}\right\|_\infty \le \frac{1}{7}(2t + 12t + 2t + 12t) = \frac{2(14t)}{7} = 4t$

(2) $$f'(x,y) = \begin{pmatrix} 2x & 2y \\ y-3x^2 & 3y^2+x \end{pmatrix}$$

$$f'(x_0,y_0) = \begin{pmatrix} 20 & 20 \\ 10-300 & 300+10 \end{pmatrix} = \begin{pmatrix} 20 & 20 \\ -290 & 310 \end{pmatrix} = 10\begin{pmatrix} 2 & 2 \\ -29 & 31 \end{pmatrix}$$

$$\left(f'(x_0,y_0)\right)^{-1} = 10^{-1}\frac{1}{(62+58)}\begin{pmatrix} 31 & -2 \\ 29 & 2 \end{pmatrix}$$

$$= \frac{1}{1200}\begin{pmatrix} 31 & -2 \\ 29 & 20 \end{pmatrix} = N$$ ✓

Then The equation we look for the zeros of is

$$F\begin{pmatrix} x \\ y \end{pmatrix} \equiv \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x^2+y^2-200 \\ y^3+xy-x^3 \end{pmatrix} = 0$$

$$\equiv \begin{pmatrix} x \\ y \end{pmatrix} - f\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\frac{DF}{D\binom{x}{y}} = I - N(x,y)$$

Using newton-rapson w/ a constant slope to solve this problem ~~we~~ iteratively we would compute $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_0)}$

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} - \left(\frac{DF}{DV}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}\right)^{-1} F\begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

$$= \begin{pmatrix} x_n \\ y_n \end{pmatrix} - \left(I - N(x_0,y_0)\right)^{-1}\left(\begin{pmatrix} x_n \\ y_n \end{pmatrix} - \begin{pmatrix} x^2+y^2-200 \\ y^3+xy-x^3 \end{pmatrix}\right)$$

$$S'\binom{x}{y} = I - (I - N)^{-1}\left(I - \begin{pmatrix} 2x & 2y \\ y-3x^2 & 3y^2+x \end{pmatrix}\right)$$

$$S'\binom{x}{y} = I - (I - N)^{-1}\left(I - N + N - \begin{pmatrix} 2x & 2y \\ y-3x^2 & 3y^2+x \end{pmatrix}\right)$$

$$=$$

$$= -(I-N)^{-1}\left(N - \begin{pmatrix} & \\ & \end{pmatrix}\right)$$

$$I - N = \begin{pmatrix} 1 - \frac{31}{1200} & \frac{-2}{1200} \\ \frac{29}{1200} & 1 - \frac{20}{1200} \end{pmatrix} = \begin{pmatrix} \frac{1169}{1200} & \frac{-2}{1200} \\ \frac{29}{1200} & \frac{1180}{1200} \end{pmatrix}$$

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} \quad \text{if} \quad f = g - x, \quad f' = g' - 1$$

$$= x_{n-1} - \frac{g-x}{g'-1}$$

Assuming for some reason that we can simply iterate $x_{n+1} = x_n - (f'(x_0,y_0))^{-1} f(x_n,y_n)$ to find a fixed point of f & not a zero which is what I don't understand I get ...

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \underbrace{\begin{pmatrix} x_n \\ y_n \end{pmatrix} - N_0 \begin{pmatrix} x_n^2 + y_n^2 - 200 \\ y_n^3 + x_n y_n - x_n^3 \end{pmatrix}}$$

$$\equiv S\begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

$$S'\begin{pmatrix} x \\ y \end{pmatrix} = I - N \begin{pmatrix} 2x & 2y \\ y - 3x^2 & 3y^2 + x \end{pmatrix}$$

$$S'\begin{pmatrix} x_0 \\ x_0 \end{pmatrix} = I - N \begin{pmatrix} 20 & 20 \\ -290 & 310 \end{pmatrix} = 0$$

$$S'(10+\underline{X}, 10+\underline{Y}) = I - N \begin{pmatrix} 20 + 2\underline{X} & 2(10+\underline{Y}) \\ 10 + \underline{Y} - 3(10+\underline{X})^2 & 3(10+\underline{Y})^2 + 10 + \underline{X} \end{pmatrix}$$

$$= I - N \begin{pmatrix} 20 + 2\underline{X} & 20 + 2\underline{Y} \\ 10 - 300 + \underline{Y} - 60\underline{X} - 3\underline{X}^2 & 300 + 60\underline{Y} + 3\underline{Y}^2 + 10 + \underline{X} \end{pmatrix}$$

$$= I - N \left[\begin{pmatrix} 20 & 20 \\ -290 & \cancel{300}\ 310 \end{pmatrix} + \begin{pmatrix} 2\underline{X} & 2\underline{Y} \\ \underline{Y} - 60\underline{X} - 3\underline{X}^2 & 60\underline{Y} + 3\underline{Y}^2 + \underline{X} \end{pmatrix} \right.$$

$$= -N \begin{pmatrix} 2\underline{X} & 2\underline{Y} \\ \underline{Y} - 60\underline{X} - 3\underline{X}^2 & 60\underline{Y} + 3\underline{Y}^2 + \underline{X} \end{pmatrix}$$

$$= \frac{1}{1200} \begin{pmatrix} 31 & -2 \\ 29 & 2 \end{pmatrix} \begin{pmatrix} 2\underline{X} & 2\underline{Y} \\ \underline{Y} - 60\underline{X} - 3\underline{X}^2 & 60\underline{Y} + 3\underline{Y}^2 + \underline{X} \end{pmatrix}$$

$$= \frac{1}{1200} \begin{pmatrix} 62\underline{X} - 2\underline{Y} + 120\underline{X} + 6\underline{X}^2 & 62\underline{Y} - 120\underline{Y} - 6\underline{Y}^2 - 2\underline{X} \\ 58\underline{X} + 2\underline{Y} - 120\underline{X} - 6\underline{X}^2 & 58\underline{Y} + 120\underline{Y} + 6\underline{Y}^2 + 2\underline{X} \end{pmatrix}$$

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$$\|S'(10+X, 10+Y)\|_\infty \leq 62t + 2t + 120t + 6t^2$$
$$+ 62t + 2t + 120t + 6t^2$$

if $\left\|\binom{X}{Y}\right\|_\infty < t$

$$= 124t + 4t + 140t + 12t^2$$

$$= 168t + 12t^2$$

All $\div$ 1200

$$\therefore \|S'(10+X, 10+Y)\|_\infty \leq .14t + \frac{1}{100}t^2$$

(3) $f(v) = Mv + g(v) \underset{\text{Solve}}{=} 0$

$$v_{n+1} = v_n - \cancel{} \left(f'(v_n)\right)^{-1} f(v_n)$$

Using the constant slope newton-rapson method

$$v_{n+1} \doteq v_n - \left(f'(v_0)\right)^{-1} f(v_n)$$

$$f'(v) = M + g'(v)$$

$$f'(v_0) = M$$

$$v_{n+1} = \underbrace{v_n - M^{-1} f(v_n)}_{S(v_n)} = v_n - M^{-1}\left(Mv_n + g(v_n)\right)$$

$$= \underbrace{-M^{-1} g(v_n)}_{S(v)}$$

For $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} -37x + 9y + x^5 + y^5 + 25 \\ 4x - 28y + x^3 y^3 + 18 \end{pmatrix}$

$$= \begin{pmatrix} -37 & 9 \\ 4 & -28 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x^5 + y^5 + 25 \\ x^3 y^3 + 18 \end{pmatrix}$$

Then

$$S(v) = -M^{-1} g(v)$$

$$= -\frac{1}{(37(28)-36)}\begin{pmatrix} -28 & -9 \\ -4 & -37 \end{pmatrix}\begin{pmatrix} x^5+y^5+25 \\ x^3y^3+18 \end{pmatrix}$$

$$= \frac{+1}{10^3}\begin{pmatrix} +28 & +9 \\ 4 & 37 \end{pmatrix}\begin{pmatrix} x^5+y^5+25 \\ x^3y^3+18 \end{pmatrix}$$

So that

$$S'(v) = 10^{-3}\begin{pmatrix} 28 & 9 \\ 4 & 37 \end{pmatrix}\begin{pmatrix} 5x^4 & 5y^4 \\ 3x^2y^3 & 3x^3y^2 \end{pmatrix}$$

$$= 10^{-3}\begin{pmatrix} 28\cdot 5x^4 + 27x^2y^3 & 28\cdot 5y^4 + 27x^3y^2 \\ 20x^4 + 3\cdot 37x^2y^3 & 20y^4 + 3\cdot 37x^3y^2 \end{pmatrix}$$

$$= 10^{-3}\begin{pmatrix} 140x^4 + 27x^2y^3 & 140y^4 + 27x^3y^2 \\ 20x^4 + 111x^2y^3 & 20y^4 + 111x^3y^2 \end{pmatrix}$$

$$\left\| S'\begin{pmatrix} x \\ y \end{pmatrix} \right\|_\infty = 10^{-3}\,\mathrm{Max}\left\{ |140x^4+27x^2y^3| + |140y^4+27x^3y^2|,\ |20x^4+111x^2y^3| + |20y^4+111x^3y^2| \right\}$$

$$\leq 10^{-3} \text{Max} \left\{ 140t^4 + 27t^5 + 140t^4 + 27t^5 , \right.$$

$$\left. 20t^4 + 111t^5 + 20t^4 + 111t^5 \right\}$$

$$= 10^{-3} \text{Max} \left\{ \underline{280t^4} + 54t^5 , 40t^4 + \underline{\underline{222t^5}} \right\}$$

$$\leq 10^{-3} \left(280t^4 + 222t^5 \right) = .28t^4 + .222t^5$$

A possible $\phi(t)$ function could be

$$\phi(t) = \phi_0 + \frac{.28}{5} t^5 + \frac{.222}{6} t^6 = \phi_0 + .056t^5 + .037t^6$$

How do I determine ϕ_0?

(4) $$Z(x) = \int_0^x (y(t)+t)^2 \, dt$$

consider

$$~~\text{x}~~ \quad y_{n+1}(x) = \underbrace{\int_0^x (y_n(t)+t)^2 \, dt}_{\equiv \cancel{S(y_n)}\; S(y_n)} \quad \text{then}$$

If this iteration converges ~~to~~ to a limit fn $y(x)$ then $y(x)$ must satisfy

$$y(x) = \int_0^x (y(t)+t)^2 \, dt$$ + thus $y(x)$ is a ~~soluti~~ solution to the following

The properties ~~we~~ we seek for our iteration fn. $\phi(t)$ are

i) $|\phi(t_1) - \phi(t_0)| \geq \|y_1 - y_0\| \overset{?}{\Longleftrightarrow} |t_1 - \overset{0}{\cancel{t_0}}| \geq \|y_1 - y_0\|$

ii) $\|\phi'(t)\| \geq \|S'(y)\|$

w/ $\cancel{y_0}$E $y_0(x) = 0$

$$y_1(x) = \int_0^x t^2 \, dt = \left.\frac{t^3}{3}\right|_0^x = \frac{1}{3}x^3$$

Thus

$$\|y_1 - y_0\| = \|\tfrac{1}{3}x^3\| = \frac{a^3}{3}$$

condition (a) requires that $\|y_1 - y_0\| \le \phi(0)$

$$\| \qquad \frac{a^3}{3}$$

set $\phi(0) = \frac{a^3}{3}$

condition (b) requires $S'(y)$

$$S(y+h) - S(y) = \int_0^x \left((y+h+t)^2 - (y+t)^2\right) dt$$

$$= \int_0^x \left(\cancel{(y+t)^2} + 2(y+t)h + h^2 - \cancel{(y+t)^2}\right) dt$$

$$\Rightarrow S'(y)h\cdot(x) = 2\int_0^x (y+t)h \, dt$$

$$\Rightarrow \|S'(y)\| = 2\sup\left[\int_0^x |y+t| \, dt\right]$$

$\Rightarrow$ if

$$\|y - y_0\| \le t \Rightarrow \|S'(y)\| \le \phi(t)$$

since $y_0 \equiv 0$

$\Rightarrow$ if $\|y\|_\infty \le t \Rightarrow \|S'(y)\| \le \phi(t)$

Then $\|S(y)\| \leq 2\sup_{x\in[0,a]} \int_0^x (s+t)\,dt$

$$= 2\sup_{x\in[0,a]} \left(st + \tfrac{t^2}{2}\right)\Big|_0^x$$

$$= 2\sup_{x\in[0,a]} sx + \tfrac{x^2}{2} = 2(sa + \tfrac{a^2}{2}) = 2sa + a^2$$

Then we require $\phi'(t) = 2ta + a^2$

So $\phi(t) = \cancel{\frac{t^2}{2}a + \frac{a^2}{2}t + \phi_0}$ $t^2a + a^2t + \phi_0$

or $\phi(t) = \cancel{\frac{t^2}{2}a + \frac{a^2}{2}t + \frac{a^3}{3}}$ $t^2a + a^2t + \frac{a^3}{3}$

Considering $\underbrace{\frac{du}{dx} - (x+u)^2}_{} = 0 \qquad u(0) = 0$

$$F(u) = 0$$

Then the newton-ryson iteration method for looking for the roots of F is

$$u_{n+1} = u_n - \frac{F(u_n)}{F'(u_n)}$$

$$\overset{\sim}{\Rightarrow} \quad u_{n+1} = u_n - (F'(u_0))^{-1} F(u_n)$$ using the constant slope approximation

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consider what $F(u)$ would be

$$F(u+h) - F(u) = \cancel{\frac{du}{dx}} + \frac{dh}{dx} - (x+u+h)^2 + \quad - \cancel{\frac{du}{dx}} + (x+u)^2$$

$$= \frac{dh}{dx} - \cancel{(x+u)^2} - 2(x+u)h - h^2 + \cancel{(x+u)^2}$$

$$= \frac{dh}{dx} - 2(x+u)h - h^2$$

$$\cong F'(u)\cdot h\,(x) = \frac{dh}{dx} - 2(x+u)h \triangleq k(x)$$

$$F'(u_0)\cdot h\,(x) = \frac{dh}{dx} = k(x) \qquad h = u_{n+1} - u_n$$

$$h = \int_0^x k(t)\,dt$$

Then $\left(F'(u_0)\right)^{-1} = \int_0^x [\quad]\,dt$

So that the iteration of u becomes

$$u_{n+1} = u_n - \int_0^x \left(\frac{du_n}{dt} - (t+u_n)^2 \right) dt$$

$$= \cancel{u_n} - \cancel{u_n} + \underbrace{u_n(0)}_{0} + \int_0^x (t+u_n)^2\,dt$$

$$\therefore\ u_{n+1} = \int_0^x (t+u_n)^2\,dt \qquad u_1 = 0 \quad \checkmark.$$

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(8) $$F(y) = b - y + axy^2 + a\int_1^x y(s)^2\,ds$$

$$F(y+h) - F(y)$$

$$= -(\cancel{y}+h) + ax(y+h)^2 + a\int_1^x (y+h)^2\,ds$$

$$+\cancel{y} - axy^2 - a\int_1^x y(s)^2\,ds$$

$$= -h + ax(\cancel{y^2} + 2yh + h^2) + a\int_1^x (\cancel{y^2} + 2hy + h^2)\,ds - \cancel{axy^2} - a\int_1^x \cancel{y^2}\,ds$$

$$= -h + ax(2yh + h^2) + a\int_1^x (2hy + h^2)\,ds$$

$$\cong F'(y)h\cdot(x) = -h + 2axyh + 2a\int_1^x yh\,ds$$

What about ?

... Assume incorrect ...

$$|\phi(0) - 0| \geq \|y_1 - y_0\|$$

$y_1 = b$ $\therefore$ take $\phi(0) = b$

+ ~~...~~ y_0 I get $F'(y_0)h\cdot(x) = -h$

$$\therefore h(x) = -F'(y_0)h\cdot(x)$$

$\therefore$ require $|\phi'(t)| \geq \|S'(y)\|$ when $\|y - y_0\| < t$

But $y_0 \equiv 0$ $\therefore$ when $\|y\| < t$

$$\|S'(y)h\cdot(x)\| \leq \sup_{x \in [0,1]} \left\{ 2axt + a2t(x-1) \right\}$$

$$= \sup_{x \in [0,1]} \left\{ 2axt + 2at(x-1) \right\}$$

$$2axt + 2axt - 2at$$

$$4axt - 2at = 4at - 2at = 2at$$

$\therefore$ Set $\phi'(t) = 2at$

$$\phi(t) = at^2 + b$$

Finish ...

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(b) $F(y) = Ly - g(y)$

$$F(y+h) - F(y) = \cancel{Ly} + Lh - g(y+h) - \cancel{Ly} + g(y)$$

$$= Lh - g(y+h) + g(y)$$

$$\underbrace{F(y+h) - F(y)} = Lh - \cancel{g(y)} - g'(y)h + O(h^2) + \cancel{g(y)}$$

$$\Rightarrow F'(y)\cdot h(x) = Lh - g'(y)h$$

$$F'(y_0)\cdot h(x) = Lh$$

$$h = L^{-1}\{F'(y_0)\cdot h(x)\}$$

Then $y_{n+1} = y_n - \dfrac{F(y_n)}{F'(y_n)}$

$$y_{n+1} = y_n - (F'(y_0))^{-1} F(y_n)$$

$$= y_n - L^{-1}(F(y_n)) = y_n - L^{-1}(Ly_n - g(y_n))$$

$$= \underbrace{L^{-1}g(y_n)}_{S(y)}$$

$$h(x) + \int_0^x h(s)\,ds = k(x)$$

$$\Leftrightarrow \frac{dh}{dx} + h(x) = \frac{dk}{dx}$$

mult by e^x

$$\underbrace{e^x \frac{dh}{dx} + e^x h} = e^x \frac{dk}{dx}$$

$$\frac{d}{dx}(e^x h) = e^x \frac{dk}{dx}$$

$$h = e^{-x} \int_0^x e^s \frac{dk}{ds} \cdot ds + h(0)$$

$$= e^{-x} \left[\cancel{e} \, e^s k(s) \Big|_0^x - \int_0^x e^s k(s) \, ds \right] + h(0)$$

$$= e^{-x} \left[e^x k(x) - k(0) - \int_0^x e^s k(s) \, ds \right] + h(0)$$

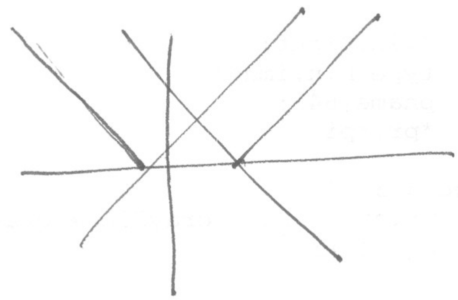
$$= k(x) - e^{-x} \int_0^x e^s k(s) \, ds - e^{-x} \underbrace{k(0) + h(0)}$$

Why not present?

. . .

Ex:

$$\min_{m\in\mathbb{R}} \sup_{x\in[0,1]} |u(x)+mv(x)| = \min_{m\in\mathbb{R}} \sup_{x\in[0,1]} |1+mx|$$



$$= \min \left\{ \underset{m\in\mathbb{R}^+}{1+m}, \quad \underset{m\in\mathbb{R}^-}{} \right.$$

$$\min_{m\in\mathbb{R}} \sup \left\{ 1, \frac{-1}{m}, |1+m| \right\}$$

?

① O, B, C

$$O + sB + tC = (s+t, t, s)$$

$$(s+t, t, s) \circ (1,0,1) = 0$$

$$s+t+s = 0$$

$$2s+t = 0$$

to be in the plan OBC we must have the form $(s+t, t, s)$

nearest to $A = (0,1,1)$

$$\Rightarrow \min_{s,t} \|(s+t,t,s) - (0,1,1)\|_2^2 = (s+t)^2 + (t-1)^2 + (s-1)^2$$

min when $\frac{\partial d}{\partial t} = 0 \quad 2(s+t) + 2(t-1) = 0$

$$\frac{\partial d}{\partial s} = 0 \Rightarrow 2(s+t) + 2(s-1) = 0$$

$$= \quad s + 2t = 1$$

$$2s + t = 1$$

$$\cancel{\begin{pmatrix} s \\ t \end{pmatrix} = \frac{\quad}{\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}}} \qquad s = \frac{\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}}{(-3)} = \frac{1}{3}$$

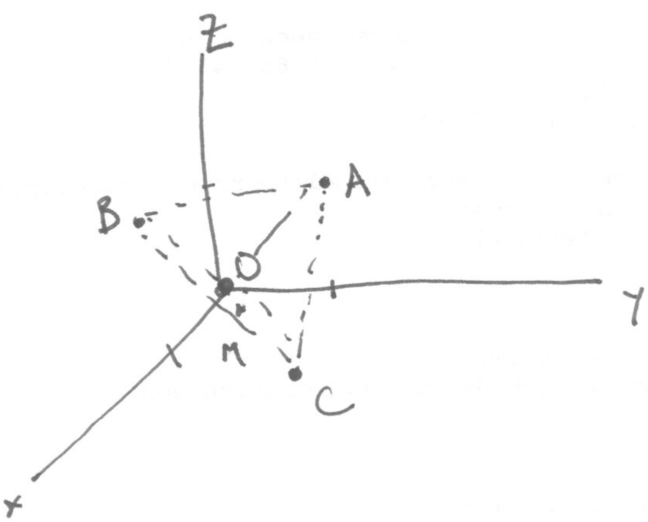
$$t = \frac{\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}}{-3} = \frac{1}{3}$$

$$\|OM\|_2 = \sqrt{\frac{1}{9} + \frac{1}{9} + \frac{4}{9}} = \sqrt{\frac{2}{3}}$$

$$\|AM\|_2 = \cancel{\sqrt{\frac{1}{9} + \left(1 - \frac{1}{3}\right)^2}}$$

$$= \sqrt{\frac{4}{9} + \left(1 - \frac{1}{3}\right)^2 + \left(1 - \frac{1}{3}\right)^2} = \sqrt{\frac{4}{9} + \frac{4}{9} + \frac{4}{9}}$$

$$= \sqrt{\frac{4}{3}}$$



$$\|OM\|_2^2 + \|AM\|_2^2 = 2\ ?$$

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(2)

$$d_1^2(t) = (3-t)^2 + (4-t)^2 + (5-t)^2$$

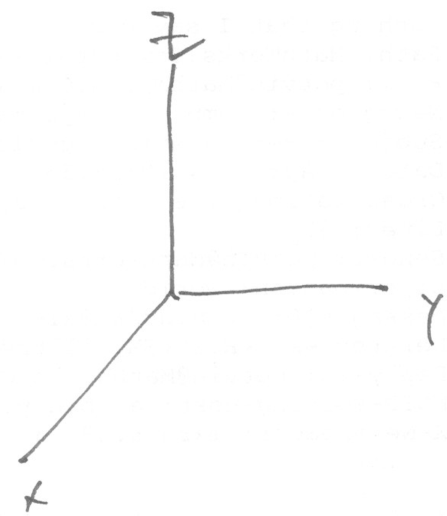
$$\frac{d(d_1^2(t))}{dt} = 2 \cdot 3 \qquad = 0$$

$\cancel{(3-t)}(-1)$

$$3-t+4-t+5-t = 0$$

$$13-3t=0$$

$$t = 13/3$$



$$d_2^2(t) = (8-t)^2 + (4-t) \ldots$$

Same solution.

point C in the plane of OAB is the

Bisector of line AB.

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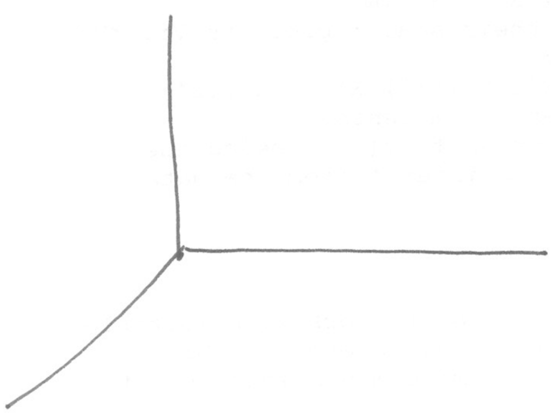
(3) $V = \cancel{\quad}$

$$V = c_1 p_1 + c_2 p_2 + c_3 p_3$$

$p_1 \circ$

$p_1 \circ V = c_1 \qquad c_2 = V \circ p_2 \qquad c_3 = p_3 \circ V$

$$V = (V \circ p_1) p_1 + (V \circ p_2) p_2 + (V \circ p_3) p_3$$



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① i	x_i	$f(x_i)$	$g(x_i)$	$e^{-x_i^2/8}$	$a+b(x_i^2-2)$	$1-(x_i^2/8)$
1	-2	1	2			
2	-1	1	-1			
3	0	1	-2			
4	1	1	-1			
5	2	1	2			

$$(\vec{f},\vec{g}) = 2-1-\cancel{2}-1+\cancel{2} = 0 \checkmark$$

Require ~~a~~ coefficients of a & b ∋

$$\sum$$

$$e^{-x_i^2/8} \cong a + b(x_i^2-2)$$

$$\Leftrightarrow \min_{a,b} \sum_{i=1}^{5}\left(e^{-x_i^2/8} - a - b(x_i^2-2)\right)^2$$

let $v = \left(e^{-x_1^2/8}, e^{-x_2^2/8}, \dots, e^{-x_5^2/8}\right)$

$$= a(1,1,\dots,1) + b(x_1^2-2, x_2^2-2, \dots, x_5^2-2)$$

Then $V = aP_1 + bP_2$

& since $(P_1,P_2)=0$

$(p_1,v) = a(p_1,p_1)$ & $(p_2,v) = b(p_2,p_2)$

∋

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$$a = \frac{(p_{11} v)}{(p_{11} p_1)} \quad + \quad b = \frac{(p_{21} v)}{(p_{21} p_2)}$$

$a =$

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(2)

x_i	1	$x_i^2 - 3.5$
0	1	
1	1	
2	1	
3	1	

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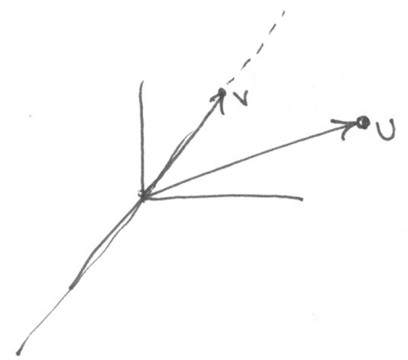
$$\int_0^{\pi/2} \left(\sin\theta - s\theta - t(\theta^3 - k\theta) \right)^2 d\theta$$

$$= \int_0^{\pi/2} \left(\sin\theta + (-s+tk)\theta - t\theta^3 \right)^2 d\theta$$

$$= \int_0^{\pi/2} \Big(\sin^2\theta + (-s+tk)\theta\sin\theta - t\theta^3\sin\theta + \cancel{(-s+tk)^2\theta^2} + \sin\theta(-s+tk)\theta$$

$$+ (-s+tk)^2\theta^2 - t(-s+tk)\theta^4 - t\theta^3\sin\theta - t(-s+tk)\theta^4 + t^2\theta^6 \Big) d\theta$$

1



Require $\underset{t\in\mathbb{R}}{\operatorname{Arg\,Min}}\, d(u,tv) \cong \underset{t\in\mathbb{R}}{\operatorname{Arg\,Min}}\, d(u,tv)^2 = (u-tv, u-tv)$

$$= (u, u-tv) - t(v, u-tv) = (u,u) - t(u,v) - t(v,u) + t^2(v,v)$$

$$= (u,u) - 2t(u,v) + t^2(v,v)$$

$$\frac{d}{dt}(\qquad) = 0$$

$$-2(u,v) + 2t(v,v) = 0 \Rightarrow t = \frac{(u,v)}{(v,v)}$$

$$\therefore \text{distance}^2_{\text{Min}} = \cancel{(u,u) - 2(u,v} \quad (u,u) - 2\frac{(u,v)}{(v,v)}(u,v) + \frac{(u,v)^2}{(v,v)^2}(v,v)$$

$$= (u,u) - 2\frac{(u,v)^2}{(v,v)} + \frac{(u,v)^2}{(v,v)}$$

$$= \frac{(v,v)(u,u) - 2(u,v)^2 + (u,v)^2}{(v,v)} = \frac{(u,u)(v,v) - (u,v)^2}{(v,v)}$$

(2) $d_{L_2}(\sin x, mx)^2 = \int_0^{\pi/2} (\sin x - mx)^2\, dx$ + to minimize w.r.t. $\underline{m}$.

$\sin x \cong mx$ expanded in series of orthogonal functions ...

$$(d_{L^2})^2 = \int_0^{\pi/2} (\sin^2 x - 2mx\sin x + m^2x^2)\, dx$$

$$= \int_0^{\pi/2} \sin^2 x\, dx - 2m\int_0^{\pi/2} x\sin x\, dx + m^2\int_0^{\pi/2} x^2\, dx$$

$$\| $$

$$= \int_0^{\pi/2}\left(\frac{1-\cos(2x)}{2}\right)dx - 2m\left[x(-\cos x\Big|_0^{\pi/2} + \int_0^{\pi/2}\cos x\, dx\right] + m^2\frac{x^3}{3}\Big|_0^{\pi/2}$$

$$= \frac{1}{2}\int_0^{\pi/2}(1-\cos(2x))\, dx - 2m\left[-\frac{\pi}{2}\cdot 0 + 0 + \sin x\Big|_0^{\pi/2}\right] + \frac{m^2}{3}\frac{\pi^3}{8}$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - \frac{\sin(2x)}{2}\Big|_0^{\pi/2}\right] - 2m + \frac{m^2}{3}\cdot\frac{\pi^3}{8}$$

$$= \frac{\pi}{4} - 2m + \frac{m^2}{24}\pi^3$$

$$\frac{d}{dm} \Rightarrow -2 + \frac{2m}{24}\pi^3 = 0 \Rightarrow m = \frac{24}{\pi^3}$$

$\Rightarrow$ ~~B~~ Best approximation in ~~P_2~~ $\mathcal{L}_2[0, \frac{\pi}{2}]$ sinx is

by $\quad = \frac{24}{\pi^3}x$

w/ $\mathcal{L}_2[0, \frac{\pi}{2}]$ distance of

$$\left[d_{L_2}\left(\sin x, \frac{24}{\pi^3}x\right)\right]^2 = \frac{\pi}{4} - \frac{48}{\pi^3} + \frac{\pi^3}{24}\left(\frac{24^2}{(\pi^3)^2}\right)$$

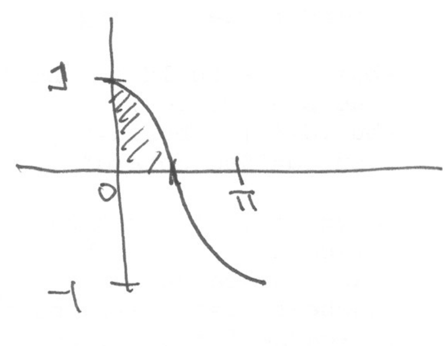
$$= \frac{\pi}{4} - \frac{48}{\pi^3} + \frac{24}{(\pi^3)}$$

Using the arguments from problem #1

$$m = \frac{(u,v)_{L_2}}{(v,v)_{L_2}} \quad \text{w/ } u = \sin x, \; v = x$$

$$\stackrel{?}{=} \frac{(\sin x, x)_{L_2}}{(x,x)_{L_2}} = \frac{\int_0^{\pi/2} x \sin x \, dx}{\left(\frac{1}{3}\frac{\pi^3}{8}\right)} = \frac{+\int_0^{\pi/2} \cos x \, dx}{\left(\frac{1}{3}\frac{\pi^3}{8}\right)}$$

$$= \frac{24}{\pi^3} \quad \text{Yes!!}$$



+ Again from problem #1

$$d^2_{\min(L_2)} = \cancel{(v,v)}\cancel{(u,u) - 2(u,v)^2 + (u,v)}$$

$$(u,u) = \int_0^{\pi/2} \sin^2 x \, dx = \frac{1}{2}\left(\frac{\pi}{2}\right) = \frac{\pi}{4}$$

so

$$d_{min}^2 = \frac{\left(\frac{\pi}{4}\right)\cdot\frac{1}{3}\cdot\frac{\pi^3}{8} - 1^2}{\frac{1}{3}\frac{\pi^3}{8}}$$

$$= \frac{\pi}{4} - \frac{24}{\pi^3}$$

(3) . . .

(4) $$\cos\theta = \frac{(u,v)}{(u,u)(v,v)} = \frac{\int_{-1}^{+1}(a+bx^2)(cx+kx^3)\,dx}{\int_{-1}^{+1}(a+bx)^2 dx \cdot \int_{-1}^{+1}(cx+kx^3)^2 dx}$$

$$= \frac{\int_{-1}^{+1}\left(acx + akx^3 + \cancel{bx^3}\, bcx^3 + bkx^5\right) dx}{\cancel{\int_{-1}^{+1}} \left.\frac{(a+bx)^3}{3\cdot b}\right|_{-1}^{+1} \cdot \int_{-1}^{+1}\left(c^2x^2 + 2ckx^4 + k^2x^6\right) dx}$$

$$= \frac{\left. ac\frac{x^2}{2} \right|_{-1}^{+1}}{\left(\frac{(a+b)^3}{3b} - \frac{(a-b)^3}{3b}\right)\left(\left.\frac{c^2x^3}{3} + \frac{2ckx^5}{5} + \frac{k^2x^7}{7}\right|_{-1}^{+1}\right)} = \ldots$$

(5) $d^2(f,g) = (f-g, f-g)_{L^2}$ $\qquad d(f,g) = \sqrt{(f-g,f-g)_{L^2}}$

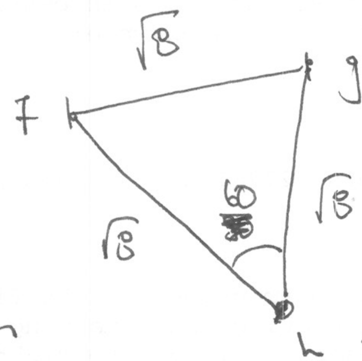
$$= \int_{-1}^{+1} 2^2\,dx = 4\cdot 2 = 8$$

$$d^2(f,h) = \int_{-1}^{+1}(1+3x)^2\,dx = \int_{-1}^{+1}(1+6x+9x^2)\,dx = 2 + 3\cancel{x^2}^{\,0} + 2\cdot\frac{9}{3}x^3\Big|_0^{+1}$$

$$= 2 + 6 = 8$$

$$d^2(g,h) = \int_{-1}^{+1}(-1+3x)^2\,dx = \int_{-1}^{+1}(1-6x+9x^2)\,dx = 2 + \frac{9}{3}x^3\cdot 2\Big|_0^{+1}$$

$$= 2 + 6 = 8.$$



eq Δ

$(g,h) =$

$= (g,g)^{1/2}(h,h)^{1/2}\cos\theta$

$$x^T y = x\circ y = \|x\|\cdot\|y\|\cos\theta$$

$$d(f,f) = \sqrt{(f,f)_{L^2}} = \|f\|_{L^2}$$

$=$

$$(g,g)_{L^2} = 4\cdot(2) = 8$$

$$(h,h)_{L^2} = \int_{-1}^{+1}(1+6x+9x^2)\,dx = 8$$

So

$$(g,h) \overset{?}{=} 8 \cdot \cos 60^\circ = 4$$

Check

$$(g,h) = \int_{-1}^{+1} (2(1+3x))\,dx = 2 \cdot 2 = 4 \checkmark$$

(6) $f_n(x) = x^n$

$$\|f_n\|_{L^2} = \sqrt{\int_0^{+1} (x^n)(x^n)\,dx} = \sqrt{\int_0^1 x^{2n}\,dx}$$

$$= \sqrt{\left.\frac{x^{2n+1}}{2n+1}\right|_0^1} = \sqrt{\frac{1}{2n+1}} = \frac{1}{\sqrt{2n+1}}$$

$$\|f_n\|_{C[0,1]} = \max_{x \in [0,1]} |f_n| = 1$$

$f_n \to 0$ in L^2

but $f_n \not\to 0$ in $C[0,1]$

(7) $$\|f_n\|_{L^2}^2 = \int_0^1 n^{1/4} e^{-nx}\,dx = \left. n^{1/4} \frac{e^{-nx}}{(-n)} \right|_0^1 = -n^{-3/4}\left(e^{-n} - 1\right)$$

$\therefore \quad \|f_n\|_{L^2} \longrightarrow 0 \quad$ as $\quad n \to \infty.$

$$\|f_n\|_{C[0,1]} = \max_{x\in[0,1]} |n^{1/4} e^{-nx}| = n^{1/4}$$

$\therefore \quad \|f_n\|_{C[0,1]} \longrightarrow +\infty \quad$ As $\quad n \to \infty$

$\therefore$ in $L^2[0,1]$ f_n tends to 0.

in $C[0,1]$ f_n has no limit.

(8) $f(x) = a \cdot 1 \qquad g(x) = \cos x$

$\therefore$ in span L^2 $\quad a = \dfrac{(u,v)_{L^2}}{(v,v)_{L^2}} = \dfrac{(\cos x, 1)_{L^2}}{(1,1)_{L^2}} = \dfrac{\int_{-\pi/2}^{\pi/2} \cos x \, dx}{2(\frac{\pi}{2})}$

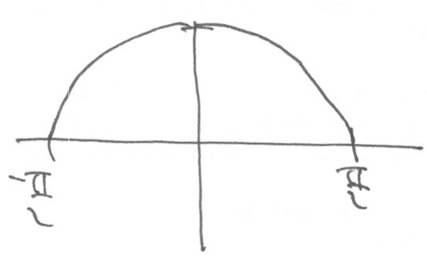
By problem 1

$$= \frac{\sin x \Big|_{-\pi/2}^{\pi/2}}{\pi} = \frac{2}{\pi}$$

$\therefore$ Best fit in L^2 space is $\dfrac{2}{\pi}$

Assuming this procedure works for $C[0,1]$ ~~eto~~ clo

$$a = \frac{(\cos x, 1)_{C[\ ]}}{(1,1)_{C[0,\ ]}} = \frac{\max_{[-\frac{\pi}{2}, \frac{\pi}{2}]} |\cos x|}{1}$$



$$= 1$$

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$\therefore$ Best fit to $\cos x$ in $C[-\frac{\pi}{2}, \frac{\pi}{2}]$ space is 1.

(9) ax,

$$a = \frac{(x, \sin x)}{(x, x)} = \frac{\int_{-\frac{\pi}{2}}^{+\pi/2} (x \sin x)\, dx}{\int_{-\frac{\pi}{2}}^{\pi/2} x^2\, dx}$$

$\sin x^\circ \cong \frac{x}{60}$ $\quad$ $\underline{x}$ $\underline{\text{in}}$ $\underline{\text{degrees}}$ $\quad$ $x^\circ = \frac{180}{\pi} x^r$

$\Vert$

$\cancel{\sin \frac{\pi}{180} x} =$ $\quad$ $\sin x = \frac{180}{60\pi} x$

$= \frac{3}{\pi} x$

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(1) eq (1)

$$c_r = \left(\frac{2}{\pi}\right)\int_0^{\pi} f(x)\sin rx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} (\pi^2 x - x^3)\sin rx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \pi^2 x \sin rx \, dx - \frac{2}{\pi}\int_0^{\pi} x^3 \sin rx \, dx$$

$$= \cancel{2}\, 2\pi\left[\frac{-\pi r\cos(r\pi) + \sin(\pi r)}{r^2}\right]$$

$$-\frac{2}{\pi}\left[\frac{6\pi r\cos(r\pi) - \pi^3 r^3\cos(\pi r) - 6\sin(\pi r) + 3\pi^2 r^2\sin(\pi r)}{r^4}\right]$$

$$= 2\pi\left[-\frac{\pi}{r}(-1)^r\right] - \frac{2}{\pi}\left[\frac{6\pi}{r^3}(-1)^r - \frac{\pi^3}{r^3}(-1)^r\right]$$

$$= \cancel{\frac{-2\pi^2(-1)^r}{r}} - \frac{12(-1)^r}{r^3} + \cancel{\frac{2\pi^2}{r}(-1)^r} = \frac{12(-1)^{r+1}}{r^3}$$

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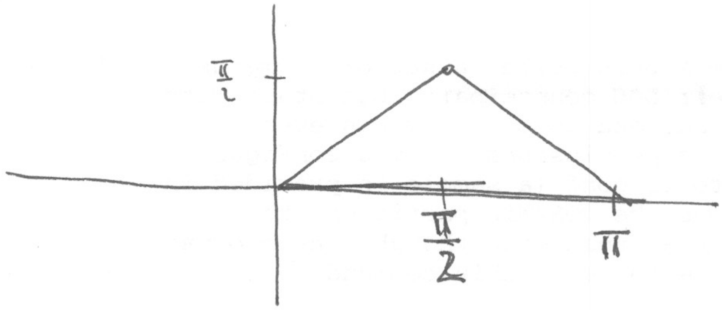
(2)



a $\quad c_r = \frac{2}{\pi}\int_0^{\pi} f(x)\sin rx\,dx \qquad r = 1, 2, 3, \ldots$

$$= \frac{2}{\pi}\int_0^{\pi/2} x\sin rx\,dx + \frac{2}{\pi}\int_{\pi/2}^{\pi}(\pi - x)\sin rx\,dx$$

$$= \frac{2}{\pi}\left[-x\frac{\cos rx}{r}\Big|_0^{\pi/2} + \frac{1}{r}\int_0^{\pi/2}\cos rx\,dx\right] + \frac{2}{\pi}\left[-(\pi-x)\frac{\cos rx}{r}\Big|_{\pi/2}^{\pi} + - \int_{\pi/2}^{\pi}\frac{\cos rx}{r}\,dx\right]$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{2r}(-1)^r + \frac{1}{r}\frac{\sin rx}{r}\Big|_0^{\pi/2}\right] + \frac{2}{\pi}\left[0 + \frac{\left(\frac{\pi}{2}\right)}{r}\cos\left(r\frac{\pi}{2}\right) - \frac{1}{r}\frac{\sin(rx)}{r}\Big|_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi}{2r}(-1)^{r+1} + \frac{1}{r^2}\sin\left(\frac{r\pi}{2}\right)\right] + \frac{2}{\pi}\left[\frac{1}{r}\frac{\pi}{2}\cos\left(r\frac{\pi}{2}\right) - \frac{1}{r^2}\left(0 - \sin\left(r\frac{\pi}{2}\right)\right)\right]$$

$$=$$

$$\frac{1}{r}(-1)^{r+1} + \frac{2}{\pi}\frac{1}{r^2}$$

r	$\sin\left(\frac{r\pi}{2}\right)$
1	1
2	0
3	-1
4	0

(-1)

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$$C_r = \frac{2}{\pi}\int_0^{\pi/2} x\sin(rx)\,dx + \frac{2}{\pi}\int_{\pi/2}^{0} v\sin(r(\pi-v))\,dv\,(-1)$$

$$= \frac{2}{\pi}$$

$V = \pi - x \quad dx = -dv$

$x = \pi - v$

$$= \frac{2}{\pi}\int_0^{\pi/2} x\sin(rx)\,dx + \frac{2}{\pi}\int_0^{\pi/2} x\sin(r(\pi-x))\,dx$$

$$x\left[\underbrace{\sin(r\pi)}_{0}\cos(rx) - \cos(r\pi)\sin(rx)\right]$$

$$= \frac{2}{\pi}\int_0^{\pi/2} x\sin(rx)\,dx - \frac{2}{\pi}(-1)^r\int_0^{\pi/2} x\sin(rx)\,dx$$

$$= \frac{2}{\pi}\left[1+(-1)^{r+1}\right]\int_0^{\pi/2} x\sin(rx)\,dx$$

$$= \frac{2}{\pi}\left(1-(-1)^r\right)\left[-x\frac{\cos(rx)}{r}\Big|_0^{\pi/2} + \int_0^{\pi/2}\frac{\cos(rx)}{r}\,dx\right]$$

$$= \frac{2}{\pi}\left(1-(-1)^r\right)\left[-\frac{\pi}{2}\frac{1}{r}\cos\left(r\frac{\pi}{2}\right) + \frac{1}{r^2}\sin(rx)\Big|_0^{\pi/2}\right]$$

$$= \frac{2}{\pi}\left(1-(-1)^r\right)\left[-\frac{\pi}{2}\frac{1}{r}\cos\left(r\frac{\pi}{2}\right) + \frac{1}{r^2}\sin\left(r\frac{\pi}{2}\right)\right]$$

$$\cos\left(r\frac{\pi}{2}\right) = \{\overset{1}{0}, \overset{2}{-1}, \overset{3}{0}, \overset{4}{+1}, 0, \overset{6}{-1}, \dots\} \overset{?}{=} \frac{1}{2}\left(1+(-1)^r\right)(-1)^{r/2}$$

$$\sin\left(r\frac{\pi}{2}\right) = \left\{ \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 0 & -1 & 0, \cdots \end{array} \right\} =$$

$$\frac{1}{2}\left(1-(-1)^r\right)(-1)^{\frac{r+1}{2}}$$

$$\therefore \quad c_r = \frac{2}{\pi}\left(1-(-1)^r\right)\bigg| -\frac{\pi}{2}\frac{1}{r}$$

Assuming the result is correct

The sum is absolutely convergent since $1/n^2$ converges absolutely

I believe it also converges uniformly since .. By Weierstrass

M test $\quad |c_r \sin(rx)| \leq M_r \quad + \quad \sum M_r < \infty.$

Now the derivative would neither converge absolutely nor

uniformly.

(3)

$$b_r = \frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi}{2}-\theta\right)\sin(r\theta)\,d\theta$$

$$= \frac{2}{\pi}\cdot\frac{\pi}{2}\int_0^{\pi}\sin(r\theta)\,d\theta - \frac{2}{\pi}\int_0^{\pi}\theta\sin(r\theta)\,d\theta$$

$$= -\frac{\cos(r\theta)}{r}\Big|_0^{\pi} - \frac{2}{\pi}\left[-\theta\frac{\cos(r\theta)}{r}\Big|_0^{\pi} + \int_0^{\pi}\frac{\cos(r\theta)}{r}\,d\theta\right]$$

$$= -\frac{1}{r}((-1)^r - 1) - \frac{2}{\pi}\left[-\frac{\pi(-1)^r}{r} + \frac{1}{r}\frac{\sin(r\theta)}{r}\Big|_0^{\pi}\right]$$

$$= -\frac{1}{r}((-1)^r-1) + \frac{2(-1)^r}{r} - \frac{2}{\pi}\frac{1}{r^2}\left[0\right]$$

$$= \frac{-(-1)^r+1+2(-1)^r}{r} = \frac{(-1)^r+1}{r} =$$

$$f(\theta) = \frac{\pi}{2}-\theta \cong \frac{4}{\pi}\sum_{r=0}^{\infty}\frac{1}{(2r+1)^2}\cos(2r+1)\theta$$

let $x = \cos\theta$

Then

$$\frac{\pi}{2} - \cos^{-1}x = \sin^{-1}x = \frac{4}{\pi}\sum_{r=0}^{\infty}\frac{1}{(2r+1)^2}T_{2r+1}(x)$$

$$\underset{\vec{x}\in\mathbb{R}^n}{\text{Max}} \sum a_r x_r \quad \to \quad \sum x_r^2 = R^2 = \text{constant}$$

Using lagrange multipliers :

$$\underset{\vec{x}\in\mathbb{R}^n}{\text{Max}}\, f(\vec{x}) \quad \text{w/} \quad f(\vec{x}) = \sum a_r x_r$$

$$\frac{df}{dt} = \sum a_r \dot{x}_r = 0$$

$$\& \quad \sum_r 2 x_r \dot{x}_r = 0$$

$\Rightarrow$ Both must be proportional

$$\sum a_r \dot{x}_r = \lambda \sum x_r \dot{x}_r$$

$$\to a_r = \lambda x_r \Rightarrow \cancel{x_r} = \quad x_r = \frac{a_r}{\lambda}$$

so $\sum x_r^2 = R^2$

$$\Rightarrow \frac{1}{\lambda^2}\sum a_r^2 = R^2$$

$$\to \lambda = \frac{\pm\|a\|_2}{R}$$

so $\cancel{\underset{\vec{x}\in\mathbb{R}^n}{\text{Extreem}}} \frac{\sum a_r^2}{\lambda}$

$$\underset{\vec{x}\in\mathbb{R}^n}{\text{Extreem}} \sum a_r x_r =$$

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$$= \frac{\sum a r^2}{\lambda} = \pm \frac{\|a\|^2}{\frac{\|a\|}{R}} = \pm \|a\| \cdot R$$

$$= \pm \|a\| \cdot \|x\|$$