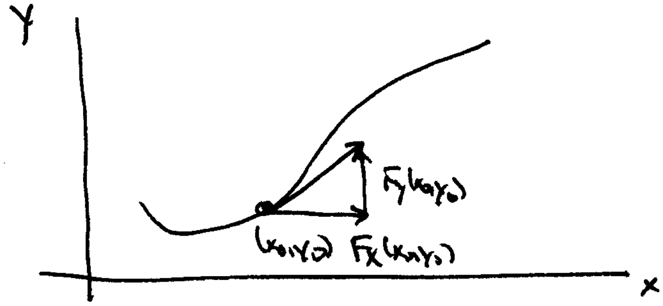
(1-6)



(a) So $\frac{dy}{dx}(x_0,y_0) = \frac{F_y(x_0,y_0)}{F_x(x_0,y_0)}$

(b) From 1-1

(a) $\frac{dy}{dx} = \frac{x}{y} = y\,dy = x\,dx \Rightarrow \frac{y^2}{2} = \frac{x^2}{2} + C$ ✓

(b) $\frac{dy}{dx} = \frac{y/\sqrt{2}}{y/\sqrt{2}} = 1 \Rightarrow y = x + C$ ✓

(c) $\frac{dy}{dx} = \frac{-y}{x} \Rightarrow -\frac{1}{y}dy = \frac{dx}{x} \Rightarrow -\ln y = \ln x + C$

$\Rightarrow \ln y^{-1} = \ln x + \ln C_2$

$\Rightarrow y^{-1} = x C_2 \Rightarrow C = xy$ ✓

(d) $\frac{dy}{dx} = 0 \Rightarrow y = C$ ✓

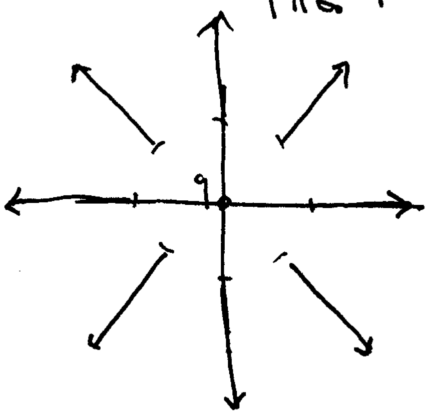
(e) $\frac{dy}{dx} = \frac{x}{0} \Rightarrow \frac{dx}{dy} = \frac{F_x}{F_y} = 0 \Rightarrow x = C$ ✓

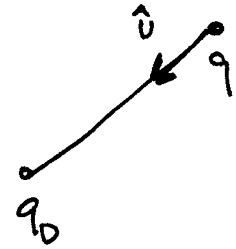
(f) $\frac{dy}{dx} = \frac{x/\sqrt{}}{y/\sqrt{}} =$ same as (a) ✓ (g) $\frac{dy}{dx} = x \Rightarrow y = \frac{x^2}{2} + C$ ✓

(h) $\frac{dy}{dx} = \frac{y}{1} \Rightarrow \frac{dy}{y} = dx \Rightarrow \ln y = x + C \Rightarrow y = Ce^x$

Pg 9 schey

(1-2) $E(r) = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{u}$





Here I pick any plane through the origin.

(1-3)

(a) $F(x,y) = \frac{x\hat{i} + y\hat{j}}{\sqrt{x^2+y^2}}$ ✓

(b) $F(x,y) = (x+y)^2 \frac{(\hat{i}+\hat{j})}{\sqrt{2}}$ ✓

(c) $F(x,y) = r \cdot \left[\frac{y\hat{i} - x\hat{j}}{\sqrt{x^2+y^2}} \right] = y\hat{i} - x\hat{j}$

(d) $F(x,y,z) = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2+y^2+z^2}}$ ✓

(1-4) $r = \hat{i} a\cos(\omega t) + \hat{j} b\sin(\omega t)$

(a) $r = \|r\| = \sqrt{a^2\cos^2(\omega t) + b^2\sin^2(\omega t)}$ ✓

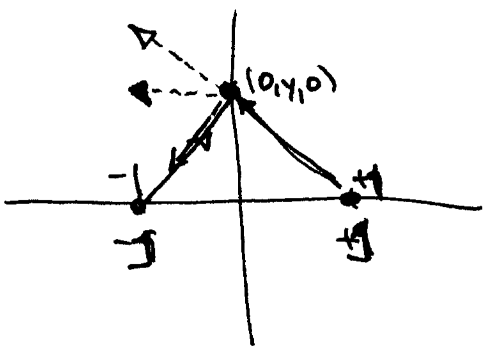
(b) $v = \dot{r} = -\hat{i} a\omega \sin(\omega t) + b\omega \hat{j}\cos(\omega t)$ ✓

$a = \dot{v} = \ddot{r} = -\hat{i} a\omega^2 \cos(\omega t) - b\omega^2 \hat{j}\sin(\omega t)$

(c) $x(t) = a\cos(\omega t)$ $y(t) = b\sin(\omega t)$

eliminating t gives $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$

(1-5)



$$E(r) = \frac{1}{4\pi\epsilon_0}\sum_{l=1}^{2}\frac{q_l}{|r-r_l|^2}\hat{u}_l$$

$$\vec{E}(r) = \frac{1}{4\pi\epsilon_0}\left[\frac{+1}{\left(\sqrt{1^2+y^2+0^2}\right)^2}\frac{-\hat{i}+y\hat{j}}{\sqrt{1+y^2}} + \frac{-1}{\left(\sqrt{(1-0)^2+y^2}\right)^2}\frac{(+\hat{i}+y\hat{j})}{\sqrt{1+y^2}}\right]$$

$$= \frac{1}{4\pi\epsilon_0}\left[\frac{1}{(1+y^2)^{3/2}}\right](-\hat{i}+\cancel{y\hat{j}}-\hat{i}-\cancel{y\hat{j}}) = \frac{-2\hat{i}}{4\pi\epsilon_0(1+y^2)^{3/2}}$$ ✓

Pg 52 Schey

11-1
$$\hat{n} = \frac{-\hat{i}\frac{\partial f}{\partial x} - \hat{j}\frac{\partial f}{\partial y} + \hat{k}}{\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}}$$

(a) $z = 2 - x - y$

$$\frac{\partial f}{\partial x} = -1$$

$$\frac{\partial f}{\partial y} = -1$$

So $$\hat{n} = \frac{-\hat{i}(-1) - \hat{j}(-1) + \hat{k}}{\sqrt{1+1+1}} = \frac{1}{\sqrt{3}}(\hat{i} + \hat{j} + \hat{k})$$ ✓

(b) $z = (x^2+y^2)^{1/2}$

$$\frac{\partial f}{\partial x} = \frac{1}{2}(x^2+y^2)^{-1/2}(2x) = \frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{\partial f}{\partial y} = \frac{1}{2}(x^2+y^2)^{-1/2}(2y) = \frac{y}{\sqrt{x^2+y^2}}$$

So $$\hat{n} = \frac{\frac{-x}{\sqrt{x^2+y^2}}\hat{i} - \frac{y}{\sqrt{x^2+y^2}}\hat{j} + \hat{k}}{\sqrt{\frac{x^2}{(x^2+y^2)} + \frac{y^2}{(x^2+y^2)} + 1}} = \frac{1}{\sqrt{2}\sqrt{x^2+y^2}}\left(-x\hat{i} - y\hat{j} + \sqrt{x^2+y^2}\,\hat{k}\right)$$

$$= \frac{-1}{\sqrt{2}\,z}\left(x\hat{i} + y\hat{j} - z\hat{k}\right)$$

(c) $z = f(x,y) = (1-x^2)^{1/2}$

$$\frac{\partial f}{\partial x} = \frac{-2x}{2(1-x^2)^{1/2}} = \frac{x}{\sqrt{1-x^2}}$$

$$\frac{\partial f}{\partial y} = 0$$

So
$$\hat{n} = \frac{\frac{-x}{\sqrt{1-x^2}}\hat{\imath} + \hat{k}}{\sqrt{\frac{x^2}{1-x^2}+1}} = \frac{\frac{-x}{\sqrt{1-x^2}}\hat{\imath} + \hat{k}}{\sqrt{\frac{\cancel{x^2}+1-\cancel{x^2}}{1-x^2}}}$$

$$= \sqrt{1-x^2}\left[\frac{-x}{\sqrt{1-x^2}}\hat{\imath} + \hat{k}\right]$$

$$= -x\hat{\imath} + \sqrt{1-x^2}\,\hat{k} = -x\hat{\imath} + z\hat{k}$$

(d) $z = f(x,y) = \left(1 - \frac{x^2}{a^2} - \frac{y^2}{a^2}\right)^{1/2}$

$$\frac{\partial f}{\partial x} = \frac{1}{2}\frac{1}{\left(1-\frac{x^2}{a^2}-\frac{y^2}{a^2}\right)^{1/2}}\left(-\frac{2x}{a^2}\right) = \frac{-x}{a^2\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{a^2}}} = \frac{-x}{a^2 z}$$

$$\frac{\partial f}{\partial y} = \frac{1}{2}\frac{1}{\left(1-\frac{x^2}{a^2}-\frac{y^2}{a^2}\right)^{1/2}}\left(-\frac{2y}{a^2}\right) = \frac{-y}{a^2\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{a^2}}} = \frac{-y}{a^2 z}$$

$\therefore$ Now $\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2} = \sqrt{1+\frac{x^2}{a^4z^2}+\frac{y^2}{a^4z^2}}$

$=$

So

$$\hat{n} = \frac{-\frac{\partial f}{\partial x}\hat{i} - \frac{\partial f}{\partial y}\hat{j} + \hat{k}}{\sqrt{\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+1}} = \frac{+\frac{x}{a^2z}\hat{i} + \frac{y}{a^2z}\hat{j} + \hat{k}}{\sqrt{1+\frac{x^2}{a^4z^2}+\frac{y^2}{a^4z^2}}}$$

+ since $\cancel{a^2z^2 = a^2+x^2+y^2}$ $\quad z^2 = 1-\frac{x^2}{a^2}-\frac{y^2}{a^2}$

$\Rightarrow$ $\cancel{x^2+y^2 = a^2(z^2-1)}$ $\iff$ $a^2z^2 = a^2-x^2-y^2$

So the square root becomes $\iff$ $x^2+y^2 = (1-z^2)a^2$

$$\cancel{\sqrt{1+\frac{x^2}{a^4z^2}+\frac{y^2}{a^4z^2}} = \sqrt{1+\frac{1}{a^4z^2}(a^2(z^2-1))}} = \cancel{\sqrt{1+\frac{z^2-1}{a^2z^2}}}$$

$$= \cancel{\frac{\sqrt{a^2z^2+z^2-1}}{a|z|}}$$

$$= \cancel{\frac{\sqrt{(a^2+1)z^2-1}}{a|z|}}$$

So

$$\hat{n} = \sqrt{1+\frac{x^2}{a^4z^2}+\frac{y^2}{a^4z^2}} = \sqrt{1+\frac{x^2+y^2}{a^4z^2}}$$

$$= \sqrt{1+\frac{(1-z^2)a^2}{a^4z^2}} = \sqrt{1+\frac{(1-z^2)}{a^2z^2}}$$

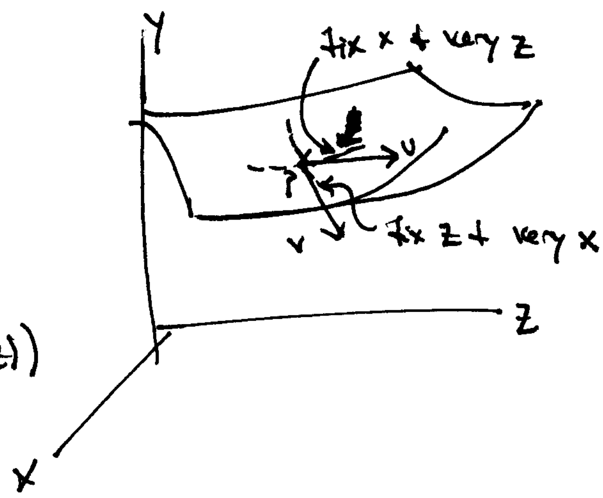
$$= \sqrt{\frac{a^2 z^2 + 1 - z^2}{a^2 z^2}} = \frac{\sqrt{1 + (a^2 - 1) z^2}}{a|z|}$$

Then $$\hat{n} = \frac{\dfrac{x}{a^2 z}\hat{\imath} + \dfrac{y}{a^2 z}\hat{\jmath} + \hat{k}}{\dfrac{\sqrt{1 + (a^2-1) z^2}}{a|z|}} = \cancel{\frac{x}{a}}.$$

$$= \frac{1}{\sqrt{1 + (a^2-1) z^2}}$$

II-3

$\hat{n} = \frac{N}{|N|} = \frac{u \times v}{|u \times v|}$ with $y = g(x,z)$



w/ $-u = \vec{r}(x,y,z) - \vec{r}(x,y,z+dz)$

$\underset{dz \ll 1}{\Rightarrow}$

$= (0, g(x,z) - g(x,z+dz), z-(z+dz))$

$= (0, -\frac{\partial g}{\partial z}dz, -dz)$

$= dz(0, -\frac{\partial g}{\partial z}, -1)$

(should have taken the negative of the choice)

For $\vec{v} = \vec{r}(x+dx,y,z) - \vec{r}(x,y,z)$

$= (x+dx-x, g(x+dx,z) - g(x,z), 0)$

$= (dx, \frac{\partial g}{\partial x}dx, 0) = dx(1, \frac{\partial g}{\partial x}, 0)$

So $N = u \times v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & +\frac{\partial g}{\partial z}dz & dz \\ dx & \frac{\partial g}{\partial x}dx & 0 \end{vmatrix}$

$= \hat{i}(-\frac{\partial g}{\partial x}dxdz) - \hat{j}(-dxdz) + \hat{k}(-\frac{\partial g}{\partial z}dzdx)$

11-2

(a) let ~~f~~ $z = f(x,y) = \frac{d}{c} - \frac{b}{c}y - \frac{a}{c}x$

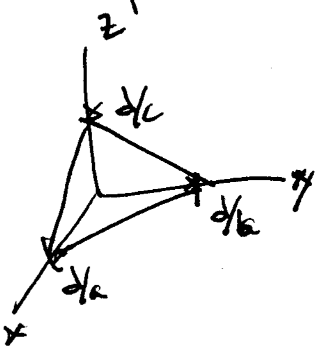
Then $\hat{n} \equiv \pm \dfrac{-\hat{\imath}\frac{\partial f}{\partial x} - \hat{\jmath}\frac{\partial f}{\partial y} + \hat{k}}{\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}}$

w/ $\frac{\partial f}{\partial x} = -\frac{a}{c}$ $\quad \frac{\partial f}{\partial y} = -\frac{b}{c}$ so

$$\hat{n} = \pm\left(\frac{-\hat{\imath}\left(-\frac{a}{c}\right) - \hat{\jmath}\left(-\frac{b}{c}\right) + \hat{k}}{\sqrt{1 + \frac{a^2}{c^2} + \frac{b^2}{c^2}}}\right) = \pm\left(\frac{\frac{a}{c}\hat{\imath} + \frac{b}{c}\hat{\jmath} + \hat{k}}{\sqrt{1 + \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2}}\right)$$

$$= \pm \frac{a\hat{\imath} + b\hat{\jmath} + c\hat{k}}{\sqrt{a^2 + b^2 + c^2}} \quad ✓$$

(b) The constant d is simply a shift in the location of the intercept of the plane, ~~i.e. the plane goes through~~ i.e. the plane goes thru the 3 points $\left(\frac{d}{a}, 0, 0\right), \left(0, \frac{d}{b}, 0\right), \left(0, 0, \frac{d}{c}\right)$ + thus changing the value of d will only shift the location of the plane + not its Normal.



So $\vec{N} = dx\,dz\left[-\frac{\partial g}{\partial x}\hat{\imath} + \hat{\jmath} - \hat{k}\frac{\partial g}{\partial z}\right]$

So upon normalizing we get

$$\hat{n} = \pm\frac{\left(-\frac{\partial g}{\partial x}\hat{\imath} + \hat{\jmath} - \frac{\partial g}{\partial z}\hat{k}\right)}{\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + 1 + \left(\frac{\partial g}{\partial z}\right)^2}}$$

By similar arguments if $x = h(y,z)$

$$\hat{n} = \pm\frac{\left(\hat{\imath} - \frac{\partial h}{\partial y}\hat{\jmath} - \frac{\partial h}{\partial z}\hat{k}\right)}{\sqrt{1 + \left(\frac{\partial h}{\partial y}\right)^2 + \left(\frac{\partial h}{\partial z}\right)^2}}$$

With the plane $ax + by + cz = d$ we have

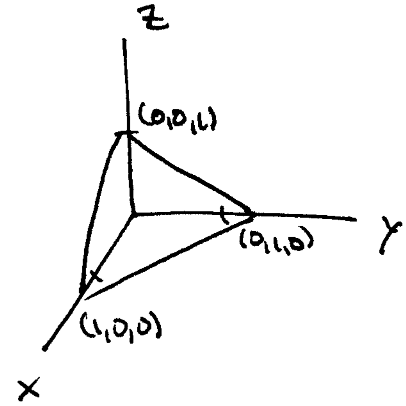
$$y = g(x,z) = \frac{1}{b}(d - ax - cz)$$

So $\frac{\partial g}{\partial x} = -\frac{a}{b}$ $\frac{\partial g}{\partial z} = -\frac{c}{b}$

So $$\hat{n} = \pm\frac{\left(\frac{a}{b}\hat{\imath} + \hat{\jmath} + \frac{c}{b}\hat{k}\right)}{\sqrt{\frac{a^2}{b^2} + 1 + \frac{c^2}{b^2}}} = \pm\frac{(a\hat{\imath} + b\hat{\jmath} + c\hat{k})}{\sqrt{a^2 + b^2 + c^2}}$$

S 52 Schey

11-4 $\quad \iint_S f(x,y,z)\,dS$



(a) $\quad f(x,y,z) = z$

$$dS = \frac{dx\,dy}{|\hat{n}\circ\hat{k}|}$$

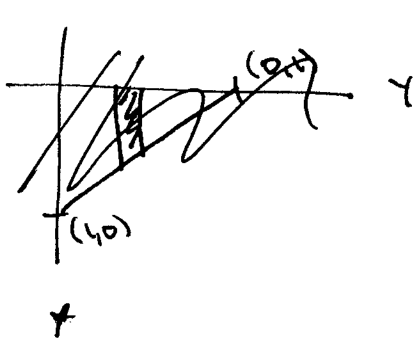
w/ $\quad z = g(x,y) = 1-x-y$

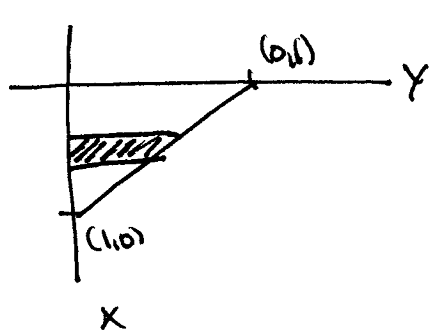
$$\hat{n} = \pm\frac{\left(-\frac{\partial z}{\partial x}\hat{i} - \frac{\partial z}{\partial y}\hat{j} + \hat{k}\right)}{\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}} = \cdots$$

But $\quad \hat{n}\circ\hat{k} = \dfrac{\pm 1}{\sqrt{1+(\;)^2+(\;)^2}} = \dfrac{\pm 1}{\sqrt{1+1+1}} = \dfrac{\pm 1}{\sqrt{3}}$

So $\quad dS = \sqrt{3}\,dx\,dy$

So $$\iint_S f(x,y,z)\,dS = \int_{x=0}^{1}\int_{y=0}^{1-x} z\sqrt{3}\,dy\,dx = \sqrt{3}\int_0^1\int_0^{1-x}(1-x-y)\,dy\,dx$$





$$\sqrt{3}\int_{x=0}^{1}\left((1-x)y-\frac{y^2}{2}\right|_{y=0}^{1-x}dx = \sqrt{3}\int_0^1\left((1-x)^2-\frac{1}{2}(1-x)^2\right)dx$$

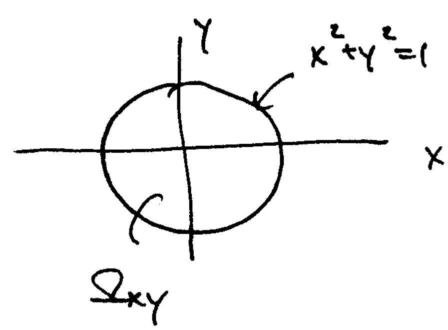
$$=+\frac{\sqrt{3}}{2}\int_0^1(1-x)^2\,dx = \frac{\sqrt{3}}{2}\,\frac{(1-x)^3}{3}(-1)\Big|_0^1 = \frac{-1}{2\sqrt{3}}\left[0-\frac{1}{3}\right] = \frac{1}{6\sqrt{3}}$$

(b) $f(x,y,z) = \dfrac{1}{1+4(x^2+y^2)}$

$z = x^2+y^2 \qquad z=0 \;\&\; z=1$

$z=0$ is $(x,y)=0$ only

$z=1$ is $x^2+y^2=1$ a circle



$$\iint_S f(x,y,z)\,dS$$

$$|\hat{n}\cdot\hat{k}| = \frac{1}{\sqrt{1+\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial y}\right)^2}} = \frac{1}{\sqrt{1+(2y)^2+(2x)^2}} = \frac{1}{\sqrt{1+4(x^2+y^2)}}$$

so w/ $dS = \dfrac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = \sqrt{1+4(x^2+y^2)}\,dx\,dy$ this integral becomes

$$\iint_{\Omega_{xy}} \frac{\sqrt{1+4(x^2+y^2)}}{1+4(x^2+y^2)}\,dx\,dy = \iint_{\Omega_{xy}} \frac{dx\,dy}{\sqrt{1+4(x^2+y^2)}}$$

convert to radial coordinates

we get w/ $dxdy = rdrd\theta$ we get

$$\int_{\theta=0}^{2\pi}\int_{r=0}^{1}\frac{r\,dr\,d\theta}{\sqrt{1+4r^2}} = \frac{2\pi}{2}\int_0^1 \frac{dv}{\sqrt{1+4v}} = \pi\frac{(1+4v)^{1/2}}{4}(2)\Big|_0^1$$

let $v = r^2$

$dv = 2r\,dr$

$$= \frac{\pi}{2}$$

Pg 52 Schey

(II-4)

(c) $G(x,y,z) = (1-x^2-y^2)^{3/2}$

with $z = (1-x^2-y^2)^{1/2}$

Now
$$\iint_S G(x,y,z)\, dS = \iint_{R_{xy}} G(x,y,z(x,y)) \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

Now
$$\hat{n} = \pm \frac{\left(\frac{\partial z}{\partial x}\hat{i} + \frac{\partial z}{\partial y}\hat{j} - \hat{k}\right)}{\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}}$$

For this problem
$$\frac{\partial z}{\partial x} = \frac{1}{2}(1-x^2-y^2)^{-1/2}(-2x)$$
$$\frac{\partial z}{\partial y} = \frac{-y}{(1-x^2-y^2)^{1/2}}$$

So
$$1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2 = 1+\frac{x^2}{1-x^2-y^2}+\frac{y^2}{1-x^2-y^2} = \cancel{x^2}\,\frac{1-x^2-y^2+x^2+y^2}{1-x^2-y^2}$$
$$= \frac{1}{1-x^2-y^2}$$

Thus
$$\hat{n} = \pm \frac{\left(\frac{-x}{(1-x^2-y^2)^{1/2}}\hat{i} - \frac{y}{(1-x^2-y^2)^{1/2}}\hat{j} - \hat{k}\right)}{\frac{1}{\sqrt{1-x^2-y^2}}}$$

For a normal vector pointing upward we would take the negative sign.

Then $|\hat{n}\cdot\hat{k}| = \sqrt{1-x^2-y^2}$

so that

$$\iint_S G(x,y,z)\, dS = \iint_{R_{xy}} \frac{(1-x^2-y^2)^{3/2}\, dx\,dy}{(1-x^2-y^2)^{1/2}} = \iint_{R_{xy}} (1-x^2-y^2)\, dx\,dy$$

converting to polar $dx\,dy = r\,dr\,d\theta$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{1} (1-r^2)\, r\,dr\,d\theta = 2\pi\left(\frac{r^2}{2} - \frac{r^4}{4}\right)\Big|_0^1 = 2\pi\left(\frac{1}{2}-\frac{1}{4}\right) = \cancel{2\pi\left(\frac{3}{6}-\frac{2}{6}\right)}$$

$$= \cancel{\frac{2\pi}{6} = \frac{\pi}{3}} = 2\pi\left(\frac{1}{4}\right) = \frac{\pi}{2}.$$

pg 53 Schey

II-5 Evaluate $\iint_S F \circ \hat{n}\, dS$

(a) $F(x,y,z) = x\hat{\imath} - z\hat{k}$

For the plane $z = 1 - \frac{x}{2} - \frac{y}{2}$

The normal is $$\hat{n} = \pm \frac{\left(\frac{\partial z}{\partial x}\hat{\imath} + \frac{\partial z}{\partial y}\hat{\jmath} - \hat{k} \right)}{\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}}$$

w/ An upward facing normal take the ~~positive~~ minus sign

Now $\frac{\partial z}{\partial x} = -\frac{1}{2}$ $\quad \frac{\partial z}{\partial y} = -\frac{1}{2}$

so the denominator is $\sqrt{1 + \frac{1}{4} + \frac{1}{4}} = \sqrt{\frac{6}{4}} = \frac{\sqrt{6}}{2}$

$$\hat{n} = \frac{\frac{1}{2}\hat{\imath} + \frac{1}{2}\hat{\jmath} + \hat{k}}{\frac{\sqrt{6}}{2}} = \frac{1}{\sqrt{6}}(\hat{\imath} + \hat{\jmath} + 2\hat{k})$$

Then $\vec{F} \circ \hat{n} = \frac{1}{\sqrt{6}}x - \frac{2}{\sqrt{6}}z = \frac{1}{\sqrt{6}}(x - 2z) = \frac{1}{\sqrt{6}}(x - 2 + x + y)$

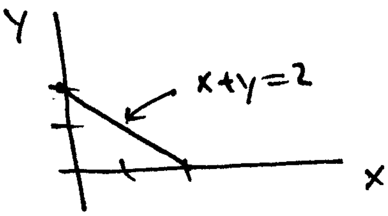
$$= \frac{1}{\sqrt{6}}(2x + y - 2)$$

$$dS = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = \frac{dx\,dy}{\left(\frac{2}{\sqrt{6}}\right)} = \frac{\sqrt{6}}{2}\,dx\,dy$$

So $\quad \text{Flux} = \displaystyle\iint_{\Omega_{xy}} \frac{1}{\sqrt{6}}(2x+y-2)\frac{\sqrt{6}}{2}\,dx\,dy = \frac{1}{2}\iint_{\Omega_{xy}}(2x+y-2)\,dx\,dy$

To evaluate the region of the xy plane in the 1st quadrant we are bounded by the x + y axis + the line z = 0 or

$$x+y=2$$



So $\quad \text{Flux} = \displaystyle\frac{1}{2}\int_{x=0}^{2}\int_{y=0}^{2-x}(2x+y-2)\,dy\,dx = \frac{1}{2}\int_{x=0}^{2}\left(2xy+\frac{y^2}{2}-2y\right)\Big|_0^{2-x}dx$

$$= \frac{1}{2}\int_{x=0}^{2}\left(2x(2-x)+\frac{(2-x)^2}{2}-2(2-x)\right)dx$$

$$= \frac{1}{2}\int_{x=0}^{2}\left(4x-2x^2+\frac{1}{2}(4-4x+x^2)-4+2x\right)dx$$

$$= \frac{1}{2}\int_{x=0}^{2}\left(4x-2x^2+\cancel{2}-\cancel{2x}+\frac{x^2}{2}-\cancel{4}+\cancel{2x}\right)dx$$

$$= \frac{1}{2}\int_{x=0}^{2}\left(-2+4x-\frac{3}{2}x^2\right)dx$$

$$= \frac{1}{2}\left[-2x + 2x^2 - \frac{1}{2}x^3 \right|_0^2$$

$$= \frac{1}{2}\left[-4 + 8 - 4\right] = 0 .$$

(b) $F(x,y,z) = \hat{\imath}x + y\hat{\jmath} + z\hat{k}$

w/ S the hemisphere $z = \sqrt{a^2 - x^2 - y^2}$

$$\iint_S F \circ \hat{n}\, dS = \iint_{\Omega_{xy}} F \circ \hat{n} \cdot \frac{dx\,dy}{|\hat{n} \circ \hat{k}|}$$

w/ $$\hat{n} = \frac{-\frac{\partial z}{\partial x}\hat{\imath} - \frac{\partial z}{\partial y}\hat{\jmath} + \hat{k}}{\sqrt{1 + (\ )^2 + (\ )^2}}$$

$$\frac{\partial z}{\partial x} = \frac{-x}{\sqrt{a^2 - x^2 - y^2}} \qquad \frac{\partial z}{\partial y} = \frac{-y}{\sqrt{a^2 - x^2 - y^2}}$$

so $$\sqrt{1 + (\ )^2 + (\ )^2} = \sqrt{1 + \frac{x^2}{a^2 - x^2 - y^2} + \frac{y^2}{a^2 - x^2 - y^2}}$$

$$= \sqrt{\frac{a^2 - \cancel{x^2} - \cancel{y^2} + \cancel{x^2} + \cancel{y^2}}{a^2 - x^2 - y^2}} = \sqrt{\frac{a^2}{a^2 - x^2 - y^2}}$$

$$= \frac{a}{\sqrt{a^2 - x^2 - y^2}} = \frac{a}{z}$$

Then

$$\hat{n} = \frac{-\left(\frac{-x}{z}\right)\hat{\imath} - \left(\frac{-y}{z}\right)\hat{\jmath} + \hat{k}}{(a/z)} = \frac{x\hat{\imath} + y\hat{\jmath} + z\hat{k}}{a}$$

Then $F(x,y,z)\circ\hat{n} = \dfrac{x^2+y^2+z^2}{a}$

So $$\iint_S F\circ\hat{n}\,dS = \iint_{R_{xy}}\left(\frac{x^2+y^2+z^2}{a}\right)\frac{dx\,dy}{|\hat{n}\circ k|} = \iint_{R_{xy}}\left(\frac{x^2+y^2+z^2}{a}\right)\frac{dx\,dy}{\left(\frac{1}{(a/z)}\right)}$$

$\frac{z}{a}$

$$= \iint_{R_{xy}}\left(\frac{x^2+y^2+z^2}{z}\right)dx\,dy$$

evaluating on $z=\sqrt{a^2-x^2-y^2}$ the above becomes

$$= \iint_{R_{xy}}\frac{a^2}{\sqrt{a^2-x^2-y^2}}\,dx\,dy$$ converting to polar gives

$$= a^2\int_{\theta=0}^{2\pi}\int_{r=0}^{a}\frac{r\,dr\,d\theta}{\sqrt{a^2-r^2}} = 2\pi a^2\,\frac{(a^2-r^2)^{1/2}(2)}{(-2)}\Bigg|_0^a$$

$$= -2\pi a^2\left(0-a\right) = 2\pi a^3$$

PS 52 Skey

II-5

(c) $\vec{F}(x,y,z) = \hat{j}y + \hat{k}$

w/ S the portion of the parabaloid $z = 1 - x^2 - y^2$

above the x-y plane

1st calculate the normal

$$\hat{n} = \frac{-\frac{\partial z}{\partial x}\hat{i} - \frac{\partial z}{\partial y}\hat{j} + \hat{k}}{\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}}.$$

$\frac{\partial z}{\partial x} = -2x$ + $\frac{\partial z}{\partial y} = -2y$ Now so the normal becomes

$$\hat{n} = \frac{2x\hat{i} + 2y\hat{j} + \hat{k}}{\sqrt{1 + 4x^2 + 4y^2}}.$$

$\left\{ w/ \ 4(x^2+y^2) = 4(1-z) \right\}$

Thus $F \circ \hat{n} = \dfrac{2y^2+1}{\sqrt{1+4x^2+4y^2}}$

Thus

$$\iint_S F\circ n \, dS = \iint_{\Omega_{xy}} F\circ\hat{n} \frac{dx\,dy}{|\hat{n}\circ\hat{k}|} = \iint_{\Omega_{xy}} \frac{2y^2+1}{\sqrt{1+4x^2+4y^2}} \left(\sqrt{1+4x^2+4y^2}\right) dx\,dy$$

$$= \iint_{\Omega_{xy}} (2y^2+1)\,dx\,dy$$

Now the domain in the x-y plane is $z = 0 = 1-x^2-y^2$ so converting to polar one obtains:

$$= \int_{r=0}^{1}\int_{\theta=0}^{2\pi} (2r^2\sin^2\theta + 1)\, r\,d\theta\,dr$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{1} (2r^3\sin^2\theta + r)\,dr\,d\theta = \int_{\theta=0}^{2\pi} \frac{r^4}{2}\sin^2\theta + \frac{r^2}{2}\Big|_0^1\,d\theta$$

$$= \frac{1}{2}\int_{\theta=0}^{2\pi} (\sin^2\theta + 1)\,d\theta = \frac{1}{2}\int_0^{2\pi}\left[\left(\frac{1-\cos(2\theta)}{2}\right)+1\right]d\theta$$

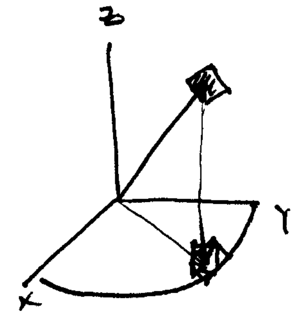
$$= \frac{1}{2}\int_0^{2\pi}\left(\frac{3}{2} - \frac{1}{2}\cos(2\theta)\right)d\theta = \frac{1}{4}\left[3\cdot 2\pi - \cancel{\frac{\sin(2\theta)}{2}}\Big|_0^{2\pi}\right]$$

$$= \frac{3\pi}{2}$$

pg 53 Schey

II-6

$$z = (R^2 - x^2 - y^2)^{1/2}$$



$$\sigma(x,y,z) = \left(\frac{\sigma_0}{R^2}\right)(x^2+y^2)$$

$$M = \iint_S \sigma(x,y,z)\,dS = \iint_{\Omega_{xy}} \frac{\sigma_0}{R^2}(x^2+y^2)\,\frac{dx\,dy}{|\hat{n}\circ\hat{k}|}$$

w/ $\hat{n} = \dfrac{\hat{i}x + \hat{j}y + z\hat{k}}{R}$ (from many other exercises) we get that

$|\hat{n}\circ\hat{k}| = \dfrac{z}{R}$ so

$$M = \frac{\sigma_0}{R^2}\iint_{\Omega_{xy}} \frac{(x^2+y^2)R}{z}\,dx\,dy = \frac{\sigma_0}{R}\iint_{\Omega_{xy}}\left(\frac{x^2+y^2}{z}\right)dx\,dy$$

$$= \frac{\sigma_0}{R}\iint_{\Omega_{xy}} \frac{(x^2+y^2)}{\sqrt{R^2-x^2-y^2}}\,dx\,dy$$

converting to polar to evaluate this integral we obtain.

$$= \frac{\sigma_0}{R}\iint_{\Omega_{xy}} \frac{r^2}{\sqrt{R^2-r^2}}\,r\,dr\,d\theta = \frac{\sigma_0}{R}\int_{\theta=0}^{2\pi}\int_{r=0}^{R} \frac{r^3}{\sqrt{R^2-r^2}}\,dr\,d\theta$$

$$= \frac{2\pi \sigma_0}{R} \int_{r=0}^{R} \frac{r^3}{\sqrt{R^2 - r^2}}\, dr$$

$$= \frac{2\pi\sigma_0}{R} \int_{r=0}^{R} \frac{r^2 \; r\,dr}{\sqrt{R^2 - r^2}}$$

let $v = r^2$

$dv = 2r\,dr$

$$= \frac{2\pi\sigma_0}{R} \int_0^{R^2} \frac{v\left(\frac{dv}{2}\right)}{\sqrt{R^2 - v}} = \frac{\pi\sigma_0}{R} \int_0^{R^2} \frac{v\,dv}{\sqrt{R^2 - v}}$$

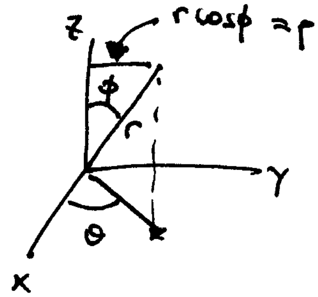
Thus

$$M = \frac{\pi\sigma_0}{R} \int_0^{R^2} \frac{v\,dv}{\sqrt{R^2 - v}} = \frac{\pi\sigma_0}{R} \left[v \left. \frac{(R^2 - v)^{1/2}}{(-1)(1/2)} \right|_0^{R^2} - \frac{1}{(-1)(1/2)} \int_0^{R^2} (R^2 - v)^{1/2}\, dv \right]$$

$$= \frac{\pi\sigma_0}{R}(2) \int_0^{R^2} (R^2 - v)^{1/2}\, dv = \frac{2\pi\sigma_0}{R} \left. \frac{(R^2 - v)^{3/2}}{(3/2)} (-1) \right|_0^{R^2}$$

$$= \frac{2\pi\sigma_0}{R}\left(\frac{2}{3}\right)\left[(R^2)^{3/2} \right] = \frac{4\pi}{3}\sigma_0 R^2$$

(II-7)



$$I_z = \int \rho^2 \, dM$$

$$= \int_{\Omega} \rho^2 \, \sigma(x,y,z) \, dS$$

$$= \int_{\Omega_{xy}} \rho^2 \frac{\sigma_0}{R^2}(x^2+y^2) \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

$$\hat{n} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{R} \quad \text{so} \quad \hat{n}\cdot\hat{k} = \frac{z}{R}$$

$$\therefore \; I_z = \int_{\Omega_{xy}} \rho^2 \frac{\sigma_0}{R^2} \rho^2 \frac{R}{z} \, dx\,dy = \int_0^{2\pi}\int_0^R \rho^4 \frac{\sigma_0}{R^2}\cdot\frac{R}{z} \, \rho \, d\rho \, d\theta$$

$$= 2\pi \frac{\sigma_0}{R} \int_0^R \frac{\rho^5}{z} \, d\rho = \frac{2\pi\sigma_0}{R} \int_0^R \frac{\rho^5 \, d\rho}{\sqrt{R^2-\rho^2}}$$

let $v = \rho^2$ so

$dv = 2\rho \, d\rho$

$$I_z = \frac{2\pi\sigma_0}{R} \int_0^{R^2} \frac{v^2 \, \tfrac{1}{2} \, dv}{\sqrt{R^2 - v}} = \frac{\pi\sigma_0}{R} \int_0^{R^2} \frac{v^2 \, dv}{\sqrt{R^2 - v}}$$

So $$\frac{R I_z}{\pi \sigma_0} = \frac{2v^2(R^2-v)^{1/2}}{(-1)}\Big|_0^{R^2} + 2^2\int_0^{R^2} v(R^2-v)^{1/2}\,dv$$

$$= 4\int_0^{R^2} v(R^2-v)^{1/2}\,dv$$

$$= \cancel{4\left[\frac{v(R^2-v)^{3/2}}{(2/3)}(-1)\Big|_0^{R^2} + \frac{3}{2}(R^2-v)\Big|_0^{R^2}\right]}$$

$$= \cancel{4\left[-\frac{3}{2}R^2\right.}$$

$$= 4\left[\frac{v(R^2-v)^{3/2}}{(2/3)}(-1)\Big|_0^{R^2} + \frac{3}{2}\int_0^{R^2}(R^2-v)^{3/2}\,dv\right]$$

$$= 6\,\frac{(R^2-v)^{5/2}}{(5/2)}(-1)\Big|_0^{R^2}$$

$$= \frac{12}{5}R^5$$

So $$I_z = \frac{12\pi}{5}\sigma_0 R^4$$