Solutions to the Problems in Adaptive Signal Processing by Bernard Spiegel and Samuel D. Stearns

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To my family.

Introduction

This is a solution manual to some of the problems in the excellent textbook:

Calculus of Finite Differences and Difference Equations
by Murray Spiegel

I'm currently working aggressively on finishing more of the problems in this book. In the meantime I'm publishing my partial results for any student who does not want to wait for the full book to be finished.

One of the benefits of this manual is that I heavily use the `R` statistical language to perform any of the needed numerical computations (rather than do them "by-hand"). Thus if you work though this manual you will be learning the `R` language at the same time as you learn statistics. The `R` programming language is one of the most desired skills for anyone who hopes to use data/statistics in their future career. The `R` code can be found at the following location:

`https://waxworksmath.com/Authors/N_Z/Spiegel/spiegel.html`

As a final comment, I've worked hard to make these notes as good as I can, but I have no illusions that they are perfect. If you feel that that there is a better way to accomplish or explain an exercise or derivation presented in these notes; or that one or more of the explanations is unclear, incomplete, or misleading, please tell me. If you find an error of any kind – technical, grammatical, typographical, whatever – please tell me that, too. I'll gladly add to the acknowledgments in later printings the name of the first person to bring each problem to my attention.

Chapter 1: The Difference Calculus

Notes on Factorial Polynomials

Here I think the expansion of x^4 in terms of falling factorial functions is wrong. The expression given in the book is

$$x^4 = x^{(4)} + 7x^{(3)}h + 6x^{(2)}h^2 + x^{(1)}h^3 \,,$$

and I think it should be

$$x^4 = x^{(4)} + 6x^{(3)}h + 7x^{(2)}h^2 + x^{(1)}h^3 \,.$$

We will prove this by expanding the right-hand-side of this last expression. We have

$$\begin{aligned}
\text{RHS} &= x(x-h)(x-2h)(x-3h) + 6x(x-h)(x-2h)h + 7x(x-h)h^2 + xh^3 \\
&= x((x-h)(x-2h)(x-3h) + 6(x-h)(x-2h)h + 7(x-h)h^2 + h^3) \\
&= x((x-h)(x^2-5hx+6h^2) + 6h(x^2-3hx+2h^2) + 7xh^2 - 7h^3 + h^3) \\
&= x(x^3 - 5hx^2 + 6h^2x - hx^2 + 5h^2x - 6h^3 + 6hx^2 - 18h^2x + 12h^3 + 7xh^2 - 7h^3 + h^3) \\
&= x(x^3) = x^4 \,,
\end{aligned}$$

as we were trying to show.

Supplementary Problem 1.46

Part (a): We have

$$\mathcal{S}(1+\sqrt{x}) = (1+\sqrt{x})^2 = 1 + 2\sqrt{x} + x \,.$$

Part (b): We have

$$\begin{aligned}
(2\mathcal{S} + 3D)(x^2 - x) &= 2(x^2-x)^2 + 3(2x-1) \\
&= 2(x^4 - 2x^3 + x^2) + 6x - 3 \\
&= 2x^4 - 4x^3 + 2x^2 + 6x - 3 \,.
\end{aligned}$$

Part (c): We have

$$\mathcal{S}D(3x+2) = \mathcal{S}3 = 9 \,.$$

Part (d): We have

$$D\mathcal{S}(3x+2) = D(3x+2)^2 = D(9x^2+12x+4) = 18x + 12 \,.$$

Part (e): We have

$$\begin{aligned}(\mathcal{S}^2+2\mathcal{S}-3)(2x-1) &= (2x-1)^4+2(2x-1)^2-3(2x-1)\\ &= (2x)^3+4(2x)^3(-1)+\left(\begin{array}{c}4\\2\end{array}\right)(2x)^2(-1)^2+4(2x)(-1)^3+1+2(4x^2-4x+1)-3(2x-1)\\ &= 16x^4-32x^3+32x-22x+6\,,\end{aligned}$$

when we simplify.

Part (f): We have

$$(D+2)(\mathcal{S}-3)x^2=(D+2)(x^4-3x^2)=4x^3-6x+2x^4-6x^2=2x^4+4x^3-6x^2-6x$$

Part (g): We have

$$\begin{aligned}(\mathcal{S}-3)(D+2)x^2 &= (\mathcal{S}-3)(2x+2x^2)=(2x+2x^2)^2-6x-6x^2\\ &= 4x^4+8x^3-2x^2-6x\,.\end{aligned}$$

Part (h): We have

$$\begin{aligned}(xD)^3\mathcal{S}(x+1) &= (xD)^3(x+1)^2\\ &= (xD)^2xD(x+1)^2=(xD)^2x(2(x+1))=(xD)^2(2x^2+2x)\\ &= (xD)(xD(2x^2+2x))=(xD)(x(4x+2))=(xD)(4x^2+2x)\\ &= x(8x+2)=8x^2+2x=2x(4x+1)\,.\end{aligned}$$

Part (i): We have

$$x^3D^3\mathcal{S}(x+1)=x^3D^3(x+1)^2=x^3D^2(2(x+1))=2x^3D^2(x+1)=2x^3D(1)=0\,.$$

Part (j): We have

$$\begin{aligned}(x\mathcal{S}-\mathcal{S}x)D\mathcal{S}x^2 &= (x\mathcal{S}-\mathcal{S}x)Dx^4\\ &= (x\mathcal{S}-\mathcal{S}x)4x^3\\ &= x\mathcal{S}(4x^3)-\mathcal{S}x(4x^3)\\ &= x(16x^6)-\mathcal{S}(4x^4)\\ &= 16x^7-16x^8=16x^7(1-x)\,.\end{aligned}$$

Supplementary Problem 1.47

Part (a): We can show that $\mathcal{S}$ is not linear by the observation that

$$\mathcal{S}(x+y)=(x+y)^2=x^2+y^2+2xy=\mathcal{S}(x)+\mathcal{S}(y)+2xy\neq\mathcal{S}(x)+\mathcal{S}(y)\,,$$

in general.

Part (b): Now if $\mathcal{S}^{-1}$ exists then $\mathcal{S}^{-1}f = g$ if and only if

$$\mathcal{S}g = f\,,$$

or

$$g^2 = f\,.$$

This means that $g = \pm\sqrt{f}$. Note that if $f < 0$ then g will not exist and in addition as there are two solutions $\pm\sqrt{f}$ we see that g is not unique.

Supplementary Problem 1.48

We have

$$\mathcal{S}^3 f = \mathcal{S}^2\mathcal{S}f = \mathcal{S}^2 f^2 = \mathcal{S}\mathcal{S}f^2 = \mathcal{S}f^4 = f^8\,,$$

and

$$C^2 f = CCf = Cf^3 = (f^3)^3 = f^9\,.$$

As $f^8 \neq f^9$ in general these two operators are *not* equal.

Supplementary Problem 1.49

Part (a): The operator $\alpha\mathcal{S}$ is to square and then multiply by α while the operator $\mathcal{S}\alpha$ is to multiply by α and then to square.

Part (b): To see if these are commutative we compute

$$\begin{aligned}\alpha\mathcal{S}f &= \alpha f^2\\ \mathcal{S}\alpha f &= (\alpha f)^2 = \alpha^2 f^2\,,\end{aligned}$$

which are *not* equal and these operators are not commutative.

Supplementary Problem 1.50

The operator $(xD)^4$ applied to f gives

$$\begin{aligned}(xD)^4 f &= (xD)^3 x f' \\ &= (xD)^2 x(f' + xf'') \\ &= (xD)(xD)(xf' + x^2 f'') \\ &= (xD)x(f' + xf'' + 2xf'' + x^2 f''') \\ &= xD(xf' + 3x^2 f'' + x^3 f''') \\ &= x(f' + xf'' + 6xf'' + 3x^2 f''' + 3x^2 f''' + x^3 f^{(iv)}) \\ &= x(f' + 7xf'' + 6x^2 f''' + x^3 f^{(iv)}) \\ &= xf' + 7x^2 f'' + 6x^3 f''' + x^4 f^{(iv)} .\end{aligned}$$

While

$$x^4 D^4 f = x^4 f^{(iv)} .$$

From the above calculation we see that this is *not* equal to $(xD)^4 f$.

Supplementary Problem 1.51

Part (a): We have

$$\begin{aligned}(D^2 x - xD^2) f &= DDxf - xf'' = D(f + xf') - xf'' \\ &= f' + f' + xf'' - xf'' = 2f' = 2Df .\end{aligned}$$

Part (b): We have

$$\begin{aligned}(D^3 x - xD^3) f &= D^3(xf) - xf''' \\ &= D^2(f + xf') - xf''' \\ &= D(f' + f' + xf'') - xf''' \\ &= 2f'' + f'' + xf''' - xf''' \\ &= 3f'' = 3D^2 f .\end{aligned}$$

If we generalize these results we would conclude

$$D^n x - xD^n = nD^{n-1} . \qquad (1)$$

To prove that this is true we will assume that Equation 1 is true and apply D to both sides to get

$$D^{n+1} x - D^n - xD^{n+1} = nD^n ,$$

or

$$D^{n+1} x - xD^{n+1} = (n+1)D^n ,$$

which shows that Equation 1 is true for $n+1$.

Supplementary Problem 1.52

Part (a): We have

$$\begin{aligned}\Delta(2x-1)^2 &= (2(x+h)-1)^2-(2x-1)^2 = 4(x+h)^2-4(x+h)+1-4x^2+4x-1\\ &= 8xh+4h^2-4h\,.\end{aligned}$$

Part (b): We have

$$E(\sqrt[3]{5x-4}) = \sqrt[3]{5(x+h)-4} = \sqrt[3]{5x+5h-4}\,.$$

Part (c): We have

$$\begin{aligned}\Delta^2(2x^2-5x) &= \Delta(2(x+h)^2-5(x+h)-2x^2+5x)\\ &= \Delta(2x^2+4xh+2h^2-5x-5h-2x^2+5x)\\ &= \Delta(4hx+2h^2-5h)\\ &= 4h(x+h)-4hx = 4h^2\,.\end{aligned}$$

Part (d): We have

$$\begin{aligned}3E^2(x^2+1) &= 3E((x+h)^2+1) = 3((x+2h)^2+1)\\ &= 3(x^2+4xh+4h^2+1) = 3x^2+12xh+12h^2+3\,.\end{aligned}$$

Part (e): If we recall that $\Delta+1 = E-1+1 = E$ we have

$$(\Delta+1)^2(x+1)^2 = E^2(x+1)^2 = E((x+h+1)^2) = (x+2h+1)^2\,.$$

Part (f): We have

$$\begin{aligned}(xE^2+2xE+1)x^2 &= xE(x+h)^2+2x(x+h)^2+x^2\\ &= x(x+2h)^2+2x(x+h)^2+x^2\\ &= x(x^2+4xh+4h^2)+2x(x^2+2xh+h^2)+x^2\\ &= x^3+4x^2h+4h^2x+2x^3+4x^2h+2xh^2+x^2\\ &= 3x^3+(8h+1)x^2+6h^2x\,.\end{aligned}$$

Part (g): We have

$$\Delta^2E^3x = \Delta^2(x+3h) = \Delta((x+4h)-(x+3h)) = \Delta h = 0\,.$$

Part (h): We have

$$
\begin{aligned}
(3\Delta + 2)(2E - 1)x^2 &= (3\Delta + 2)(2(x+h)^2 - x^2) \\
&= (3\Delta + 2)(2x^2 + 4xh + 2h^2 - x^2) \\
&= (3\Delta + 2)(x^2 + 4xh + 2h^2) \\
&= 3((x+h)^2 + 4(x+h)h + 2h^2 - x^2 - 4xh - 2h^2) + 2x^2 + 8xh + 4h^2 \\
&= 3(2xh + 5h^2) + 2x^2 + 8xh + 4h^2 \\
&= 2x^2 + 14xh + 19h^2 \,.
\end{aligned}
$$

Part (i): We have

$$
\begin{aligned}
(2E - 1)(3\Delta + 2)x^2 &= (2E - 1)(3(x+h)^2 - x^2) + 2x^2) \\
&= (2E - 1)(3(x^2 + 2xh + h^2 - x^2) + 2x^2) \\
&= (2E - 1)(3(2xh + h^2) + 2x^2) \\
&= (2E - 1)(2x^2 + 6xh + 3h^2) \\
&= 2(2(x+h)^2 + 6(x+h)h + 3h^2) - 2x^2 - 6xh - 3h^2 \\
&= 2(2x^2 + 4xh + 2h^2 + 6xh + 6h^2 + 3h^2) - 2x^2 - 6xh - 3h^2 \\
&= 2(2x^2 + 10xh + 11h^2) - 2x^2 - 6xh - 3h^2 \\
&= 4x^2 + 20xh + 22h^2 - 2x^2 - 6xh - 3h^2 \\
&= 2x^2 + 14xh + 19h^2 \,.
\end{aligned}
$$

Note that this equals the expression in Part (i) as it should since the two operators are commutative.

Part (j): We have

$$
\begin{aligned}
(x\Delta E)^2 x^2 &= (x\Delta E)(x\Delta E)x^2 \\
&= (x\Delta E)(x\Delta(x+h)^2) \\
&= (x\Delta E)(x((x+2h)^2 - (x+h)^2)) \\
&= (x\Delta E)(x(x^2 + 4xh + 4h^2 - (x^2 + 2xh + h^2)) \\
&= (x\Delta E)(x(2xh + 3h^2)) \\
&= (x\Delta E)(2hx^2 + 3h^2x) \\
&= x\Delta(2h(x+h)^2 + 3h^2(x+h)) \\
&= x\Delta[2h(x^2 + 2xh + h^2) + 3h^2x + 3h^3] \\
&= x\Delta[2hx^2 + 4h^2x + 2h^3 + 3h^2x + 3h^3] \\
&= x\Delta[2hx^2 + 7h^2x + 5h^2] \\
&= x[2h[(x+h)^2 - x^2] + 7h^2h] \\
&= x[2h(x^2 + 2xh + h^2 - x^2) + 7h^3] \\
&= x[4h^2x + 2h^3 + 7h^3] \\
&= 4h^2x^2 + 9h^3x \,.
\end{aligned}
$$

Supplementary Problem 1.53

Part (a): Note that each of the two sides can be written as

$$
\begin{aligned}
(E-2)(\Delta+3) &= E\Delta + 3E - 2\Delta - 6 \\
(\Delta+3)(E-2) &= \Delta E - 2\Delta + 3E - 6\,.
\end{aligned}
$$

For these two be equal we must have $E\Delta = \Delta E$. Lets consider each of the operators applied to a function f. We have

$$
\begin{aligned}
E\Delta f &= E(f(x+h) - f(x)) = f(x+2h) - f(x+h) \\
\Delta E f &= \Delta f(x+h) = f(x+2h) - f(x+h)\,.
\end{aligned}
$$

As these are equal we have shown that $(E-2)(\Delta+3) = (\Delta+3)(E-2)$.

Part (b): Now

$$
(E-2x)(\Delta+3x) = E\Delta + 3Ex - 2x\Delta - 6x^2\,,
$$

and

$$
(\Delta+3x)(E-2) = \Delta E - 2\Delta + 3xE - 6x\,.
$$

As we know that $E\Delta = \Delta E$ for these to be equal we must have

$$
3Ex - 2x\Delta - 6x^2 = -2\Delta + 3xE - 6x\,.
$$

To show that these are *not* equal lets apply them both to a simple function $f(x) = 1$. We find

$$
\begin{aligned}
(3Ex - 2x\Delta - 6x^2)1 &= 3Ex - 6x^2 = 3(x+h) - 6x^2 \\
(-2\Delta + 3xE - 6x)1 &= 0 + 3x - 6x = -3x\,.
\end{aligned}
$$

As these two expressions are not equal in general the original operator expressions are not equal either.

Supplementary Problem 1.54

We have

$$
\begin{aligned}
E(f(x) + g(x)) &= f(x+h) + g(x+h) = Ef(x) + Eg(x) \\
E(\alpha f(x)) &= \alpha f(x+h) = \alpha E f(x)\,.
\end{aligned}
$$

Supplementary Problem 1.55

Part (a): Note that

$$
\begin{aligned}
\Delta^2 f(x) &= \Delta(f(x+h) - f(x)) = f(x+2h) - f(x+h) - f(x+h) + f(x)) \\
&= f(x+2h) - 2f(x+h) + f(x)\,.
\end{aligned}
$$

Thus we can see that

$$\Delta^2(af(x) + bg(x)) = a\Delta^2 f(x) + b\Delta^2 g(x)\,,$$

and Δ^2 is a linear operator.

Part (b): As $E^2 f(x) = f(x + 2h)$ we also see that E^2 will be a linear operator.

Part (c): Yes both will be linear operators.

Supplementary Problem 1.56

To show this we can compute

$$\begin{aligned}
(E-1)^3 f(x) &= (E-1)^2(E-1)f(x) = (E-1)^2(f(x+h) - f(x)) \\
&= (E-1)(f(x+2h) - f(x+h) - f(x+h) + f(x)) \\
&= (E-1)(f(x+2h) - 2f(x+h) + f(x)) \\
&= (f(x+3h) - 2f(x+2h) + f(x+h)) - (f(x+2h) - 2f(x+h) + f(x)) \\
&= f(x+3h) - 3f(x+2h) + 3f(x+h) - f(x) \\
&= (E^3 - 3E^2 + 3E - 1)f(x)\,.
\end{aligned}$$

Supplementary Problem 1.57

Part (a): From the definition we have that

$$\Delta\cos(rx) = \cos(r(x+h)) - \cos(rx)\,.$$

Now to prove the desired expression we will use the identity

$$\sin(\theta_1) - \sin(\theta_2) = 2\sin\left(\frac{\theta_1 - \theta_2}{2}\right)\cos\left(\frac{\theta_1 + \theta_2}{2}\right)\,. \tag{2}$$

To show that this is true we will expand each factor in the right-hand-side (RHS) of the above. Doing this we have

$$\begin{aligned}
\text{RHS} &= \left(\sin\left(\frac{\theta_1}{2}\right)\cos\left(\frac{\theta_2}{2}\right) - \sin\left(\frac{\theta_2}{2}\right)\cos\left(\frac{\theta_1}{2}\right)\right)\left(\cos\left(\frac{\theta_1}{2}\right)\cos\left(\frac{\theta_2}{2}\right) - \sin\left(\frac{\theta_2}{2}\right)\sin\left(\frac{\theta_1}{2}\right)\right) \\
&= \sin\left(\frac{\theta_1}{2}\right)\cos\left(\frac{\theta_1}{2}\right)\cos^2\left(\frac{\theta_2}{2}\right) - \sin^2\left(\frac{\theta_1}{2}\right)\sin\left(\frac{\theta_2}{2}\right)\cos\left(\frac{\theta_2}{2}\right) \\
&- \sin\left(\frac{\theta_2}{2}\right)\cos^2\left(\frac{\theta_1}{2}\right)\cos\left(\frac{\theta_2}{2}\right) + \sin\left(\frac{\theta_1}{2}\right)\sin^2\left(\frac{\theta_2}{2}\right)\cos\left(\frac{\theta_1}{2}\right) \\
&= \frac{1}{2}\sin(\theta_1)\cos^2\left(\frac{\theta_2}{2}\right) - \frac{1}{2}\sin^2\left(\frac{\theta_1}{2}\right)\sin(\theta_2) - \frac{1}{2}\sin(\theta_2)\cos\left(\frac{\theta_1}{2}\right) + \frac{1}{2}\sin(\theta_1)\sin^2\left(\frac{\theta_2}{2}\right) \\
&= \frac{1}{2}\sin(\theta_1) - \frac{1}{2}\sin(\theta_2)\,,
\end{aligned}$$

as we were to show.

Next using the fact that $\cos(x) = \sin(\pi - x)$ we can write

$$\cos(\theta_1) - \cos(\theta_2) = \sin(\pi - \theta_1) - \sin(\pi - \theta_2)\,.$$

Next using Equation 2 on the above differences of sins we get

$$\begin{aligned}
\cos(\theta_1) - \cos(\theta_2) &= 2\sin\left(\frac{\pi - \theta_1 - \pi + \theta_2}{2}\right)\cos\left(\frac{\pi - \theta_1 + \pi - \theta_2}{2}\right) \\
&= 2\sin\left(\frac{\theta_2 - \theta_1}{2}\right)\cos\left(\frac{2\pi - \theta_1 - \theta_2}{2}\right) \\
&= 2\sin\left(\frac{\theta_2 - \theta_1}{2}\right)\cos\left(\pi - \left(\frac{\theta_1 + \theta_2}{2}\right)\right) \\
&= 2\sin\left(\frac{\theta_2 - \theta_1}{2}\right)\sin\left(\frac{\theta_1 + \theta_2}{2}\right)\,.
\end{aligned}$$

Thus using this we have

$$\begin{aligned}
\Delta\cos(rx) &= \cos(r(x+h)) - \cos(rx) \\
&= 2\sin\left(\frac{rx - rx - rh}{2}\right)\sin\left(\frac{rx + rh + rx}{2}\right) \\
&= -2\sin\left(\frac{rh}{2}\right)\sin\left(\frac{2rx + rh}{2}\right) \\
&= -2\sin\left(\frac{rh}{2}\right)\sin\left(r\left(x + \frac{h}{2}\right)\right)\,,
\end{aligned}$$

which is the desired expression.

Part (b): For this we have

$$\Delta\ln(x) = \ln(x+h) - \ln(x) = \ln\left(1 + \frac{h}{x}\right)\,.$$

Part (c): For this we have

$$\Delta\log_b(x) = \log_b(x+h) - \log_b(x) = \log_b\left(1 + \frac{h}{x}\right)\,.$$

Supplementary Problem 1.58

Part (a): For this note that

$$(D + \Delta)f(x) = f'(x) + f(x+h) - f(x) = (\Delta + D)f(x)\,.$$

Part (b): For this note that

$$D\Delta f(x) = D(f(x+h) - f(x)) = f'(x+h) - f'(x) = \Delta f'(x) = \Delta D f(x)\,.$$

Supplementary Problem 1.59

Part (a): First we will compute $(D\Delta)f(x)$ as

$$(D\Delta)f(x) = D(f(x+h) - f(x)) = f'(x+h) - f'(x)\,.$$

Then $(D\Delta)Ef(x)$ is

$$(D\Delta)Ef(x) = (D\Delta)f(x+h) = f'(x+2h) - f'(x+h)\,.$$

Next we compute $(\Delta E)f(x)$ as

$$(\Delta E)f(x) = \Delta f(x+h) = f(x+2h) - f(x+h)\,.$$

Then $D(\Delta E)f(x)$ is

$$D(\Delta E)f(x) = f'(x+2h) - f'(x+h)\,,$$

which is equal to the previous expression, verifying that $(D\Delta)E = D(\Delta E)$.

Part (b): Yes because

$$\begin{aligned} DEf(x) &= Df(x+h) = f'(x+h) \\ EDf(x) &= Ef'(x) = f'(x+h)\,, \end{aligned}$$

are equal.

Supplementary Problem 1.60

Part (a): Recalling that $\Delta x = h$ we have

$$\begin{aligned} \lim_{\Delta x \to 0} \frac{\Delta}{\Delta x}[x(2-x)] &= \lim_{h \to 0} \frac{(x+h)(2-x-h) - x(2-x)}{h} \\ &= \lim_{h \to 0} \frac{x(2-x) - xh + h(2-x) - h^2 - x(2-x)}{h} \\ &= \lim_{h \to 0}(-x + 2 - x - h) = 2 - 2x = 2(1-x)\,. \end{aligned}$$

Also we have

$$D[x(2-x)] = (2-x) - x = 2 - 2x\,,$$

showing the desired equivalence.

Part (b): Recall that

$$\Delta^2 f(x) = f(x+2h) - 2f(x+h) + f(x)\,.$$

Using this result we have

$$\begin{aligned}
\Delta^2(x(2-x)) &= (x+2h)(2-x-2h) - 2(x+h)(2-x-h) + x(2-x) \\
&= (x+2h)(2-2h-x) - 2(x+h)(2-h-x) + x(2-x) \\
&= (2-2h)x - x^2 + 2h(2-2h) - 2hx - 2x(2-h-x) - 2h(2-h-x) + 2x - x^2 \\
&= -x^2 + (2-4h)x + 4h(1-h) - 2(2-h)x + 2x^2 - 2h(2-h) + 2hx + 2x - x^2 \\
&= -2h^2 \,,
\end{aligned}$$

when we simplify. Using $\Delta x = h$ this means that

$$\lim_{\Delta x \to 0} \frac{\Delta^2}{\Delta x^2}[x(2-x)] = \lim_{h \to 0} \left(\frac{-2h^2}{h^2} \right) = -2 \,.$$

Supplementary Problem 1.61

Part (a): We have

$$d(x^3 - 3x^2 + 2x - 1) = (3x^2 - 6x + 2)dx \,.$$

Part (b): We have

$$d^2(3x^2 + 2x - 5) = d(6x + 2)dx = 6(dx)^2 \,.$$

Supplementary Problem 1.62

This is a consequence of the fact that "the limit operation" is distributive over addition.

Supplementary Problem 1.63

To prove this we can use the product rule in the form

$$\begin{aligned}
D(f(x)g(x)^{-1}) &= f'(x)g(x)^{-1} - f(x)g(x)^{-2}g'(x) \\
&= \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{g(x)^2} \\
&= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} = \frac{Df(x)g(x) - f(x)Dg(x)}{g(x)^2} \,.
\end{aligned}$$

Supplementary Problem 1.64

We have

$$\begin{aligned}\lim_{\Delta x \to 0} \frac{\Delta \cos(rx)}{\Delta x} &= \lim_{h \to 0} \frac{\cos(r(x+h)) - \cos(rx)}{h} \\ &= \lim_{h \to 0} \frac{\cos(rx)\cos(rh) - \sin(rx)\sin(rh) - \cos(rx)}{h} \\ &= \lim_{h \to 0} \frac{\cos(rx)(\cos(rh) - 1) - \sin(rx)\sin(rh)}{h} \\ &= \lim_{h \to 0} \frac{\cos(rx)(-r\sin(rh)) - \sin(rx) r\cos(rh)}{1} \\ &= \lim_{h \to 0} (-r\sin(rx)\cos(rh)) = -r\sin(rx)\,,\end{aligned}$$

as we were to show.

Supplementary Problem 1.65

We have

$$\begin{aligned}\lim_{\Delta x \to 0} \frac{\Delta b^x}{\Delta x} &= \lim_{h \to 0} \frac{b^{x+h} - b^x}{h} \\ &= b^x \lim_{h \to 0} \left(\frac{b^h - 1}{h} \right) \\ &= b^x \lim_{h \to 0} \left(\frac{\ln(b) b^h}{1} \right) = \ln(b) b^x\,.\end{aligned}$$

Here we have used L'Hôpital's rule and the derivative of b^h with respect to h.

Here we derived what the derivative of b^h with respect to h is. Starting with $y = b^h$ or $\ln(y) = h\ln(b)$ and by taking the h derivative of both sides we get $\frac{1}{y}\frac{dy}{dh} = \ln(b)$. Solving this for $\frac{dy}{dh}$ we get

$$\frac{dy}{dh} = y\ln(b) = \ln(b) b^h\,,$$

as we were to show.

Supplementary Problem 1.66

Note that $\Delta f(x) = f(x+h) - f(x)$ so that

$$\Delta^2 f(x) = f(x+2h) - f(x+h) - f(x+h) + f(x) = f(x+2h) - 2f(x+h) + f(x)\,,$$

which is the expression we are taking the limit of in the expression for $D^2 f(x)$.

Supplementary Problem 1.67

Because

$$D^2 f(x) = \lim_{\Delta \to 0} \frac{1}{(\Delta x)^2} \Delta^2 f(x) \,,$$

and an application of another "D" operator would mean that

$$D^3 f(x) = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \Delta \left(\frac{1}{(\Delta x)^2} \Delta^2 f(x) \right) = \lim_{\Delta x \to 0} \frac{1}{(\Delta x)^3} \Delta^3 f(x) \,.$$

This in tern means that

$$\begin{aligned} D^3 f(x) &= \lim_{h \to 0} \frac{1}{h^3} \Delta(f(x+2h) - 2f(x+h) + f(x)) \\ &= \lim_{h \to 0} \frac{f(x+3h) - 2f(x+2h) + f(x+h) - f(x+2h) + 2f(x+h) - f(x)}{h^3} \\ &= \lim_{h \to 0} \frac{f(x+3h) - 3f(x+2h) + 3f(x+h) - f(x)}{h^3} \,. \end{aligned}$$

Supplementary Problem 1.68

To work the parts of this problem recall that $\Delta x^{(r)} = r x^{(r-1)} h$.

Part (a): Using the above we have

$$\Delta(3x^{(5)} + 5x^{(4)} - 7x^{(2)} + 3x^{(1)} + 6) = (15x^{(4)} + 20x^{(3)} - 14x^{(1)} + 3)h \,.$$

Part (b): Using the above we have

$$\frac{\Delta}{\Delta x}(x^{(-3)} - 3x^{(-2)}) = \frac{(-3x^{(-4)} + 6x^{(-3)})h}{h} = -3x^{(-4)} + 6x^{(-3)} \,.$$

Part (c): Using the above we have

$$\Delta \left(\frac{x^{(2)} + x^{(-2)}}{2} \right) = \frac{1}{2}(2x^{(1)} - 2x^{(-3)})h = (x^{(1)} - x^{(-3)})h \,.$$

Supplementary Problem 1.69

Part (a): We have

$$\Delta(2x^{(-3)} - 3x^{(-2)} + 4x^{(2)}) = (-6x^{(-4)} + 6x^{(-3)} + 8x^{(1)})h \,,$$

and thus

$$\Delta^2(2x^{(-3)} - 3x^{(-2)} + 4x^{(2)}) = (24x^{(-5)} - 18x^{(-4)} + 8)h^2 \,.$$

Part (b): We have

$$\Delta(x^{(4)} + x^{(-4)}) = (4x^{(3)} - 4x^{(-5)})h\,,$$

and

$$\Delta^2(x^{(4)} + x^{(-4)}) = (12x^{(2)} + 20x^{(-6)})h^2\,,$$

and

$$\Delta^3(x^{(4)} + x^{(-4)}) = (24x^{(1)} - 120x^{(-7)})h^3\,.$$

Thus

$$\frac{\Delta^3(x^{(4)} + x^{(-4)})}{\Delta x^3} = \frac{\Delta^3(x^{(4)} + x^{(-4)})}{h^3} = 24x^{(1)} - 120x^{(-7)}\,.$$

Supplementary Problem 1.70

Part (a): As one method to solve this we can replace each monomial with its factorial polynomial expression and then simplifying. Doing this we would get

$$3x^2 - 5x + 2 = 3(x^{(2)} + x^{(1)}h) - 5x^{(1)} + 2 = 3x^{(2)} + (3h - 5)x^{(1)} + 2\,.$$

As another method to solve this problem we can express the factorial polynomial form of the polynomial and seek to determine its coefficients. For example we know that we can write

$$3x^2 - 5x + 2 = A_0x^{(2)} + A_1x^{(1)} + A_2 = A_0x(x - h) + A_1x + A_2\,.$$

If we take $x = 0$ then $A_2 = 2$. Putting this into the above and simplifying we get

$$3x^2 - 5x = A_0x(x - h) + A_1x\,.$$

If we divide this expression by x we get

$$3x - 5 = A_0(x - h) + A_1\,. \tag{3}$$

In this if we take $x = h$ we get

$$3h - 5 = A_1\,.$$

Putting this value of A_1 into Equation 3 gives

$$3x - 5 = A_0(x - h) + 3h - 5\,,$$

which simplifies to

$$3(x - h) = A_0(x - h)\,.$$

If we divide this expression by $x - h$ we get $A_0 = 3$.

All of this together means that we have shown

$$3x^2 - 5x + 2 = 3x^{(2)} + (3h - 5)x^{(1)} + 2\,.$$

Part (b): Writing this polynomial in terms of factorial monomials with unknown coefficients we would have

$$\begin{aligned}2x^4 + 5x^2 - 4x + 7 &= A_0 x^{(4)} + A_1 x^{(3)} + A_2 x^{(2)} + A_3 x^{(1)} + A_4 \\ &= A_0 x(x-h)(x-2h)(x-3h) + A_1 x(x-h)(x-2h) + A_2 x(x-h) + A_3 x + A_4 \,.\end{aligned}$$

Taking $x = 0$ we see that $A_4 = 7$. Using this value in the above, simplifying and then dividing by x we get

$$2x^3 + 5x - 4 = A_0(x-h)(x-2h)(x-3h) + A_1(x-h)(x-2h) + A_2(x-h) + A_3 \,.$$

If we let $x = h$ we get $A_3 = 2h^3 + 5h - 4$. Using this in the above we get

$$2x^3 + 5x = A_0(x-h)(x-2h)(x-3h) + A_1(x-h)(x-2h) + A_2(x-h) + 2h^3 + 5h \,, \quad (4)$$

or moving things to the left-hand-side

$$2(x^3 - h^3) + 5(x-h) = A_0(x-h)(x-2h)(x-3h) + A_1(x-h)(x-2h) + A_2(x-h) \,. \quad (5)$$

If we divide this by $x - h$ we get

$$2(x^2 + xh + h^2) + 5 = A_0(x-2h)(x-3h) + A_1(x-2h) + A_2 \,. \quad (6)$$

In this expression we now take $x = 2h$ to get

$$A_2 = 2(4h^2 + 2h^2 + h^2) + 5 = 14h^2 + 5 \,.$$

If we put this into Equation 6 we get

$$2x^2 + 2xh - 12h^2 = A_0(x-2h)(x-3h) + A_1(x-2h) \,.$$

Lets divide both sides by $x - 2h$. Then since the left-hand-side can be written as

$$2x^2 + 2hx - 12h^2 = (x-2h)(2x+6h) \,,$$

we get

$$2x + 6h = A_0(x-3h) + A_1 \,. \quad (7)$$

If we let $x = 3h$ we get $A_1 = 12h$. If we put that expression for A_1 back into Equation 7 we get

$$2(x-3h) = A_0(x-3h) \,,$$

This means that $A_0 = 2$.

Taking what we know about A_0, A_1, A_2, A_3, and A_4 we have shown that

$$2x^4 + 5x^2 - 4x + 7 = 2x^{(4)} + 12hx^{(3)} + (14h^2 + 5)x^{(2)} + (2h^3 + 5h - 4)x^{(1)} + 7 \,.$$

Supplementary Problem 1.71

Part (a): Using the definition of Δ we can write

$$\Delta(x^4 - 2x^2 + 5x - 3) = (x+h)^4 - x^4 - 2((x+h)^2 - x^2) + 5(x + h - x)\,.$$

Simplifying this into a polynomial in x we get

$$4hx^3 + 6h^2x^2 + (4h^3 - 4h)x + (h^4 - 2h^2 + 5h)\,.$$

If we divide by $\Delta x = h$ we get

$$\frac{\Delta}{\Delta x}(x^4 - 2x^2 + 5x - 3) = 4x^3 + 6hx^2 + (4h^2 - 4)x + (h^3 - 2h + 5)\,.$$

Part (b): As this is the second difference of the original polynomial in Part (a) taking the Δ operator on the result from the above gives

$$12hx^2 + 24h^2x + 14h^3 - 4h\,.$$

Then dividing this by $\Delta x = h$ gives

$$12x^2 + 24hx + 14h^2 - 4\,.$$

Supplementary Problem 1.72

Recall the definition

$$f(x)^{(m)} = f(x)f(x-h)f(x-2h)\cdots f(x-[m-1]h)\,. \tag{8}$$

Part (a): Using the above we have

$$\begin{aligned}(2x-1)^{(4)} &= (2x-1)(2(x-h)-1)(2(x-2h)-1)(2(x-3h)-1)\\ &= (2x-1)(2x-1-2h)(2x-1-4h)(2x-1-6h)\,.\end{aligned}$$

If $h = 2$ this is given by

$$(2x-1)^{(4)} = (2x-1)(2x-5)(2x-9)(2x-13)\,.$$

Part (b): Using the above we have

$$\begin{aligned}(3x+5)^{(3)} &= (3x+5)(3(x-h)+5)(3(x-2h)+5)\\ &= (3x+5)(3x+5-3h)(3x+5-6h)\,.\end{aligned}$$

If $h = 1$ this is given by

$$(3x+5)^{(3)} = (3x+5)(3x+2)(3x-1)\,.$$

For the next part of this problem recall that

$$f(x)^{(-m)} = \frac{1}{f(x+h)f(x+2h)\cdots f(x+mh)}\,. \tag{9}$$

Part (c): Using the above we have

$$(4x-5)^{(-2)} = \frac{1}{(4(x+h)-5)(4(x+2h)-5)} = \frac{1}{(4x-5+4h)(4x-5+8h)}\,.$$

If $h = 1$ this is

$$\frac{1}{(4x-1)(4x+3)}\,.$$

Part (d): Using the above we have

$$(5x+2)^{(-4)} = \frac{1}{(5(x+h)+2)(5(x+2h)+2)(5(x+3h)+2)(5(x+4h)+2)}$$

If $h = 2$ this is

$$\frac{1}{(5x+12)(5x+22)(5x+32)(5x+42)}\,.$$

Supplementary Problem 1.73

Recall that

$$f(x)^{(m)} = f(x)f(x-h)f(x-2h)\cdots f(x-[m-1]h)\,,$$

and thus

$$\begin{aligned}(ax+b)^{(m)} &= (ax+b)(a(x-h)+b)(a(x-2h)+b)\cdots(a(x-[m-1]h)+b)\\ &= (ax-b)(ax+b-ah)(ax+b-2ah)\cdots(ax+b-[m-1]ah)\,.\end{aligned} \tag{10}$$

From this we see that the right-hand-side has m factors.

Part (a): Notice that we have three factors so $m = 3$ and that we can write this product as

$$\begin{aligned}(3x-2)(3x+5)(3x+12) &= (3x+12)(3(x-h)+12)(3(x-2h)+12)\\ &= (3x+12)(3x+12-3h)(3x+12-6h)\,.\end{aligned}$$

To have this expression have a *second* factor that matches the books given expression we would need to have $12 - 3h = 5$ so $h = \frac{7}{3}$.

We can also check that when $h = \frac{7}{2}$ we have $12 - 6h = -2$ which will means that the third factor matches as well. Thus the given expression is equal to $(3x+12)^{(3)}$ with $h = \frac{7}{2}$.

Part (b): If we write the products from the largest value to the smallest value we get

$$(11+2x)(8+2x)(5+2x)(2+2x)\,.$$

Note that there are $m = 4$ products in the above. From the largest factor we take

$$ax + b = 11 + 2x\,,$$

thus we may take $a = 2$ and $b = 11$. From the smallest factor we take

$$2 + 2x = ax + b - [m-1]ah = 2x + 11 - [4-1](2)h\,.$$

Solving this we find $h = \frac{3}{2}$. Thus we have shown that this is $(11+2x)^{(4)}$ with $h = \frac{3}{2}$.

Part (c): Recall that

$$(ax+b)^{(-m)} = \frac{1}{(ax+b+ah)(ax+b+2ah)\cdots(ax+b+mah)}\,.$$

Note that the denominator of the above has m factors and they are ordered in "increasing" order.

Now considering

$$\frac{1}{x(x+2)(x+4)}\,.$$

Thus to match the "smallest" factor we need to have $ax + b + ah = x$ so we will take $a = 1$ and then must have $b + h = 0$. Then to match the "largest" factor we also need to have

$$ax + b + mah = x + 4\,,$$

or

$$b + 3h = 4\,.$$

Thus as $b = -h$ this means that $h = 2$ and so $b = -2$. All of this taken together mean that the above is given by $(x-2)^{(-3)}$ with $h = 2$.

Part (d): With

$$\frac{1}{(2x-1)(2x+3)(2x+7)(2x+11)}\,,$$

we have $m = 4$ factors. From the "smallest" factor we will need to have $ax+b+ah = 2x-1$ so that we should have $a = 2$ and

$$b + 2h = -1 \quad \text{so} \quad b = -1 - 2h\,.$$

The "largest" factor means that

$$ax + b + mah = 2x + 11\,,$$

or

$$-1 - 2h + 4(2)h = 11\,,$$

which means that $h = 2$ and then $b = -1 - 4 = -5$. Thus this expression is

$$(2x-5)^{(-4)}\,,$$

with $h = 2$.

Supplementary Problem 1.74

Part (a): Now if $m \geq 0$ we have

$$\begin{aligned}
\Delta(px+q)^{(m)} &= \Delta[(px+q)(p(x-h)+q)(p(x-2h)+q)\cdots(p(x-[m-1]h)+q)] \\
&= [(p(x+h)+q)(px+q)(p(x-h)+q)\cdots(p(x-[m-2]h)+q)] \\
&- [(px+q)(p(x-h)+q)(p(x-2h)+q)\cdots(p(x-[m-1]h)+q)] \\
&= [(p(x+h)+q](px+q)^{(m-1)} - (px+q)^{(m-1)}[p(x-[m-1]h)+q] \\
&= (px+q)^{(m-1)}[px+ph+q-px+p(m-1)h-q] \\
&= pmh(px+q)^{(m-1)}\,.
\end{aligned}$$

This means that as $\Delta x = h$ we have

$$\frac{\Delta(px+q)^{(m)}}{\Delta x} = pm(px+q)^{(m-1)}\,,$$

as we were to show.

Part (b): Now if $m \leq -1$ then

$$(px+q)^{(-m)} = \frac{1}{(p(x+h)+q)(p(x+2h)+q)(p(x+3h)+q)\cdots(p(x+mh)+q)}\,.$$

This means that

$$\begin{aligned}
\Delta(px+q)^{(-m)} &= \frac{1}{(p(x+2h)+q)(p(x+3h)+q)\cdots(p(x+[m+1]h)+q)} \\
&- \frac{1}{(p(x+h)+q)(p(x+2h)+q)(p(x+3h)+q)\cdots(p(x+mh)+q)}\,.
\end{aligned}$$

This is the fraction

$$\frac{1}{(p(x+2h)+q)(p(x+3h)+q)\cdots(p(x+mh)+q)}\,,$$

multiplied by the expression

$$\frac{1}{p(x+[m+1]h)+q} - \frac{1}{p(x+h)+q} = \frac{-pmh}{(p(x+[m+1]h)+q)(p(x+h)+q)}\,.$$

The total expression for $\Delta(px+q)^{(-m)}$ is then a fraction with a numerator $-pmh$ and a denominator that is the product

$$(p(x+h)+q)(p(x+2h)+q)\cdots(p(x+mh)+q)(p(x+[m+1]h)+q)\,.$$

Notice that

$$\frac{1}{(p(x+h)+q)(p(x+2h)+q)\cdots(p(x+mh)+q)(p(x+[m+1]h)+q)} = (px+q)^{(-(m+1))}\,,$$

and thus we have shown that

$$\Delta(px+q)^{(-m)} = -pmh(px+q)^{(-(m+1))}\,.$$

This means that as $\Delta x = h$ we have

$$\frac{\Delta(px+q)^{(-m)}}{\Delta x} = -pm(px+q)^{(-(m+1))}\,,$$

as we were to show.

Supplementary Problem 1.75

Part (a): There are probably several ways to write this but one way is to recognize that it is $(x^2-1)x^{(-3)}$ with $h = 2$.

Part (b): In working this problem I'll assume that the last factor in the denominator is $2x+7$ rather than $2x+9$. Now there are probably several ways to write this but if we desire to have

$$\frac{1}{(2(x+h)+1)(2(x+2h)+1)(2(x+3h)+1)} = \frac{1}{(2x+3)(2x+5)(2x+7)}\,,$$

we would want to have

$$\begin{aligned} 2h+1 &= 3 \\ 4h+1 &= 5 \\ 6h+1 &= 7\,. \end{aligned}$$

This can be made true if $h = 1$. In that case the given expression can be written as

$$(2x+1)(2x+1)^{(-3)}\,.$$

Supplementary Problem 1.76

For this problem we will use

$$s_k^{n+1} = s_{k-1}^n - n s_k^n\,, \tag{11}$$

with the "boundary" conditions that $s_n^n = 1$ and $s_k^n = 0$ for $k \le 0$ and $k \ge n+1$.

If we take $n = 1$ then we have $s_n^n = s_1^1 = 1$ which we can check is correct by noting that

$$x^{(1)} = s_1^1 x^1 h^{1-1} = x^1\,,$$

which is true.

Next if we use Equation 11 with $n = 1$ and $k = 1$ we get

$$s_1^2 = s_0^1 - 1s_1^1 = -1 .$$

If we use Equation 11 with $n = 1$ and $k = 2$ we get

$$s_2^2 = s_1^1 - 1s_2^1 = 1 .$$

We can check this is correct by considering

$$x^{(2)} = s_1^2 x^1 h^{2-1} + s_2^2 x^2 h^{2-2} = -x^1 h + x^2 = x^2 - xh = x(x - h) ,$$

which is true.

Next if we use Equation 11 with $n = 2$ and $k = 1$ we get

$$s_1^3 = s_0^2 - 2s_1^2 = 0 - 2(-1) = 2 .$$

If we use Equation 11 with $n = 2$ and $k = 2$ we get

$$s_2^3 = s_1^2 - 2s_2^2 = -1 - 2(1) = -3 .$$

If we use Equation 11 with $n = 2$ and $k = 3$ we get

$$s_3^3 = s_2^2 - 2s_3^2 = 1 - 0 = 1 .$$

We can check this is correct by considering

$$\begin{aligned} x^{(3)} &= 2x^1 h^2 - 3x^2 h^1 + 1x^3 \\ &= x^3 - 3hx^2 + 2h^2 x \\ &= x(x^2 - 3hx + 2h^2) \\ &= x(x - h)(x - 2h) , \end{aligned}$$

which is true.

Next if we use Equation 11 with $n = 3$ and $k = 1$ we get

$$s_1^4 = s_0^3 - 3s_1^3 = 0 - 3(2) = -6 .$$

If we use Equation 11 with $n = 3$ and $k = 2$ we get

$$s_2^4 = s_1^3 - 3s_2^3 = 2 - 3(-3) = 11 .$$

If we use Equation 11 with $n = 3$ and $k = 3$ we get

$$s_3^4 = s_2^3 - 3s_3^3 = -3 - 3(1) = -6 .$$

If we use Equation 11 with $n = 3$ and $k = 4$ we get

$$s_4^4 = s_3^3 - 3s_4^3 = 1 - 0 = 1 .$$

We can check this is correct by considering

$$\begin{aligned} x^{(4)} &= s_1^4 x^1 h^3 + s_2^4 x^2 h^2 + s_3^4 x^3 h^1 + s_4^4 x^4 h^0 \\ &= -6xh^3 + 11x^2 h^2 - 6x^3 h + x^4 \\ &= x(x^3 - 6hx^2 + 11h^2 x - 6h^3) \\ &= x(x - h)(x^2 - 5hx + 6h^2) \\ &= x(x - h)(x - 2h)(x - 3h) , \end{aligned}$$

which is true.

Supplementary Problem 1.77

We will start with Eq. 32 or

$$S_k^{n+1} = S_{k-1}^n + kS_k^n \,, \tag{12}$$

where $S_n^n = 1$ and $S_k^n = 0$ when $k \leq 0$ or $k \geq n+1$.

Now starting with $n = 1$ we have $S_1^1 = 1$. We can check that this is correct by considering

$$x^1 = S_1^1 x^{(1)} h^{1-1} = x^{(1)} \,,$$

which is true.

If we take $n = 1$ in Equation 12 then for $k = 1$ this is

$$S_1^2 = S_0^1 + 1S_1^1 = 1 \,.$$

For $k = 2$ this is

$$S_2^2 = S_1^1 + 2S_2^1 = 1 + 0 = 1 \,.$$

We can check that these are correct by computing

$$\begin{aligned} x^2 &= S_1^2 x^{(1)} h^{2-1} + S_2^2 x^{(2)} h^0 \\ &= xh + x^{(2)} = xh + x(x-h) = x^2 \,, \end{aligned}$$

which is true.

If we take $n = 2$ in Equation 12 then for $k = 1$ this is

$$S_1^3 = S_0^2 + 1S_1^2 = 1 \,.$$

For $k = 2$ this is

$$S_2^3 = S_1^2 + 2S_2^2 = 1 + 2(1) = 3 \,.$$

For $k = 3$ this is

$$S_3^3 = S_2^2 + 3S_3^2 = 1 \,.$$

We can check that these are correct by computing

$$\begin{aligned} x^3 &= S_1^3 x^{(1)} h^2 + S_2^3 x^{(2)} h^1 + S_3^3 x^{(3)} h^0 \\ &= xh^2 + 3x^{(2)} h + x^{(3)} \\ &= xh^2 + 3x(x-h)h + x(x-h)(x-2h) \\ &= xh^2 + 3x^2 - 3xh^2 + (x^2 - hx)(x-2h) \\ &= 3hx^2 - 2h^2 x + x^3 - 2hx^2 - hx^2 + 2h^2 x = x^3 \,, \end{aligned}$$

which is true.

Supplementary Problem 1.78

The method of Problem 1.25 is to use "long division" to express the polynomial x^6 in terms of factorial polynomials $x^{(k)}$ as

$$x^n = \sum_{k=1}^{n} S_k^n x^{(k)} h^{n-k} \,. \tag{13}$$

In a format similar to that presented in the book we have

	x^6	x^5	x^4	x^3	x^2	x	1
1	1	0	0	0	0	0	0
		1	1	1	1	1	
2	1	1	1	1	1	1	
		2	6	14	30		
3	1	3	7	15	31		
		3	18	75			
4	1	6	25	90			
		4	40				
5	1	10	65				
		5					
6	1	15					

The last element in each "row" is S_k^6 for $1 \le k \le 6$. Reading these elements we have

$$x^6 = x^{(6)} + 15x^{(5)} + 65x^{(4)} + 90x^{(3)} + 31x^{(2)} + x^{(1)} \,.$$

Supplementary Problem 1.79

Note that the definition of the Stirling numbers of the first kind satisfy

$$x^{(n)} = \sum_{k=1}^{n} s_k^n x^k \,.$$

When $h = 1$ and we take $x = 1$ then as $x^{(n)} = 0$ if $n \ge 2$ the left-hand-side of the above is zero and we have

$$0 = \sum_{k=1}^{n} s_k^n \,,$$

as we were to show. Note that looking at the numbers s_k^n in Appendix A summing "across rows" gives zero.

Supplementary Problem 1.80

Part (a): From the definition of the Stirling numbers of the first kind we have

$$x^{(n)} = \sum_{k=1}^{n} s_k^n x^k \,,$$

where

$$x^{(n)} = x(x-1)(x-2)\cdots(x-[n-1])\,.$$

If we take $x = -1$ then the left-hand-side of the above is

$$(-1)^{(n)} = (-1)(-2)(-3)\cdots(-(n-1))(-n) = (-1)^n n!\,,$$

while the right-hand-side is

$$\sum_{k=1}^{n} s_k^n (-1)^k = -s_1^n + s_2^n - s_3^n + \cdots + (-1)^n s_n^n\,.$$

If we set these two expressions equal to each other and multiply everything by -1 we get

$$s_1^n - s_2^n + s_3^n - \cdots + (-1)^{n-1} s_n^n = (-1)^{n-1} n!\,, \tag{14}$$

as we were to show.

Part (b): Looking at the numbers in Appendix A we might form the hypothesis that if n is odd then

$$\begin{aligned} s_k^n &< 0 \quad \text{for } k \text{ even} \\ s_k^n &> 0 \quad \text{for } k \text{ odd}\,, \end{aligned}$$

while if n is even then

$$\begin{aligned} s_k^n &< 0 \quad \text{for } k \text{ odd} \\ s_k^n &> 0 \quad \text{for } k \text{ even}\,. \end{aligned}$$

Then in both cases Equation 14 will simplify to the desired expression.

Supplementary Problem 1.81

To start we will quote the Gregory-Newton formula

$$f(x) = f(a) + \frac{\Delta f(a)}{\Delta x}\frac{(x-a)^{(1)}}{1!} + \frac{\Delta^2 f(a)}{\Delta x^2}\frac{(x-a)^{(2)}}{2!} + \cdots + \frac{\Delta^n f(a)}{\Delta x^n}\frac{(x-a)^{(n)}}{n!} + R_n\,. \tag{15}$$

Part (a): If we have $f(x) = 3x^2 - 5x + 2$ with $h = 1$ and $a = 0$ then $f(a) = 2$ and we compute

$$\begin{aligned} \Delta f(x) &= 3(x+h)^2 - 5(x+h) - 3x^2 + 5x \\ &= 3(x^2 + 2xh + h^2) - 3x^2 - 5x - 5h + 5x = 6xh + 3h^2 - 5h\,. \end{aligned}$$

This means that $\Delta f(a) = 3h^2 - 5h$. Next we compute

$$\begin{aligned}
\Delta^2 f(x) &= 6h(x+h) - 6hx = 6hx + 6h^2 - 6hx = 6h^2 \\
\Delta^2 f(a) &= 6h^2 .
\end{aligned}$$

Finally we have $\Delta^3 f(x) = 0$. Using all of these in the Gregory-Newton formula we get

$$\begin{aligned}
f(x) &= 2 + \frac{(3h^2 - 5h)}{h}\frac{x^{(1)}}{1!} + \frac{6h^2}{h^2}\frac{x^{(2)}}{2!} + 0 \\
&= 2 + (3h-5)x^{(1)} + 3x^{(2)} .
\end{aligned}$$

If $h = 1$ the above is

$$f(x) = 2 - 2x^{(1)} + 3x^{(2)} .$$

We can check that the above is correct by expanding the right-hand-side as

$$2 - 2x + 3x(x-1) = 2 - 2x + 3x^2 - 3x = 2 - 5x + 3x^2 .$$

Part (b): Next for $f(x) = 2x^4 + 5x^2 - 4x + 7$ with $h = 1$ and $a = 0$ we first evaluate that $f(a) = -7$. Next we compute

$$\begin{aligned}
\Delta f(x) &= 2[(x+h)^4 - x^4] + 5[(x+h)^2 - x^2] - 4h \\
&= 2[4x^3h + 6x^2h^2 + 4xh^3 + h^4] + 5[2hx + h^2] - 4h \\
&= 8hx^3 + 12h^2x^2 + (8h^3 + 10h)x + 2h^4 + 5h^2 - 4h .
\end{aligned}$$

This means that

$$\Delta f(a) = 2h^4 + 5h^2 - 4h .$$

Next we compute

$$\begin{aligned}
\Delta^2 f(x) &= 8h((x+h)^3 - x^3) + 12h^2((x+h)^2 - x^2) + (8h^3 + 10h)h \\
&= 8h(3x^2h + 3xh^2 + h^3) + 12h^2(2hx + h^2) + 8h^4 + 10h^2 \\
&= 24h^2x^2 + 24h^3x + 8h^4 + 24h^3x + 12h^4 + 8h^4 + 10h^2 \\
&= 24h^2x^2 + 48h^3x + 28h^4 + 10h^2 .
\end{aligned}$$

This means that

$$\Delta^2 f(a) = 28h^4 + 10h^2 .$$

Next we compute

$$\Delta^3 f(x) = 24h^2(2xh + h^2) + 48h^3(h) = 48h^3x + 72h^4 .$$

This means that

$$\Delta^3 f(a) = 72h^4 .$$

Next we compute

$$\Delta^4 f(x) = 48h^4 .$$

Which means that

$$\Delta^4 f(a) = 48h^4 \,.$$

Finally we have $\Delta^5 f(x) = 0$.

All of this taken together means that we can write $f(x)$ using the Gregory-Newton formula as

$$\begin{aligned}
f(x) &= 7 + \left(\frac{2h^4 + 5h^2 - 4h}{h}\right)\left(\frac{x^{(1)}}{1!}\right) + \left(\frac{28h^4 + 10h^2}{h^2}\right)\left(\frac{x^{(2)}}{2!}\right) + \left(\frac{72h^4}{h^3}\right)\left(\frac{x^{(3)}}{3!}\right) + \left(\frac{48h^4}{h^4}\right)\left(\frac{x^{(4)}}{4!}\right) + 0 \\
&= 7 + (2h^3 + 5h - 4)x^{(1)} + (14h^2 + 5)x^{(2)} + 12hx^{(3)} + 2x^{(4)} \,.
\end{aligned}$$

Lets verify that this is correct. If we call the right-hand-side of the above RHS then we can expand the factorial monomials $x^{(m)}$ to get

$$\begin{aligned}
\text{RHS} &= 7 + (2h^3 + 5h - 4)x + (14h^2 + 5)x(x - h) + 12hx(x - h)(x - 2h) + 2x(x - h)(x - 2h)(x - 3h) \\
&= 7 + (2h^3 + 5h - 4)x + (14h^2 + 5)x(x - h) + x(x - h)(x - 2h)[12h + 2x - 6h] \\
&= 7 + (2h^3 + 5h - 4)x + (14h^2 + 5)x(x - h) + x(x - h)(x - 2h)(6h + 2x) \\
&= 7 + (2h^3 + 5h - 4)x + x(x - h)[14h^2 + 5 + 6hx - 12h^2 + 2x^2 - 4hx] \\
&= 7 + (2h^3 + 5h - 4)x + x(x - h)[2x^2 + 2hx + 2h^2 + 5] \\
&= 7 + x[2h^3 + 5h - 4 + 2x^3 + 2hx^2 + 2h^2x + 5x - 2hx^2 - 2h^2x - 2h^3 - 5h] \\
&= 7 + x[-4 + 2x^3 + 5x] = 2x^4 + 5x^2 - 4x + 7 \,,
\end{aligned}$$

as expected.

Supplementary Problem 1.82

From the form of R_n we see that $R_n \equiv 0$ if $f(x)$ is a function such that $f^{(n+1)}(x) = 0$ for all x. This means that $f(x)$ must be a nth order polynomial.

Supplementary Problem 1.83

To start we will assume that $h \neq 1$ and $a \neq 0$ with $x = a+h$. Then using symbolic operations we have

$$\begin{aligned}
f(x) &= f(a + h) = E^h f(a) = (1 + \Delta)^h f(a) \\
&= \left(1 + h\Delta + \frac{h(h-1)}{2!}\Delta^2 + \frac{h(h-1)(h-2)}{3!}\Delta^3 + \cdots\right) f(a) \\
&= 1 + h\Delta f(a) + \frac{h(h-1)}{2!}\Delta^2 f(a) + \frac{h(h-1)(h-2)}{3!}\Delta^3 f(a) + \dots \\
&= 1 + \Delta f(a)h^{(1)} + \frac{\Delta^2 f(a)}{2!}h^{(2)} + \frac{\Delta^3 f(a)}{3!}h^{(3)} + \dots \,.
\end{aligned}$$

From the definition of x we have that $h = x - a$ so the above becomes

$$f(x) = 1 + \Delta f(a)(x-a)^{(1)} + \frac{\Delta^2 f(a)}{2!}(x-a)^{(2)} + \frac{\Delta^3 f(a)}{3!}(x-a)^{(3)} + \dots\,, \tag{16}$$

which is the **Gregory-Newton** formula.

Supplementary Problem 1.84

Now since $f(x) \equiv a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n$ is $O(x^n)$ polynomial if we were to write it using the Gregory-Newton formula Equation 16 we will have a_0 be the coefficient of the $x^{(n)}$ factor monomial. From the Gregory-Newton formula Equation 16 itself this means that

$$\frac{\Delta^n f(x)}{\Delta x^n}\frac{1}{n!} = a_0\,.$$

Thus

$$\Delta^n[a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n] = n!a_0h^n\,,$$

which is what we wanted to show.

Supplementary Problem 1.85

If we take the limit as $h \to 0$ on both sides of the Gregory-Newton formula Equation 16 then because

$$\lim_{h\to 0}\frac{\Delta^n f(a)}{\Delta x^n} = \frac{d^n}{dx^n}f(a)\,,$$

and

$$\lim_{h\to 0} x^{(n)} = x^n\,,$$

we get

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a)^1 + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n\,,$$

with R_n given by

$$R_n = \frac{f^{(n+1)}(\xi)(x-a)^{n+1}}{(n+1)!}\,.$$

This is the Taylor series expansion.

Supplementary Problem 1.87

We will use Leibnitz's rule to evaluate

$$\Delta^3(x^2 \cdot 2^x)\,.$$

Recall Leibniz's rule for $n = 3$ for the differences of a product of two function fg is given by

$$\Delta^3(fg) = f\Delta^3 g + \left(\begin{array}{c} 3 \\ 1 \end{array} \right) \Delta f(\Delta^2 Eg) + \left(\begin{array}{c} 3 \\ 2 \end{array} \right) \Delta f(\Delta E^2 g) + \left(\begin{array}{c} 3 \\ 3 \end{array} \right) \Delta^3 f(E^3 g)\,.$$

For this problem we will take $f(x) = x^2$ and $g(x) = 2^x$. Then to use the above we need to compute differences of $f(x) = x^2$ as

$$\begin{aligned} \Delta x^2 &= 2hx + h^2 \\ \Delta^2 x^2 &= 2h^2 \\ \Delta^3 x^2 &= 0\,. \end{aligned}$$

Also we need to compute differences of $g(x) = 2^x$ as

$$\begin{aligned} \Delta 2^x &= 2^{x+h} - 2^x = 2^x(2^h - 1) \\ \Delta^2 2^x &= 2^x(2^h - 1)^2 \\ \Delta^3 2^x &= 2^x(2^h - 1)^3\,. \end{aligned}$$

This means that

$$\Delta^3(x^2 2^x) = x^2 2^x(2^h - 1)^3 + 3(2hx + h^2)2^{x+h}(2^h - 1)^2 + 3(2h^2)2^{x+2h}(2^h - 1) + 1(0)\,.$$

If $h = 1$ this becomes

$$\begin{aligned} \Delta^3(x^2 2^x) &= x^2 2^x + 3(2x + 1)2^{x+1} + 6 \cdot 2^{x+2} = x^2 2^x + 6(2x + 1)2^x + 24 \cdot 2^x \\ &= (x^2 + 12x + 30)2^x\,. \end{aligned}$$

Supplementary Problem 1.88

Leibniz's rule in this case is given by

$$\begin{aligned} \Delta^n(xa^x) &= x\Delta^n a^x + \left(\begin{array}{c} n \\ 1 \end{array} \right) \Delta x(\Delta^{n-1} E a^x) + \left(\begin{array}{c} n \\ 2 \end{array} \right) \Delta^2 x(\Delta^{n-2} E^2 a^x) + \cdots \\ &= xa^x(a^h - 1)^n + nh\Delta^{n-1} a^{x+h} + 0 \\ &= xa^x(a^h - 1)^n + nha^h a^x(a^h - 1)^{n-1} \\ &= xa^x(a^h - 1)^n + nh(a^h - 1)^{n-1} a^{x+h}\,. \end{aligned}$$

Here we have used the fact that $\Delta^n x = 0$ for $n \geq 2$.

Supplementary Problem 1.89

Leibniz's rule in this case is given by

$$\Delta^n(x^2 a^x) = x^2 \Delta^n a^x + \left(\begin{array}{c} n \\ 1 \end{array} \right) \Delta x^2(\Delta^{n-1} E a^x) + \left(\begin{array}{c} n \\ 2 \end{array} \right) \Delta^2 x^2(\Delta^{n-2} E^2 a^x) + 0\,.$$

Here we have used the fact that $\Delta^n x^2 = 0$ for $n \geq 3$. Next we compute

$$\begin{aligned}
\Delta x^2 &= 2xh + h^2 \\
\Delta^2 x^2 &= 2h^2 \\
\Delta^3 x^2 &= 0\,.
\end{aligned}$$

This means that we have

$$\begin{aligned}
\Delta^n(x^2 a^x) &= x^2 a^x (a^h - 1)^n + n(2xh + h^2) a^h (a^h - 1)^{n-1} a^x + \frac{n(n-1)}{2}(2h^2) a^{2h} (a^h - 1)^{n-2} a^x \\
&= x^2 a^x (a^h - 1)^n + (2xh + h^2) n (a^h - 1)^{n-1} a^{x+h} + h^2 n(n-1) a^{2h} (a^h - 1)^{n-2} a^x \\
&= (a^h - 1)^n x^2 a^x + 2nh(a^h - 1)^{n-1} x a^{x+h} \\
&+ h^2 n (a^h - 1)^{n-1} a^{x+h} + h^2 n(n-1) a^{2h} (a^h - 1)^{n-2} a^x \\
&= (a^h - 1)^n x^2 a^x + 2nh(a^h - 1)^{n-1} x a^{x+h} + nh^2 (a^h - 1)^{n-2} a^{x+h} [(a^h - 1) + (n-1) a^h] \\
&= (a^h - 1)^n x^2 a^x + 2nh(a^h - 1)^{n-1} x a^{x+h} + nh^2 (a^h - 1)^{n-2} a^{x+h} (n a^h - 1)\,.
\end{aligned}$$

Supplementary Problem 1.90

If we write down Leibnitz's rule for differences and divide it by $\Delta x^n = h^n$ and then take the limit of both sides as $h \to 0$ we get Leibnitz's rule for derivatives.

Supplementary Problem 1.91

Part (a): Recall that $\nabla f(x) = f(x) - f(x-h)$ and thus

$$\begin{aligned}
\nabla f(x) &= (2x^2 + 3x - 5) - (2(x-h)^2 + 3(x-h) - 5) \\
&= 2[x^2 - (x-h)^2] + 3[x - (x-h)] \\
&= 2[2hx - h^2] + 3h = 4hx - 2h^2 + 3h\,.
\end{aligned}$$

Part (b): Recall that $\delta f(x) = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)$ and thus

$$\begin{aligned}
\delta f(x) &= 2\left[\left(x + \frac{h}{2}\right)^2 - \left(x - \frac{h}{2}\right)^2\right] + 3\left[\left(x + \frac{h}{2}\right) - \left(x - \frac{h}{2}\right)\right] \\
&= 2\left[x^2 + hx + \frac{h^2}{4} - x^2 + hx - \frac{h^2}{4}\right] + 3h \\
&= 4hx + 3h\,.
\end{aligned}$$

Part (c): Using the above expression for $\nabla f(x)$ we have

$$\begin{aligned}
\nabla^2 f(x) &= \nabla(\nabla f(x)) = \nabla(4hx - 2h^2 + 3h) \\
&= 4hx - 4h(x-h) = 4h^2\,.
\end{aligned}$$

Part (d): Using the above expression for $\delta f(x)$ we have

$$\delta^2 f(x) = \delta(\delta f(x)) = \delta(4hx + 3h) = 4h\left[\left(x + \frac{h}{2}\right) - \left(x - \frac{h}{2}\right)\right]$$
$$= 4h(h) = 4h^2\,.$$

Supplementary Problem 1.92

We want to evaluate

$$(\nabla^2 - 3\nabla\delta + 2\delta^2)(x^2 + 2x)\,.$$

To do that first we evaluate

$$\begin{aligned}\nabla(x^2 + 2x) &= x^2 + 2x - (x-h)^2 - 2(x-h)\\ &= x^2 + 2x - (x^2 - 2hx + h^2) - 2x + 2h\\ &= 2hx - h^2 + 2h\,.\end{aligned}$$

Now using this result we evaluate

$$\nabla^2(x^2 + 2x) = 2hx - (2h(x-h)) = 2h^2\,.$$

Next we compute

$$\begin{aligned}\delta(x^2 + 2x) &= \left(x + \frac{h}{2}\right)^2 + 2\left(x + \frac{h}{2}\right) - \left(x - \frac{h}{2}\right)^2 - 2\left(x - \frac{h}{2}\right)\\ &= \left(x^2 + hx + \frac{h^2}{4}\right) + 2\left(x + \frac{h}{2}\right) - \left(x^2 - hx + \frac{h^2}{4}\right) - 2\left(x - \frac{h}{2}\right)\\ &= 2hx + 2h\,.\end{aligned}$$

Using this result we have

$$\delta^2(x^2 + 2x) = 2h\left(x + \frac{h}{2}\right) - 2h\left(x - \frac{h}{2}\right) = h^2 + h^2 = 2h^2\,.$$

In addition we have

$$\nabla\delta(x^2 + 2x) = \nabla(2hx + 2h) = 2hx - 2h(x-h) = 2h^2\,.$$

Using these "parts" we can compute

$$(\nabla^2 - 3\nabla\delta + 2\delta^2)(x^2 + 2x) = 2h^2 - 3(2h^2) + 2(2h^2) = 0\,.$$

Supplementary Problem 1.93

Part (a): We desire to prove that

$$\nabla^2 = (\Delta E^{-1})^2 = \Delta^2 E^{-2} .$$

To start that process, first recall that $\nabla f(x) = f(x) - f(x-h)$ so that the left-hand-side of the above can be expressed as

$$\nabla^2 f(x) = f(x) - f(x-h) - (f(x-h) - f(x-2h)) = f(x) - 2f(x-h) + f(x-2h) .$$

Next consider

$$\begin{aligned}(\Delta E^{-1})^2 f(x) &= (\Delta E^{-1})(\Delta E^{-1}) f(x) = (\Delta E^{-1})\Delta f(x-h) \\ &= (\Delta E^{-1})(f(x) - f(x-h)) = \Delta(f(x-h) - f(x-2h)) \\ &= f(x) - f(x-h) - (f(x-h) - f(x-2h)) \\ &= f(x) - 2f(x-h) + f(x-2h) ,\end{aligned}$$

which is the same expression we derived for $\nabla^2 f(x)$ showing that $\nabla^2 = (\Delta E^{-1})^2$.

Finally lets evaluate

$$\begin{aligned}\Delta^2 E^{-2} f(x) &= \Delta^2 f(x-2h) \\ &= \Delta(f(x-h) - f(x-2h)) = f(x) - f(x-h) - (f(x-h) - f(x-2h)) \\ &= f(x) - 2f(x-h) + f(x-2h) ,\end{aligned}$$

which is the same expression we derived for $\nabla^2 f(x)$ and $(\Delta E^{-1})^2 f(x)$.

Part (b): We have shown above that

$$\nabla^n = \Delta^n E^{-n} , \tag{17}$$

for $n = 2$. For $n = 1$ note that

$$\nabla f(x) = f(x) - f(x-h) ,$$

while

$$\Delta E^{-1} f(x) = \Delta f(x-h) = f(x) - f(x-h) ,$$

which equals $\nabla f(x)$ from above. This shows that $\nabla = \Delta E^{-1}$ which is Equation 17 for $n = 1$. Thus Equation 17 holds for $1 \leq n \leq 2$.

Perhaps a simpler way to show some of the above is to note that

$$\begin{aligned}\nabla f(x) &= f(x) - f(x-h) \\ &= f(x-h+h) - f(x-h) \\ &= E^{-1}(f(x+h) - f(x)) \\ &= E^{-1}\Delta f(x) ,\end{aligned}$$

and that

$$\nabla f(x) = f(x) - f(x-h) = \Delta f(x-h) = \Delta E^{-1} f(x)\,.$$

Thus we have that

$$\nabla = E^{-1}\Delta = \Delta E^{-1}\,.$$

Then to prove Equation 17 for general n we will use induction and some of the results above. We have

$$\begin{aligned}
\nabla^{n+1} &= \nabla\nabla^n = \nabla(\Delta^n E^{-n}) \\
&= (\Delta E^{-1})(\Delta^n E^{-n}) = \Delta\Delta^n E^{-1} E^{-n} \\
&= \Delta^{n+1} E^{-(n+1)}\,,
\end{aligned}$$

as we were to show. Here we have used the fact that $E^{-1}\Delta^n = \Delta^n E^{-1}$ which we can get by applying the identity $E^{-1}\Delta = \Delta E^{-1}$ multiple (n) times.

Supplementary Problem 1.94

Evaluating $\nabla\delta f(x)$ we have

$$\begin{aligned}
\nabla\delta f(x) &= \nabla\left(f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)\right) \\
&= f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right) - \left(f\left(x-\frac{h}{2}\right) - f\left(x-\frac{3h}{2}\right)\right) \\
&= f\left(x+\frac{h}{2}\right) - 2f\left(x-\frac{h}{2}\right) + f\left(x-\frac{3h}{2}\right)\,.
\end{aligned}$$

Evaluating $\delta\nabla f(x)$ we have

$$\begin{aligned}
\delta\nabla f(x) &= \delta(f(x) - f(x-h)) \\
&= f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right) - \left(f\left(x-\frac{h}{2}\right) - f\left(x-\frac{3h}{2}\right)\right) \\
&= f\left(x+\frac{h}{2}\right) - 2f\left(x-\frac{h}{2}\right) + f\left(x-\frac{3h}{2}\right)\,.
\end{aligned}$$

As we see that $\nabla\delta f(x) = \delta\nabla f(x)$ we have that the two operators are commutable.

Supplementary Problem 1.95

We are asked to prove that

$$E = \left(\frac{\delta}{2} + \sqrt{1 + \frac{\delta^2}{4}}\right)^2\,.$$

To start this process we will manipulate this expression some to "remove" the square root in favor of an identity that involves only products of what I'll call "simple" operators. Note that since E is an increment operator factors like E^{α} are not really complicated since $E^{\alpha} f(x) = f(x+\alpha)$. This is different from operators like

$$\sqrt{1+\frac{\delta^2}{4}}\,,$$

which would have to be expressed in their infinite Taylor series expansion as a function of δ^2.

With that motivation we can write the original expression as

$$E^{1/2} - \frac{\delta}{2} = \sqrt{1+\frac{\delta^2}{4}}\,,$$

or squaring

$$\left(E^{1/2} - \frac{\delta}{2}\right)^2 = 1 + \frac{\delta^2}{4}\,. \tag{18}$$

Now the left-hand-side (LHS) of this is

$$\text{LHS} = \left(E^{1/2} - \frac{\delta}{2}\right)^2 = \left(E^{1/2} - \frac{\delta}{2}\right)\left(E^{1/2} - \frac{\delta}{2}\right) = E - \frac{1}{2}E^{1/2}\delta - \frac{1}{2}\delta E^{1/2} + \frac{1}{4}\delta^2\,.$$

Note that $E^{1/2}\delta = \delta E^{1/2}$ as

$$\begin{aligned} E^{1/2}\delta f(x) &= E^{1/2}\left(f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)\right) = f(x+h) - f(x) = \Delta f(x) \\ \delta E^{1/2} f(x) &= \delta f\left(x+\frac{h}{2}\right) = f(x+h) - f(x) = \Delta f(x)\,. \end{aligned}$$

This means that as operators we have

$$-\frac{1}{2}E^{1/2}\delta - \frac{1}{2}\delta E^{1/2} = -\Delta = -(E-1) = -E+1\,.$$

Therefore we have shown that the left-hand-side takes the form

$$\begin{aligned} \text{LHS} &= E - \frac{1}{2}E^{1/2}\delta - \frac{1}{2}\delta E^{1/2} + \frac{1}{4}\delta^2 \\ &= E - E + 1 + \frac{1}{4}\delta^2 = 1 + \frac{1}{4}\delta^2\,, \end{aligned}$$

the same as the right-hand-side of Equation 18.

Supplementary Problem 1.96

From what we want to prove we have

$$\begin{aligned} \nabla\Delta f(x) &= \nabla(f(x+h) - f(x)) \\ &= f(x+h) - f(x) - (f(x) - f(x-h)) \\ &= f(x+h) - 2f(x) + f(x-h)\,. \end{aligned}$$

Next we have

$$\begin{aligned}\Delta\nabla f(x) &= \Delta(f(x) - f(x-h)) \\ &= f(x+h) - f(x) - (f(x) - f(x-h)) \\ &= f(x+h) - 2f(x) + f(x-h)\,,\end{aligned}$$

which is the same as $\nabla\Delta f(x)$.

Next lets expand

$$\begin{aligned}\delta^2 f(x) &= \delta\left(f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)\right) \\ &= (f(x+h) - f(x)) - (f(x) - f(x-h)) \\ &= f(x+h) - 2f(x) + f(x-h)\,,\end{aligned}$$

which is the same as the two expressions above.

Supplementary Problem 1.97

Both expressions are true. To show this note that

$$\begin{aligned}\delta x &= \left(x+\frac{h}{2}\right) - \left(x-\frac{h}{2}\right) = h \\ \delta y &= f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)\,.\end{aligned}$$

Thus

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \frac{dy}{dx}\,.$$

In the same way $\delta^n y$ is $n-1$ applications of δ on "δy" so the limiting procedure applied above would yield higher order derivatives.

Supplementary Problem 1.98

Part (a): We want to show that $M = \frac{1}{2}(1+E)$. From the definition of M we have

$$Mf(x) = \frac{1}{2}(f(x+h) + f(x)) = \frac{1}{2}(Ef(x) + f(x)) = \frac{1}{2}(E+1)f(x)\,.$$

Thus $M = \frac{1}{2}(1+E)$.

Next we consider the operator $E - \frac{1}{2}\Delta$ on $f(x)$. We have

$$\left(E - \frac{1}{2}\Delta\right)f(x) = f(x+h) - \frac{1}{2}(f(x+h) - f(x)) = \frac{1}{2}(f(x+h) + f(x)) = Mf(x)\,,$$

thus $E - \frac{1}{2}\Delta = M$.

Part (b): We want to show $\mu = \frac{M}{E^{1/2}}$ or $\mu E^{1/2} = M$. Consider $\mu E^{1/2}$ on $f(x)$. We find

$$\begin{aligned} \mu E^{1/2} f(x) = \mu f\left(x + \frac{h}{2}\right) &= \frac{1}{2}\left[f(x+h) + f\left(x + \frac{h}{2} - \frac{h}{2}\right)\right] \\ &= \frac{1}{2}\left[f(x+h) + f(x)\right] = Mf(x)\,. \end{aligned}$$

Supplementary Problem 1.99

Part (a): To start we will prove that $M\Delta = \Delta M$. The left-hand-side of this applied to a function $f(x)$ is

$$\begin{aligned} M\Delta f(x) &= M(f(x+h) - f(x)) \\ &= \frac{1}{2}\left[(f(x+2h) - f(x+h)) + (f(x+h) - f(x))\right] \\ &= \frac{1}{2}\left[f(x+2h) - f(x)\right]\,. \end{aligned}$$

While the right-hand-side applied to a function $f(x)$ is

$$\begin{aligned} \Delta M f(x) &= \Delta\left[\frac{1}{2}(f(x+h) + f(x))\right] \\ &= \frac{1}{2}\left[(f(x+2h) + f(x+h)) - (f(x+h) + f(x))\right] \\ &= \frac{1}{2}\left[f(x+2h) - f(x)\right]\,. \end{aligned}$$

As these two expression are equal we have shown that $M\Delta = \Delta M$.

Next we want to prove that $MD = DM$. The left-hand-side of this expression applied to a function $f(x)$ is

$$MDf(x) = Mf'(x) = \frac{1}{2}(f'(x+h) + f'(x))\,,$$

while the right-hand-side applied to $f(x)$ is

$$DMf(x) = D\left(\frac{1}{2}(f(x+h) + f(x))\right) = \frac{1}{2}(f'(x+h) + f'(x))\,.$$

As these two expressions are equal we conclude that $MD = DM$.

Finally, we want to prove that $ME = EM$. The two sides of this expression applied to a function $f(x)$ are given by

$$\begin{aligned} MEf(x) &= Mf(x+h) = \frac{1}{2}(f(x+2h) + f(x+h)) \\ EMf(x) &= E\left(\frac{1}{2}(f(x+h) + f(x))\right) = \frac{1}{2}(f(x+2h) + f(x+h))\,. \end{aligned}$$

As these two expressions are the same we conclude that $ME = EM$.

Supplementary Problem 1.100

Part (a): We want to show that

$$\Delta = \mu\delta + \frac{1}{2}\delta^2 .$$

The first term in the right-hand-side of this expression applied on a function $f(x)$ is

$$\begin{aligned}
\mu\delta f(x) &= \mu\left(f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)\right) \\
&= \frac{1}{2}(E^{1/2}+E^{-1/2})\left(f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)\right) \\
&= \frac{1}{2}(f(x+h) - f(x) + f(x) - f(x-h)) \\
&= \frac{1}{2}(f(x+h) - f(x-h)) .
\end{aligned}$$

Next notice that the second term above in the right-hand-side (on a function $f(x)$) is

$$\begin{aligned}
\frac{1}{2}\delta^2 f(x) &= \frac{1}{2}(E^{1/2}-E^{-1/2})^2 f(x) = \frac{1}{2}(E-2+E^{-1})f(x) \\
&= \frac{1}{2}(f(x+h) - 2f(x) + f(x-h)) .
\end{aligned}$$

Using both of these and adding we see that

$$\mu\delta + \frac{1}{2}\delta^2 = f(x+h) - f(x) ,$$

which is the same as the left-hand-side showing the desired identity.

Another proof of this identity uses the E notation. From what μ and δ are in terms of E we can write

$$\begin{aligned}
\mu\delta + \frac{1}{2}\delta^2 &= \frac{1}{2}(E^{1/2}+E^{-1/2})(E^{1/2}-E^{-1/2}) + \frac{1}{2}(E^{1/2}-E^{-1/2})^2 \\
&= \frac{1}{2}(E-1+1-E^{-1}) + \frac{1}{2}(E-2+E^{-1}) \\
&= \frac{1}{2}E - \frac{1}{2}E^{-1} + \frac{1}{2}E - 1 + \frac{1}{2}E^{-1} = E - 1 \equiv \Delta .
\end{aligned}$$

Supplementary Problem 1.101

Part (a): We have

$$\begin{aligned}
(A-B)(A+B) &= A(A+B) - B(A+B) \\
&= A^2 + AB - BA - B^2 \\
&= A^2 - B^2 + AB - BA .
\end{aligned}$$

Part (b): This will be true if $(AB - BA)f(x) = 0$ or $ABf(x) = BAf(x)$.

Part (c): From the above for $(\Delta - D)(\Delta + D)x^2 = (\Delta^2 - D^2)x^2$ to be true we need to have $(\Delta D - D\Delta)x^2 = 0$. These are true because

$$\begin{aligned}
\Delta D x^2 &= \Delta(2x) = 2h \\
D\Delta x^2 &= D((x+h)^2 - x^2) = 2(x+h) - 2x = 2h\,.
\end{aligned}$$

We can also compute

$$\begin{aligned}
(\Delta^2 - D^2)x^2 &= \Delta((x+h)^2 - x^2) - D(2x) \\
&= (x+2h)^2 - (x+h)^2 - (x+h)^2 + x^2 - 2 \\
&= (x+2h)^2 - 2(x+h)^2 + x^2 - 2 \\
&= x^2 + 4hx + 4h^2 - 2(x^2 + 2hx + h^2) + x^2 - 2 \\
&= 4hx + 4h^2 - 4hx - 2h^2 - 2 = 2h^2 - 2\,.
\end{aligned}$$

and

$$\begin{aligned}
(\Delta - D)(\Delta + D)x^2 &= (\Delta - D)((x+h)^2 - x^2 + 2x) \\
&= (\Delta - D)(2hx + 2x + h^2) \\
&= (2h^2 + 2h - (2h+2)) = 2h^2 - 2\,,
\end{aligned}$$

which is the same as above.

Supplementary Problem 1.102

Part (a): We have

$$\Delta \sin(px + q) = \sin(p(x+h) + q) - \sin(px + q)\,.$$

To simplify this recall the trigonometric identity

$$\sin(\theta_1) - \sin(\theta_2) = 2\sin\left(\frac{\theta_1 - \theta_2}{2}\right)\cos\left(\frac{\theta_1 + \theta_2}{2}\right)\,. \tag{19}$$

Using this we can write $\Delta \sin(px + q)$ as

$$\begin{aligned}
\Delta \sin(px + q) &= 2\sin\left(\frac{p(x+h) + q - px - q}{2}\right)\cos\left(\frac{2px + 2q + ph}{2}\right) \\
&= 2\sin\left(\frac{ph}{2}\right)\cos\left(px + q + \frac{ph}{2}\right)\,.
\end{aligned}$$

But as

$$\cos(\theta) = \sin\left(\theta + \frac{\pi}{2}\right)\,, \tag{20}$$

the above can be written

$$\Delta \sin(px + q) = 2\sin\left(\frac{ph}{2}\right)\sin\left(px + q + \frac{ph + \pi}{2}\right)\,,$$

as we were to show.

Part (b): We have

$$\Delta\cos(px+q) = \cos(p(x+h)+q) - \cos(px+q)\,.$$

To simplify this recall the trigonometric identity

$$\cos(\theta_1) - \cos(\theta_2) = -2\sin\left(\frac{\theta_1+\theta_2}{2}\right)\sin\left(\frac{\theta_1-\theta_2}{2}\right)\,. \tag{21}$$

Using this we can write $\Delta\cos(px+q)$ as

$$\begin{aligned}\Delta\cos(px+q) &= -2\sin\left(px+q+\frac{ph}{2}\right)\sin\left(\frac{ph}{2}\right)\\ &= 2\sin\left(\frac{ph}{2}\right)\left[-\sin\left(px+q+\frac{ph}{2}\right)\right]\,.\end{aligned}$$

But as

$$-\sin(\theta) = \cos\left(\theta+\frac{\pi}{2}\right)\,, \tag{22}$$

the above can be written

$$\Delta\cos(px+q) = 2\sin\left(\frac{ph}{2}\right)\cos\left(px+q+\frac{ph+\pi}{2}\right)\,,$$

as we were to show.

Supplementary Problem 1.103

Part (a-b): From the previous problem each application of Δ produces "changes" the value of q by adding $\frac{1}{2}(ph+\pi)$ to it and produces factor $2\sin\left(\frac{ph}{2}\right)$. Thus we can conclude that

$$\Delta^m\sin(px+q) = \left(2\sin\left(\frac{ph}{2}\right)\right)^m\sin\left(px+q+\frac{m}{2}(ph+\pi)\right)\,.$$

A similar argument gives the functional form for $\Delta^m\cos(px+q)$.

Supplementary Problem 1.104

The derivative we seek to evaluate can be written as

$$\frac{d^m}{dx^m}\sin(x) = \lim_{h\to 0}\frac{\Delta^m\sin(x)}{h^m}\,.$$

Using the results from the previous problem with $p=1$ and $q=0$ we have

$$\lim_{h\to 0}\frac{\Delta^m\sin(x)}{h^m} = \lim_{h\to 0}\frac{(2\sin(h/2))^m\sin(x+(m/2)(h+\pi))}{h^m}\,.$$

Now note we could show that

$$\lim_{h\to 0}\frac{2\sin(h/2)}{h}=\frac{2(h/2)}{h}=1\,.$$

This means that the limit above is equal to

$$1^m\cdot\sin\left(x+\frac{m\pi}{2}\right)=\sin\left(x+\frac{m\pi}{2}\right)\,,$$

as we were to show.

Part (b): Here the limit we need to consider

$$\frac{d^m}{dx^m}\cos(x)=\lim_{h\to 0}\frac{\Delta^m\cos(x)}{h^m}\,.$$

Using the results from the previous problem with $p=1$ and $q=0$ we have

$$\lim_{h\to 0}\frac{\Delta^m\cos(x)}{h^m}=\lim_{h\to 0}\frac{(2\sin(h/2))^m\cos(x+(m/2)(h+\pi))}{h^m}\,.$$

As before we can evaluate this limit to get

$$\frac{d^m}{dx^m}\cos(x)=\cos\left(x+\frac{m\pi}{2}\right)\,.$$

Supplementary Problem 1.105

Starting with the left-hand-side and the definition of $\left(\begin{array}{c} n \\ r \end{array}\right)$ we have

$$\begin{aligned}
\left(\begin{array}{c} n \\ r \end{array}\right)+\left(\begin{array}{c} n \\ r+1 \end{array}\right) &= \frac{n!}{(n-r)!r!}+\frac{n!}{(n-(r+1))!(r+1)!} \\
&= \frac{n!}{(n-r)!r!}+\frac{n!}{(n-r-1)!(r+1)!} \\
&= \frac{n!}{(n-r-1)!r!}\left[\frac{1}{n-r}+\frac{1}{r+1}\right] \\
&= \frac{n!}{(n-r-1)!r!}\left[\frac{r+1+n-r}{(n-r)(r+1)}\right] \\
&= \frac{(n+1)!}{(n-r)!(r+1)!}\,,
\end{aligned}$$

as we were to show.

Supplementary Problem 1.106

Part (a): From the definition of Δ we have

$$\begin{aligned}
\Delta \tan(x) &= \frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin(x)}{\cos(x)} \\
&= \frac{\sin(x+h)\cos(x) - \sin(x)\cos(x+h)}{\cos(x)\cos(x+h)} \\
&= \frac{\sin(x)\cos(x)\cos(h) + \cos^2(x)\sin(h) - \cos(x)\sin(x)\cos(h) + \sin^2(x)\sin(h)}{\cos(x)\cos(x+h)} \\
&= \frac{\sin(h)}{\cos(x)\cos(x+h)} = \frac{\tan(h)}{\frac{\cos(x)}{\cos(h)}\cos(x+h)} \\
&= \frac{\tan(h)}{\frac{\cos(x)}{\cos(h)}(\cos(x)\cos(h) - \sin(x)\sin(h))} \\
&= \frac{\tan(h)}{\cos^2(x) - \cos(x)\sin(x)\tan(h)} = \frac{\tan(h)\sec^2(x)}{1 - \tan(x)\tan(h)} \,,
\end{aligned}$$

as we were to show.

Part (b): Using the above we have

$$\frac{d}{dx}\tan(x) = \lim_{h\to 0} \frac{1}{h}\left(\frac{\tan(h)\sec^2(x)}{1-\tan(x)\tan(h)}\right) .$$

As

$$\lim_{h\to 0} \frac{\tan(h)}{h} = 1 \,,$$

we have that the above limit tends to

$$\sec^2(x) \,,$$

as we were to show.

Supplementary Problem 1.107

Part (a): Using the definition of Δ we have

$$\Delta \tan^{-1}(x) = \tan^{-1}(x+h) - \tan^{-1}(x) \,.$$

Recall that

$$\tan^{-1}(x) - \tan^{-1}(y) = \tan^{-1}\left(\frac{x-y}{1+xy}\right) .$$

This means that

$$\begin{aligned}
\Delta \tan^{-1}(x) &= \tan^{-1}\left(\frac{x+h-x}{1+(x+h)x}\right) = \tan^{-1}\left(\frac{h}{1+x^2+xh}\right) \\
&= \tan^{-1}\left(\frac{h}{x^2+hx+1}\right) ,
\end{aligned}$$

as we were to show.

Part (b): From the definition of the derivative we have

$$\frac{d}{dx}\tan^{-1}(x) = \lim_{h\to 0}\frac{1}{h}\Delta\tan^{-1}(x) = \lim_{h\to 0}\frac{1}{h}\tan^{-1}\left(\frac{h}{x^2+hx+1}\right).$$

Recall the Taylor series for $\tan^{-1}(x)$ where we have

$$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots.$$

This means that

$$\tan^{-1}\left(\frac{h}{x^2+hx+1}\right) \approx \frac{h}{x^2+hx+1},$$

which means that

$$\lim_{h\to 0}\frac{1}{h}\tan^{-1}\left(\frac{h}{x^2+hx+1}\right) \approx \lim_{h\to 0}\left(\frac{1}{x^2+hx+1}\right) = \frac{1}{1+x^2},$$

For the last part using the chain rule we have (with $v = \frac{x}{a}$) that

$$\frac{d}{dx}\tan^{-1}\left(\frac{x}{a}\right) = \frac{d}{dv}\tan^{-1}(v)\frac{dv}{dx} = \frac{1}{a}\frac{d}{dv}\tan^{-1}(v).$$

Using what we know about the derivative of $\tan^{-1}(v)$ we conclude that

$$\frac{d}{dx}\tan^{-1}\left(\frac{x}{a}\right) = \frac{1}{a(1+v^2)} = \frac{1}{a\left(1-\left(\frac{x}{a}\right)^2\right)} = \frac{a}{a^2+x^2},$$

as we were to show.

Supplementary Problem 1.108

Part (a): We start with the definition of Δ and have

$$\Delta\sin^{-1}(x) = \sin^{-1}(x+h) - \sin^{-1}(x).$$

Then using the identity

$$\arcsin(x) \pm \arcsin(y) = \arcsin(x\sqrt{1-y^2} \pm y\sqrt{1-x^2}), \tag{23}$$

we can write

$$\Delta\sin^{-1}(x) = \sin^{-1}((x+h)\sqrt{1-x^2} - x\sqrt{1-(x+h)^2}).$$

Part (b): Using this we would then have

$$\frac{d}{dx}\sin^{-1}(x) = \lim_{h\to 0}\frac{\Delta\sin^{-1}(x)}{h} = \lim_{h\to 0}\left(\frac{\sin^{-1}((x+h)\sqrt{1-x^2} - x\sqrt{1-(x+h)^2})}{h}\right).$$

This is a limit of the type 0/0 and we would normally use L'Hospital's rule to evaluate it. Here we will use a Taylor expansions. To do this recall that when $x \ll 1$ we have

$$\sin^{-1}(x) = x + \frac{x^3}{6} + \frac{3x^5}{40} + \cdots \,, \tag{24}$$

This means that for small h keeping only one term we have

$$\frac{\sin^{-1}((x+h)\sqrt{1-x^2} - x\sqrt{1-(x+h)^2})}{h} \approx \frac{(x+h)\sqrt{1-x^2} - x\sqrt{1-(x+h)^2}}{h} \,.$$

Next note that

$$\sqrt{1-(x+h)^2} = \sqrt{1-x^2-2xh-h^2} \approx \sqrt{1-x^2-2xh} = \sqrt{1-x^2}\sqrt{1-\frac{2x}{1-x^2}h} \,.$$

Then using the Taylor series

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \cdots \,, \tag{25}$$

we have that

$$\begin{aligned} \sqrt{1-(x+h)^2} &\approx \sqrt{1-x^2}\left[1 - \frac{x}{1-x^2}h + O(h^2)\right] \\ &= \sqrt{1-x^2} - \frac{x}{\sqrt{1-x^2}}h + O(h^2) \,. \end{aligned}$$

Using all of this we can now evaluate

$$\begin{aligned} \lim_{h\to 0} \frac{\Delta \sin^{-1}(x)}{h} &= \lim_{h\to 0} \frac{(x+h)\sqrt{1-x^2} - x\sqrt{1-x^2} + \frac{x^2}{\sqrt{1-x^2}}h}{h} \\ &= \sqrt{1-x^2} + \frac{x^2}{\sqrt{1-x^2}} = \frac{1-x^2+x^2}{\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}} \,, \end{aligned}$$

as we were to show.

For the last part using the chain rule we have (with $v = \frac{x}{a}$) that

$$\frac{d}{dx}\sin^{-1}\left(\frac{x}{a}\right) = \frac{d}{dv}\sin^{-1}(v)\frac{dv}{dx} = \frac{1}{a}\frac{d}{dv}\sin^{-1}(v) \,.$$

Using what we know about the derivative of $\sin^{-1}(v)$ we conclude that

$$\begin{aligned} \frac{d}{dx}\sin^{-1}\left(\frac{x}{a}\right) &= \frac{1}{a\sqrt{1-v^2}} = \frac{1}{a\sqrt{1-\left(\frac{x}{a}\right)^2}} \\ &= \frac{1}{\sqrt{a^2-x^2}} \,, \end{aligned}$$

as we were to show.

Supplementary Problem 1.109

Part (a): Recall that

$$x^{(n)} = \sum_{k=1}^{n} s_k^n x^k h^{n-k} .$$

If $h = 1$ this is

$$x^{(n)} = \sum_{k=1}^{n} s_k^n x^k .$$

Lets take k derivatives of both sides of this expression, divide by $k!$, and evaluate at $x = 0$. Then on the right-hand-side we get s_k^n and the identity

$$s_k^n = \frac{1}{k!} D^k x^{(n)}\Big|_{x=0} ,$$

as we were to show.

Part (b): Recall that

$$x^n = \sum_{k=1}^{n} S_k^n x^{(k)} h^{n-k} .$$

If $h = 1$ this is

$$x^n = \sum_{k=1}^{n} S_k^n x^{(k)} . \tag{26}$$

Now note that

$$\Delta^k x^{(p)} = 0 ,$$

if $p < k$. If $k = p$ we have

$$\Delta^k x^{(k)} = k! ,$$

when $h = 1$ and if $k < p$ we have

$$\Delta^k x^{(p)} = O(x^{(p-k)}) ,$$

which vanishes when $x = 0$. Thus to derive the desired result for this part we start with Equation **??**, then we take Δ^k of both sides, divide by $k!$, and evaluate at $x = 0$. From the above this gives

$$S_k^n = \frac{1}{k!} \Delta^k x^n\Big|_{x=0} ,$$

as we were to show.

Supplementary Problem 1.110

From Part (b) in the previous problem we have

$$S_k^n = \frac{1}{k!} \Delta^k x^n\Big|_{x=0} .$$

Recalling that when $\Delta = E - 1$ we can write the above as

$$\begin{aligned} S_k^n &= \frac{1}{k!} (E-1)^k x^n \Big|_{x=0} \\ &= \frac{1}{k!} \left[\sum_{p=0}^{k} \left(\begin{array}{c} k \\ p \end{array} \right) E^p (-1)^{k-p} \right] x^n \Bigg|_{x=0} . \end{aligned}$$

Now when $h = 1$ we can write this as

$$\begin{aligned} S_k^n &= \frac{(-1)^k}{k!} \sum_{p=0}^{k} \left(\begin{array}{c} k \\ p \end{array} \right) (-1)^p (x+p)^n \Bigg|_{x=0} \\ &= \frac{(-1)^k}{k!} \sum_{p=0}^{k} \left(\begin{array}{c} k \\ p \end{array} \right) (-1)^p p^n , \end{aligned}$$

which is what we wanted to show.

Supplementary Problem 1.111

Part (a): To prove this we have

$$E[f(x)g(x)] = f(x+h)g(x+h) = E[f(x)]E[g(x)] .$$

Part (b): To prove this we have

$$E[f(x)]^n = f(x+h)^n = (E[f(x)])^n .$$

Part (c): To prove this we have

$$\begin{aligned} E^m[f_1(x)f_2(x)\cdots f_n(x)] &= f_1(x+mh)f_2(x+mh)\cdots f_n(x+mh) \\ &= E^m f_1(x) E^m f_2(x) \cdots E^m f_n(x) . \end{aligned}$$

Supplementary Problem 1.112

To show that $x^{(m)}x^{(n)} \neq x^{(m+n)}$ we can evaluate both sides for particular values of $m = 1$ and $n = 2$. If $m = 1$ and $n = 2$ then

$$x^{(m)}x^{(n)} = x^{(1)}x^{(2)} = x \cdot x(x-h) = x^2(x-h) ,$$

while

$$x^{(m+n)} = x^{(3)} = x(x-h)(x-2h) .$$

These are not equal for all x.

Supplementary Problem 1.113

Using operator arithmetic we have that

$$\begin{aligned}\Delta^n[f(x)g(x)] &= (E-1)^n(f(x)g(x)) \\ &= \left(\sum_{k=0}^{n}\binom{n}{k}E^k(-1)^{n-k}\right)f(x)g(x) \\ &= \sum_{k=0}^{n}(-1)^{k+n}\binom{n}{k}f(x+kh)g(x+kh)\,,\end{aligned}$$

which is what we were to show. This is different than Liebnitz's rule in that the above involves only products increments of $f(x)$ and $g(x)$ where as Liebnitz's rule involves products of *differences* i.e. products involving the Δ operator.

Supplementary Problem 1.114

Using the previous problem with $g(x) \equiv 1$ we have that

$$\Delta^n f(x) = \sum_{k=0}^{n}(-1)^{k+n}\binom{n}{k}f(x+kh)\,.$$

If $h = 1$ this is

$$\Delta^n f(x) = \sum_{k=0}^{n}(-1)^{k+n}\binom{n}{k}f(x+k)\,.$$

If we take $f(x) = x^n$ this becomes

$$\Delta^n x^n = \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}(x+k)^n\,.$$

Now if we take $x = 0$ in the above notice that the right-hand-side RHS is equal to

$$\begin{aligned}\text{RHS} &= \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}k^n \\ &= n^n - n(n-1)^n + \binom{n}{2}(n-2)^n - \binom{n}{3}(n-3)^n + \cdots\,,\end{aligned}$$

which is the right-hand-side of the expression we are trying to derive.

We now need to evaluate the left-hand-side LHS i.e. $\Delta^n x^n|_{x=0}$ is equal to. From miscellaneous problem 1.41 when

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots\,,$$

we have that

$$\Delta^n f(x) = n!a_0h^n\,.$$

If $a_0 = 1$ and $h = 1$ we see that

$$\text{LHS} = \Delta^n x^n|_{x=0} = n!\,.$$

Supplementary Problem 1.115

Part (a): Using Taylor series in "operator form" we have

$$Ef(x) = f(x+h) = \left[1 + hD + \frac{h^2D^2}{2!} + \frac{h^3D^3}{3!} + \cdots\right] f(x) .$$

This means that

$$Ef(x) - f(x) = \left(hD + \frac{h^2D^2}{2!} + \frac{h^3D^3}{3!} + \cdots\right) f(x) ,$$

and thus that

$$\Delta \equiv \left(hD + \frac{h^2D^2}{2!} + \frac{h^3D^3}{3!} + \cdots\right) .$$

Part (b-c): Given the expression for Δ above we would have

$$\Delta^2 = \left(hD + \frac{h^2D^2}{2!} + \frac{h^3D^3}{3!} + \cdots\right)^2 ,$$

and

$$\Delta^3 = \left(hD + \frac{h^2D^2}{2!} + \frac{h^3D^3}{3!} + \cdots\right)^3 .$$

In the `python` code `problem_115.py` we symbolically evaluate both of these giving results that match those in the book.

Supplementary Problem 1.116

Part (a): To evaluate $\Delta^2 f(x)$ lets use the result of the previous problem where we derived an expression for Δ^2 in terms of D^k. Notice that

$$D^k(3x^3 - 2x^2 + 4x - 6) = 0 ,$$

if $k \geq 4$. Thus for $f(x) = 3x^3 - 2x^2 + 4x - 6$ we have that

$$\Delta^2 f(x) = (h^2D^2 + h^3D^3)f(x) .$$

We now compute

$$\begin{aligned} D(3x^3 - 2x^2 + 4x - 6) &= 9x^2 - 4x + 4 \\ D^2(3x^3 - 2x^2 + 4x - 6) &= 18x - 4 \\ D^3(3x^3 - 2x^2 + 4x - 6) &= 18 . \end{aligned}$$

This means that

$$\Delta^2(3x^3 - 2x^2 + 4x - 6) = h^2(18x - 4) + h^3(18) = 18h^3 + 18h^2x - 4h^2 .$$

Lets check these results by direct evaluation. We have

$$
\begin{aligned}
\Delta(3x^3 - 2x^2 + 4x - 6) &= 3((x+h)^3 - x^3) - 2((x+h)^2 - x^2) + 4h \\
&= 3(3x^2h + 3xh^2 + h^3) - 2(2hx + h^2) + 4h \\
&= 9hx^2 + 9h^2x + 3h^3 - 4hx - 2h^2 + 4h \\
&= 9hx^2 + (9h^2 - 4h)x + 3h^3 - 2h^2 + 4h\,.
\end{aligned}
$$

Using this we can compute

$$
\begin{aligned}
\Delta^2(3x^3 - 2x^2 + 4x - 6) &= 9h((x+h)^2 - x^2) + (9h^2 - 4h)h \\
&= 9h(2hx + h^2) + 9h^3 - 4h^2 \\
&= 18h^3 + 18h^2x - 4h^2\,,
\end{aligned}
$$

which is the same as we derived earlier.

Part (b): Here we want to evaluate

$$
\Delta^3((x^2 + x)^2) = \Delta^3(x^4 + 2x^3 + x^2)\,.
$$

Using the expansion for Δ^3 in terms of D^k derived in the previous problem but keeping only the nonzero terms (when applied to $x^4 + 2x^3 + x^2$) gives

$$
\Delta^3 = h^3D^3 + \frac{3h^4D^4}{2}\,.
$$

Thus we need to compute

$$
\begin{aligned}
D(x^4 + 2x^3 + x^2) &= 4x^3 + 6x^2 + 2x \\
D^2(x^4 + 2x^3 + x^2) &= 12x^2 + 12x + 2 \\
D^3(x^4 + 2x^3 + x^2) &= 24x + 12 \\
D^4(x^4 + 2x^3 + x^2) &= 24 \\
D^k(x^4 + 2x^3 + x^2) &= 0 \quad \text{for} \quad k \geq 5\,.
\end{aligned}
$$

Using these we compute

$$
\Delta^3((x^2 + x)^2) = h^3(24x + 12) + \frac{3}{2}h^4(24) = 24h^3x + 12h^3 + 36h^4\,.
$$

Lets check these results by direct evaluation. In this case to do that we will write each of monomial x^k in $x^4 + 2x^3 + x^2$ in terms of falling factorial functions $x^{(l)}$. Note the comment at the start of this chapter on the expansion of x^4. We have

$$
\begin{aligned}
x^4 + 2x^3 + x^2 &= (x^{(4)} + 6hx^{(3)} + 7h^2x^{(2)} + h^3x^{(1)}) \\
&\quad + 2(x^{(3)} + 3hx^{(2)} + h^2x^{(1)}) \\
&\quad + (x^{(2)} + hx^{(1)}) \\
&= x^{(4)} + (6h + 2)x^{(3)} + (7h^2 + 6h + 1)x^{(2)} + (h^3 + 2h^2 + h)x^{(1)}\,.
\end{aligned}
$$

Then using $\Delta x^{(m)} = mx^{(m-1)}h$ we have

$$\Delta(x^4+2x^3+x^2) = 4hx^{(3)} + (18h^2+6h)x^{(2)} + (14h^3+12h^2+h)x^{(1)} + (h^4+2h^3+h^2)$$
$$\Delta^2(x^4+2x^3+x^2) = 12h^2x^{(2)} + (36h^3+12h^2)x^{(1)} + (14h^4+12h^3+h^2)$$
$$\Delta^3(x^4+2x^3+x^2) = 24h^3x^{(1)} + (36h^4+12h^3)\,.$$

This last expression agrees with what we computed earlier.

Supplementary Problem 1.117

The $U(t)$ function in Problem 1.33 is given by

$$U(t) = f(t) - p_n(t) - K(x)(t-x_0)(t-x_1)\cdots(t-x_n)\,,$$

and by construction has $n+2$ roots for $t \in \{x, x_0, x_1, \cdots, x_n\}$. Without loss of generally lets assume that the points fall in the order listed i.e. that

$$x \le x_0 \le x_1 \le x_2 \le \cdots \le x_n\,.$$

Rolle's theorem gives us that between any two of these roots say the $n+1$ pairs

$$(x, x_0)\,, (x_0, x_1)\cdots(x_{n-1}, x_n)\,,$$

there exist $n+1$ points y_j such that $U'(y_j) = 0$ for $0 \le j \le n$ and due to the ordering of the pairs above we also have that

$$y_0 \le y_1 \le y_2 \le \cdots \le y_n\,.$$

Here

$$\begin{aligned} y_0 &\in (x, x_0) \\ y_1 &\in (x_0, x_1) \\ &\vdots \\ y_n &\in (x_{n-1}, x_n)\,. \end{aligned}$$

If we consider Rolle's theorem for the pairs (y_j, y_{j+1}) we can get n roots z_k such that $U''(z_k) = 0$ for $0 \le k \le n-1$. We will continue this pattern to get the desired result.

The pattern is then

- There exists $n+2$ roots of $U(t)$.
- There exists $n+1$ roots of $U'(t)$.
- There exists n roots of $U''(t)$.
- Continuing
- There exist two roots of $U^{(n)}(t)$.
- There exist one roots of $U^{(n+1)}(t)$.

Chapter 2: Applications of the Difference Calculus

Supplementary Problem 2.50

Part (a):

References

[1] G. Corliss. Which root does the bisection algorithm find? *SIAM Review*, 19(2):325–327, 1977.

[2] W. Ferrar. *A text-book of convergence*. The Clarendon Press, 1938.