John Waether
#574199

Math 361

1 2 4 5 6 7 8 9 10 ✓

Ex 1.4.1

By expansion about the row or column w/ all but 1 zero, as many times as nessassary we get $\det(G) = g_{11} g_{22} \cdots g_{nn}$

Ex 1.42

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 3 & 1 & -1 & 0 \\ 4 & 1 & -3 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 2 \\ 9 \end{bmatrix}$$

$$y_1 = 2/2 = 1$$

$$y_2 = \frac{b_2 - g_{21} y_1}{g_{22}} = \frac{3-(-1)(1)}{2} = 2$$

$$y_3 = \frac{b_3 - g_{31} y_1 - g_{32} y_2}{g_{33}} = \frac{2-3(1)-1(2)}{-1} = 3$$

$$y_4 = \frac{b_4 - g_{41} y_1 - g_{42} y_2 - g_{43} y_3}{g_{44}} = \frac{9-4(1)-1(2)-(-3)(3)}{3} = \frac{12}{3} = 4$$

Ex 1.4.4

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 3 & 1 & -1 & 0 \\ -4 & 1 & -3 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 2 \\ 9 \end{bmatrix}$$

$$y_1 = 2/2 = 1$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 3 & 3 \end{bmatrix} \begin{bmatrix} y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 9 \end{bmatrix} - \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix} y_1 = \begin{bmatrix} 3 \\ 2 \\ 9 \end{bmatrix} - \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix} 1 = \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix}$$

$$y_2 = 4/2 = 2$$

$$\begin{bmatrix} -1 & 0 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} (2) = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$$

$$y_3 = -3/-1 = 3$$

$$[3][y_4] = [3] + 3y_3 \Rightarrow y_4 = 4.$$

Ex 1.4.5

Given g_{ij} $1 \le i, j \le n$

b_i $1 \le i \le n$

Begin:

for $j = 1 : n-1$

if $g_{jj} = 0$ set error

else $b_j = b_j / g_{jj}$

for $i = j+1 : n$

$b_i = b_i - b_j g_{ij}$

End

The b contains y values.

Ex 1.4.6

For $b_j = b_j / g_{jj}$ have n flops. ~~total~~

have $n-j$ flops $1 \le j \le n$

+ so $\sum_{j=1}^{n} n-j = n(n-1) - \sum_{j=1}^{n} j = n(n-1) - \frac{n(n-1)}{2} \Big|_{1}^{n+1}$

$= n(n-1) - \frac{(n+1)(n)}{2} = n\left[n-1 - \frac{n+1}{2}\right]$

$= n\left[\frac{2n-2-n-1}{2}\right] = \frac{n}{2}(n-3) \approx O\left(\frac{n^2}{2}\right)$

Ex 1.4.7

Ex 1.4.7

In row-oriented substitution we 1st solve for x_1
then x_2 using x_1
then x_3 using x_1, x_2
then

x_n using $x_1, \ldots, x_{n-1}$

the column subtractions (in col-oriented subtraction)
occur one at a time i.e.
In row-oriented

$$a_{11}x_1 = b_1 \Rightarrow x_1 = b_1/a_{11}$$

$$a_{21}x_1 + a_{22}x_2 = b_2 \Rightarrow x_2 = \frac{b_2 - a_{21}x_1}{a_{22}}$$

$$\Rightarrow x_2 = \frac{b_2 - a_{21}\left(\frac{b_1}{a_{11}}\right)}{a_{22}}$$

& in col-oriented we

solve for $x_1 = b_1/a_{11}$

then get $a_{22}x_2 = b_2 - a_{21}(b_1/a_{11})$

& solve for x_2

In summary using col-oriented we do some of the subtractions in $y_i = \dfrac{b_i - \sum_{j=1}^{i-1} a_{ij} y_j}{a_{ii}}$

. each time we go through the loop:
The row-oriented does all subtractions at once

Ex 1.4.8

$$\begin{bmatrix} \hat{G} & 0 \\ \vec{h}^T & g_{nn} \end{bmatrix} \begin{bmatrix} \hat{y} \\ y_n \end{bmatrix} = \begin{bmatrix} \hat{b} \\ b_n \end{bmatrix}$$

given $y_1 \ldots y_{i-1}$ get y_i

$$\Rightarrow y_i = \frac{b_i - \vec{h}^T \hat{y}}{a_{ii}}$$

Begin:

for $i = 1:n$ Do
- If $a_{ii} = 0$ (set error)
- for $j = 1$ to $i-1$ do
  - $b_i \leftarrow b_i - a_{ij} y_j$
- $b_i \leftarrow b_i / a_{ii}$

Ex 1.4.9

$u_{11}x_1 + u_{12}x_2 + \cdots + u_{1,n-1}x_{n-1} + u_{1n}x_n = y_1$

$\vdots \qquad = \vdots$

$u_{n-2,n-2}x_{n-2} + u_{n-2,n-1}x_{n-1} + u_{n-2,n}x_n = y_{n-2}$

$u_{n-1,n-1}x_{n-1} + u_{n-1,n}x_n = y_{n-1}$

$u_{nn}x_n = y_n$

row oriented back subs

for $i = n : -1 : 1$ Do

if $u_{ii} = 0$ (set error flag)

for $j = n : (-1) : i+1$ (none done for $i = n$)

$y_i \leftarrow y_i - u_{i,j}x_j$

$x_i = \frac{y_i}{u_{ii}}$

Ex 1.4.10 col. oriented.

factor as such

$U\vec{x} = \vec{y}$

$$= \begin{bmatrix} \hat{U}_{n-1 \times n-1} & h \\ 0 & U_{nn} \end{bmatrix} \begin{bmatrix} \hat{x} \\ x_n \end{bmatrix} = \begin{bmatrix} \hat{y} \\ y_n \end{bmatrix}$$

$\Rightarrow \hat{U}\hat{x} + h x_n = \hat{y}$

$+ \quad U_{nn} x_n = y_n$

Solve for x_n

The $\hat{U}\vec{x} = \hat{y} - h x_n$ is a $n-1 \times n-1$ upper Δ system when we make the same substitutions.

So let $y_1 = y_1 - U_{1n} x_n$

$y_2 = y_2 - U_{2n} x_n$

put x_n in $\overset{y_{n-1}}{y_n}$

Begin:

for $i = n : -1 : 1$ Do
  If $u_{ii} = 0$ set flag
  $y_i \leftarrow y_i / u_{ii}$
  for $j = i-1 : -1 : 1$ Do
    $y_j \leftarrow y_j - u_{ji} y_i$

Values of $\vec{x}$ stored in $y_1 \cdots y_n$

Check n:

$y_n \leftarrow y_n / u_{nn}$ ✓ $n-1 (-1) 1$

$y_{n-1} \leftarrow y_{n-1} - u_{n-1,n} x_n$

$y_1 \leftarrow y_1 - u_{1,n} x_n$

Check $n-1$

$$y_{n-1} = \frac{y_{n-1}}{u_{n-1,n-1}} \qquad \begin{bmatrix} u_{11} & \cdots & u_{1n-1} \\ & \ddots & \\ & & u_{n-1,n-1} \end{bmatrix} = \begin{pmatrix} y_1 \\ \\ y_{n-1} \end{pmatrix}$$

$y_{n-2} = y_{n-2} - u_{n-2,n-1} x_{n-1}$

Ex 1.7.15   15 ✓   16 ✓   14   1 ✓   2 ✓   3 ✓   John Watkins

(a) For $n=2$ $\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}^{-1} = \begin{bmatrix} a_{11}^{-1} & -a_{11}^{-1} a_{12} a_{22}^{-1} \\ 0 & a_{22}^{-1} \end{bmatrix}$

So upper Δ matrix $(n=2)$ has upper Δ matrix as inverse

Assume true for $n=k$.

Show for $n=k+1$

$$\begin{bmatrix} a_{11} & \cdots & \cdots & a_{1k+1} \\ 0 & a_{22} & & \\ 0 & 0 & & \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & a_{k+1,k+1} \end{bmatrix} = \begin{bmatrix} a_{11} & h^T \\ 0_c & A \end{bmatrix}$$ ✓

w/ A as (k,k) matrix

Now inverse of above is $\begin{bmatrix} a_{11}^{-1} & a_{11}^{-1} h^T A^{-1} \\ 0 & A^{-1} \end{bmatrix}$

Now A^{-1} is upper Δ by hypothesis & $\therefore$ Matrix above is upper Δ.

Have proven by math induction

(b) V unit upper Δ then V^{-1} is unit upper Δ.

As inverse of Any Δ matrix (upper or lower) must have its diagonal elements to be the recipicals of the corresponding element in the original matrix. i.e. More details appropriate here,

$a_{ii}^{-1} = 1/a_{ii}$ Thus if $a_{ii} \equiv 1$ then a_{ii}^{-1} inverse diag elts are all one. (This can be proven w/ induction as above).

Ex 1.7.16

(a) V + W upper Δ then VW is upper Δ.

$n=2$:

$$\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12}+a_{12}b_{22} \\ 0 & a_{22}b_{22} \end{bmatrix}$$

Assume for $n=k$

for $n=k+1$

$$\begin{bmatrix} a_{11} & \cdots & a_{1,k+1} \\ 0 & a_{22} & \\ \vdots & 0 & \ddots \\ 0 & & a_{k+1,k+1} \end{bmatrix}\begin{bmatrix} b_{11} & \cdots & b_{1,k+1} \\ 0 & \ddots & \vdots \\ 0 & & b_{k+1,k+1} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & \vec{a}^T \\ 0_c & \mathbb{A}_{(k\times k)} \end{bmatrix}\begin{bmatrix} b_{11} & b^T \\ 0_c & \mathbb{B}_{n\times n} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11} & a_{11}b^T + a^T\mathbb{B}_{n\times n} \\ 0 & \mathbb{A}\cdot\mathbb{B} \end{bmatrix}$$ As $\mathbb{A}\cdot\mathbb{B}$ is upper Δ see that Induction step holds.

$\therefore$ By induction product of upper Δ matrices is upper Δ.

(b) Let U + V be Unit upper Δ

$\Rightarrow$ ith row of $U = [0 \cdots \overset{i}{1}\ U_{i,i+1} \cdots U_{i,n}]$

ith col of $V = [V_{1,i}, \cdots 1, 0 \cdots 0]$

So the diag elt is Product$_{ii}$ = $U \circ V = 1$ + each diag elt is 1 $\therefore$ product is unit upper Δ.

Ex. 1.7.17

(a) Let U + V be lower Δ.

Then U^T & V^T are upper Δ

$U^T V^T = (VU)^T$ is upper Δ

take transpose of both sides

$\underset{\text{lower } \Delta}{VU} = (U^T V^T)^T = V \cdot U$

Both lower Δ.

(b) As Transpose does not affect the diagonal elt-
The unitary products must still hold.

John Weatherwax
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Ex 15.1

1 ✓ 3 ✓ 21 ✓ 26 27 ✗

$A = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}$

(a) $\vec{x}^T A \vec{x} = [x_1 \; x_2]\begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 4x_1^2 + 9x_2^2 \geq 0$

w/ $= 0$ iff $\vec{x} = 0$.

(b) $A = GG^T$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \\ \vdots & & & \\ a_{n1} & & & \end{bmatrix} = \begin{bmatrix} g_{11} & 0 & \cdots & 0 \\ g_{21} & g_{22} & 0 & \cdots 0 \\ g_{31} & g_{32} & g_{33} & \end{bmatrix} \begin{bmatrix} g_{11} & g_{21} & g_{31} & \cdots \\ 0 & g_{22} & g_{32} & \cdots \\ \vdots & 0 & & \\ 0 & & & \end{bmatrix}$$

$g_{11} = \pm\sqrt{a_{11}}$ take positive one.

$g_{11} = 2$

Then $g_{i1} = \dfrac{a_{i1}}{g_{11}} \quad i = 2, \ldots, n$

$a_{11} = g_{11}^2$

So $g_{21} = \dfrac{a_{21}}{g_{11}} = \dfrac{0}{2} = 0.$

$a_{12} = g_{11} g_{21}$

$a_{22} = g_{21}^2 + g_{22}^2$

$a_{i2} = g_{i1} g_{21} + g_{i2} g_{22}$

$a_{22} = 0 + g_{22}^2 \Rightarrow g_{22} = +\sqrt{a_{22}} = 3.$

- Need $g_{12} = 0$.
- So $A = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = GG^T$

  Cholesky factor is $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

(c) $G_2 = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}$ $G_3 = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$ $G_4 = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}$

(d) we can change the sign of any of the elts on the diagonal

$$\underbrace{2 \cdots 2}_{n \text{ diag elts}} = 2^n \text{ lower } \Delta \text{ matrices}$$

✓

Ex 1.5.3

$$A = \begin{bmatrix} 16 & 4 & 8 & 4 \\ 4 & 10 & 8 & 4 \\ 8 & 8 & 12 & 10 \\ 4 & 4 & 10 & 12 \end{bmatrix}$$

$$g_{jj} = \sqrt{a_{jj} - \sum_{k=1} g_{jk}^2}$$

$$g_{ij} = \frac{\left(a_{ij} - \sum_{k=1}^{j-1} g_{ik} g_{jk}\right)}{g_{jj}}$$

$$g_{11} = \sqrt{16} = 4$$

$$g_{21} = 4/4 = 1$$

$$g_{31} = 8/4 = 2$$

$$g_{41} = 4/4 = 1$$

$$g_{22} = \sqrt{a_{22} - g_{21}^2} = \sqrt{10 - 1^2} = 3$$

$$g_{32} = \frac{a_{32} - g_{31} g_{21}}{g_{22}} = \frac{8 - 2(1)}{3} = 2$$

$$g_{42} = \frac{a_{42} - g_{41} g_{21}}{g_{22}} = \frac{4 - 1(1)}{3} = 1.$$

$$g_{33} = \sqrt{a_{33} - g_{31}^2 - g_{32}^2} = \sqrt{12 - 4 - 4}$$

$$= 2$$

$$g_{43} = \frac{a_{43} - g_{41} g_{31} - g_{42} g_{32}}{g_{33}}$$

$$g_{43} = \frac{10 - 1(2) - 1(2)}{2} = 3$$

$$g_{44} = \sqrt{a_{44} - g_{41}^2 - g_{42}^2 - g_{43}^2} = \sqrt{12 - 1^2 - 1^2 - 3^2} = 1$$

$$\Rightarrow G = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 2 & 2 & 2 & 0 \\ 1 & 1 & 3 & 1 \end{bmatrix}$$

$GG^T\vec{x} = \vec{b}$

$\vec{y} = G^T\vec{x}$

$$\begin{bmatrix} 4 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 2 & 2 & 2 & 0 \\ 1 & 1 & 3 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 32 \\ 26 \\ 38 \\ 30 \end{bmatrix}$$

$$y_1 = 8$$

$$8 + 3y_2 = 26 \Rightarrow 3y_2 = 18 \Rightarrow y_2 = 6$$

$$2y_1 + 2y_2 + 2y_3 = \cancel{16} + 12 + 2y_3 = 38 \Rightarrow y_3 = 5.$$

$$8 + 6 + 15 + y_4 = 30 \Rightarrow y_4 = 1$$

$$\begin{bmatrix} 4 & 1 & 2 & 1 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \\ 5 \\ 1 \end{bmatrix}$$

$x_4 = 1$

$2x_3 + 3 = 5 \Rightarrow x_3 = 1$

$3x_2 + 2 + 1 = 6 \Rightarrow x_2 = 1$

$4x_1 + 1 + 2 + 1 = 8 \Rightarrow x_1 = 1$

$$\vec{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Ex 1.5.21

A pos def $\Rightarrow$ $x^T A x > 0$

pick e_i for x

$e_i^T A e_i = a_{ii} > 0$ ✓

Ex 1.5.26

Show Cholesky factor is unique for $n=1$ trivial. $[a_{11}] \Rightarrow [\pm\sqrt{a_{11}}][\pm\sqrt{a_{11}}]$ w + taken

Assume for n. Let A be positive definite $n+1 \times n+1$ matrix.

+ Assume A has 2 factorization; i.e

$A = GG^T = HH^T$

$$\Rightarrow \begin{bmatrix} A_{11} & A_{21}^T \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} G_{11} & 0 \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} G_{11}^T & G_{21}^T \\ 0 & G_{22}^T \end{bmatrix}$$

$\Rightarrow A_{11} = G_{11} G_{11}^T$ + G_{nn} is unique by induction. if A $n \times n$

$A_{21} = G_{21} G_{11}^T$ + $A_{22} = G_{21} G_{21}^T + G_{22} G_{22}^T$.

$A_{21}^T = G_{11} G_{21}^T$

· Then G_{11} is unique by induction

+ G_{21}^T is unique because G_{11} is invertible (+ ∴ 1-to-1)

So if G_{11} is $n \times n$ then G_{22} is 1×1

+ $G_{22}G_{22}^T = G_{22}^2 = A_{22} - G_{21}G_{21}^T$

$$\Rightarrow G_{22} = +\sqrt{A_{22} - G_{21}G_{21}^T} \text{ is unique}$$

As G is unique partitioning H in the same way as G

gives

$$G_{11} = H_{11}$$
$$G_{21} = H_{21}$$
$$G_{22} = H_{22}$$

+ By induction G is unique

Ex 1.5.27

A pos def. $\Rightarrow A = GG^T$ w/ All pos entries on diagonals

$$|A| = |G||G^T| = \left(\prod_{i=1}^{n} g_{ii}\right)^2 > 0$$

John Weatherwax

Ex 2.2.2

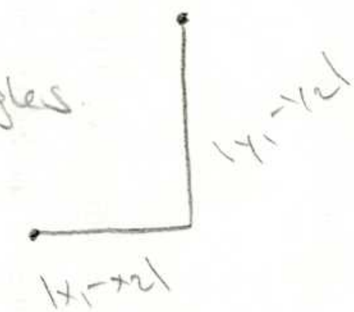
$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$

$\|x\|_1 \geq 0$ & $\|x\|_1 = 0$ iff $x = 0$

$$\|\alpha x\|_1 = \sum_{i=1}^{n} |\alpha x_i| = |\alpha| \sum_i |x_i| = |\alpha| \|x\|_1$$

$$\|x+y\|_1 = \sum |x_i + y_i| \leq \sum_i (|x_i| + |y_i|) = \|x\|_1 + \|y\|_1$$

Ex 2.2.3 distance between 2 pt is distance taking only right angles.



Ex 2.2.4

1) $\|x\|_\infty \geq 0$ $\quad \|x\|_\infty = 0$ iff $x = 0$

2) $\|\alpha x\|_\infty = \max_{1 \leq i \leq n} |\alpha x_i| = |\alpha| \max_{1 \leq i \leq n} |x_i| = |\alpha| \|x\|_\infty$

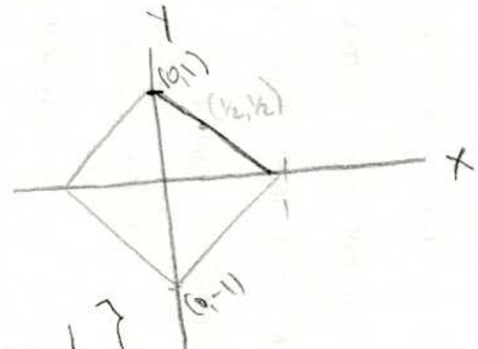
3) $\|x+y\|_\infty = \max\limits_{1\le i\le n} |x_i+y_i| \le \max\limits_{1\le i\le n} (|x_i|+|y_i|)$

$\le \max\limits_{1\le i\le n} |x_i| + \max\limits_{1\le i\le n} |y_i| = \|x\|_\infty + \|y\|_\infty$

Ex 2.2.5 I checked all of these using mathematica

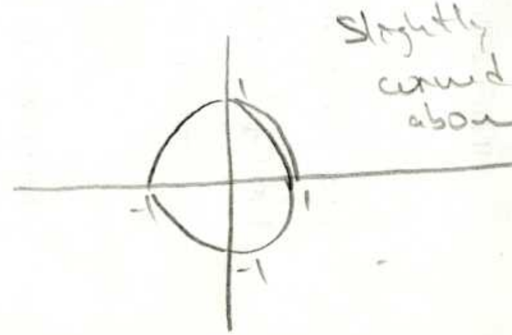
$P=1 \; \{x \in \mathbb{R}^2 \mid \|x\|_1 = 1\} \Rightarrow \{x \in \mathbb{R}^2 \mid |x|+|y| = 1\}$

$\Rightarrow$



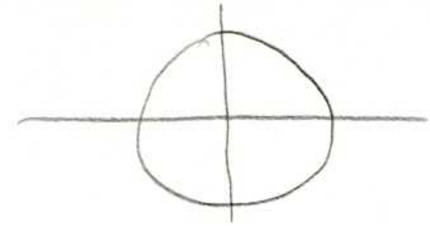
$P = 3/2 \Rightarrow \{x \in \mathbb{R}^2 \mid \sqrt[2/3]{|x|^{3/2}+|y|^{3/2}} = 1\}$

$\Rightarrow \{x \in \mathbb{R}^2 \mid |x|^{3/2}+|y|^{3/2} = 1\}$



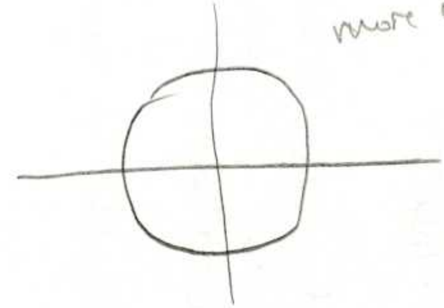
Slightly more curved than above

$p = 2$ $\{x \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \Rightarrow$



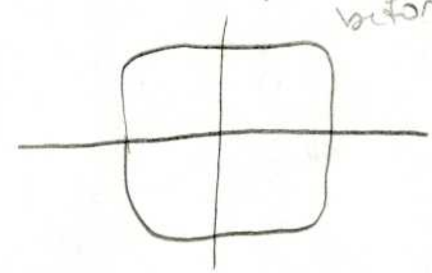
$p = 3 \Rightarrow \{x \in \mathbb{R}^2 \mid |x|^3 + |y|^3 = 1\}$

more square

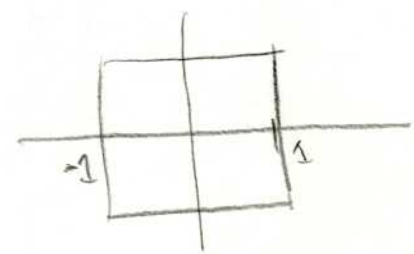


$p = 10 \Rightarrow \{x \in \mathbb{R}^2 \mid |x|^{10} + |y|^{10} = 1\}$

more square than before



$p = \infty \Rightarrow \{x \in \mathbb{R}^2 \mid \max_{1 \le i \le n} |x_i| = 1\} \Rightarrow$



Ex 2.2.6 $\quad A = GG^T$

(a)

$$\|x\|_A = (x^TAx)^{1/2} = (x^TGG^Tx)^{1/2}$$

$$= \left[(G^Tx)^T \cdot (G^Tx)\right]^{1/2} = \|G^Tx\|_2$$

(b) Know 1) $\|G^Tx\|_2 \geq 0 \quad \forall\, G^Tx$ or $\forall\, x$

$+ \quad \|G^Tx\|_2 = 0$ iff $G^Tx = 0$ $\quad$ As G^T is invertible

$\Leftrightarrow x = 0$

2) $\|\alpha x\|_A = \|G^T(\alpha x)\|_2$

$$= |\alpha|\,\|G^Tx\|_2 = |\alpha|\,\|x\|_A$$

3) $\|x+y\|_A = \|G^T(x+y)\|_2 = \|G^Tx + G^Ty\|_2$

$$\leq \|G^Tx\|_2 + \|G^Ty\|_2$$

$$= \|x\|_2 + \|y\|_2$$

Ex 2.2.7

Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Then $\|A\| = 1$ $\|B\| = 1$

$A \cdot B = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ & $\|AB\| = 2$

But $\|AB\| = 2 \not\leq \|A\| \cdot \|B\| = 1 \cdot 1$

Ex 2.2.8 (a) $\|A(cx)\| = \|cAx\| = |c|\,\|Ax\|$
& $\|cx\| = |c|\,\|x\|$ } so ratio is unchanged by scalar multiplication

(b) $\|A\| = \max\limits_{x \neq 0} \dfrac{\|Ax\|}{\|x\|} = \max\limits_{x \neq 0} \left\| A\left(\frac{x}{\|x\|}\right) \right\|$

But As x runs through $\mathbb{R}^n$ $x \neq 0$
$\frac{x}{\|x\|}$ runs through $\mathbb{R}^n$ w/ $\|x\| = 1$

$$\Rightarrow \quad \|A\| = \max_{\|x\|=1} \|Ax\|$$

<u>Ex 2.2.9</u> $\quad \|A\|_F = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2} \quad \|A\|_2 = \max_{x \neq 0} \dfrac{\|Ax\|_2}{\|x\|_2}$

(a) $$\|I\|_F = \left(\sum_{i=1}^{n} 1^2 \right)^{1/2} = n^{1/2}$$

$$\|I\|_2 = \max_{x \neq 0} \frac{\|Ix\|_2}{\|x\|_2} = 1$$

(b) $$\|A\|_2^2 = \max_{\|x\|_2 = 1} \|Ax\|_2^2 = \max_{\|x\|_2=1} \sum_{i=1}^{n} \left| \sum_{j=1}^{n} a_{ij} x_j \right|^2$$

By Cauchy-Schwarz inequality $\left| \sum_{j=1}^{n} a_{ij} x_j \right|^2 \leq \sum_{j=1}^{n} a_{ij}^2 \sum_{j=1}^{n} x_j^2$

As $\|x\|_2 = 1$

$$\Rightarrow \left(\sum_{j=1}^{n} x_j^2 \right)^{1/2} = 1$$

Ex 2.2.10 Pr : $\|A\|_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|$

Consider

$$\|Ax\|_\infty = \max_{1\le i\le n} |Ax|_i = \max_{1\le i\le n} \left|\sum_{j=1}^{n} a_{ij}x_j\right| \le \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}||x_j|$$

$$\le \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}| \left(\max_{1\le k\le n} |x_k|\right) \le \|x\|_\infty \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|$$

$\Rightarrow$ $$\frac{\|Ax\|_\infty}{\|x\|_\infty} \le \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|$$

Let $\max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|$ occur at the kth row.

Then $Ax_k = \sum_{j=1}^{n} a_{kj}x_j = \sum_{j=1}^{n} |a_{kj}|$ if x_j is chosen so that

+ As $\vec{x} = (x_j)_{j=1}^{n}$ picked as $\longrightarrow$ $x_j = \begin{cases} -1 & \text{if } a_{kj} \text{ is neg} \\ 1 & \text{if } a_{kj} \text{ is positive} \end{cases}$

$\|x\|_\infty = 1$.

• One [illegible] $\frac{\|Ax\|_\infty}{\|x\|_\infty} = \sum_{j=1}^{n} |a_{kj}| = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|$

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OCT 03 1995

Ex 2.3.1 ✓

(a) Show $\kappa(A) = \kappa(A^{-1})$

$\kappa(A) = \|A\| \cdot \|A^{-1}\|$

$\kappa(A^{-1}) = \|A^{-1}\| \cdot \|(A^{-1})^{-1}\| = \|A\| \cdot \|A^{-1}\|$

(b) $\kappa(cA) = \|cA\| \cdot \|(cA)^{-1}\| = |c| \, \|A\| \cdot \frac{1}{|c|} \|A^{-1}\| = \kappa(A)$

1. ✓
2. ✓
3. ✓
4. ✓
5. ✓
7.
8. ✓
9.

Ex 2.3.2 ✓

Pr:

$\text{maxmag}(A) = \max\limits_{x \neq 0} \frac{\|Ax\|}{\|x\|}$ & $\text{minmag}(A^{-1}) = \min\limits_{x \neq 0} \frac{\|A^{-1}x\|}{\|x\|}$

Pf

$\text{maxmag}(A) = \max\limits_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \max\limits_{v \neq 0} \frac{\|v\|}{\|A^{-1}v\|} = \frac{1}{\min\limits_{v \neq 0} \frac{\|A^{-1}v\|}{\|v\|}} = \frac{1}{\text{minmag}(A^{-1})}$

$Ax = v$

$x = A^{-1}v$

$\text{minmag}(A^{-1}) = \min\limits_{x \neq 0} \frac{\|A^{-1}x\|}{\|x\|} = \min\limits_{v \neq 0} \frac{\|v\|}{\|Av\|} = \frac{1}{\max\limits_{v \neq 0} \frac{\|Av\|}{\|v\|}} = \frac{1}{\text{maxmag}(A^{-1})}$

$A^{-1}x = v$

$x = Av$

Ex 2.3.3 $\quad k(A) = \|A\| \cdot \|A^{-1}\|$

Then $\dfrac{\text{max mag}(A)}{\text{min mag}(A)} = \|A\| \, \text{max mag}(A^{-1})$ from Ex 2.3.2 ✓

$$= \|A\| \cdot \|A^{-1}\|$$ ✓

Ex 2.3.4 $\quad A_\epsilon = \begin{bmatrix} \epsilon & 0 \\ 0 & \epsilon \end{bmatrix}$

$$\|A_\epsilon\| = \max_{x \neq 0} \frac{\|A_\epsilon x\|}{\|x\|} = \max_{(x_1, x_2) \neq 0} \frac{\|(\epsilon x_1, \epsilon x_2)\|}{\|x\|} = |\epsilon| \max_{x \neq 0} \frac{\|x\|}{\|x\|} = |\epsilon| = \epsilon$$

$A_\epsilon^{-1} = \begin{bmatrix} 1/\epsilon & 0 \\ 0 & 1/\epsilon \end{bmatrix}$ so By above $\|A_\epsilon^{-1}\| = 1/\epsilon$

$\therefore \; k(A_\epsilon) = \epsilon(1/\epsilon) = 1$ & obviously $\det A_\epsilon = \epsilon^2$

<u>Ex 2.3.2</u> <u>Pr</u>: $\frac{\|\delta b\|}{\|b\|} \le k(A) \frac{\|\delta x\|}{\|x\|}$

$$Ax = b \Rightarrow x = A^{-1}b$$

$$\|x\| \le \|A^{-1}\| \, \|b\| \qquad (1)$$

& $A(x+\delta x) = b + \delta b$

$$A\delta x = \delta b$$

$$\|A\| \cdot \|\delta x\| \geq \|\delta b\| \Rightarrow \|\delta x\| \geq \frac{\|\delta b\|}{\|A\|}$$

$$\Rightarrow \frac{1}{\|\delta x\|} \le \frac{\|A\|}{\|\delta b\|} \qquad (2)$$

multiply we get

$$\frac{\|x\|}{\|\delta x\|} \le \overbrace{\|A^{-1}\| \cdot \|A\|}^{k(A)} \frac{\|b\|}{\|\delta b\|}$$

$$\Rightarrow \frac{\|\delta b\|}{\|b\|} \le k(A) \frac{\|\delta x\|}{\|x\|}$$

equality holds when $\|x\| = \|A^{-1}\| \cdot \|b\|$ or b is in direction of maximum magnification of A^{-1}.

& $\|A\| \cdot \|\delta x\| = \|\delta b\|$ when δx is in direction of maximal magnification of A. Then

(δb in dir of min mag for A^{-1})

$$\frac{\|\delta b\|}{\|b\|} = k(A) \cdot \frac{\|\delta x\|}{\|x\|}$$

Ex 2.3.6

$$r(\hat{x}) = A\hat{x} - b = A(x+\delta x) - b = A\delta x = \delta b$$

So $\dfrac{\|r(\hat{x})\|}{\|b\|} = \dfrac{\|\delta b\|}{\|b\|} \le k(A)\dfrac{\|\delta x\|}{\|x\|}$

So $\dfrac{\|r(\hat{x})\|}{\|b\|} \le k(A)\dfrac{\|\delta x\|}{\|x\|}$

As Above is sharp we can get $\dfrac{\|r(\hat{x})\|}{\|b\|} = k(A)\dfrac{\|\delta x\|}{\|x\|}$

& thus the residual can be very much non zero.

If the matrix is well conditioned the $k(A)$ is fairly small & by above the residual must be small & we cannot get a super large $\dfrac{\|r(\hat{x})\|}{\|b\|}$ w/ a well conditioned matrix

Ex 2.3.7 $\hat{x} = \begin{bmatrix} 20.97 \\ -18.99 \end{bmatrix}$ Exaple 2.3.6 $A = \begin{bmatrix} 1000 & 999 \\ 999 & 998 \end{bmatrix}$

$b = \begin{bmatrix} 1999 \\ 1997 \end{bmatrix}$

w/ solution $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$r(\hat{x}) = A\hat{x} - b = \begin{bmatrix} 1000 & 999 \\ 999 & 1000 \end{bmatrix}\begin{bmatrix} 20.97 \\ -18.97 \end{bmatrix} - \begin{bmatrix} 1999 \\ 1997 \end{bmatrix}$$

$$= \begin{bmatrix} 19.9 \\ -17.97 \end{bmatrix}$$

$$\frac{\|r(\hat{x})\|_2}{\|b\|_2} = \frac{26.81}{2.8256 \times 10^3} = 9.489 \times 10^{-3}$$

Ex 2.3.8 $\quad A = \begin{bmatrix} 375 & 374 \\ 752 & 750 \end{bmatrix} \begin{matrix} 749 \\ 1502 \end{matrix}$

(a) $A^{-1} = \frac{1}{2}\begin{bmatrix} 750 & -374 \\ -752 & 375 \end{bmatrix} \begin{matrix} 562 \\ 563.5 \end{matrix}$ $\quad k_\infty(A) = \|A\|_\infty \|A^{-1}\|_\infty$

$= (1502)(563.5)$

$= 846380$ ill conditioned.

(b) To make $\frac{\|\delta x\|}{\|x\|}$ large relative to $\frac{\|\delta b\|}{\|b\|}$

~~$\delta A = A$~~

$\delta x = A^{-1}\delta b \quad Ax = b$

pick δb in direction of maxmag(A^{-1}) ~~minmag for A~~

& pick b in direction of minmag(A^{-1})

Note $A\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}749\\1502\end{bmatrix}$

So $\|A\begin{pmatrix}1\\1\end{pmatrix}\|_\infty = 1502 = \|A\|_\infty$ so $\begin{pmatrix}1\\1\end{pmatrix}$ is max magnified by A

eg $b = \begin{pmatrix}749\\1502\end{pmatrix}$ is in minimum mag(A^{-1})

for δb $\begin{pmatrix}-1\\1\end{pmatrix}$ is in direction of maxmag of A^{-1}.

Now solve $\begin{bmatrix} 375 & 374 \\ 752 & 750 \end{bmatrix}\begin{bmatrix}\hat{x}_1\\\hat{x}_2\end{bmatrix} = \begin{bmatrix}749\\1502\end{bmatrix} + .01\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}7.4899\times10^2\\1502.01\end{bmatrix}$

d $\begin{bmatrix}\hat{x}_1 \\ \hat{x}_2\end{bmatrix} = x + \delta x$ $\qquad$ δx sol + $A\delta x = \delta b = .01\begin{bmatrix}-1 \\ 1\end{bmatrix}$

$$\delta x = \begin{bmatrix}-5.620 \\ 5.635\end{bmatrix}$$

So $\begin{bmatrix}\hat{x}_1 \\ \hat{x}_2\end{bmatrix} = \begin{bmatrix}-4.620 \\ 6.635\end{bmatrix}$ very different

$$\frac{\|\delta b\|_\infty}{\|b\|_\infty} = \frac{(.01)}{1502} = 6.65\times10^{-6}$$

$$\frac{\|\delta x\|_\infty}{\|x\|_\infty} = \frac{6.635}{1} = 6.635 \;!$$

$\delta x = A^{-1}\delta b \qquad \delta b = A\delta x$

$x = A^{-1}b \qquad b = Ax$

(c) want $\frac{\|\delta x\|_\infty}{\|x\|_\infty}$ small + $\frac{\|\delta b\|_\infty}{\|b\|_\infty}$ large

want x in direction of min mag of A $\Rightarrow$ x in direc of max mag A^{-1}

+ δx in direction of max mag of A.

pick $x = \begin{pmatrix}-1 \\ 1\end{pmatrix}$ then $b = \begin{pmatrix}-1 \\ -2\end{pmatrix}$

pick $\delta x = .01\begin{pmatrix}1 \\ 1\end{pmatrix}$

The $\delta b = A\delta x = \begin{bmatrix} 7.49 \\ 15.02 \end{bmatrix}$

So $\dfrac{\|\delta x\|_\infty}{\|x\|_\infty} = \dfrac{(.01)}{1} = .01$

$$\frac{\|\delta b\|_\infty}{\|b\|_\infty} = \frac{15.02}{2} = 7.51$$

Ex 2.3.9

Pr: $k_\infty(A) = k_1(A^T)$

$$(A^T)^{-1} = (A^{-1})^T$$

$$a^{T(-1)}_{ji} = a^{-1\,T}_{ij}$$

$$k_\infty(A) = \|A\|_\infty \cdot \|A^{-1}\|_\infty = \left(\max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|\right)\left(\max_{1 \le i \le n} \sum_{j=1}^{n} |a^{-1}_{ij}|\right)$$

$$= \left(\max_{1 \le i \le n} \sum_{j=1}^{n} |a^T_{ji}|\right)\left(\max_{1 \le i \le n} \sum_{j=1}^{n} (a^T_{ji})^{-1}\right)$$

$$= \|A^T\|_1 \ \|(A^T)^{-1}\|_1$$

$$= k_1(A)$$

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Ex 3.2.1–7 3.2.8–10.

✓

Ex 3.2.1

(a) Q orth $\Rightarrow QQ^T = I$

$\Rightarrow Q^T = Q^{-1}$

so $(Q^{-1})^T = Q$

so $Q^{-1} \cdot (Q^{-1})^T = Q^{-1} \cdot Q = I$

(b) Q_1 & Q_2 orthog $Q_1^{-1} = Q_1^T$ $Q_2^{-1} = Q_2^T$

$(Q_1 Q_2)^{-1} = Q_2^{-1} Q_1^{-1} = Q_2^T Q_1^T = (Q_1 Q_2)^T$

Ex 3.2.2 Q orth $Q^{-1} = Q^T$

or $QQ^T = I$

$\Rightarrow |Q||Q^T| = 1$

$\Rightarrow |Q|^2 = 1 \Rightarrow |Q| = \pm 1$

Ex 3.2.3

(a) $Q \Rightarrow (Qx, Qy) = (x, y)$

$$\Rightarrow (Qx)^T Qy = x^T Q^T Q y = x^T y \quad \forall\, x + y.$$

Let $x = e_i$ + $y = e_j$

The $e_i^T Q^T Q e_j = Q^T Q_{ij}$ ij-th component of $Q^T Q$

By hypothesis
$$= e_i^T e_j = \delta_{ij}$$

So As $Q^T Q_{ij} = \delta_{ij}$

$$\Rightarrow Q^T Q = I.$$

Ex 3.2.4

If Q orthog $\|Q\|_2 = ?$ $\quad QQ^T = I$

$$\|Q\|_2 = \max_{x \neq 0} \frac{\|Qx\|_2}{\|x\|_2} = \max_{x \neq 0} \frac{\|x\|_2}{\|x\|_2} = 1$$

$$\|Q^{-1}\|_2 = \max_{x \neq 0} \frac{\|Q^{-1}x\|_2}{\|x\|_2} = \max_{x \neq 0} 1 = 1$$

As Q^{-1} is also orthogonal.

So $k_2(Q) = 1 \cdot 1 = 1$.

<u>Ex 3.2.5</u> verify every rotator is an orthogonal matrix w/ determinant 1.

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$QQ^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

& $Q^T Q = I$ also so $Q^{-1} = Q^T$ $\therefore$ Q is orthogonal.

As Q is orthog $|Q| = \pm 1$

$|Q| = \cos^2\theta + \sin^2\theta = 1$

$Q^{-1} = Q^T$ as above

& Q^{-1} corresponds to rotating in the opposite direction

Ex 3.2.6

(a) geometrically fact that $Q^T\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}y\\0\end{pmatrix}$ means we can always rotate the vector so that it lies along the x axis only.

& so $y = \|x\|_2$ makes sense for the distance along the x-axis.

(b) $\cos\theta = \dfrac{x_1}{\sqrt{x_1^2+x_2^2}}$ $\quad\sin\theta = \dfrac{x_2}{\sqrt{x_1^2+x_2^2}}$

$$y = \cos\theta\, x_1 + \sin\theta\, x_2 = \frac{x_1^2}{\sqrt{x_1^2+x_2^2}} + \frac{x_2^2}{\sqrt{x_1^2+x_2^2}} = \sqrt{x_1^2+x_2^2} = \|\vec{x}\|_2$$

Ex 3.2.7

find $Q^T \ni Q^TA = R$ upper Δ.

get Q^T so that $Q^T\begin{bmatrix}2\\5\end{bmatrix} = \begin{bmatrix}*\\0\end{bmatrix}$

get $\cos\theta = \dfrac{2}{\sqrt{2^2+5^2}} = \dfrac{2}{\sqrt{29}}$ Then $Q = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$

$\sin\theta = \dfrac{5}{\sqrt{2^2+5^2}} = \dfrac{5}{\sqrt{29}}$

So $Q = \begin{bmatrix} 2/\sqrt{29} & -5/\sqrt{29} \\ 5/\sqrt{29} & 2/\sqrt{29} \end{bmatrix} = \frac{1}{\sqrt{29}} \begin{bmatrix} 2 & -5 \\ 5 & +2 \end{bmatrix}$

$6 + 35$

$R = Q^T A = \frac{1}{\sqrt{29}} \begin{bmatrix} 2 & 5 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$ $\quad -15 + 14 = -1$

$= \frac{1}{\sqrt{29}} \begin{bmatrix} 29 & 41 \\ 0 & -1 \end{bmatrix}$

So

$A = QR = \frac{1}{\sqrt{29}} \begin{bmatrix} 2 & -5 \\ 5 & 2 \end{bmatrix} \frac{1}{\sqrt{29}} \begin{bmatrix} 29 & 41 \\ 0 & -1 \end{bmatrix}$

$\Rightarrow Rx = Q^T b = \frac{1}{\sqrt{29}} \begin{bmatrix} 2 & 5 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 12 \\ 29 \end{bmatrix} = \frac{1}{\sqrt{29}} \begin{bmatrix} 169 \\ -2 \end{bmatrix}$ $\quad 24 + \quad = 169$

$-60 \quad = -2$

Then solve $Rx =$

$\Rightarrow \quad \frac{1}{\sqrt{29}} (29x_1 + 41x_2) = \frac{1}{\sqrt{29}} 169$

& $\quad \frac{1}{\sqrt{29}} (-x_2) = \frac{1}{\sqrt{29}} (-2) \quad \Rightarrow \quad x_2 = 2$

$$29x_1 + 82 = 169$$

$$x_1 = 3.$$

$$\vec{x} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

<u>Ex 3.2.8</u> Q

$$Q = \begin{array}{r} \\ \\ \\ i \to \\ \\ j \to \\ \\ \end{array} \begin{bmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & c & \cdots & -s & \\ & & & & \ddots & & \\ & & & s & \cdots & c & \\ & & & & & & \ddots \end{bmatrix}$$

$$\begin{array}{cc} \uparrow & \uparrow \\ i & j \end{array}$$

(next pg)

$$|Q| = |1\cdots1| \begin{vmatrix} c & \cdots & -s & & & \\ & 1 & & & & \\ & & \ddots & & & \\ 0 & & & 1 & & \\ +s & & & & c & \\ 0 & & & & & 1 \\ & & & & & & \ddots \\ & & & & & & & 1 \end{vmatrix}$$

by expanding about 1st $n-i$ columns

$$= c \begin{vmatrix} 1 & & & 0 \\ & \ddots & & \\ & & 1 & \\ 0 & & & c \\ & & & & 1 \\ & & & & & \ddots \end{vmatrix} + (-1)^{j-i+1}(-s) \begin{vmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & 0 & \cdots \\ +s & & & & 1 & 0 \\ 0 & & & & & 1 \\ \vdots & & & & & & \ddots \\ & & & & & & & 1 \end{vmatrix}$$

$$= c^2 + (-1)^{j-i+1}(-s)(s)(-1)^{j-i+1-1}$$

2nd term by expanding about row w $+s$.

$$= c^2 + s^2 = 1$$

$$
QQ^T = \begin{array}{c} \\ i \rightarrow \\ \\ j \rightarrow \\ \\ \end{array}
\begin{bmatrix}
1 & & & & \\
 & c & \cdots & -s & \\
 & \vdots & 1 & \vdots & \\
 & s & \cdots & c & \\
 & & & & 1
\end{bmatrix}
\begin{bmatrix}
1 & & & & \\
 & c & \cdots & s & \\
 & \vdots & \ddots & \vdots & \\
 & -s & \cdots & c & \\
 & & & & 1
\end{bmatrix}
$$

$$
\begin{array}{ccccc} & \uparrow & & \uparrow & \\ & i & & j & \end{array}
\qquad
\begin{array}{c} \uparrow \\ i \end{array}
$$

$$
= \begin{array}{c} \\ i \rightarrow \\ \\ \\ \end{array}
\begin{bmatrix}
1 & & & & \\
 & c^2+s^2 & 0 \cdots & cs-sc & \\
 & \vdots & \ddots & \vdots & \\
 & sc-cs & \cdots & s^2+c^2 & \\
 & & & & 1
\end{bmatrix} = I
$$

$$
\begin{array}{ccccc} & \uparrow & & \uparrow & \\ & i & & j & \end{array}
$$

& correspondingly $Q^TQ = I$

so Q is orthogonal.

<u>Ex 3.2.9</u>

Let Q be the plane rotator. Qx & Q^Tx alter only i, j entries of x

As from row 1 to $i-1$ we have nothing but the identity in both

& from row $j+1$ to n " " " " " " "

& from $i+1$ to $j-1$ it is the identity we get that,

Qx is not altered except in ith & jth positions & those elts

become

$$(Qx)_i = cx_i - sx_j$$
$$(Qx)_j = sx_i + cx_j \quad \& \quad \begin{bmatrix} c & -s \\ s & c \end{bmatrix}\begin{bmatrix} x_i \\ x_j \end{bmatrix}$$

for Q^T

$$(Q^Tx)_i = cx_i + sx_j$$
$$(Q^Tx)_j = -sx_i + cx_j \approx \begin{bmatrix} c & s \\ -s & c \end{bmatrix}\begin{bmatrix} x_i \\ x_j \end{bmatrix} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}^T\begin{bmatrix} x_i \\ x_j \end{bmatrix}$$

Ex 3.2.10

(a) $QA = Q[\tilde{a}_1\ \tilde{a}_2 \cdots \tilde{a}_n]$ $\quad \tilde{a}_i$ cols of A.

$\Rightarrow [Q\tilde{a}_1\ Q\tilde{a}_2 \cdots Q\tilde{a}_n]$

& $Q\tilde{a}_k$ only alters the i, j elts of $\tilde{a}_k$ so only i & j th rows of A are changed.

To compute the i, j th rows of QA.

$$Q\tilde{a}_1 = \begin{bmatrix} \tilde{a}_{11} \\ \vdots \\ c\tilde{a}_{1i} - s\tilde{a}_{1j} \\ \vdots \\ s\tilde{a}_{1i} + c\tilde{a}_{1j} \\ \vdots \end{bmatrix} \qquad Q\tilde{a}_2 = \begin{bmatrix} \tilde{a}_{21} \\ \vdots \\ c\tilde{a}_{2i} - s\tilde{a}_{2j} \\ \vdots \\ s\tilde{a}_{2i} + c\tilde{a}_{2j} \\ \vdots \end{bmatrix}$$

w/ dots representing no change in value
+ $\tilde{a}_{ki}$ = ith elt in $\tilde{a}_k$ col.

So ith row of QA

(next pg)

becomes $C\begin{pmatrix} \tilde{a}_{1i} \\ \tilde{a}_{2i} \\ a_{3i} \\ \vdots \\ \tilde{a}_{ni} \end{pmatrix}^T - S\begin{pmatrix} \tilde{a}_{1j} \\ \tilde{a}_{2j} \\ \vdots \\ \tilde{a}_{nj} \end{pmatrix}^T$

+ jth row of QA becomes

$$S\begin{pmatrix} \tilde{a}_{1i} \\ \tilde{a}_{2i} \\ \vdots \\ \tilde{a}_{ni} \end{pmatrix}^T + C\begin{pmatrix} \hat{a}_{1j} \\ \hat{a}_{2j} \\ \vdots \\ \tilde{a}_{nj} \end{pmatrix}^T$$

Both linear combinations of rows i + j.

(b) BQ only elts changed by Q in BQ are ones in i + j th columns of B by reasoning as above (identity in every spot of Q but i + jth columns)

To get what they are consider ith column of BQ.

ith column 1st elt $b_{1i}c + sb_{1j}$

2nd elt $b_{2i}c + b_{2j}s$

$\vdots$

nth elt $b_{ni}c + b_{nj}s$

$$\approx c\begin{pmatrix} \text{ith} \\ \text{column} \\ \text{of } B \end{pmatrix} + s\begin{pmatrix} \text{jth} \\ \text{column} \\ \text{of } B \end{pmatrix}$$

For jth column of BQ

1st elt $b_{1i}(-s) + b_{1j}(c)$

2nd $b_{2i}(-s) + b_{2j}(c)$

$\vdots$

nth elt $b_{ni}(-s) + b_{nj}(c)$

$$\approx -s\begin{pmatrix} \text{ith} \\ \text{column} \\ \text{of } B \end{pmatrix} + c\begin{pmatrix} \text{jth} \\ \text{column} \\ \text{of } B \end{pmatrix}$$

Which are both linear combinations of ith & jth columns of B

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Ex 3.2.12

12. ✓
17. ✓
22. ✓

each rotator requires 4 multiplications + 2 additions

$$Q_{n,n-1}^T \;\text{———}\; Q_{n2}^T \cdots Q_{42}^T Q_{32}^T \; Q_{n1}^T \cdots Q_{31}^T Q_{21}^T b$$

$n-2$ Q's $\qquad$ $n-1$ Q's

So total # of multiplications is

$$4\sum_{k=1}^{n-1}(n-k) = 4\left(n(n-1) - \sum_{k=1}^{n-1} k\right)$$

$$= 4n^2 - 4n - \frac{4(n-1)(n)}{2}$$

$$= 4n^2 - 4n - 2(n)(n-1)$$

$$= 4n^2 - 4n - 2n^2 + 2n = 2n^2 - 2n$$

$\approx 2n^2$ when n large

Additions become

$$2\sum_{k=1}^{n-1}(n-k) = 2n^2 - 2n - (n-1)(n)$$

$$= n[2n - 2 - n + 1] = n^2 - n \approx n^2 \text{ when } n \text{ large}$$

Ex 3.2.13

$\|u\|_2 = 1$ Defn $P = uu^T$

(a) $Pu = u(u^T \cdot u) = 1 \cdot u = u$.

(b) $Pv = uu^T v = u(u^T v) = 0 \cdot u = 0$ if $u^T v = 0 = (u,v)$

(c) $P^2 = u \cdot u^T \cdot u \cdot u^T = u \cdot u^T = P$

(d) $P^T = (uu^T)^T = u^T u = P$

Ex 3.2.14

$u \ni \|u\|_2 = 1$ $Q \in \mathbb{R}^{n \times n}$ by $Q = I - 2uu^T = I - 2P$

(a) $Qu = u - 2Pu = u - 2u = -u$

(b) $Qv = v - 2Pv = v$ if $(u,v) = 0$

(c) $Q^T = I^T - 2P^T = I - 2P = Q$

(d) $Q \cdot Q^T = Q \cdot Q = (I-2P)(I-2P) = I - 2P - 2P + 4P^2$

$= I - 4P + 4P = I$

(e) As $Q^{-1} = Q^T = Q$ done

I used Ex 3.2.13 to do some of the steps.

Ex 3.2.17 next pg

Ex 3.2.17

(a) $Q \ni Qx \longrightarrow [-6, 0, 0, 0, 0]^T \quad x = [3\ 4\ 1\ 3\ 1]^T$

for + w/o scaling

$$\sigma = (+1)\|x\|_2 = (+1)\sqrt{3^2+4^2+1+9+1} = 6$$

+ $u_1 = x_1 + \sigma = 3 + 6 = 9$

$u_2 = x_2 = 4$

$u_3 = x_3 = 1$

$u_4 = x_4 = 3 \qquad \gamma = \frac{1}{\sigma(u_1)} = \frac{1}{6(9)} = \frac{1}{54}$

$u_5 = x_5 = 1$

So $Q = I - \gamma u u^T$

(i) (ii)

$$= I - \frac{1}{54}\begin{bmatrix} 9 \\ 4 \\ 1 \\ 3 \\ 1 \end{bmatrix}\begin{bmatrix} 9 & 4 & 1 & 3 & 1 \end{bmatrix} = I - \frac{1}{54}\begin{bmatrix} 81 & 36 & 9 & 27 & 9 \\ 36 & 16 & 4 & 12 & 4 \\ 9 & 4 & 1 & 3 & 1 \\ 27 & 12 & 3 & 9 & 3 \\ 9 & 4 & 1 & 3 & 1 \end{bmatrix}$$

(ii) cont

$$Q = \begin{bmatrix} -\frac{1}{2} & -\frac{2}{3} & -\frac{1}{6} & -\frac{1}{2} & -\frac{1}{6} \\ -\frac{2}{3} & \frac{19}{27} & -\frac{2}{27} & -\frac{2}{9} & -\frac{2}{27} \\ -\frac{1}{6} & -\frac{2}{27} & \frac{53}{54} & -\frac{1}{18} & -\frac{1}{54} \\ -\frac{1}{2} & -2/9 & -\frac{1}{18} & 5/6 & -1/18 \\ -1/6 & -2/27 & -1/54 & -1/18 & \frac{53}{54} \end{bmatrix}$$

(b) (i) $Qa = a - \frac{1}{54} uu^T a = a - \frac{1}{54}(u^T a)u$

$u^T a = 9\cdot 0 + 4\cdot 2 + 1\cdot 1 + -1\cdot 3 + 0\cdot 1 = 6$

$\Rightarrow Qa = a - \frac{1}{9}u = \left[0 - \frac{9}{9}, 2 - \frac{4}{9}, 1 - \frac{1}{9}, -1 - \frac{3}{9}, 0 - \frac{1}{9}\right]^T$

$= \left[-1, \frac{14}{9}, \frac{8}{9}, -\frac{4}{3}, -\frac{1}{9}\right]^T$

(ii) $Q_{\text{from above}}\, a = \begin{bmatrix} \text{from above} \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{4}{3} - \frac{1}{6} + \frac{1}{2} \\ \frac{2(19)}{27} + \frac{-2}{27} + \frac{2}{9} \\ -\frac{4}{27} + \frac{53}{54} + \frac{1}{18} \\ -\frac{4}{9} - \frac{1}{18} - \frac{5}{6} \\ -\frac{4}{27} - \frac{1}{54} + \frac{1}{18} \end{bmatrix} = \begin{bmatrix} -1 \\ 14/9 \\ 8/9 \\ -4/3 \\ -1/9 \end{bmatrix}$ ✓

✓

Ex 3.2.22:

(a) $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$

$Q^T A = \begin{bmatrix} -\sqrt{1^2+1^2} & \tilde{a}_{12} \\ 0 & \tilde{a}_{22} \end{bmatrix}$

$Q^T = I - \gamma u u^T$ w/ $u_1 = x_1 + \sigma = 1+\sqrt{2}$, $\gamma = \frac{1}{\sigma u_1} = \frac{1}{\sqrt{2}(1+\sqrt{2})}$

$u_2 = x_2 = 1$

So $Q^T = I - \gamma u u^T = I - \frac{1}{\sqrt{2}(1+\sqrt{2})} \begin{bmatrix} 1+\sqrt{2} \\ 1 \end{bmatrix} \begin{bmatrix} 1+\sqrt{2} & 1 \end{bmatrix}$ ✓

$= I - \frac{1}{\sqrt{2}(1+\sqrt{2})} \begin{bmatrix} (1+\sqrt{2})^2 & (1+\sqrt{2}) \\ (1+\sqrt{2}) & 1 \end{bmatrix} = I - \frac{1}{\sqrt{2}} \begin{bmatrix} (1+\sqrt{2}) & 1 \\ 1 & 1/_{1+\sqrt{2}} \end{bmatrix}$

$= \begin{bmatrix} \cancel{1} - \frac{1}{\sqrt{2}} - \cancel{1} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 1 - \frac{1}{(1+\sqrt{2})\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & -1 \\ -1 & \sqrt{2} - \frac{1}{1+\sqrt{2}} \end{bmatrix}$

$= \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & -1 \\ -1 & \frac{\sqrt{2}(1+\sqrt{2}) - 1}{1+\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$

Ex 3.2.22 cont

so $Q = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$

$R = Q^T A = \begin{bmatrix} -\sqrt{2} & \tilde{a}_{12} \\ 0 & \tilde{a}_{22} \end{bmatrix}$ w/ $\begin{pmatrix} \tilde{a}_{12} \\ \tilde{a}_{22} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$= \frac{1}{\sqrt{2}}\begin{pmatrix} -5 \\ 1 \end{pmatrix}$

$\Rightarrow R = \begin{bmatrix} -\sqrt{2} & -5/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} -2 & -5 \\ 0 & 1 \end{bmatrix}$

To make R all positive mult by $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

Then

$Q = \hat{Q} D = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

& $R = D\hat{R} = \frac{1}{\sqrt{2}}\begin{bmatrix} 2 & -5 \\ 0 & 1 \end{bmatrix}$

Ex 3.2.23

$Q = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ Then geometrically $Q^{-1} = Q$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+bc & ab+db \\ ca+cd & cb+d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow \quad a = -d \qquad a^2+bc = 1$

$Q = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$ + $Q^{-1} = Q^T$ As reflectors are orthogonal

$Q^T Q = \begin{bmatrix} a & c \\ b & -a \end{bmatrix}\begin{bmatrix} a & b \\ c & -a \end{bmatrix} = \begin{bmatrix} a^2+c^2 & ab-ca \\ ba-ac & b^2+a^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$a^2+c^2 = 1 \qquad b = c$

call $a \Leftrightarrow c$
$c \Leftrightarrow s$ to match books notation

Then

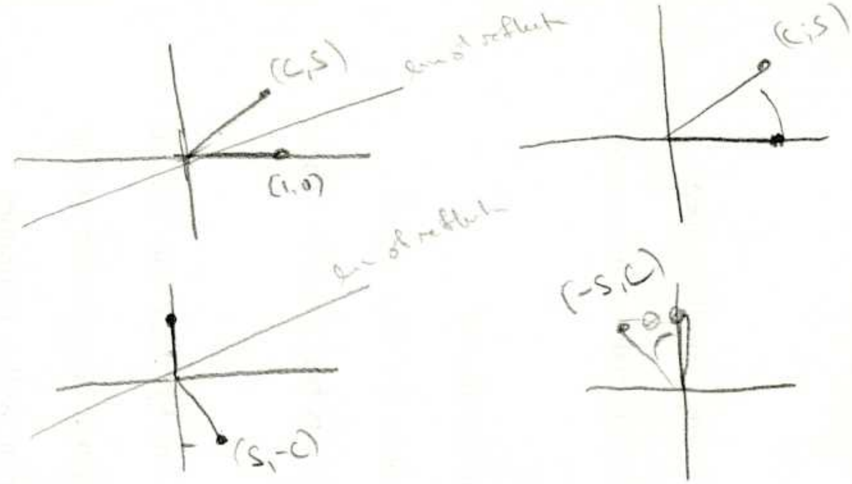
$Q = \begin{bmatrix} c & s \\ s & -c \end{bmatrix}$ w/ $c^2+s^2 = 1$

on unit vectors

$$\begin{bmatrix} c & s \\ s & -c \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} c \\ s \end{bmatrix}$$

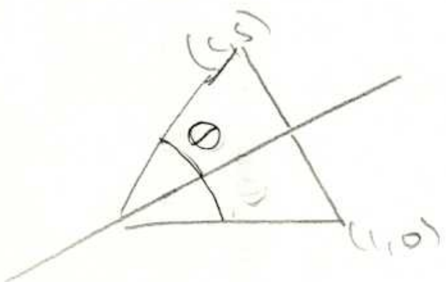
$$\begin{bmatrix} c & s \\ s & -c \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} s \\ -c \end{bmatrix}$$

V.S rotator



line of reflection must bisect

both ∡s pt (0,0) slope = geometrically bisecting 1st ∡



Slope =

Ex 3.2.24 $\quad Qx = y \quad$ w/ $\quad Q = I - ruu^T$

$$(I - ruu^T)x = y$$

$$x - ru(u^Tx) = y$$

$$x - r(u^Tx)u = y$$

$$x - y = r(u^Tx)u \implies u = \frac{x-y}{r(u^Tx)} = k(x-y)$$

So u must be a multiple of $x-y$

John Weatherwax
574199

Do Gram Schmit on $A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & -1 & 2 \\ 1 & -1 & 2 \\ -1 & 1 & 1 \end{bmatrix}$ (classical G.S.)

Let $v_1 = [1\ 1\ 1\ -1]^T$ $v_2 = [2\ -1\ -1\ 1]^T$
$v_3 = [-1\ 2\ 2\ 1]^T$

$r_{11} = \|v_1\|_2 = \sqrt{4} = 2$

Define $q_1 = \frac{v_1}{r_{11}} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$

r_{12}? $\tilde{q}_2 = v_2 - r_{12} q_1$ dotting w/ q_1 gives

$r_{12} = (v_2, q_1)$

$= 2\left(\frac{1}{2}\right) + \frac{1}{2}(-1) + \frac{1}{2}(-1) + 1\left(-\frac{1}{2}\right) = -\frac{1}{2}$

So $\tilde{q}_2 = \begin{bmatrix} 2 \\ -1 \\ -1 \\ 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix} = \begin{bmatrix} 9/4 \\ -3/4 \\ -3/4 \\ 3/4 \end{bmatrix}$

$r_{22} = \|\tilde{q}_2\|_2 = \frac{3\sqrt{3}}{2}$

$q_2 = \frac{\tilde{q}_2}{r_{22}} = \frac{2}{3\sqrt{3}} \begin{bmatrix} \cancel{9}^{3}/\cancel{4}_{2} \\ -\cancel{3}/\cancel{4}_{2} \\ -\cancel{3}/\cancel{4}_{2} \\ \cancel{3}/\cancel{4}_{2} \end{bmatrix} = \frac{1}{2\sqrt{3}} \begin{bmatrix} 3 \\ -1 \\ -1 \\ 1 \end{bmatrix}$

Now get r_{13} r_{23} →

$\tilde{q}_3 = v_3 - r_{13} q_1 - r_{23} q_2$ dot w/ q_1 1st

$0 = (v_3, q_1) - r_{13} \Rightarrow r_{13} = (v_3, q_1)$

$= -\frac{1}{2} + 1 + 1 - \frac{1}{2} = 1$

dot w/ q_2 2nd.

$0 = (v_3, q_2) - r_{23} \Rightarrow r_{23} = (v_3, q_2)$

$= \frac{1}{2\sqrt{3}}(-3 - 2 - 2 + 1) = -\frac{3}{\sqrt{3}} = -\sqrt{3}$

so $\tilde{q}_3 = \begin{bmatrix} -1 \\ 2 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix} + \sqrt{3}\frac{1}{2\sqrt{3}} \begin{bmatrix} 3 \\ -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 - \frac{1}{2} + 3/2 \\ 2 - 1/2 - 1/2 \\ 2 - 1/2 - 1/2 \\ 1 + 1/2 + 1/2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}$

$r_{33} = \|\tilde{q}_3\|_2 = \sqrt{6}$ $q_3 = \frac{\tilde{q}_3}{r_{33}} = \frac{1}{\sqrt{6}} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}$

So $A = QR$

$$\Rightarrow A = \begin{bmatrix} 1/2 & 3/2\sqrt{3} & 0 \\ 1/2 & -1/2\sqrt{3} & 1/\sqrt{6} \\ 1/2 & -1/2\sqrt{3} & 1/\sqrt{6} \\ -1/2 & 1/2\sqrt{3} & 2/\sqrt{6} \end{bmatrix} \begin{bmatrix} 2 & -1/2 & 1 \\ 0 & \frac{3\sqrt{3}}{2} & -\sqrt{3} \\ 0 & 0 & \sqrt{6} \end{bmatrix}$$

Do Modified GS on $A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & -1 & 2 \\ 1 & -1 & 2 \\ -1 & 1 & 1 \end{bmatrix}$

$v_1 = [1\ 1\ 1\ -1]^T$
$v_2 = [2\ -1\ -1\ 1]^T$
$v_3 = [-1\ 2\ 2\ 1]^T$

$r_{11}' = \|v_1\|_2 = 2$

$q_1' = \dfrac{v_1}{r_{11}'} = [1/2\ \ 1/2\ \ 1/2\ \ -1/2]^T$

dot q_1' w/ v_2 & v_3 & remove those comp.

$r_{1j}' = (v_j, q_1')$ $\quad j = 2, 3$

$r_{12}' = (v_2, q_1') = 1 - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} = -\frac{1}{2}$

$r_{13}' = (v_3, q_1') = -\frac{1}{2} + 1 + 1 - \frac{1}{2} = 1$

Then

$v_j^{(1)} = v_j - r_{1j}' q_1'$ $\quad j = 2, 3$

$$v_2^{(1)} = v_2 + \tfrac{1}{2}(q_1') = \begin{bmatrix} 2 \\ -1 \\ -1 \\ 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 2 + 1/4 \\ -1 + 1/4 \\ -1 + 1/4 \\ 1 - 1/4 \end{bmatrix} = \begin{bmatrix} 9/4 \\ -3/4 \\ -3/4 \\ 3/4 \end{bmatrix}$$

$$v_3^{(1)} = v_3 - 1 \cdot q_1' = \begin{bmatrix} -1 \\ 2 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix} = \begin{bmatrix} -3/2 \\ 3/2 \\ 3/2 \\ 3/2 \end{bmatrix}$$

$$r_{22}' = \|v_2^{(1)}\|_2 = \frac{1}{4}\sqrt{9 \cdot 9 + 9 + 9 + 9} = \frac{3}{4}\sqrt{12} = \frac{3\sqrt{3}}{2}$$

$$q_2' = \frac{v_2^{(1)}}{r_{22}'} = \frac{2}{3\sqrt{3}} \begin{bmatrix} 9/4^2 \\ -3/4^2 \\ -3/4^2 \\ 3/4^2 \end{bmatrix} = \frac{1}{2\sqrt{3}} \begin{bmatrix} 3 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

$$r_{23}' = (v_3^{(1)}, q_2')$$

$$= \frac{1}{2\sqrt{3}}\left(-9/2 - 3/2 - 3/2 + 3/2\right) = -\frac{3}{\sqrt{3}} = -\sqrt{3}$$

$$v_3^{(2)} = v_3^{(1)} - r_{23}' q_2' = \begin{bmatrix} -3/2 \\ 3/2 \\ 3/2 \\ 3/2 \end{bmatrix} + \frac{\sqrt{3}}{2\sqrt{3}} \begin{bmatrix} 3 \\ -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}$$

$$r_{33}' = \|v_3^{(2)}\|_2 = \sqrt{6} \qquad q_3' = \frac{v_3^{(2)}}{r_{33}} = \frac{1}{\sqrt{6}} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}$$

So $A = Q'R'$

$$= \begin{bmatrix} \frac{1}{2} & \frac{3}{2\sqrt{3}} & 0 \\ \frac{1}{2} & \frac{-1}{2\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{2} & \frac{-1}{2\sqrt{3}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{2} & \frac{1}{2\sqrt{3}} & \frac{2}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 2 & -1/2 & 1 \\ 0 & \frac{3\sqrt{3}}{2} & -\sqrt{3} \\ 0 & 0 & \sqrt{6} \end{bmatrix}$$

John Weathorwax

Ex 4.2 1-4, 6-9

All submitted
#6 ✓
#7 ✓
#8 ✓
#9 ✓

Ex 4.2.1

(a) $v_1, v_2 \in g_\lambda$

$$A(v_1+v_2) = Av_1 + Av_2 = \lambda v_1 + \lambda v_2 = \lambda(v_1+v_2) \Rightarrow v_1+v_2 \in g_\lambda.$$

(b) $v_1 \in g_\lambda$

$$A(cv_1) = cAv_1 = c\lambda v_1 = \lambda(cv_1) \Rightarrow cv_1 \in g_\lambda$$

4.2.2

$$\overline{p(z)} = \overline{\sum_{k=0}^{n} a_k \lambda^k} = \sum_{k=0}^{n} \overline{a_k}\,\bar{\lambda}^k = \sum_{k=0}^{n} a_k \bar{\lambda}^k \quad \text{if } a_k\text{'s} \in \mathbb{R}$$

If $p(z) = 0$

Then $\overline{p(z)} = p(\bar{z}) = \bar{0} = 0$

& If $p(\bar{z}) = 0$

$\overline{p(\bar{z})} = p(\bar{\bar{z}}) = 0 \Rightarrow p(z) = 0$

(b) $Av = \lambda v$

take complex of $Av = \lambda v$

$\bar{A}\bar{v} = \bar{\lambda}\bar{v} \qquad A \in \mathbb{R}^{n \times n}$

$\Rightarrow A\bar{v} = \bar{\lambda}\bar{v}$ i.e $\bar{v}$ is e.value for $\bar{\lambda}$ Thus $\bar{\lambda}$ is an e.value of A also.

Ex 4.2.3

(a) $\det(A - \lambda I) = 0$

$\Rightarrow \det(A - \lambda I)^T = \det(A^T - \lambda I^T) = \det(A^T - \lambda I)$

So if λ makes $\det(A - \lambda I)$ then λ makes $\det(A^T - \lambda I) = 0$

(b) λ e.value of A iff $\bar{\lambda}$ e.value ~~of~~ A^*

Pf.

$\det(A - \lambda I) = 0$

$\overset{\text{iff}}{\Leftrightarrow} \det(A - \lambda I)^*$

$\overset{\text{iff}}{\Leftrightarrow} \det(A^* - (\lambda I)^*) \Leftrightarrow \det(\,\,- \bar{\lambda} I) = 0$

Thus λ is evalue of A iff $\bar{\lambda}$ evalue of A^*.

Ex 4.2.4

(a) Thm 4.2.2. (λ e.value of A iff $\mathcal{N}(\lambda I - A) \neq \{0\}$)

If A is upper Δ. then

$$(\lambda I - A) = \begin{bmatrix} \lambda - a_{11} & \text{---} & \text{---} \\ & \lambda - a_{22} & \text{---} \\ & & \ddots \\ & & & \lambda - a_{nn} \end{bmatrix}$$

for this matrix to have non zero Null space $\mathcal{N}(\lambda I - A)$ must have

$\lambda - a_{11}, \lambda - a_{22} \cdots\cdots \lambda - a_{nn} = 0$ i.e. $\lambda = a_{11}, a_{22} \text{———} a_{nn}$

$\therefore$ eivalue are $a_{11} \cdots a_{nn}$

(b) $\det(\lambda I - A) = (\lambda - a_{11})(\lambda - a_{22}) \cdots (\lambda - a_{nn}) = 0$

$\Rightarrow \lambda = a_{11}, \cdots, a_{nn}$ $\therefore$ evalues are $a_{11} \cdots\text{—} a_{nn}$

Ex 4.2.6

$I \in \mathbb{C}^{n \times n}$

$\det(I - \lambda I) = (1-\lambda)^n = 0$. ✓

$\Rightarrow \lambda = 1$. only one eigenvalue repeated n times.

Solve $(I - I)v = 0$ thus any vector is a eigen vector

so pick $e_1, \ldots, e_n$ As eigenvectors (standard basis).

Ex 4.2.7

(a) $\det(A - \lambda I) = \det\begin{pmatrix} -\lambda & 1 \\ 0 & -\lambda \end{pmatrix} = \lambda^2 - 0 = 0 \Rightarrow \lambda = 0$

Solve $(A - 0I)v = 0$

$Av = 0$

$\Rightarrow v_2 = 0$ v_1 arb $\Rightarrow$ vector space $\langle (1,0) \rangle$ but no other vector

$\therefore$ A is defective

(b) Zero matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\det(A - \lambda I) = \lambda^2 - 0 = 0 \Rightarrow \lambda = 0$

+ e.vectors are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ + $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

(c) Show matrix $\begin{bmatrix} c & 1 & & \\ & c & 1 & \\ & & \ddots & 1 \\ & & & c \end{bmatrix}$ is defective

As this matrix is upper Δ we have c as only eigenvalue

get $v \ni$

$$\begin{bmatrix} c & 1 & & \\ & c & 1 & \\ & & \ddots & 1 \\ & & & c \end{bmatrix} v = cv$$

$\Rightarrow$ Solve $\begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & \ddots & 1 \\ & & & & 0 \end{bmatrix} v = 0$

$v = \begin{bmatrix} v_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ v_1 arb all rest must be zero $\therefore$ A is defective.

4.28

$$A = \begin{bmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & \ddots & \ddots & \\ & & & & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & & -a_{n-1} \end{bmatrix}$$

✓

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda & -1 & & & & \\ & \lambda & -1 & & 0 & \\ & & \lambda & -1 & & \\ & 0 & & \ddots & -1 & \\ & & & & \lambda & -1 \\ a_0 & a_1 & a_2 & \cdots & a_{n-2} & \lambda + a_{n-1} \end{bmatrix}_{n \times n} = 0$$

expand about bottom row

$$\Rightarrow a_0 \cancel{(-1)}^{n+1} \cancel{(-1)}^{n-1} + a_1 \cancel{(-1)}^{n+2} \cdot \cancel{(-1)}^{n-2} \lambda + a_2 \cancel{(-1)}^{n+3} \lambda^2 \cancel{(-1)}^{n-3} + \cdots + a_{n-2} \cancel{(-1)}^{n+(n-1)} \cancel{(-1)} \lambda^{n-2}$$

$$+ (\lambda + a_{n-1})(-1)^{2n} \lambda^{n-1}$$

$$= a_0 + a_1 \lambda + a_2 \lambda^2 + \cdots + a_{n-2} \lambda^{n-2} + a_{n-1} \lambda^{n-1} + \lambda^n.$$

Ex 4.2.9 If λ evalue of $A = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & & \\ & & & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix}$

(a)

λ sat.

$a_0 + a_1\lambda + \cdots a_{n-1}\lambda^{n-1} + \lambda^n = 0.$

so

$$A \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \\ \lambda^{n-2} \\ \lambda^{n-1} \end{bmatrix} = \begin{bmatrix} \lambda \\ \lambda^2 \\ \lambda^3 \\ \vdots \\ \lambda^{n-1} \\ -a_0 - a_1\lambda - a_2\lambda^2 \cdots - a_{n-1}\lambda^{n-1} \end{bmatrix} = \begin{bmatrix} \lambda \\ \lambda^2 \\ \vdots \\ \lambda^{n-1} \\ \lambda^n \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{n-1} \end{bmatrix}$$

$\therefore$ $[1 \ \lambda \ \cdots \ \lambda^{n-1}]$ is evector w/ evalue λ.

(b) Let v satisfy $(\lambda I - A)v = 0.$

$$\Rightarrow \begin{bmatrix} \lambda & -1 & & & \\ & \lambda & -1 & & \\ & & \lambda & -1 & \\ & & & \lambda & -1 \\ a_0 & a_1 & a_2 & a_3 \cdots & \lambda + a_{n-1} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = 0.$$

$\lambda v_1 - v_2 = 0 \Rightarrow v_2 = \lambda v_1$

$\lambda v_2 - v_3 = 0 \Rightarrow v_3 = \lambda v_2 = \lambda^2 v_1$

$\vdots$

$\lambda v_i - v_{i+1} = 0 \Rightarrow v_{i+1} = \lambda v_i \Rightarrow v_i = \lambda^{i-1} v_1 \qquad 1 \le i \le n-1$

$\lambda v_{n-1} - v_n = 0 \Rightarrow v_n = \lambda v_{n-1}$

$a_0 v_1 + a_1 v_2 + \cdots \quad a_{n-2} v_{n-1} + (\lambda + a_{n-1}) v_n = 0.$

$= a_0 v_1 + a_1 \lambda v_1 + \cdots \quad a_{n-2}\lambda^{n-2} v_1 + (\lambda + a_{n-1})\lambda^{n-1} v_1 = 0.$

$\Rightarrow v_1 (a_0 + a_1\lambda + \cdots \; a_{n-2}\lambda^{n-2} + a_{n-1}\lambda^{n-1} + \lambda^n) = 0$ as λ sat $p(\lambda) = 0$

So evector is $\vec{v} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} v_1 \\ \lambda v_1 \\ \vdots \\ \lambda^{n-1} v_1 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{n-1} \end{bmatrix}$ i.e. all

evectors are multiples of $[1 \; \lambda \; \cdots \; \lambda^{n-1}]^T$

Ex 5.6.1

(a) $Ax = \lambda Bx$

B^{-1} $\quad B^{-1}Ax = \lambda x$

(b) $AB^{-1}(Bx) = \lambda(Bx)$

(c) $B^{-1}A \approx AB^{-1}$

$B(B^{-1}A)B^{-1} = AB^{-1}$

$\therefore$ $B^{-1}A$ is sim to AB^{-1}

(d) $\det(\lambda B - A) = 0$ $\quad \det B \neq 0.$

$\det(B(\lambda I - B^{-1}A)) = 0$

$\det B \cdot \det(\lambda I - B^{-1}A) = 0$ char. eq for $B^{-1}A$.

$$\det B^{-1} \cdot \det(\lambda B - A)$$

$$\Rightarrow \det((\lambda B - A)B^{-1}) = 0$$

$$\Rightarrow \det(\lambda I - AB^{-1}) = 0.$$

John Weatherwax

Ex 6.1.1 $A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$ w $Q = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$

1. ✓
2. *
3. *
4. *
5. ✓

$$Q^T A Q = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} a & b \\ b & d \end{bmatrix} \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$$

$$= \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} ac+bs & -as+bc \\ bc+ds & -bs+dc \end{bmatrix}$$

$$= \begin{bmatrix} ac^2+cbs+sbc+ds^2 & -asc+bc^2-bs^2+sdc \\ -sac-bs^2+bc^2+cds & as^2-sbc-cbs+dc^2 \end{bmatrix}$$

$$= \begin{bmatrix} c^2a+s^2d+2csb & (c^2-s^2)b+cs(d-a) \\ (c^2-s^2)b+cs(d-a) & c^2d+as^2-2csb \end{bmatrix}$$

Q^TAQ diag iff $(c^2-s^2)b+cs(d-a) = 0$ (1)

$+$ $(c^2-s^2)b+cs(d-a) = 0$ (2)

w/ $\hat{t} = \frac{2b}{a-d}$ eq (1) becomes $(c^2-s^2)\hat{t} \quad -2cs = 0$

iff $\hat{t} = \frac{2cs}{(c^2-s^2)}$

& eq (2) becomes $\frac{(c^2-s^2)}{a-d}2b + 2cs(-1) = 0$

$\Rightarrow (c^2-s^2)\hat{t} - 2sc = 0$

iff $\hat{t} = \frac{2sc}{(c^2-s^2)}$

<u>Ex 6.1.2</u>

(a) Show given identity $\frac{\sin 2\theta}{1+\cos 2\theta} = \frac{2\sin\theta\cos\theta}{1+\cos^2\theta - \sin^2\theta} = \frac{2\sin\theta\cancel{\cos\theta}}{2\cos^{\cancel{2}}\theta} = \tan\theta$.

(b) $\theta \in (-\pi/4, \pi/4)$

$\cos 2\theta = \frac{1}{\sqrt{1+\tan^2 2\theta}}$ & $\sin 2\theta = \cos 2\theta \tan 2\theta$

Pf.

$1+\tan^2 2\theta = \sec^2 2\theta$

$\frac{1}{\sqrt{\sec^2 2\theta}} = \frac{1}{|\sec 2\theta|} = \frac{1}{\sec 2\theta}$ As $-\frac{\pi}{2} \le 2\theta \le \frac{\pi}{2}$ & $\cos 2\theta > 0$ then

$$= \cos 2\theta \quad \checkmark$$

Pf II

$$\cos 2\theta \tan 2\theta = \cos 2\theta \frac{\sin 2\theta}{\cos 2\theta} = \sin 2\theta \quad \checkmark$$

(c) so $\tan\theta = \dfrac{\sin 2\theta}{1+\cos 2\theta} = \dfrac{\cos 2\theta \tan 2\theta}{1 + \dfrac{1}{\sqrt{1+\tan^2 2\theta}}}$

$$= \frac{\cos 2\theta \tan 2\theta \sqrt{1+\tan^2 2\theta}}{1+\sqrt{1+\tan^2 2\theta}}$$

$$= \frac{\cancel{\cos 2\theta} \cancel{\sec 2\theta} \tan 2\theta}{1+\sqrt{1+\tan^2 2\theta}} = \frac{\tan 2\theta}{1+\sqrt{1+\tan^2 2\theta}}$$

w/ $\tan 2\theta = \hat{t}$

$\tan\theta = t$ $\Rightarrow$ $\dfrac{\hat{t}}{1+\sqrt{1+\hat{t}^2}} = t$

Ex 6.1.3:

from $\hat{t} = \frac{2b}{a-d} \Rightarrow (a-d)\hat{t} = 2b$

$$a\hat{t} - d\hat{t} = 2b$$

$$\frac{a\hat{t} - 2b}{\hat{t}} = d$$

So $c^2 a + s^2 d + 2csb$

$$= c^2 a + s^2\left[a - \frac{2b}{\hat{t}}\right] + 2csb$$

$$= a - \frac{2bs^2}{\hat{t}} + 2csb = a + b\left[-\frac{\cancel{2}s^2}{\frac{\cancel{2}cs}{(c^2-s^2)}} + 2cs\right]$$

$$= a + b\left[\left(\frac{s}{c}\right)(s^2 - c^2) + 2cs\right]$$

$$= a + b\left[\frac{s^3}{c} - sc + 2cs\right]$$

$$= a + b\left[s^3 + cs\right] = a + b\left(\frac{s}{c}\right)\left[s^2 + c^2\right]$$

$$= a + bt$$

From $\hat{t} = \dfrac{2b}{a-d}$

$$a = \frac{2b + d\hat{t}}{\hat{t}} = \frac{2b}{\hat{t}} + d$$

Then $c^2 d + s^2 a - 2csb$

$$c^2 d + \frac{2bs^2}{\hat{t}} + ds^2 - 2csb$$

$$= d - b\left[\frac{-2s^2}{\frac{2sc}{c^2-s^2}} + 2cs\right] = d - bt$$ from other pg.

Ex 6.1.4

(a) $$\frac{1}{\sqrt{1+\cot^2 2\theta}} = \frac{1}{|\csc 2\theta|} = \frac{1}{\csc 2\theta} \quad \theta > 0$$

$$= \frac{-1}{\csc 2\theta} \quad \theta < 0$$

$\Rightarrow$

$$= \begin{cases} \sin 2\theta & \theta > 0 \\ -\sin 2\theta & \theta < 0 \end{cases}$$

$$\Rightarrow \quad \sin 2\theta = \begin{cases} \dfrac{1}{\sqrt{1+\cot^2 2\theta}} & \theta > 0 \\ \dfrac{-1}{\sqrt{1+\cot^2 2\theta}} & \theta < 0 \end{cases}$$

$$\sin 2\theta \cot 2\theta = \sin 2\theta \frac{\cos 2\theta}{\sin 2\theta} = \cos 2\theta$$

(b) know $\tan\theta = \dfrac{\sin 2\theta}{1+\cos 2\theta} = \dfrac{\sin 2\theta}{1+\sin 2\theta \cot 2\theta} =$

If $\theta \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$ & $\theta > 0$

$\Rightarrow \sin 2\theta = \dfrac{1}{\sqrt{1+\cot^2 2\theta}}$ & $\cot 2\theta > 0$ if $t = \cot 2\theta$

$\operatorname{sign}(t) = +1$

If $\theta \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$ & $\theta < 0$

$\sin 2\theta = \dfrac{-1}{\sqrt{1+\cot^2 2\theta}}$ & $\cot 2\theta < 0$ As $\sin 2\theta < 0$ & $2\theta < 0$

So $\operatorname{sign}(t) = -1$

$\therefore \sin 2\theta = \dfrac{\operatorname{sign}(\cot 2\theta)}{\sqrt{1+\cot^2 2\theta}}$

$$\therefore \tan\theta = \frac{\dfrac{\operatorname{sign}(\cot 2\theta)}{\sqrt{1+\cot^2 2\theta}}}{1 + \dfrac{\operatorname{sign}(\cot 2\theta)}{\sqrt{1+\cot^2 2\theta}} \cdot \cot 2\theta} \cdot \frac{\sqrt{1+\cot^2 2\theta}}{\sqrt{1+\cot^2 2\theta}}$$

$$\Rightarrow \tan\theta = \frac{\operatorname{sign}(\cot 2\theta)}{\sqrt{1+\cot^2 2\theta} + \operatorname{sign}(\cot 2\theta)\cot 2\theta}$$

w/ $\hat{k} = \cot 2\theta$

$$\Rightarrow \tan\theta = \frac{\operatorname{sign}(\hat{k})}{\sqrt{1+\hat{k}^2} + |\hat{k}|}$$

Ex 6.1.5

(a) $\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ using 6.1.2, 6.1.5, + 6.1.7.

$$\hat{t} = \frac{2(1)}{2-3} = -2 \qquad (6.1.2)$$

$$t = \frac{\hat{t}}{1+\sqrt{1+\hat{t}^2}} = \frac{-2}{1+\sqrt{5}} = \frac{-2(1-\sqrt{5})}{(1-5)} = \frac{(1-\sqrt{5})}{2} \qquad (6.1.5)$$

So $$Q^T A Q = \begin{bmatrix} a+tb & 0 \\ 0 & d-tb \end{bmatrix} = \begin{bmatrix} 2+\frac{(1-\sqrt{5})}{2}(1) & 0 \\ 0 & 3-\frac{(1-\sqrt{5})}{2}(1) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5-\sqrt{5}}{2} & 0 \\ 0 & \frac{6-1+\sqrt{5}}{2} \end{bmatrix} = \begin{bmatrix} \frac{5-\sqrt{5}}{2} & 0 \\ 0 & \frac{5+\sqrt{5}}{2} \end{bmatrix}$$

Check $\lambda^2 - \text{trac}(A)\lambda + \det A = 0$ for 2×2's

$$\lambda^2 - 5\lambda + 5 = 0$$

$$\lambda = \frac{5 \pm \sqrt{25 - 4(5)}}{2} = \frac{5 \pm \sqrt{5}}{2}$$

(b) $\begin{bmatrix} 5 & 6 \\ 6 & 1 \end{bmatrix}$

$$\hat{t} = \frac{2b}{a-d} = \frac{2(6)}{5-1} = 3$$

$$t = \frac{\hat{t}}{1+\sqrt{1+\hat{t}^2}} = \frac{3}{1+\sqrt{1+3^2}} = \frac{3}{1+\sqrt{10}} = \frac{3(1-\sqrt{10})}{1-10} = \frac{-1}{3}(1-\sqrt{10})$$

So $$Q^T A Q = \begin{bmatrix} a+tb & 0 \\ 0 & d-tb \end{bmatrix} = \begin{bmatrix} 5 + \frac{-1}{\cancel{3}}(1-\sqrt{10})\cancel{6}^{2} & 0 \\ 0 & 1 + +\frac{1}{\cancel{3}}(1-\sqrt{10})\cancel{6}^{2} \end{bmatrix}$$

$$= \begin{bmatrix} 5 - 2 + 2\sqrt{10} & 0 \\ 0 & 1 + 2 - 2\sqrt{10} \end{bmatrix} = \begin{bmatrix} 3 + 2\sqrt{10} & 0 \\ 0 & 3 - 2\sqrt{10} \end{bmatrix}.$$

Check: $\lambda^2 - \operatorname{tr} A \, \lambda + \det(A) = 0$

$$\lambda^2 - 6\lambda + (5 - 36) = 0$$

$$\lambda^2 - 6\lambda - 31 = 0$$

$$\lambda = \frac{6 \pm \sqrt{36 - 4(1)(-31)}}{2} = \frac{6 \pm \sqrt{36 + 124}}{2}$$

$$= \frac{6 \pm \sqrt{160}}{2} = \frac{6 \pm 4\sqrt{10}}{2} = 3 \pm 2\sqrt{10}$$

(c) for (a) & (b) using (6.1.8) $t = \dfrac{\operatorname{sign}(\hat{t})}{|\hat{k}| + \sqrt{1 + \hat{t}^2}}$

$$\hat{t} = \frac{a - d}{2b}$$

for (a) $= \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$

For (a) $\hat{t} = \frac{2-3}{2(1)} = \frac{-1}{2}$

$$t = \frac{\operatorname{sign}(\hat{t})}{|\hat{t}| + \sqrt{1+\hat{t}^2}} = \frac{-1}{(1/2) + \sqrt{1+(1/4)}} = \frac{-1}{1/2 + \frac{\sqrt{5}}{\sqrt{4}}}$$

$$= \frac{-1}{\frac{1+\sqrt{5}}{2}} = \frac{-2}{1+\sqrt{5}} = \frac{-2(1-\sqrt{5})}{1-5} = \frac{1-\sqrt{5}}{2}$$

Then

$$Q^T A Q = \begin{bmatrix} a+tb & 0 \\ 0 & d-tb \end{bmatrix} = \begin{bmatrix} 2+\left(\frac{1-\sqrt{5}}{2}\right)1 & 0 \\ 0 & 3-\left(\frac{1-\sqrt{5}}{2}\right)(1) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4+1-\sqrt{5}}{2} & 0 \\ 0 & \frac{6-1+\sqrt{5}}{2} \end{bmatrix} = \begin{bmatrix} \frac{5-\sqrt{5}}{2} & 0 \\ 0 & \frac{5+\sqrt{5}}{2} \end{bmatrix}$$ Same!!

for (b) $A = \begin{bmatrix} 5 & 6 \\ 6 & 1 \end{bmatrix}$ $\qquad t = \dfrac{\text{sign}(\hat{k})}{|\hat{k}| + \sqrt{1 + |\hat{k}|^2}}$

$$\hat{k} = \frac{a-d}{2b} = \frac{5-1}{2(6)} = \frac{4}{2 \cdot 6} = \frac{1}{3}$$

$$t = \frac{+1}{(1/3) + \sqrt{1 + 1/9}} = \frac{1}{\frac{1+\sqrt{10}}{3}} = \frac{3}{1+\sqrt{10}} = \frac{3(1-\sqrt{10})}{1-10}$$

$$= \frac{-(1-\sqrt{10})}{3} = \frac{\sqrt{10}-1}{3}$$

so $$Q^T A Q = \begin{bmatrix} 5 + \frac{(\sqrt{10}-1)}{\cancel{3}}(\cancel{6})^{2} & 0 \\ 0 & 1 - \frac{(\sqrt{10}-1)}{\cancel{3}}(\cancel{6})^{2} \end{bmatrix}$$

$$= \begin{bmatrix} 5-2+2\sqrt{10} & 0 \\ 0 & 1-2\sqrt{10}+2 \end{bmatrix} = \begin{bmatrix} 3+2\sqrt{10} & 0 \\ 0 & 3-2\sqrt{10} \end{bmatrix}$$ same ✓

(7.17)

(a) $A = U\Sigma V^T$

$A^T = V\Sigma^T U^T = V\Sigma U^T$

$A^T A = V\Sigma U^T U \Sigma V^T$

$= V\Sigma^2 V^T$

$AA^T = V\Sigma U^T U\Sigma V^T = V\Sigma^2 V^T$

(b) $v_1 \cdots v_m$ $\quad V = [v_1, \cdots, v_m]$

$A^T A v_j = V\Sigma^2 V^T v_j$

But $V^T \begin{pmatrix} v_{1j} \\ \vdots \\ v_{mj} \end{pmatrix} =$ row j of V^T

$= \begin{pmatrix} v_{11} & \cdots & v_{m1} \\ \vdots & & \vdots \\ v_{1m} & & v_{mm} \end{pmatrix}^T \begin{pmatrix} v_{1j} \\ \vdots \\ v_{mj} \end{pmatrix}$

$\cancel{A = U\Sigma V^{-1}}$

$$= \begin{pmatrix} v_{11} & & v_{1m} \\ \vdots & & \\ v_{m1} & \cdots & v_{mn} \end{pmatrix} \begin{pmatrix} v_{1j} \\ \vdots \\ v_{mj} \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow j\text{th} = \hat{e}_j$$

$$\therefore A^T A \vec{v}_j = V \Sigma^2 \hat{e}_j = V \sigma_j^2 \hat{e}_j = \sigma_j^2 V \hat{e}_j$$

$$= \sigma_j^2 \vec{v}_j$$

$\therefore$ $\vec{v}_j$ is an evector of $A^T A$ w/ evalue σ_j^2

$AA^T \vec{v}_j = V \Sigma^2 V^T \vec{v}_j$

But $V^T \vec{v}_j = \hat{e}_j$

& $\Sigma^2 \hat{e}_j = \sigma_j^2 \hat{e}_j$

$V \hat{e}_j = \vec{v}_j$ jth col of V.

$= AA^T \vec{v}_j = \sigma_j^2 \vec{v}_j$. ✓

Ex 7.1.11 $\quad A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$

$A^T A = \underset{3\times 2}{\begin{bmatrix} 1 & 2 \\ 2 & 0 \\ 0 & 2 \end{bmatrix}} \underset{2\times 3}{\begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}} = \begin{bmatrix} 5 & 2 & 4 \\ 2 & 4 & 0 \\ 4 & 0 & 4 \end{bmatrix}$

1) Gram Schmidt — I, & II.
2) Jacobi *
3) SVD *

$\cancel{\lambda_1} = \cancel{0}$ $\qquad \sigma_1^2 = 9 \qquad \sigma_2^2 = 4 \qquad \sigma_3^2 = 0.$

$$\begin{array}{ccc} 2 & 3 & 1 \\ \begin{pmatrix} 1 \\ .4 \\ .8 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \\ -.5 \end{pmatrix} & \begin{pmatrix} -1 \\ .5 \\ 1 \end{pmatrix} \end{array}$$

$$\begin{pmatrix} 5 \\ 2 \\ 4 \end{pmatrix} \leftrightarrow \begin{pmatrix} 10 \\ 4 \\ 8 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \checkmark \qquad \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

$$\frac{1}{\sqrt{5}}\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \qquad \frac{1}{3}\begin{pmatrix} \cancel{9}-2 \\ 1 \\ 2 \end{pmatrix} \checkmark$$

Same!

7.1.12 $A = \underset{1\times 2}{[3 \;\; 4]} \in \mathbb{R}^{1\times 2}$

$$\underset{1\times2 \cdot 2\times1}{AA^T} = [3 \;\; 4]\begin{bmatrix} 3 \\ 4 \end{bmatrix} = 9 + 16 = 25 = \sigma_1^2$$

$\sigma_1 = 5$

evectors of $AA^T = 25$ is $[1]$ $\qquad A = U\Sigma V^T$

$\therefore$ $\qquad AA^T = U\Sigma \underline{V^T V}\Sigma U^T$

$\qquad\qquad U\Sigma^2 U^T$

$\Rightarrow AA^T = [1][25][1]^T$. $\quad$ know $U = [1]$

want $A = U\Sigma V^T$.

$\therefore V^T = \Sigma^{-1} U^T A$

$V = A^T U(\Sigma^{-1})^T = A^T \underline{U}\Sigma^{-1}$

$\therefore V = \cancel{8} \begin{bmatrix} 3 \\ 4 \end{bmatrix}[1][1/5] = \frac{1}{5}\begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$\therefore$

$A = [3\ 4] = [1]\cdot[5]\begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$

7.1.13 $\quad A_{3\times 2} \qquad \underset{2\times 3\,\cdot\,3\times 2}{A^T A} = \underset{2\times 2}{} \quad ✓ \quad A = U\Sigma V^T$

$$A^T A = V \Sigma^2 V^T.$$

$$\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & +3 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 9+8 & 6+6+-4 \\ 6+6-4 & 4+9+4 \end{bmatrix}$$

$$= \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}$$

$$\det(A - \lambda I) = \lambda^2 - 2(17)\lambda + (17^2 - 8^2) = 0$$

$$\lambda^2 - 34\lambda + 225 = 0$$

$$\lambda = \frac{34 \pm \sqrt{(34)^2 - 4(1)(225)}}{2} = \frac{34 \pm 16}{2} = 25, 9$$

$\Rightarrow \quad \sigma_1^2 = 25 \qquad \sigma_2^2 = 9$

$\sigma_1 = 5 \qquad \sigma_2 = 3.$

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$v_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

$$\therefore \cancel{V = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}} \quad \cancel{V^S} \quad \cancel{V = [}$$

$$\therefore \quad V = \begin{bmatrix} 1/\sqrt{2} & \frac{1}{\sqrt{2}} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

Now $\quad U = \quad AV\Sigma^{-1} = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$

$$= [$$

Ex 7.1.14

$A = \begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix}$

$A = U \Sigma V^T$

$AA^T = U \Sigma^2 U^T$

$AA^T = \begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 10 & 20 \\ 20 & 40 \end{bmatrix}$

$\sigma_1^2 = 50 \qquad \sigma_2^2 = 0$

$\vec{x}_1 = \begin{pmatrix} .5 \\ 1 \end{pmatrix} \qquad \vec{x}_2 = \begin{pmatrix} 1 \\ -.5 \end{pmatrix}$

$\vec{x}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \sim \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad x_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \sim \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$\therefore \quad U = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \qquad \Sigma = \begin{bmatrix} \sqrt{50} & 0 \\ 0 & 0 \end{bmatrix}$

$V^T = \Sigma^{-1} U^T A$

$V = A^T U \Sigma^{-1} = \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \frac{1}{\sqrt{5}} \cancel{\begin{bmatrix} \ & \ \\ 0 & 0 \end{bmatrix}} = \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{50} & 0 \\ 0 & 0 \end{bmatrix}$

$$V = \frac{1}{\sqrt{5}}\begin{bmatrix} 15 & 0 \\ 5 & 0 \end{bmatrix}\begin{bmatrix} 1/\sqrt{50} & 0 \\ 0 & 0 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} 15/\sqrt{50} & 0 \\ 5/\sqrt{50} & 0 \end{bmatrix}$$

$$= \frac{\sqrt{5}\,\cancel{5}}{\cancel{\sqrt{5}}\sqrt{50}}\begin{bmatrix} 3 & 0 \\ 1 & 0 \end{bmatrix} = \frac{1}{\sqrt{10}}\begin{bmatrix} 3 & 0 \\ 1 & 0 \end{bmatrix}$$

<u>So</u>:

$$A = \frac{1}{\sqrt{5}}\left[\begin{array}{c|c} 1 & 2 \\ 2 & -1 \end{array}\right]\left[\begin{array}{c|c} \sqrt{50} & 0 \\ \hline 0 & 0 \end{array}\right]\frac{1}{\sqrt{10}}\left[\begin{array}{cc} 3 & 1 \\ \hline 0 & 0 \end{array}\right]$$ check

w/ hats

$$A = \frac{1}{\cancel{\sqrt{5}}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}\left[\overset{1}{\cancel{\sqrt{50}}}\right]\frac{1}{\cancel{\sqrt{10}}}\begin{bmatrix} 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix}$$ ✓ yes

$A = U\Sigma V^T$

$$\text{minmag}(A) = \min_{\|x\|=1} \|Ax\|_2 =$$

$\cancel{\Sigma} = \cancel{U} V^T A V$

$$= \min_{\|y\|=1}$$

7.3.4

$A = U\Sigma V^T$

$A^T = V\Sigma U^T$

$A^T A = V\Sigma^2 V^T \leftarrow$ As σ_1^2 is SVD for $A^T A$

$\|A^T \cancel{A}\|_2 = \sigma_1^2$

$\|A\|_2^2 = \sigma$ so $\kappa(A^T A) = \dfrac{\sigma_1^2}{\sigma_r^2} = \kappa(A)^2$.

(b) $M = GG^T$. Show $\|M\|_2 = \|G\|_2^2$

know $\|GG^T\|_2^2 =$

Math 324 Problems on LU Factorization Sept. 8, 1995

Due w/ 1.7.15, 16, 1

Monday.

Due: Monday, Sept. 11, 1995

1. Let

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 0 & 4 \\ -2 & 5 & -4 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix}.$$

a) Find an LU factorization of A without using row interchanges.
b) Solve $A\mathbf{x} = \mathbf{b}$ by solving i) $L\mathbf{y} = \mathbf{b}$, and ii) $U\mathbf{x} = \mathbf{y}$.

2. Find the $PA = LU$ decomposition for the matrix in Problem 1 and use it to solve $A\mathbf{x} = \mathbf{b}$.

3. Consider the tridiagonal matrix

$$A = \begin{bmatrix} a_1 & c_1 & & & & \\ b_2 & a_2 & c_2 & & & \\ & b_3 & a_3 & c_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & & & c_{n-1} \\ & & & & b_n & a_n \end{bmatrix}.$$

a) Write an efficient algorithm for reducing A to packed LU form. Assume no row or column interchanges are needed or used.

b) Determine the flop count for your program.

Math 324 **Test 1 - Sample Problems** **September 11, 1995**

Test Date: Friday, September 15. Please bring paper.
Covers: Assignments through 9/6/95.
In problems asking for a flop count, find the dominant term, such as $n^3/3$, for the number of multiplications and/or divisions.

1. Let

$$G = \begin{bmatrix} -3 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & -5 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} -3 \\ 4 \\ -4 \end{bmatrix}.$$

Solve $G\mathbf{x} = \mathbf{b}$ by the column-oriented method.

2. Let

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 4 & 1 & 1 \\ -6 & -1 & -6 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 3 \\ 5 \\ -4 \end{bmatrix}.$$

a) Find an LU factorization of A without using row interchanges.
b) Solve $A\mathbf{x} = \mathbf{b}$ by solving i) $L\mathbf{y} = \mathbf{b}$, and ii) $U\mathbf{x} = \mathbf{y}$.

3. Repeat Problem 1 but using partial pivoting. That is, in (a) find $PA = LU$, and use this to solve the system in (b).

4. Let A be an $n \times n$ matrix and let $\mathbf{x} \in \mathbf{R^n}$. Show that $\mathbf{x}^T A^T A\mathbf{x} = 0$ if and only if $A\mathbf{x} = 0$.

5. Find the flop count for the following algorithm.

For $k = n : -1 : 1$
 $b_{kk} = 1/a_{kk}$ 1 flop
 For $i = k-1 : -1 : 1$ k flops + 1
 $$b_{ik} = -\left(\sum_{j=i+1}^{k} a_{ij} b_{jk}\right)/a_{ii}$$
 end;
end;

6. Suppose the $n \times n$ matrix Q has column representation $Q = [\mathbf{q}_1 \ \ \mathbf{q}_2 \ \ \ldots \ \ \mathbf{q}_n]$. Let I be the $n \times n$ identity matrix. Show that $Q^T Q = I$ if and only if for each $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$, it is true that $\mathbf{q}_i^T \mathbf{q}_j = \delta_{ij}$, where $\delta_{ii} = 1$ and $\delta_{ij} = 0$ if $i \neq j$. *Hint.* Use block multiplication.

7. Let $\mathbf{x}$ be an $n \times 1$ vector with $\mathbf{x}^T\mathbf{x} = 1$. Define $H = I - 2\mathbf{x}\mathbf{x}^T$. Show that

a) H is symmetric,

b) $H^2 = I$,

c) $H^T H = I$.

8. (*Continuation of Problem 3 on Homework dated Sept. 8.*) Let

$$A = \begin{bmatrix} a_1 & c_1 & & & & \\ b_1 & a_2 & c_2 & & & \\ & b_2 & a_3 & c_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & & & c_{n-1} \\ & & & & b_{n-1} & a_n \end{bmatrix}.$$

a) Use the LU factorization previously obtained to write an efficient algorithm for solving $A\mathbf{x} = \mathbf{b}$ by first solving $L\mathbf{y} = \mathbf{b}$ and then solving $U\mathbf{x} = \mathbf{y}$.

b) Determine the flop count for your algorithm.

3 a.) Packed LU $= \left[\begin{array}{cc|c} -6 & -1 & -6 \\ -1/3 & 2/3 & -3 \\ -2/3 & 1/2 & -3/2 \end{array}\right]$ $P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ b.) $Ly = Pb \Rightarrow [-4 \ -5/3 \ 3/2]^T$

$Ux = y \Rightarrow [2 \ -2 \ -1]^T$

Math 324 **Test 1 - Sample Problems** **September 11, 1995**

Test Date: Friday, September 15. Please bring paper.

Covers: Assignments through 9/6/95.

In problems asking for a flop count, find the dominant term, such as $n^3/3$, for the number of multiplications and/or divisions.

1. Let

Ans. $x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$$G = \begin{bmatrix} -3 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & -5 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -3 \\ 4 \\ -4 \end{bmatrix}.$$

$\left[\begin{array}{c|cc} -3 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 5 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \\ -4 \end{bmatrix} \Rightarrow -3x_1 = -3$ $(x_1 = 1)$

and $\begin{bmatrix} 2 \\ 3 \end{bmatrix} x_1 + \begin{bmatrix} 1 & 0 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} x_2 \\ x_3 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 0 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -7 \end{bmatrix} \Rightarrow$

Solve $G\mathbf{x} = \mathbf{b}$ by the column-oriented method.

$x_2 = 2$

$4x_2 + 5x_3 = -7 \Rightarrow x_3 = 3$

2. Let

a) Packed LU $= \left[\begin{array}{c|cc} 2 & 1 & -1 \\ 2 & -1 & 3 \\ -3 & -2 & -3 \end{array}\right]$

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 4 & 1 & 1 \\ -6 & -1 & -6 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 3 \\ 5 \\ -4 \end{bmatrix}.$$

b) 1) $Ly = b \Rightarrow y = [3 \ -1 \ 3]^T$

2) $Ux = y \Rightarrow x = [2 \ -2 \ -1]^T$

a) Find an LU factorization of A without using row interchanges.

b) Solve $A\mathbf{x} = \mathbf{b}$ by solving i) $L\mathbf{y} = \mathbf{b}$, and ii) $U\mathbf{x} = \mathbf{y}$.

3. Repeat Problem 1 but using partial pivoting. That is, in (a) find $PA = LU$, and use this to solve the system in (b).

4. Let A be an $n \times n$ matrix and let $\mathbf{x} \in \mathbf{R}^n$. Show that $\mathbf{x}^T A^T A\mathbf{x} = 0$ if and only if $A\mathbf{x} = 0$.

5. Find the flop count for the following algorithm.

Let $y = Ax = [y_1 \ y_2 \cdots y_n]^T$.

$x^T A^T A x = (Ax)^T(Ax) = y^T y = y_1^2 + \cdots + y_n^2$

$= 0$ iff $y = 0$

$= 0$ iff $Ax = 0$

For $k = n : -1 : 1$
  $b_{kk} = 1/a_{kk}$
  For $i = k-1 : -1 : 1$
    $b_{ik} = -\left(\sum_{j=i+1}^{k} a_{ij} b_{jk}\right)/a_{ii}$
  end;
end;

$\sum_{i=1}^{k-1}(k-i)$

Total Flops $= \sum_{k=1}^{n}\left(1 + \sum_{i=1}^{k-1}(k-i)\right) \approx \sum_{k=1}^{n} \frac{k^2}{2} = \frac{1}{2}\cdot\frac{n^3}{3} = \frac{n^3}{6}$

6. Suppose the $n \times n$ matrix Q has column representation $Q = [\mathbf{q}_1 \ \mathbf{q}_2 \ \ldots \ \mathbf{q}_n]$. Let I be the $n \times n$ identity matrix. Show that $Q^T Q = I$ if and only if for each $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$, it is true that $\mathbf{q}_i^T \mathbf{q}_j = \delta_{ij}$, where $\delta_{ii} = 1$ and $\delta_{ij} = 0$ if $i \neq j$. *Hint.* Use block multiplication.

$Q^T Q = \begin{bmatrix} q_1^T \\ \vdots \\ q_n^T \end{bmatrix} [q_1 \cdots q_n]$

$= \begin{bmatrix} q_1^T q_1 & \cdots & q_1^T q_n \\ \vdots & & \\ q_n^T q_1 & \cdots & q_n^T q_n \end{bmatrix}$

$= I$ iff

$q_i^T q_j = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$

$= \delta_{ij}$

7. Let $\mathbf{x}$ be an $n \times 1$ vector with $\mathbf{x}^T\mathbf{x} = 1$. Define $H = I - 2\mathbf{x}\mathbf{x}^T$. Show that

a) H is symmetric, $H^T = (I - 2xx^T)^T = I^T - 2x^{TT}x^T = I - 2xx^T = H$

b) $H^2 = I$, $HH = (I - 2xx^T)(I - 2xx^T) = I - 2xx^T - 2xx^T + 4x\underbrace{(x^Tx)}_{=1}x^T = I$

c) $H^T H = I$. use (a) + (b)

8. (*Continuation of Problem 3 on Homework dated Sept. 8.*) Let

Packed LU Steps

for $i = 2:n$
  $b_i = b_i / a_{i-1}$;
  $a_i = a_i - b_i c_{i-1}$;
end

$2n-2$ flops

$$A = \begin{bmatrix} a_1 & c_1 & & & \\ b_2\ \cancel{b_1} & a_2 & c_2 & & \\ & b_3\ \cancel{b_2} & a_3 & c_3 & \\ & & \ddots & \ddots & \ddots \\ & & & & c_{n-1} \\ & & & b_n\ \cancel{b_{n-1}} & a_n \end{bmatrix}.$$

a) Use the LU factorization previously obtained to write an efficient algorithm for solving $A\mathbf{x} = \cancel{\mathbf{b}}\ d$ by first solving $L\mathbf{y} = \cancel{\mathbf{b}}\ d$ and then solving $U\mathbf{x} = \mathbf{y}$.

b) Determine the flop count for your algorithm.

Solve $Ly = d$ $(d \leftarrow y)$

for $i = 2:n$
  $d_i = d_i - b_i d_{i-1}$;
end;

$n-1$ flops

Solve $Ux = y$ $(= d)$ $(d \leftarrow x)$

$d_n = d_n / a_n$;
for $i = n-1 : -1 : 1$
  $d_i = (d_i - c_i d_{i+1})/a_i$;
end

$2n-1$ flops

Math 324 Handout

(1)

(a) $A = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 0 & 4 \\ -2 & 5 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix}$

$m_{21} = 4/2 = 2$

$m_{31} = -2/2 = -1$

$$\Rightarrow \begin{bmatrix} 2 & -1 & 3 \\ 2 & 2 & -2 \\ -1 & 4 & -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & -1 & 3 \\ 2 & 2 & -2 \\ -1 & 2 & 3 \end{bmatrix} \qquad m_{32} = 4/2 = 2$$

$$\Rightarrow A = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 0 & 4 \\ -2 & 5 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \\ 0 & 2 & -2 \\ 0 & 0 & 3 \end{bmatrix}$$

(b) Solve $A\vec{x} = \vec{b}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \\ 0 & 2 & -2 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix}$$

1st $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix} \Rightarrow$ $y_1 = 0$
$y_2 = 0$
$y_3 = -3$

Then solve

$$\begin{bmatrix} 2 & -1 & 3 \\ 0 & 2 & -2 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix}$$

$$x_3 = -1$$

$$2x_2 - 2(-1) = 0$$

$$2x_2 = -2$$

$$x_2 = -1$$

$$2x_1 - 1(-1) + 3(-1) = 0$$

$$2x_1 = -1 + 3 = 2$$

$$x_1 = 1$$

So $\vec{x} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

(2) I Assume that we are to use pivoting.

$$\begin{bmatrix} 2 & -1 & 3 \\ 4 & 0 & 4 \\ -2 & 5 & -4 \end{bmatrix} \longrightarrow \begin{bmatrix} 4 & 0 & 4 \\ 2 & -1 & 3 \\ -2 & 5 & -4 \end{bmatrix} \qquad I_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$m_{21} = 2/4 = 1/2$$

$$m_{31} = -2/4 = -1/2$$

$$\Rightarrow \begin{bmatrix} 4 & 0 & 4 \\ 1/2 & -1 & 1 \\ -1/2 & 5 & -2 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 4 & 0 & 4 \\ -1/2 & 5 & -2 \\ 1/2 & -1 & 1 \end{bmatrix} \qquad I_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 4 & 0 & 4 \\ -1/2 & 5 & -2 \\ 1/2 & -1/5 & 3/5 \end{bmatrix} \qquad m_{32} = -1/5$$

So

$PA = LU$

$$\Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & -1 & 3 \\ 4 & 0 & 4 \\ -2 & 5 & -4 \end{bmatrix}\vec{x} = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 1/2 & -1/5 & 1 \end{bmatrix}\begin{bmatrix} 4 & 0 & 4 \\ 0 & 5 & -2 \\ 0 & 0 & 3/5 \end{bmatrix}\vec{x}$$

$$LU\vec{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 1/2 & -1/5 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 0 \end{bmatrix}$$

$$\Rightarrow y_1 = 0$$
$$y_2 = -3$$
$$-\frac{1}{5}(-3) + y_3 = 0 \Rightarrow y_3 = -\frac{3}{5}$$

$$\begin{bmatrix} 4 & 0 & 4 \\ 0 & 5 & -2 \\ 0 & 0 & 3/5 \end{bmatrix}\vec{x} = \begin{bmatrix} 0 \\ -3 \\ -3/5 \end{bmatrix}$$

$$\frac{3}{\cancel{5}}x_3 = -\frac{3}{\cancel{5}}$$

$$x_3 = -1$$

$$5x_2 - 2(-1) = -3$$

$$5x_2 = -5 \Rightarrow x_2 = -1$$

$$4x_1 + 4(-1) = 0 \Rightarrow x_1 = 1$$

$$\vec{x} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$ Same as before

③

(a) Begin

for $i = 1$ to $n-1$ do

$a_{i+1,i} \leftarrow \dfrac{a_{i+1,i}}{a_{i,i}}$ 1 flop

$a_{i+1,i+1} \leftarrow a_{i+1,i+1} - (a_{i+1,i})(a_{i,i+1})$ 1 flop

End

Comments L stored in matrix A. so is U.

flop count

2 flops for $i = 1$ to $n-1$

$2(n-1)$ total flops used

Better to write algm in terms of a_i, b_i, c_i's.
You need only store the arrays $\vec{a}, \vec{b}, \vec{c}$.