(2.6.2)

$$x(t) = x(t+T)$$

$$x(t) \neq x(t+s) \quad \forall\ 0 < s < T$$

then consider

$$\int_t^{t+T} f(x)\frac{dx}{dt}\,dt = \int_{x(t)}^{x(t+T)} f(u)\,du = \int_{x(t)}^{x(t)} f(u)\,du = 0$$

$$\longrightarrow$$

$u = x(t)$

$du = \frac{dx}{dt}\cdot dt$

But ~~$\int_t^{t+T} f(x)\frac{dx}{dt}dt = \int_t^{t+T} f(x)dx$~~ integral is

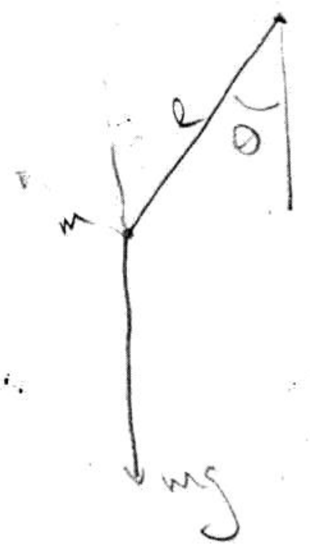
~~$\int_{s=t}^{s=t+T} f(x)\frac{dx}{ds}ds =$~~

But $\frac{dx}{dt} = f(x)$ By Nonlinear

$$\Rightarrow 0 = \int_t^{t+T} f^2(x(t))\,dt \Rightarrow f^2(x(t)) = 0$$

$$\Rightarrow f = 0 \quad \rightarrow\leftarrow.$$

Pg 36 Stogatz:

(a) 

$I\ddot{\theta} = \tau$ w/ damping $-b\dot{\theta}$

$m\ell^2 \ddot{\theta} = mg\ell \sin\theta - b\dot{\theta}$

$\Rightarrow \dot{\theta} = \left(\frac{mg\ell}{b}\right) \sin\theta - \frac{m\ell^2}{b} \ddot{\theta}$

If ℓ is choosen very small $\Rightarrow \ell^2 \approx 0$

the above equation becomes

$\dot{\theta} = \frac{mg\ell}{b} \sin\theta$ or scalled $\dot{\theta} = \sin\theta$

(b) ?

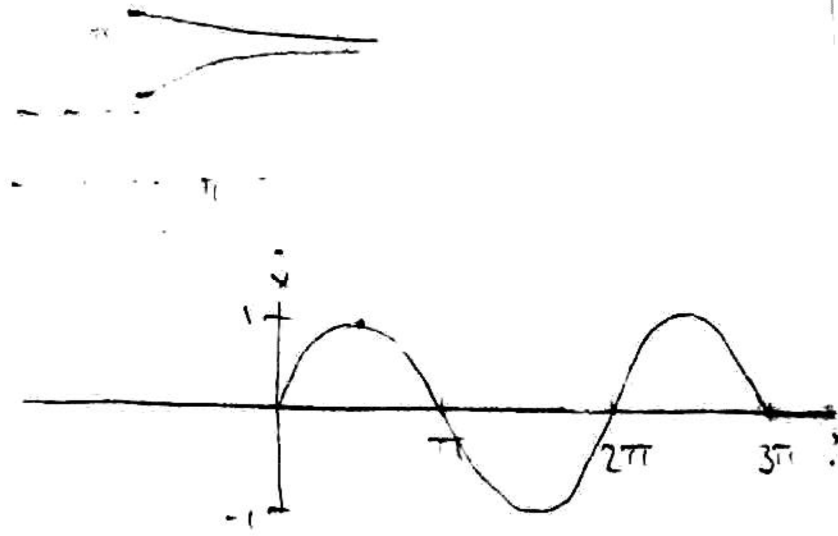
(a) $\dot{x} = \sin x$

$\ddot{x} = \dot{x}\cos x$

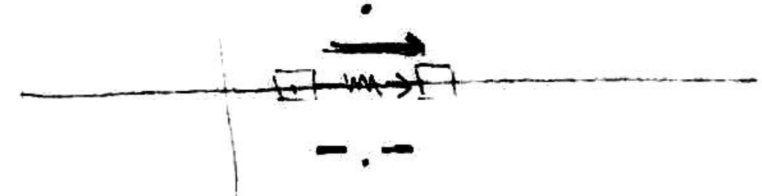
Newtons equation $\Rightarrow \dot{x} = \sin x$

w/ Forcing fn $\dot{x}\cos x$

$\therefore \ddot{x} = \dot{x}\cos x$ or $\Rightarrow \dot{x} = \sin x$.



Consider 2 masses connected by a spring



Let $v_{cm} = \text{const } v$.

Let the $r_{cm} = v_{cm}t + v_0$

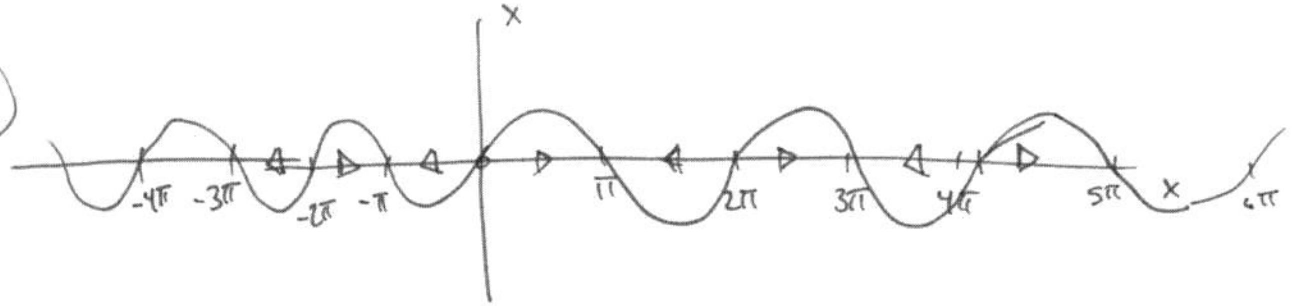
Let $v_{1,2} =$

θ
π-θ
mg

Pg 36 Strogatz

2.1

2.1.1



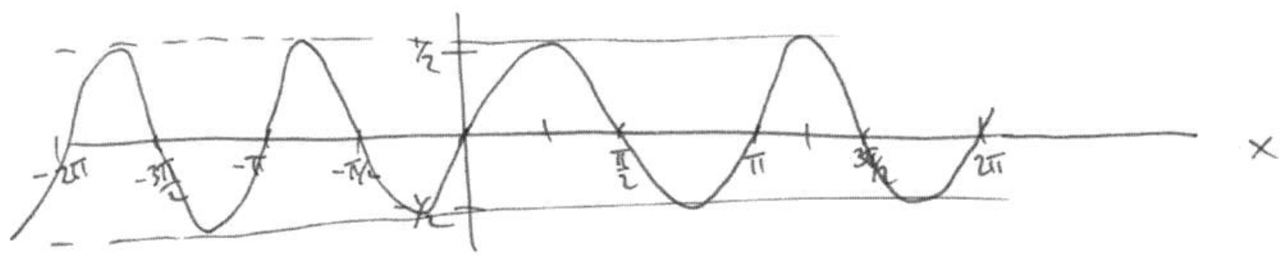
fixed pts $\quad n\pi \quad n \in \mathbb{Z}$

2.1.2 at $\frac{\pi}{2}, \; 2\pi + \frac{\pi}{2}, \; 4\pi + \frac{\pi}{2},$

$-4\pi + \frac{\pi}{2}, \; -2\pi + \frac{\pi}{2}$ $\quad$ or $\quad 2\pi k + \frac{\pi}{2} \quad k \in \mathbb{Z}.$

2.1.3 (a) $\ddot{x} = \cos x \cdot \dot{x} = \cos x \sin x = \frac{1}{2}\sin 2x$

(b)



Max $\ddot{x}$ when $x = \frac{\pi}{4}, \; \pi + \frac{\pi}{4}, \; 2\pi + \frac{\pi}{4}, \; \ldots$

$-\pi + \pi/4, \; -2\pi + \pi/4, \; \ldots$

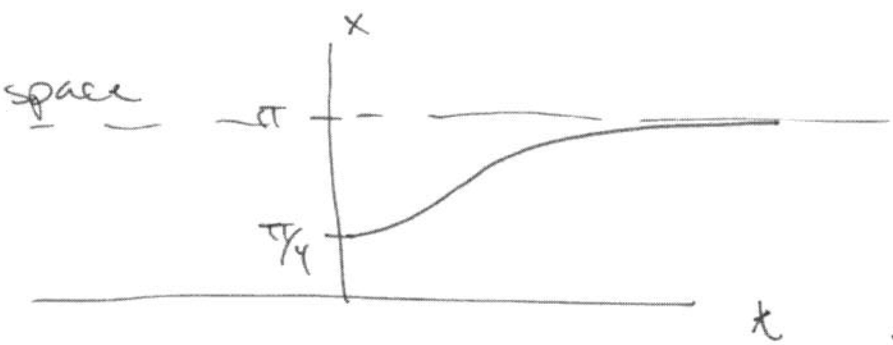
or when $x = n\pi + \pi/4 \quad n \in \mathbb{Z}$

2.1.4 $\quad t = \ln\left|\dfrac{\csc x_0 + \cot x_0}{\csc x + \cot x}\right|$

(a) $x_0 = \pi/4 \quad \Rightarrow \quad t = \ln\left|\dfrac{\sqrt{2}+1}{\csc x + \cot x}\right|$

$$\Rightarrow \left|\frac{1+\sqrt{2}}{\frac{1}{\sin x} + \frac{\cos x}{\sin x}}\right| = e^t$$

from phase space

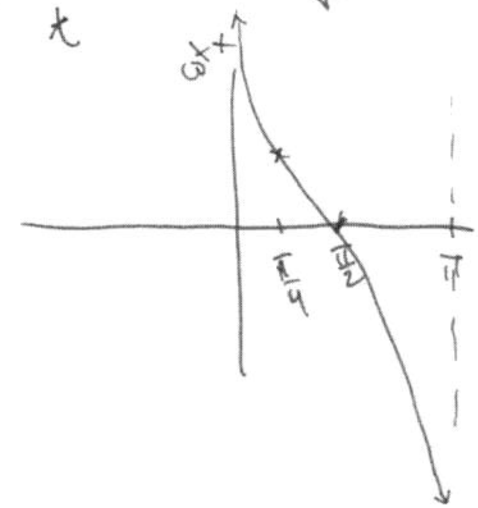


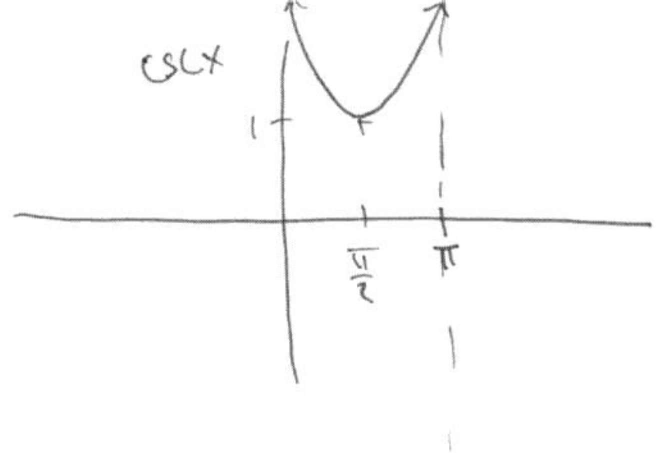
Thus $x > \pi/4$

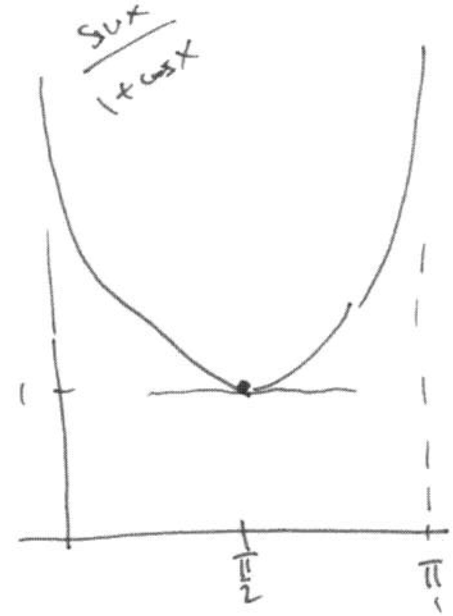
$\forall\, t$

$\forall \qquad x < \pi.$

$\therefore \sin x > 0 \quad \forall\, x$ under consideration







$\therefore$ $\csc x + \cot x$ between 0 & π is

for range from $\pi/2$ to π

consider $0 < \frac{\sin x}{1+\cos x} \longrightarrow \lim_{x\to\pi} \frac{\sin x}{1+\cos x} = \lim_{x\to\pi} \frac{\cos x}{-\sin x} = +\infty$

$\frac{d}{dx} \frac{\sin x}{1+\cos x} \neq 0$

$\checkmark$ $+$ $\forall x \in (\frac{\pi}{2}, \pi)$

$\therefore$

---

just plot $(1+\sqrt{2})\frac{\sin x}{1+\cos x}$

get graph looking like above

$\left(\frac{\sin x}{1+\cos x}\right) \overset{??}{\geq} 1$

* consider $$\frac{\sin x}{1+\cos x} = \frac{2\sin x/2 \cos x/2}{1+\cos^2 x/2 - \sin^2 x/2}$$

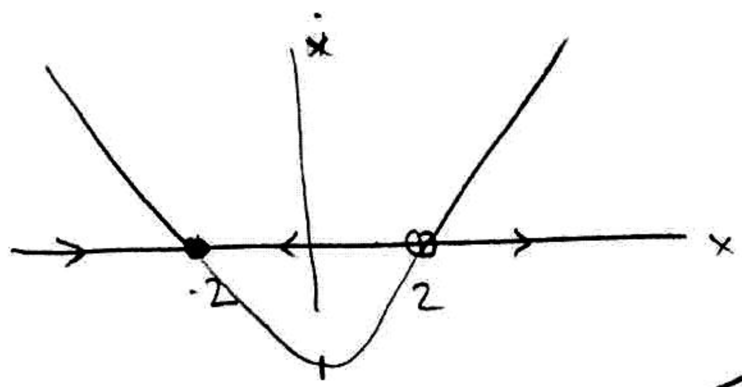
$$= \frac{2\sin x/2 \cos x/2}{2\cos^2 x/2} = \tan x/2$$

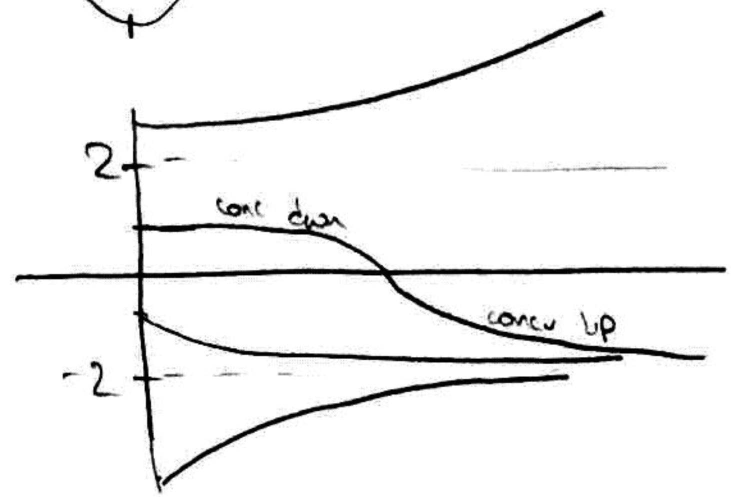
$\therefore$ $$x = 2\tan^{-1}\left[\frac{e^t}{1+\sqrt{2}}\right]$$

Pg 37 Strogatz

2.2.1

$$\dot{x} = 4x^2 - 16$$
$$= 4(x^2 - 4)$$

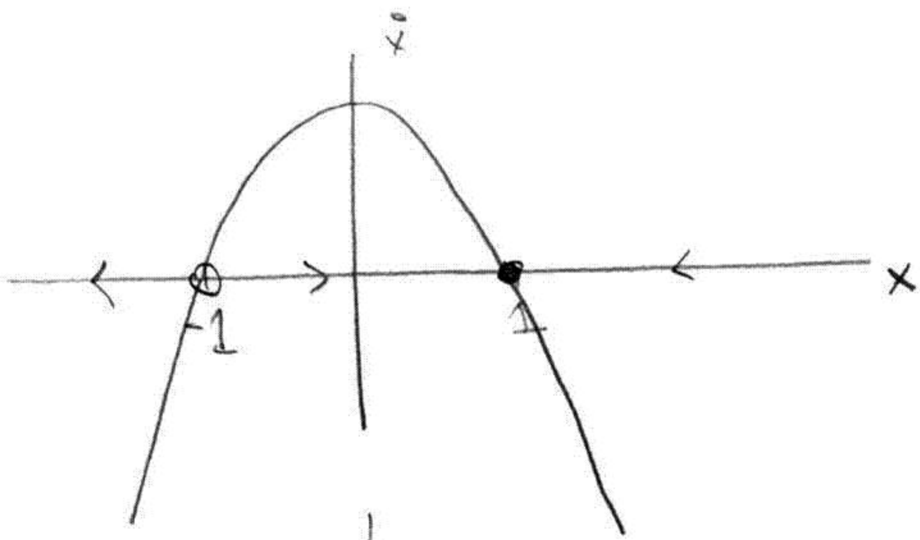




$$\frac{dx}{4(x-2)(x+2)} = dt$$

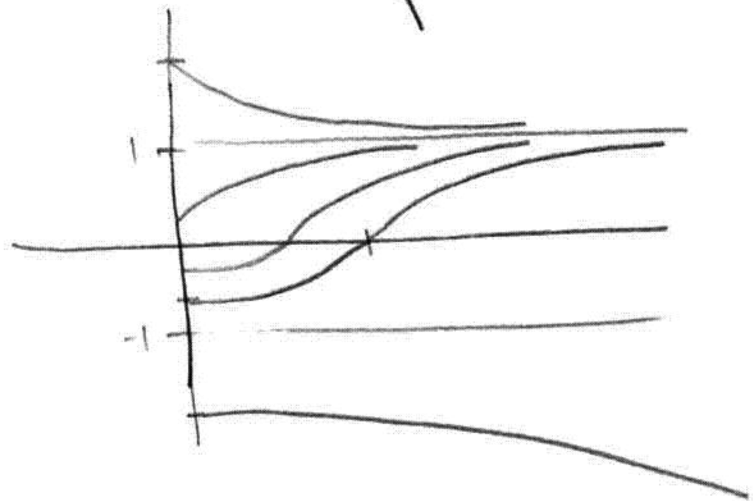
2.2.2

$\dot{x} = 1 - x^{14}$



$$\frac{dx}{(x+1)(-1+x)(\quad)(\quad)}$$

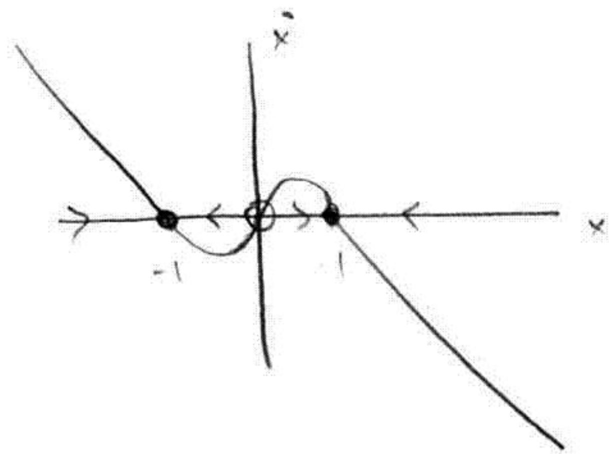
Huge Partial fractions.

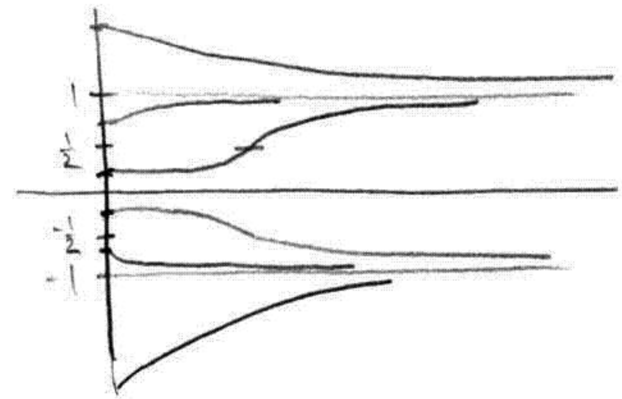


2.2.3

$\dot{x} = x(1-x)(1+x)$

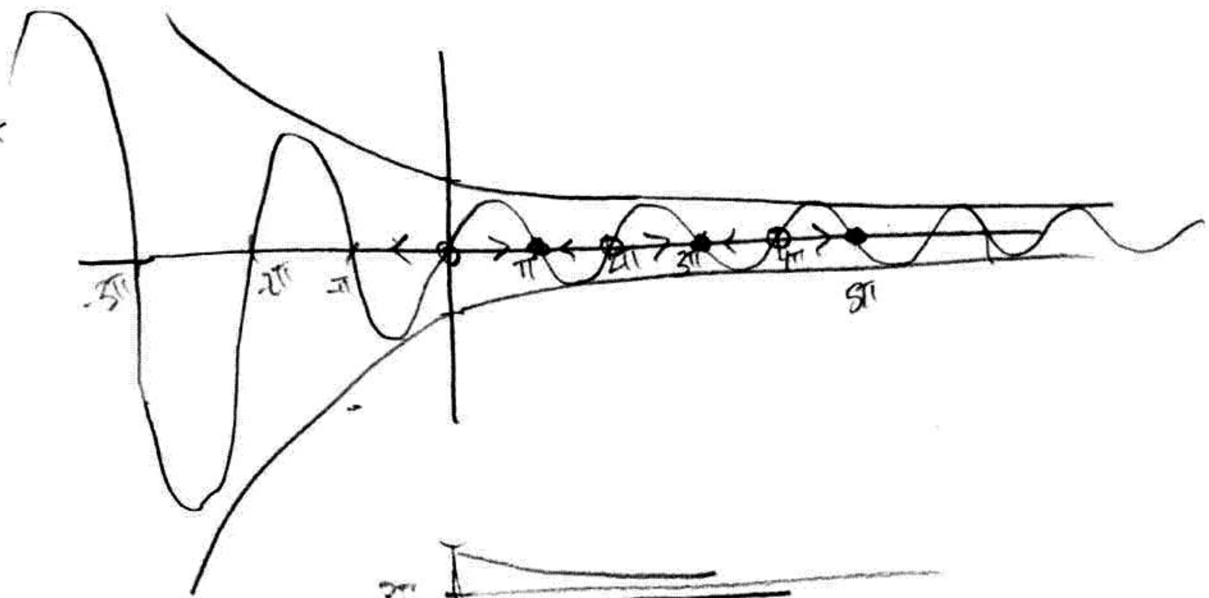
P.F How integrate





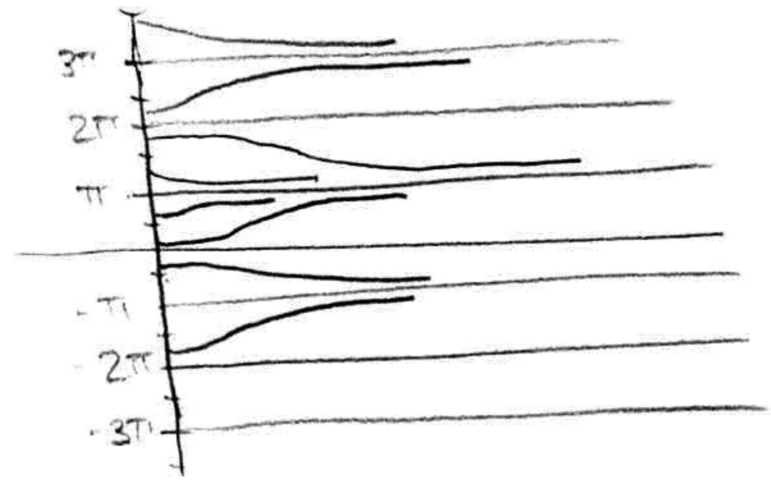
2.2.4

$\dot{x} = e^{-x} \sin x$



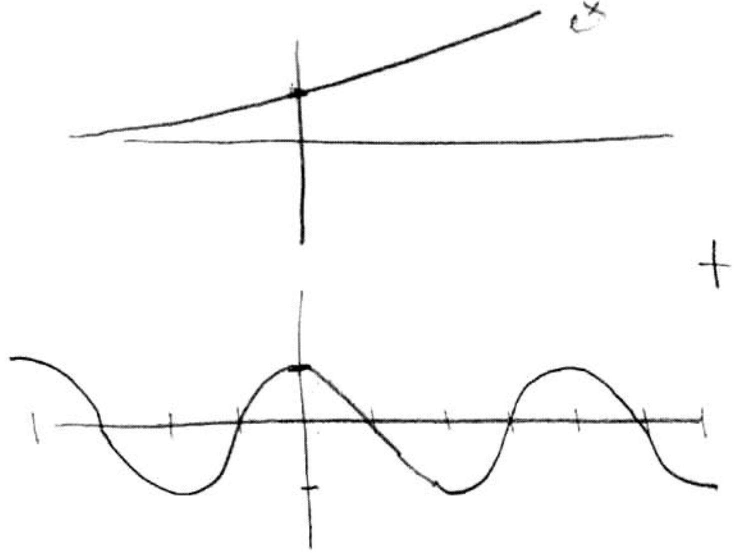
$$\frac{dx}{e^{-x} \sin x} = dt$$

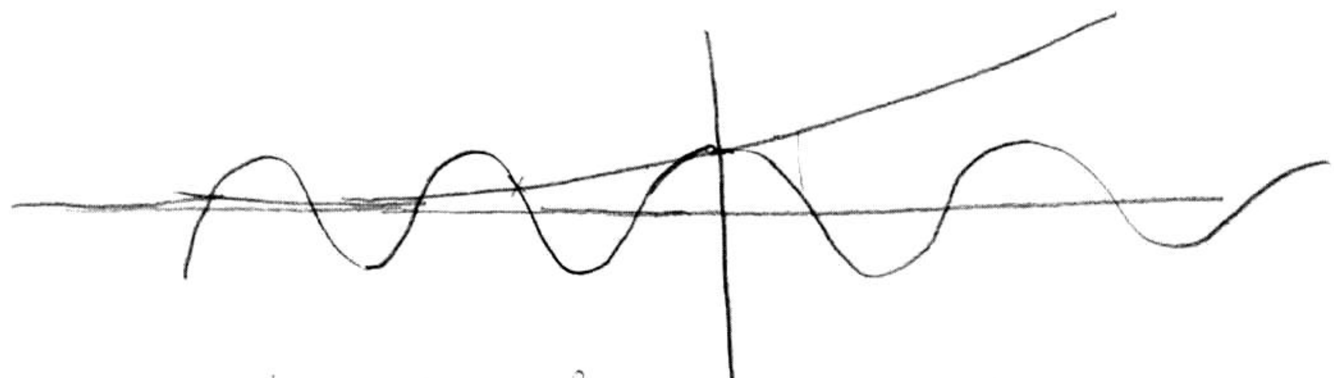
Can't do?



2.2.7 $\dot{x} = e^x - \cos x$

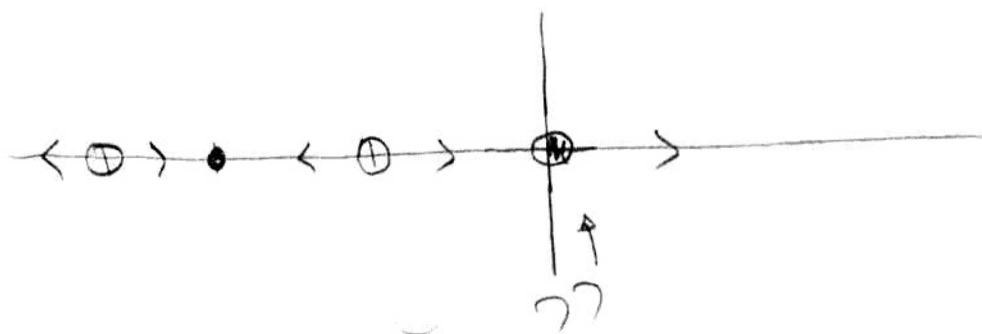
$x = 1$ sol $\dot{x} = 0$





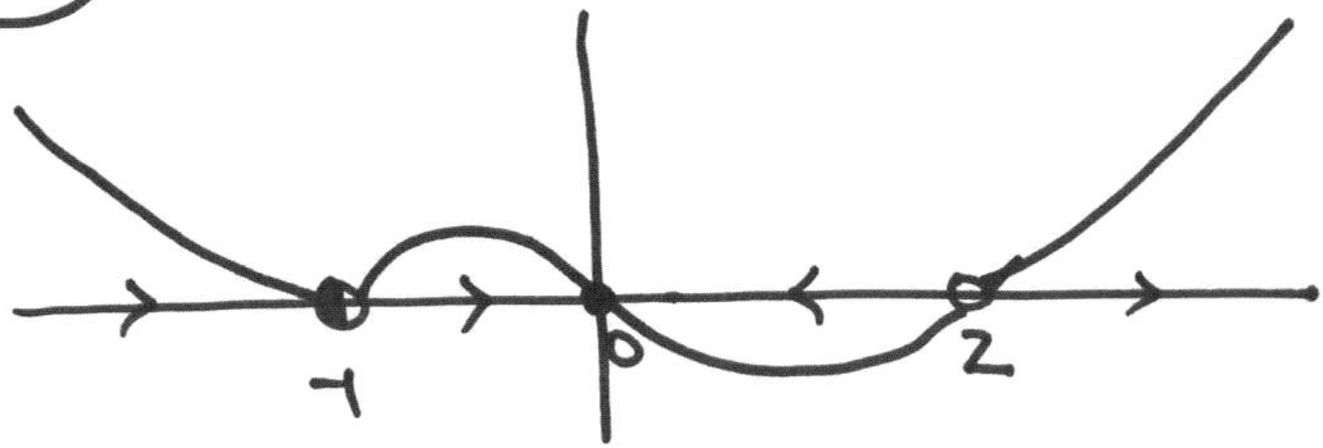
$\dot{x} < 0 \mid \dot{x} > 0 \mid \dot{x} < 0 \mid \dot{x} > 0 \mid \longrightarrow e^x \geq \cos x$

$\therefore \dot{x} > 0$



Pg 37 Strogatz

2.2.8

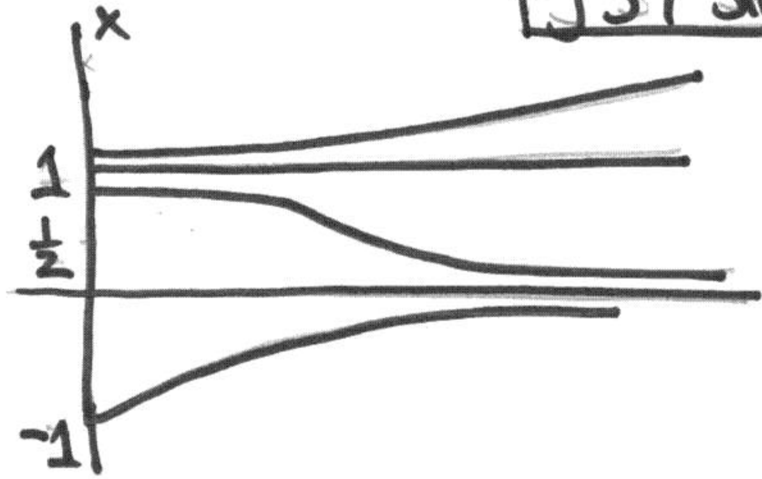


Let $f(x) = Ax(x+1)^2(x-2)$ w/ $A > 0$

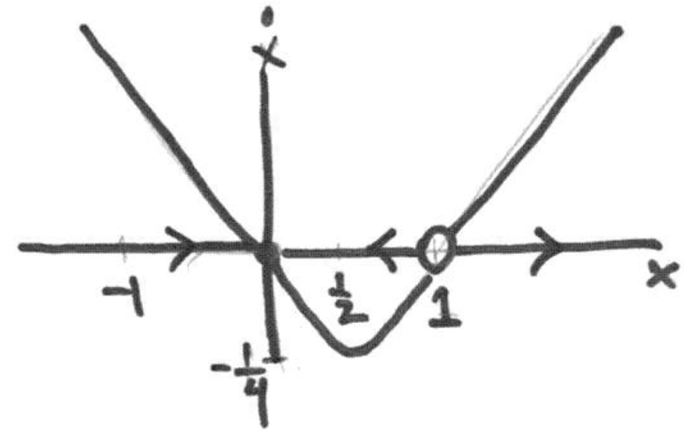
Then $\dot{x} = f(x)$ is ODE.

2.29

PJ 37 Strogatz



$\approx$



$\dot{x} = f(x)$ where $f(x) = (x-\frac{1}{2})^2 - c$

to find c $\quad x=0 \Rightarrow (\frac{1}{2})^2 - c = 0 \Rightarrow c = 1/4$

Check at $x=1 \Rightarrow f(1) = 0$ ✓

$\dot{x} = (x-\frac{1}{2})^2 - \frac{1}{4}$

2.2.13 (a) $\dot{v} = g - \frac{k}{m} v^2$

$$\frac{dv}{g - \frac{k}{m} v^2} = dt$$

$$\frac{1}{\left(\frac{k}{m}\right)} \int \frac{dv}{\left(\frac{mg}{k} - v^2\right)} = \frac{m}{k} \int \frac{dv}{\left(\sqrt{\frac{mg}{k}} - v\right)\left(\sqrt{\frac{mg}{k}} + v\right)} = \frac{m}{k} \int \left[\frac{\frac{1}{2\sqrt{\frac{mg}{k}}}}{\sqrt{\frac{mg}{k}} - v} + \frac{\frac{1}{2\sqrt{mg/k}}}{\sqrt{\frac{mg}{k}} + v} \right]$$

$$= \frac{m/k}{2\sqrt{\frac{mg}{k}}} \int \left[\frac{1}{\sqrt{\frac{mg}{k}} - v} + \frac{1}{\sqrt{\frac{mg}{k}} + v} \right] dv$$

$$= \frac{\sqrt{m/k}}{2\sqrt{g}} \left[-\ln\left(\sqrt{\frac{mg}{k}} - v\right) + \ln\left(\sqrt{\frac{mg}{k}} + v\right) \right] = \frac{1}{2\sqrt{g}} \sqrt{\frac{m}{k}} \ln\left(\frac{\sqrt{\frac{mg}{k}} + v}{\sqrt{\frac{mg}{k}} - v} \right)$$

$= t - t_0$

evaluate t_0 $\quad \dfrac{1}{2\sqrt{g}}\sqrt{\dfrac{m}{k}}\,\ln\left(1\right) = 0 - t_0 \Rightarrow t_0 = 0$

$\ln\left(\quad\quad\right) = \dfrac{2\sqrt{g}}{\sqrt{m/k}}\,t$ $\qquad$ let $\alpha = \dfrac{2\sqrt{g}}{\sqrt{m/k}} = 2\sqrt{\dfrac{gk}{m}}$

$$\frac{\sqrt{\frac{mg}{k}} + v}{\sqrt{\frac{mg}{k}} - v} = e^{\alpha t}$$

$$\sqrt{\frac{mg}{k}} + v = e^{\alpha t}\left(\sqrt{\frac{mg}{k}} - v\right) =$$

$$\left(1 + e^{\alpha t}\right)v = \left(e^{\alpha t} - 1\right)\sqrt{\frac{mg}{k}}$$

$$v(t) = \sqrt{\frac{mg}{k}} \left(\frac{e^{\frac{\alpha t}{2}} - e^{\frac{-\alpha t}{2}}}{e^{\alpha t/2} + e^{-\alpha t/2}} \right) = \sqrt{\frac{mg}{K}} \frac{\sinh\left(\frac{\alpha t}{2}\right)}{\cosh(\alpha t/2)}$$

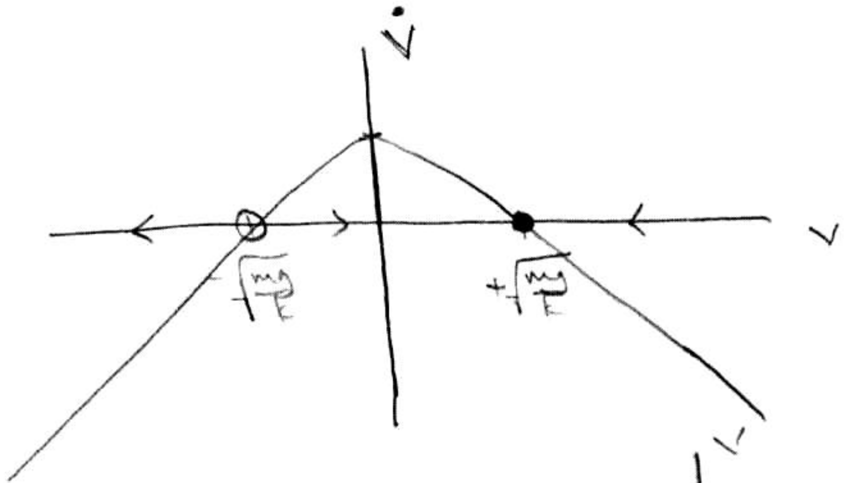
$$\therefore \quad v(t) = \sqrt{\frac{mg}{K}} \tanh\left(\frac{\sqrt{g}}{\sqrt{m/k}} t \right)$$

(b) $$\lim_{t\to\infty} v(t) = \sqrt{\frac{mg}{k}} \lim_{t\to\infty} \frac{\sinh\left(\sqrt{\frac{gk}{m}}\, t\right)}{\cosh\left(\sqrt{\frac{gk}{m}}\, t\right)}$$ let $\alpha' = \sqrt{\frac{gk}{m}}$

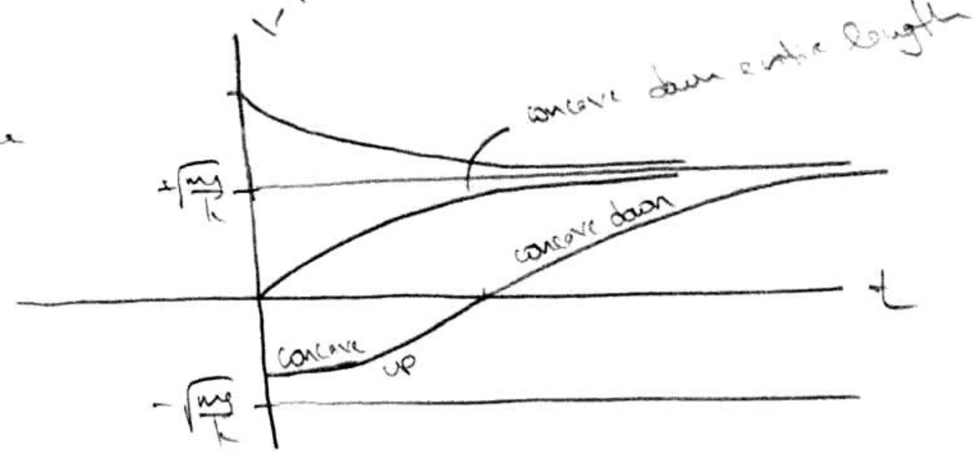
$$= \sqrt{\frac{mg}{k}} \lim_{t\to\infty} \left(\frac{e^{\alpha t} - e^{-\alpha t}}{e^{\alpha t} + e^{-\alpha t}} \cdot \frac{e^{-\alpha t}}{e^{-\alpha t}} \right)$$

$$= \sqrt{\frac{mg}{k}}\ 1$$

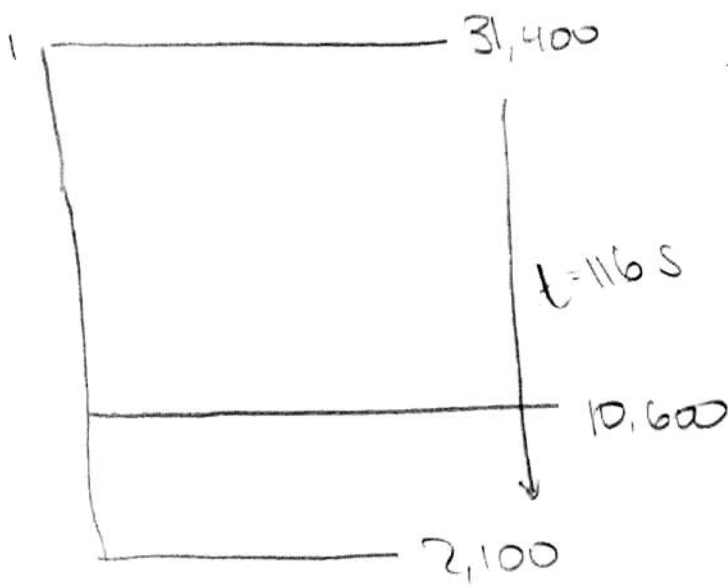
(c) $\dot{v} = g - \frac{k}{m}v^2 = \frac{k}{m}\left(\frac{m}{k}g - v^2\right) = \frac{k}{m}\left(\sqrt{\frac{mg}{k}} - v\right)\left(\sqrt{\frac{mg}{k}} + v\right)$



+ thus solutions look like



(d)



$\langle m \rangle = \frac{261.2}{32.2} = 8.11$ slug

V_{avg} = Avg termal velocity $= \dfrac{31{,}400 - 2{,}100}{116} = 252.58$ ft/s

(e) let $s(t)$ be formula for exact distance fallen

$s(0) = 0$ $\qquad\qquad s(t) = \int v(t)\,dt + S_0$

$\dot{s}(t) = v(t) = \dfrac{\sqrt{\frac{mg}{k}}\,(e^{\alpha t} - 1)}{e^{\alpha t} + 1}$

$$s(t) = \sqrt{\frac{mg}{k}}\left[\frac{1}{\alpha}\ln(e^{\alpha t}+1) - \int \frac{dt}{e^{\alpha t}+1}\right] + S_0$$

$$= \sqrt{\frac{mg}{k}}\left[\frac{1}{\alpha}\ln(e^{\alpha t}+1) - \int \frac{e^{-\alpha t}}{1+e^{-\alpha t}}\right]$$

$$= \sqrt{\frac{mg}{k}}\left[\frac{1}{\alpha}\ln(e^{\alpha t}+1) + \frac{1}{\alpha}\ln(1+e^{-\alpha t})\right]$$

$$= \frac{1}{\alpha}\sqrt{\frac{mg}{k}}\left[\ln\left((1+e^{\alpha t})(1+e^{-\alpha t})\right)\right] + S_0$$

$$= \frac{1}{2}\,\frac{k}{m}\left[\ln\left(1+1+2\cosh(\alpha t)\right)\right] + S_0$$

$$2 + 2\cosh(\alpha t) = 2(1+\cosh(\alpha t))$$

$$\frac{1+\cos(2x)}{2} = \cos^2 x$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$\cosh(2x) = \cosh^2 x + \sinh^2 x$$

$$\Rightarrow \quad \cosh(2x) + 1 = 2\cosh^2 x$$

$$\cosh^2 x = \frac{1}{2}(\cosh(2x)+1)$$

$\therefore$ Above is $4\cosh^2\left(\frac{\alpha}{2}t\right)$

$$\therefore \; s(t) = \frac{1}{2}\frac{k}{m}\ln\left(4\cosh^2\left(\tfrac{\alpha}{2}t\right)\right) + S_0$$

$$= \frac{k}{m}\ln\left(4\cosh\left(\tfrac{\alpha}{2}t\right)\right) + S_0$$

$$= \frac{k}{m}\ln 4 + S_0 + \frac{k}{m}\ln\left(\cosh\left(\sqrt{\tfrac{gk}{m}}\,t\right)\right)$$

$$S(0) = 0 \Rightarrow S_0 = -\frac{k}{m}\ln 4$$

$$\therefore \; s(t) = \frac{k}{m}\ln\left(\cosh\left(\sqrt{\tfrac{gk}{m}}\,t\right)\right) \qquad V^2 = \frac{mg}{k}$$

$$= \frac{V^2}{g}\ln\left(\cosh\left(\frac{gt}{V}\right)\right)$$

$S = 29{,}300 \qquad t = 116 \qquad g = 32.2$

$$29{,}300 = \frac{v^2}{32.2} \ln\left(\cosh\left(\frac{32.2}{v} \cdot 116\right)\right) \longleftarrow \text{solve numerically.}$$

Suppose $v \approx v_{Avg}$ Then $\frac{gt}{v}$

put v_{Avg} in for v^2

$$\ln\left(\cosh\left(\frac{g}{v} t\right)\right) = \cdots \quad \Rightarrow \frac{g}{v} t = 15.48 \gg 1$$

$\therefore$ use Approx $\ln(\cosh x) \approx x - \ln 2$ for $x \gg 1$

Show $\displaystyle \lim_{x \to \infty} \frac{\ln(\cosh x)}{x - \ln 2} = 1$

$$= \lim_{x \to \infty} \frac{\frac{1}{\cosh x}(\sinh x)}{1} = 1 \checkmark$$

$$\ln(\cosh x) \xrightarrow{x \gg 1} \ln(\cosh(\tfrac{1}{t})) = x - \ln 2$$

How derive expansion?

If so $\frac{V^2}{g}\left(\frac{gt}{V} - \ln 2\right) = S$ Can solve for V.

2.31

(a) ✓

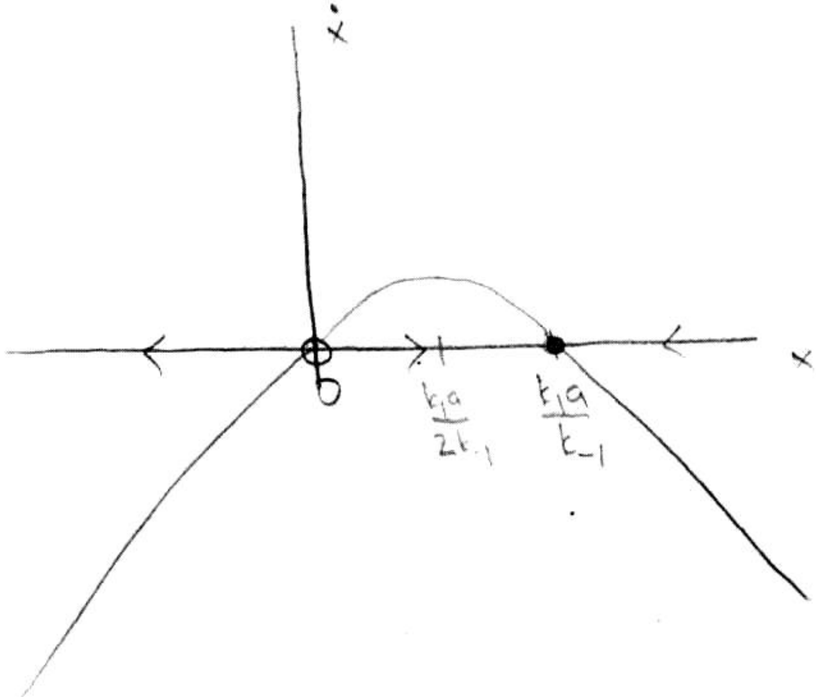
(b) $\dot{N} = rN\left(1 - \frac{N}{k}\right) = rN - \frac{r}{k}N^2$ Bernoulli

$x = \frac{1}{N}$ $\dot{x} = -\frac{1}{N^2}\dot{N}$ Bernoulli subs.

2.3.2

$\dot{x} = k_1 a x - k_{-1} x^2 = x(k_1 a - k_{-1} x)$ $\qquad k_1, a, k_{-1} > 0$

(a)



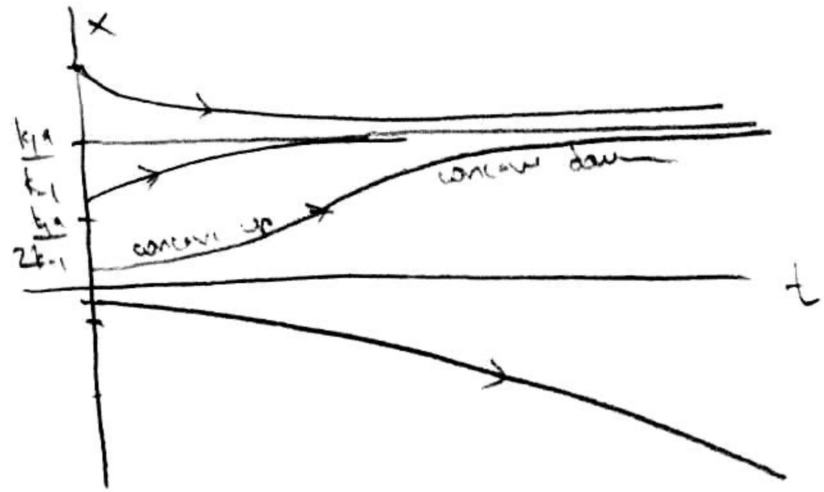
Fixed points $x = 0$ & $x = \frac{k_1 a}{k_{-1}}$

$x = 0$ unstable

$x = \frac{k_1 a}{k_{-1}}$ stable

(over)

(b)



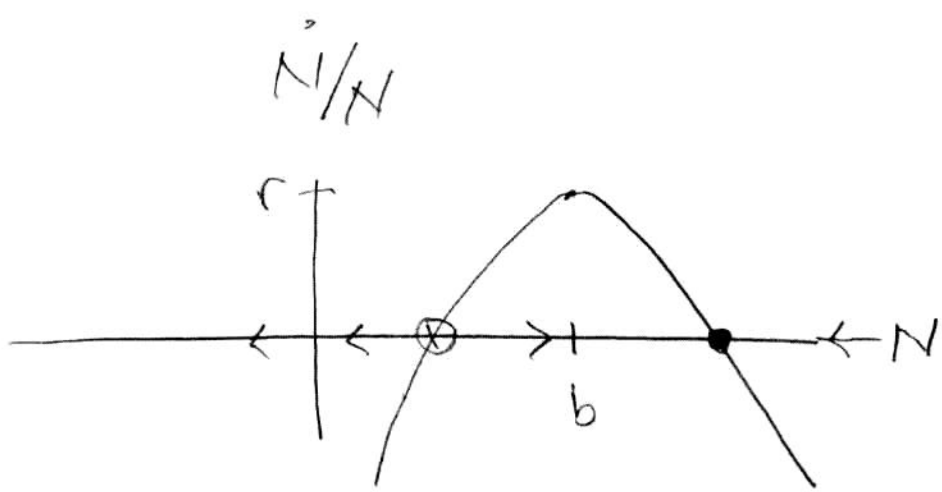
2.3.4

(a) $\frac{\dot{N}}{N} = r - a(N-b)^2$

$N \ll 1$ $\qquad \frac{\dot{N}}{N} \approx r - ab < 0$ if Appr

$N \gg 1$ $\qquad \frac{\dot{N}}{N} = r - aN^2 < 0$ $\quad N \gg 1$

$r, a, b > 0$



Must have $\frac{\dot{N}}{N}$ to the right side of the origin

Pg 40 ~~B~~ Strogatz

2.4.1 $\dot{x} = x(1-x) = x - x^2$

St pt $x = 0$ $x = 1$

$f'(0) = 1 - 2x = 1 > 0$ unstable

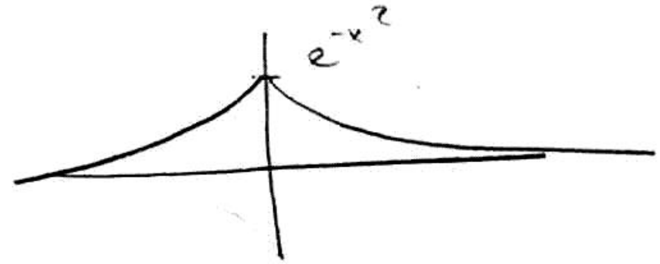
$f'(1) = -1 < 0$ stable

2.4.3 $\dot{x} = \tan x$ on $-\frac{\pi}{2}, \frac{\pi}{2}$

$f'(x) = -\sec^2 x$

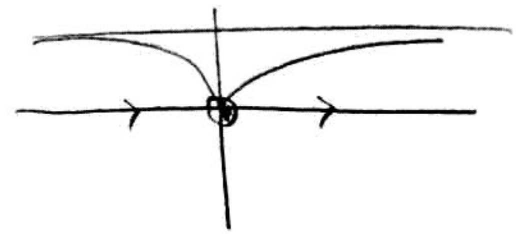
$f'(0) = -1$ Asy stable.

2.45 $1-e^{-x^2}$



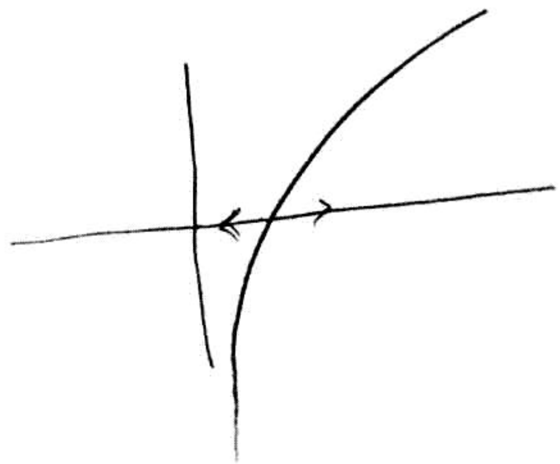
$f' = 2xe^{-x^2}$

$f'(0) = 0$.



$\frac{1}{2}$ stable

2.46 $\dot{x} = \ln x$



$f'(x) = \frac{1}{x}$

$f'(1) = 1 > 0$

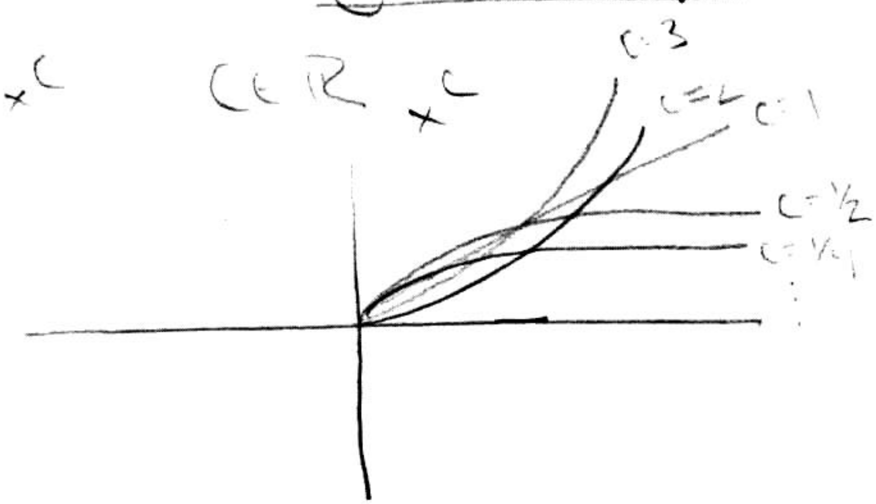
unstable

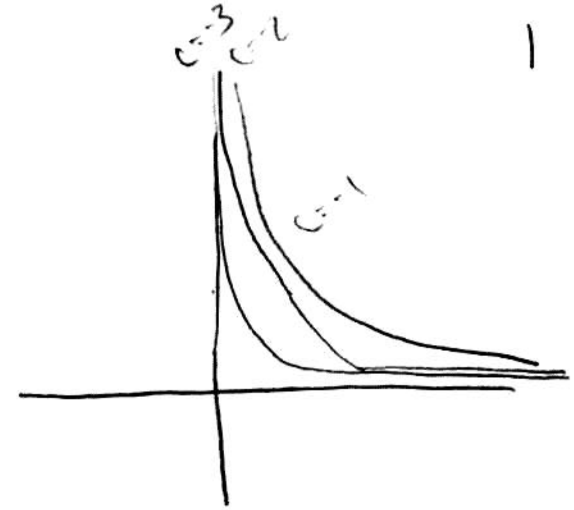
Pg 40 Strogatz

1

(2.5.1) $\dot{x} = -x^c \qquad c \in \mathbb{R}$

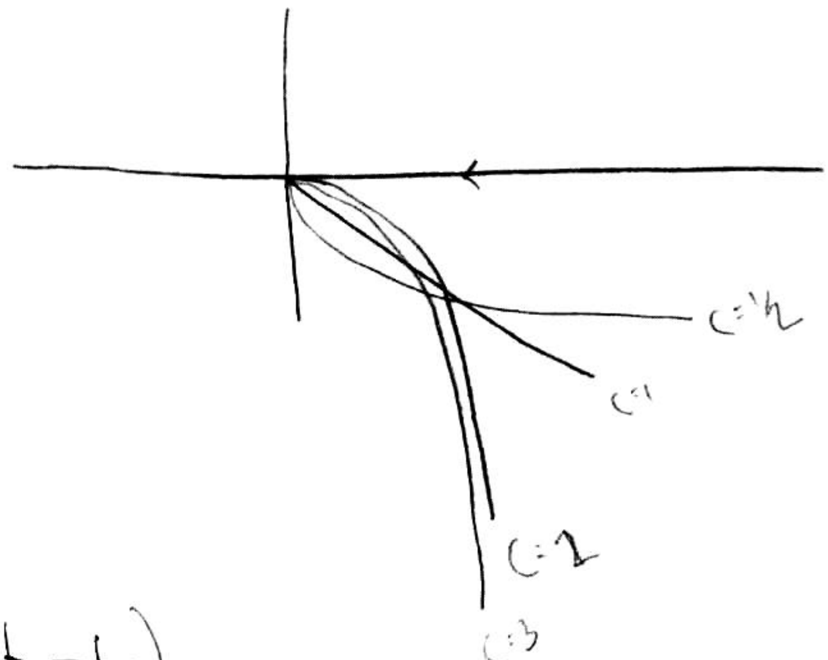
(a)

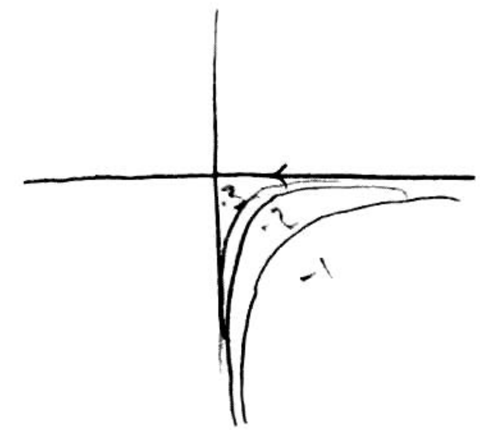




|| $\quad c > 0$

$c < 0$





(b) $x(t) \overset{?}{=} 0$

$$\frac{dx}{x^c} = -t$$

All values of C

$c \neq 1 \qquad \dfrac{x^{(-c+1)}}{(-c+1)} = -(t - t_0)$

$$x^{-c+1} = (1-c)(t_0 - t)$$

$$\Delta t = t(1) - t(0)$$

$$= \frac{1^{-c+1}}{1-c}$$

2.52

$$\dot{x}_2 = 1 + x_2^{10} > 1 + x_1^2 = \dot{x}_1$$

$$\dot{x}_1 = 1 + x_1^2$$

$$\frac{dx_1}{1+x_1^2} = dt$$

$$\tan^{-1} x_1 = t - t_0$$

$$x_1(t) = \tan(t - t_0) \qquad t = t_0 + \frac{\pi}{2} \quad \nearrow \infty$$

$\therefore$ $x_2 \nearrow \infty$ finite time

Pg 40 Strogatz

2.5.3

$\dot{x} = rx + x^3 \qquad r > 0$

$$\frac{dx}{x(r+x^2)} = \frac{dx}{x(\sqrt{r}+x)(-\sqrt{r}+x)} = dt$$

get

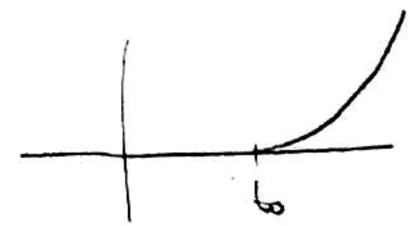
Pg 40 Strogatz 1

(2.5.4) $\dot{x} = x^{1/3}$

$x^{-1/3}dx = dt$

$\frac{3}{2}x^{2/3} = t - t_0$

$x = \left(\frac{2}{3}(t-t_0)\right)^{3/2} = \left(\frac{2}{3}\right)^{3/2}(t-t_0)^{3/2}$



Thus let $x_{sol} = \begin{cases} 0 & t < t_0 \\ \left(\frac{2}{3}\right)^{3/2}(t-t_0)^{3/2} & t > t_0 \end{cases}$

Now this ~~solution~~ is continuous at $t = t_0$

& the derivative exists at t_0 & $\therefore$ x_{sol} is differentiable

$$x'_{sol}\Big|_{t_0} = \left(\frac{2}{3}\right)^{3/2}\left(\frac{3}{2}\right)^{1}(t-t_0)^{1/2}\Big|_{t_0} = 0$$

As t_0 is arbitrary we have an entire family of curves & thus ∞ many solutions.

Pg 41 Strogatz

2.6.1 $m\ddot{x} = -kx$ is not a 1-D system. 1-D systems can only have 1st derivatives. Letting $y = \dot{x}$

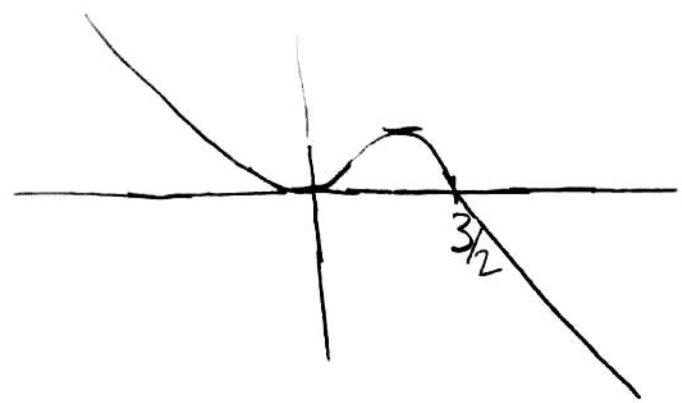
Then $\dot{x} = y$
+ $\dot{y} = -\frac{k}{m}x$ is the 2D system that DE represents.

Pg 42 Strogatz

$$\frac{dV}{dx} = -f$$

2.7.1 $\dot{x} = x(1-x)$

$$V = -\int x(\cancel{1}-x) = -\frac{x^2}{2} + \frac{x^3}{3} = x^2\left(-\frac{1}{2} + \frac{x}{3}\right)$$

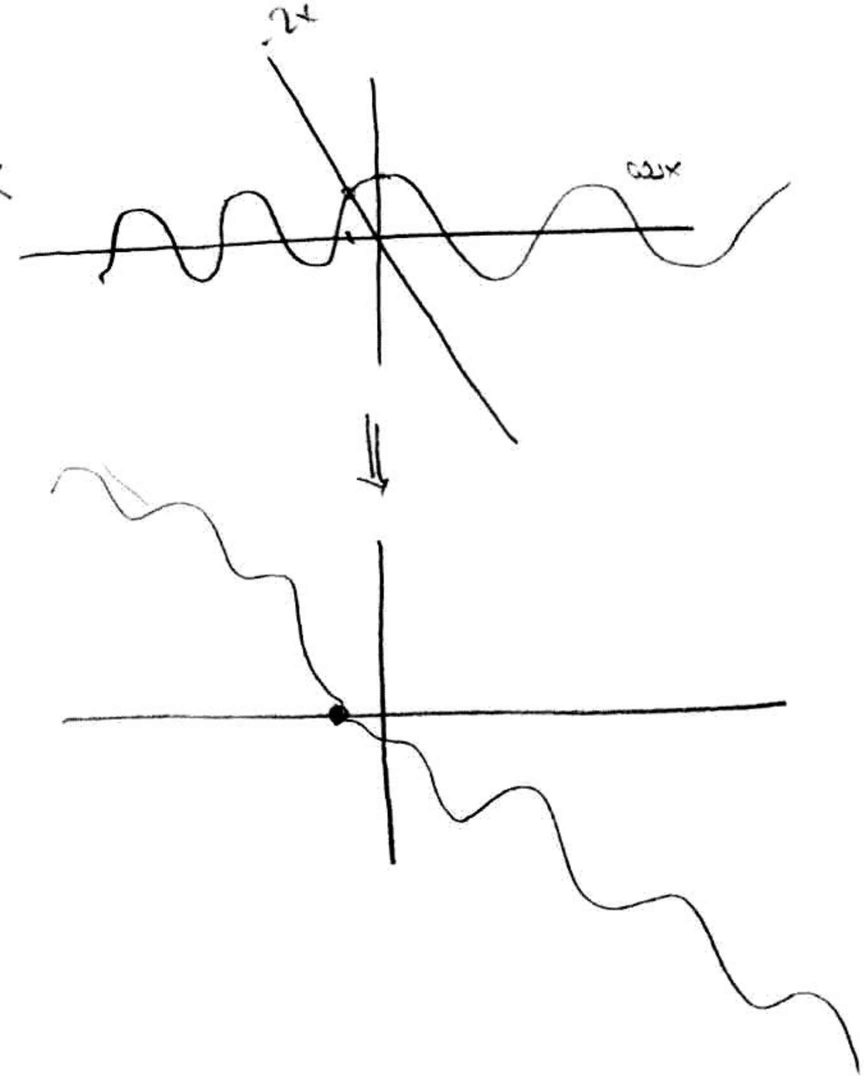


2.7.4

$\dot{x} = 2 + \sin x$

$f(x) = 2 + \sin x$

$V = -\int f = -2x + \cos x$



??

P.J 62 Strogatz

$$b\dot{\phi} = mg\sin\phi\Big(\underbrace{\frac{r\omega^2}{g}}_{r}\cos\phi - 1\Big) =: f(\phi)$$

If $r > 1$ 2 fixed pts $\phi = 0, \pi$.

Also $\cos\phi^* = \frac{1}{r}$

Is $\phi = 0, \pi, \pm\cos^{-1}(1/r)$ stable or unstable?

Analize the linear system

$$f'(\phi) = mg\cos\phi\big(r\cos\phi - 1\big) + mg\sin\phi(-r\sin\phi)$$

$$f'(0) = mg(r-1) \begin{cases} < 0 & r < 1 \text{ stable} \\ > 0 & r > 1 \text{ unstable} \end{cases}$$

$$f'(\pi) = -mg(-r-1) = mg(r+1) > 0 \quad \forall\, r > 0.$$

Always unstable

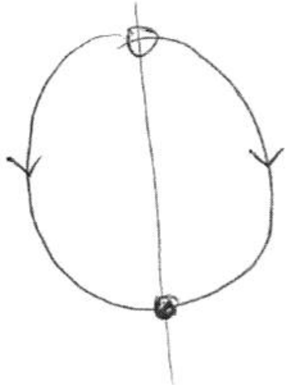
$$f'(\pm\cos^{-1}(1/r)) = mg\left(\tfrac{1}{r}\right)\cdot 0 - mg\sin\phi^*\, r\sin\phi^*$$

$$= -mgr\sin^2\phi^* = -mgr(1-\cos^2\phi^*)$$

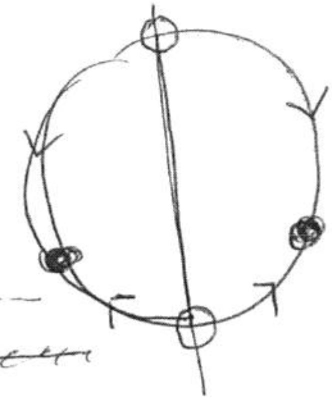
$$= -mgr\left(1 - \tfrac{1}{r}\right) = +mg(1-r)$$

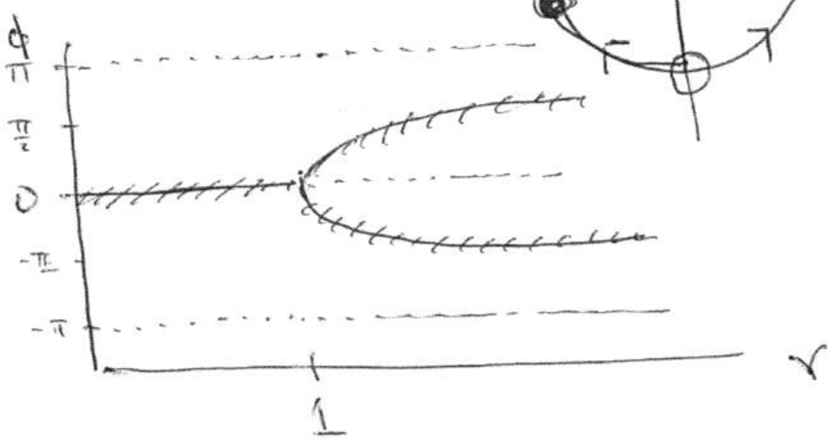
$$= \begin{cases} > 0 & r < 1 \text{ unstable} \\ < 0 & r > 1 \text{ stable} \end{cases}$$

Thus $r < 1$ 2 fixed pts $0, \pi$, 0 is stable
π " unstable



If $r > 1$ 4 fixed pts
Both are stable ϕ^*, 0 is unstable
π " "

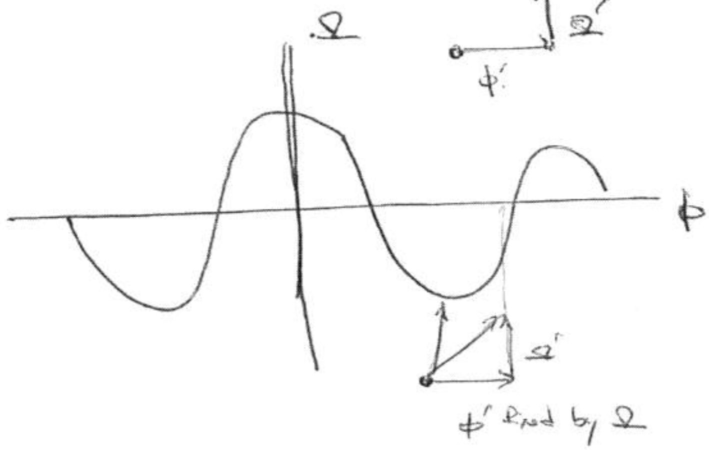




pg 67-68 Strogatz

$\phi' = \Omega$

$\Omega' = \frac{1}{\epsilon}(f(\phi) - \Omega)$



Draw graph of $\Omega = f(\phi)$

See Fig 3.5.8.

Then if initially $f(\phi) \neq \Omega$ (say a pt off this line

As $\epsilon \to 0$ ~~$\Omega' \to \infty$~~ say $f(\phi) > \Omega$ i.e. below

graph $f(\phi)$ Then $\Omega' \to +\infty$ goes st to $\Omega = f(\phi)$

once $f = \Omega$ $\Omega' = 0$. Then $\phi' = \Omega$ go left or

right (depending on IC) to reach the fixed pt. So on $\Omega = f(\phi)$

evolve according to $\phi' = \Omega = f(\phi)$ 1st order system

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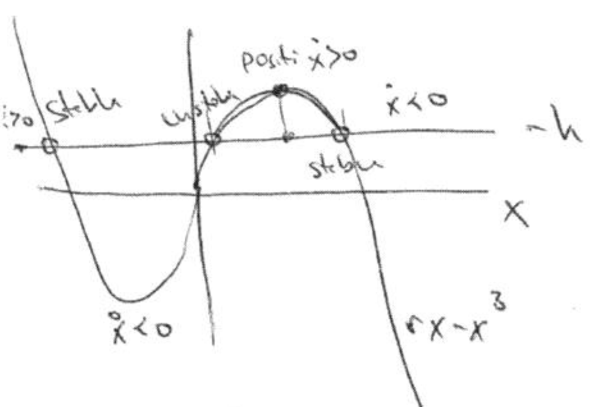
pg 72. Strogatz

$r > 0$ 3 intersections in $\dot{x} = h + rx - x^3$ plotting

$-h + rx - x^3$ $rx - x^3 - (-h)$



$r < 0$

pg 75 Strogatz

$$\frac{\dot{N}}{B} = \frac{R}{B} N\left(1 - \frac{N}{K}\right) - \frac{N^2}{A^2 + N^2}$$

let $x = N/A$

$\dot{x} = \dot{N}/A$

$$\frac{A\dot{x}}{B} = \frac{R}{B} Ax\left(1 - \frac{Ax}{K}\right) - \frac{A^2x^2}{A^2 + A^2x^2}$$

$$\frac{A}{B}\dot{x} = \frac{R}{B} A x\left(1 - \frac{Ax}{K}\right) - \frac{x^2}{1+x^2}$$

$$\tau = \frac{Bt}{A} \quad + \quad r = \frac{RA}{B} \qquad k = \frac{K}{A}$$

eq (3)

$$x\left(r\left(1 - \frac{x}{k}\right) - \frac{x}{1+x^2}\right) = 0 \qquad \rightarrow x = 0$$

or $$\frac{x}{1+x^2} = r\left(1 - \frac{x}{k}\right)$$

for $x = 0$ let $f(x) = rx\left(1 - \frac{x}{k}\right) - \frac{x^2}{1+x^2}$

$$f'(x) = r\left(1 - \frac{x}{k}\right) + rx\left(-\frac{1}{k}\right) - \frac{2x}{1+x^2} + \frac{2x^3}{(1+x^2)^2}$$

$f'(0) = r > 0$ unstable

pg 76 Strogatz

Ref

$f_1(x) = \frac{x}{1+x^2}$

Slope at 0 is $f'(x) \sim \frac{1}{1+x^2}\Big|_{x=0} = 1 > 0.$

$f_2(x) = r\left(1 - \frac{x}{k}\right)$

$f_2(k) = 0 \qquad f_2(0) = r$

---

pg 77 Strogatz

$$\frac{1}{1+x^2} - \frac{2x^2}{(1+x^2)^2} = \frac{1+x^2-2x^2}{(1+x^2)^2} =$$

$$r - \frac{r}{k}x = \frac{x}{1+x^2}$$

$$r + \frac{x(1-x^2)}{(1+x^2)^2} = \frac{x}{1+x^2}$$

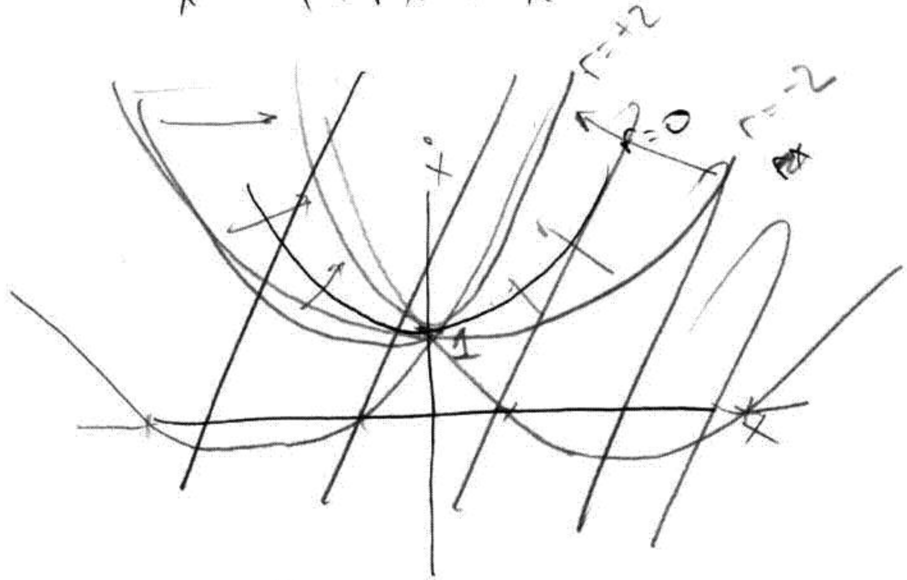
$$r = \frac{-\cancel{x} + x^3 + \cancel{x} + x^3}{(1+x^2)^2} = \frac{2x^3}{(1+x^2)^2}$$

$$\Rightarrow \quad k = -r\frac{(1+x^2)^2}{1-x^2} = -\frac{2x^3}{(1+x^2)^2}\frac{(1+x^2)^2}{1-x^2} = \frac{2x^3}{x^2-1}$$

Pg 79 Strogatz

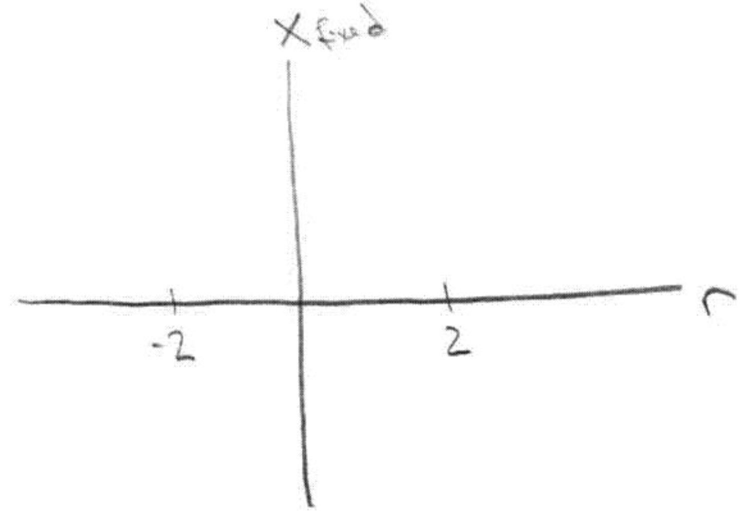
3.1.1

$$\dot{x} = 1 + rx + x^2 = 0 \Rightarrow x = \frac{-r \pm \sqrt{r^2-4}}{2} = \frac{-r}{2} \pm \frac{\sqrt{r^2-4}}{2}$$



$-2 < r < 2$ No

Critical pts

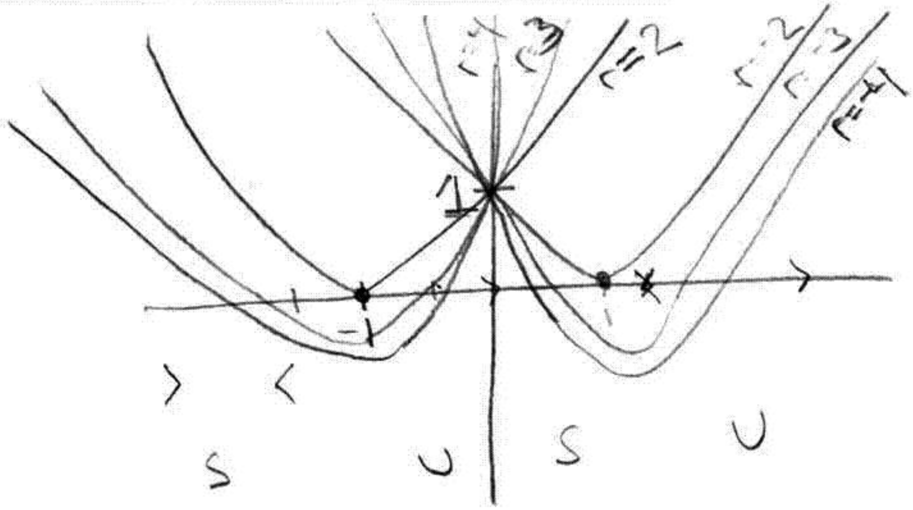


$r > 2$

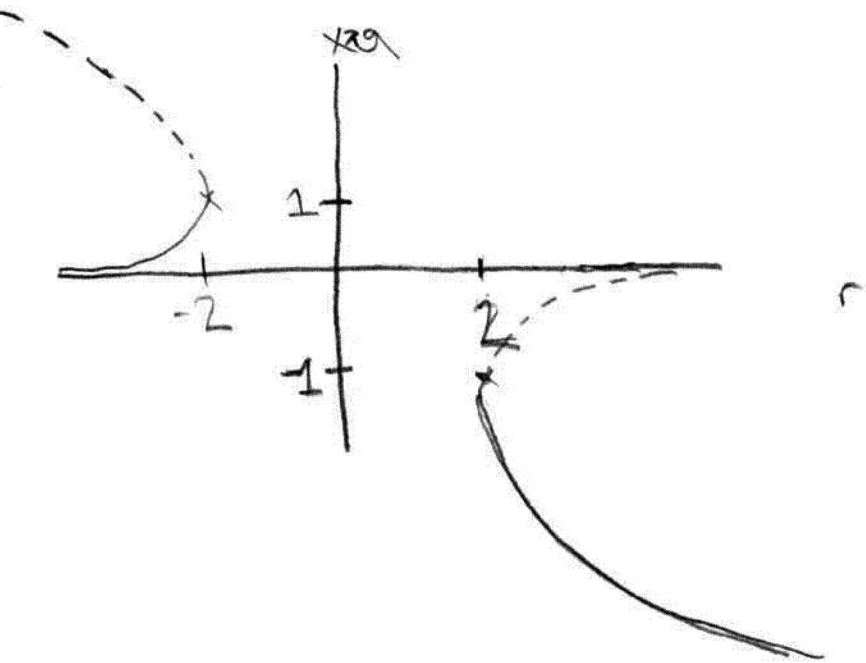
Both roots are negative

$r < -2$

" " " pos.



Thus

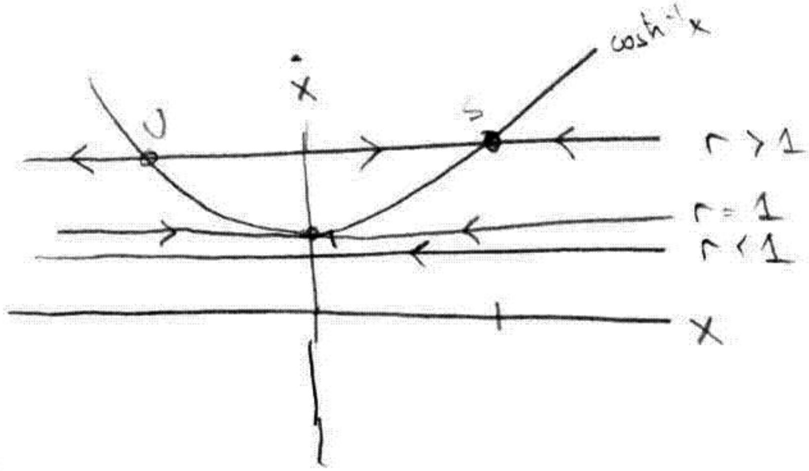


$$x_{eq} = -\frac{r}{2} \pm \sqrt{\frac{r^2-4}{2}}$$

3.1.2

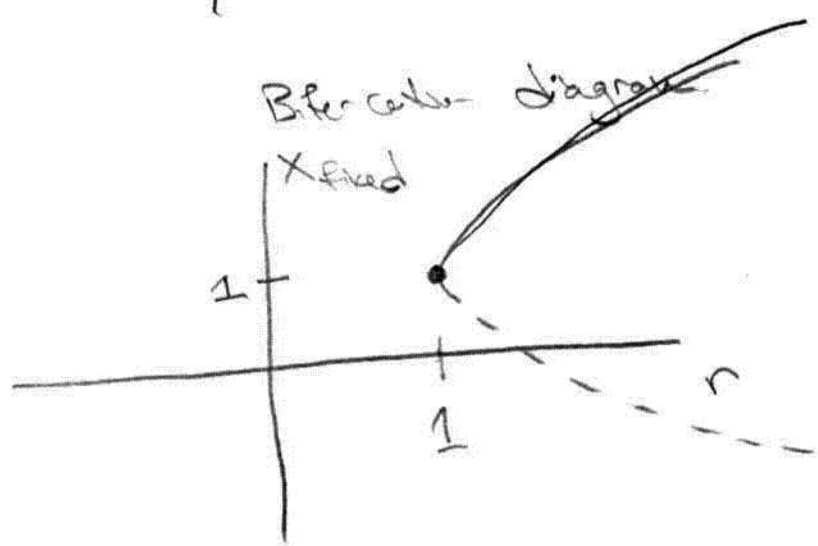
$\dot{x} = r - \cosh x = 0$

$\Rightarrow \; x = \cosh^{-1} r$



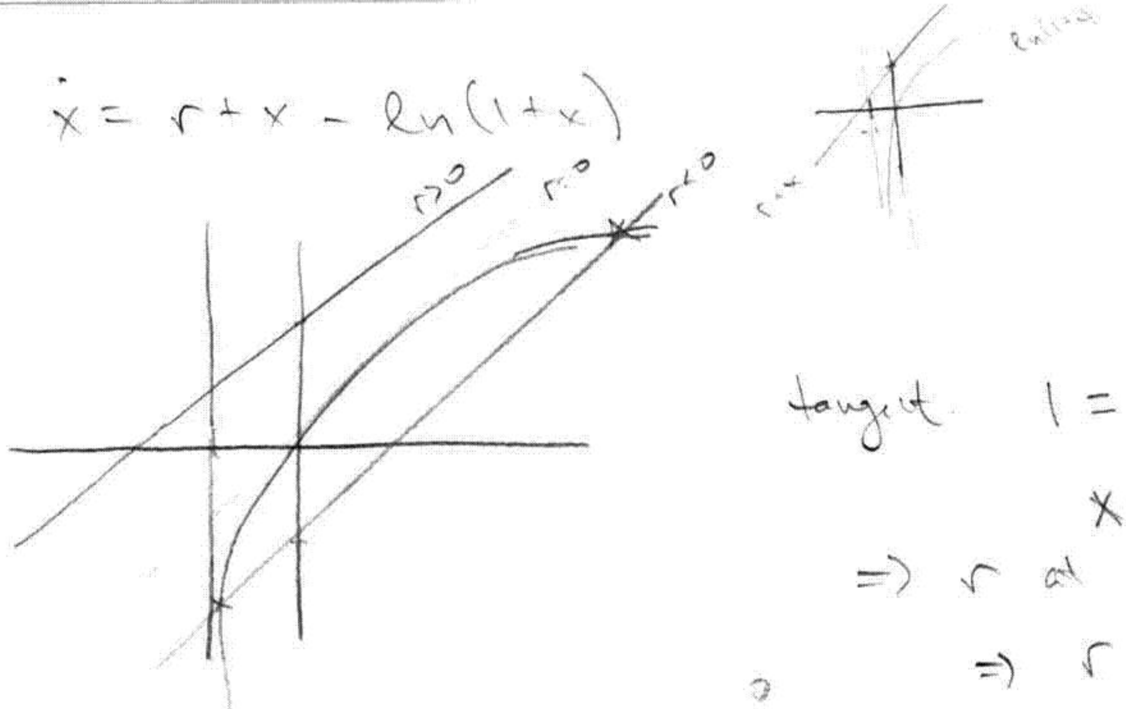
$\underline{r = 1 \quad 1 \text{ critical value}}$

Bifurcation diagram



$x = \cosh^{-1}$

3.1.3 $\dot{x} = r + x - \ln(1+x)$

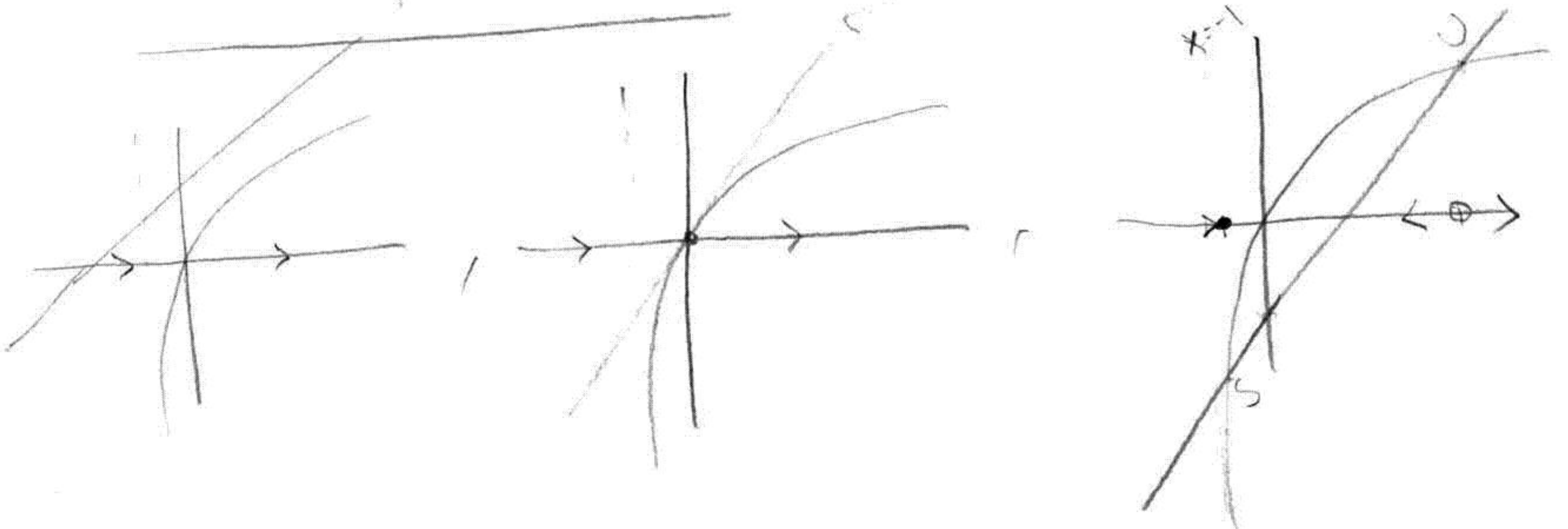


tangent $1 = \frac{+1}{1+x}$

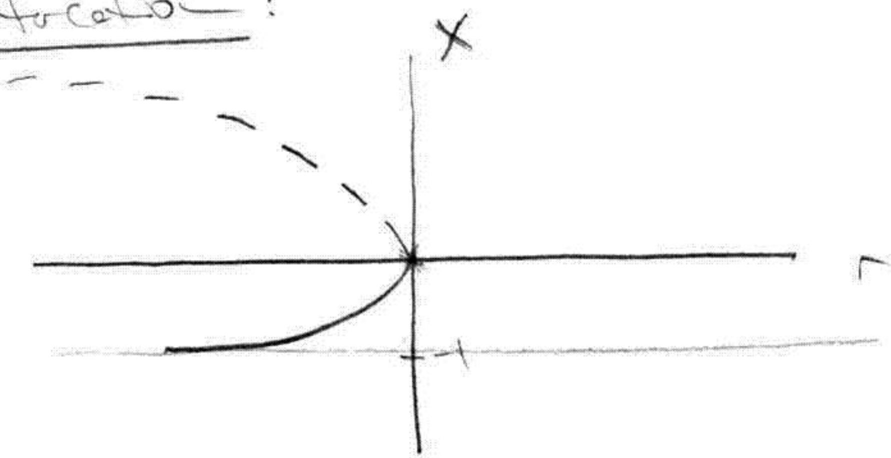
$x=0$

$\Rightarrow r$ at tangent

$\Rightarrow r = 0.$



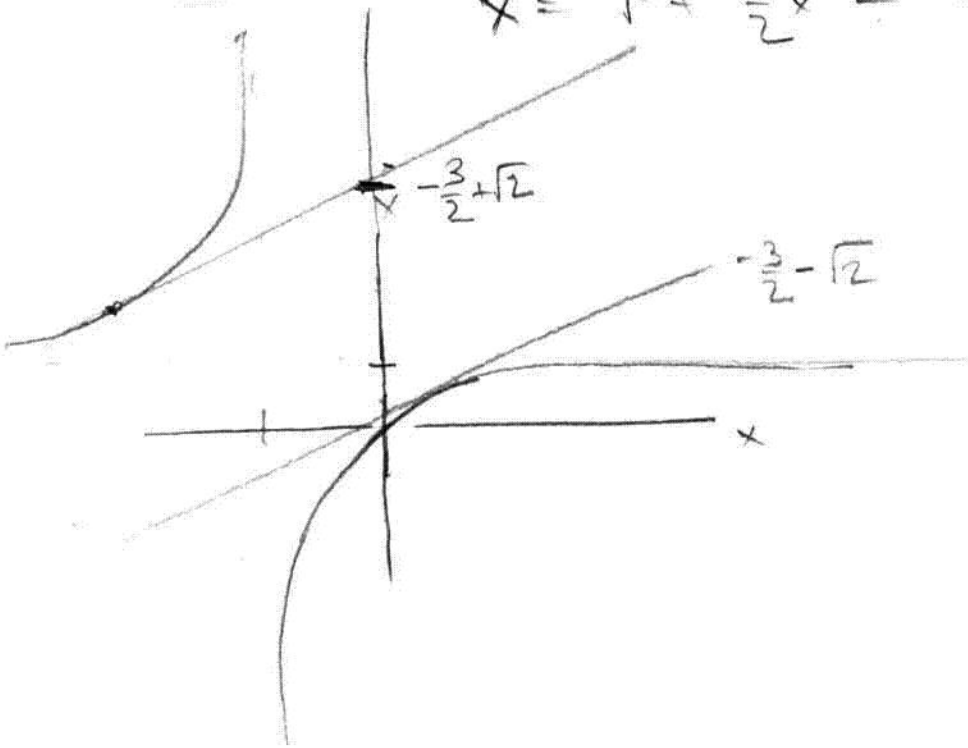
Byfurcation:



eg $r = \ln(1+x) - x$

(3.1.4)

$$\dot{x} = r + \frac{1}{2}x - \frac{x}{1+x}$$



If Byfurcation occ. then

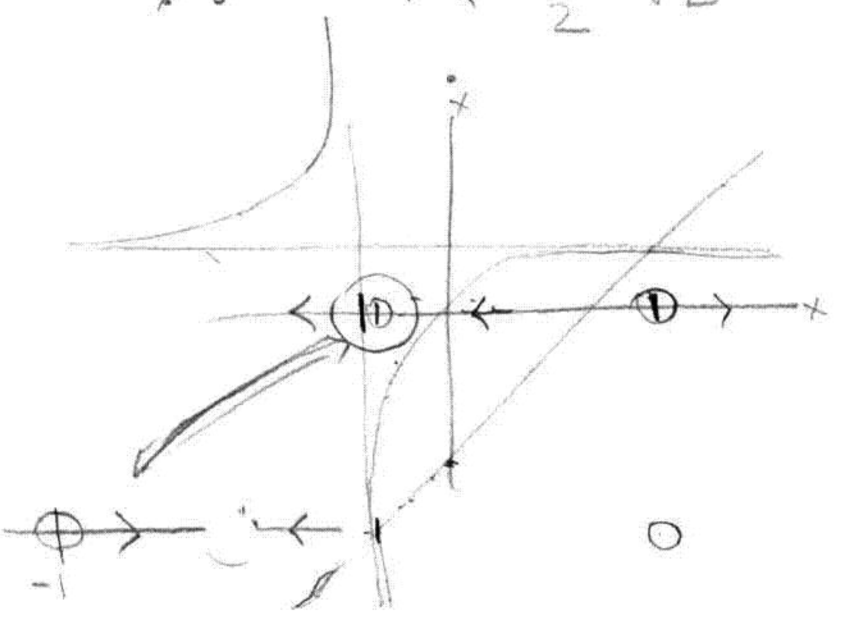
$$\frac{1}{2} = \frac{d}{dx}\left(\frac{x}{1+x}\right)$$

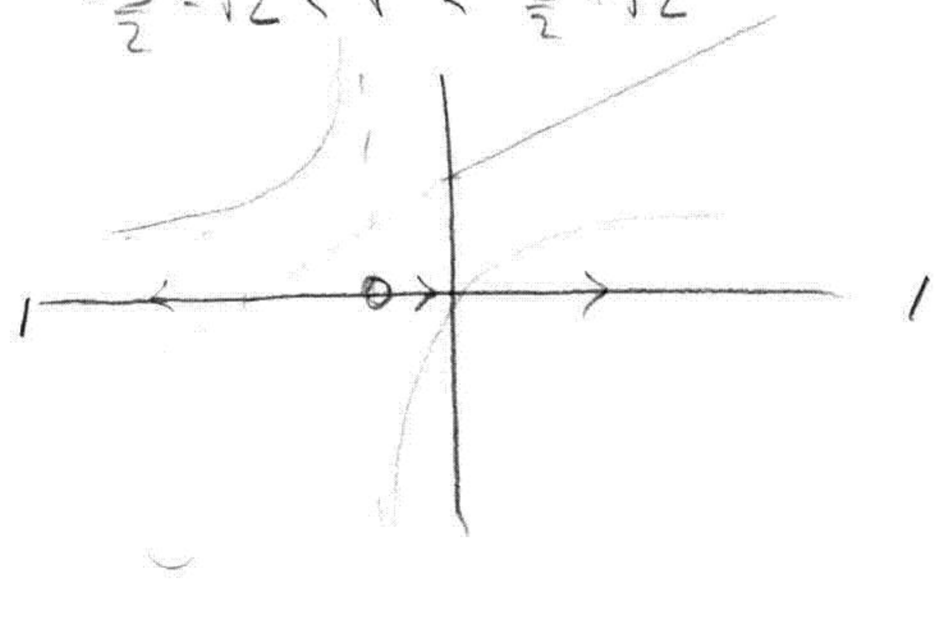
$$\Rightarrow x = \frac{-2 \pm \sqrt{2}}{2}$$

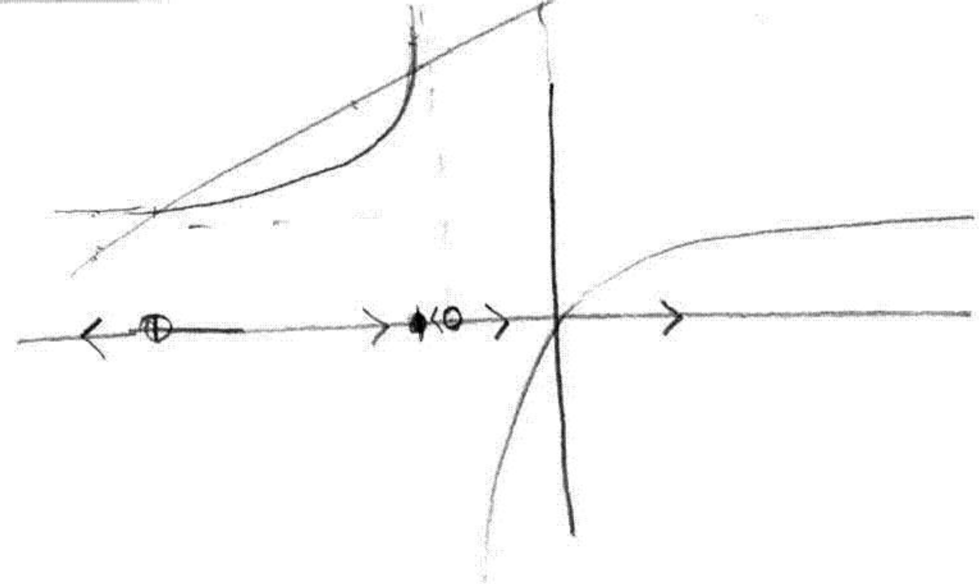
Then $r = -\frac{3}{2} - \sqrt{2},\ -\frac{3}{2} + \sqrt{2}$

$\therefore$ $r < -\frac{3}{2} - \sqrt{2}$

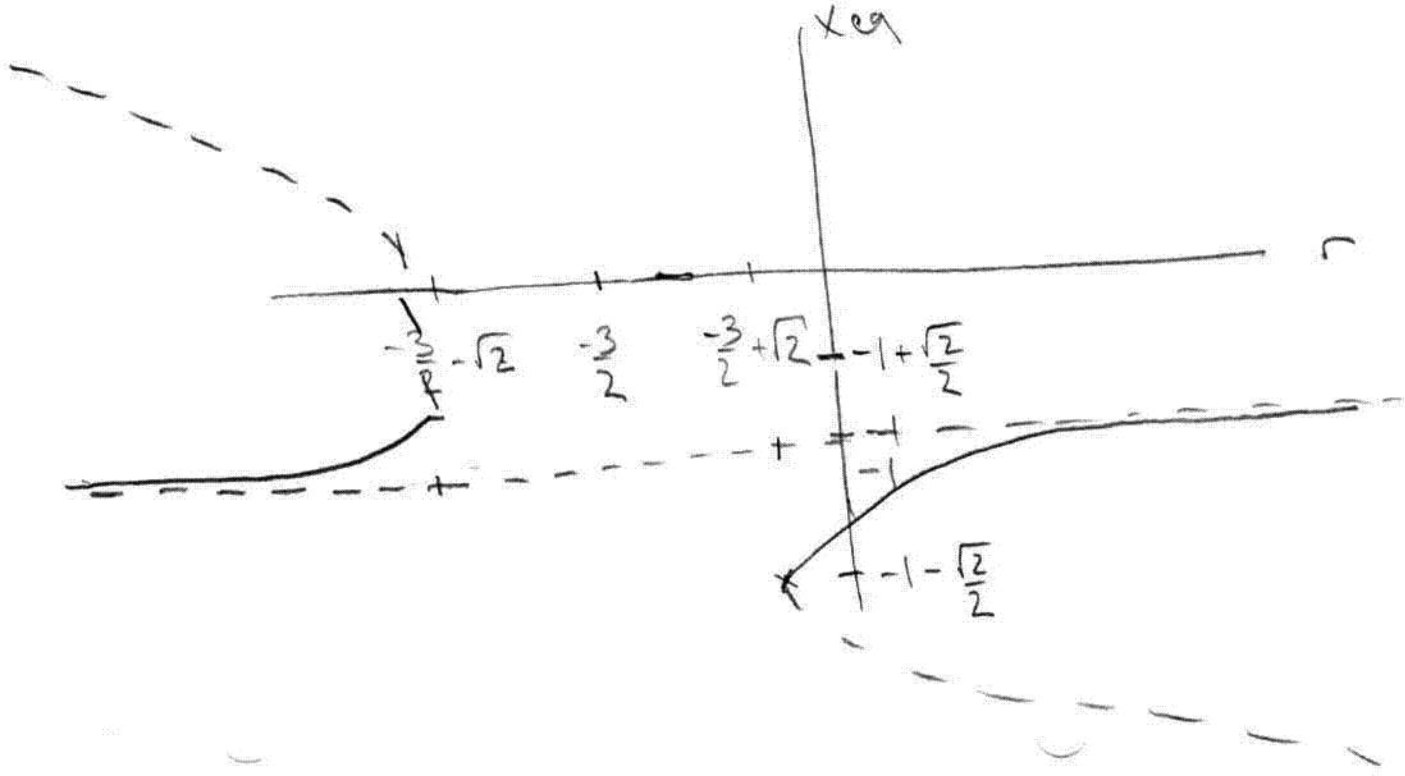
$-\frac{3}{2} - \sqrt{2} < r < -\frac{3}{2} + \sqrt{2}$







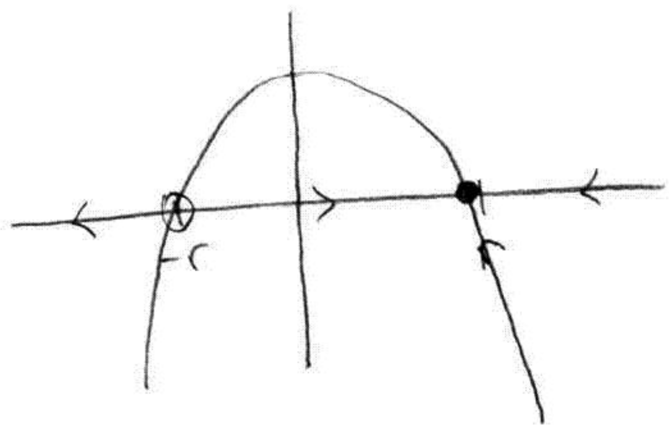
Bifurcation diagram

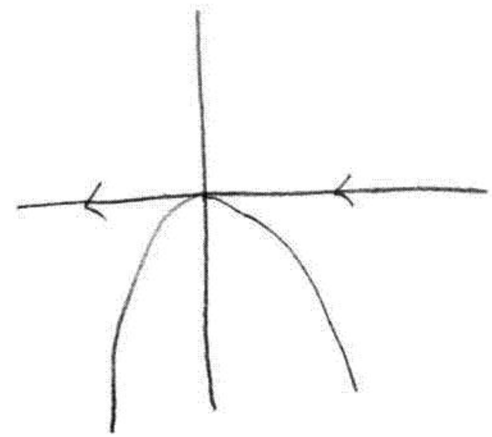


Pg 80 Strogatz

3.1.5

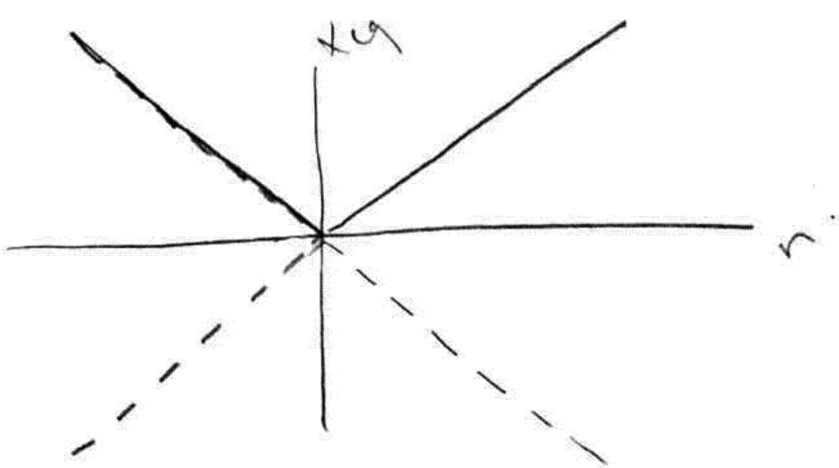
(a) $\dot{x} = r^2 - x^2 = (r-x)(r+x)$





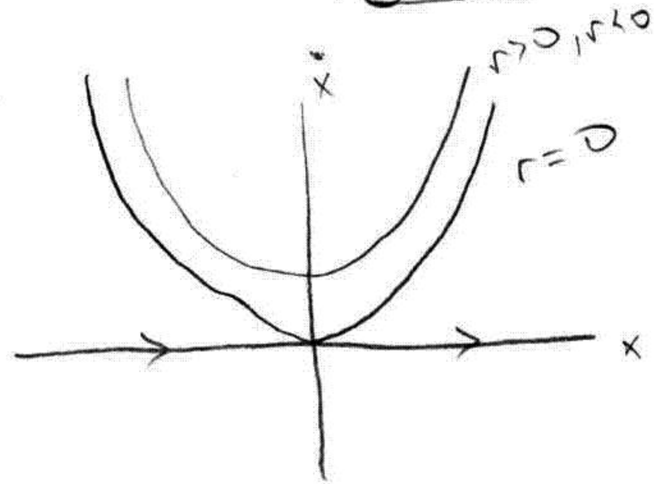
10, -10

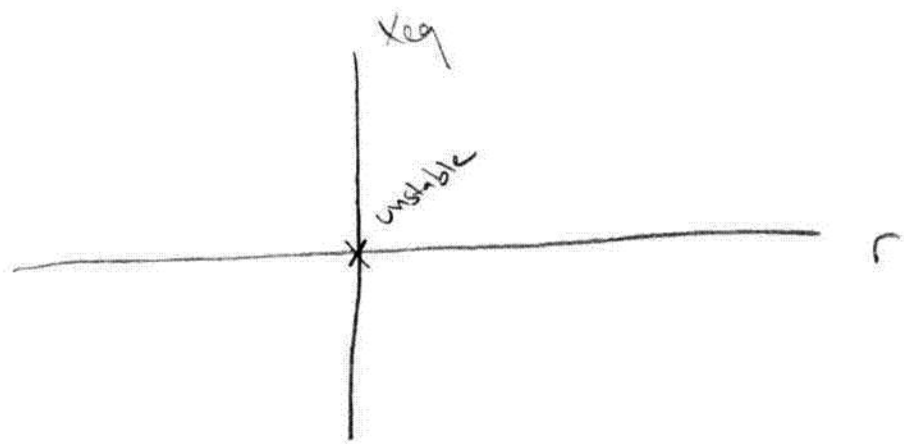
$r<0$ same



Pg 80 Strogatz

(b) $\dot{x} = r^2 + x^2$

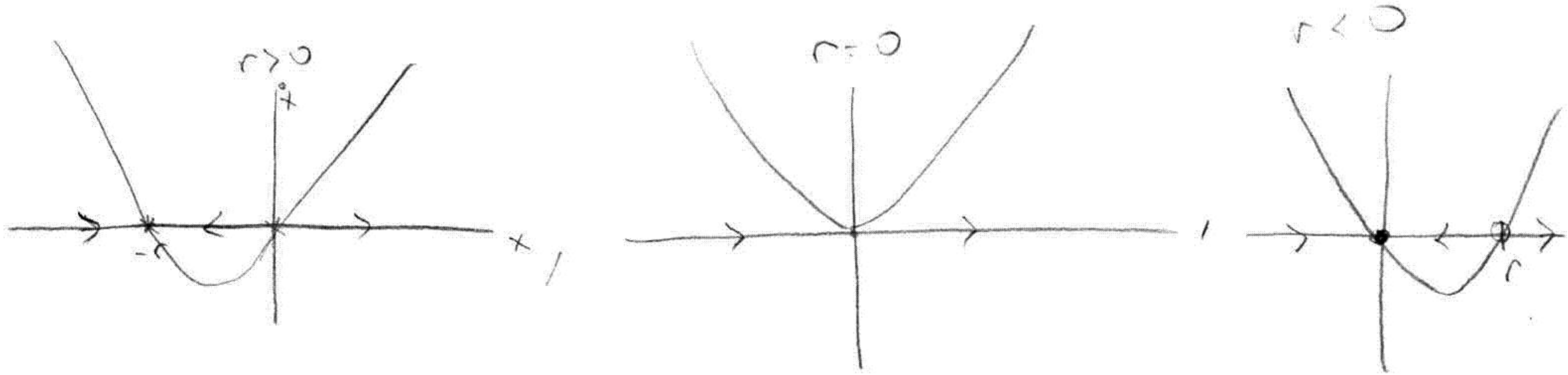




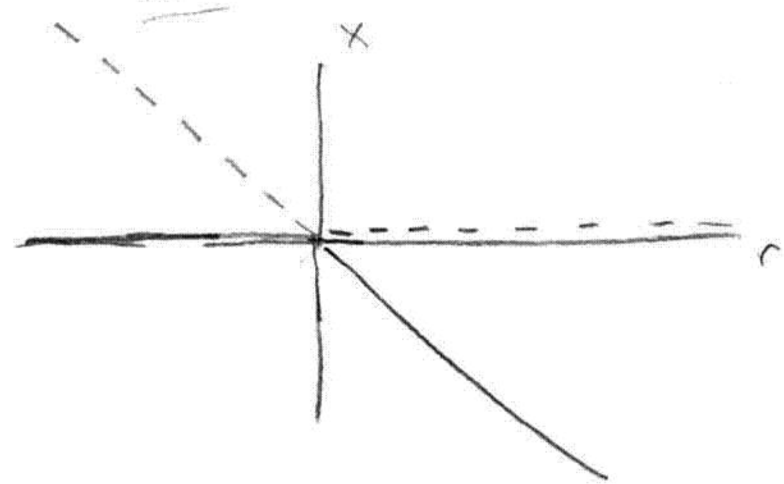
Pg 80 Strog

3.21

$\dot{x} = rx + x^2 = x(x+r)$



BfD

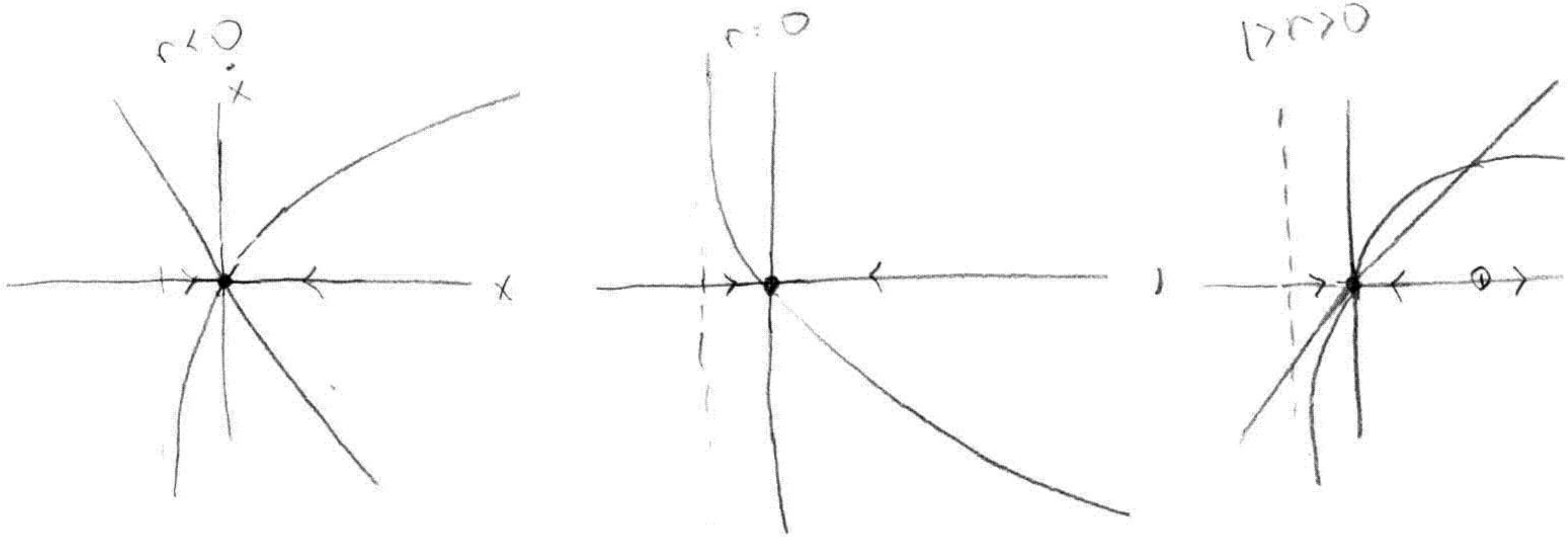


Transcritical.

3.3.2

Pg 30 Strog

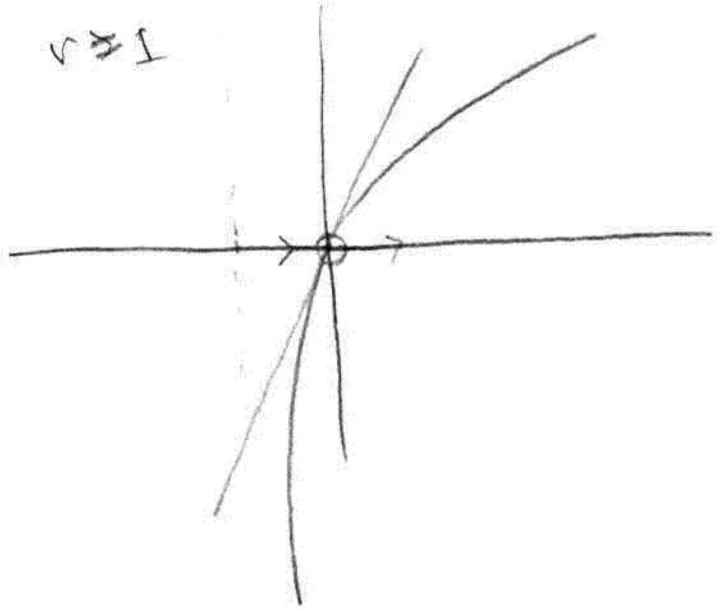
$\dot{x} = rx - \ln(1+x)$



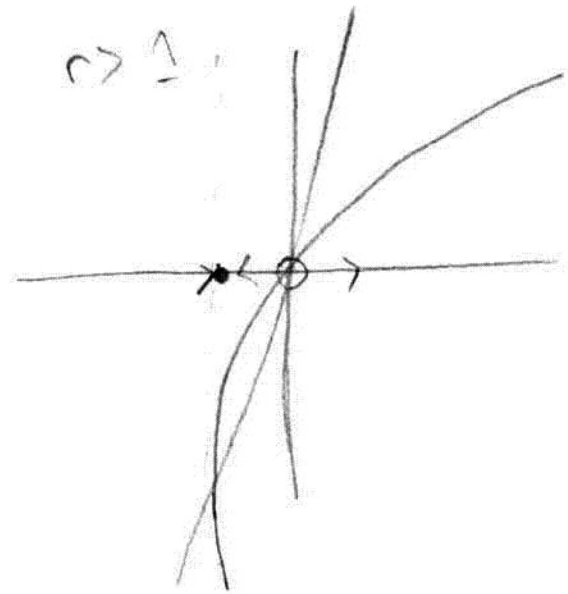
pt of tangency $r = \frac{1}{1+x}$ At $x=0$?

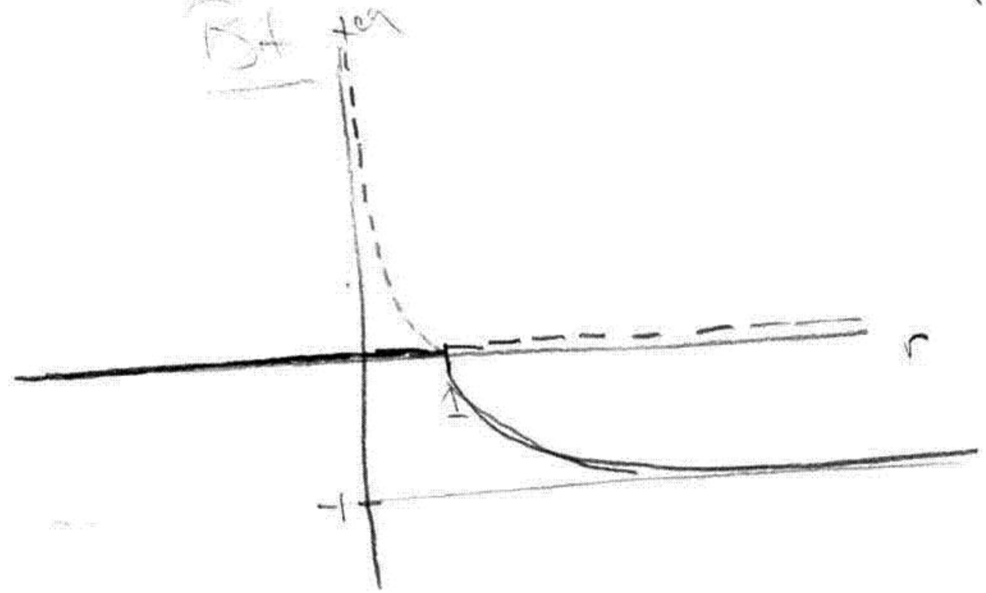
$\Rightarrow r = 1$

$r \geq 1$



$r > 1$





Translated Bf

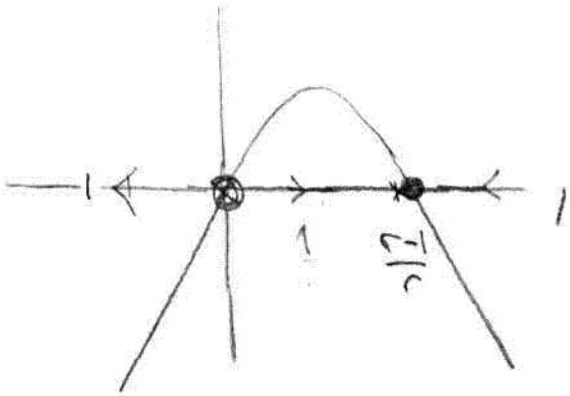
3.2.3

$$\dot{x} = x - rx(1-x) = x(1-r(1-x)) = x(1-r+rx)$$

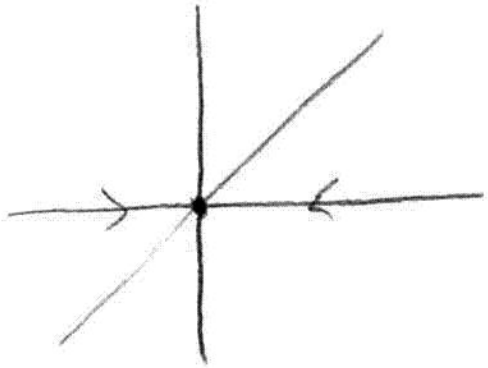
$$\Rightarrow x = 0 \quad \text{or} \quad \frac{1}{r} = 1-x$$

$$x = 1 - \frac{1}{r} = \frac{r-1}{r}$$

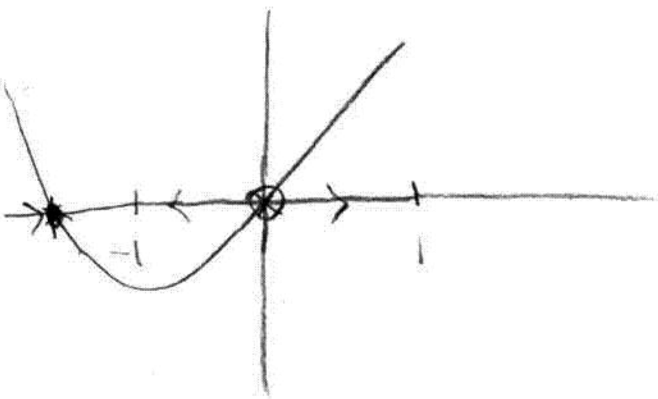
$\underline{r<0}$



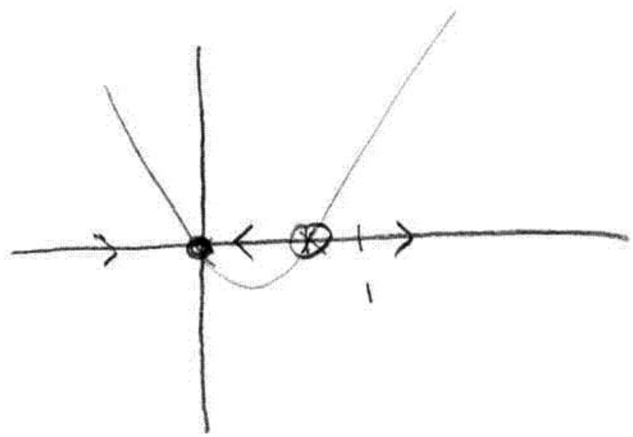
$\underline{r=0}$

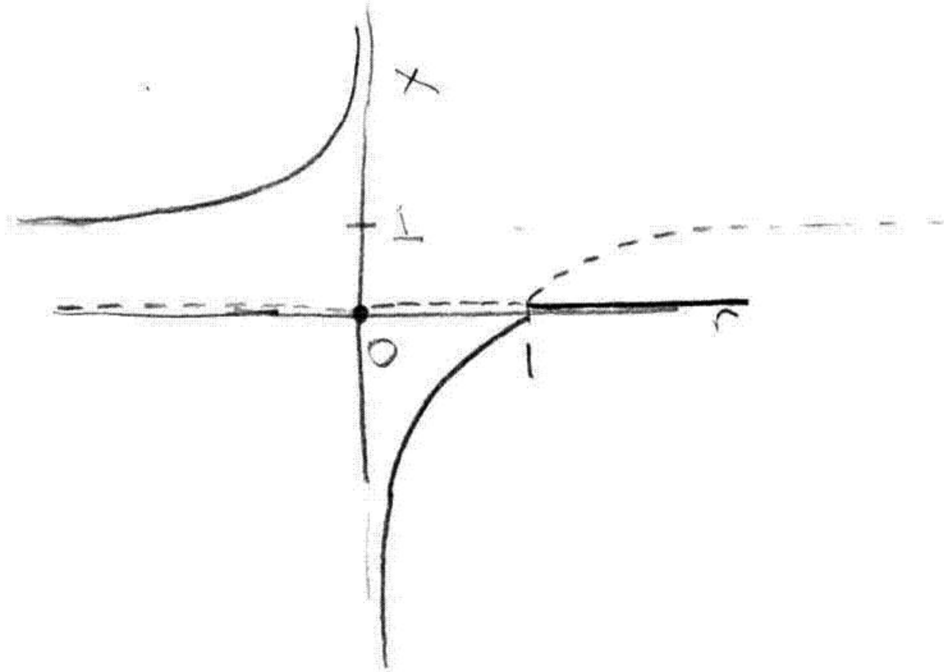


$0<r<1$



$r>1$

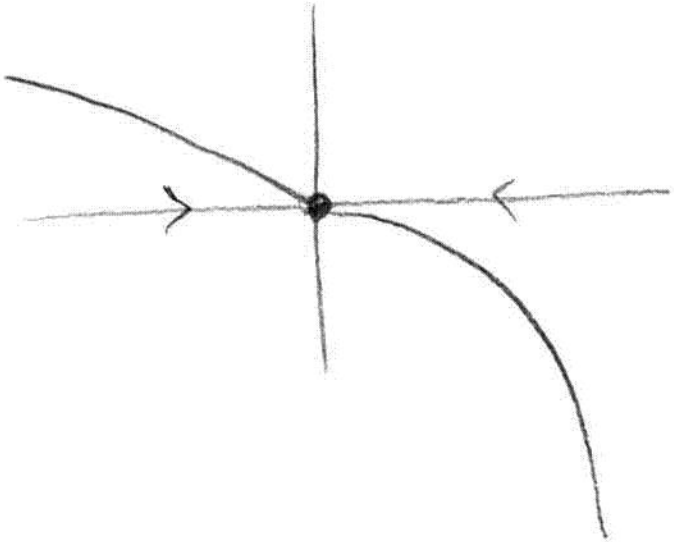



x
1
0
1
r

3.2.4 $\dot{x} = x(r - e^x) = 0$

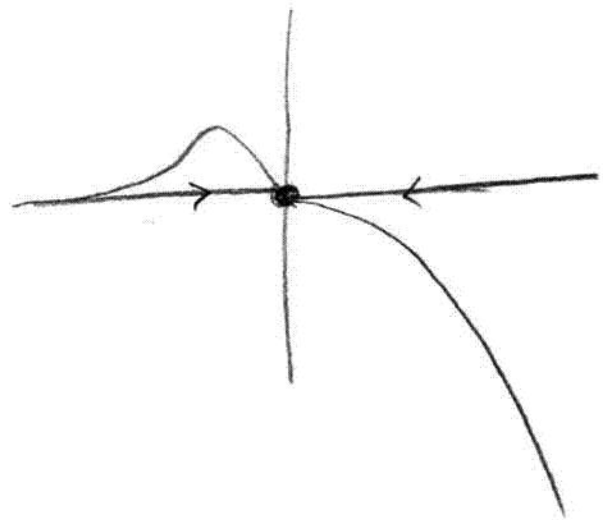
$\Rightarrow x = 0$ & $r = e^x$

$r<0$



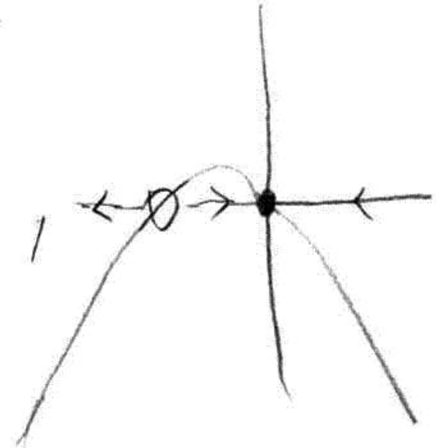
1

$r=0$

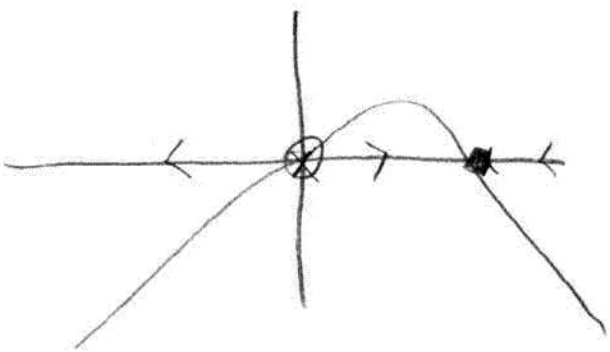


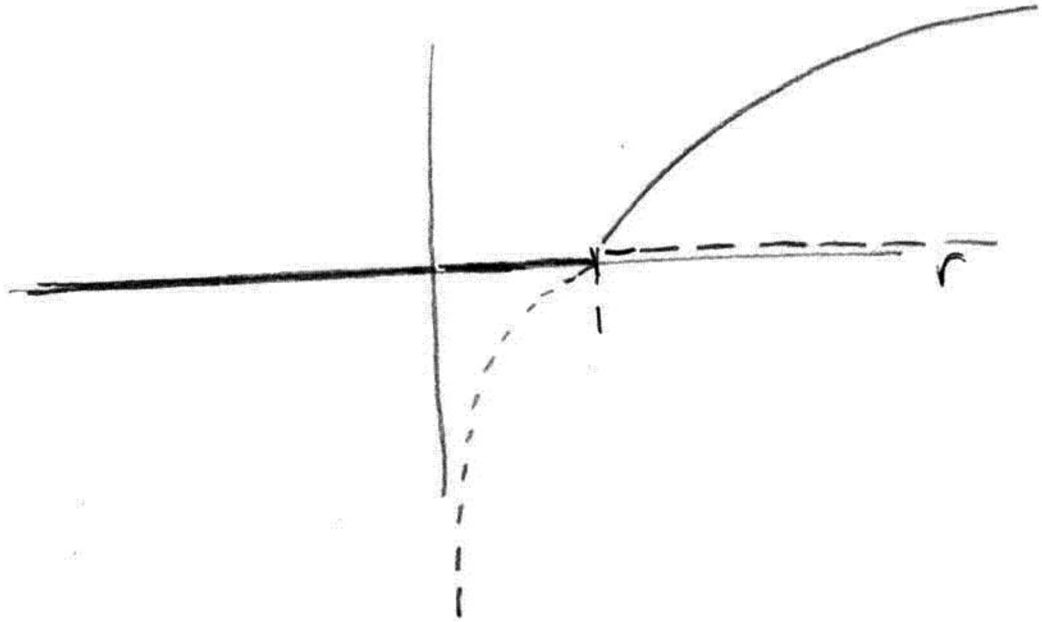
$1>r>0$

1



$r>1$




r

3.2.5

x = concent of x in system　　a = concent of A

　　　　　　　　　　　　　　　　b " " B

$\dot{x}_1 = k_1 a x - k_{-1} x^2$　rate of eq 1

$\dot{x}_2 = k_2 b x$

$\dot{x}_2$ = Rate of eq 2 =　　$\dot{x} = k_1 a x - k_{-1} x^2 + k_2 b x$

$= (k_1 a + k_2 b) x - k_{-1} x^2$

$X \underset{k_{-1}}{\rightleftharpoons}$ or, $X \rightarrow$ or

$\dot{x} = k_1 a x - k_{-1} x^2 + k_2 b x$

$= (k_1 a + k_2 b) x - k_{-1} x^2$

$x = 0$ stable $\Rightarrow \left.\frac{d\dot{x}}{dx}\right|_{x=0} < 0$

$\frac{d\dot{x}}{dx} = k_1 a + k_2 b - 2k_{-1} x \Big|_0$　　$k_1 a + k_2 b < 0$

(b) $\mathring{x} = (k_1 a - k_2 b) x - x^2 k_{-1}$

Then $\left.\frac{d\mathring{x}}{dx}\right|_{x=0} = k_1 a - k_2 b < 0$

$\Rightarrow k_2 b > k_1 a$ rate of production of b is greater than that of a

John Weatherwax

Pg 80 Strogatz

1

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$x = X + bX^3 + O(X^4)$

5/5

(a) $X = x - bX^3 - O(X^4)$

$= x - b(x - bX^3 - O(X^4))^3 - O((x - bX - O(X^4))^4)$

$= x - b(x^3) - O(x^4)$

$= x - bx^3 + O(x^4) \quad \therefore \quad c = -b$

(b) $\dot{x} = \dot{X} + 3bX^2\dot{X} + O(X^4)$

$= (RX - X^2 + aX^3 + O(X^4)) + 3b(x - bx^3 + O(x^4))^2(RX - X^2 + aX^3 + O(X^4))$

$+ O(x^4)$

$= R(x - bx^3 + O(x^4)) - (x - bx^3 + O(x^4))^2 + a(x - bx^3 + O(x^4))^3 + O(x^4)$

$+ 3b(x - bx^3 + O(x^4))^2(RX - X^2 + aX^3 + O(X^4)) + O(x^4)$

$$= Rx - Rbx^3 - x^2 + ax^3 + O(x^4)$$

$$+ 3b(x - bx^3 + O(x^4))^2 (Rx - Rbx^3 - x^2 + ax^3 + O(x^4))$$

$$= Rx - Rbx^3 - x^2 + ax^3 + O(x^4) + 3b(x^2 + O(x^4))(Rx - Rbx^3 - x^2 + ax^3 + O(x^4))$$

$$= Rx - Rbx^3 - x^2 + ax^3 + O(x^4) + 3bRx^3$$

$$= Rx - x^2 + (a + 2bR)x^3 + O(x^4)$$

$k = a + 2bR$ want $= 0$

(c) $\Rightarrow b = -\frac{a}{2R}$

(d) No, one would just have to take a higher power of x to complete the calculation i.e. $\dot{X} = -X^2 + aX^3 + bX^4 + O(X^5)$ (when $R = 0$)

pg 81 Strogatz

3.3.1

(a) $\dot{N} \approx 0$

$$\Rightarrow (Gn+f)N = p$$

$$\Rightarrow N = \frac{p}{Gn+f}$$

Then $\dot{n} = \frac{Gn\,p}{Gn+f} - kn = n\left[\frac{Gp}{Gn+f} - k\right]$

(b) $\dot{n} = 0 \Rightarrow n\left(\frac{Gp}{Gn+f} - k\right) = 0$

$n = 0$ or $Gp = kGn + kf$

$$\frac{Gp - kf}{kG} = n$$

Stability of $n^* = 0$ depends on sign of $\left.\frac{d\dot{n}}{dn}\right|_{n=0}$

$$\left.\frac{d\dot{n}}{dn}\right|_{n=0} = -k + \frac{G P_c}{f} = 0$$

$$P_c = \frac{kf}{G}$$

Thus n^* is stable when $p < \frac{kf}{G}$

& unstable when $p > \frac{kf}{G}$

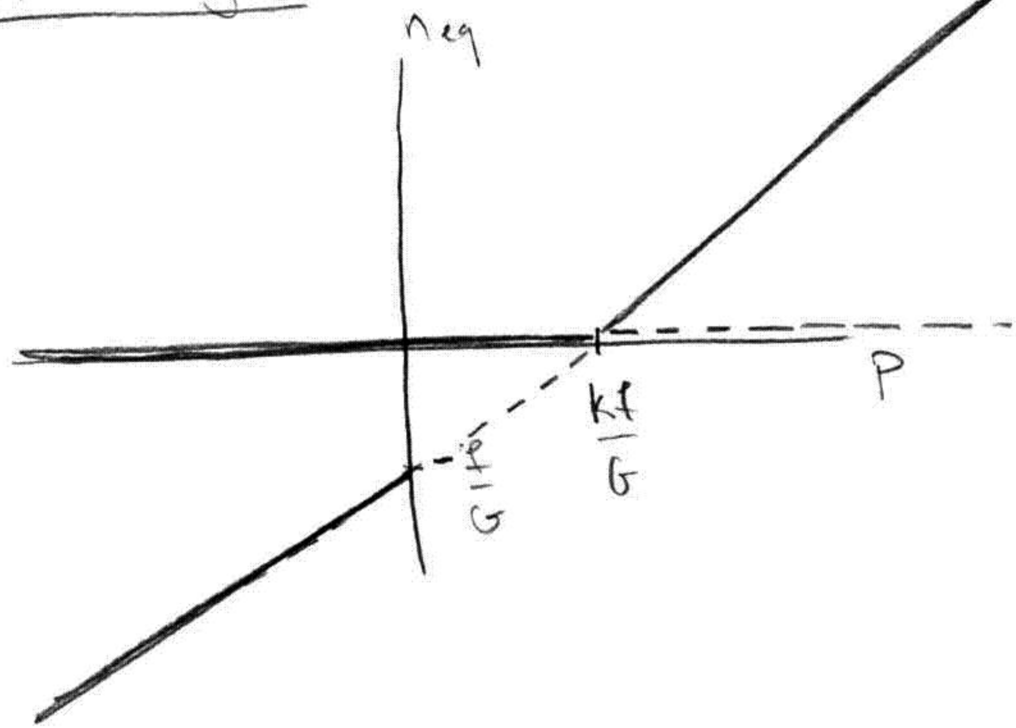
(c) Write n_{eq} As $n_{eq} = \frac{p}{k} - \frac{f}{G}$

Then stability $\left.\frac{d\dot{n}}{dn}\right|_{n_{eq} = \frac{p}{k} - \frac{f}{G}} = -k + \frac{fk^2}{Gp}$

> 0 $\quad 0 < p < \frac{kf}{G}$

< 0 & stable $\quad p > \frac{kf}{G}$ or $p < 0$

Bifurcation Diagram:



Thus $P = \frac{kf}{G}$ is a transcritical bifurcation.

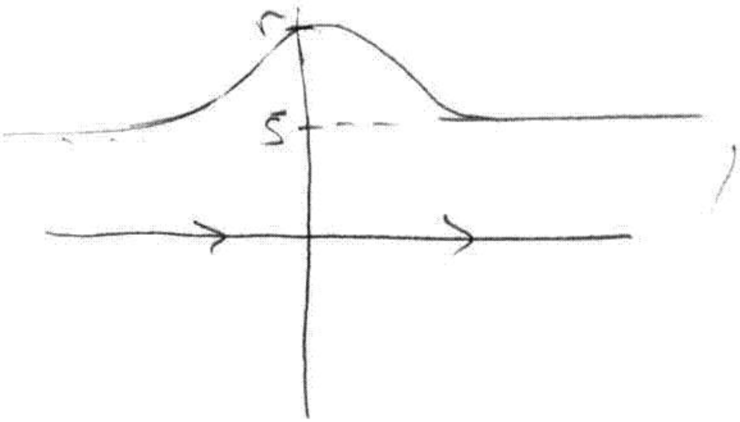
(d)

1

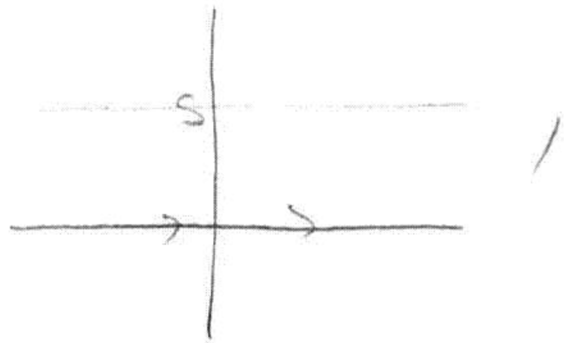
Pg 83 Strogatz

3.4.7 $\quad \dot{x} = 5 - re^{-x^2} = 0 \Rightarrow r = 5e^{x^2}$

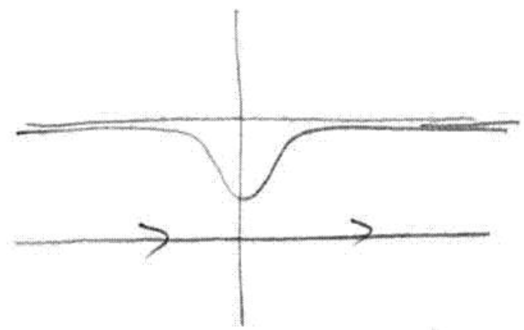
$r < 0$



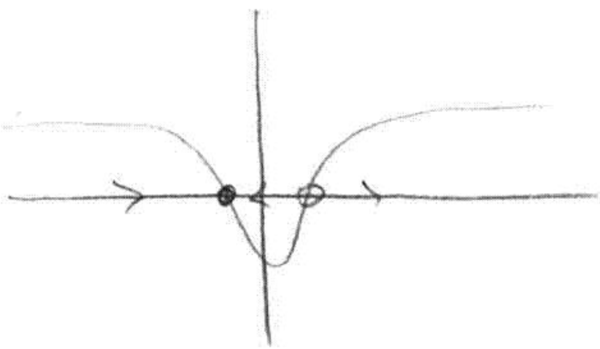
$r = 0$



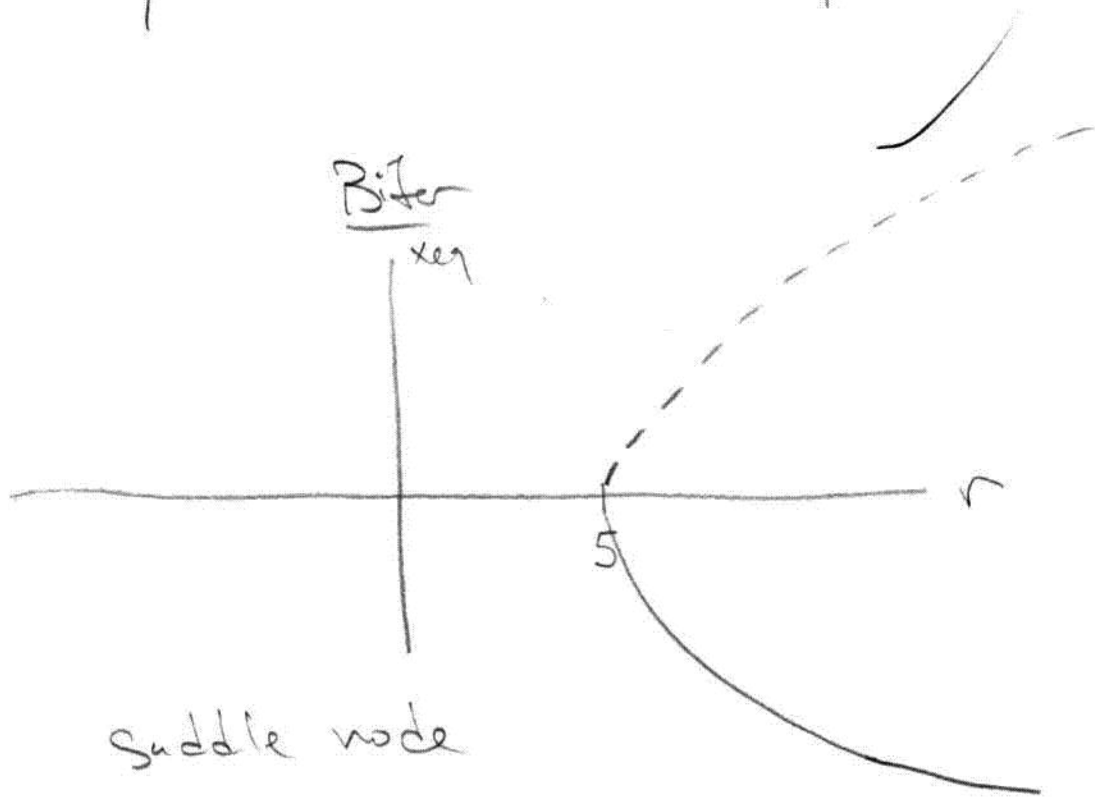
$5 > r > 0$



$r > 5$



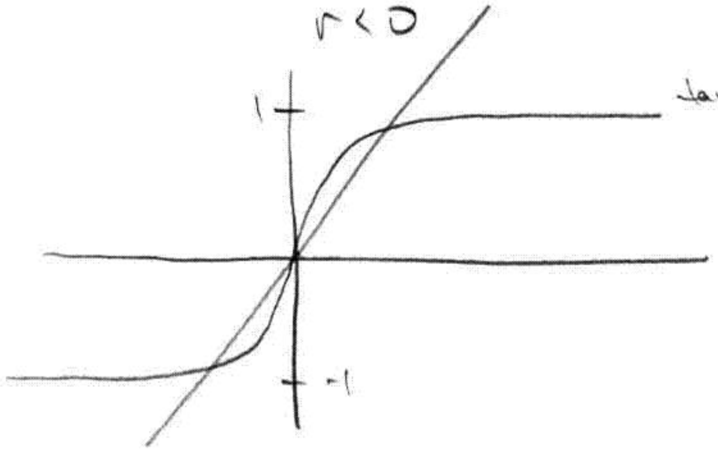
Bifer



saddle node

3.4.9 $\dot{x} = x + \tanh(rx) = x - \tanh(-rx)$

$r < 0$



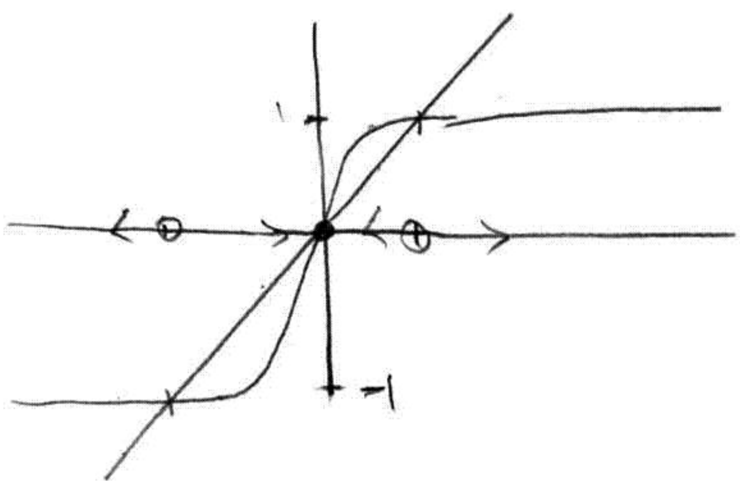
what value of r is $\tanh(rx)$ tangent to x at $x=0$

$$1 \overset{?}{=} \frac{d}{dx}\left(\tanh(-rx)\right)\Big|_{x=0} = -r$$

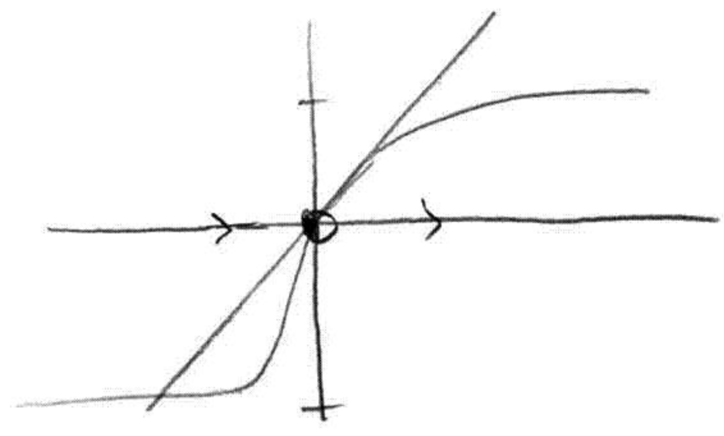
$\Rightarrow r = -1$

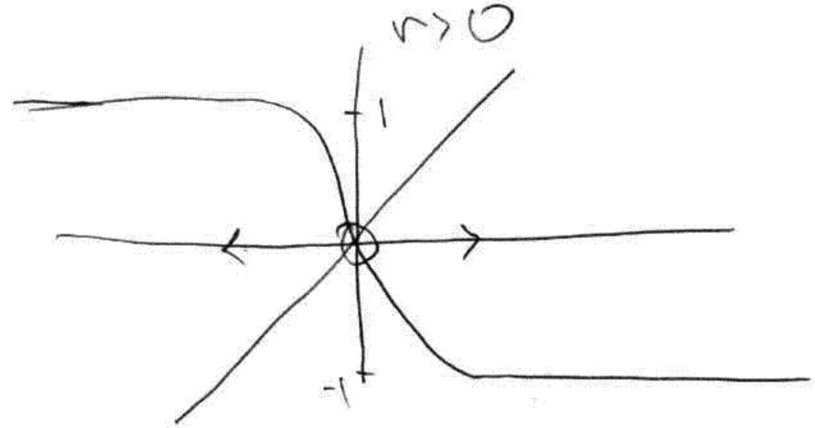
Thus plot

$r < -1$

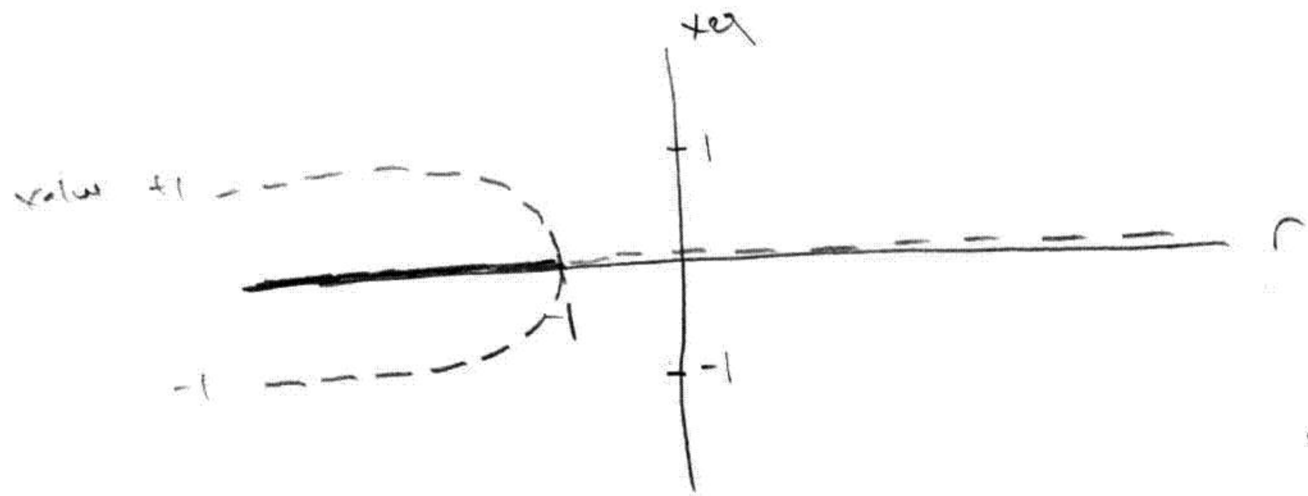


$-1 > r > 0$





Bifurcation diagram



hard pitchfork
or subcritical
bifurcation.

3.4.10 $\dot{x} = rx + \frac{x^3}{1+x^2} = 0$

$\Rightarrow x\left(r + \frac{x^2}{1+x^2}\right) = 0 \Rightarrow x = 0$ or $r = \frac{-x^2}{1+x^2} \Leftrightarrow r = (-1-r)x^2$

$\Rightarrow x_{eq} = \pm\sqrt{\frac{-r}{1+r}}$

only exists when $0 < \frac{-r}{1+r}$

or $-1 < r < 0$

Check stability at $x = 0$

$\frac{d}{dx}\left(rx + \frac{x^3}{1+x^2}\right)\Big|_{x=0} = r$ < 0 $x=0$ stable

> 0 $x=0$ unstable

Check stability at $x = \pm\sqrt{\frac{-r}{1+r}}$

$\frac{d}{dx}\left(rx + \frac{x^3}{1+x^2}\right)\Big|_{x=\pm\sqrt{\frac{-r}{1+r}}} = -2r(1+r) > 0$ on $-1 < r < 0$

for both signs

$\therefore$ Both curves are unstable

2

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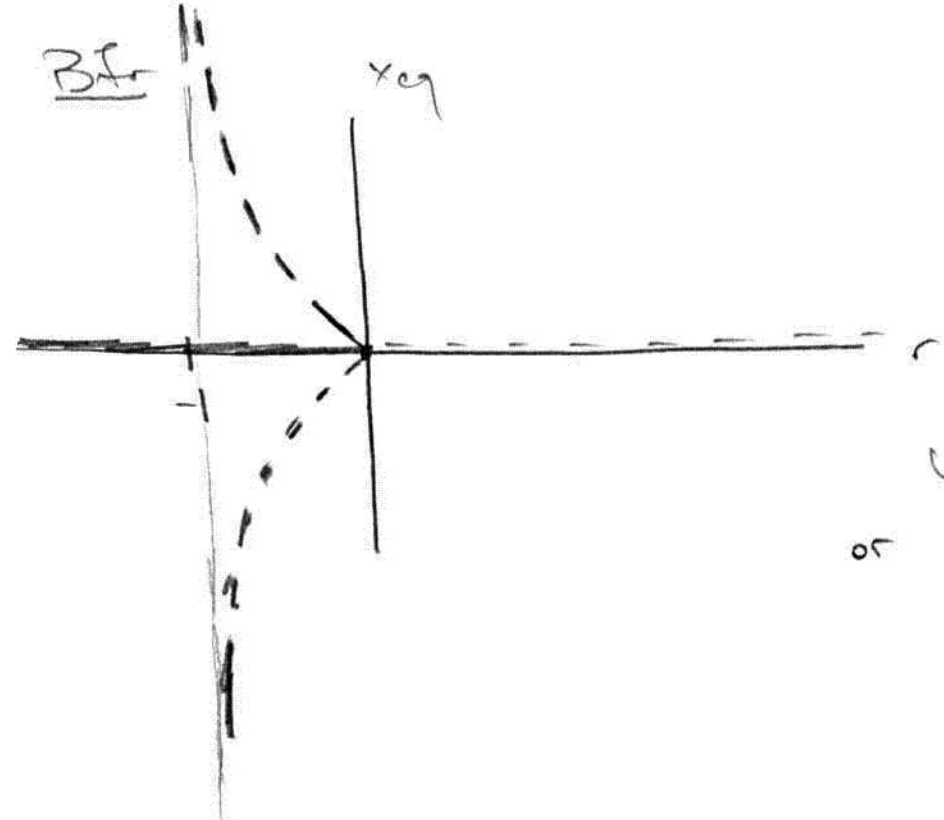
<u>Bifn</u>



Which is a hard pitchfork
or subcritical bifurcation

3.4.14

$\dot{x} = 0$

(a) $\Rightarrow \quad x(r + x^2 - x^4) = 0$

$\Rightarrow \quad x = 0$ or $\quad x^4 - x^2 - r = 0$

$$x^2 = \frac{1 \pm \sqrt{1^2 + 4r}}{2}$$

$$\Rightarrow \quad x = \frac{\pm\sqrt{1 \pm \sqrt{1+4r}}}{\sqrt{2}}$$

Thus $\quad x_1 = \dfrac{+\sqrt{1 + \sqrt{1+4r}}}{\sqrt{2}}$ requirement that $1+4r > 0$

$\Rightarrow r > -1/4$

$x_2 = \dfrac{-\sqrt{1 + \sqrt{1+4r}}}{\sqrt{2}}$ ''

$x_3 = \dfrac{+\sqrt{1 - \sqrt{1+4r}}}{\sqrt{2}}$ requirement $1 - \sqrt{1+4r} > 0$

$1 > 1+4r \Rightarrow r < 0$

$\therefore$ total $\quad -1/4 < r < 0$

$$x_4 = -\frac{\sqrt{1-\sqrt{1+4r}}}{\sqrt{2}}$$ requirement $-\frac{1}{4} < r < 0$

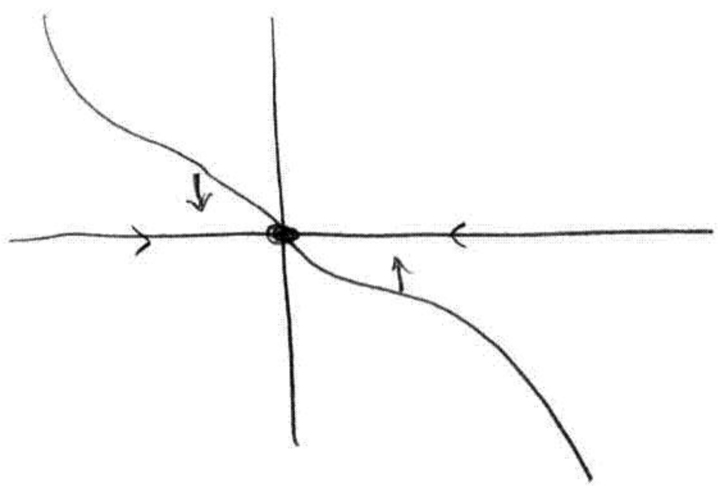
$+ \; x_5 \equiv 0$

(b) will have different vector fields in the following ranges

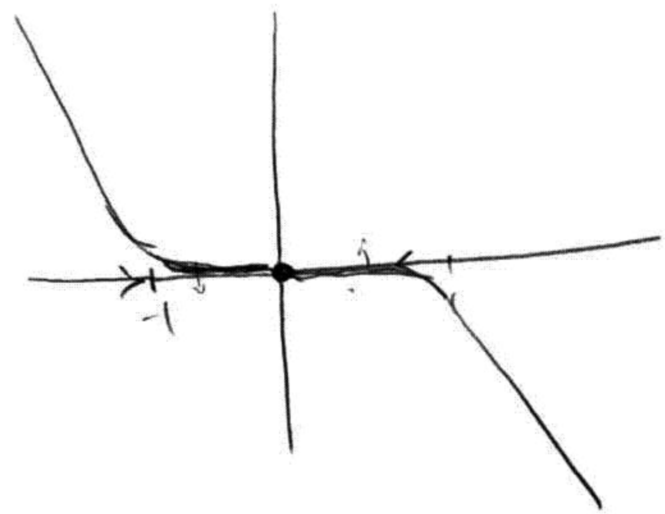
$r < -\frac{1}{4}$, $-\frac{1}{4} < r < 0$, $0 < r$

All plots done w/ computer

<u>$r < -\frac{1}{4}$</u> choose $r = -2$

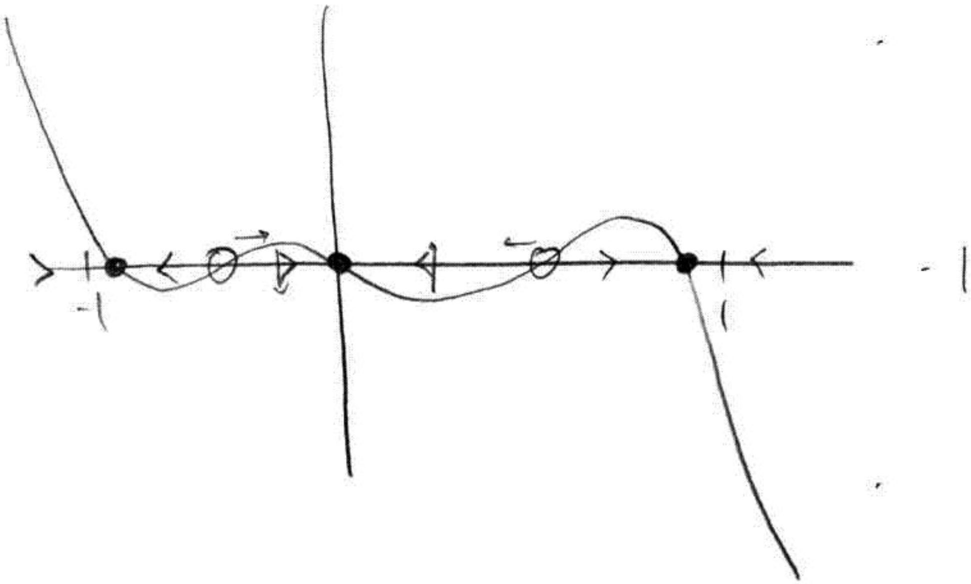


<u>$r = -\frac{1}{4}$</u>

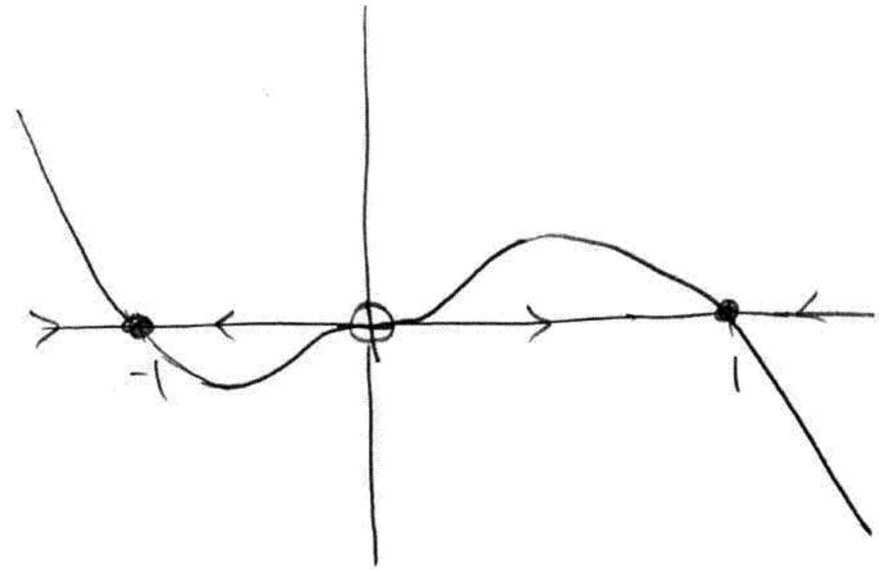


About to break through

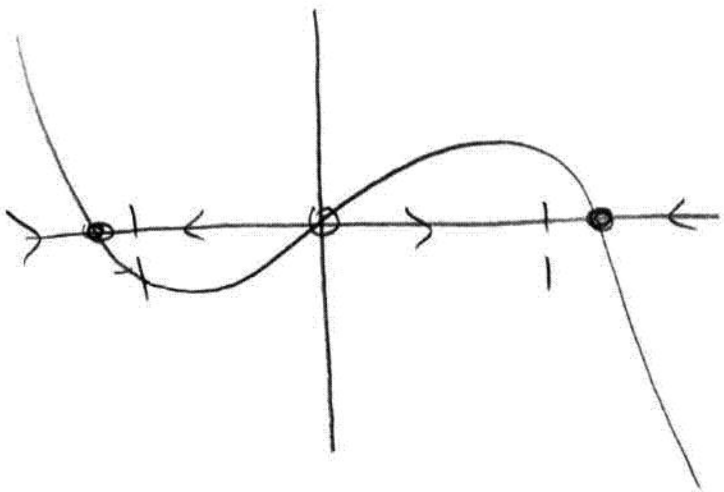
$-1/4 < r < 0$ say $r = -1/8$



$r = 0$



$r > 0$ say $r = 1$



(c) Non zero saddle fixed pts are born

when $r = -1/4$

Pg 83 Strogatz

3.4.15 $\quad \dot{x} = -\dfrac{dV}{dx}$

$$\frac{dV}{dx} = -(rx + x^3 - x^5) \Rightarrow V = -r\frac{x^2}{2} - \frac{x^4}{4} + \frac{x^6}{6}$$

$$= \frac{x^2}{6}\left[x^4 - \frac{3}{2}x^2 - 3r\right]$$

local min of $V \Rightarrow \dfrac{dV}{dx} = 0$

or $\quad x(r + x^2 - x^4) = 0$

$x = 0 \quad + \quad x^4 - x^2 - r = 0$

$$\Rightarrow x^2 = \frac{1 \pm \sqrt{1+4r}}{2}$$

$$\Rightarrow x = \pm \frac{\sqrt{1 \pm \sqrt{1+4r}}}{\sqrt{2}}$$

Require for 5 max/min $\quad 1+4r > 0$

$\Rightarrow \quad 1 > -4r$

$r > -1/4$

$+ \quad 1-\sqrt{1+4r} > 0 \Rightarrow 1 > 1+4r$

$r > 0 \quad \therefore$ combined $\quad -\frac{1}{4} < r$

Then $V(\pm\infty) = +\infty \quad \therefore$ critical pts of V must go min, max, min,

max, min.

$$\therefore V_{min} = V\left(-\frac{\sqrt{1+\sqrt{1+4r}}}{\sqrt{2}}\right) = V(x=0) = V\left(+\frac{\sqrt{1+\sqrt{1+4r}}}{\sqrt{2}}\right)$$

$$V(0) = 0 = \frac{1}{\cancel{16}}\left(\frac{1+\sqrt{1+4r}}{\cancel{2}}\right)\left[\left(\frac{1+\sqrt{1+4r}}{2}\right)^2 - \frac{3}{2}\left(\frac{1+\sqrt{1+4r}}{2}\right) - 3r\right]$$

$$\Rightarrow \quad 0 = -\frac{1}{4} - 2r - \frac{\sqrt{1+4r}}{4}$$

$$\Rightarrow \quad r = -\frac{3}{16}$$ which is in the valid range

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1

3.2.7 $\dot{X} = RX - X^2 + a_n X^n + O(X^{n+1})$ $\quad n \geq 3$

let $x = X + b_n X^n + O(X^{n+1})$

Then $X = x - b_n X^n - O(X^{n+1})$

$$= x - b_n\left(x - b_n X^n - O(X^{n+1})\right)^n - O(x^{n+1})$$

$$= x - b_n(x^n) + O(x^{n+1})$$

Now: $\dot{x} = \dot{X} + b_n n X^{n-1}\dot{X} + O(X^{n+1})$

$$= (RX - X^2 + a_n X^n) + b_n n X^{n-1}(RX - X^2 + a_n X^n) + O(X^{n+1})$$

$$= RX - X^2 + a_n X^n + b_n n R X^n + O(X^{n+1})$$

$$= R(x - b_n x^n) - (x - b_n x^n)^2 + (a_n + b_n n R)(x - b_n x^n)^n + O(x^{n+1})$$

$$\dot{x} = R(x - b_n x^n) - x^2 + (a_n + b_n n R)(x^n) + O(x^{n+1})$$

$$= Rx - x^2 + (a_n + b_n n R - R b_n) x^n + O(x^{n+1})$$

pick $b_n \ni \quad a_n + b_n(nR - R) = 0$

$$\text{or} \quad b_n = \frac{-a_n}{R(n-1)} = \frac{a_n}{R(1-n)}$$

3.3.2 Assume P

(a) $\dot{P} \approx 0 \Rightarrow P = ED$

$\dot{D} \approx 0 \Rightarrow \lambda + 1 = D + \lambda EP$

solve for P & D

$$P = \frac{E(1+\lambda)}{1+\lambda E^2}$$

$$D = \frac{1+\lambda}{1+\lambda E^2}$$

$$\therefore \quad \dot{E} = k(P-E) = kE\left[\frac{1+\lambda}{1+\lambda E^2} - 1\right] = \frac{kE\lambda(1-E^2)}{1+E^2\lambda}$$

(b) Fixed pts $E \Rightarrow \dot{E} = 0$ (investigate $\lambda = 0$ & $1 + E^2\lambda = 0 \Rightarrow \lambda = \frac{-1}{E^2}$)

$\Rightarrow E = 0 \quad E = \pm 1$

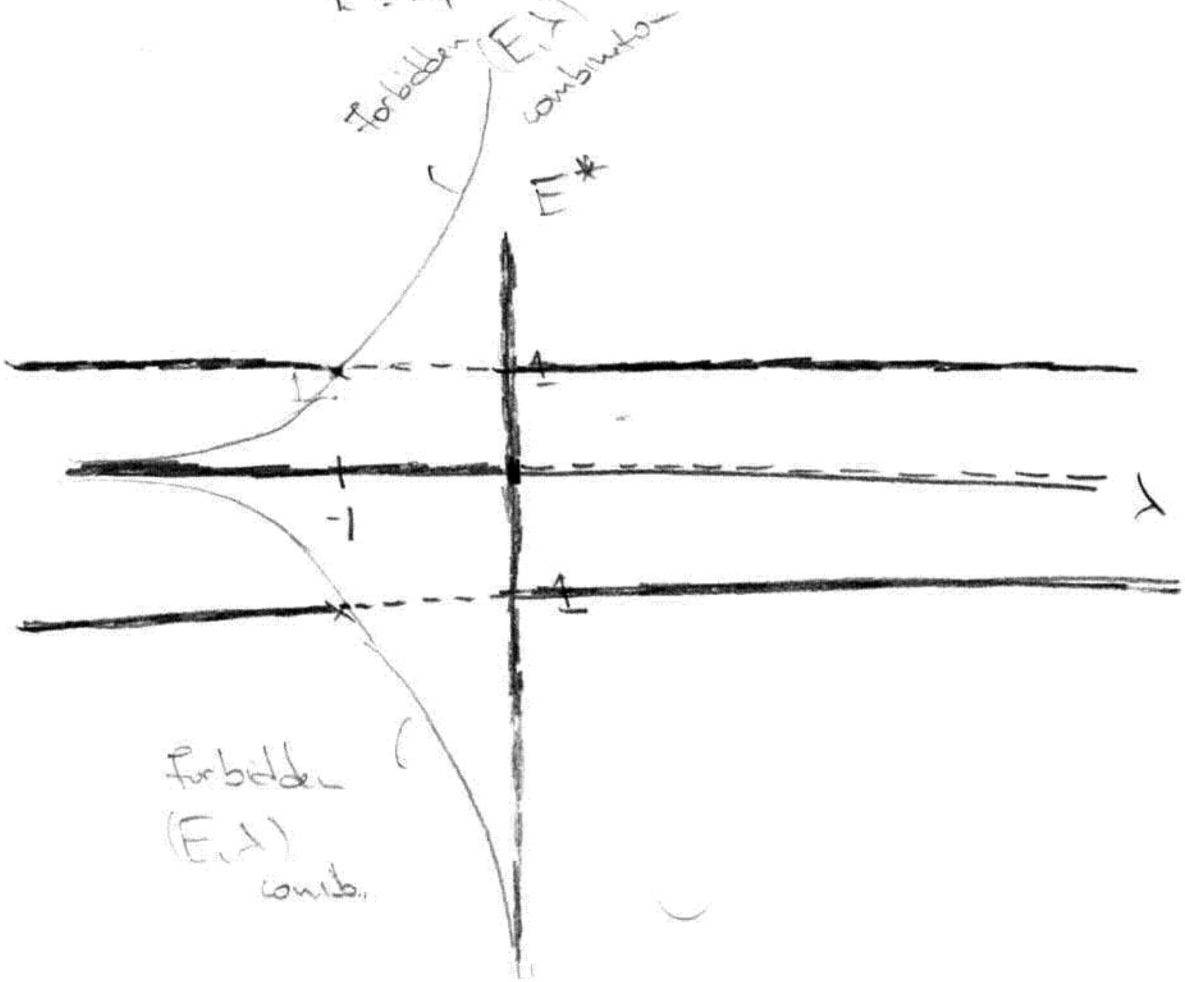
(c) Stability at $E = 0$

$$\Rightarrow \frac{d}{dE}\left(\frac{E\lambda(1-E^2)}{1+E^2\lambda}\right)\Big|_{E=0} = \lambda$$

Stab at $E = 1$

$$\frac{d}{dE}\left(\frac{E\lambda(1-E^2)}{1+E^2\lambda}\right)\Bigg|_{E=+1} = \frac{-2\lambda}{1+\lambda}$$

$$\text{"} \quad \Bigg|_{E=-1} = \frac{-2\lambda}{1+\lambda}$$



$\lambda = 0 \quad \dot{E} = 0 \ \forall E$

Stable $\forall E$

As any change in E no matter how large leaves stability the same

$$\lambda = -\frac{1}{E^2} \Rightarrow \frac{1}{(-\lambda)} = E^2$$

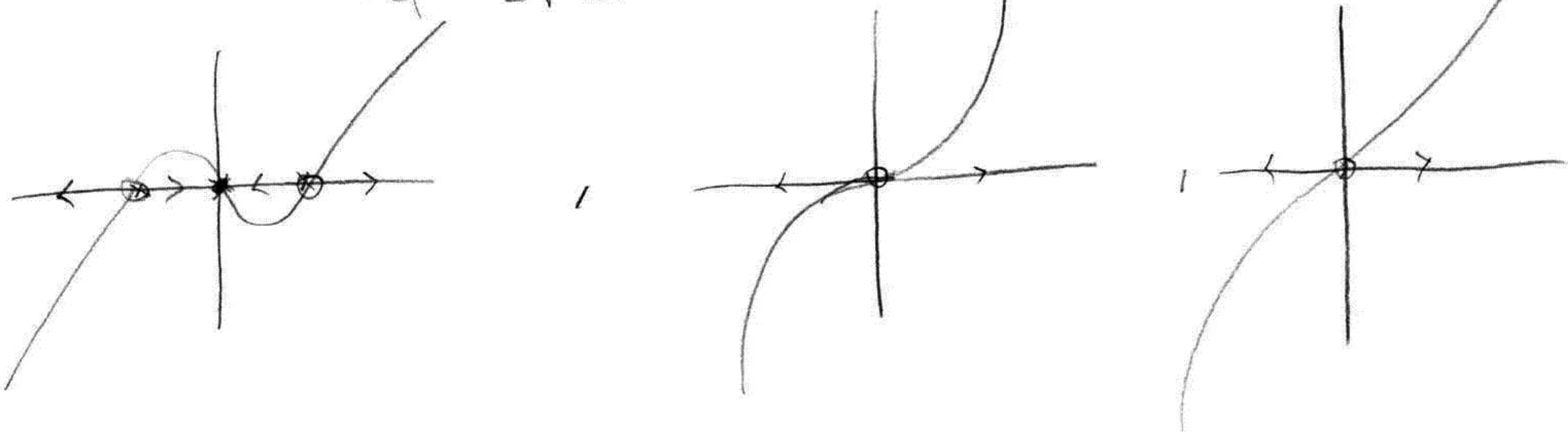
$$\Rightarrow E = \pm\frac{1}{\sqrt{-\lambda}} \quad (\lambda < 0)$$

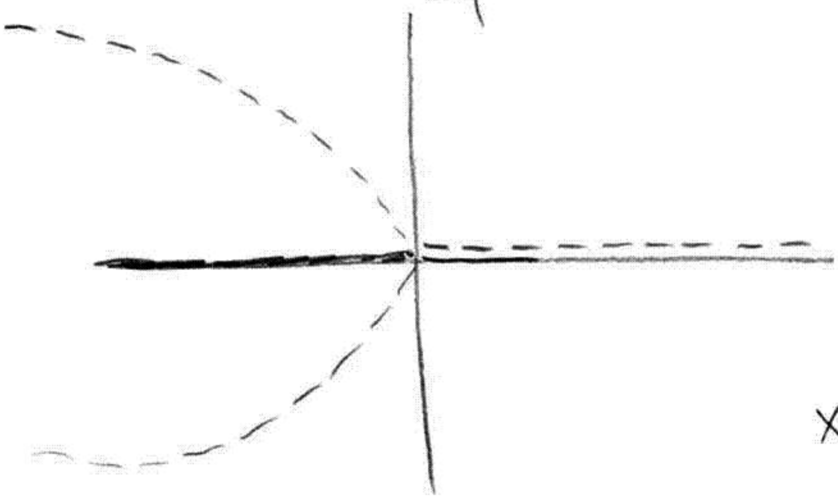
These physically "unpassable" regions i.e. if $E^* = 1$ then you cannot take $\lambda <$ than -1 else go to $\mathring{E} = \infty$ range

Pg 82 Strogatz

3.4.1 $\dot{x} = rx + \frac{1}{4}x^3 = x\left(r + \frac{1}{4}x^2\right) = \frac{x}{4}\left(4r + x^2\right)$

$r<0$ $\quad x_{eq} = \pm\sqrt{-2r}$ $\qquad r=0 \qquad r>0$





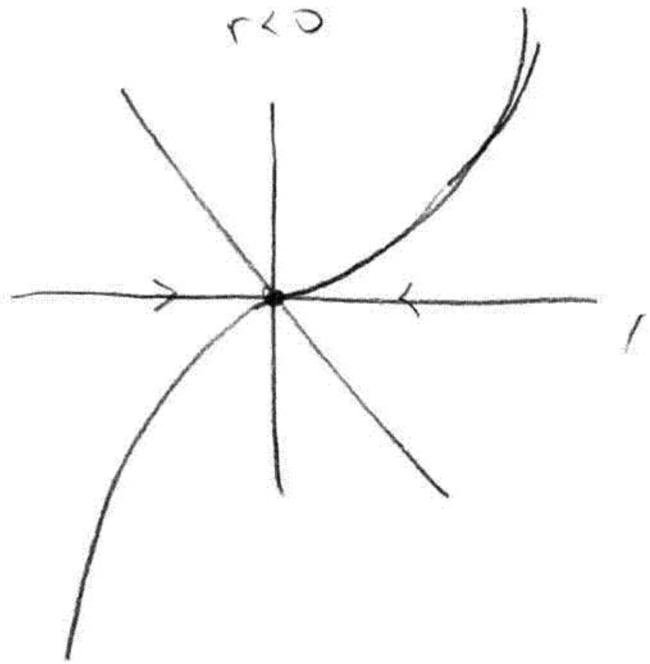
hard, subcritical pitchfork bifurcation

$x_{eq} = \pm\sqrt{-2r} \quad r<0$

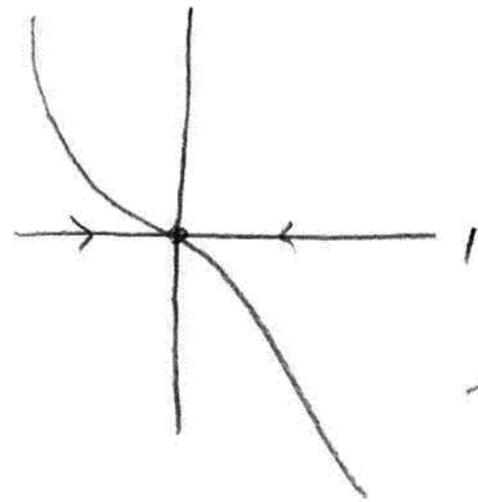
Pg 82 Strogatz

3.4.2 $\dot{x} = rx - \sinh x$

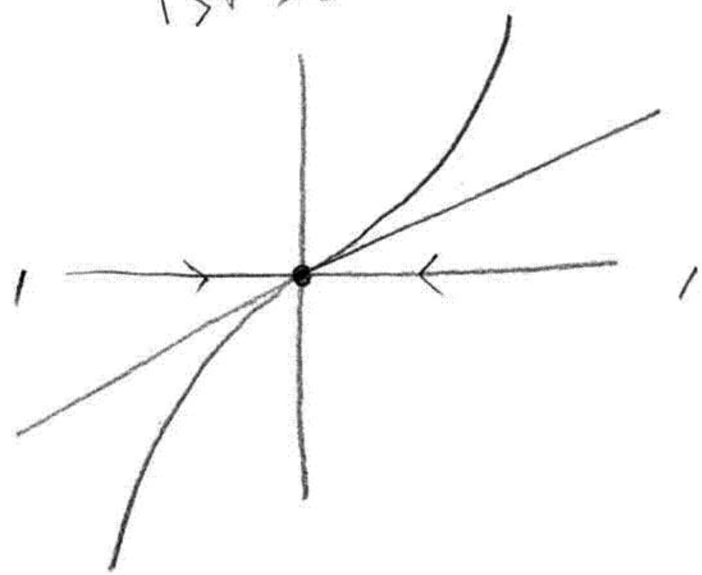
$r<0$



$r=0$



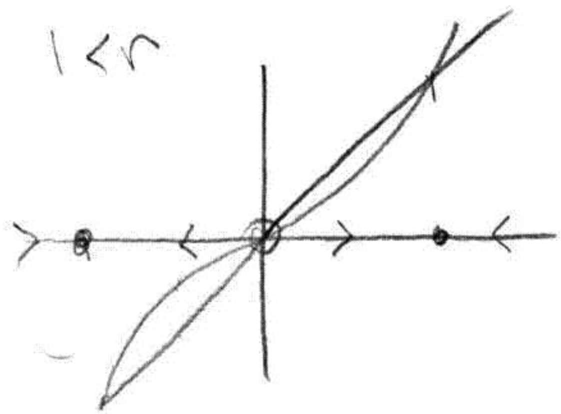
$1>r>0$

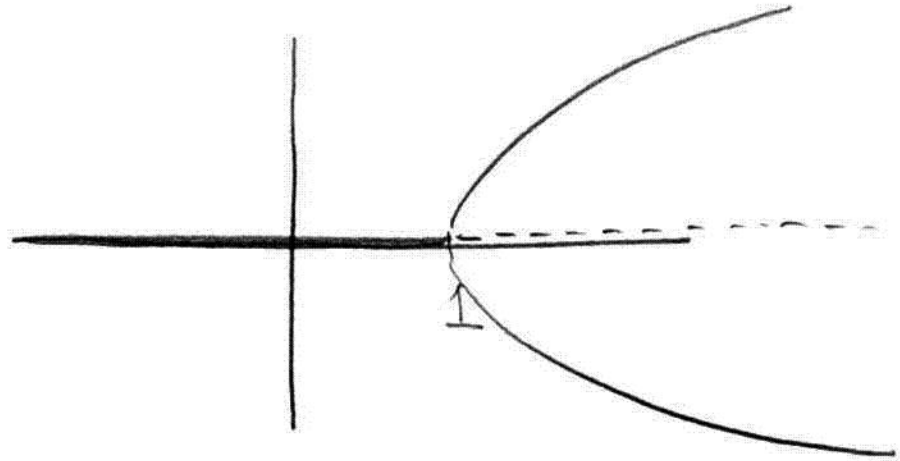


Check when tangent $r = +\cosh x$ But must be tangent at 0

$\Rightarrow r = 1$

$1<r$





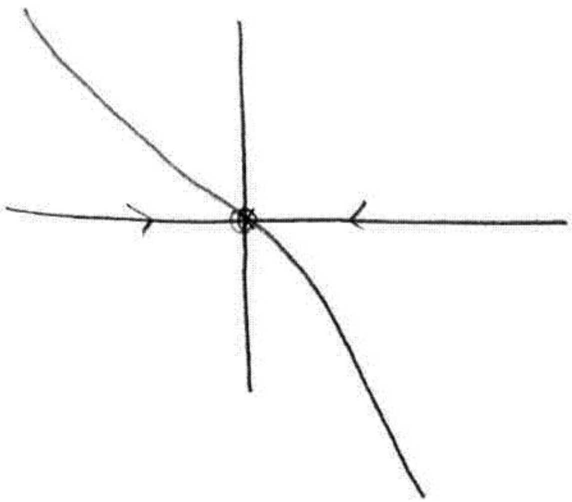
"soft"

super critical bif.

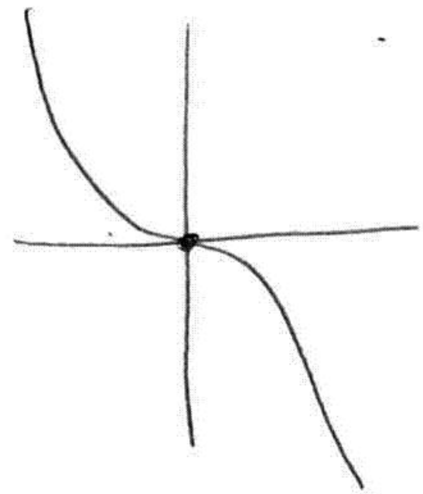
Pg 32 Strogatz

(3.4.3) $\dot{x} = rx - 4x^3 = x(r - 4x^2)$

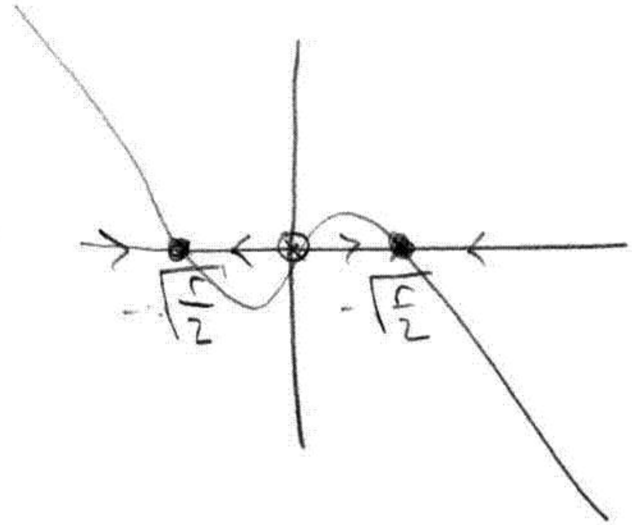
$r < 0$

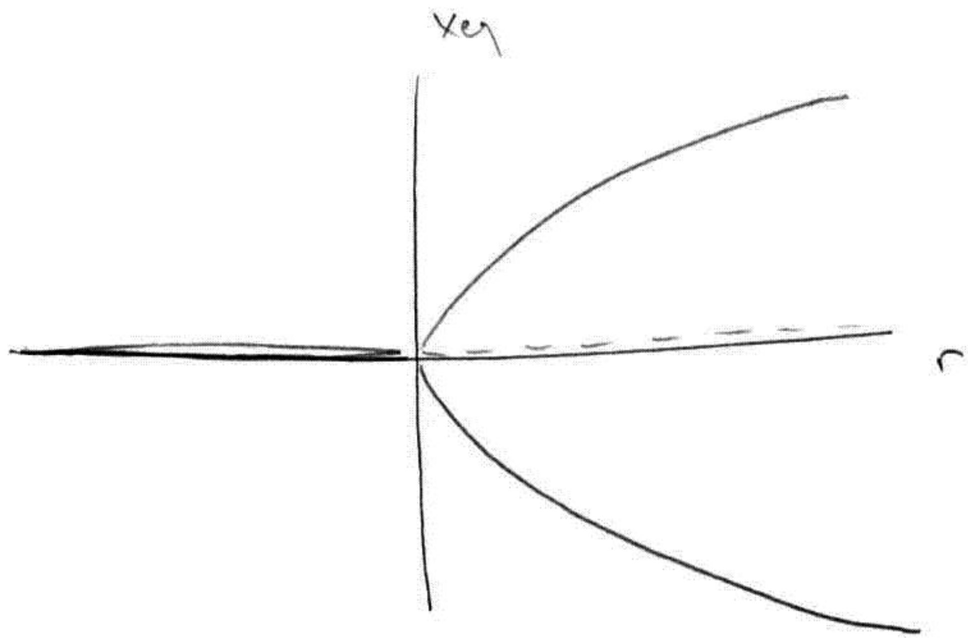


$r = 0$



$r > 0$





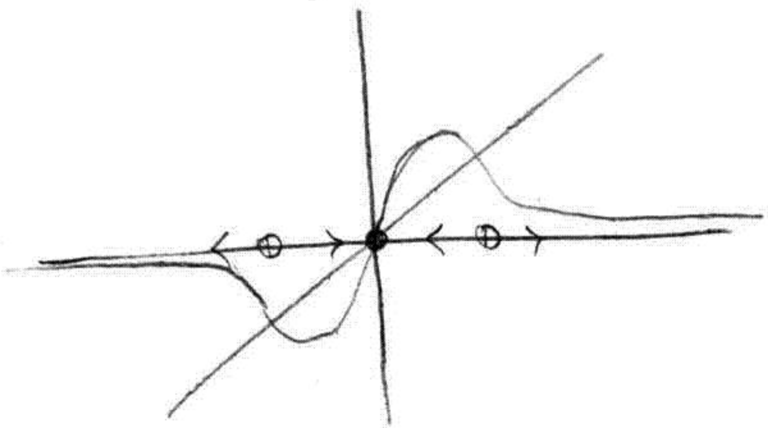
soft
or
supercritical
bifurcation

Pg 82 Strogatz

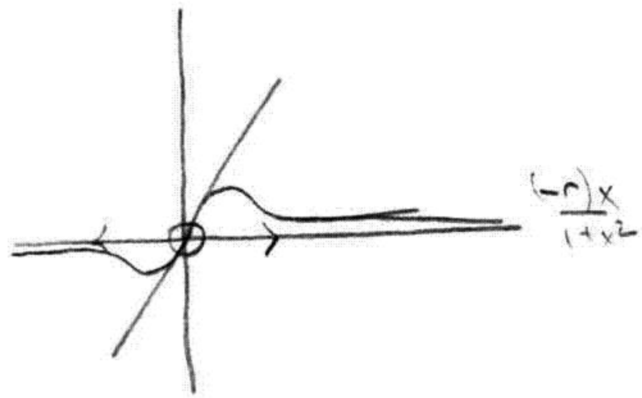
3.4.4

$$\dot{x} = x + \frac{rx}{1+x^2} = x - (-r)\frac{x}{1+x^2}$$

$r < -1$



$r = -1$

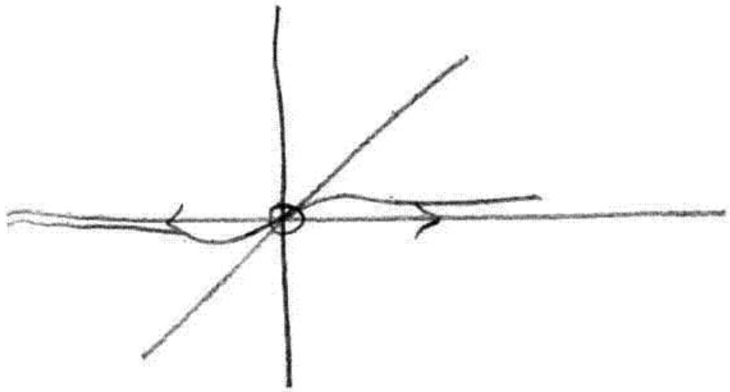


What value of r does tangency occur at $x=0$

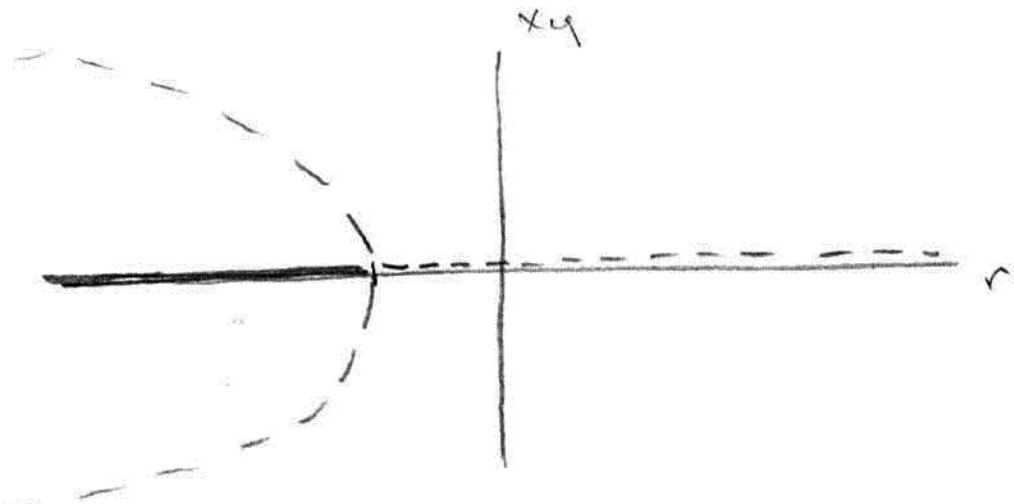
$$1 = -r \left.\frac{d}{dx}\left(\frac{x}{1+x^2}\right)\right|_{x=0} = -r(1)$$

$$\Rightarrow r = -1$$

$\underline{r > -1}$



<u>Bifer</u>



Pitchfork Bif

eq of wru

$$x\left(1 + \frac{r}{1+x^2}\right) = 0$$

$$\Rightarrow \quad -1 - x^2 = r \text{ or } \quad x = 0$$

$$x^2 =$$

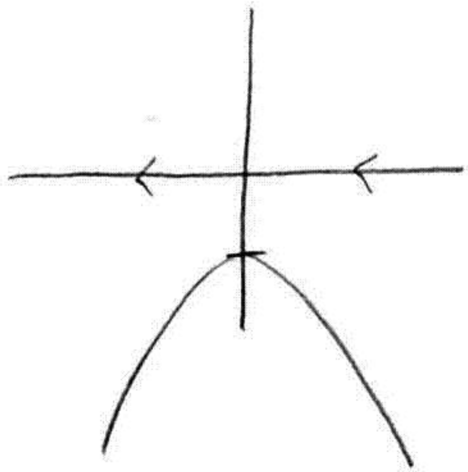
$$x = \pm\sqrt{1-r}$$

Pg 83 Strog:

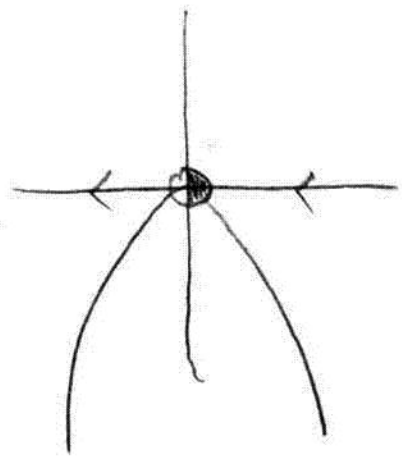
3.4.5 $\dot{x} = r - 3x^2 = 0$

$\Rightarrow \pm\sqrt{\frac{r}{3}}$

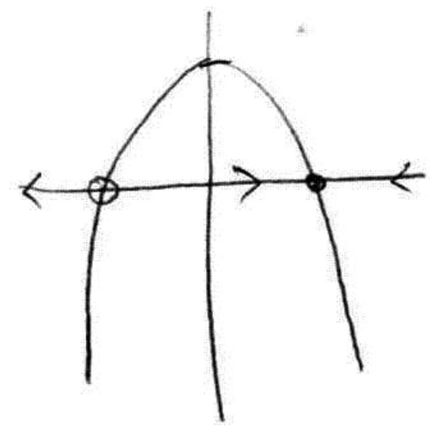
$r < 0$



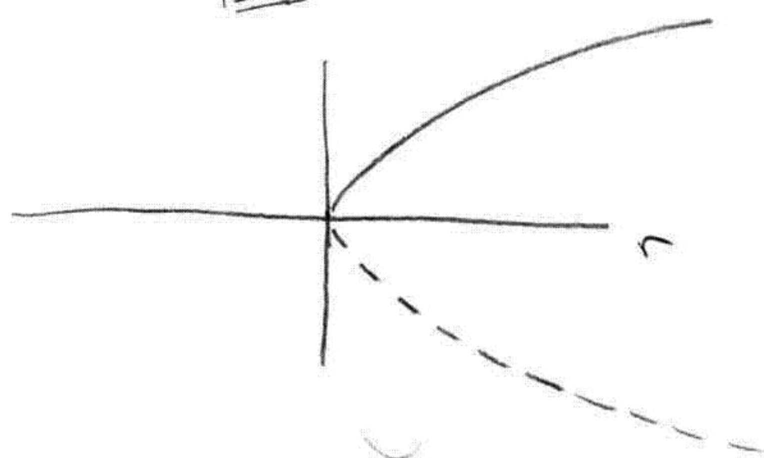
$r = 0$



$r > 0$



Bifur



"Saddle node bifurcation"

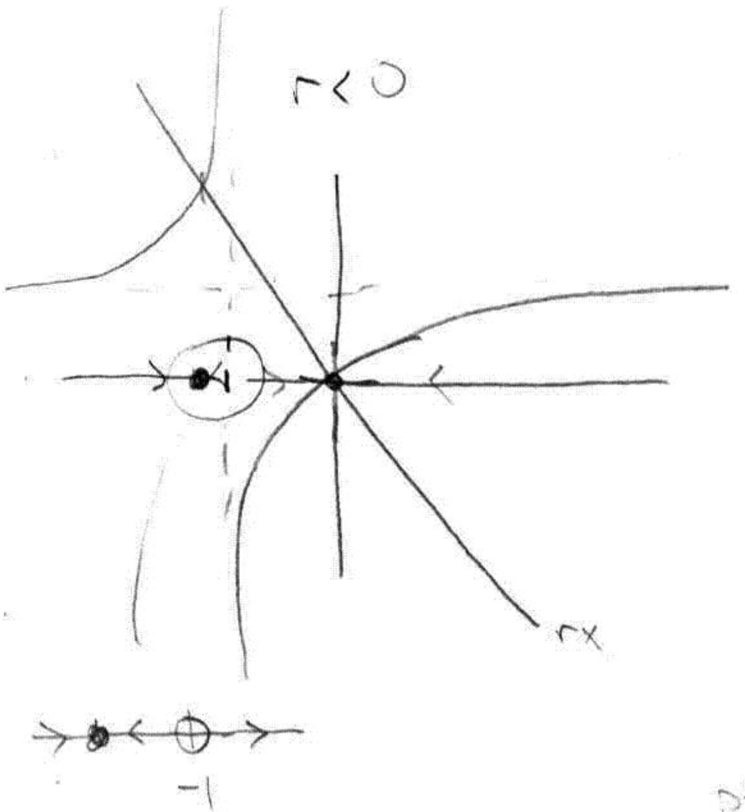
Pg 33 Stro

3.4.6 $\dot{x} = rx - \frac{x}{1+x} = x\left[r - \frac{1}{x+1}\right] = 0$

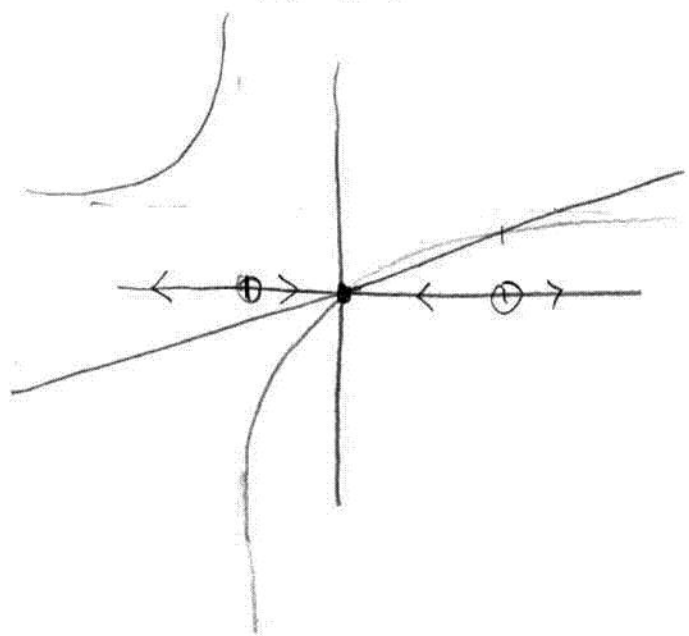
$\Rightarrow x = 0$ or $rx + r = 1$

$x = \frac{1-r}{r}$ 1,0

$r < 0$

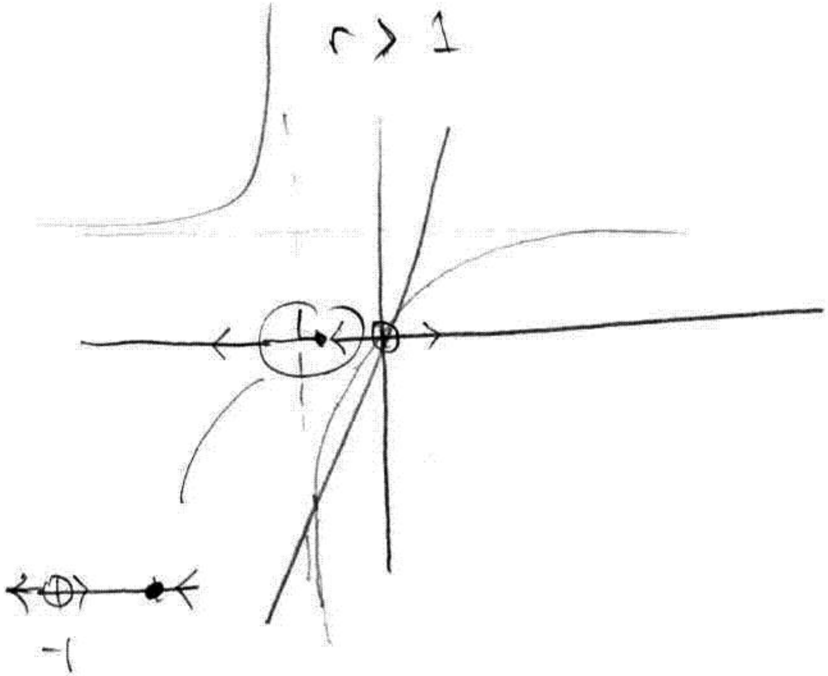


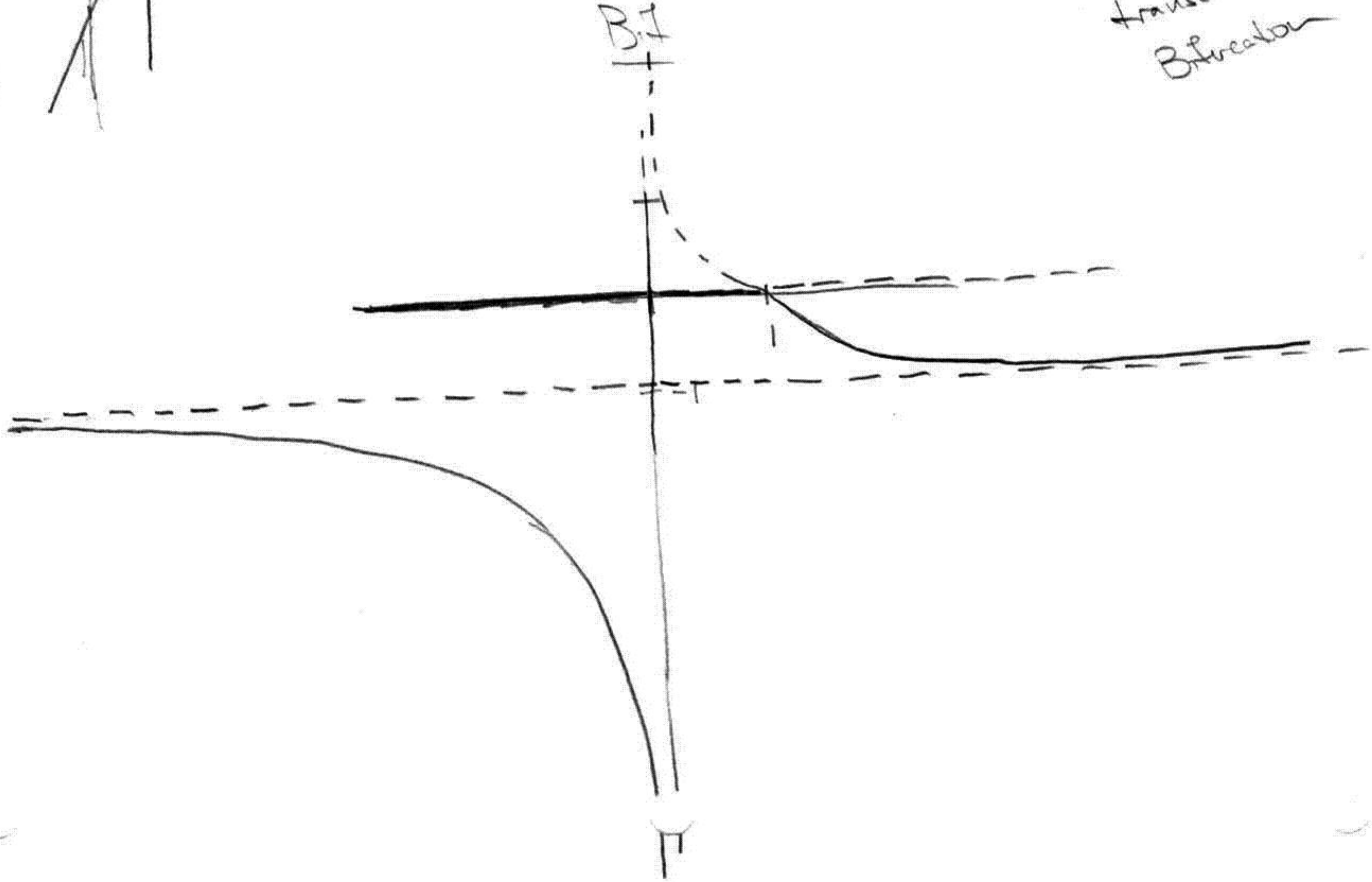
$0 < r < 1$



pt of tangency? $r = \frac{d}{dx}\left(\frac{x}{1+x}\right)\Big|_{x=0} = \underline{1}$

$r > 1$





transcritical Bifurcation

3.4.8

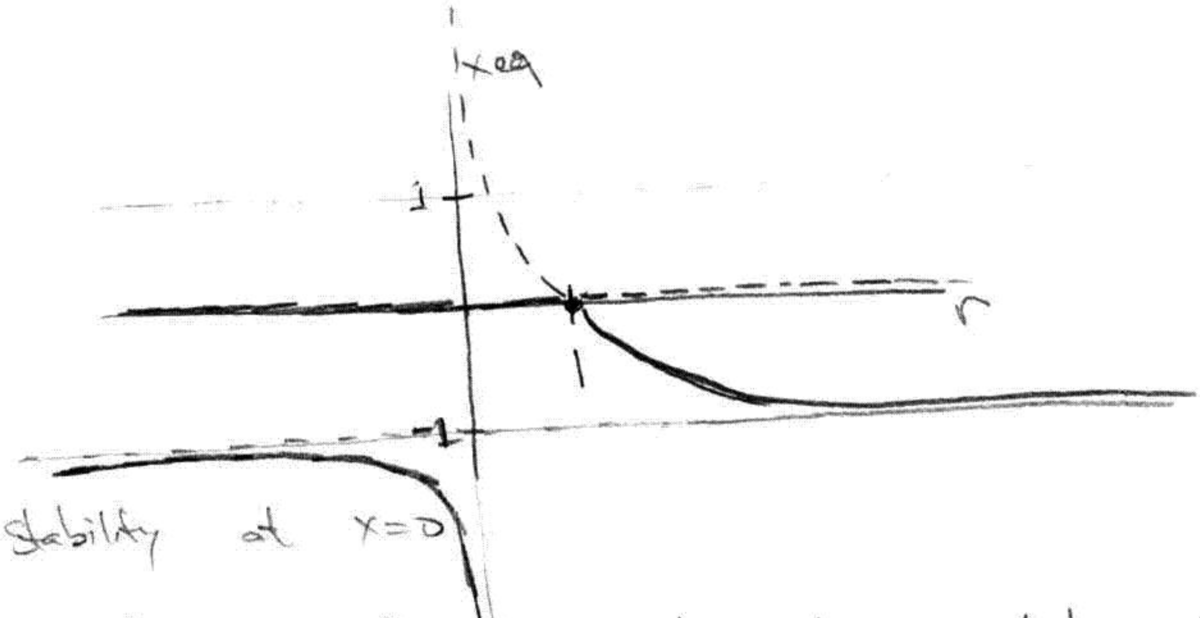
$$\dot{x} = rx - \frac{x}{1+x} = 0$$

$$= x\left(r - \frac{1}{1+x}\right) = 0 \quad \Rightarrow x = 0 \text{ or } r = \frac{1}{1+x}$$

$$\Downarrow$$

$$x = \frac{1}{r} - 1$$

$$x = \frac{1-r}{r}$$

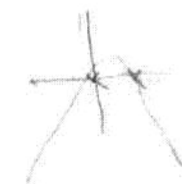


Stability at $x=0$

$$D\left(rx - \frac{x}{1+x}\right)\Big|_{x=0} = r-1 \quad \begin{array}{lll} <0 & r<1 & \text{stable} \\ >0 & r>1 & \text{unstable} \end{array}$$

Stability at $x = \frac{1-r}{r}$

$$D\left(rx - \frac{x}{1+x}\right)\Big|_{x=\frac{1-r}{r}} = (1-r)r \quad \begin{array}{lll} <0 & \begin{array}{l} -\infty < r < 0 \\ 1 < r < \infty \end{array} & \text{stable} \\ >0 & 0 < r < 1 & \text{unstable} \end{array}$$



$r = 1$ is a transcritical bifurcation.

John Weatherwax

3.5.7

Pg 85 Strogatz

1

(a) N has dimensions of animals (like 10 toads)

k " " " " " " "

r " " " animals/time " " toads/hr.

(Next pg)

Pg 35 Strogatz

3.5.7

(b) $\dot{N} = rN\left(1 - \frac{N}{k}\right)$

Let $N = x\xi$ $\quad$ ξ, η constants to be determined latter

& $t = \tau\eta$ $\quad$ & x, τ dimentionless

$\Rightarrow$ eq becomes $\dfrac{\xi\, dx}{\eta\, d\tau} = r\xi x\left(1 - \dfrac{\xi x}{k}\right)$

$\Rightarrow \dfrac{dx}{d\tau} = \eta r x\left(1 - \dfrac{\xi x}{k}\right)$ $\quad$ pick $\xi = k$

& $\eta = \dfrac{1}{r}$

$\Rightarrow \dfrac{dx}{d\tau} = x(1-x)$ $\quad$ & $N(0) = \xi x(0) = N_0 = \xi x_0$

$\Rightarrow x(0) = x_0$

Then $N = kx \Rightarrow x = \dfrac{N}{k}$ $\quad$ $x_0 = \dfrac{N_0}{k}$

$t = \dfrac{\tau}{r} \Rightarrow \tau = tr$

(c) To make $u_0 = 1$ As $N = u\xi$

$N(0) = u(0)\xi = N_0 \Rightarrow \xi = N_0$

$\therefore N = N_0 u$

& keep $\tau = tr$

to get $\frac{N_0}{(1/r)} \frac{du}{d\tau} = r N_0 u \left(1 - \frac{N_0 u}{k}\right)$

$\Rightarrow \frac{du}{d\tau} = u\left(1 - \left(\frac{N_0}{k}\right)u\right)$

(d) In the nondimensionalization II the quantity $\frac{N_0}{k}$ appears in the D.E rather than in the I.C. thus if we were also told something about $\frac{N_0}{k}$ like $\frac{N_0}{k} \ll 1$ It might be easier to simplify the D.E & then solve it rather than solve the D.E. & then simplify w/ $\frac{N_0}{k} \ll 1$

3.5.8

$$\dot{u} = au + bu^3 - cu^5 \qquad b, c > 0$$

let $u = Ux$ w/ x & τ nondimensional

$t = T\tau$

$$\Rightarrow \frac{U}{T}\frac{dx}{d\tau} = aUx + bU^3x^3 - cU^5x^5$$

$$\frac{dx}{d\tau} = Tax + bU^2Tx^3 - cU^4Tx^5$$

to get desired form must have $bU^2T = 1$

& $cU^4T = 1$

$\div$ both eq $\quad \frac{c}{b}U^2 = 1$

$$U = \pm\sqrt{b/c}$$

Then $T = \frac{1}{bU^2} = \frac{1}{b}\left(\frac{c}{b}\right) = \frac{c}{b^2}$

$x = \frac{u}{U}$ w/ $U = \pm\sqrt{b/c}$

& $\tau = \frac{t}{T}$ w/ $T = c/b^2$

Then $r = Ta = \frac{ca}{b^2}$

3.7.1

Pg 89 Strogatz

1

$$\frac{dx}{dJ} = rx\left(1 - \frac{x}{k}\right) - \frac{x^2}{1+x^2}$$

to evaluate stability at $x = 0$

$$\frac{d}{dx}\left(rx\left(1 - \frac{x}{k}\right) - \frac{x^2}{1+x^2}\right)\Bigg|_{x=0} = r - \frac{2rx}{k} + \frac{2x^3}{(1+x^2)^3} - \frac{2x}{1+x^2}\Bigg|_{x=0}$$

$$= r > 0 \quad \forall x$$

$\therefore$ $x = 0$ is unstable

Pg 37 Strogatz

3.6.6

(a) Supercritical bifurcation $g > 0$

$\tau \dot{A} = \epsilon A - gA^3 = 0$ for eq.

$= A^*(\epsilon - gA^{*2}) = 0$

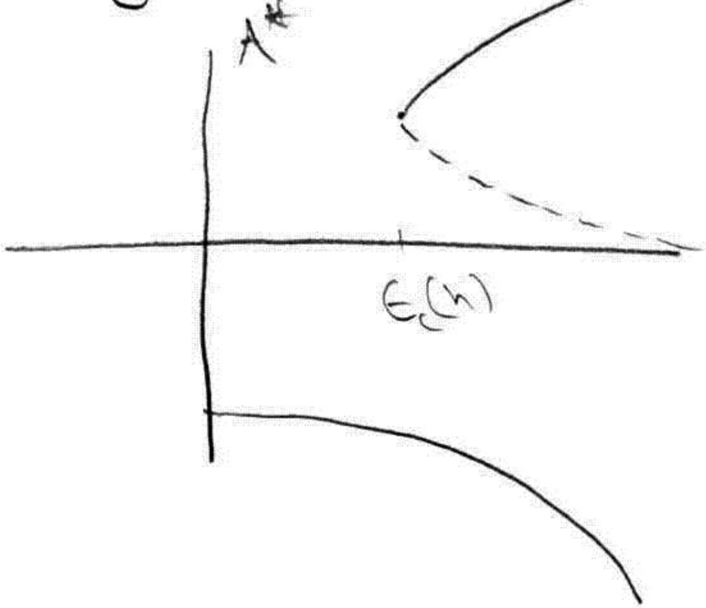
$\Rightarrow A^* = 0 \Rightarrow \epsilon = gA^{*2}$ or $A^* \propto \epsilon^{1/2}$ $\Rightarrow \beta$ exactly equals $1/2$

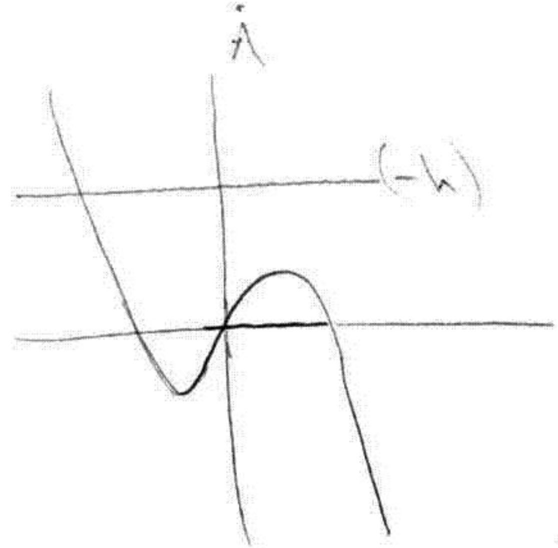
(b) $\tau \dot{A} = \epsilon A - kA^5$ $k > 0$ $g = 0$

$\tau \dot{A} = 0$ for eq & bifc

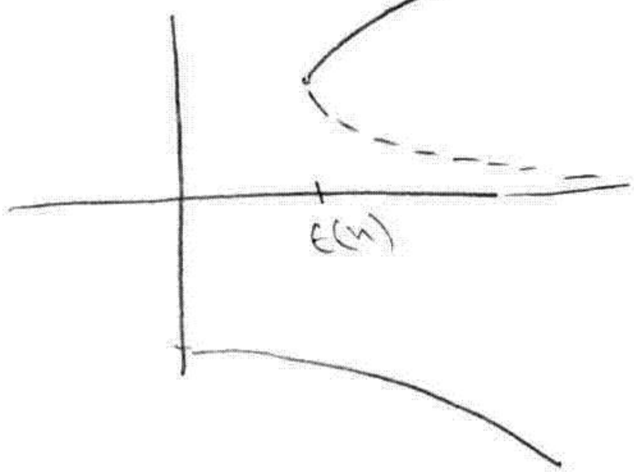
$\Rightarrow A^*(\epsilon - kA^{*4}) = 0 \Rightarrow A^* = \pm \left(\frac{\epsilon}{k}\right)^{1/4}$

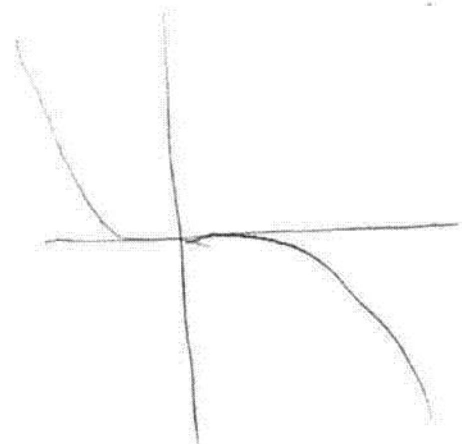
(c) $\underline{g > 0}$

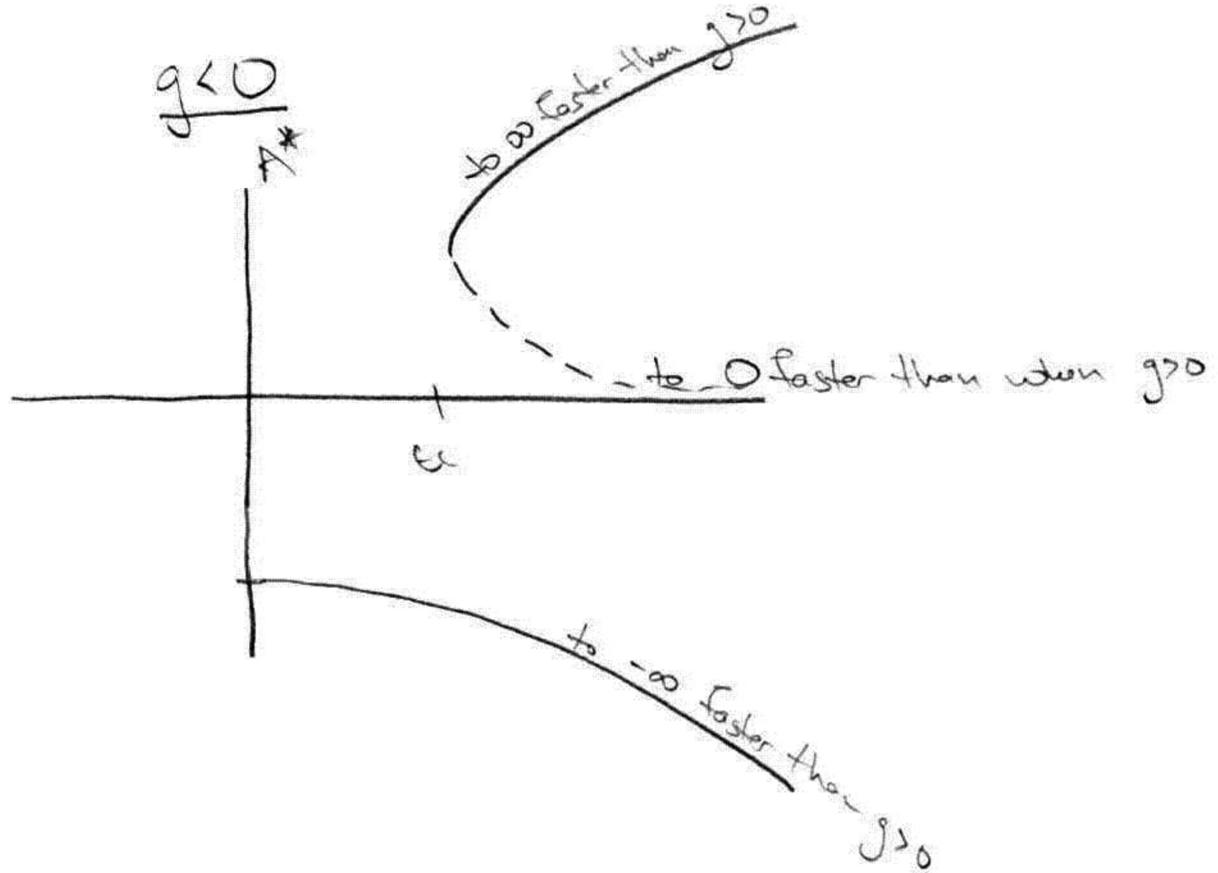




$\underline{g = 0}$






g<0
A*
to ∞ faster than g>0
to 0 faster than when g>0
t_c
to -∞ faster than g>0

(d) Plot $J\dot{A} = h + \epsilon A + |g|A^3 - kA^5 \equiv f(A)$ & $h > 0$

$A \to -\infty \qquad f(A) \longrightarrow +\infty$

$A \to +\infty \qquad f(A) \longrightarrow -\infty$

Look for critical pts

$$f'(A) = \epsilon + 3|g|A^2 - 5kA^4 = 0$$

$$5kA^4 - 3|g|A^2 - \epsilon = 0$$

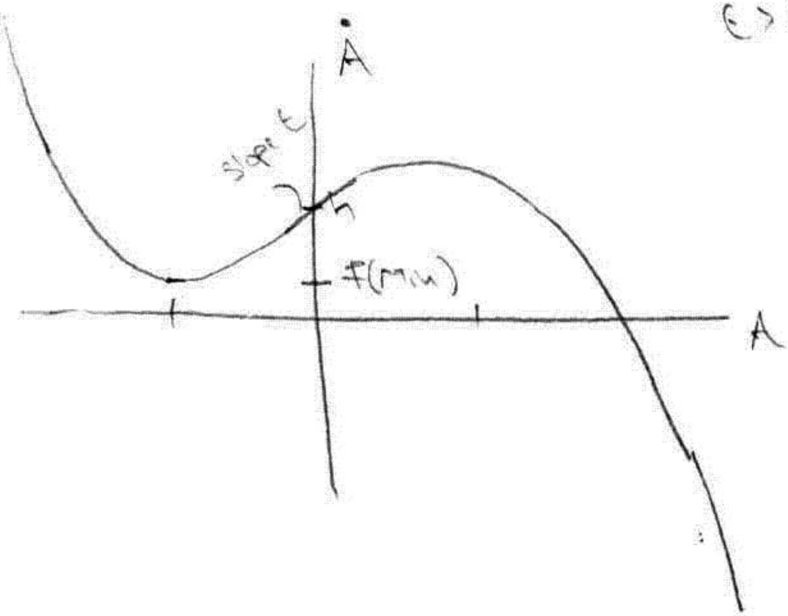
$$A^2 = \frac{3|g| \pm \sqrt{(3|g|)^2 + 4(5k)\epsilon}}{2(5k)}$$

only 2 non complex roots

← + sign in there

look at $\dot{A}$ v.s A

$\epsilon > 0$



Now evaluate $f(A_{min})$ to see if f is above or below axis.

$$f(A) = h + A\left(\epsilon + |g|A^2 - kA^4\right)$$

$$\therefore \quad f(A_{min}) = h + A_{min}\left[\frac{3g^2 + 20k\epsilon + g\sqrt{9g^2 + 20k\epsilon}}{25k}\right] \quad \text{when substituted}$$

Now $A_{min} < 0$ thus if ϵ is small $A_{min}^2 \approx \dfrac{3|g|}{5k} \Rightarrow A_{min} \approx -\sqrt{\dfrac{3|g|}{5k}}$

Then $f(A_{min}) \approx h - \dfrac{6}{25}\sqrt{\dfrac{3}{5}}\, g\left(\dfrac{g}{h}\right)^{3/2}$

ϵ large

$$f(A_{min}) \approx h - \left(\sqrt{\frac{3g + \sqrt{\quad}}{10k}}\right)\left[3g^2 + 20k\epsilon\right.$$

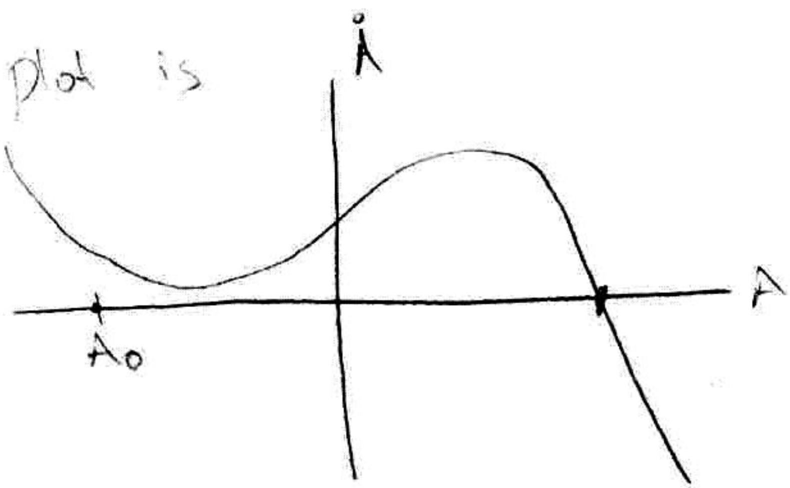
$$\approx h - O(\epsilon^{1/4})O(\epsilon^{1})$$

$$\approx h - O(\epsilon^{5/4}) \quad \text{thus}$$

for large enough ϵ $f(A_{min}) < 0$

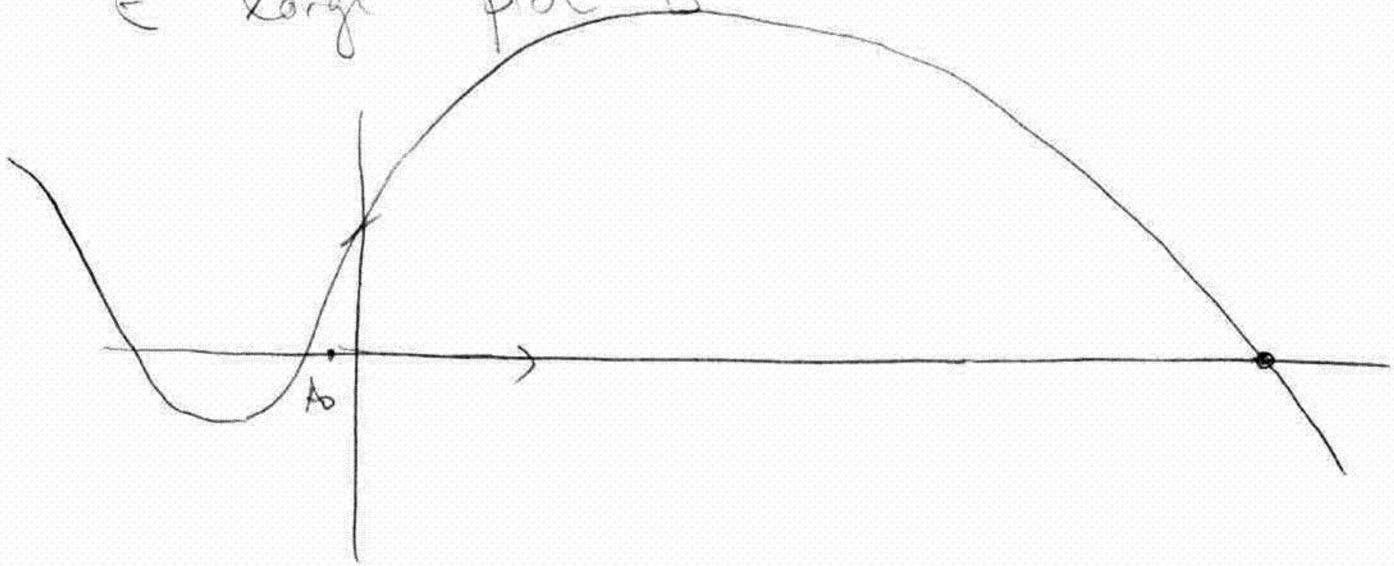
Thus for ϵ small but not small enough to make $f(A_{min}) < 0$

Plot is



Thus A_0 starts < 0 & goes through a "bottle neck" at Min of $f(A)$ then speeds rapidly up to final equilibrium pt.

ϵ large plot is



Thus A_0 quickly heads to the equilibrium location + then stays there. There is no "bottle neck"

John Weatherwax

Pg 113 Strogatz

5/5

4.1.1

$\sin(a\theta)$ must be 2π periodic & thus

$$\sin(a(\theta+2\pi)) = \sin a\theta \cos 2\pi a + \cos a\theta \sin 2\pi a = \sin(a\theta)$$

$$\Rightarrow \left.\begin{array}{l} \cos 2\pi a = +1 \\ \sin 2\pi a = 0 \end{array}\right\} \Rightarrow a = \text{an integer}$$

pg 113 Strogatz

4.1.7 $\dot{\theta} = \sin(k\theta) = 0$

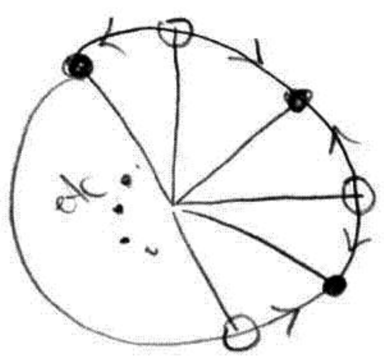
$\Rightarrow k\theta = n\pi \Rightarrow \theta = \frac{n\pi}{k}$ $n \in$ integers

As $\sin(k\theta)$ is 2π periodic unique fixed points occur for $n = 0, \ldots, 2k-1$

Then stability $\frac{d}{d\theta}(\sin k\theta) = k\cos(k\theta)$

which alternates stability & $\theta = 0$ is unstable thus points alternate from unstable to stable

Phase portrait on the circle looks as follows

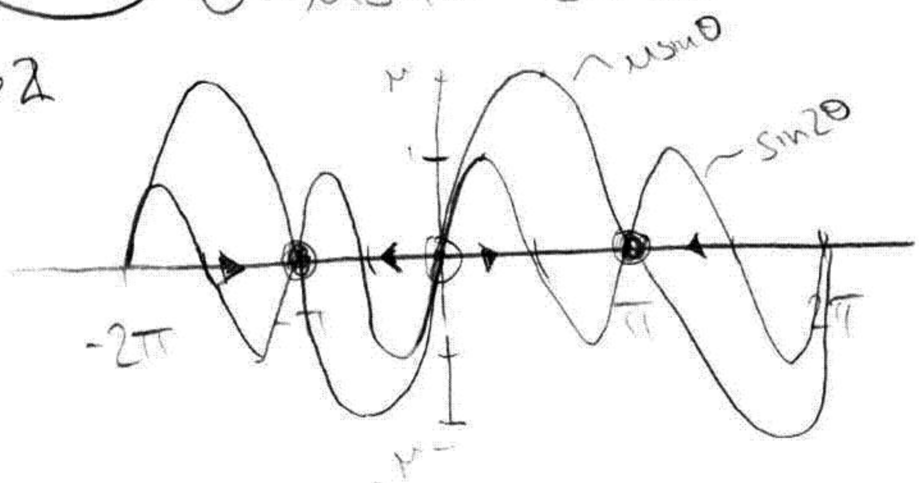


w/ $2k$ fixed points & thus $2k$ slices

Pg 115 Strogatz

4.3.3 $\dot{\theta} = \mu \sin\theta - \sin 2\theta$

$\mu > 2$



Intersections $\quad \mu \sin\theta = 2\sin\theta\cos\theta$

$\sin\theta\,[\mu - 2\cos\theta] = 0 \qquad \sin\theta = 0$

$\theta = 0, \pi$

$\cos\theta = \frac{\mu}{2}$

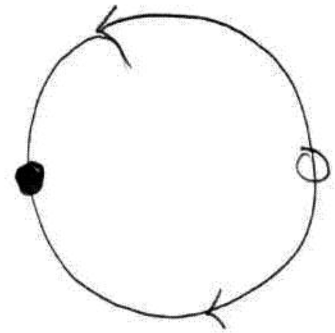
(No solution as $\mu > 2$)

$\mu > 2$

$\theta = 0$ unstable

$\theta = \pi$ stable

$\mu > 2$



$0 < \mu < 2$

Then sol are $\theta = 0, \pi$ + $\cos\theta = \frac{\mu}{2}$

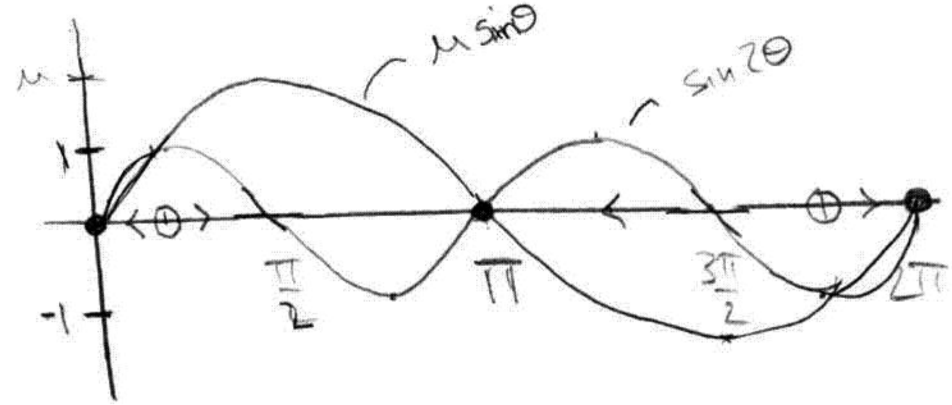
or $\theta = \cos^{-1}\left(\frac{\mu}{2}\right)$ $\exists$ 2 pts

$0 < \theta_1 < \frac{\pi}{2}$ + $\frac{3\pi}{2} < \theta_2 < 2\pi$

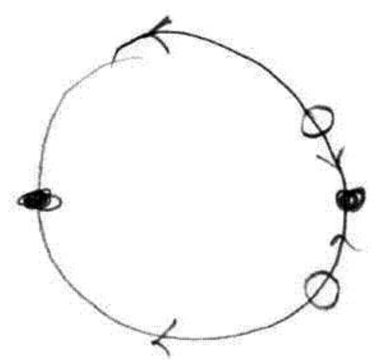
(If $\mu = 2$ $\theta_1 = 0$

If $\mu = 0$ $\theta_1 = \frac{\pi}{2}$, $\theta_2 = \frac{3\pi}{2}$)

Thus graph looks like



Phase diagram on circle looks like



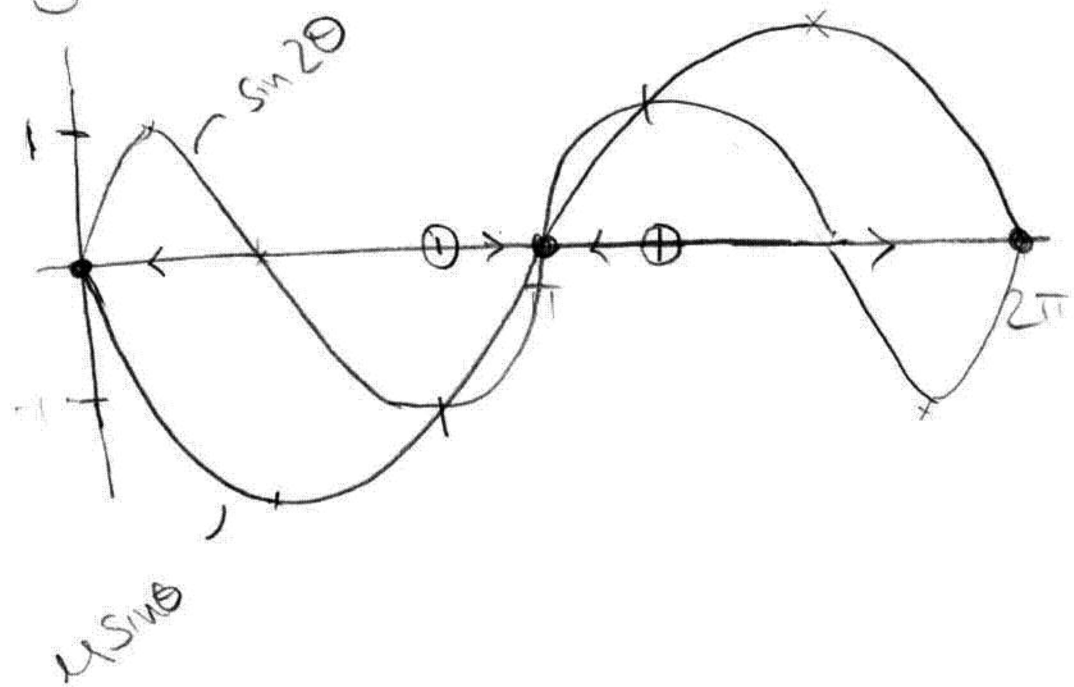
For $-2 < \mu < 0$

$\cos\theta = \frac{\mu}{2}$ gives 2 pts in $\pi < \theta_1 < \frac{\pi}{2}$

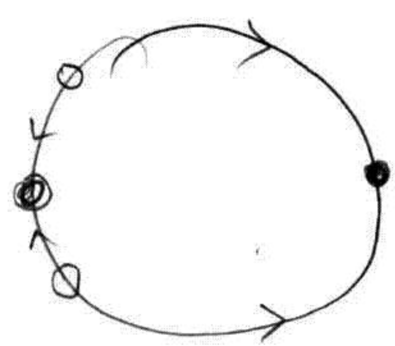
$+ \quad \pi < \theta_2 < \frac{3\pi}{2}$

That are simply π added to pts found before thus

Phase diagram looks like

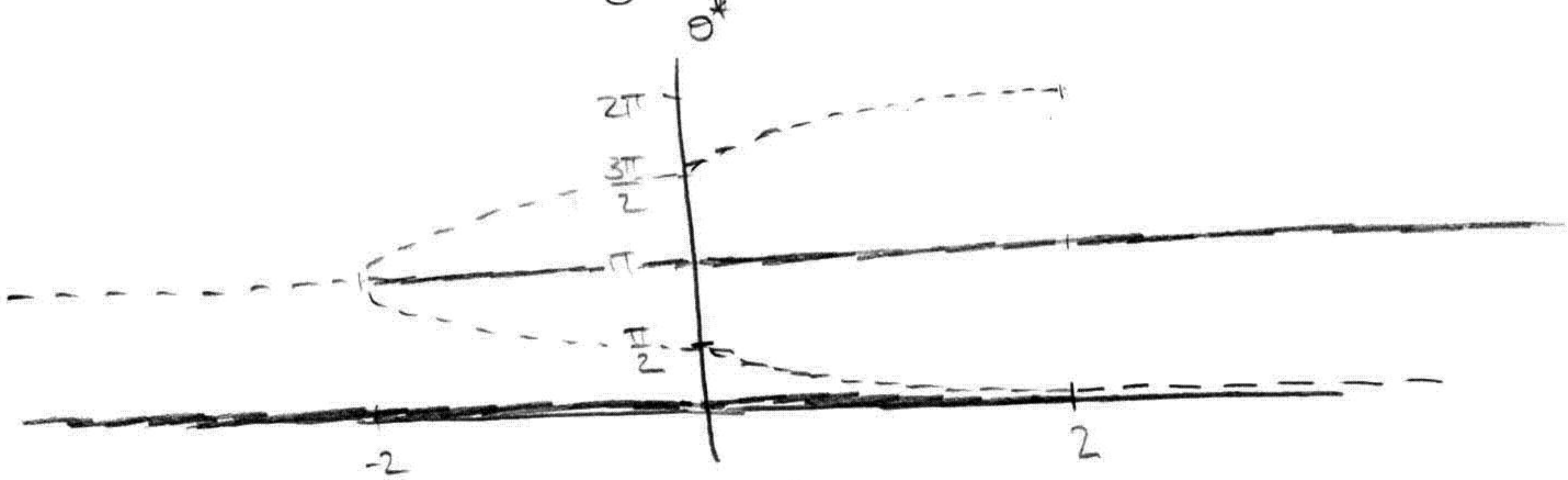


Thus phase diagram on circle looks like



for $\mu < -2$ No other phase points exist besides $0, \pi$

Thus Bifurcation diagram looks as follows



Thus $\mu = -2$ is a Saddle node bifurcation

5.1.10

(a) $\dot{x} = y$, $\dot{y} = -4x$ $= \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

$\tau = 0$

$\Delta = 4$

Thus $(0,0)$ is a center & $\therefore$ only Liapunov stable

(b) $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

$\tau = 0$ $(0,0)$ is saddle point Not stable at all

$\Delta = -2$

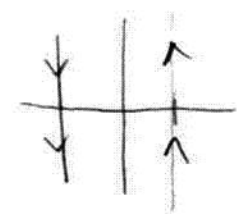
(c) $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

$\tau = 0$

$\Delta = 0$

Sol $x = \text{const}$

$y = (\text{const})t$



$(0,0)$ is Liapunov stable

(d) $\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$

$\left.\begin{array}{l}\tau = -1\\ \Delta = 0\end{array}\right\} \Rightarrow$ (0,0) is stable node + in this case ~~is~~

Liapunov stable.

(e) $\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}-1 & 0\\ 0 & -5\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$

$\tau = -6$ $\qquad \tau^2 - 4\Delta = 36 - 20 = 16 > 0$ stable node

$\Delta = 5$

asymptotically stable

f) $\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$

$\tau^2 - 4\Delta = 0$ $\therefore$ star ~~or~~ degenerate node

$\tau = 2$

$\Delta = 1$ $\qquad$ As this is multiple of Identity.

This is a star

But $\dot{x} = x \Rightarrow x = Ce^{+t}$
$\dot{y} = y \Rightarrow y = Ce^{+t}$ $\Big\}$ thus $(0,0)$ is not stable

As all points move away from there

Thus $(0,0)$ is none of the given options.

PJ 142 Strogatz

1

John Weatherwax

5.2.2

5/5

(a) $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}}_{=A} \begin{pmatrix} x \\ y \end{pmatrix}$

evalues

$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & -1 \\ 1 & +1-\lambda \end{vmatrix} = (1-\lambda)^2 + 1 = 0$

$= 1 - 2\lambda + \lambda^2 + 1 = 0$

$\Rightarrow \lambda = 1 \pm i$

evectors

$$\begin{pmatrix} 1-(1+i) & -1 \\ 1 & 1-(1+i) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & -i \\ 1 & -i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow \quad v_1 = i v_2$ pick $\bar{v} = \begin{pmatrix} i \\ 1 \end{pmatrix}$

other evector

$$\begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & i \\ 1 & i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

pick $\bar{v} = \begin{pmatrix} -i \\ 1 \end{pmatrix}$

(b) $$\bar{x}(t) = c_1 e^t e^{it}\begin{pmatrix} i \\ 1 \end{pmatrix} + c_2 e^t e^{-it}\begin{pmatrix} -i \\ 1 \end{pmatrix}$$

$$= c_1 e^t(\cos t + i\sin t)\begin{pmatrix} i \\ 1 \end{pmatrix} + c_2 e^t(\cos t - i\sin t)\begin{pmatrix} -i \\ 1 \end{pmatrix}$$

$$= e^t\begin{bmatrix} c_1(i\cos t - \sin t) \\ c_1(\cos t + i\sin t) \end{bmatrix} + e^t\begin{bmatrix} c_2(-i\cos t - \sin t) \\ c_2(\cos t - i\sin t) \end{bmatrix}$$

$$= e^t\begin{bmatrix} -c_1\sin t - c_2\sin t \\ c_1\cos t + c_2\cos t \end{bmatrix} + e^t i\begin{bmatrix} c_1\cos t - c_2\cos t \\ c_1\sin t - c_2\sin t \end{bmatrix}$$

$$= (c_1+c_2)e^t\begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} + i(c_1-c_2)e^t\begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$$

$$= C_1 e^t\begin{pmatrix} \cos t \\ \sin t \end{pmatrix} + C_2 e^t\begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} \quad \text{w/ } \begin{matrix} C_1 = i(c_1-c_2) \\ C_2 = c_1+c_2 \end{matrix}$$

6.1.7

Null cline is $\dot{x} = 0$ or $x = -e^{-y}$

to determine stable manifold attempt to solve the DE

$$\dot{x} = x + e^{-y}$$
$$\dot{y} = -y$$

$$\frac{dy}{dx} = \frac{-y}{x + e^{-y}}$$

$$\Leftrightarrow \quad y\,dx + (x + e^{-y})\,dy = 0$$

$$M_y = 1 \qquad N_x = 1 \quad \text{exact}$$

$$\exists\, \phi \ni \frac{\partial \phi}{\partial x} = y \quad \Rightarrow \quad \phi = xy + \phi_1(y)$$

$$\frac{\partial \phi}{\partial y} = x + \phi_1'(y) = x + e^{-y}$$

$$\Rightarrow \quad \phi = xy - e^{-y} = \text{const}$$

to cross x-axis $y = 0$

$\Rightarrow \quad 0 = C + 1$

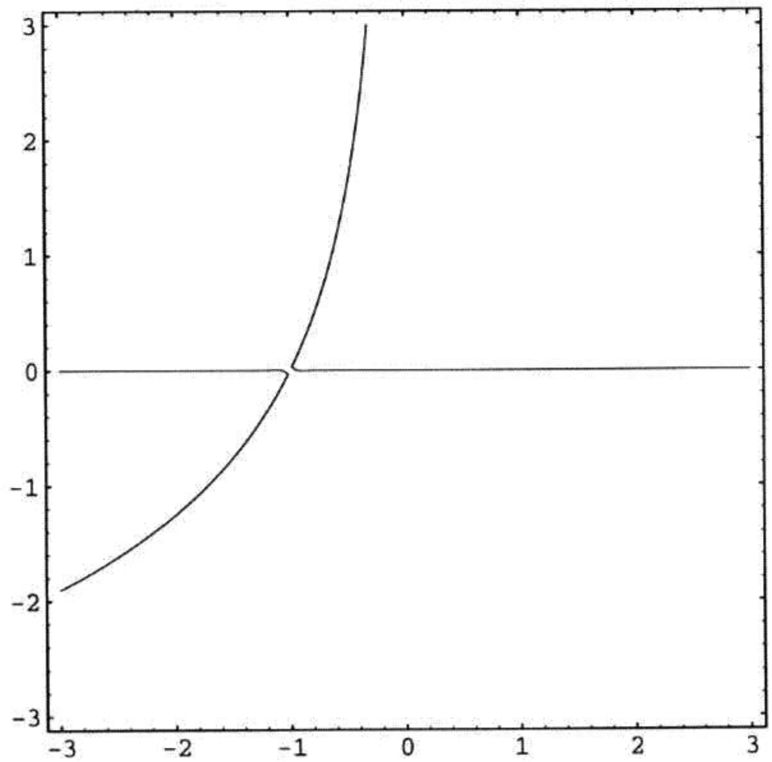
$\Rightarrow$ const $= -1$

$\Rightarrow$ curve of stable manifold is $xy = -1 + e^{-y}$

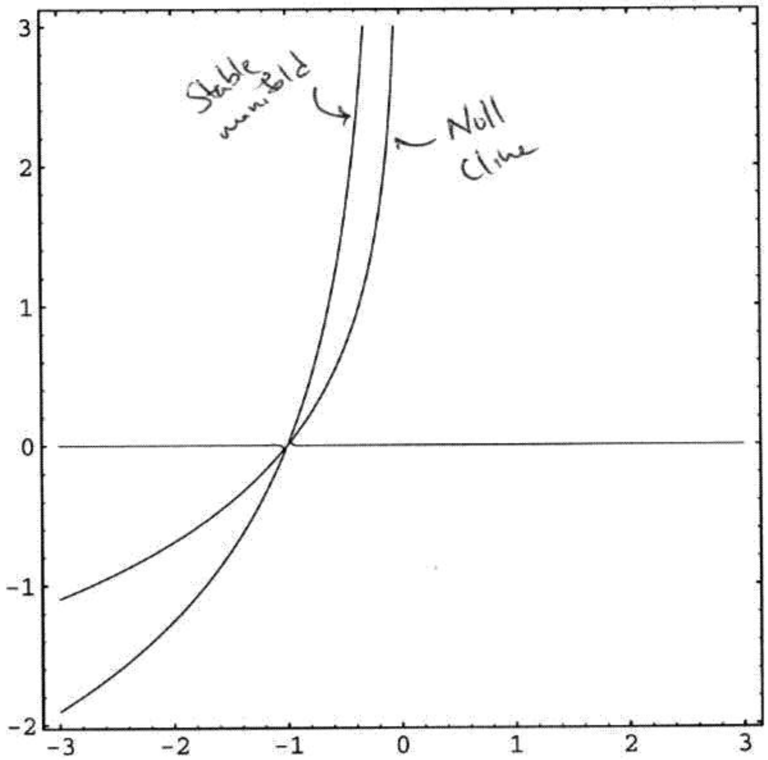
Both are ploted by the computer & thus one can see the results.

6.1.7

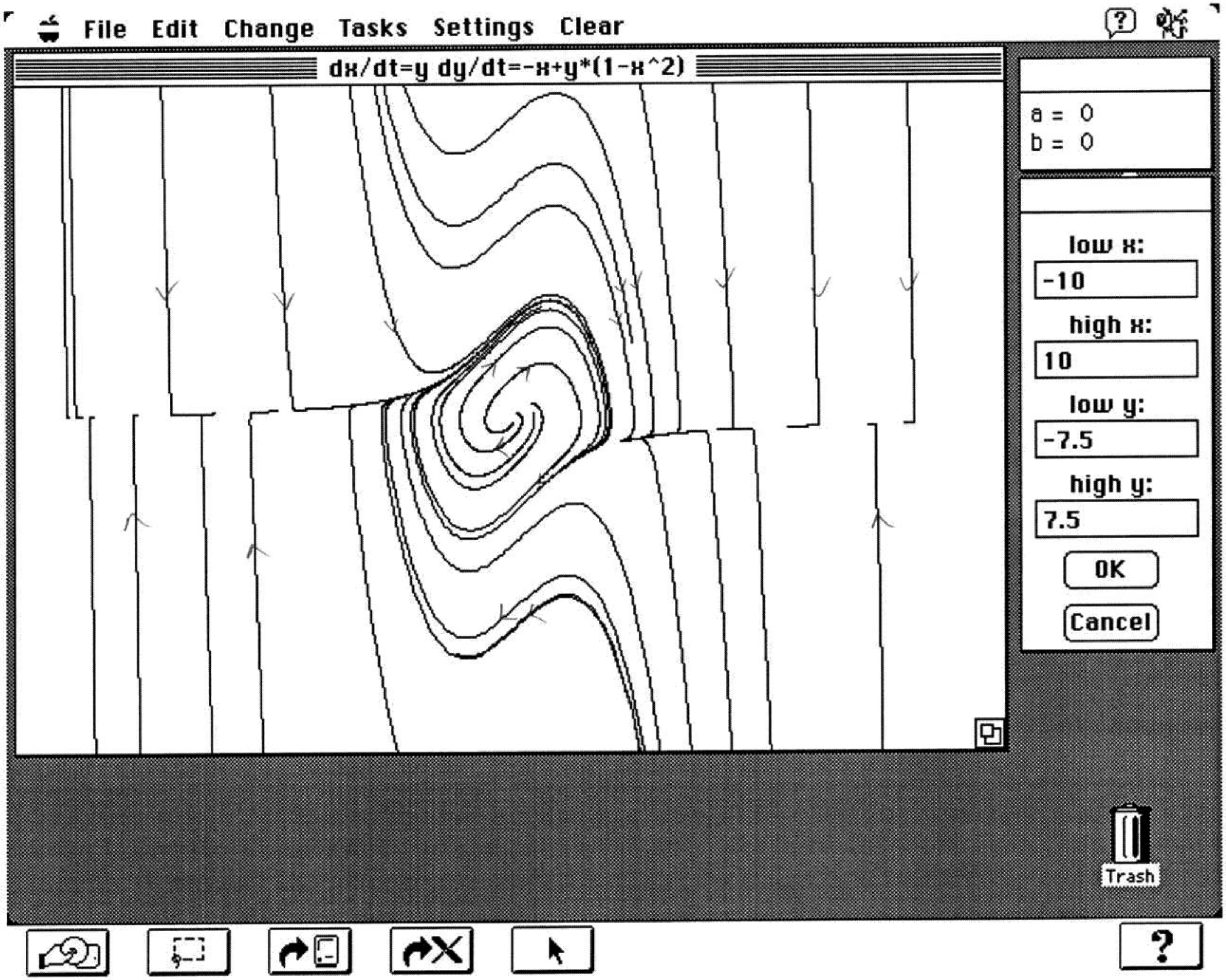

```
ContourPlot[x y-E^(-y),{x,-3,3},{y,-3,3},
    Contours->{-1},
    ContourShading->False,
    PlotPoints->100];
```



```
Show[Out[5],Out[8]];
```

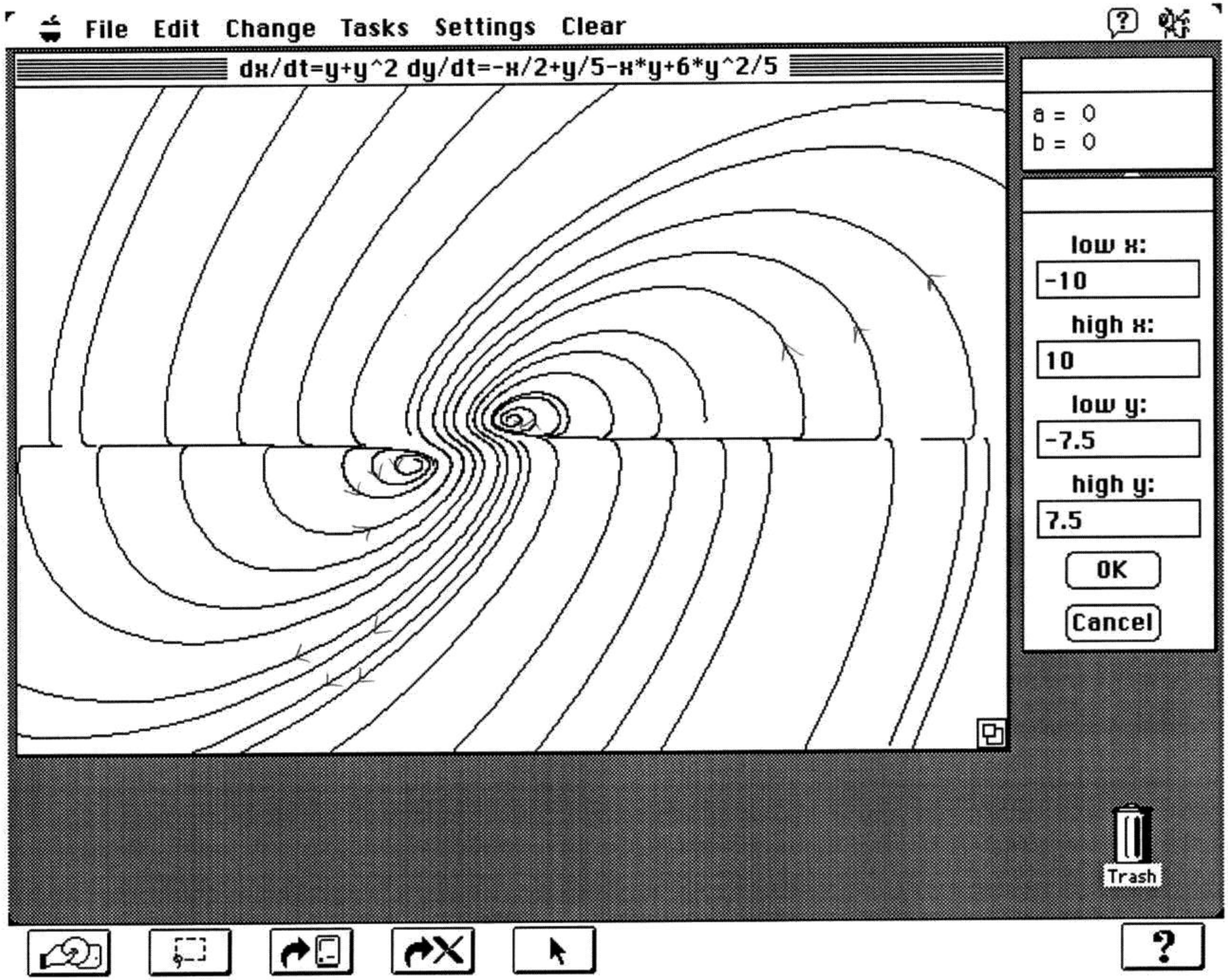


6.1.8



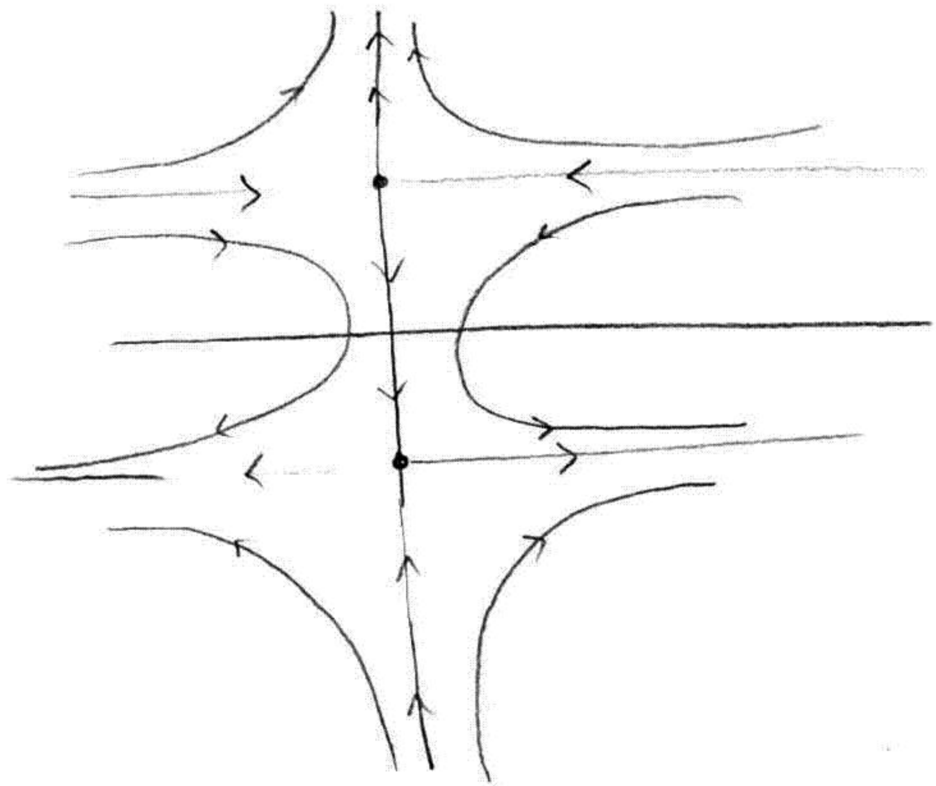
File Edit Change Tasks Settings Clear
dx/dt=y dy/dt=-x+y*(1-x^2)
a = 0
b = 0
low x:
-10
high x:
10
low y:
-7.5
high y:
7.5
OK
Cancel
Trash

6.1.10

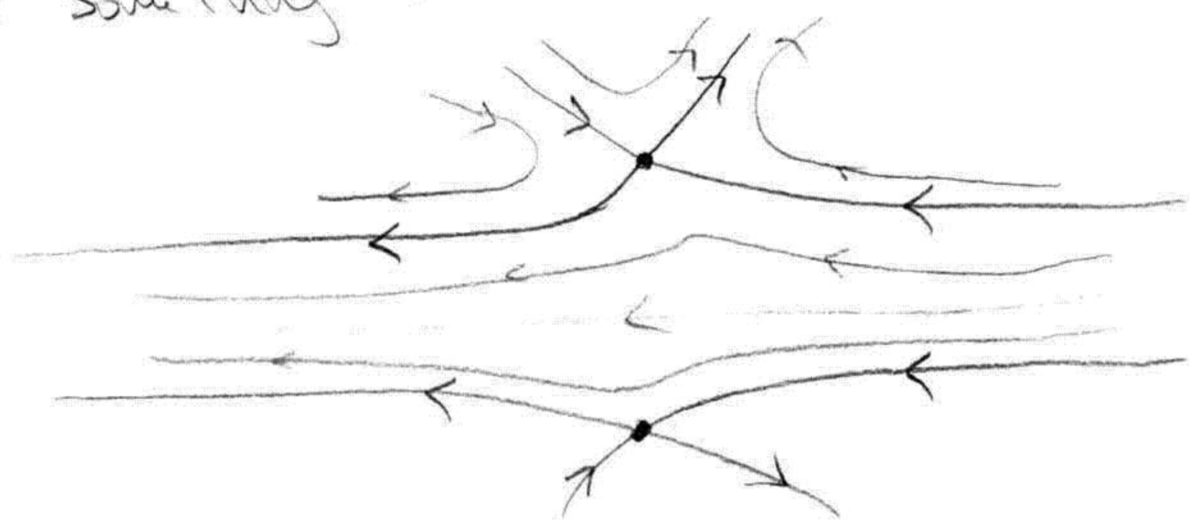

File Edit Change Tasks Settings Clear
dx/dt=y+y^2 dy/dt=-x/2+y/5-x*y+6*y^2/5
a = 0
b = 0
low x:
-10
high x:
10
low y:
-7.5
high y:
7.5
OK
Cancel
Trash
?

6.1.12

(a)



(b) Something like the following



pg 183 Strogatt

1

6.3.10

$$\dot{x} = xy$$
$$\dot{y} = x^2 - y^2$$

(a)

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y & x \\ 2x & -2y \end{pmatrix}_{(x,y)=(0,0)} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$\therefore$ $\Delta = 0$ $\tau = 0$ Non isolated fixed pt as $\Delta = 0$

(b) If $x \neq 0$ & $y \neq 0$ then either $\dot{x} \neq 0$ or $\dot{y} \neq 0$

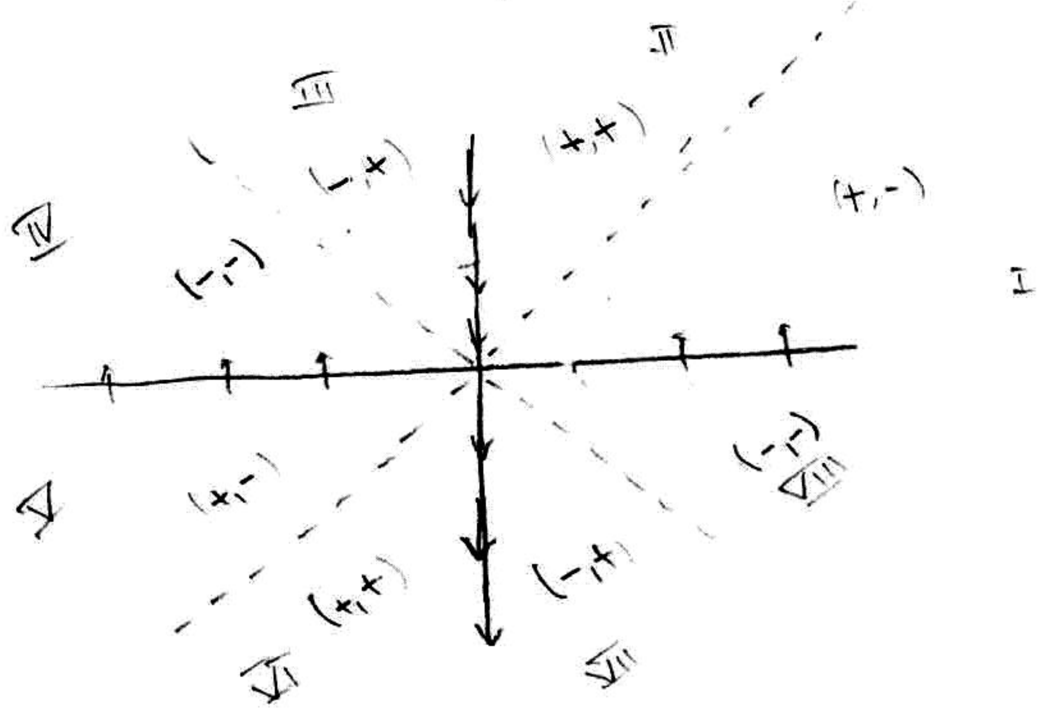
Thus in a small Nbhd of $(0,0)$ there is no other fixed pt as $\dot{x} \neq 0$ & $\dot{y} \neq 0$ thus $(0,0)$ is isolated

(c) Null clines $x = 0$ & $y = 0$ $\Rightarrow$ $\dot{x} = 0$

$\dot{y} = 0 \Rightarrow x^2 = y^2 \Rightarrow x = \pm y.$

If $x, y \in$ 1st quad $\Rightarrow$ $x < y$ $\quad \dot{x} > 0 \quad \dot{y} < 0$

$x > y \quad \dot{x} > 0 \quad \dot{y} > 0$



As we cross into the eight regions one of the inequalities will flip thus in region

III $\quad \dot{x} < 0 \quad \dot{y} > 0$

IV $\quad \dot{x} < 0 \quad \dot{y} < 0$

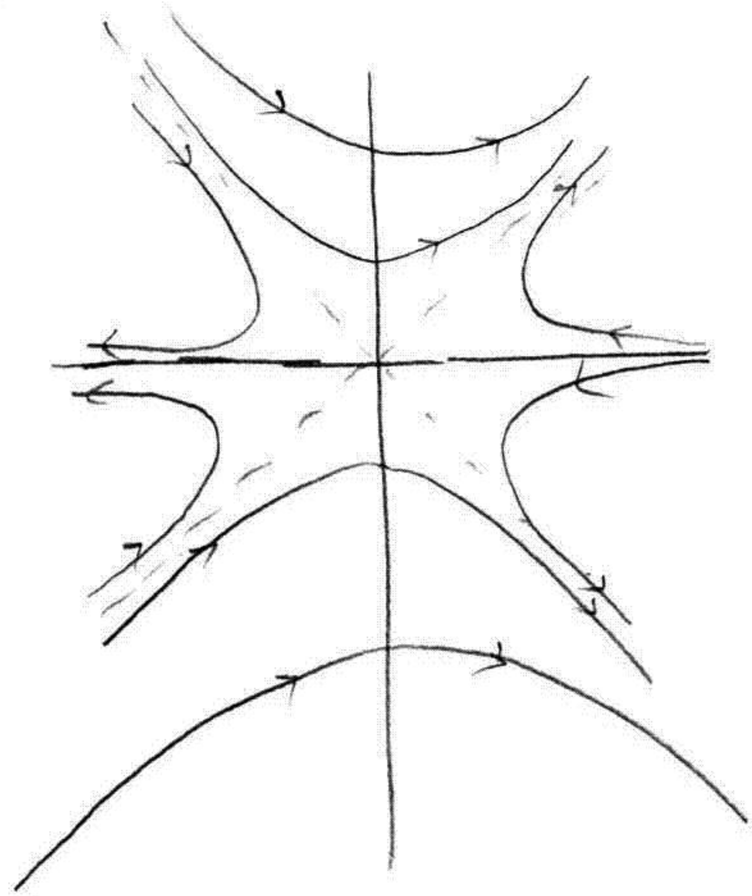
V $\quad \dot{x} > 0 \quad \dot{y} < 0$

VI $\quad \dot{x} > 0 \quad \dot{y} > 0$

VII $\quad \dot{x} < 0 \quad \dot{y} > 0$

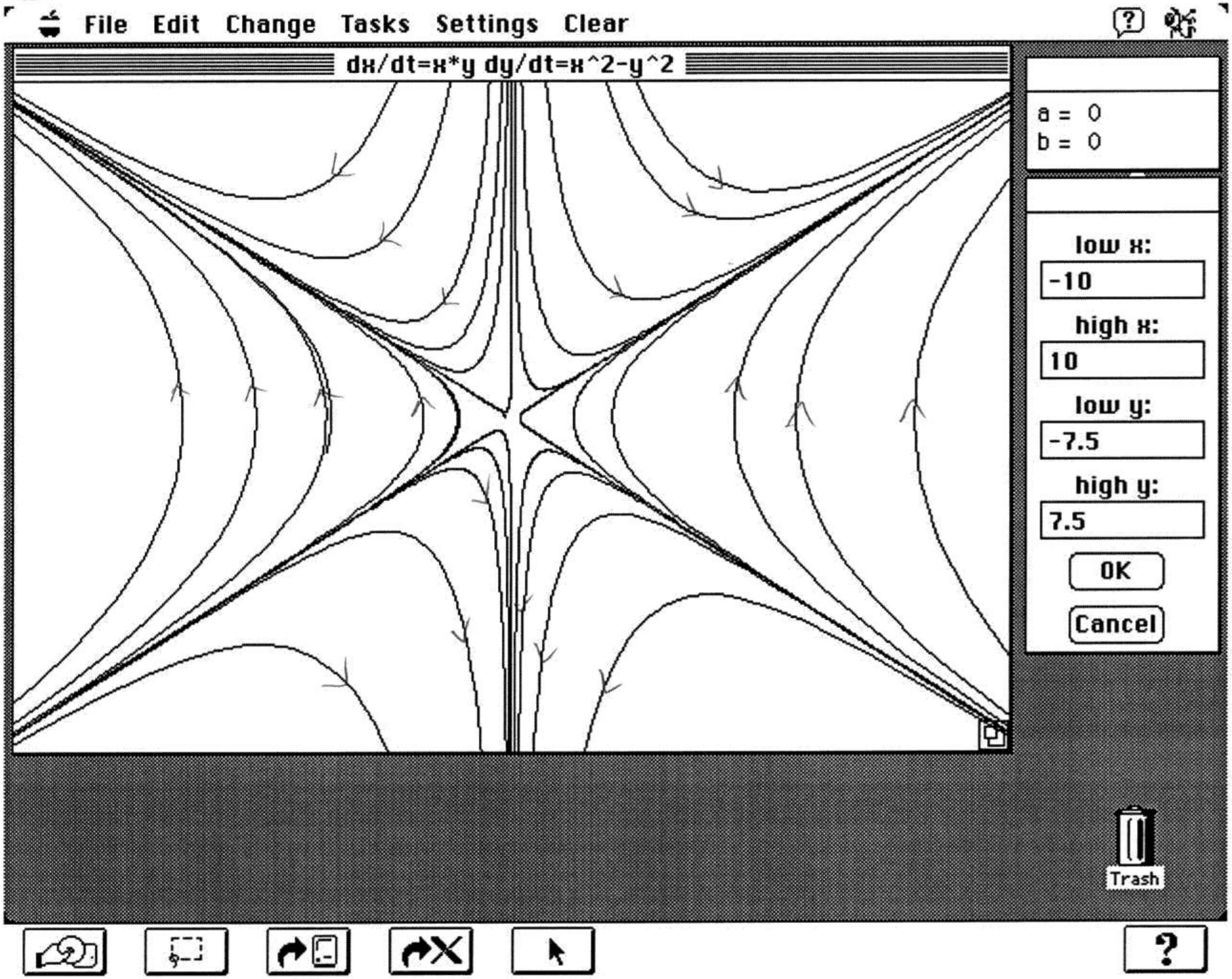
VIII $\quad \dot{x} < 0 \quad \dot{y} > 0$

c)



6.3.10

(d)



File Edit Change Tasks Settings Clear
dx/dt=x*y dy/dt=x^2-y^2
a = 0
b = 0
low x:
-10
high x:
10
low y:
-7.5
high y:
7.5
OK
Cancel
Trash
?

Pg 166 Strogatz

$$J(0,0) \neq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

e. vectors

e. values $\lambda^2 - 0\lambda + -1 = 0$

$\lambda = \pm 1$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = +1\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$\Rightarrow \quad v_2 = v_1 \qquad$ let $\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$v_1 = v_2$

$2\cos x = -\cos y$

$2\cos y = -\cos x \Rightarrow \cos x = -2\cos y$

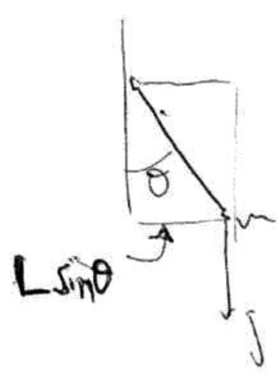
$\Rightarrow -4\cos y = -\cos y \Rightarrow -3\cos y = 0 \Rightarrow \cos y = 0$

$$J = \begin{pmatrix} 2\sin x & \sin y \\ \sin x & 2\sin y \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix}$$

$(x = \frac{-\pi}{2}, y = \frac{-\pi}{2})$

τ $\quad \tau = I\alpha$

$$mL^2 \frac{d^2\theta}{dt^2} = -L\sin\theta \, mg$$



$$\frac{L}{g}\frac{d^2\theta}{dt^2} + \sin\theta = 0$$

$$\frac{d}{dt} = \omega \frac{d}{d\tau}$$

$$\frac{d^2}{dt^2} = \omega^2 \frac{d^2}{d\tau^2} = \frac{g}{L}\frac{d^2}{d\tau^2}$$

$$J = \begin{pmatrix} 0 & 1 \\ -\cos\theta & 0 \end{pmatrix}$$

$$J(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad J(\pi,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$\tau = 0 \qquad \Delta = 1$

$$\lambda^2 - 0\lambda + -1 = 0$$

$$\lambda = \pm 1$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

take $v_1 = 1 = v_2$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = -\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$v_2 = -v_1$ take $v_1 = 1$

$v_1 = -v_2$ $\qquad v_2 = -1$

$$\vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

John Weatherwax

6.5.19

$$\dot{R} = aR - bRF$$
$$\dot{F} = -cF + dRF$$

5 (a) a rep. growth of rabbits due to reproduction

c " dealth " foxes due to old age

b relates how foxes eat rabbits $\propto$ (# of encounters RF)

d " " foxes grow when there is plenty of food

No account of growth of foxes to reproduction

No dealth of rabbits due to old age

(b) let $R = k_1 x$ $F = k_2 y$ $t = T\tau$

Then

$$\frac{k_1}{T}\frac{dx}{d\tau} = ak_1 x - bk_1k_2 xy$$

& $\frac{dx}{dJ} = aTx - bTk_2xy$ pick $T = \frac{1}{a}$

Then $k_2 = \frac{b}{a}$

& $\frac{k_2}{T}\frac{dy}{dJ} = -ck_2y + dk_1k_2xy$

$\Rightarrow \frac{dy}{dJ} = -cTy + dk_1Txy = dk_1Ty\left(x - \frac{c\cancel{T}}{dk_1\cancel{T}}\right)$

pick $k_1 = \frac{c}{d}$

& $\mu = \frac{c}{a}$ Then equations become

$$\frac{dx}{dJ} = x(1-y) \quad \& \quad \frac{dy}{dJ} = \mu y(x-1)$$

(c) $\frac{dy}{dx} = \frac{\mu y(x-1)}{x(1-y)} \Rightarrow \frac{(1-y)}{y}\,dy = \frac{\mu(x-1)}{x}\,dx$

$\Rightarrow \ln y - y + C = \mu(x - \ln x)$

Thus $\underbrace{\mu(x - \ln x) + (y - \ln y)}_{= f(x,y)} = C$ is conserved

d) w/ I.C $C = f(x_0, y_0)$ As $\mu = \frac{c}{a}$ pick $\mu = 10^{-1}$

(ratio of foxes death to rabbit births)

Then contour plots of

$\mu(x - \ln x) + (y - \ln y) = \mu(x_0 - \ln x_0) + (y_0 - \ln y_0)$ give for

various initial conditions (see Mathematica at pt)

for $x > 0$ & $y > 0$ the only critical pt is at $(1,1)$

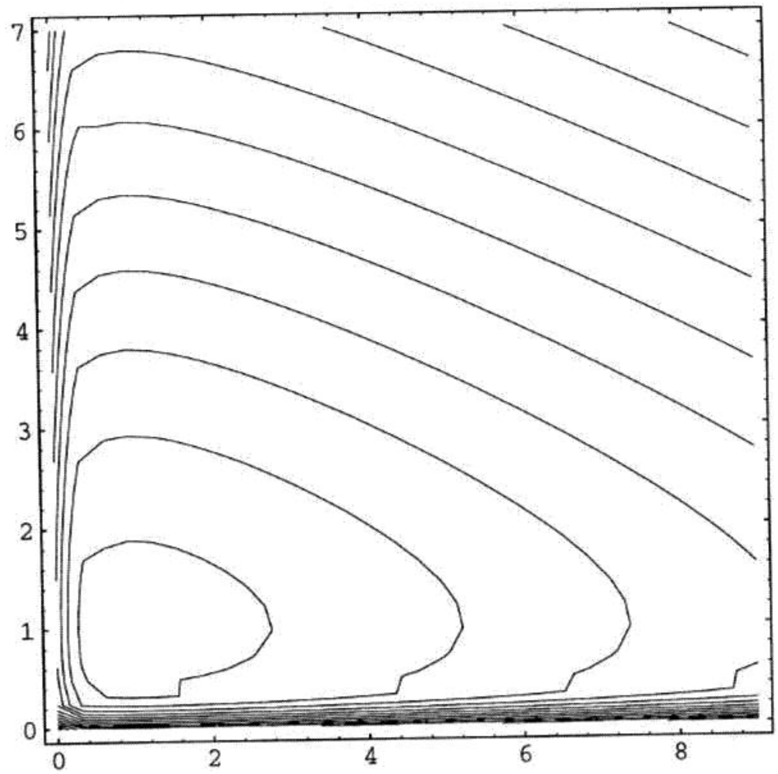
Then I claim around $(1,1)$ $C(x,y)$ is a local min &

thus $\forall$ i.c. in 1st quadrant we have a periodic orbit

for $C(x,y) = \mu(x - \ln x) + (y - \ln y)$

$C_x = \mu(1 - \frac{1}{x})$ $C_{xx} = \mu\left(\frac{1}{x^2}\right)$ $C_{xy} = 0$

```
In[13]:=
  ContourPlot[3^(-1) (x-Log[x])+(y-Log[y]),{x,.001,9},{y,.001,7},
     ContourShading->False,
     Contours->15,
     PlotPoints->30];
```



Consider 2nd D test for $C(x,y)$

$$C_{xx}C_{yy} - C_{xy}^2 = \mu\left(\frac{1}{x^2}\right)\frac{1}{y^2}\Big|_{(1,1)}$$

< 0 saddle

> 0 & $C_{xx} > 0$ min

> 0 & $C_{xx} < 0$ max

$= \mu > 0$ & $C_{xx} > 0$ $\therefore$ $C(1,1)$

is minimum & thus all level curves are closed orbits.

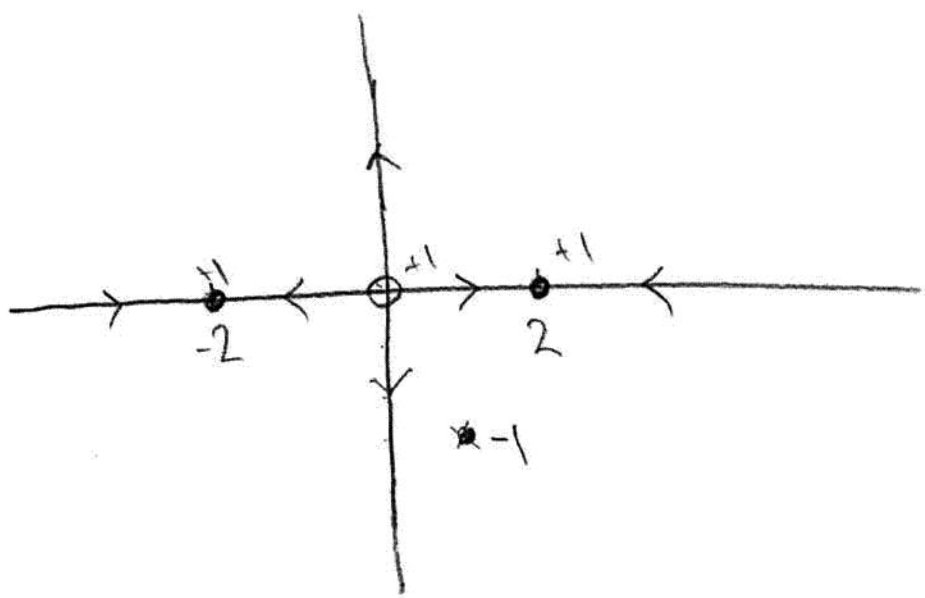
6.8.7

$\dot{x} = x(4-y-x^2)$
$\dot{y} = y(x-1)$

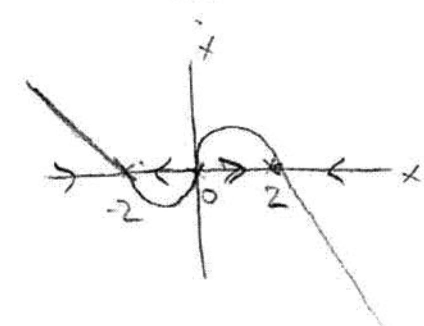
fixed pts $x=0,\ y=0$

$y = x^2-4$ & $x=1 \Rightarrow y=-3$ $(1,-3)$

$y = x^2-4$ & $y=0 \Rightarrow x=\pm 2$ $(2,0),\ (-2,0)$



$y=0$
$\dot{x} = x(4-x^2)$
$= x(2-x)(2+x)$



1

$P_1 = (0,0)$

$P_2 = (2,0)$

$P_3 = (-2,0)$

$P_4 = (1,-3)$

Need stability of P_4

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \overset{lin}{\approx} \begin{bmatrix} 4-y-3x^2 & -x \\ y & x-1 \end{bmatrix}_{(x,y)=(1,-3)} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} 7-3 & -1 \\ -3 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ -3 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$\tau = 4 \qquad \Delta = -3 \Rightarrow$ Saddle pt

Thus has index -1

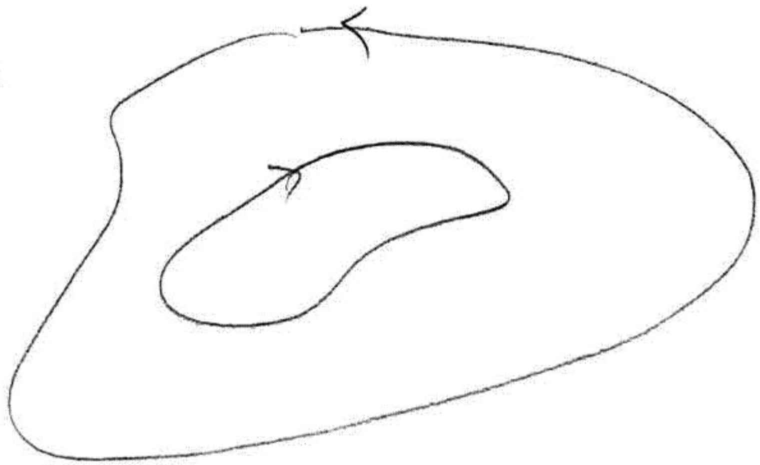
Any closed orbit must have index 1 thus must encircle one fixed pt w/ index +1 or 3 fixed pts 1 w/ index −1 + 2 w/ index +1

If closed orbit encircles any of the pts w/ index +1 + thus is it is crossi either the x or y-axes ruled out due to non uniqueness of orbits when they cross axes. Same results follow for any orbit enclosing

3 pts.

Any orbit not encircling a fixed pt cannot be closed orbit due to theorm

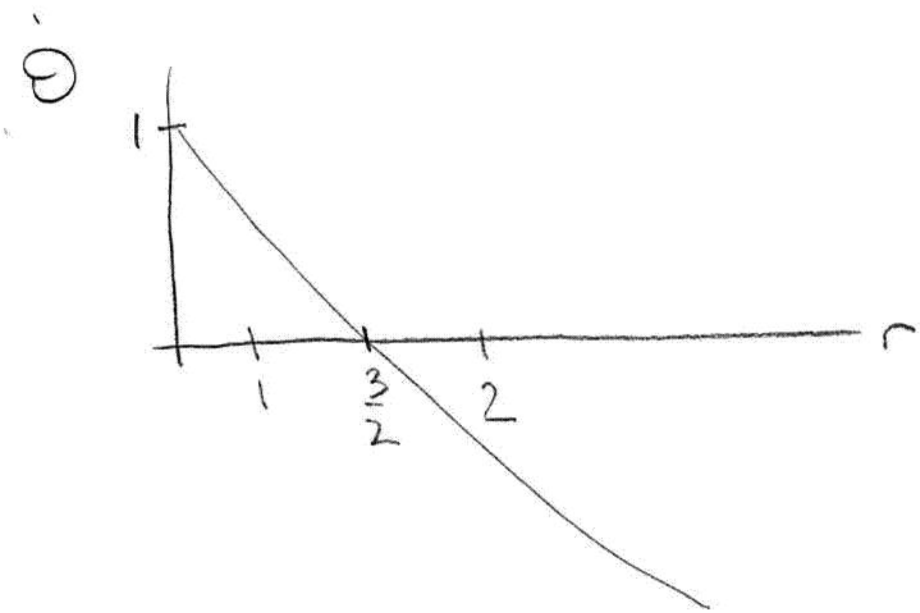
6.8.9



want $\dot{\Theta}$ that is positive from $r=0$ to $r \approx$ "r_1"

+ $\dot{\Theta}$ that goes through 0 from $r \approx$ "r_1" to $r \approx$ "r_2"

then get $\dot{\Theta}$ to be neg from $r \approx$ "r_2" to ∞

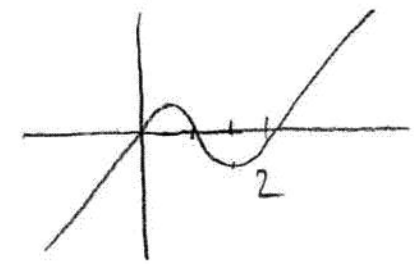


let $\dot{\Theta} = -\frac{1}{3/2}(r - 3/2)$

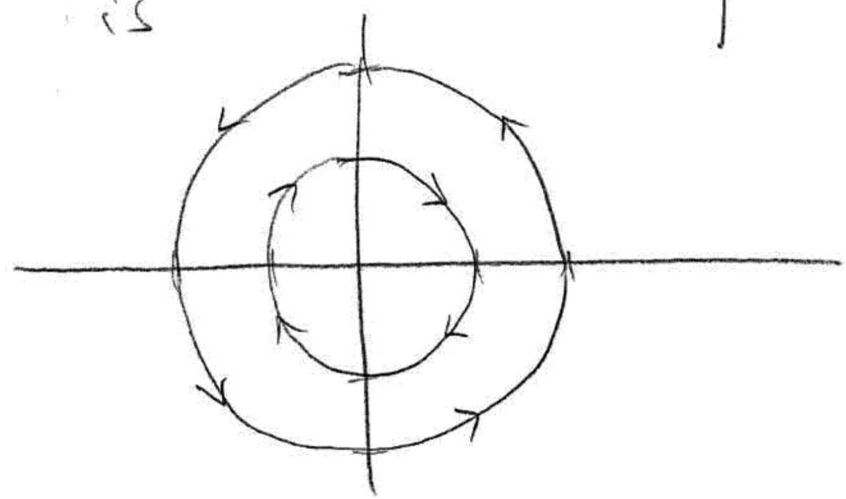
$= -\frac{2}{3}(r - 3/2)$

want $\dot{r} = 0$ on $r_1 = 1$ + $r_2 = 2$

Thus try $\dot{r} = r(r-1)(r-2)$



Then plot is



Note this satisfies the conditions w/o any critical pts

Pg 204 Strogatz

$$r(1-r^2)+\mu r\cos\theta \geq r[(1-r^2)+\mu\cos\theta] >$$

$$> r[(1-r^2)+-1(\mu)] > 0$$

$$1-r^2-\mu > 0$$

$$r^2 < 1-\mu \qquad r < \sqrt{1-\mu}$$

Also $\quad r(1-r^2)+\mu r\cos\theta \leq r(1-r^2)+\mu \underset{\text{on } r_{max}}{r} < 0$

$$1-r^2+\mu < 0$$

$$r^2 > 1+\mu$$

$$r > \sqrt{1+\mu}$$

Pg 206

on $\dot{x}=0$ null cline

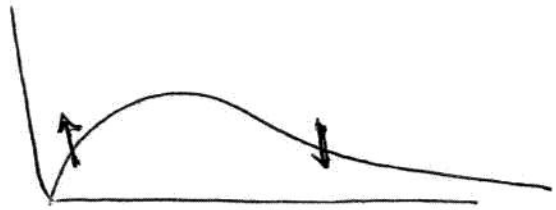
$$y = \frac{x}{(a+x^2)}$$

Then $\quad \dot{y} = b - a\left(\frac{x}{a+x^2}\right) - \frac{x^2 x}{a+x^2}$

$$= \frac{b(a+x^2)-ax-x^3}{a+x^2}$$

$\Rightarrow \quad x \to \infty \qquad \dot{y} < 0$

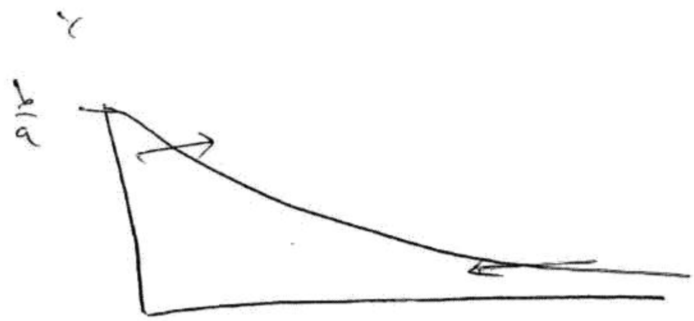
$x \to 0 \qquad \dot{y} = ab > 0$



on $\dot{y} = 0$ null cline

$$b - ay - x^2 y = 0$$

$$\frac{b}{a + x^2} = y$$



$$\dot{x} = -x + \frac{ab}{a+x^2} + \frac{x^2 b}{a+x^2}$$

$$= \frac{-ax - x^3 + ab + bx^2}{a + x^2}$$

As $x \to 0 \qquad \dot{x} > 0$

$x \to \infty \qquad \dot{x} < 0$

---

pg 207 Strogatz

On horiz + vertical sides Say I: $y = 0$

$\dot{x} = -x$ All vectors pt 

$\dot{y} = b$

Say II $\quad x = 0 \quad$ then $\quad \dot{x} = ay > 0$

$\dot{y} = b - ay > 0$ iff $y < b/a$

Then vectors pt 

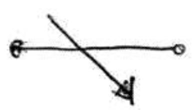
Region III $\quad y = b/a$

$$\dot{x} = -x + b + x^2 b/a > 0 \quad \text{if} \quad x^2 - \frac{a}{b}x + a > 0$$

$$\dot{y} = -x^2(b/a) < 0$$

$$x = \frac{\frac{a}{b} \pm \sqrt{\frac{a^2}{b^2} - 4a^2}}{2}$$

Follows that vectors look like



from fact that $\dot{x} > 0,\ \dot{y} < 0$ in This Region of space split by the nullclines.

For Region V $\quad x =$ $\qquad x, y \to +\infty$

$$\dot{x} \approx x^2 y \qquad \frac{\dot{y}}{\dot{x}} = -1$$

$$\dot{y} \sim -x^2 y$$

$$\dot{x} - (-\dot{y}) =$$

$$\dot{x} = -x + ay + x^2 y = 0$$

$$\dot{y} = b - ay - x^2 y = 0$$

Add $\quad = \quad x = b$

Then $\cancel{y + x^2 y}$ $\quad (a + x^2)y = x$

$$y = \left.\frac{x}{a + x^2}\right|_{x=b} = \frac{b}{a + b^2}$$

$$A(x^*, y^*) = \begin{pmatrix} -1 + \frac{2b^2}{a+b^2} & a + b^2 \\ -\frac{2b^2}{a+b^2} & -a - b^2 \end{pmatrix}$$

$$\det A(x^*,y^*) = a+b^2 - \cancel{2}b^2 + \cancel{2}b^2$$

$$= a+b^2$$

$$\tau = \frac{-(a+b^2)(a+b^2)}{a+b^2} - \frac{a+b^2}{a+b^2} + \frac{2b^2}{a+b^2}$$

$$= \frac{-a^2 - 2ab^2 - b^4 - a - b^2 + 2b^2}{a+b^2}$$

$$= \frac{-a^2 - 2ab^2 - b^4 - a + b^2}{a+b^2}$$

$$= -\frac{(b^4 + (2a-1)b^2 + (a+a^2))}{a+b^2} = 0 \text{ if}$$

$$\Rightarrow b^2 = \frac{-(2a-1) \pm \sqrt{(2a-1)^2 - 4a(1+a)}}{2}$$

$$= \frac{-(2a-1) \pm \sqrt{\cancel{4a^2} - 4a + 1 - \cancel{4a^2} - 4a}}{2}$$

$$= \frac{1}{2}\left(1 - 2a \pm \sqrt{1-8a}\right)$$

Pg 209 Strogatz

$$b^2 = \frac{1}{2}\left(1 - 2a \pm \sqrt{1-8a}\right) \qquad \text{pick } a = 0,\ b = 1/2$$

in one region of a-b plane. Stable or unstable?

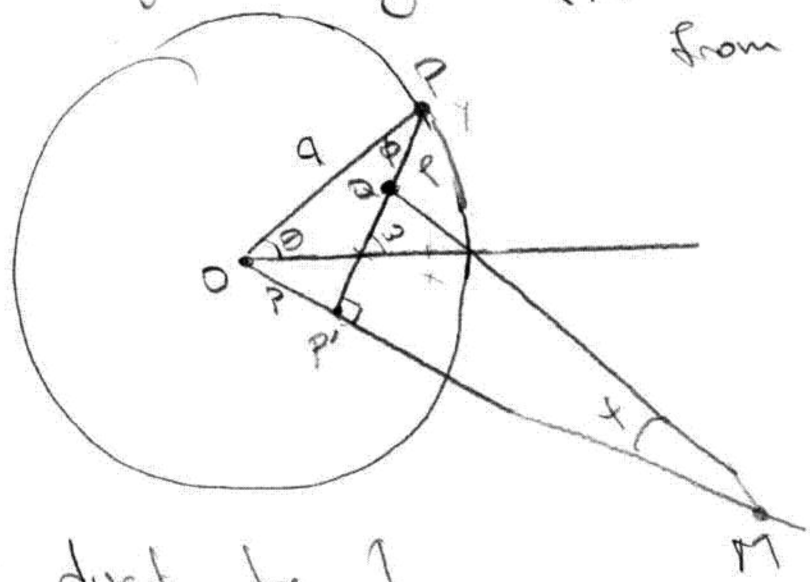
$$\tau = -\frac{(b^4 + -b^2)}{b^2} = -\left(b^2 - 1\right) = -\left(\tfrac{1}{4} - 1\right) = \tfrac{3}{4}$$

Pg 241 Strogatz

$$F(x) = \int_0^x f(u)\,du$$

$$\frac{dF}{dt} = f(x)\dot{x} = -\text{damping force}$$

7.1.9 consider following drawing: (Note this was obtained from Davis 1962 pp 113-125) 1



$OM = ka$

+ let speed of duck be 1
+ that of dog be k
+ Radius of disk a.

Then eq of tangent PQ can be written as

$$x \sin\omega - y\cos\omega = p = a\sin(\omega-\theta) \qquad (2)$$

+ Normal through pt Q has the following

$$x\cos\omega + y\sin\omega = a\cos(\omega-\theta) - p \qquad (3)$$

consider the following $\quad \sin\omega\, dx - \cos\omega\, dy = 0$

Statement that $\quad \frac{dy}{dx} = \tan\omega$.

& Note that $\quad ds = \left(1 + \left(\frac{dy}{dx}\right)^2\right)^{1/2} dx = \sec\omega\, dx = ka\, d\theta$

& that $\quad ds = \left(1 + \left(\frac{dx}{dy}\right)^2\right)^{1/2} dy = \csc\omega\, dy = ka\, d\theta$

Thus $dx = \cos\omega\, ka\, d\theta$

& $dy = \sin\omega\, ka\, d\theta$

Thus it follows that $\quad \cos\omega\, dx + \sin\omega\, dy = ka\, d\theta$

let ω be fn of x & y & form diff eq of (2) & (3)

to obtain

$$\sin\omega\, dx + x\cos\omega\, d\omega - dy\cos\omega + y\sin\omega\, d\omega = a\cos(\omega-\theta)(d\omega - d\theta)$$

$$\Rightarrow \quad \sin\omega\, dx - \cos\omega\, dy + (x\cos\omega + y\sin\omega)\,d\omega = a\cos(\omega-\theta)(d\omega - d\theta) \quad (6)$$

& $\cos\omega\, dx + \sin\omega\, dy - (x\sin\omega - y\cos\omega)\,d\omega$ (7)

$$= -a\sin(\omega-\theta)(d\omega - d\theta) - dp$$

(6) Becomes

$$0 + (a\cos(\cancel{\omega}-\theta) - p)\,d\omega = a\cos(\omega-\theta)(\cancel{d\omega} - d\theta)$$

$$\Rightarrow \quad p\,d\omega = a\cos(\omega-\theta)\,d\theta$$

& (7) Becomes

$$ka\,d\theta - a\sin(\cancel{\omega}-\theta)\,d\omega = -a\sin(\omega-\theta)(\cancel{d\omega} - d\theta) - dp$$

$$ka\,d\theta = a\sin(\omega-\theta)\,d\theta - dp$$

let $\phi = \omega - \theta$

to get

$$\rho(d\phi + d\theta) = a\cos\phi \, d\theta \Rightarrow \rho \frac{d\phi}{d\theta} = a\cos\phi - \rho$$

& $$ka\,d\theta = a\sin\phi\, d\theta - d\rho \Rightarrow \frac{d\rho}{d\theta} = a\sin\phi - ka$$

equations asked for are w/ $a = 1$ $k = 1$ & $\rho = R$

$$R\frac{d\phi}{d\theta} = \cos\phi - R$$

$$\frac{dR}{d\theta} = \sin\phi - 1$$

$$\frac{dR}{d\phi} = \frac{R(\sin\phi - 1)}{(\cos\phi - R)} \Rightarrow (\cos\phi - R)\,dR - R(\sin\phi - 1)\,d\phi = 0$$

Check if exact

$-\sin\phi \overset{?}{=} -\sin\phi + 1$ Almost w/ integrating factor.

But neither $\cos\phi - R$ or $-R(\sin\phi - 1)$ is independent of R or ϕ.

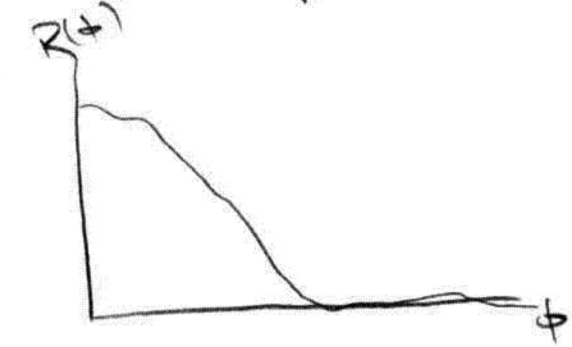
$\therefore$ can't use integrating factor. Can't solve

is $R = 0$ ever?

Numerically integrated w/ dog position ½ distance from duck & also w/ distance .9 from duck & answer yes & from .99 from duck

So yes dog catches duck (See Mathematica output)

Plot looked qualitatively



(b) These we gotten in part (a)

or

$$R\frac{d\phi}{d\theta} = \cos\phi - R$$

$$\frac{dR}{d\theta} = \sin\phi - k$$

(c) $k = 1/2$ get the following $R\frac{d\phi}{d\theta} = \cos\phi - R$

$$\frac{dR}{d\theta} = \sin\phi - \frac{1}{2}$$

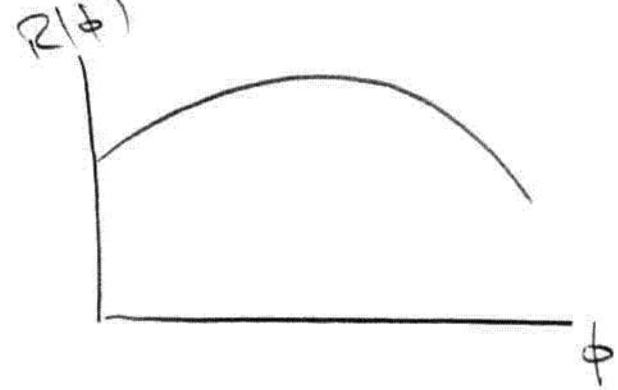
or for path

$$\frac{dR}{d\phi} = \frac{R(\sin\phi - 1/2)}{\cos\phi - R}$$

Numerically plotted get following plot (See Mathematica) 7

Apparently the dog circles the duck

Qualitatively



7.2.16

$$\dot{r} = R(r)$$
$$\dot{\theta} = 1$$

Then $\nabla \circ (\dot{\vec{x}}) = \frac{1}{r}\frac{\partial}{\partial r}(rR) + 0 \overset{set}{=} r-1$ pos $r>1$

$$\Rightarrow \frac{\partial(rR)}{\partial r} = r^2 - r$$

$$\Rightarrow rR = \frac{r^3}{3} - r^2 \Rightarrow R = \frac{r^2}{3} - r$$
$$= \frac{r}{3}(r-3)$$

Thus let system be $\dot{r} = \frac{r}{3}(r-3)$

$$\dot{\theta} = 1$$

Then $r=3$ is a closed orbit + region

John Weatherwax.

7.2.5

(a) $\dot{\vec{x}}$ grad system $\Rightarrow$ $\dot{\vec{x}} = -\nabla V$

$$= -\frac{\partial V}{\partial x}\hat{i} + -\frac{\partial V}{\partial y}\hat{j} = f(x,y)\hat{i} + g(x,y)\hat{j}$$

Thus $$\frac{\partial f}{\partial y} = -\frac{\partial^2 V}{\partial y \partial x} = -\frac{\partial^2 V}{\partial x \partial y} = \frac{\partial g}{\partial x}$$

(b) We are told that f & g are smooth & w/o singularities

so the answer is yes.

4

7.2.6 (a)

$$\dot{x} = y^2 + y\cos x \qquad \dot{y} = 2xy + \sin x \overset{\text{set}}{\equiv} -\frac{\partial V}{\partial y}$$

$$\Rightarrow V = -xy^2 - y\sin x + g(x)$$

Then $$-\frac{\partial V}{\partial x} = y^2 + y\cos x + g' = y^2 + y\cos x$$

$$\Rightarrow g = \text{const.}$$

$$\therefore V = -xy^2 - y\sin x + C$$

7.3.1

$$\dot{x} = x - y - x(x^2 + 5y^2)$$
$$\dot{y} = x + y - y(x^2 + y^2)$$

(a) At $(0,0)$

$$M = \begin{bmatrix} 1-(x^2+5y^2)-x(2x) & -1-x(10y) \\ 1-y(2x) & 1-(x^2+y^2)-y(2y) \end{bmatrix}_{(0,0)}$$

$$= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \qquad \tau = 2 \quad \Delta = 2$$

Thus $(0,0)$ is an unstable spiral.

(b) $r\dot{r} = x\dot{x} + y\dot{y} = x(x - y - x(x^2+5y^2)) + y(x + y - y(x^2+y^2))$

$$= x^2 - \cancel{xy} - x^2(x^2+5y^2) + \cancel{xy} + y^2 - y^2(x^2+y^2)$$
$$= x^2 - x^4 - 5x^2y^2 + y^2 - y^2x^2 - y^4$$

putting $x = r\cos\theta$ $y = r\sin\theta$ & simplifying we get

using Mathematica

$$r^2\left(1 - 3r^2 + \frac{1}{2}r^2\cos(4\theta)\right)$$

As $\sin^2(2\theta) = \dfrac{1-\cos(4\theta)}{2}$ The Above becomes

$$r^2\left(1 - \frac{3}{2}r^2 + \frac{1}{2}r^2 - r^2\sin^2(2\theta)\right)$$

$$= r^2\left(1 - r^2 - r^2\sin^2(2\theta)\right) \Rightarrow \dot{r} = r\left(1 - r^2 - r^2\sin^2(2\theta)\right)$$

$$\dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2} = \frac{r^2 + 4xy^3}{r^2} = 1 + 4r^2\cos\theta\sin^3\theta$$

(c) want $\dot{r} > 0$

$$\Rightarrow \quad 1 - r^2\left(1 + \sin^2(2\theta)\right) > 0$$

$$\Rightarrow \quad r^2 < \frac{1}{1 + \sin^2(2\theta)} \quad \Leftarrow \text{was min at } \frac{1}{2}$$

Thus pick $r^2 < \frac{1}{2} \Rightarrow r < \frac{1}{\sqrt{2}} \approx .707$ take $r \simeq .706$

(d) want $\dot{r} < 0$

$$\Rightarrow \quad r^2 > \frac{1}{1 + \sin^2(2\theta)} \quad \Leftarrow \text{has Max at } 1$$

pick $r > 1$ say $r \simeq 1.01$

(e) As region has no fixed points Poincaré - Bendixson Theorem can be applied directly

7.3.11

(a) As $r\dot{r} = x\dot{x} + y\dot{y}$

& $\dot{\theta}r^2 = x\dot{y} - y\dot{x}$

we get $x\dot{x} + y\dot{y} = r^2(1-r^2)[r^2\sin^2\theta + (r^2\cos^2\theta - 1)^2]$

$x\dot{y} - y\dot{x} = r^2[r^2\sin^2\theta + (r^2\cos^2\theta - 1)^2]$

$x\dot{x} + y\dot{y} = (x^2+y^2)(1-x^2-y^2)(y^2 + (x^2-1)^2]$

$x\dot{y} - y\dot{x} = (x^2+y^2)[y^2 + (x^2-1)^2]$

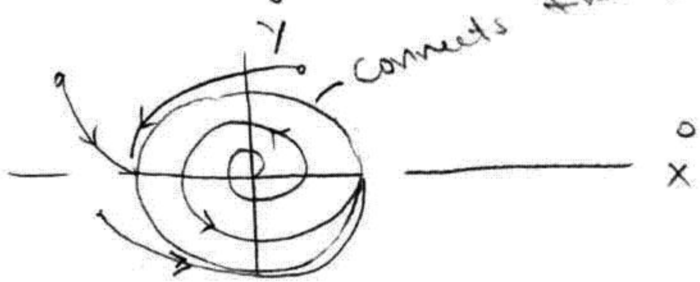
$\Rightarrow \quad \dot{x} = (1-2x^2+x^4+y^2)(x - x^3 - y - xy^2)$

$\dot{y} = (1-2x^2+x^4+y^2)(x+y-x^2y-y^3)$

Source at $(0,0)$

Fixed pts at $\begin{cases}(0,1)\\(-1,0)\end{cases}$

plot looks as follows



Trajectories seem to be pulled to fixed pts $(0,1)$ & $(-1,0)$

Phase portrait vanishes when

$$r^2 \sin^2\theta + (r^2\cos^2 x - 1)^2 = 0$$

$$\Rightarrow \quad y^2 + (x^2-1)^2 = 0$$

$$\Rightarrow \quad y = 0 \ \ \& \ \ x = \pm 1 \quad \Rightarrow \quad r = 1 \quad \theta = 0, \pi$$

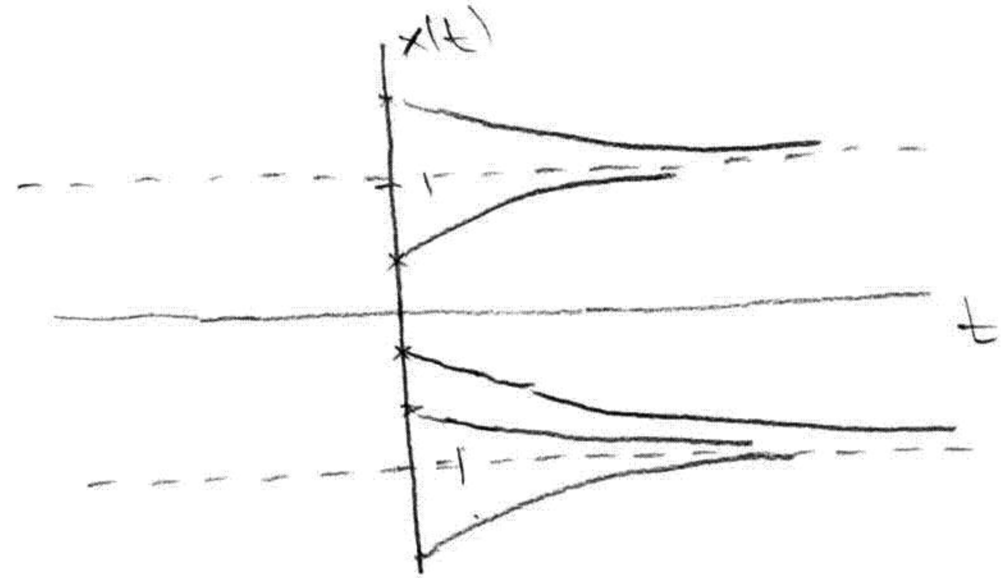
See sol

(b) Qualitatively $x(t) \longrightarrow \begin{cases} +1 \\ -1 \end{cases}$ from any starting pt

from computer plots

$x(t) \longrightarrow +1$ $x(0) < 0$ & $x(t) \to -1$ if $x(0) > 0$

Thus $x(t)$ v.s t would look like



7.5.5 $\quad \ddot{x} + \mu(|x|-1)\dot{x} + x = 0.$

$\underline{x>0}$ $\quad \ddot{x} + \mu(x-1)\dot{x} + x = 0$

$\underline{x<0}$ $\quad \ddot{x} + \mu(-x-1)\dot{x} + x = 0.$

Note $\quad \ddot{x} + \mu(x-1)\dot{x} = \frac{d}{dt}\left(\dot{x} + \mu\left(\frac{x^2}{2} - x\right)\right)$

& $\quad \ddot{x} + \mu(-x-1)\dot{x} = \frac{d}{dt}\left(\dot{x} + \mu\left(-\frac{x^2}{2} - x\right)\right)$

Let $\quad F_1(x) = \frac{x^2}{2} - x$

$\quad F_2(x) = -\frac{x^2}{2} - x$

$\underline{x>0}$ Let $\omega_1 = \dot{x} + \mu F_1(x)$ $\qquad$ $\underline{x<0}$ $\omega_2 = \dot{x} + \mu F_2(x)$

Then $\quad \dot{\omega}_1 = -x$ $\qquad$ $\dot{\omega}_2 = -x$

$$\dot{x} = \omega_1 - \mu F_1(x)$$
$$\dot{\omega}_1 = -x$$

$$y_1 = \frac{\omega_1}{\mu}$$

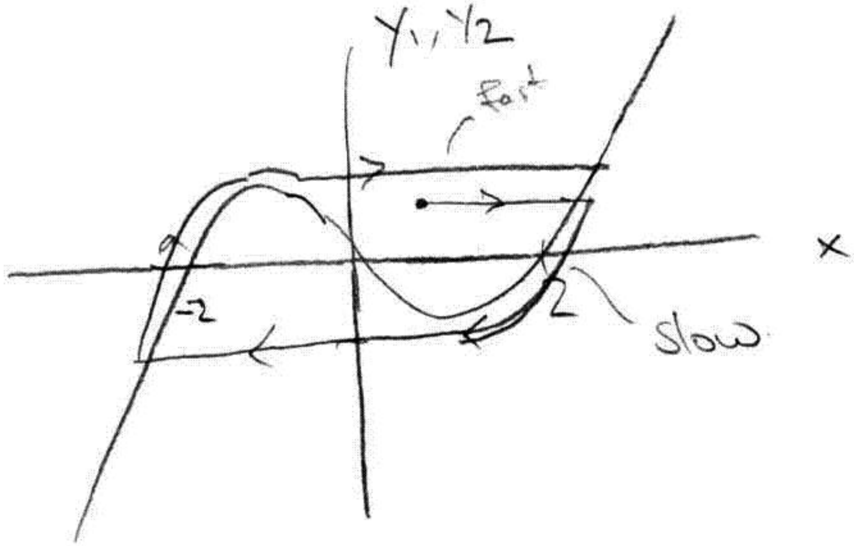
$x > 0$

$$\dot{x} = \mu(y_1 - F_1(x))$$
$$\dot{y}_1 = -\frac{1}{\mu} x$$

$x < 0$

$$\dot{x} = \mu(y_2 - F_2(x))$$
$$\dot{y}_2 = -\frac{1}{\mu} x$$

Plot two nullclines $F_1(x)$ $x > 0$ + $F_2(x)$ $x < 0$



Thus compute time required to go along 2 slow branches

from $y = y_{max}$ of $F_2(x)$ $\quad F_2' = -x-1 = 0 \Rightarrow x = -1$

$\Rightarrow F_2(-1) = \frac{-1}{2} + 1 = \frac{1}{2}$

to $y = y_{min}$ of $F_1(x)$ $\quad F_1' = x - 1 = 0 \Rightarrow x = 1$

$F_1(1) = -\frac{1}{2}$

Then

$$\tau = 2\int_{y_{max} = \frac{1}{2}}^{y_{min} = -\frac{1}{2}} dt$$

$\dot{x} \approx 0 \Rightarrow y_1 \approx F_1(x)$

$\frac{dy_1}{dt} = F_1'(x)\,\dot{x} = -\frac{1}{\mu} x$

$\Rightarrow F_1'(x)\frac{dx}{dt} = \frac{-1}{\mu} x$

$\Rightarrow dt = \frac{F_1'(x)(-\mu)\,dx}{x}$

$$\tau = 2\int_{x = F_1^{-1}(\frac{1}{2})}^{1}$$

$\Rightarrow \frac{1}{2} = \frac{x^2}{2} - x \Rightarrow x = 1 + \sqrt{2}$

Then

$$\tau = 2\mu \int_1^{1+\sqrt{2}} \frac{F_1'(x)}{x}\,dx = 2\mu \int_1^{1+\sqrt{2}} \frac{x-1}{x}\,dx = 2\mu\,(x - \ln x)\Big|_1^{1+\sqrt{2}}$$

$$= 2\mu\left(\sqrt{2} - \ln(1+\sqrt{2})\right)$$

2

$1 < r < \infty$ Satisfies Dulac's Criterion for $\nabla \circ (\dot{x})$

$(g \equiv 1)$ is always pos there

But $\exists$ 3 is closed orbit inside R thus Dulac's is false

John Weatherwax

7.3.10

$$\dot{x} = ax + by - (x^2+y^2)x$$
$$\dot{y} = cx + dy - (x^2+y^2)y$$

Then Around fixed pt $(0,0)$ Eqn is

$\dot{\vec{x}} = A\vec{x}$ Thus $(0,0)$ is stable spiral if $\alpha < 0$

(b) else $(0,0)$ is unstable spiral if $\alpha > 0$

Now $r\dot{r} = x\dot{x} + y\dot{y} = ax^2 + bxy - x^2 r^2$

$+ cxy + dy^2 - y^2 r^2$

$= ar^2\cos^2\theta + br^2\cos\theta\sin\theta - r^2 r^2$

$+ cr^2\cos\theta\sin\theta + dr^2\sin^2\theta$

$$\dot{r} = r\left[a\cos^2\theta + d\sin^2\theta + (c+b)\cos\theta\sin\theta\right] - r^3$$

Thus

$$\mathring{r} = -r^3 + r\left[a\cos^2\theta + d\sin^2\theta + (c+b)\cos\theta\sin\theta\right]$$

& for large r $\mathring{r} \approx O(r^3)$ Thus for large enough r

All trajectories point in

Thus an application of Poincaré-Bendixson gives us that

when $\alpha > 0$ $\exists$ at least one limit cycle (As $(0,0)$ is a

repellor)

If $\alpha < 0$ claim no periodic orbit exists consider $|A-\lambda I| = 0$

Now $\lambda^2 - (a+d)\lambda + (ad-bc) = 0$

$$\Rightarrow \lambda = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$

$$= \frac{(a+d)}{2} \pm \frac{i}{2}\sqrt{4(ad-bc) - (a+d)^2} \equiv \alpha \pm i\omega$$

$$= \alpha = \frac{a+d}{2} \quad \text{if} \quad \alpha < 0 \Rightarrow a+d < 0$$

Then consider

$$\nabla \cdot (\dot{\vec{x}}) = a - 3x^2 - y^2 + d - x^2 - 3y^2$$

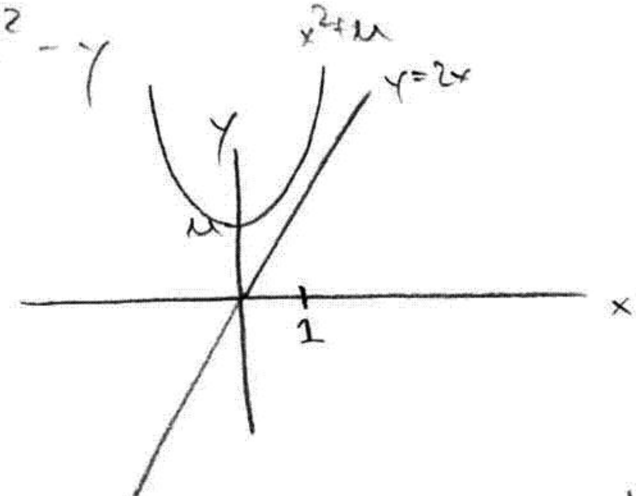
$$= (a+d) - 4x^2 - 4y^2 < 0 \quad \forall \ (x,y) \in \mathbb{R}^2$$

Thus By Dulac's Criterion As $\nabla \cdot (\dot{\vec{x}})$ is one sign

Then there are no closed orbits in $\mathbb{R}^2$. As was to show

1

8.1.6 $\dot{x} = y - 2x \qquad \dot{y} = \mu + x^2 - y$



(a) $\dot{x} = 0 \Rightarrow y = 2x$

$\dot{y} = 0 \Rightarrow y = x^2 + \mu$

(b) want μ to decrease until it becomes small enough

so that parabola crosses line $y = 2x$

1st it becomes tangent

$$2x = 2 \Rightarrow x = 1$$

The $\left.x^2 + \mu_c\right|_{x=1} = \left.2x\right|_{x=1} \Rightarrow \mu_c = 1$

As μ decreases below 1 2 fixed pts Appear

Appea at $2x = x^2 + \mu$

$$x = \frac{2 \pm \sqrt{4 - 4\mu}}{2} = 1 \pm \sqrt{1-\mu}$$

Classify stability of 2 pts

$$J = \begin{bmatrix} -2 & 1 \\ 2x & -1 \end{bmatrix}$$

$$J(1 \pm \sqrt{1-\mu}) = \begin{bmatrix} -2 & 1 \\ 2(1 \pm \sqrt{1-\mu}) & -1 \end{bmatrix}$$

$$\Delta_{\pm} = 3 - 2(1 \pm \sqrt{1-\mu}) = 1 \mp 2\sqrt{1-\mu}$$

$$\tau_{\pm} = -3$$

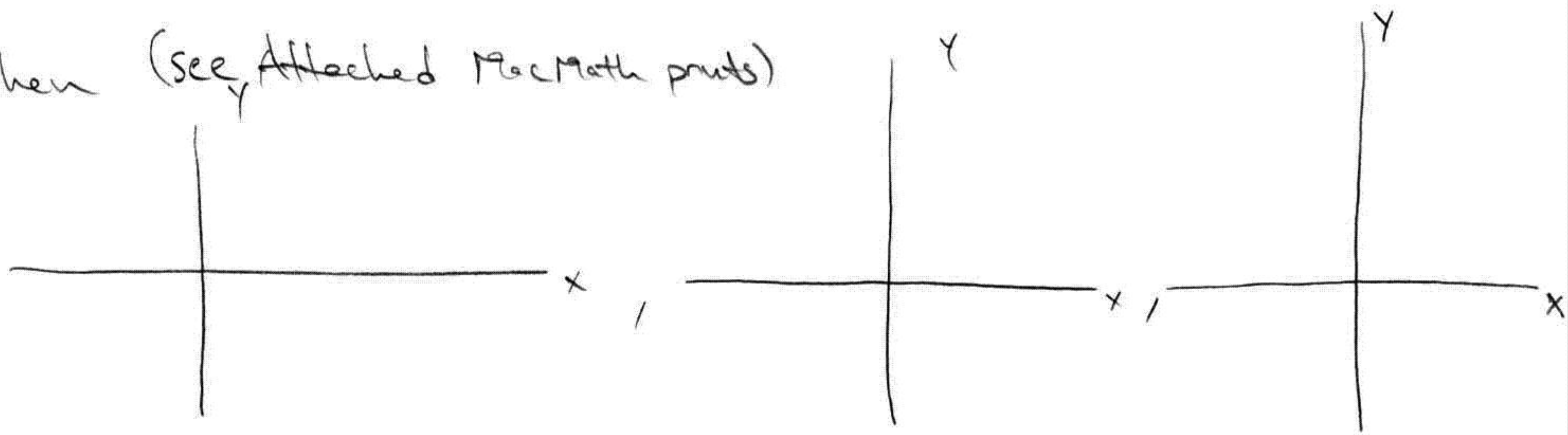
Then $1+\sqrt{1-\mu}$ is a saddle

+ $1-\sqrt{1-\mu}$ is a stable node

$$\begin{aligned} &T^2-4\Delta \\ &= 9-4(1+2\sqrt{1-\mu}) \\ &= 5-8\sqrt{1-\mu} > 0 \end{aligned}$$

for $\mu \approx 1$

(c) Then (see Attached MacMath prints)

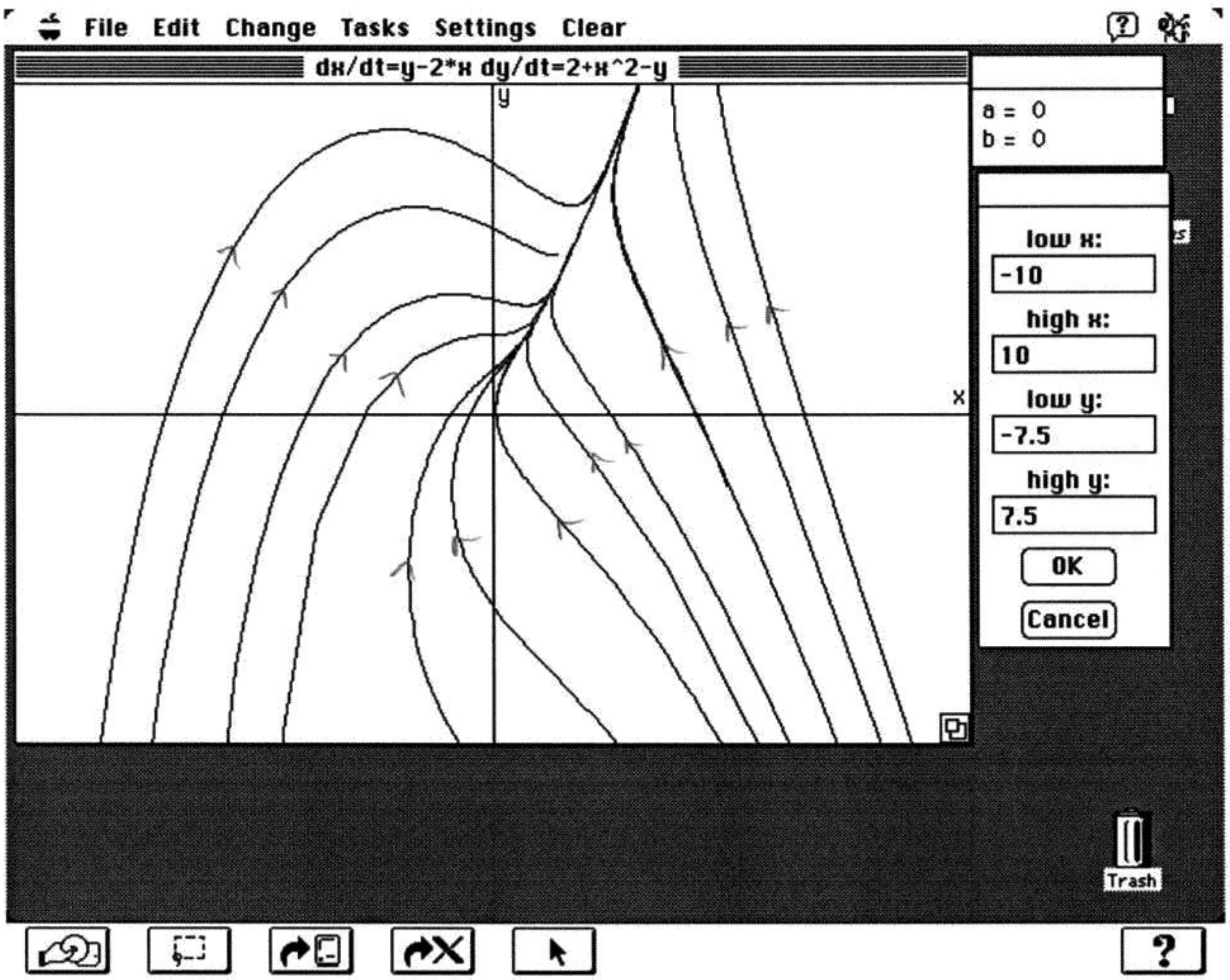


$\mu > 1$ $\mu = 1$ $\mu < 1$

$\mu = 2$ $\mu = 0$

8.1.6

$\mu = 2$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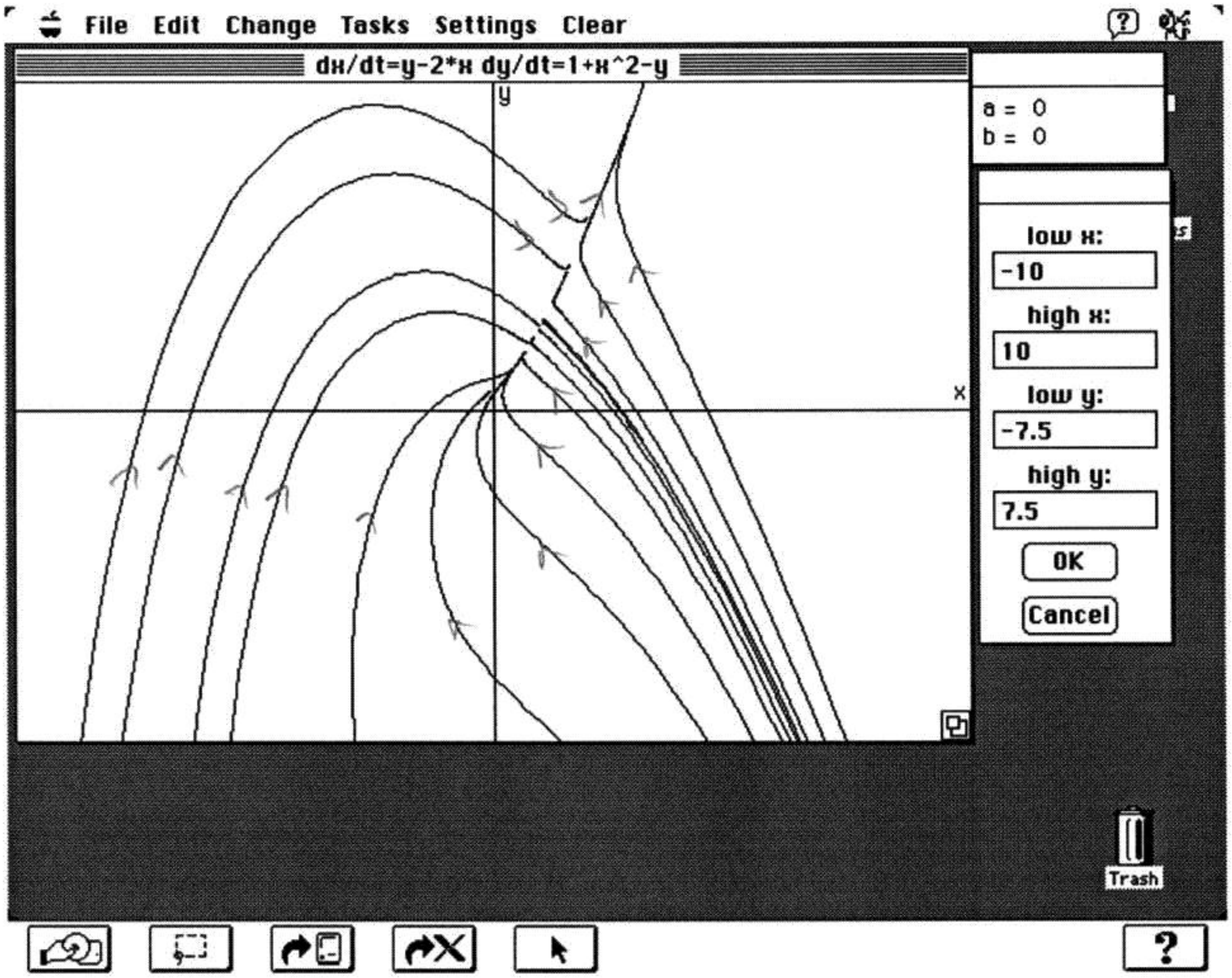
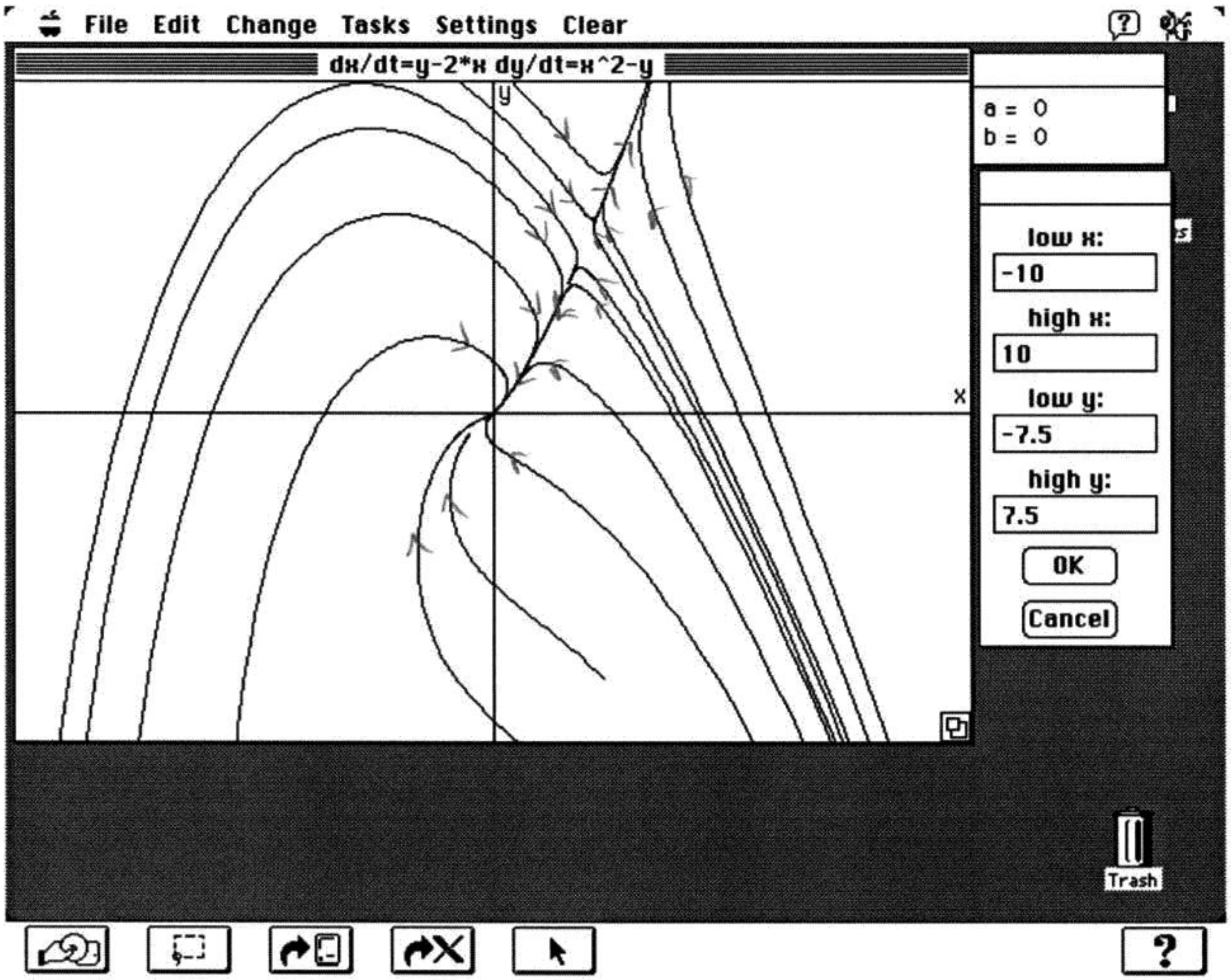
dx/dt=y-2*x dy/dt=2+x^2-y
y
x
a = 0
b = 0
low x:
-10
high x:
10
low y:
-7.5
high y:
7.5
OK
Cancel
Trash

$\mu = 1$


File Edit Change Tasks Settings Clear
dx/dt=y-2*x dy/dt=1+x^2-y
y
x
a = 0
b = 0
low x:
-10
high x:
10
low y:
-7.5
high y:
7.5
OK
Cancel
Trash
?

u=0



File Edit Change Tasks Settings Clear
dx/dt=y-2*x dy/dt=x^2-y
y
x
a = 0
b = 0
low x:
-10
high x:
10
low y:
-7.5
high y:
7.5
OK
Cancel
Trash
?

8.1.11

$$\dot{u} = a(1-u) - uv^2$$

$$\dot{v} = uv^2 - (a+k)v$$

look for CPs at Null clines

$$a(1-u) - uv^2 = 0 \Rightarrow u = \frac{a}{a+v^2}$$

$$uv^2 - (a+k)v = 0 \Rightarrow v \neq 0 \Rightarrow u = \frac{a+k}{v}$$

$$v = 0$$

Thus on fixed pt $(1,0)$ (Always present independent of k & a)

Then another is $\frac{a}{a+v^2} = \frac{a+k}{v} \Rightarrow v = \frac{a \pm \sqrt{a^2 - 4a(a+k)^2}}{2(a+k)}$

then for there to be 2 fixed points we require

$$a^2 - 4a(a+k)^2 = 0$$

$\Rightarrow \quad a \neq 0$

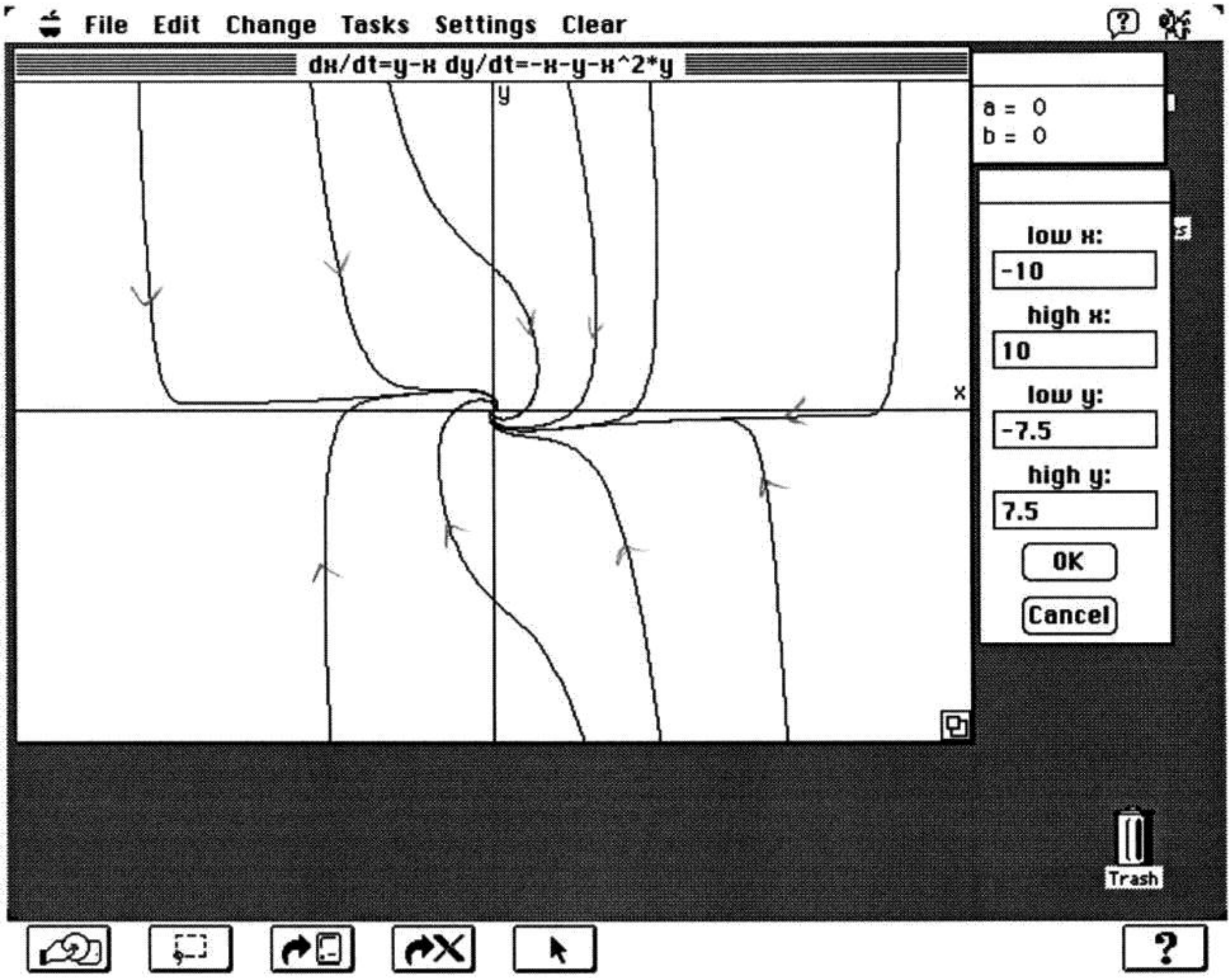
$$a - 4(a+k)^2 = 0 \quad \Rightarrow \quad k = -a \pm \frac{\sqrt{a}}{2}$$

Thus for $-a - \frac{\sqrt{a}}{2} < k < -a + \frac{\sqrt{a}}{2}$ or $0 < k < -a + \frac{\sqrt{a}}{2}$

There are 2 critical pts where before there were none. Thus a Saddle pt Bifercation

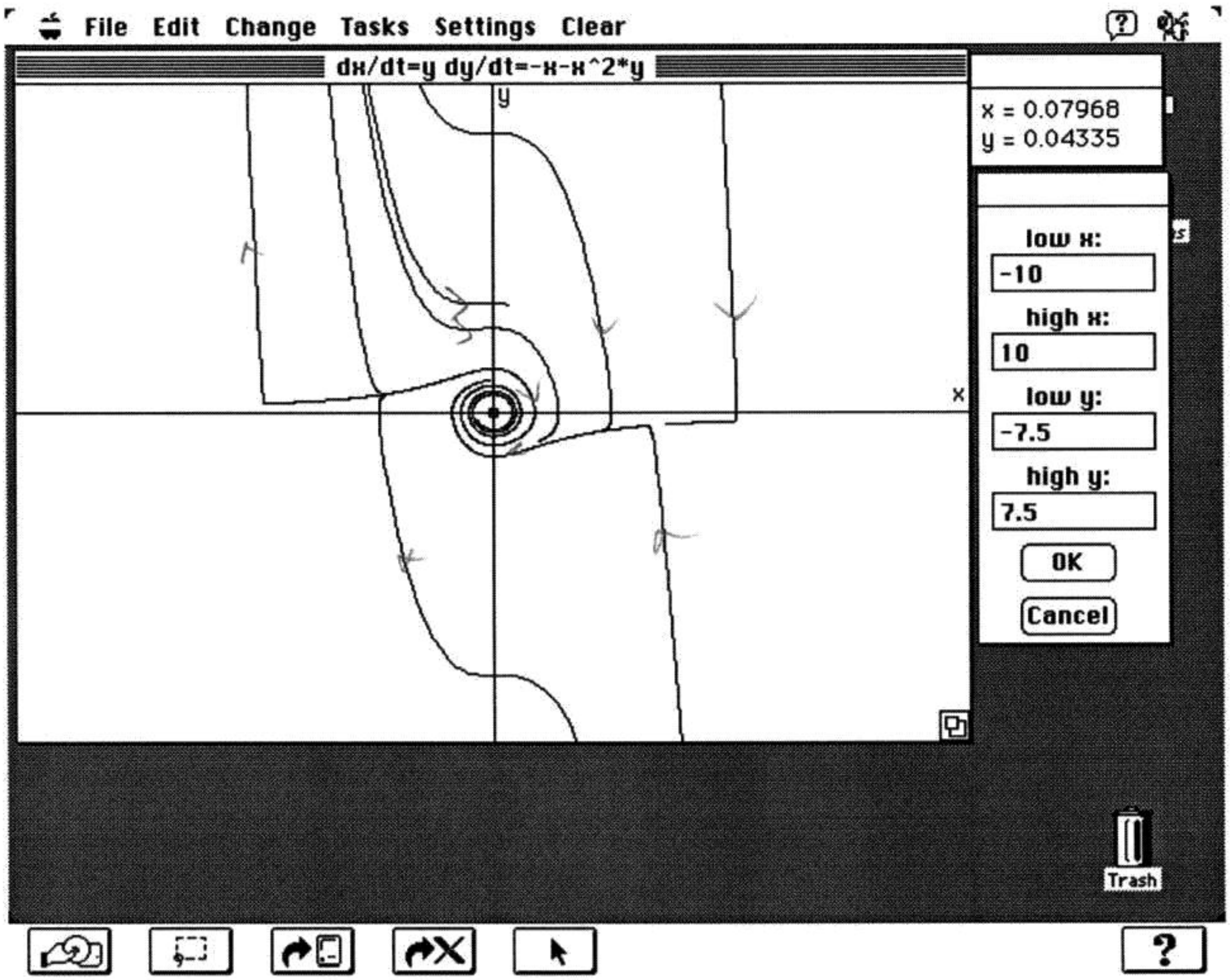
(8.25) consider the following plots

$\mu = -1$

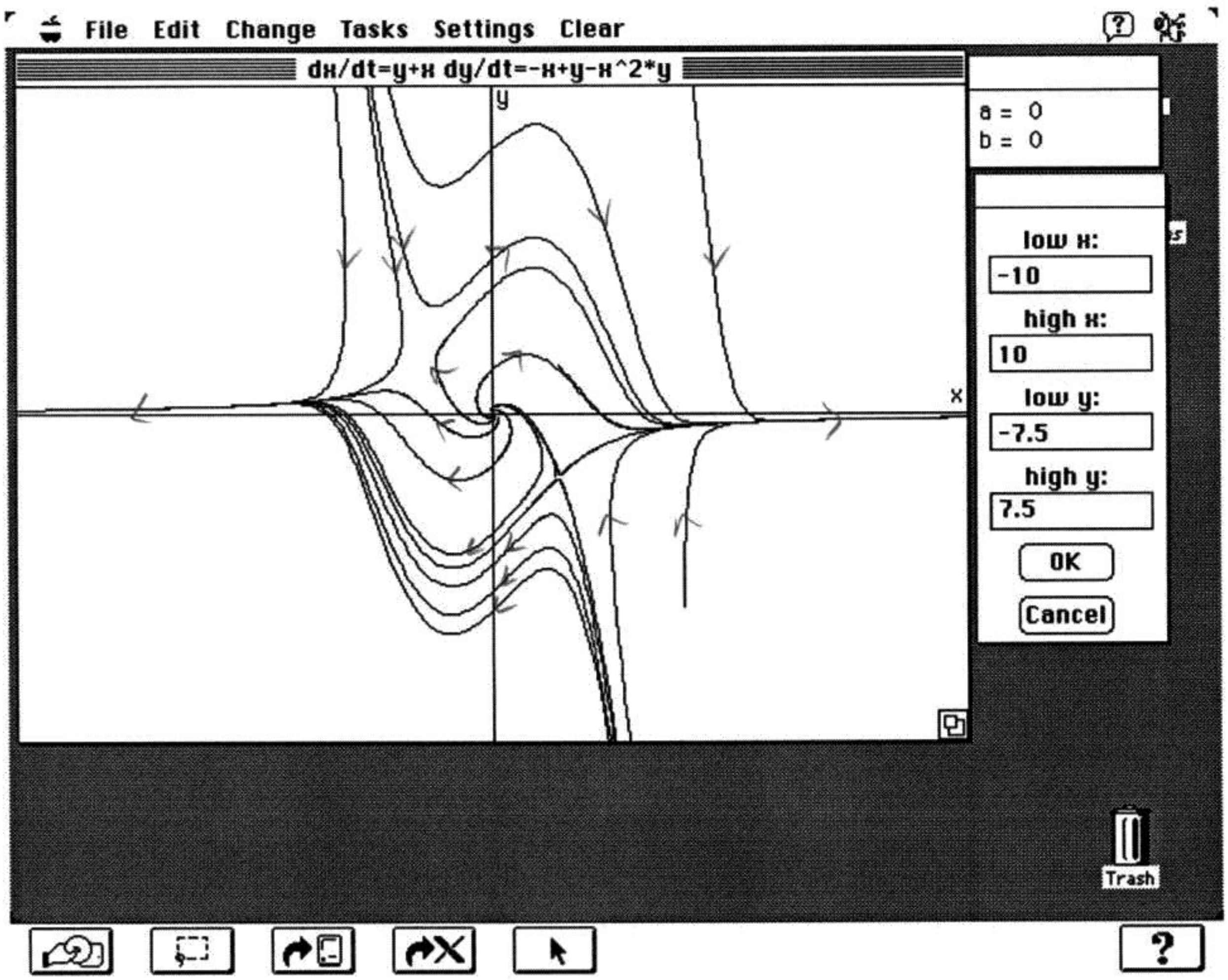


3.2.5

$\mu = 0$


File Edit Change Tasks Settings Clear
dx/dt=y dy/dt=-x-x^2*y
y
x
x = 0.07968
y = 0.04335
low x:
-10
high x:
10
low y:
-7.5
high y:
7.5
OK
Cancel
Trash

$\mu = 1$



we see that (0,0) $\mu = 0$ is a super critical bifercation

8.2.11

Let $y = \dot{x}$

Then $\ddot{x} = \dot{y} = x^3 - x - \mu y$

$\therefore$ $\dot{x} = y$, $\dot{y} = x^3 - x - \mu y$ is system to consider

(a) consider $(0,0)$

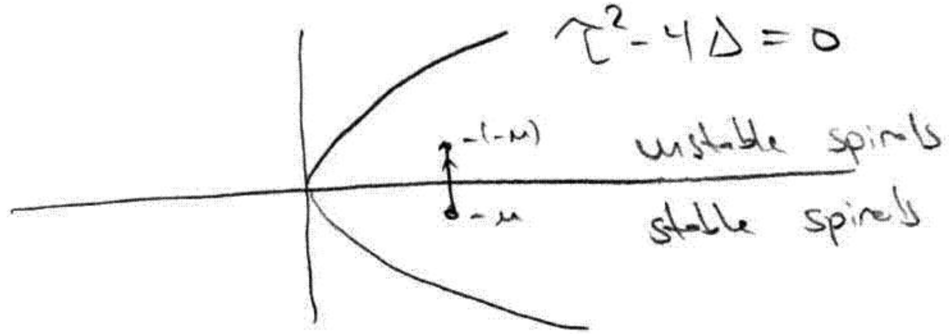
$$J = \left.\begin{bmatrix} 0 & 1 \\ 3x^2 - 1 & -\mu \end{bmatrix}\right|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -1 & -\mu \end{bmatrix} \qquad \begin{array}{l} \tau = -\mu \\ \Delta = 1 > 0 \end{array}$$

Now If $\mu > 0$

Then $(0,0)$ is a stable spiral but As $\mu \searrow$ to 0

then goes negative then $\tau > 0$ + we get a unstable spiral

See



(b) See computer plots & Notice that when $\mu = 0$

we have $\ddot{x} + x - x^3 = 0$

which is a conservative system As

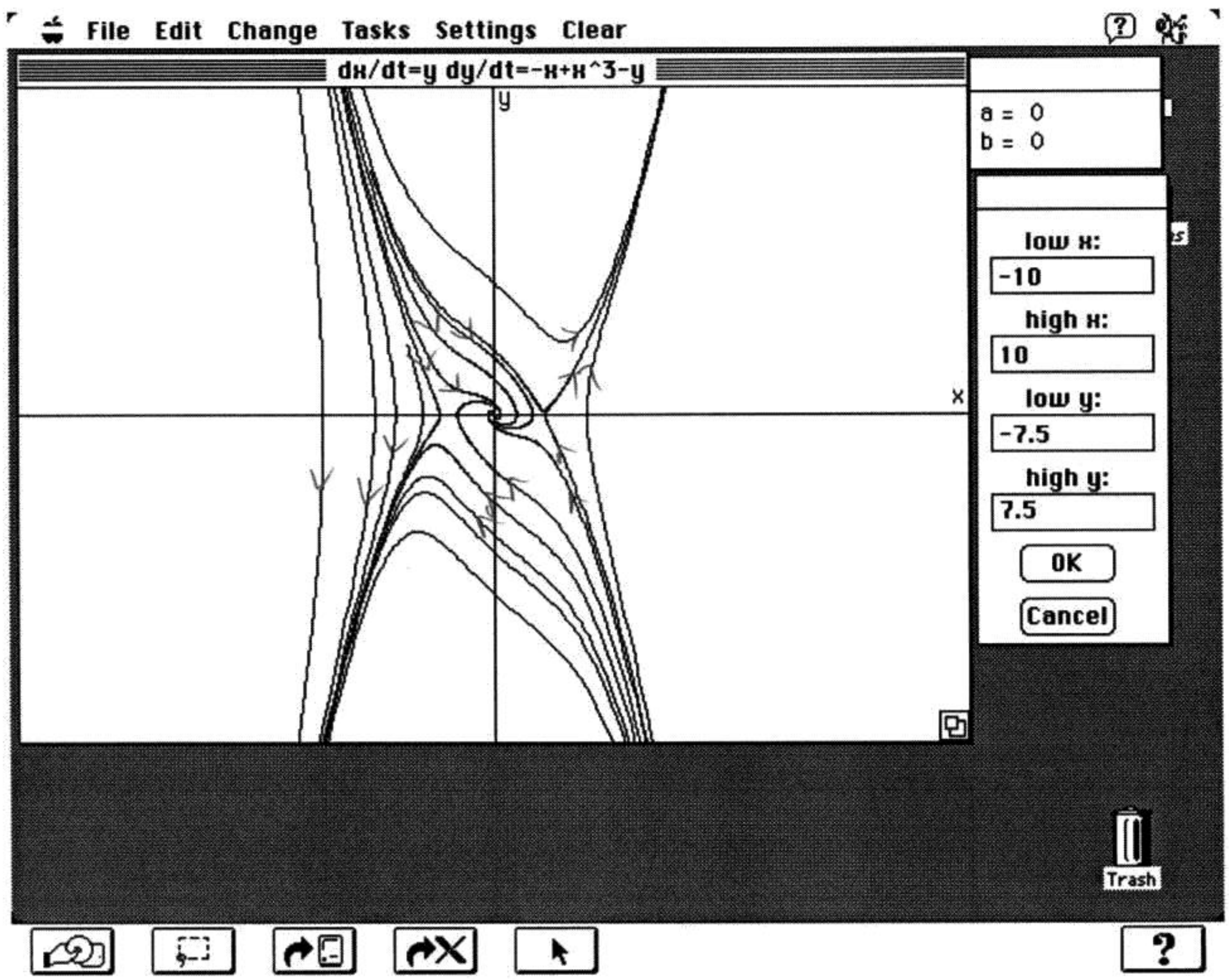
$$E = \frac{1}{2}\dot{x}^2 - \frac{x^2}{2} + \frac{x^4}{4} = \frac{1}{2}y^2 - \frac{x^2}{2} + \frac{x^4}{4}$$ is conserved

& For $(x,y) \approx (0,0)$ represents a continuous family of closed

orbits & not a limit cycle thus Not a Hopf bifurcation

8.2.11

$\mu = 1$



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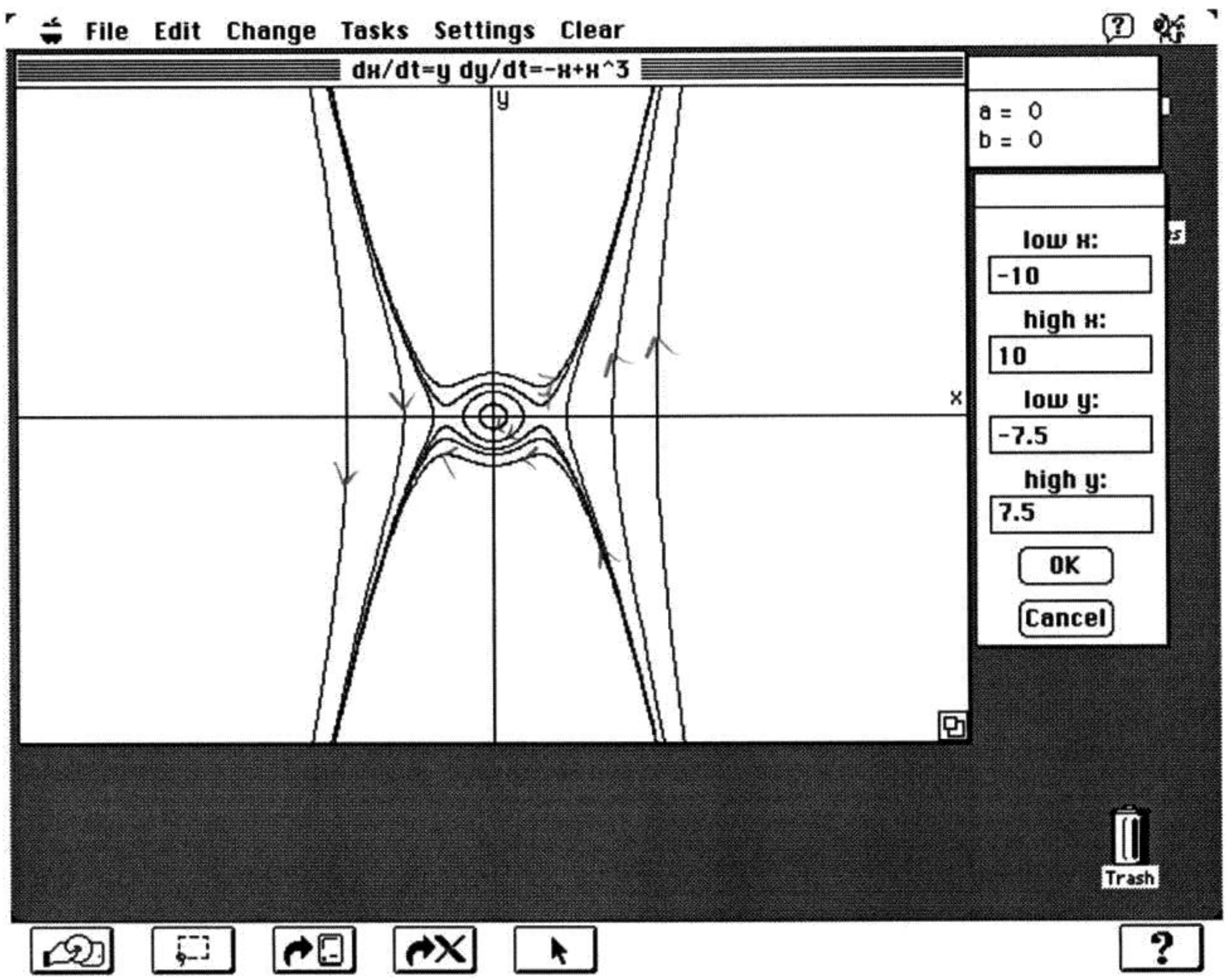
dx/dt=y dy/dt=-x+x^3-y
y
x
a = 0
b = 0
low x:
-10
high x:
10
low y:
-7.5
high y:
7.5
OK
Cancel
Trash

8.2.11

$\mu = 0$



File
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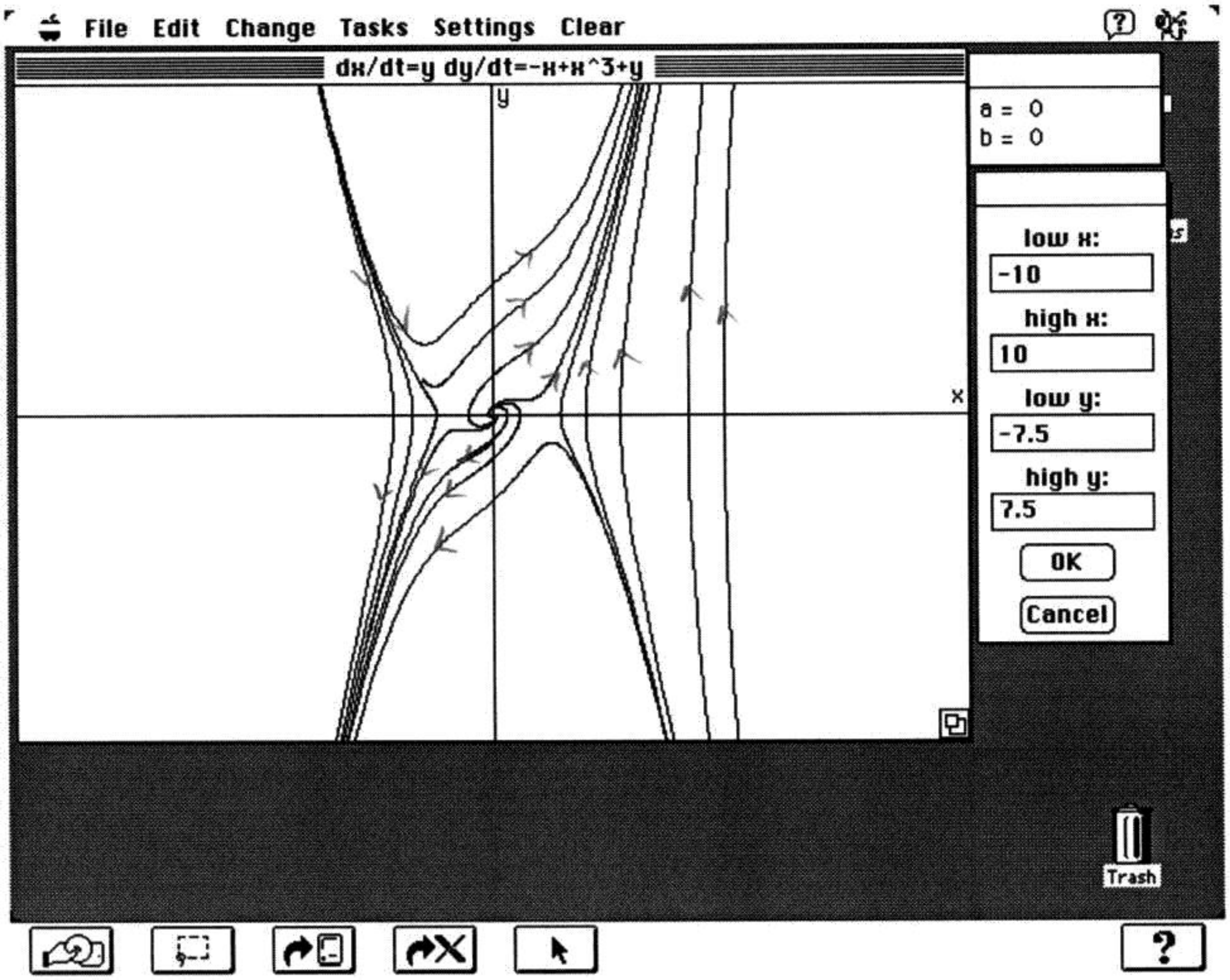
Settings
Clear
dx/dt=y dy/dt=-x+x^3
y
x
a = 0
b = 0
low x:
-10
high x:
10
low y:
-7.5
high y:
7.5
OK
Cancel
Trash

8.2.11

$\mu = -1$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dx/dt=y dy/dt=-x+x^3+y
y
x
a = 0
b = 0
low x:
-10
high x:
10
low y:
-7.5
high y:
7.5
OK
Cancel
Trash

John Wrotherwax

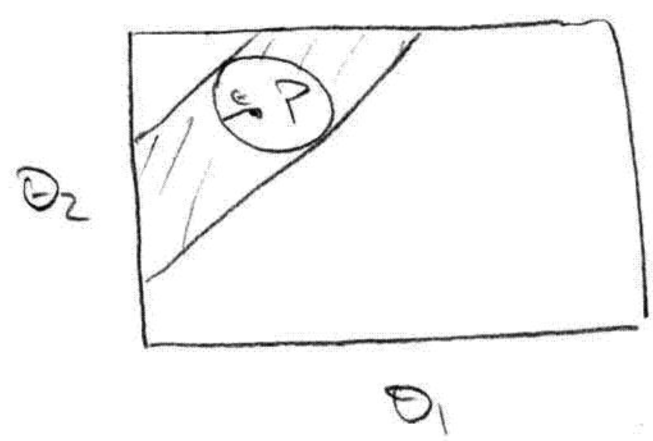
8.6.3

Assume the opposite $\Rightarrow \exists$ a pt P, $\epsilon > 0$ + an Initial condition Q

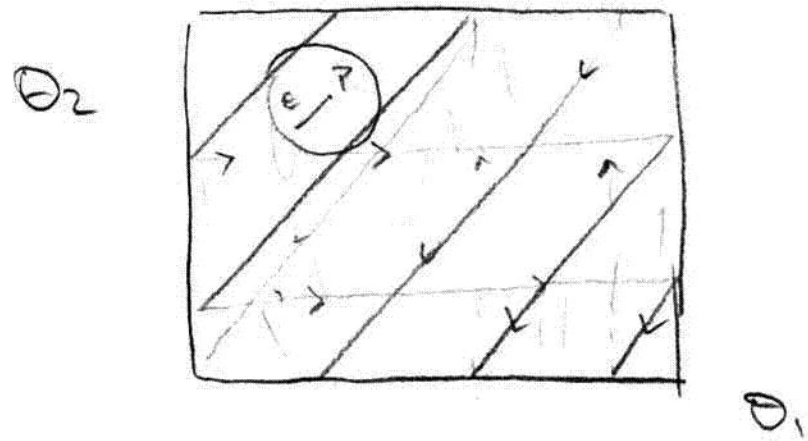
$\Rightarrow$ No trajectory starting at Q intersects this ϵ ball

Consider phase space w/ periodic Boundary conditions

3



Now trajectories in this phase space are straight lines w/ slopes $\frac{\omega_1}{\omega_2}$ But from Above picture No trajectories can enter the shaded region. But By periodic Boundary conditions this region is extended to the following



In other words the entire square is filled w/ forbidden trajectories (if a trajectory was in there it would eventually end up in sphere around P) But this gives no place for the trajectories to go $\rightarrow\leftarrow$ Thus trajectory must be dense.

Note the first involve rational values of w and thus close or repeat

In[45]:=

```
w = {3,2/3,5/3};
Table[
    ParametricPlot[{Sin[t],Sin[w[[i]] t]},
        {t,0,8 Pi},
        PlotPoints->1000,
        PlotLabel->w[[i]]
    ],
    {i,1,Length[w]}
];
```

In[41]:=

```
w = {Sqrt[2],Pi,1/2 (1+Sqrt[5])};
Table[
    ParametricPlot[{Sin[t],Sin[w[[i]] t]},
        {t,0,16 Pi},
        PlotPoints->1000,
        PlotLabel->w[[i]]
    ],
    {i,1,Length[w]}
];
```

Pg 296 Strogatz

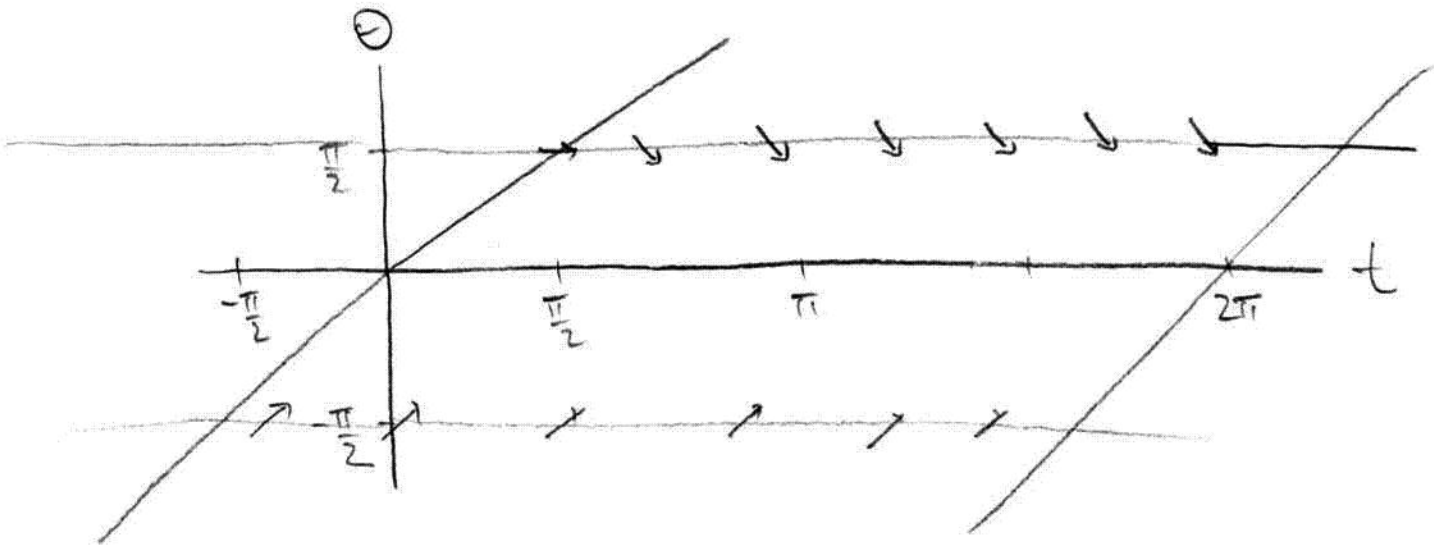
8.7.5

$\dot{t} = 1$

$\dot{\theta} = \sin t - \sin\theta$

Nullclines $\sin t = \sin\theta$

$\Rightarrow t = \theta + 0 \bmod 2\pi$



on $\theta = \frac{\pi}{2}$ DE becomes $\dot{t} = 1$

$\dot{\theta} = \sin t - 1 \leq 0 \quad \forall t$

Thus flow points in

on $\theta = -\frac{\pi}{2}$ DE becomes $\dot{t} = 1$
$\dot{\theta} = \sin t + 1 \geq 0 \quad \forall t$

Thus flow pts in

let Poincare Map take from slanted surface $\theta = t$

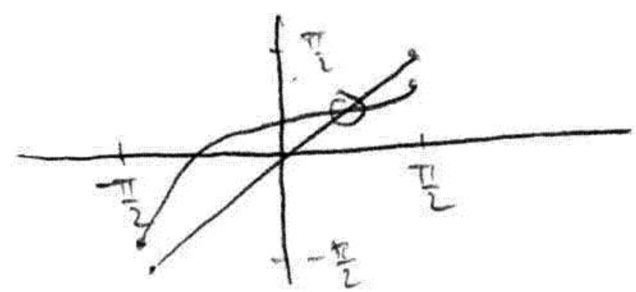
to slanted surface $\theta = t - 2\pi$

Now $P(\theta = -\pi/2)$ must be Above $-\frac{\pi}{2}$ on Right surface

+ $P(\theta = \pi/2)$ must be below $\pi/2$ on left surface

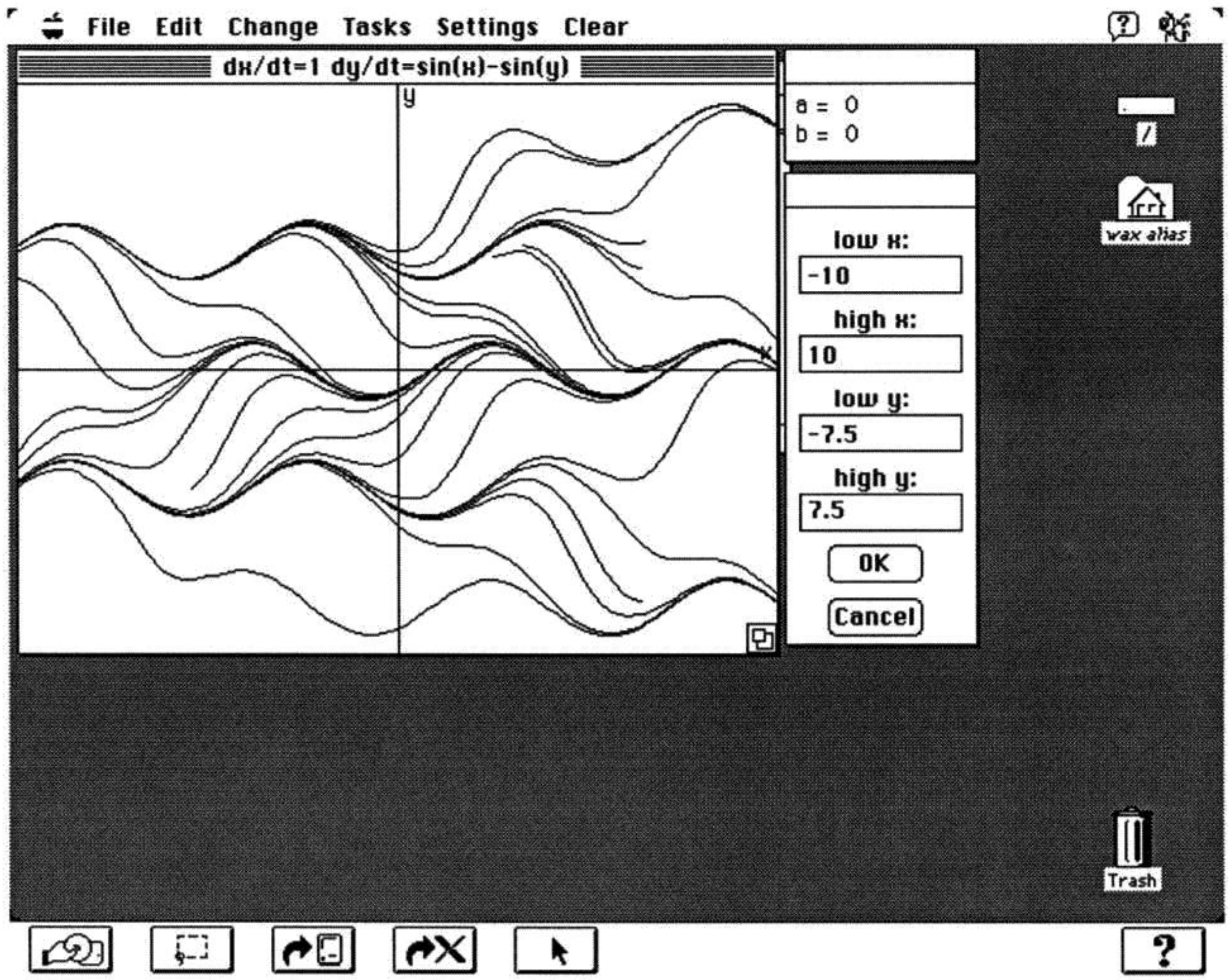
due to the way the flow is inward

Thus plot of P is something like + P must have a fixed pt



$\therefore$ a periodic orbit!

8.7.7


File Edit Change Tasks Settings Clear
dx/dt=1 dy/dt=sin(x)-sin(y)
y
x
a = 0
b = 0
low x:
-10
high x:
10
low y:
-7.5
high y:
7.5
OK
Cancel
wax alias
Trash
?