P. 8.15

$U = -\epsilon + \epsilon + \epsilon - \epsilon - \epsilon - \epsilon + \epsilon - \epsilon - \epsilon - \epsilon - \epsilon + \epsilon$

$+ \epsilon + \epsilon - \epsilon + \epsilon - \epsilon + \epsilon - \epsilon \mp \epsilon + \epsilon$

$+ \epsilon - \epsilon - \epsilon$

$= -14\epsilon + 10\epsilon = -4\epsilon$

P. 8.16 100 dipols

10^9 terms ~~of~~ per second

$Z = \sum_{\{s_i\}} e^{-\beta U}$ # states = # of terms in the sum

$2^N = 2^{100}$ terms

$\frac{2^{100}}{10^9}$ second $= 1.26 \cdot 10^{21}$ seconds $= 4.1 \cdot 10^{13}$ years

P. 8.17 $\uparrow\uparrow$; $\uparrow\downarrow$; $\downarrow\uparrow$; $\downarrow\downarrow$

$U = \quad -\epsilon \quad +\epsilon \quad +\epsilon \quad -\epsilon$

$Z = \sum_{\{s_i\}} e^{-\beta U} = 2e^{\beta\epsilon} + 2e^{-\beta\epsilon}$

$$P_{\text{state antiparallel}} = \frac{2e^{-(\beta\epsilon)}}{2e^{\beta\epsilon} + 2e^{-\beta\epsilon}} = \frac{e^{-\beta\epsilon}}{e^{\beta\epsilon} + e^{-\beta\epsilon}}$$

$$P_{\text{state parallel}} = \frac{e^{+\beta\epsilon}}{e^{\beta\epsilon} + e^{-\beta\epsilon}}$$

$$P_{\perp}(kT/\epsilon \equiv x) = \frac{e^{-(1/x)}}{e^{1/x} + e^{-1/x}} =$$

$$P_{\parallel}(kT/\epsilon \equiv x) = \frac{e^{(1/x)}}{e^{1/x} + e^{-1/x}} =$$

I will plot this vs. ϵ/kT in stead.

$$P_{\perp} = \frac{e^{-x}}{e^{x} + e^{-x}} \qquad + \qquad P_{\parallel} = \frac{e^{+x}}{e^{x} + e^{-x}}$$

$$= \frac{1}{e^{2x} + 1} \qquad\qquad P_{\parallel} = \frac{1}{1 + e^{-2x}}$$

$$\epsilon \approx O(1\,\text{ev}) \qquad\qquad x \equiv \frac{kT}{\epsilon}$$

$$k = O(8.617 \cdot 10^{-5}\ \text{ev}/k) \qquad\qquad x = T/(\epsilon/k)$$

$$\frac{\epsilon}{k} = O\left(\frac{1}{8.617 \cdot 10^{-5}\ 1/k}\right)$$

$$= O(1.1 \cdot 10^{4}) \qquad\qquad 100 < T < 1000$$

$$\Rightarrow \qquad\qquad \frac{1}{100} < x < \frac{1}{10}$$

$$\sum_{s_N} e^{\beta\epsilon s_{N-1} s_N} = \begin{cases} e^{\beta\epsilon(1)(1)} + e^{\beta\epsilon(1)(-1)} \\ e^{\epsilon\beta(-1)(1)} \end{cases} = 2\cosh(\beta\epsilon)$$

$$Z = \underbrace{\sum_{s_1}\sum_{s_2}\cdots\sum_{s_N}}_{N \text{ sums}} \underbrace{e^{\beta\epsilon s_1 s_2}\, e^{\beta\epsilon s_2 s_3}\cdots e^{\beta\epsilon s_{N-2} s_{N-1}}}_{N-1 \text{ terms}}\, \underbrace{e^{\beta\epsilon s_{N-1} s_N}}_{2\cosh(\beta\epsilon)}$$

$$=$$

$$= \sum_{s_1} \left(2\cosh(\beta\epsilon)\right)^{N-1} = 2^N \left(\cosh(\beta\epsilon)\right)^{N-1}$$

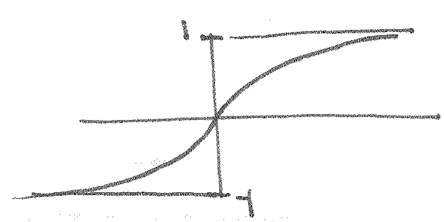
$$\approx 2^N \cosh(\beta\epsilon)^N$$

$$\bar{U} = -\frac{\partial}{\partial\beta}\ln Z = -\frac{\partial}{\partial\beta}\left(N \ln\cosh(\beta\epsilon)\right)$$

$$= \frac{-N}{\cosh(\beta\epsilon)}\sinh(\beta\epsilon)\cdot\epsilon = -N\epsilon\tanh(\beta\epsilon) \quad \text{eq } 8.44$$

$T \to \infty \Rightarrow \beta \to 0$

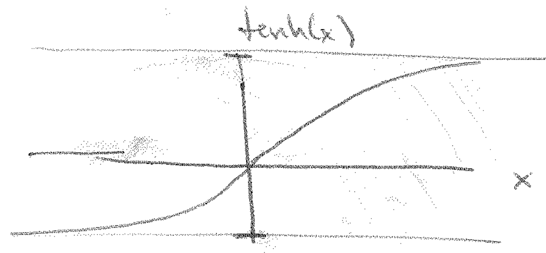
$T \to 0 \Rightarrow \beta \to \infty$

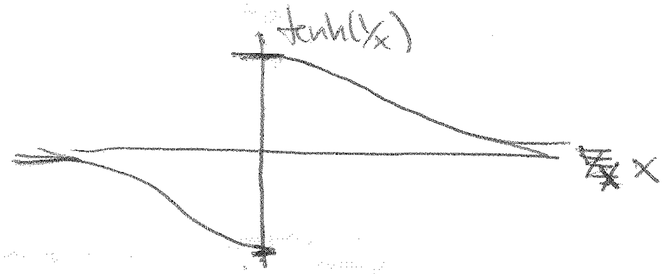


(3.19)

$\bar{U} = -\frac{\partial}{\partial \beta} \ln Z$ See notes pg 342

$U(T) = -N\epsilon \tanh(\beta\epsilon)$





P. 8.19

$kT_c = n\epsilon$

$\epsilon = \frac{kT_c}{n}$

$$\epsilon = \frac{(8.617\cdot 10^{-5}\,eV/k)(1043k)}{n} = \ldots$$

Think iron is BCC. $\Rightarrow$ $n = 8$

P.8.20

$\bar{s} = \tanh(\beta\epsilon n\bar{s})$ implicitly this is a fn of

$s = s(\beta\epsilon) = s(\frac{\epsilon}{kT})$

From $kT_c = n\epsilon$

$\frac{kT_c}{\epsilon} \sim n$ $\Rightarrow$ plot Above for $\frac{kT}{\epsilon} \sim 1-20.$

P.8.21

$$\bar{s} = \tanh(\beta\epsilon n\bar{s}) = \frac{\sinh(\beta\epsilon n\bar{s})}{\cosh(\beta\epsilon n\bar{s})}$$

$T = 0 \Rightarrow \beta = +\infty$

$$= \frac{e^{\beta\epsilon n\bar{s}} - e^{-\beta\epsilon n\bar{s}}}{e^{\beta\epsilon n\bar{s}} + e^{-\beta\epsilon n\bar{s}}} \cdot \frac{e^{-\beta\epsilon n\bar{s}}}{e^{-\beta\epsilon n\bar{s}}}$$

$=$

$\bar{s} = \cancel{f(x_0) + f'(x_0)}($

~~Expand~~ about $T = 0$ $\quad$ $\mathrm{sech}^2(\beta \epsilon n \bar{s}) \cdot \frac{(\epsilon n \bar{s})(-1)}{kT^2}$ $\quad$ A, M, J, J

$\bar{s} = 1 +$

let $\bar{s} = \tanh(\frac{1}{x})$ $\quad x \to \infty$

$\bar{s} = \tanh(x)$ $\quad x \ll 1$

Dammit how do you take the limit as $x \to \infty$ + do a taylor series type expansion?

$\bar{s} = \tanh(\beta \epsilon n \bar{s})$

$\rightarrow \beta \epsilon n \bar{s} = \tanh^{-1}(\bar{s})$

$\frac{\epsilon n \bar{s}}{kT} = \tanh^{-1}(\bar{s})$

$\epsilon n \bar{s} = kT \tanh^{-1}(\bar{s})$

$\frac{\epsilon n \bar{s}}{\tanh^{-1}(\bar{s})} = kT$

P. 8.22 $\pm \mu_B B$

$$E_\uparrow = -\epsilon \sum_{\text{neighbors}} s_{\text{neighbor}} - \mu_B B$$

$$= -\epsilon n \bar{s} - \mu_B B$$

$$E_\downarrow = +\epsilon n \bar{s} + \mu_B B$$

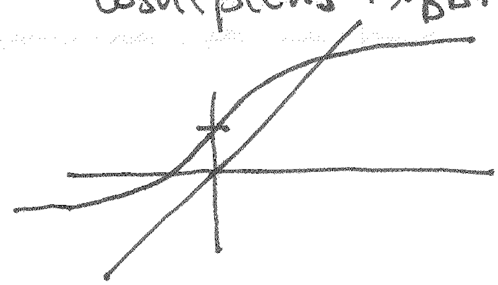
$$Z_i = e^{-\beta E_\downarrow} + e^{-\beta E_\uparrow} = e^{-\beta(\epsilon n \bar{s} + \mu_B B)} + e^{-\beta(-\epsilon n \bar{s} - \mu_B B)}$$

$$= e^{-\beta(\epsilon n \bar{s} + \mu_B B)} + e^{+\beta(\epsilon n \bar{s} + \mu_B B)}$$

$$= 2\cosh(\beta(\epsilon n \bar{s} + \mu_B B))$$

Avg $$\bar{s}_i = \frac{1}{Z_i}\left[+1 e^{+\beta(\epsilon n \bar{s} + \mu_B B)} - 1 e^{-\beta(\epsilon n \bar{s} + \mu_B B)}\right]$$

$$\bar{s} = \frac{\sinh(\beta(\epsilon n \bar{s} + \mu_B B))}{\cosh(\beta(\epsilon n \bar{s} + \mu_B B))} = \tanh(\beta \epsilon n \bar{s} + \beta \mu_B B)$$



2 variables

$\beta \epsilon n \quad + \quad \beta \mu_B B$

P. 8.23 (a) ?

(b) ?

(c) ?

P. 8.24 (a) $\tanh x \approx x - \frac{1}{3}x^3$

(b) $\bar{s} \cong \beta \epsilon n \bar{s} - \frac{1}{3}(\beta \epsilon n \bar{s})^3$

?

(c) $\chi = \frac{\partial M}{\partial B}\Big|_T$

Prob A.1

(a) $hc = (6.63\cdot 10^{-34}\,J\cdot s)(3\cdot 10^{8}\,m/s)$

$= 1.98\cdot 10^{-25}\ J\cdot m$

$1m = 10^9\ nm.$

$1J = \frac{1}{1.602\cdot 10^{-19}}\ eV$

$\therefore\ hc = \frac{1.98\cdot 10^{-25}\cdot 10^{9}}{1.602\cdot 10^{-19}}\ eV\,nm$

$= 1241.\ eV\ nm$

(b) $E_{photon} = hf = \frac{hc}{\lambda} \cong \frac{1240\ eV\cdot nm}{\lambda}$

$\therefore\ E(\text{red light}) = \frac{1240\ eV\cdot nm}{680\ nm} = 1.9\ eV$

$E(\text{blue light}) = \frac{1240\ eV\cdot nm}{450\ nm} = 2.7\ eV$

$E(\text{x-ray}) = \frac{1240\ eV\cdot nm}{.1\ nm} = 1.24\cdot 10^{4}\ eV$

$E(\text{cosmic radiation}) = \frac{1240\ eV\cdot nm}{10^6\ nm} = 1.24\cdot 10^{-3}\ eV$

$1\,mm = 10^{-3}\,m$

$= 10^{-3}\cdot 10^{9}\ nm$

$= 10^{6}\ nm$

(c) $1 \cdot 10^{-3}$ W at red laser

$E(\text{red}) \cong 2$ eV. per photon

$$1 W = 1 J/s = \frac{1}{1.602 \cdot 10^{-19}} eV/s = 6.24 \cdot 10^{18} eV/s$$

$\therefore$ ~~#photo~~ $\#\text{photons} = (3.12 \cdot 10^{18} \text{ photons}/s)(10^{-3})$

since it is a 1 milliwatt laser

$= 3.12 \cdot 10^{+16}$ photons/sec.

Prob A.2 $K_{max} = Ve = (.8)e = .8$ eV.

(a) $K_{max} = \frac{hc}{\lambda} - \phi$

$$.8 eV = \frac{1240 \text{ eV} \cdot \text{nm}}{400 \text{ nm}} - \phi$$

$\phi = 2.3$ eV.

(b) ϕ does not change

$$K_{max} = \frac{1240 \text{ eV} \cdot \text{nm}}{300 \text{ nm}} - 2.3 \text{ eV} = 1.8 \text{ eV}$$

$\rightarrow V = 1.8$ eV $\quad V(300\text{ nm}) = 1.8$ V

$V(500 \text{ nm}) = .18$ V

$V(600 \text{ nm}) = -.23$ V.

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Prob A.3 $E = hf$ $E = pc$

$f \cdot \lambda = c$

$hf = pc$

$h\frac{f}{c} = p$ $\frac{h}{\lambda} = p$ $\rightarrow$ $\lambda = \frac{h}{p}$

Prob A.4

$$E = \frac{mc^2}{\sqrt{1+(\frac{v}{c})^2}} \qquad p = \frac{mv}{\sqrt{1+(\frac{v}{c})^2}}$$

If $v = c$.

$E =$

?

Prob A.5

$$E = pc$$

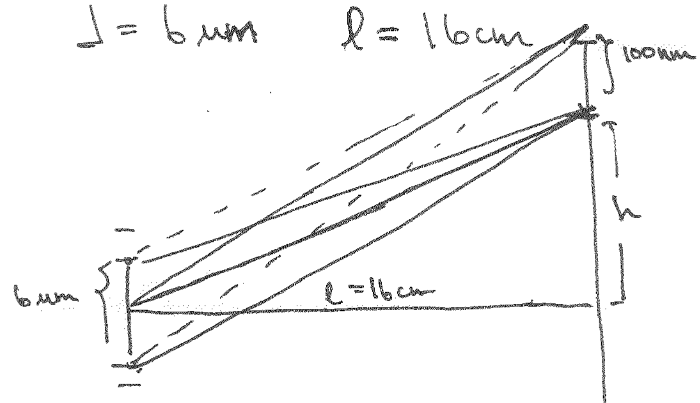
$$10^5 eV = pc \Rightarrow p = \frac{10^5 eV}{3 \cdot 10^8 \text{ m/s}}$$

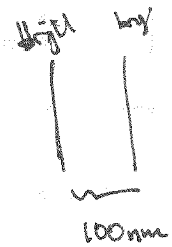
$1eV = 1.602 \cdot 10^{-19}$ J

$p = 5.34 \cdot 10^{-23}$ kg m/s $\qquad$ $1J = \text{kg} \cdot \text{m}^2/\text{s}^2$

$$\lambda = \frac{h}{p} = \frac{6.626 \cdot 10^{-34} \text{ J} \cdot \text{s}}{5.34 \cdot 10^{-23} \text{ kg m/s}} = 1.24 \cdot 10^{-11} \frac{\text{kg} \cdot \text{m}^2/\text{s}^2 \cdot \text{s}}{\text{kg m/s}} = \text{m}.$$

Prob A.6 $\quad d = 6 \text{ um} \quad l = 16 \text{cm}$





path length for 1st maximum $\cong \sqrt{h^2 + l^2}$

path " " 2nd max. $\cong \sqrt{(h+d)^2 + l^2}$

$$\lambda \cdot \lambda = \sqrt{(h+\Delta)^2 + \ell^2} - \sqrt{h^2+\ell^2}$$

$$=$$

How Do?

But given λ one has $E = eV = pc = \frac{h}{\lambda}c$ one can calculate the voltage

Prob A.7

(a) $\lambda = \frac{h}{p}$ $\quad \frac{1}{2}\frac{p^2}{m} = K.E.$

$$p = \sqrt{2mK} =$$

$m_{neutron} = ? \approx m_{proton} = 1.673 \cdot 10^{-27}$ kg

$K = 1eV = 1.602 \cdot 10^{-19}$ J.

$\therefore$ $p^2 =$ $2.3 \cdot 10^{-23}$ kgm/s

(b) $m \cong 1lb = \frac{1}{2.2}$ kg

$1kg = 2.2 lbs$

$1lb = \frac{1}{2.2}$ kg

$V \cong 90$ mi/hr $= 40$ m/s

$p = 18.1$ kgm/s

$\lambda = \frac{h}{p} = 3.6 \cdot 10^{-35}$ m

Prob A.8

$$\psi(x) = A(\cos(kx) + i\sin(kx))$$

(a) $k = \frac{2\pi}{\lambda}$ w/ λ the deBroglie wave length

$$p = \frac{h}{\lambda} \Rightarrow \lambda = \frac{h}{p}$$

$$k = \frac{2\pi}{h} \cdot p = \frac{2\pi p}{h}$$

How to justify?

(b) $|\psi(x)|^2 = |A|^2(\cos(kx) + i\sin(kx))(\cos(kx) - i\sin(kx))$

$$= |A|^2(\cos^2(kx) + \sin^2(kx)) = |A|^2$$

$$P_{ab} = \int_a^b |\psi|^2 dx = |A|^2(b-a)$$

$P_{-\infty,+\infty} = \int_{-\infty}^{+\infty} |A|^2 dx = 1$ if $A \neq$ infintesimal ~~+e~~

(d) $\frac{d\psi}{dx} = A(-\sin(kx) + i\cos(kx))k$

$$= i(\psi)k$$

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A9

(a) $\psi(x)\psi^{*} = A^2 e^{-2ax^2} = |\psi|^2$

(b) $$\int_{-\infty}^{+\infty} |\psi|^2 dx = 1$$

(c) $$\overline{x^2} = \int_{-\infty}^{+\infty} x^2 |\psi|^2 dx = \ldots$$

$$\Delta x = \sqrt{\overline{x^2} - \overline{x}^2}$$

(d) $$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ikx} A e^{ik_0 x - ax^2} dx$$

(e) $$\Delta k = \sqrt{\overline{k^2} - \overline{k}^2} = \int_{-\infty}^{+\infty} k^2 |\hat{f}(k)|^2 dk$$

(f) $\Delta p_x =$

$k = \frac{2\pi}{\lambda}$ $\qquad p = \frac{h}{\lambda} \Rightarrow p = \frac{h \cdot k}{2\pi}$

$\lambda = \frac{2\pi}{k}$ $\qquad \Delta p = \frac{h \Delta k}{2\pi} = \hbar \Delta k$

$$\Delta x \cdot \Delta p = \frac{1}{2\sqrt{a}} \cdot \hbar \cdot \sqrt{a} = \frac{\hbar}{2}$$

PJ 367 Schrod

A.11

$\sin(x) - \sin(2x)$

$\sin(x) + \sin(2x)$

$\sin(x) \cancel{\times} (2x)$

A.12 $l = 10^{-15}\ m$

$$E = \frac{h^2}{8mL^2}\left(n_x^2 + n_y^2 + n_z^2\right)$$

$$E_{min} = \frac{h^2}{8m(L)^2}(1+1+1)$$

Prob A,12

$$E = \frac{h^2}{8mL^2}(n_x^2 + n_y^2 + n_z^2)$$

Prob A,13 $E = \frac{p^2}{2m}$ but $E = pc$

(a) p_n is still quantized by quantizing the wavelength ...

$\therefore$ $E_n = p_n c = c\left(\frac{h}{\lambda_n}\right)$

$\lambda_n = \frac{2L}{n}$

$\therefore$ $E_n = ch\left(\frac{n}{2L}\right) = \frac{ch}{2L} n$

(b) $E_1 = \frac{(3 \cdot 10^8 \text{ m/s})(6.626 \cdot 10^{-34} \text{ J} \cdot \text{s})}{2(10^{-15} \text{ m})} = 9.939 \cdot 10^{-11} \text{ J}$

$= 6.02 \cdot 10^8 \text{ eV}.$

They would have to have a huge amount of energy ... prob. to much.

(c) $E_1 = 6.02 \cdot 10^8 \text{ eV}$ $\qquad 1\text{J} = F \cdot L = M\frac{L^2}{s^2}$

$\frac{E_1}{c^2} = 1.104 \cdot 10^{-27} \text{ J}/(\text{m/s})^2$

$=$ " (kg)

A.14 $$E = \frac{h^2}{8mL^2}\left(n_x^2 + n_y^2 + n_z^2\right)$$

$$n_x^2 + n_y^2 + n_z^2 \leq 15$$

$\Rightarrow$ ~~~~ $\vec{n} = (1,0,0), (0,1,0), (0,0,1)$

$(1,1,0), (1,0,1), ($

Avg # of states per unit energy increases.

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Prob. A.15 $f = 6.4 \cdot 10^{13}\ s^{-1}$

(a) $E = \frac{1}{2}hf,\ \frac{3}{2}hf,\ \frac{5}{2}hf,\ \frac{7}{2}hf,\ \frac{9}{2}hf$

$= \cancel{\frac{1}{2}(6.626 \cdot 10^{-34}\ J \cdot s)(6.4 \cdot 10^{13}\ s^{-1})},$

Now $hf = (4.136 \cdot 10^{-15}\ eV \cdot s)(6.4 \cdot 10^{13}\ s^{-1})$

$= \cancel{2.647}\ .2647\ eV$

$.1323\ eV,\ .397\ eV,\ .6617\ eV,\ \cancel{9.26\ eV}\ .926\ eV,$

$1.19\ eV$

(b) $E = hv_L = \frac{hc}{\lambda}$

$\frac{hc}{\lambda} = \frac{1}{2}hf \Rightarrow v_L = \frac{f}{2} = 3.2 \cdot 10^{13}\ s^{-1}$

$\lambda = \frac{c}{v_L} = 9.3 \cdot 10^{-6}\ m = 9.3\ \mu m$

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Prob A16

(a) ? Don't know what is being asked?

(b) ?

(c) ?

(d) ?

(e) ?

Prob A17

$$E = \frac{(2n_1+1)}{2}hf + \frac{(2n_2+1)}{2}hf$$

$$= (n_1+n_2)hf + hf$$

$$= (n_1+n_2+1)hf$$

if E is fixed how many ways can we distribute the energy

$$\underbrace{|\,|\ \ |\ \ |\,|\,|\,|\,|\ \ |}\ \ \ |$$

n_1+n_2 sticks + we are asked to select

1 location from the n_1+n_2+1 spots

$\Rightarrow \binom{n_1+n_2+1}{1}$ is the degeneracy

$= n_1+n_2+1$

Prob A7.18

$$E = \frac{(2n_1+1)\hbar\omega}{2} + \frac{(2n_2+1)\hbar\omega}{2} + \frac{(2n_3+1)\hbar\omega}{2}$$

$$= (n_1+n_2+n_3)h\nu + \frac{3}{2}\hbar\omega$$

$\binom{n_1+n_2+n_3+1}{2}$ = degeneracy for a given energy.

Mj 374 Schrade

A,19

B $\Delta E = E_2 - E_1$ $\qquad E_n = -\frac{13.6 \text{ eV}}{n^2}$

$= -\frac{13.6}{4} + 13.6 =$ $\qquad n=2$

$E_2 > E_1$ $\qquad n=1$

$\Delta E = h\nu = \frac{hc}{\lambda}$

A.20

$\pm rp$

$\psi = \psi(s) = A \sin(\lambda s)$? Don't follow

A.21

~~[scribble]~~ $n = 1, 2, 3, \ldots$

$\ell = 0, 1, 2, 3, \ldots, n-1$

$m = -\ell, -\ell+1, -\ell+2, \ldots, \ell-1, \ell$

$\underline{n=1}$:

$\ell = 0$ only
$m = 0$ only $\Big\}$ # independent states $= 1^2 = 1$ ✓

$\underline{n=2}$:

$\ell = 0, 1$

$m = 0$, $m = -1, 0, +1$

Then independent states are

(n, ℓ, m):

$(2,0,0)$, $(2,1,-1)$, $(2,1,0)$, $(2,1,+1)$

independent states $= 4 = 2^2$ ✓

$\underline{n=3}$:

$\ell = 0, 1, 2$

$m = 0$, $m = -1, 0, +1$, $m = -2, -1, 0, +1, +2$

Then independent states are

(n, ℓ, m)

$(3,0,0)$;

$(3,1,-1)$; $(3,1,0)$; $(3,1,+1)$

$(3,2,-2);\ (3,2,-1);\ (3,2,0);\ (3,2,+1);\ (3,2,+2)$

independent states

$= 5 + 3 + 1 = 9 = 3^2.$ ✓

Prob A.22

$$\epsilon = \frac{\hbar^2}{2I}$$

(a) $\frac{\hbar^2}{2I} = .00024 \text{ eV}$

$$E_{rot} = j(j+1)\cdot\epsilon$$

$$\Delta E = E(j=1) - E(j=0) = \epsilon = h\nu = \frac{hc}{\lambda}$$

$$= \frac{(4.136\cdot 10^{-15} \text{ eV}\cdot s)(3\cdot 10^{8} \text{ m/s})}{\lambda} = 2.4\cdot 10^{-5} \text{ eV}$$

(b) $I = \frac{\hbar^2}{2\epsilon} = \frac{(4.136\cdot 10^{-15} \text{ eV}\cdot s)^2}{4\pi^2} \frac{1}{2(2.4\cdot 10^{-5} \text{ eV})}$

$$= \left(\frac{1}{8\pi^2}\right)(\quad)^2 \text{ eV}\cdot s^2$$

$1 \text{eV} = 1.602\cdot 10^{-19} \text{ J}$

$1 \text{eV}\cdot s^2 = \text{J}\cdot s^2$

$[I] = \text{kg m}^2$

$[E] = [F\cdot d] = \{\text{kg}\frac{m}{s^2}\cdot m$

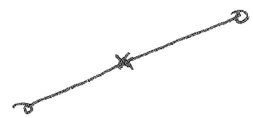
$= \text{kg}\frac{m^2}{s^2}$

$= \frac{[I]}{s^2}$

$\Rightarrow [I] = [E]\cdot s^2$ ✓

(c) $m_O \cong \dfrac{16\,g}{6.022\cdot 10^{23}} \lesssim \dfrac{8}{3}\cdot 10^{-23} < 2.6\cdot 10^{-23}\,g$

$m_C \cong \dfrac{12\,g}{6.022\cdot 10^{23}} \approx 2\cdot 10^{-23}\,g$



$$I = m_O r_{1/2}^2 + m_C r_{1/2}^2$$

$$= r_{1/2}^2 (m_O + m_C)$$

$$\Rightarrow \cancel{B_{1/2}}\; r = 2 r_{1/2} = 2\sqrt{\frac{I}{(m_O + m_C)}} \approx 2\sqrt{\frac{I}{5\cdot 10^{-23}\,g}}$$

A.23 $|J| = \sqrt{s(s+1)}\,\hbar$

$s=1/2$ $|J| = \sqrt{\frac{1}{2}\cdot\frac{3}{2}}\,\hbar$

$J_z \in \{s\hbar, (s-1)\hbar, \ldots, (-s+1)\hbar, -s\hbar\}$

$J_z = \frac{1}{2}\hbar, -\frac{1}{2}\hbar.$

$s=3/2$

$|J| = \sqrt{\frac{3}{2}\cdot\frac{5}{2}}\,\hbar$

$J_z = \frac{3}{2}\hbar, \frac{1}{2}\hbar, -\frac{1}{2}\hbar, -\frac{3}{2}\hbar.$

A.24

$E_0 = \frac{1}{2}h\nu$ $\lambda > \frac{\ell}{2}$ $\therefore \exists$ a smallest λ

$\lambda_{min} \geq \frac{\ell}{2}$ w/ ℓ = lattice spacing of 1D lattice

(a) $E_{\vec{n}} = G(n_x^2 + n_y^2 + n_z^2)$

w/ $n_i = p\frac{2\pi}{L}$

$E_{(\ell,m,n)} = G\frac{4\pi^2}{L^2}(\ell^2 + m^2 + n^2)$?

(b) How get $\sqrt{\frac{G\hbar}{c^3}}$

$[G] = \frac{N \cdot m^2}{kg^2}$ $[\hbar] = J \cdot s = kg\frac{m^2}{s^2}s = kg\frac{m^2}{s}$

$[c] = m/s$

$\therefore \left[\frac{G\hbar}{c^3}\right] = \frac{\left(\frac{N \cdot m^2}{kg^2}\right)\left(\frac{kg\, m^2}{s}\right)}{\frac{m^3}{s^3}} = \frac{N \cdot m^4\, s^2}{kg\, m^3} = \frac{kg\frac{m^2}{s^2}s^2}{kg} = m^2$ ✓

$\sqrt{\frac{(6.673 \cdot 10^{-11} N \cdot m^2/kg^2)(6.626 \cdot 10^{-34} J \cdot s)}{(3 \cdot 10^8)^3 \quad (2\pi)}}$

$$= \left(\frac{6.6^2}{3^3} \, \frac{10^{-11-34-24}}{2\pi} \right)^{1/2}$$

$$\begin{array}{r} 45 \\ 24 \\ \hline 69 \end{array}$$

$$= \left(\frac{6.6^2}{3^3 \cdot 2\pi} \right)^{1/2} \left(10^{-69} \right)^{1/2}$$

$$= \left(\frac{6.6^2}{3^3 \, 2\pi} \right)^{1/2} 10^{-34.5}$$

(c) $$\int_{\ell_p}$$

$\bar{E}$

$$\frac{J/m^3}{m/s} = \frac{\frac{N \cdot m}{m^3}}{m/s} = \frac{\frac{N}{m^2}}{m/s} = \frac{N \cdot s}{m}$$

$$= \frac{kg \, \frac{m}{s}}{m} = \frac{kg}{s}$$

(B,1) $$F(x) = \int_0^x e^{-\xi^2} d\xi = \frac{\sqrt{\pi}}{2} \operatorname{erf}(x) \quad \text{plot.}$$

$$\operatorname{erf}(x) \equiv \frac{2}{\sqrt{4}} \int_0^{\infty} e^{-\xi^2} d\xi$$

(B,2) $$\int_0^{\infty} x^2 e^{-ax^2} dx = \frac{1}{4}\sqrt{\frac{\pi}{a^3}}$$

$\frac{d}{da} \rightarrow$

$$-\int_0^{\infty} x^4 e^{-ax^2} dx = \frac{1}{4}\sqrt{\pi}\, \frac{d(a^{-3/2})}{da} = \frac{\sqrt{\pi}}{4}\left(-\frac{3}{2}\right) a^{-5/2}$$

$$\Rightarrow \int_0^{\infty} x^4 e^{-ax^2} dx = \frac{3\sqrt{\pi}}{8\, a^{5/2}}$$

(B3) (a) $= 0$

(b) $$F(x) = \int_0^x r e^{-ar^2} dr =$$

$$U = ar^2$$

$$dU = 2ar\,dr \Rightarrow r\,dr = \frac{dU}{2a}$$

$$F(x) = \int_0^{ax^2} \frac{du}{2a} e^{-u} = -\frac{1}{2a}\left(e^{-u}\right|_0^{ax^2}$$

$$= -\frac{1}{2a}\left(e^{-ax^2} - 1\right)$$

$$\therefore \quad -\frac{e^{-ax^2}}{2a} + \frac{1}{2a} = F(x)$$

(c) $\displaystyle\int_0^\infty x e^{-ax^2} dx = F(\infty) = +\frac{1}{2a}$? Can't be correct.

(d) $\displaystyle\int_0^\infty x e^{-ax^2} dx = \frac{1}{2a}$

$$\frac{1}{da} =$$

$$+\int_0^\infty x(-x^2) e^{-ax^2} dx = \frac{-1}{2a^2}$$

$$\int_0^\infty x^3 e^{-ax^2} dx = \frac{1}{2a^2}$$

B.4

$$\int_x^\infty e^{-t^2} dt = ?$$

let $s = t^2$

$ds = 2t\,dt \rightarrow \cancel{\quad} \; dt = \frac{ds}{2t} = \frac{ds}{2\sqrt{s}}$

$$= \int_{x^2}^\infty e^{-s} \cdot \frac{ds}{2\sqrt{s}}$$

$$= \frac{1}{2} \int_{x^2}^\infty s^{-1/2} e^{-s}\, ds$$

expand $s^{-1/2}$ in taylor series about x^2

$$s^{-1/2} \cong x^{-1} + \frac{d}{ds}(s^{-1/2})\Big|_{s=x^2} (s-x^2) + \frac{1}{2!}\frac{d^2}{ds^2}(s^{-1/2})\Big|_{s=x^2} (s-x^2)^2 + \cdots$$

$\cancel{\cong} \; x^{-1}$

$$\frac{d}{ds}(s^{-1/2}) = -\frac{1}{2} s^{-3/2}\Big|_{x^2} = -\frac{1}{2} x^{-3}$$

$$\frac{d^2}{ds^2}(s^{-1/2}) = \frac{3}{4} s^{-5/2}\Big|_{x^2} = \frac{3}{4} x^{-5}$$

$$s^{-1/2} = \frac{1}{x} - \frac{1}{2x^3}(s-x^2) + \frac{3}{2\cdot 4x^5}(s-x^2)^2 + O((s-x^2)^3)$$

$\therefore$

$$\frac{1}{2}\int_{x^2}^{\infty} s^{-1/2}e^{-s}\,ds = \frac{1}{2}\int_{x^2}^{\infty}\frac{1}{x}e^{-s}\,ds - \frac{1}{2\cdot 4x^3}\int_{x^2}^{\infty}(s-x^2)e^{-s}\,ds$$

$$+ \frac{3}{8x^5}\int_{x^2}^{\infty}(s-x^2)^2e^{-s}\,ds + \cdots$$

$$= \frac{1}{2x}\left[-e^{-s}\Big|_{x^2}^{\infty}\right] - \frac{1}{2\cdot 4x^3}\left[\int_{x^2}^{\infty} se^{-s}\,ds - x^2\int_{x^2}^{\infty}e^{-s}\,ds\right]$$

$$+ \frac{3}{8x^5}\left[\int_{x^2}^{\infty}(s^2-2sx^2+x^4)e^{-s}\,ds\right]$$

$$= \frac{1}{2x}\left[0+e^{-x^2}\right] \cancel{-\frac{1}{2\cdot 4x^3}}\left[-se^{-s}\Big|_{x^2}^{\infty} + \int_{x^2}^{\infty}e^{-s}\,ds\right]$$

$$+ \frac{1}{4x}\left[-e^{-s}\Big|_{x^2}^{\infty}\right. + \frac{3}{8x^5}\left[-s^2e^{-}\right.$$

$$\therefore \int_x^\infty e^{-t^2}\,dt = \frac{1}{2}\int_{x^2}^\infty \frac{1}{x} e^{-s}\,ds - \frac{1}{4x^3}\int_{x^2}^\infty (s-x^2)e^{-s}\,ds$$

$$+\frac{3}{2\cdot 8x^5}\int_{x^2}^\infty (s-x^2)^2 e^{-s}\,ds + \cdots$$

$v = s - x^2$

$dv = ds$

$s = v + x^2$

$$= \frac{1}{2x}\left(-e^{-s}\Big|_{x^2}^\infty - \frac{1}{4x^3}\int_0^\infty v e^{-v-x^2}\,dv\right.$$

$$+\frac{3}{2\cdot 8x^5}\int_0^\infty v^2 e^{-v-x^2}\,dv$$

$$= \frac{1}{2x}e^{-x^2} - \frac{e^{-x^2}}{4x^3}\left[\cancel{-ve^{-v}\Big|_0^\infty}^{\,0} + \int_0^\infty e^{-v}\,dv\right]$$

$$+\frac{3}{2\cdot 8x^5}e^{-x^2}\left[-v^2e^{-v}\Big|_0^\infty + 2\int_0^\infty ve^{-v}\,dv\right]$$

$$= \frac{e^{-x^2}}{2x} - \frac{e^{-x^2}}{4x^3} + \frac{3}{2\cdot 8x^5}e^{-x^2}\,2\left[\cancel{-ve^{-v}\Big|_0^\infty}^{\,0} + \int_0^\infty e^{-v}\,dv\right]$$

$$= e^{-x^2}\left(\frac{1}{2x} - \frac{1}{4x^3} + \frac{3}{8x^5} + \cdots\right)$$

ϵ

$\cancel{\;\;}\,x^{-7}$

B, δ

$$I(x) = \int_x^\infty t^2 e^{-t^2} dt$$

let $s = t^2$

$$ds = 2t\,dt \rightarrow dt = \frac{ds}{2t} = \frac{ds}{2s^{1/2}}$$

$$= \int_{x^2}^\infty s e^{-s} \frac{ds}{2s^{1/2}} = \frac{1}{2}\int_{x^2}^\infty s^{1/2} e^{-s} ds$$

$$s^{1/2} = x.$$

$$\frac{ds^{1/2}}{ds} = \frac{1}{2} s^{-1/2}\Big|_{x^2} = \frac{1}{2x}$$

$$\frac{d^2 s^{1/2}}{ds^2} = \frac{-1}{4} s^{-3/2}\Big|_{x^2} = \frac{-1}{4x^3}$$

$\vdots$

$$I(x) = \frac{1}{2}\int_{x^2}^\infty x e^{-s} ds + \frac{1}{2}\cdot\frac{1}{2x}\int_{x^2}^\infty (s-x^2) e^{-s} ds - \frac{1}{2\cdot 2}\frac{1}{4x^3}\int_{x^2}^\infty (s-x^2)^2 e^{-s} ds$$

$+ \cdots$

$$I(x) = \frac{x}{2} e^{-s} (-1) \Big|_{x^2}^{\infty}$$

$$+ \frac{1}{4x} \int_0^{\infty} v e^{-v-x^2} \, dv$$

$$- \frac{1}{16x^3} \int_0^{\infty} v^2 e^{-v-x^2} \, dv$$

$$= \frac{x}{2} e^{-x^2} + \frac{e^{-x^2}}{4x} \left[\cancel{-v e^{-v} \Big|_0^{\infty}}^{\,0} + \int_0^{\infty} e^{-v} \, dv \right]$$

$$- \frac{e^{-x^2}}{16x^3} \left[-v^2 e^{-v} \Big|_0^{\infty} + 2 \cancel{\int_0^{\infty} v e^{-v} \, dv}^{\,1} \right]$$

$$= \frac{x}{2} e^{-x^2} + \frac{e^{-x^2}}{4x} - \frac{e^{-x^2}}{8x^3}$$

$$= e^{-x^2} \left[\frac{x}{2} + \frac{1}{4x} - \frac{1}{8x^3} \cdots \right]$$

8.6 (e)

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt$$

(a) $$\mathrm{erf}(-\infty) = \frac{2}{\sqrt{\pi}} \int_0^{-\infty} e^{-t^2}\,dt \qquad v = -t \qquad \cancel{t = -v}$$

$$= \cancel{\frac{2}{\sqrt{\pi}} \int_0^{+\infty} e^{-v^2}\,(-dv)}$$

$$= -\frac{2}{\sqrt{\pi}} \int_{-\infty}^{0} e^{-t^2}\,dt = -\frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{\cdot\cdot}\,dt = -1$$

(b) $$\int_0^x t^2 e^{-t^2}\,dt = \int_0^x t\left(t e^{-t^2}\right) dt$$

$$= t\,\frac{(-1)}{2} e^{-t^2}\Big|_0^x + \int_0^x t e^{-t^2}\,dt$$

$$= -\frac{x}{2} e^{-x^2} + \frac{\sqrt{\pi}}{2}\,\mathrm{erf}(x)$$

(c) $$\cancel{\mathrm{erf}(x)} = \frac{\sqrt{\pi}}{2} = \int_0^x e^{-t^2}\,dt + \int_x^{\infty} e^{-t^2}\,dt$$

$$\cancel{\frac{\sqrt{\pi}}{2}\,\mathrm{erf}}(x)$$

$$1 = \underbrace{\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt}_{} + \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt$$

erf(x)

$$\therefore \ \mathrm{erf}(x) = 1 - \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt$$

$$\approx 1 - \frac{2}{\sqrt{\pi}}\left[e^{-x^2}\left(\frac{1}{2x} - \frac{1}{4x^3} \cdots\right)\right]$$

B.7 Pu: $\Gamma(n+1) = n\Gamma(n)$

$$\Gamma(n+1) = \int_0^\infty x^n e^{-x}\,dx = \cancel{\int_0^\infty x\,(x^{n-1} e^{-x})\,dx}$$

$$\cancel{v = x}$$

$$\cancel{dv = dx} \qquad \cancel{du = x^{n-1} e^{-x}}$$

let $v = x^n$ $\qquad u = -e^{-x}$

$dv = nx^{n-1}$ $\qquad du = e^{-x}\,dx$

$$= -x^n \overset{0}{\cancel{e^{-x}}}\Big|_0^\infty + \int_0^\infty n x^{n-1} e^{-x}\,dx = n\int_0^\infty x^{n-1} e^{-x}\,dx = n\Gamma(n)$$

B.8 $\Gamma(1/2) = ?$

$$\Gamma(1/2) = \int_0^\infty x^{-1/2} e^{-x}\,dx$$

let $v = x^{+1/2} \Rightarrow x = v^2$

$dv = \frac{1}{2}x^{-1/2}\,dx \Rightarrow dx = 2x^{1/2}\,dv$

$= 2v\,dv$

$$= \int_0^\infty v^{-1} e^{-v^2}\, 2v\,dv = 2\int_0^\infty e^{-v^2}\,dv$$

$$= 2 \cdot \frac{1}{2}\sqrt{\frac{\pi}{1}} = \sqrt{\pi}.$$

$$\Gamma(3/2) = 1/2\,\Gamma(1/2) = \frac{\sqrt{\pi}}{2}$$

$\Gamma(-1/2) = ?$ $\quad (-\frac{1}{2})\Gamma(-1/2) = \Gamma(1/2)$

so $\Gamma(-1/2) = -2\Gamma(1/2) = -2\sqrt{\pi}$

(B.9) $\Gamma(n+1) = \int_0^\infty x^n e^{-x}\,dx$

$\Gamma(1/3) = ?$

$$\Gamma(1/3) = \int_0^\infty x^{-2/3} e^{-x}\,dx = \cdots$$

$\Gamma(\frac{2}{3}) = ?$

$$\Gamma(2/3) = \int_0^\infty x^{-1/3} e^{-x} = \cdots$$

Check:

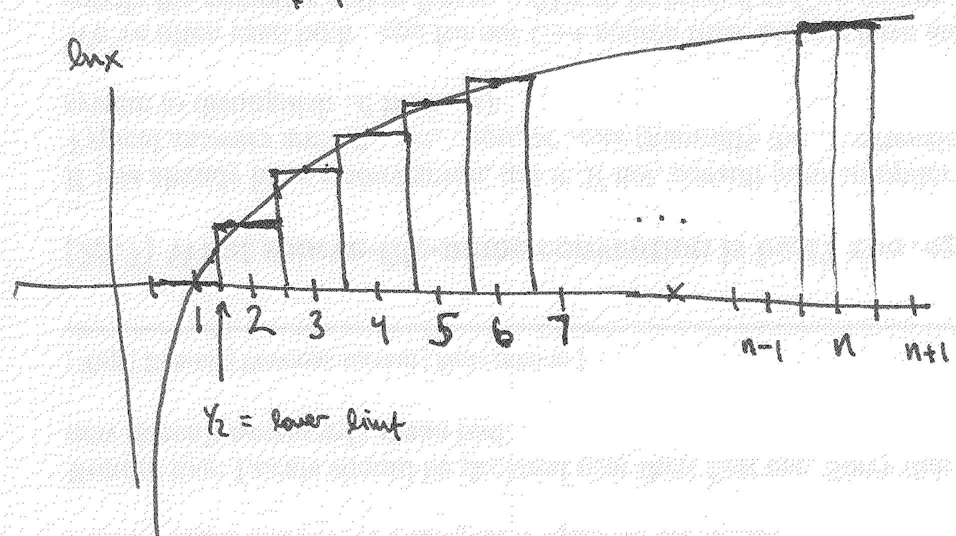
$$\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin(n\pi)}$$

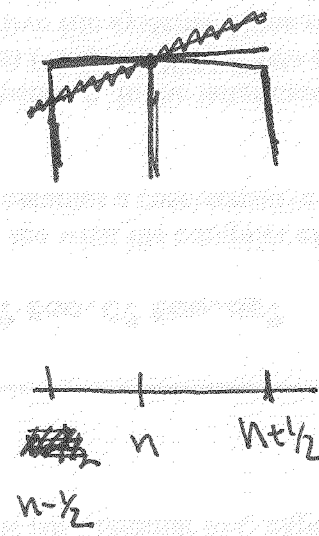
$n = 1/3 \Rightarrow$ $\Gamma(1/3)\Gamma(2/3) = \frac{\pi}{\sin(\frac{\pi}{3})} = \frac{\pi}{\frac{\sqrt{3}}{2}} = \frac{2\pi}{\sqrt{3}}$

B.10

$$\ln n! = \ln n + \ln(n-1) + \ln(n-2) + \cdots + \ln 2 + \ln 1$$

$$= \sum_{k=1}^{n} \ln k \cong \int_{1/2}^{n+1/2} \ln x \, dx$$





$$= x\ln x - x \Big|_{1/2}^{n+1/2} = \left[(n+\tfrac{1}{2})\ln(n+\tfrac{1}{2}) - (n+\tfrac{1}{2})\right] - \left[\tfrac{1}{2}\ln(\tfrac{1}{2}) - \tfrac{1}{2}\right]$$

$$= (n+\tfrac{1}{2})\ln\left(n\left(1+\tfrac{1}{2n}\right)\right) - (n+\tfrac{1}{2}) - \tfrac{1}{2}\ln\tfrac{1}{2} + \tfrac{1}{2}$$

large n approx

$$= (n+\tfrac{1}{2})\ln\left(n\left(1+\tfrac{1}{2n}\right)\right) - n - \cancel{\tfrac{1}{2}} - \tfrac{1}{2}\ln\tfrac{1}{2} + \cancel{\tfrac{1}{2}}$$

$$= \cancel{(n+\tfrac{1}{2})(\ln(n)\ldots)}$$

$$= (n+\tfrac{1}{2})\left[\ln(n) + \ln\left(1+\tfrac{1}{2n}\right)\right] - n - \tfrac{1}{2}\ln\tfrac{1}{2}$$

$$\ln(1+x) = \sum_{k=0}^{\infty} \frac{x^k}{(k+1)} \qquad \frac{1}{1-x} = \sum_{k\geq 0} x^k \qquad -\ln(1-x) = \sum_{k\geq 0} \frac{x^{k+1}}{k+1}$$

$$\therefore \ln(1+x) = \sum_{k\geq 0} \frac{(-1)^k x^{k+1}}{k+1}$$

$$\therefore \ln\left(1+\frac{1}{2n}\right) \approx \sum_{k\geq 0} \frac{(-1)^k}{(k+1)} \frac{1}{(2n)^{k+1}}$$

$\therefore$

$$\ln(n!) \cong \left(n+\tfrac{1}{2}\right)\left(\ln(n) + \sum_{k\geq 0} \frac{(-1)^k}{(k+1)} \frac{1}{(2n)^{k+1}}\right) - n - \frac{1}{2}\ln\frac{1}{2}$$

$$\approx \left(n+\tfrac{1}{2}\right)\left(\ln(n) + \frac{1}{2n}\right) - n - \frac{1}{2}\ln\frac{1}{2}$$

$$= \left(n+\tfrac{1}{2}\right)\ln(n) + \overset{\sim 0}{\cancel{\frac{1}{2}}} + \overset{\sim 0}{\cancel{\frac{1}{4n}}} - n - \overset{\sim 0}{\cancel{\frac{1}{2}\ln\frac{1}{2}}}$$

constants become insignificant for large n.

$$\therefore n! \cong n^{(n+\frac{1}{2})} \cdot e^{-n} \cdot \cancel{e^{-\frac{1}{2}\ln\frac{1}{2}}} =$$

$$n^n \cdot n^{\frac{1}{2}} e^{-n}$$ ✓ $$n! \approx \left(\frac{n}{e}\right)^n \sqrt{n} \cdot c.$$

Check

~~$\ln(0!)$~~ $= \ln(1!) = \cancel{0} = \frac{3}{2}\cancel{0} + \frac{1}{2} - 1$ can't expect to

be close to exact at 1.

B.11

$\cancel{x}\,x^n e^{-x}$ $\qquad x^n \nearrow \quad x \gg 1$

$e^{-x} \searrow \quad x \gg 1$

$$\frac{d}{dx} f = n x^{n-1} e^{-x} - x^n e^{-x} \underset{\text{set}}{=} 0$$

$$\frac{n}{x} - 1 = 0 \rightarrow n = x$$

B.12 $f_n(x) = x^n e^{-x} = e^{n \ln x - x}$ $\qquad$ let $y = x - n$

$$x^n = e^{\log x^n} = e^{n \ln x} \qquad x = y + n$$

$$= e^{n \ln(n+y) - (y+n)}$$

...

B.13 $\ln\left(1 + \frac{y}{n}\right) \cong \frac{y}{n} - \frac{1}{2}\left(\frac{y}{n}\right)^2 + \frac{1}{3}\left(\frac{y}{n}\right)^3$ keep.

$y \sim \sqrt{n}$ $\qquad \sim \frac{1}{\sqrt{n}}\,;\ \frac{1}{n}\,;\ \frac{1}{n^{3/2}}$

$$n! \cong \int_0^\infty x^n e^{-x}\,dx \cong \int_{-\infty}^{\infty} e^{n \ln n - n + n \ln\left(1 + \frac{y}{n}\right) - y}\,dy$$

$$n! \cong e^{n\ln n - n}\int_{-\infty}^{+\infty} \exp\left\{\cancel{y} - \frac{1}{2}\frac{y^2}{n} + \frac{1}{3}\frac{y^3}{n^2} - \cancel{y}\right\} dy$$

$$= e^{n\ln n - n}\int_{-\infty}^{\infty} e^{\frac{1}{3}\frac{y^3}{n^2}}\, e^{-\frac{1}{2}\frac{y^2}{n}}\, dy$$

$$\approx n^n e^{-n}\int_{-\infty}^{\infty}\left(1 + \frac{y^3}{3n^2}\right) e^{-\frac{1}{2}\frac{y^2}{n}}\, dy$$

let $\cancel{v = \frac{y}{2n}}$ $\quad y = \sqrt{2n}\cdot v \;\Rightarrow\; v = \frac{y}{\sqrt{2n}}$

$$dy = \sqrt{2n}\cdot dv$$

$$\cong n^n e^{-n}\int_{-\infty}^{\infty}\Big(1 + \underbrace{\frac{1}{3n^2}\cdot(2n)^{3/2}}\, v^3\Big) e^{-v^2}$$

this term will vanish ... should have kept a higher order ~~term~~ term. (dropping $O(y^3)$ term ...)

$$n! \cong \left(\frac{n}{e}\right)^n \int_{-\infty}^{\infty} \exp\left\{-\frac{y^2}{2n} - \frac{y^4}{4n^3}\right\} dy$$

$$= \left(\frac{n}{e}\right)^n \int_{-\infty}^{\infty} e^{-\frac{y^4}{4n^3}}\, e^{-\frac{y^2}{2n}}\, dy = \left(\frac{n}{e}\right)^n \int_{-\infty}^{\infty}\left(1 - \frac{y^4}{4n^3}\right) e^{-\frac{y^2}{2n}}\, dy$$

let $\cancel{v = \sqrt{2n}\cdot y}$ $\quad \cancel{y = \frac{v}{\sqrt{2n}}}$ $\quad y = \sqrt{2n}\cdot v$

$\cancel{dv = \sqrt{2n}\, dy}$ $\quad dy = \sqrt{2n}\cdot dv$

so
$$n! \cong \left(\frac{n}{e}\right)^n \int_{-\infty}^{\infty} \left(1 - \frac{v^4 (2n)^2}{4n^3}\right) e^{-v^2} \sqrt{2n}\cdot dv$$

$$= \left(\frac{n}{e}\right)^n \cdot \sqrt{2n} \int_{-\infty}^{\infty} \left(1 - \frac{v^4}{n}\right) e^{-v^2} dv$$

$$= \left(\frac{n}{e}\right)^n \sqrt{2n} \cdot \left[\int_{-\infty}^{+\infty} e^{-v^2} dv - \frac{1}{n} \int_{-\infty}^{+\infty} v^4 e^{-v^2} dv \right]$$

$$\left\{ \int_0^\infty x^2 e^{-ax^2} dx = \frac{1}{4}\sqrt{\frac{\pi}{a^3}} \right.$$

$$\frac{d}{da} \rightarrow -\int_0^\infty x^4 e^{-ax^2} dx = \cancel{\frac{1}{4}\sqrt{\frac{\pi}{a^3}}} \cdot \frac{1}{4}\sqrt{\pi}\ \frac{3}{2}\cdot a^{-5/2}$$

$$\Rightarrow \int_0^\infty x^4 e^{-ax^2} dx = \frac{3}{8}\sqrt{\pi}\ .$$

$$\therefore\ n! \approx \left(\frac{n}{e}\right)^n \sqrt{2n} \left[\sqrt{\pi} - \frac{1}{n}\cdot \frac{3}{4}\sqrt{\pi} \right]$$

$$\approx n^n e^{-n} \sqrt{2\pi n} \left[1 - \frac{3}{4n} \right]$$ $\sim$ error: 3 should be in denom...

(3.14) (a) B.28

$$\int_0^\pi (\sin\theta)^n \, d\theta = \frac{\sqrt{\pi}\,\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}$$

$n=0$ $\quad \pi \stackrel{?}{=} \dfrac{\sqrt{\pi}\cdot\Gamma(1/2)}{\Gamma(1)}$ ✓ $\qquad \Gamma(1)=1$

$\Gamma(1/2) = \sqrt{\pi}$

$n=1$ $\quad -\cos\theta\Big|_0^\pi \stackrel{?}{=} \dfrac{\sqrt{\pi}\,\Gamma(1)}{\Gamma(3/2)}$ $\qquad \Gamma(n+1) = n\Gamma(n)$

$$= -(-1-1) = 2 \stackrel{?}{=} \frac{\sqrt{\pi}}{\frac{1}{2}\cdot\sqrt{\pi}} = 2 \text{ ✓}$$

(b) Pr: $\displaystyle\int_0^\pi (\sin\theta)^n \, d\theta = \left(\frac{n-1}{n}\right)\int_0^\pi (\sin\theta)^{n-2}\, d\theta \qquad n \geq 2$

$$\|$$

$$\int_0^\pi (\sin\theta)^{n-2}(1-\cos^2\theta)\, d\theta = \int_0^\pi (\sin\theta)^{n-2}\, d\theta - \int_0^\pi (\sin\theta)^{n-2}\cos^2\theta\, d\theta$$

$u = \cos\theta \qquad v = \dfrac{(\sin\theta)^{n-1}}{n-1}$

$du = -\sin\theta \qquad dv = (\sin\theta)^{n-2}\cos\theta\, d\theta$

$$= \int_0^\pi (\sin\theta)^{n-2}\, d\theta - \left[\cancelto{0}{\frac{(\cos\theta)}{n-1}(\sin\theta)^{n-1}\Big|_0^\pi} + \frac{1}{n-1}\int_0^\pi (\sin\theta)^n\, d\theta\right]$$

$$\therefore \int_0^{\pi} (\sin\theta)^n d\theta = \int_0^{\pi} (\sin\theta)^{n-2} d\theta - \frac{1}{n-1}\int_0^{\pi} (\sin\theta)^n d\theta$$

$$= \left(1+\frac{1}{n-1}\right)\int_0^{\pi} (\quad)^n d\theta =$$

$$\int_0^{\pi} (\sin\theta)^n d\theta = \left(\frac{n-1}{n}\right)\int_0^{\pi} (\sin\theta)^{n-2} d\theta$$

(c) $$\int_0^{\pi} (\sin\theta)^n d\theta = \left(\frac{n-1}{n}\right)\int_0^{\pi} (\sin\theta)^{n-2} d\theta = \left(\frac{n-1}{n}\right) \frac{\sqrt{\pi}\,\Gamma\left(\frac{n-2}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{n-2}{2}+1\right)}$$

By induction

$$= \left(\frac{n-1}{n}\right)\frac{\sqrt{\pi}\,\Gamma\left(\frac{n}{2}-\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \qquad \Gamma(n+1) = n\Gamma(n)$$

$$= \frac{\left(\frac{n}{2}-\frac{1}{2}\right)\sqrt{\pi}\,\Gamma\left(\frac{n}{2}-\frac{1}{2}\right)}{\frac{n}{2}\,\Gamma\left(\frac{n}{2}\right)} = \frac{\sqrt{\pi}\,\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}$$

eq B.28 ✓

B15

$$A_d(r) = \frac{2\pi^{d/2}}{\Gamma(d/2)} r^{d-1}$$

(a) $$\int_{-\infty}^{+\infty} e^{-r^2} dr = \prod_{i=1}^{d} \left(\int e^{-x_i^2} dx_i \right) = \left(\sqrt{\pi}\right)^d = \pi^{d/2}$$

(b) $$\int_{\mathbb{R}^d} e^{-\vec{r}^2} d\vec{r} = \int_0^\infty e^{-|r|^2} (dr) \cdot A_d(1) \cdot r^{d-1}$$

are ~~...~~ of r.

All angular integrals give surface of sphere in d dimensions

$$= A_d(1) \int_0^\infty e^{-r^2} dr\, r^{d-1} \quad \cancel{A_d(1)}$$

$$= A_d(1) \int_0^\infty r^{d-1} e^{-r^2} dr$$

let $r^2 = v$

$r = \sqrt{v}$

$dr = \frac{1}{2}\frac{1}{\sqrt{v}} dv$

$$= A_d(1) \int_0^\infty v^{\left(\frac{d-1}{2}\right)} e^{-v} \frac{1}{2}\frac{1}{\sqrt{v}} dv$$

$$= \frac{A_d(1)}{2} \int_0^\infty v^{\frac{d}{2}-\frac{1}{2}} v^{-\frac{1}{2}} e^{-v} dv = \frac{A_d(1)}{2} \int_0^\infty v^{\frac{d}{2}-1} e^{-v} dv = \frac{A_d(1)}{2} \Gamma\left(\frac{d}{2}\right)$$

$$\Gamma(x) \equiv \int_0^\infty \xi^{x-1} e^{-\xi} d\xi$$

$$A_d(r) = \frac{A_d(1)\,\Gamma(d/2)}{2}$$

w/ $$A_d(1) = \frac{2\pi^{d/2}}{\Gamma(d/2)}$$

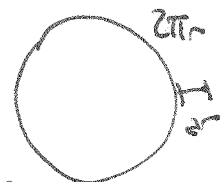
$\therefore$ $A_d(r)$ . . . somethig wrong . . .

B.1b

$$V_d(r) = \int_0^R A_d(r)\cdot dr$$

$$= \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_0^R r^{d-1}\,dr$$

$$= \frac{2\pi^{d/2}}{\Gamma(d/2)} \frac{R^d}{d}$$



$$\int_0^R 2\pi r\,dr = \frac{2\pi R^2}{2}$$

$$\int_0^R 4\pi r^2\,dr = \frac{4\pi}{3} r^3 \Big|_0^R$$

$$= \frac{4\pi}{3} R^3$$

$$\therefore \quad V_d(r) = \frac{2\pi^{d/2}}{\Gamma(d/2)} \frac{r^d}{d}$$

Check $$V_2(r) = \frac{2\pi}{\Gamma(1)} \frac{r^2}{2} = \pi r^2$$

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$$V_3(r) = \frac{2\pi^{3/2} r^3}{\Gamma(3/2)\, 3} = \frac{2\pi^{3/2} r^3}{\frac{1}{2}\Gamma(1/2) \cdot 3}$$

$$= \frac{4}{3}\pi r^3 \quad \checkmark.$$

B,17

Show B,36

$$\int_0^\infty \frac{x^n}{e^x-1}\,dx = \Gamma(n+1)\,\zeta(n+1)$$

$$+ \quad \int_0^\infty \frac{x^n}{e^x+1}\,dx = \left(1-\frac{1}{2^n}\right)\Gamma(n+1)\,\zeta(n+1)$$

~~≠ odd integer~~

consider

$$\int_0^\infty \frac{x^n}{e^x-1}\,dx = \int_0^\infty \frac{x^n e^{-x}}{1-e^{-x}}\,dx$$

$$= \int_0^\infty x^n e^{-x} \sum_{k=0}^{\infty} e^{-kx}\,dx = \sum_{k\geq 0} \int_0^\infty x^n e^{-(k+1)x}\,dx$$

Now

$$\int_0^\infty x^n e^{-(k+1)x}\,dx =$$

$$= \left.\frac{x^n e^{-(k+1)x}}{-(k+1)}\right|_0^\infty + \frac{n}{(k+1)}\int_0^\infty x^{n-1} e^{-(k+1)x}\,dx$$

$$\therefore \int_0^\infty x^n e^{-(k+1)x}\,dx = \frac{n}{k+1}\int_0^\infty x^{n-1} e^{-(k+1)x}\,dx = \frac{n(n-1)}{(k+1)^2}\int_0^\infty x^{n-2} e^{-(k+1)x}\,dx$$

$$= \cdots = \frac{n(n-1)(n-2)\cdots 1}{(k+1)^n}\int_0^\infty e^{-(k+1)x}\,dx$$

$$= \frac{n(n-1)(n-2)\cdots(n-(n-1))}{(k+1)^n}\left(\frac{-1}{(k+1)}e^{-(k+1)x}\right|_0^\infty$$

$$= \frac{n!}{(k+1)^{n+1}}$$

$$\therefore \int_0^\infty \frac{x^n}{e^x-1}\,dx = \sum_{k\geq 0} n!\,\frac{1}{(k+1)^{n+1}} = n!\sum_{k\geq 0}\frac{1}{(k+1)^{n+1}}$$

$$= n!\sum_{k\geq 1}\frac{1}{k^{n+1}} = \Gamma(n+1)\zeta(n+1) \qquad \text{B.36} \quad \checkmark$$

Consider

$$\int_0^\infty \frac{x^n}{e^x+1}\,dx = \int_0^\infty \frac{x^n e^{-x}}{1+e^{-x}}\,dx = \int_0^\infty x^n e^{-x}\sum_{k\geq 0}(-1)^k e^{-kx}\,dx$$

$$= \cancel{\int_0^\infty x^n e^{-x}} = \cancel{\int_0^\infty} - \sum_{k\geq 0}(-1)^k \int_0^\infty x^n e^{-(k+1)x}\,dx$$

From before $\displaystyle\int_0^\infty x^n e^{-(k+1)x}\,dx = \frac{n!}{(k+1)^{n+1}}$

So $\displaystyle\int_0^\infty \frac{x^n}{e^x+1}\,dx = \sum_{k\geq 0} \frac{n!\,(-1)^k}{(k+1)^{n+1}}$

$$= n!\left[\sum_{k=0,2,4,6,\ldots} \frac{1}{(k+1)^{n+1}} + \sum_{k\text{ odd}} \frac{-1}{(k+1)^{n+1}}\right]$$

$$= n!\left[\sum_{k\geq 0} \frac{1}{(2k+1)^{n+1}} - \sum_{k\geq 0} \frac{1}{(2k+2)^{n+1}}\right] \qquad \cancel{2k+1}$$

$$= n!\left[\sum_{k\geq 0} \frac{1}{(2k+1)^{n+1}} - \sum_{k\geq 0} \frac{1}{2^{n+1}(k+1)^{n+1}}\right]$$

$$= n!\left[\underbrace{\sum_{k\geq 0} \frac{1}{(2k+1)^{n+1}} + \sum_{k\geq 1} \frac{1}{(2k)^{n+1}}}_{\zeta(n+1)} - \sum_{k\geq 1} \frac{1}{(2k)^{n+1}} - \sum_{k\geq 0} \frac{1}{2^{n+1}(k+1)^{n+1}}\right]$$

$$- \frac{1}{2^{n+1}}\left[\sum_{k\geq 1} \frac{1}{k^{n+1}} + \sum_{k\geq 0} \frac{1}{(k+1)^{n+1}}\right]$$

$\cancel{\zeta(n+1)}$

$2\zeta(n+1)$

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$$= n! \left[\lambda(n+1) - \frac{1}{2^{n+1}} \left[\lambda(n+1) + \lambda(n+1) \right] \right]$$

$$= n! \, \lambda(n+1) \left[1 - \frac{1}{2^n} \right]$$

B,36

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(B,18) $$\frac{\pi}{4} = \sum_{k \text{ odd}} \frac{\sin(kx)}{k} \quad \ldots$$

(B,19) B,42

$$\frac{\pi x}{4} = \sum_{k \text{ odd}} \frac{1}{k^2}(1 - \cos kx)$$

$$\int_0^x f(x')\,dx' \to \quad \frac{\pi x^2}{8} = \sum_{k \text{ odd}} \frac{1}{k^2}\left(x - \frac{\sin kx}{k}\Big|_0^x\right)$$

$$= \sum_{k \text{ odd}} \frac{1}{k^2}\left(x - \frac{\sin kx}{k}\right) = \sum_{k \text{ odd}} \frac{1}{k^3}(kx - \sin(kx))$$

evaluate at $x = \frac{\pi}{2}$ to get ... will not get a simple sum

$$\sum_{k \text{ odd}} \frac{1}{k^3}\left(k\frac{\pi}{2} - \sin\left(k\frac{\pi}{2}\right)\right) \quad \ldots$$

integrate again

$$\frac{\pi x^3}{24} = \sum_{k \text{ odd}} \frac{1}{k^3}\left(\frac{kx^2}{2} + \frac{\cos(kx)}{k}\Big|_0^x\right)$$

$$\frac{\pi x^3}{24} = \sum_{k \text{ odd}} \frac{1}{k^2}\left(\frac{kx^2}{2} + \frac{\cos(kx)}{k} - \frac{1}{k} \right)$$

$x = \frac{\pi}{2}$

$$= \sum_{k \text{ odd}} \frac{1}{k^2} \cdot \frac{x^2}{2} + \sum_{k \text{ odd}} \frac{\cos(k\pi/2)}{k^4} - \sum_{k \text{ odd}} \frac{1}{k^4}$$

gives powers of 4.

B.20

$$\frac{\pi}{4} = \sum_{\text{odd } k} \frac{\sin(kx)}{k} = \sum_{k \text{ odd}} \frac{\sin(k\pi/2)}{k} = \sum_{k \text{ odd}} \frac{(-1)^{k+1}}{k}$$

$x = \pi/2$

B.21

replacing x w/ $-x$ gives

$$\frac{x^2 e^{-x}}{(e^{-x}+1)^2} = \frac{x^2 e^x}{(1+e^x)^2} \checkmark$$

$$\int_0^\infty \frac{x^2 e^x}{(e^x+1)^2}\,dx = \left.\frac{-x^2}{(1+e^x)}\right|_0^\infty + \int_0^\infty \frac{2x\,dx}{(1+e^x)} \cdots$$