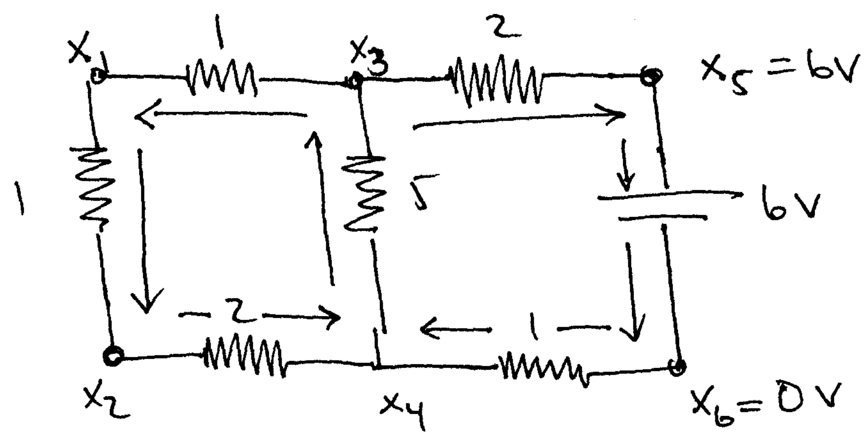
pg 15 Watkins

Ex 1.2.7



Kirchoff's current law at node now gives: Drawing an arbitrary set of consistent currents (to help me get the signs of the potentials correct)

$$-\frac{1}{1}(x_3-x_1)+\frac{1}{5}(x_4-x_3)-\frac{1}{2}(x_3-6) = 0$$

For node x_3:

or $$\frac{1}{2}(x_3-6)+(x_3-x_1)+\frac{1}{5}(x_3-x_4) = 0$$ same as in the text.

For node x_1

$$\frac{1}{1}(x_3-x_1) - \frac{1}{1}(x_1-x_2) = 0$$

or $$-2x_1 + x_2 + x_3 = 0$$ same as the 1st eq in the given system

For node x_2:

$$+\frac{1}{1}(x_1-x_2)-\frac{1}{2}(x_2-x_4)=0$$

or $x_1-\frac{3}{2}x_2+\frac{1}{2}x_4=0$ same as 2nd eq in given system

For node x_4:

$$+\frac{1}{2}(x_2-x_4)-\frac{1}{5}(x_4-x_3)+\frac{1}{1}(0-x_4)=0$$

or

$\frac{1}{2}x_2+\frac{1}{5}x_3-\left(\frac{1}{2}+\frac{1}{5}+1\right)x_4=0$ same as the 4th eq in given system

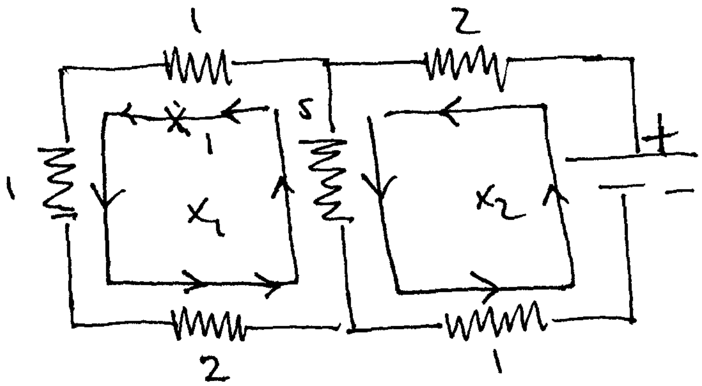
For node x_3:

$$-\frac{1}{1}(x_3-x_1)+\frac{1}{5}(x_4-x_3)-\frac{1}{2}(x_3-6)=0$$

or $-\left(1+\frac{1}{5}+\frac{1}{2}\right)x_3+x_1+\frac{1}{5}x_4=3$ or 3rd eq in given system

pg 16 watkins

Ex 1.2.9



Apply kirchoffs voltage law to this system.

For the left loop we have:

$V = IR$

$$2x_1 + 5(x_1 - x_2) + 1x_1 + 1x_1 = 0$$

or $\quad 9x_1 - 5x_2 = 0$

For the right loop we have:

$$x_2 \cdot 1 \overset{-}{\cancel{+}} 6 + 2x_2 + 5(x_2 - x_1) = 0$$

or $\quad -5x_1 + 8x_2 = +6$

Both are the same as given eqs

Since $\det(A) = 9(8) - (25) = 47 \neq 0$

this matrix is non singular

pg 19 Watkins

Ex 1.2.16

Eq 1.2.15 ~~after multiplying by h^2 gives~~ is given by

$$-\frac{1}{h^2}u_{i+1} + \frac{2}{h^2}u_i - \frac{1}{h^2}u_{i-1} + \frac{c}{2h}u_{i+1} - \frac{c}{2h}u_{i-1} + du_i = f_i$$

or grouping terms based on ~~of~~ $u_{i-1}, u_i, + u_{i+1}$

$$\left(-\frac{1}{h^2} - \frac{c}{2h}\right)u_{i-1} + \left(\frac{2}{h^2} + d\right)u_i + \left(-\frac{1}{h^2} + \frac{c}{2h}\right)u_{i+1} = f_i$$

$$\Rightarrow -\left(\frac{1}{h^2} + \frac{c}{2h}\right)u_{i-1} + \left(\frac{2}{h^2} + d\right)u_i - \left(\frac{1}{h^2} - \frac{c}{2h}\right)u_{i+1} = f_i$$

Then for $i = 1, 2, \cdots, m-1$ w/ $u_0 = 0$
$u_m = 0$

(a) with $m = 8$ $h = \frac{1}{8}$ + the above becomes

$\frac{1}{h^2} = 64$; $\frac{1}{h} = 8$

$$-(64 + 4c)u_{i-1} + (128 + d)u_i - (64 - 4c)u_{i+1} = f_i \qquad i = 1, \ldots, 7$$

So enumerating this eq for $i = 1, 2, 3, \cdots, 7$ we get

$$\begin{bmatrix} 128+d & -64+4c & & & \\ -64-4c & 128+d & -64+4c & & \\ & -64-4c & 128+d & -64+4c & \\ & & -64-4c & 128+d & -64+4c \\ & & & & \ddots \end{bmatrix}$$

This is a $m-1 \times m-1$ tridiagonal dimensional system

7x7

with main diagonal $128+d$

w/ super diagonal of $-64+4c$

& sub diagonal of $-64-4c$

(b) when $m=20$ or matrix is of size 19×19

~~w/ main diagonal~~ w/ $h = \frac{1}{20}$

$\frac{1}{h} = 20$ & $\frac{1}{h^2} = 400$

Giving a main diagonal

$d + 800$

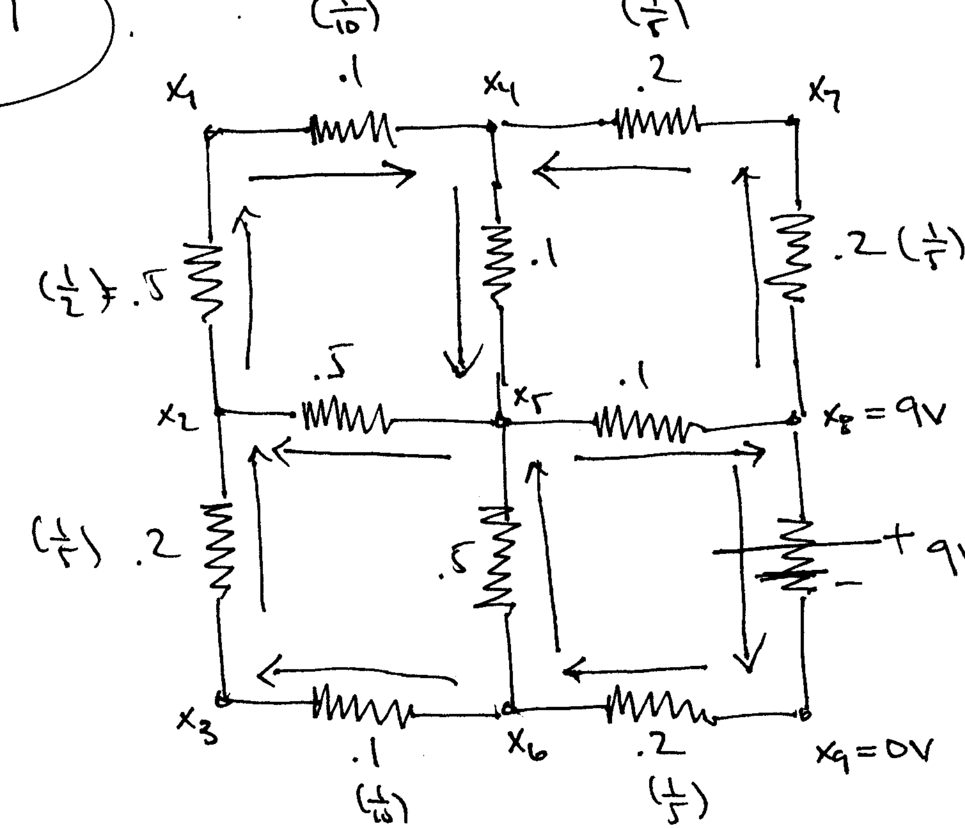
supper diagonal w/ element $-(400 - 10c)$

& subdiagonal w/ elements $-(400 + 10c)$

pg 19 heffrons

Ex 1.2.17

(a)



Assigning currents in each edge & then performing Kirchoffs current law at each Node x_i gives the following system of equations,

For x_1:

$$+2(x_2 - x_1) - 10(x_1 - x_4) = 0$$ ✓

For x_2:

$$-2(x_2 - x_1) + 2(x_5 - x_2) + 5(x_3 - x_2) = 0$$ ✓

For x_3:

$$-5(x_3 - x_2) + 10(x_6 - x_3) = 0$$ ✓

For x_4:

$$+10(x_1 - x_4) - 10(x_4 - x_5) + 5(x_7 - x_4) = 0$$

For x_5:

$$+10(x_4 - x_5) - 2(x_5 - x_2) + 10(x_5 - 9) + 2(x_6 - x_5) = 0$$

For x_6:

$$-10(x_6 - x_3) - 2(x_6 - x_5) + 5(0 - x_6) = 0$$

For x_7:

$$-5(x_7 - x_4) + 5(9 - x_7) = 0$$

Simplifying each eq we obtain the following equations:

$\underline{x_1}$: $-12x_1 + 2x_2 + 10x_4 = 0$

$\underline{x_2}$: $2x_1 - 7x_2 + 5x_3 + 2x_5 = 0$

$\underline{x_3}$: $5x_2 - 15x_3 + 10x_6 = 0$

$\underline{x_4}$: $10x_1 - 25x_4 + 10x_5 + 5x_7 = 0$

$\underline{x_5}$: $2x_2 + 10x_4 - 4x_5 + 2x_6 = 90$

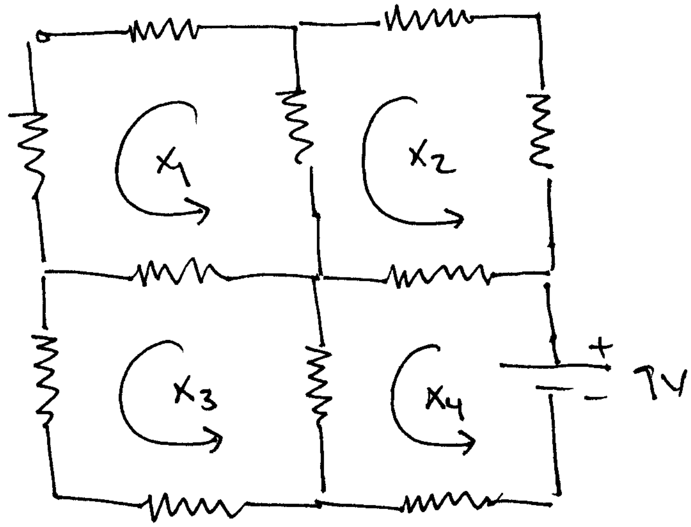
$\underline{x_6}$: $10x_3 + 2x_5 - 17x_6 = 0$

$\underline{x_7}$: $5x_4 - 10x_7 = -45$

$$
\begin{array}{c c}
 & \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} &
\begin{bmatrix}
-12 & 2 & 0 & 10 & 0 & 0 & 0 \\
2 & -7 & 5 & 0 & 2 & 0 & 0 \\
0 & 5 & -15 & 0 & 0 & 10 & 0 \\
10 & 0 & 0 & -25 & 10 & 0 & 5 \\
0 & 2 & 0 & 10 & -4 & 2 & 0 \\
0 & 0 & 10 & 0 & 2 & -17 & 0 \\
0 & 0 & 0 & 5 & 0 & 0 & -10
\end{bmatrix}
\end{array}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 90 \\ 0 \\ -45 \end{bmatrix}
$$

Ex 1.2.18

(a)



The sum of the voltage drops in each loop must be zero giving for

x_1:

$$.5(x_1) + .1(x_1) + .5(x_1 - x_3) + .1(x_1 - x_2) = 0$$

x_2:

$$.1(x_2 - x_1) + .1(x_2 - x_4) + .2x_2 + .2x_2 = 0$$

x_3:

$$.5(x_3 - x_1) + .2(x_3) + .1x_3 + .5(x_3 - x_4) = 0$$

x_4:

$$.5(x_4 - x_3) + .2x_4 - 9 + .1(x_4 - x_2) = 0$$

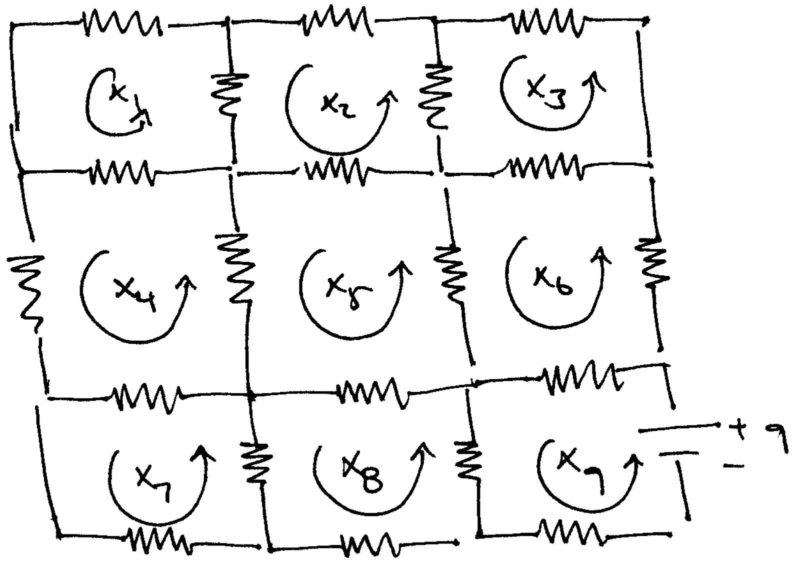
In system form

$$
\begin{aligned}
1.2x_1 - .1x_2 - .5x_3 &= 0 \\
-.1x_1 + .6x_2 \qquad - .1x_4 &= 0 \\
-.5x_1 \qquad + 1.3x_3 - .5x_4 &= 0 \\
-.1x_2 - .5x_3 + .8x_4 &= 9
\end{aligned}
$$

$$
\begin{bmatrix} 1.2 & -.1 & -.5 & 0 \\ -.1 & .6 & 0 & -.1 \\ -.5 & 0 & 1.3 & -.5 \\ 0 & -.1 & -.5 & .8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 9 \end{bmatrix}
$$

Ex 1.2.19

pg 20 batteries



x_1: $1x_1 + (x_1 - x_4) + x_1 - x_2 = 0$

x_2: $x_2 - x_1 + x_2 - x_5 + x_2 - x_3 + x_2 = 0$

x_3: $x_3 + x_3 - x_2 + x_3 - x_6 = 0$

x_4: $x_4 + x_4 - x_7 + x_4 - x_5 + x_4 - x_1 = 0$

x_5: $x_5 - x_4 + x_5 - x_8 + x_5 - x_6 + x_5 - x_2 = 0$

x_6: $x_6 - x_3 + x_6 - x_5 + x_6 - x_9 + x_6 = 0$

x_7: $x_7 - x_4 + x_7 + x_7 - x_8 = 0$

x_8: $x_8 + x_8 - x_9 + x_8 - x_5 + x_8 - x_7 = 0$

x_9: $x_9 + (x_9 - x_8) + x_9 - x_6 - 9 = 0$

$$
\begin{array}{ll}
\underline{x_1} & 3x_1 - x_2 - x_4 = 0 \\
\underline{x_2} & -x_1 + 4x_2 - x_3 - x_5 = 0 \\
\underline{x_3} & -x_2 + 3x_3 - x_6 = 0 \\
\underline{x_4} & -x_1 + 4x_4 - x_5 - x_7 = 0 \\
\underline{x_5} & -x_2 - x_4 + 4x_5 - x_6 - x_8 = 0 \\
\underline{x_6} & -x_3 - x_5 + 4x_6 - x_9 = 0 \\
\underline{x_7} & -x_4 + 3x_7 - x_8 = 0 \\
\underline{x_8} & -x_5 - x_7 + 4x_8 - x_9 = 0 \\
\underline{x_9} & -x_6 - x_8 + 3x_9 = 9
\end{array}
$$

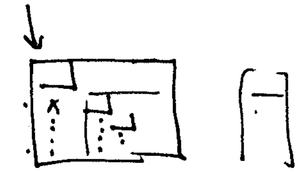
$$
\begin{array}{l}
x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9
\end{array}
\left[\begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array} \right.
$$

Giving

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 3 & 1 & -1 & 0 \\ 4 & 1 & -3 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ -1+4 \\ 3+2-3 \\ 4+2-\underbrace{9+12}_{3} \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 2 \\ 9 \end{bmatrix}$$

Ex 1.3.12 Column oriented forward substitution $Ay = b$ on entry b contains y on exit.

For $j = 1, 2, \cdots, n$
  if $A(j,j) == 0$ set error flg exit
  $b(j) = b(j)/A(j,j)$
  for $i = j+1, j+2, \cdots, n$
    $b(i) = b(i) - A(i,j)*b(j)$
  end
end



Ex 1.3.14 To count the operation count the inner for-
(a) loop takes 2 flops + is executed $n-(j+1)+1 = n-j$ times
The outer loop is executed once for every $j = 1, 2, \cdots, n$ $\therefore$ total flop count

$$\sum_{j=1}^{n} \sum_{i=j+1}^{n} 2 = \sum_{j=1}^{n} 2(n-j) = 2\sum_{j=1}^{n-1} j = 2\left(\tfrac{1}{2}\right)n(n-1) = n(n-1)$$

The same as row oriented forward substitution

Ex 1.3.11 our system is

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 3 & 1 & -1 & 0 \\ 4 & 1 & -3 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 2 \\ 9 \end{bmatrix}$$

Using column oriented forward substitution 1st solves for

$y_1 = 2/2 = 1$ + then reduces the order of the matrix by 1

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & -3 & 3 \end{bmatrix} \begin{bmatrix} y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 9 \end{bmatrix} - 1 \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix}$$ ✓

Then repeating this process gives y_2

$y_2 = 4/2 = 2$ +

$$\begin{bmatrix} -1 & 0 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$$ ✓

Now $y_3 = 3$

$[3]\, y_4 = \cancel{5 - 3(-3) = 14}\ 3 - (-3)y_3 = 3 - (-3)3 = 12$

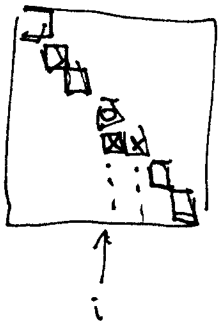
$\Rightarrow y_4 = \cancel{\frac{14}{3}}\ \frac{12}{3} = 4$

Put the solution back in to see if we get the RHS

pg 31 Watkins

Ex 1.3.24

Assume without loss of generality our matrix G is lower ~~D~~ triangle, then If $g_{ii} = 0$ then column i + $i+1$ have the same # of Nonzero elements namely $n-i$. Now column n has only 1 number in the last component. ~~+~~ Thus the span of c_n = span of e_n. The 1st + 2nd column are equivalent to the span of $e_n + e_{n-1}$. ~~Then the~~ Similarly the span of $\{c_n, c_{n-1}, \ldots, c_{n-(i+1)}\}$ equals the span of $\{e_n, e_{n-1}, \ldots, e_{n-i}\}$



But span $\{c_n, c_{n-1}, \ldots, c_{n-(i+1)}, c_{n-i}\}$ is just adding another vector of length $n-i$ to the set span of $\{e_n, e_{n-1}, \ldots, e_{n-i}\}$ + does not change the span. Continuing we see that

$\dim(\text{span}\{c_1, c_2, \ldots, c_n\}) = n-1 \neq n$.

Thus this matrix ~~cannot~~ ~~be~~ must be singular.

Ex 1.3.25

$$
\begin{array}{cccccccc}
 & 1 & 2 & 3 & \ldots & n-2 & n-1 \\
+ & n-1 & n-2 & n-3 & \ldots & 2 & 1 \\
\hline
 & n & n & n & \ldots & n & n
\end{array}
$$

$$2\sum_{k=1}^{n-1} k = n(n-1)$$

$$\therefore \sum_{k=1}^{n-1} = \frac{1}{2}n(n-1)$$

Ex 1.3.26

When $n=1$ our sum becomes

$$\sum_{i=1}^{0} i = 0 \stackrel{?}{=} 0$$

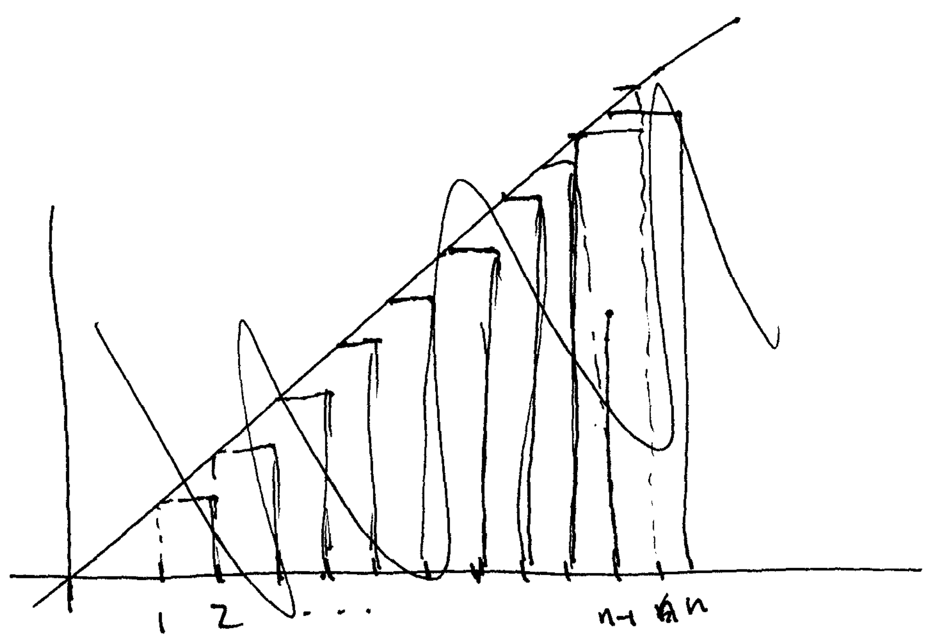
If our sum holds for $n=k$ so

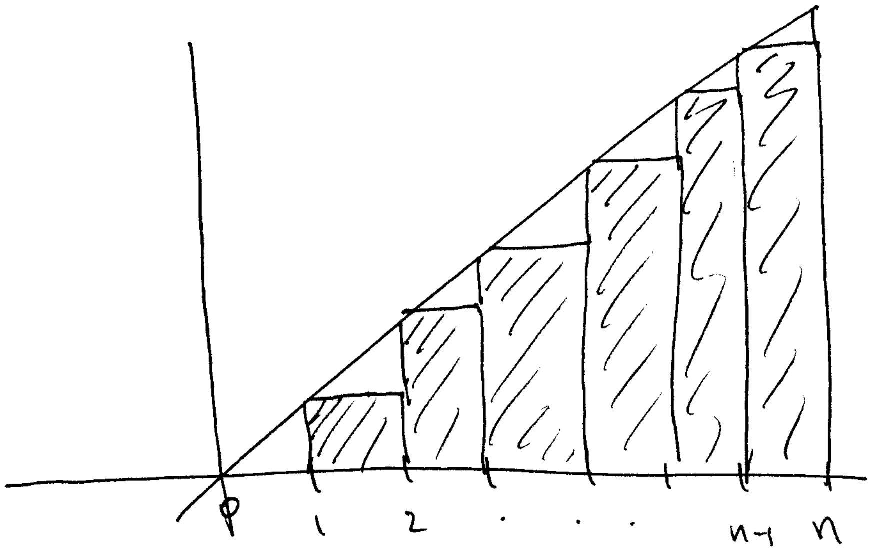
$$\sum_{i=1}^{k-1} i = \frac{k(k-1)}{2}$$

Then $\underbrace{\sum_{i=1}^{k-1} i + k}$ $= \frac{k(k-1)}{2} + k = \frac{k(k-1)}{2} + \frac{2k}{2} = \frac{k^2+k}{2} = \frac{1}{2}k(k+1)$

$$\sum_{i=1}^{k} i = \frac{1}{2}k(k+1)$$

Ex 1.3.28





Then $\sum_{i=1}^{n-1} i$ is the sum of the blocks shaded above

which is less then the area under the full Δ

i.e. $\int_0^n x\,dx$

But is greater then the area under the curve between 0 + n-1

$\therefore \quad \int_0^{n-1} x\,dx \le \sum_{i=1}^{n-1} i \le \int_0^{n} x\,dx$

or $\quad \frac{x^2}{2}\Big|_0^{n-1} \le \sum_{i=1}^{n-1} i \le \frac{x^2}{2}\Big|_0^{n}$

$\Rightarrow \quad \frac{(n-1)^2}{2} \le \sum_{i=1}^{n-1} i \le \frac{n^2}{2}$

$$\Rightarrow \quad \frac{1}{2}(n^2 - 2n + 1) \le \sum_{i=1}^{n-1} i \le \frac{n^2}{2}$$

$$\Rightarrow \quad \frac{1}{2}n^2 + O(n) \le \sum_{i=1}^{n-1} i \le \frac{1}{2}n^2 .$$

For $\sum_{i=1}^{n} i$ is just the above shifted by 1.

pg 32 watkins

Ex 1.3.29 The partitioned matrix in this case is given by

$$\begin{bmatrix} \hat{G} & 0 \\ h^T & g_{nn} \end{bmatrix} \begin{bmatrix} \hat{y} \\ y_n \end{bmatrix} = \begin{bmatrix} \hat{b} \\ b_n \end{bmatrix}$$

Which give the eqs

$$\hat{G}\hat{y} = \hat{b}$$

$$\& \quad h^T\hat{y} + g_{nn}y_n = b_n$$

Since to solve for y_n we must 1st compute $\hat{y}$. So an ~~algorithm~~ algorithm would be

(1) $\hat{y} = \hat{G}^{-1}\hat{b}$

(2) $y_n = \dfrac{b_n - h^T\hat{y}}{g_{nn}}$

with step 1 done by recursion. Below is a recursive algorithm in matlab for this partitioning.

```
y = rec_forwd_sub(A,b)
  ~~[illegible]~~
  ~~[illegible]~~:
if (length(b)==1)
  y = b/A;
  return
end
```

% to extract terms for convenience:

```
Ghat = A(1:end-1,1:end-1);
htrans = A(end,1:end-1);
gnn = A(end,end);
bhat = b(1:end-1);
bn = b(end);

%
~~[illegible]~~ = rec_fwd_sub(A(1:end-1, 1:end-1), b(1:end-1));
b(1:end-1) =
```

$$b(\text{end}) = \frac{b(\text{end}) - \cancel{\text{htrans}} \; A(\text{end},1{:}\text{end}-1) * b(1{:}\text{end}-1);}{A(\text{end},\text{end})}$$

(b) To write a Non-recursive version of the algorithm:

Given $y_1, y_2, \cdots, y_{i-1}$ we can calculate y_i as

$$y_i = \frac{b_i - h^T(y_1, y_2, \cdots, y_{i-1})}{g_{ii}}$$

$$y_i = \frac{1}{g_{ii}}\left(b_i - \sum_{k=1}^{i-1} g_{ik} y_k\right) \qquad i = 1, 2, 3, \cdots, n$$

which is row oriented forward substitution & the algorithm in 1.3.5 is what we would implement this

pg 35 Watkins

The ~~ ~~ Cholesky factor R is upper triangular

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} & \cdots & r_{1n} \\ 0 & r_{22} & r_{23} & \cdots & r_{2n} \\ 0 & 0 & r_{33} & \cdots & r_{3n} \\ \vdots & & & & \\ 0 & \cdots & & & r_{nn} \end{bmatrix}$$

So $A = R^T R$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & & & & \\ \vdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} r_{11} & 0 & 0 & \cdots & 0 \\ r_{12} & r_{22} & 0 & & \vdots \\ r_{13} & r_{23} & r_{33} & & \\ \vdots & \vdots & \vdots & & \\ r_{1n} & r_{2n} & r_{3n} & & r_{nn} \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & \cdots & r_{1n} \\ 0 & r_{22} & r_{23} & \cdots & r_{2n} \\ 0 & & & & \\ \vdots & & & & \\ 0 & \cdots & & & r_{nn} \end{bmatrix}$$

~~The 3rd row times the~~

We effectively work by rows computing ~~r_{11}, r_{12}~~. the elements $r_{11}, r_{12}, r_{13}, \ldots, r_{1n}$

Then $r_{22}, r_{23}, r_{24}, r_{25}, \ldots, r_{2n}$, $r_{33}, r_{34}, r_{35}, \ldots, r_{3n}$, etc.

After computing $r_{11}, r_{12}, r_{13}, \ldots, r_{1n}, r_{22}, r_{23}, \ldots r_{2n}$ (the 1st two rows)

we multiply the 3rd row of R^T by R. First multiply by the 3rd ~~row~~ column

giving $r_{13}^2 + r_{23}^2 + r_{33}^2 = a_{33}$ since r_{13} + r_{23} are known

From this equation we can compute ~~r_{33}~~ $r_{33} = +\sqrt{a_{33} - r_{13}^2 - r_{23}^2}$

Now with r_{33} we know the entire 3rd row of R^T. multiply this

row by all ~~rows~~ columns j of R such that $j > 3$ giving

$$a_{3j} = r_{13} r_{1j} + r_{23} r_{2j} + r_{33} r_{3j} \qquad j > 3$$

Thus $r_{3j} = \dfrac{a_{3j} - r_{13} r_{1j} - r_{23} r_{2j}}{r_{33}}$ $\quad j > 3$

To calculate the ith row of R we assume that we have the all of the i-1 th rows computed.

$$a_{ij} = r_{1i} r_{1j} + r_{2i} r_{2j} + r_{3i} r_{3j} + r_{4i} r_{4j} + \cdots + r_{i-1,i} r_{i-1,j} + r_{ii} r_{ij} \qquad j \geq i \qquad *$$

If $i = j$ the above becomes

$$a_{ii} = r_{1i} r_{1i} + r_{2i} r_{2i} + r_{3i} r_{3i} + \cdots + r_{ii} r_{ii}$$

giving to $r_{ii} = + \sqrt{a_{ii} - \sum_{k=1}^{i-1} r_{ki}^2}$ which is eq 1.4.13

Given r_{ii} use eq $*$ to solve for r_{ij} for $j \geq i$

$$r_{ij} = \left(a_{ij} - \sum_{k=1}^{i-1} r_{ki} r_{kj} \right) \Big/ r_{ii} \qquad j > i \quad j = i+1, i+2, \cdots, n$$

Ex 1.4.15

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}$$

(a) Consider $y^T A y = 4y_1^2 + 9y_2^2 > 0$ since it is the sum of squares each of which must be positive. Also A is symmetric + real $\therefore$ A is positive definite

(b) For such a simple matrix we can write out the Cholesky product

$$A = R^T R$$

~~so $r_{11}^2 = 4 \Rightarrow r_{11} = +2$~~

~~comparing the element $a_{12} = 0 = r_{11} r_{12}$~~

$$\begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix} = \begin{bmatrix} r_{11} & 0 \\ r_{12} & r_{22} \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix} = \begin{bmatrix} r_{11}^2 & r_{11} r_{12} \\ r_{11} r_{12} & r_{12}^2 + r_{22}^2 \end{bmatrix}$$

So $r_{11}^2 = 4 \Rightarrow r_{11} = 2$

From the 1st row $0 = r_{11} r_{12} = 2 r_{12} \Rightarrow r_{12} = 0$

From the second row $9 = r_{22}^2 \Rightarrow r_{22} = 3$

Thus $$\begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

If we require that the diagonal elements of R to be positive we will have the unique Cholesky decomposition. By selecting non positive terms we'll obtain other decomposition

$$\begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}$$

(d) We have 2 choices for each sign at the diagonal giving 2^n possible total choices. The one corresponding to a diagonal with all positive entries is the unique Cholesky factor.

pg 37 Watkins

Ex 1.4.16

For $i = 1, 2, \cdots, n$

~~[illegible]~~

for $k = 1, 2, \cdots, i-1$ (* Not executed when $i = 1$)

$a_{ii} = a_{ii} - a_{ki}^2$ % ~~$a_{ii} = a_{ii} - a_{ki}^2$~~

end /* k loop */

if ($a_{ii} \leq 0$) error: matrix is not positive definite

$a_{ii} = +\sqrt{a_{ii}}$

for $j = i+1, i+2, \cdots, n$ (not executed when $i = n$)

for $k = 1, 2, \cdots, i-1$ (not executed when $i = 1$)

$a_{ij} =$ ~~$a_{ij} - a_{ki} a_{kj}$~~ $a_{ij} - a_{ki} a_{kj}$

end /* k loop */

$a_{ij} = a_{ij} / a_{ii}$

end /* j loop */

end /* i loop */

~~Ex 1.4.18~~

$$A = \begin{bmatrix} 4 & -2 & 4 & 2 \\ -2 & 10 & -2 & -7 \\ 4 & -2 & 8 & 4 \\ 2 & -7 & 4 & 7 \end{bmatrix}$$

$$r_{11} = +\sqrt{a_{11}} = 2$$

$$r_{12} = \frac{a_{12}}{r_{11}} = \frac{-2}{2} = -1$$

$$r_{13} = \frac{a_{13}}{r_{11}} = \frac{4}{2} = 2$$

$$r_{14} = \frac{a_{14}}{r_{11}} = \frac{2}{2} = 1$$

$$r_{22} = +\sqrt{10 - r_{12}^2} = \sqrt{10-1} = +3$$

$$r_{23} = (a_{23} - r_{12}r_{13})/r_{22} = (-2-(-1)(2))/3 = 0$$

$$r_{24} = (a_{24} - r_{12}r_{14})/r_{22} = (-7-(-1)(1))/3 = -2$$

$$r_{33} = +\sqrt{8 - r_{13}^2 - r_{23}^2} = \sqrt{8-4-0} = 2$$

$$r_{34} = (a_{34} - r_{13}r_{14} - r_{23}r_{24})/r_{33} =$$

$$= (4 - 2(1) - 0)/2 = 1$$

$$r_{44} = \sqrt{7 - r_{14}^2 - r_{24}^2 - r_{34}^2} = \sqrt{7-1-4-1}$$

$$= 1$$

pg 38 Watkins

Ex 1.4.19

$$A = \begin{bmatrix} 4 & -2 & 4 & 2 \\ & 10 & -2 & -7 \\ & & 8 & 4 \\ & & & 7 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -1 & 2 & 1 \\ & 3 & 0 & 2 \\ & & 2 & 1 \\ & & & 1 \end{bmatrix}$$

Ex 1.4.21 Using the erasure method we have

(a)

$$A = \begin{bmatrix} 16 & 4 & 8 & 4 \\ & 10 & 8 & 4 \\ & & 12 & 10 \\ & & & 12 \end{bmatrix} \rightarrow \begin{bmatrix} 4 & 1 & 2 & 1 \\ & 3 & 2 & 1 \\ & & 2 & (10-2-2)/2 = 3 \\ & & & 1 \end{bmatrix}$$

The diagonal elements are obtained by the ~~[illegible]~~ square root of the element a_{ii} minus the square of all r_{ki}'s above this element.

The non diagonal elements r_{ij} are a_{ij} minus the product of the two r's in the columns i + j divided by the diagonal element.

Since we never had $a_{ii} \leq 0$ ~~[illegible]~~ our original A is positive definite

(b) $Ax = R^T R x = b$

let $y = Rx$

to have $R^T y = b$

$$\Rightarrow \begin{bmatrix} 4 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 2 & 2 & 2 & 0 \\ 1 & 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 32 \\ 26 \\ 38 \\ 30 \end{bmatrix}$$

$$y_1 = 8 \; ; \; y_2 = \frac{26-8}{3} = 6 \; ; \; y_3 = \frac{38 - 2(8) - 2(6)}{2} = \frac{22-12}{2} = 5$$

$$y_4 = 30 - 8 - 6 - 3(5) = 30 - 14 - 15 = 1$$

Then solve

$$Rx = y \Rightarrow \begin{bmatrix} 4 & 1 & 2 & 1 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \\ 5 \\ 1 \end{bmatrix}$$

$$\Rightarrow x_4 = 1 \; ; \; x_3 = 1 \; ; \; x_2 = 1 \; ; \; x_1 = 1$$

Ex 1.4.22 Using the erasure method we have the following attempted Cholesky factorization

$$A = \begin{bmatrix} 9 & 3 & 3 \\ 3 & 10 & 7 \\ 3 & 5 & 9 \end{bmatrix} \rightarrow$$

is Not symmetric so it ~~cannot~~ cannot be SPD.

$$B = \begin{bmatrix} 4 & 2 & 6 \\ & 2 & 5 \\ & & 29 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 3 \\ & 1 & 5-12=-7 \\ & & \sqrt{29-49-9} \end{bmatrix}$$

$= \sqrt{-29}$ which is negative

implying that ~~there can be no~~ this matrix is Not SPD.

$$C = \begin{bmatrix} 4 & 4 & 8 \\ & -4 & 1 \\ & & 6 \end{bmatrix} \to \begin{bmatrix} 2 & 2 & 4 \\ & \sqrt{-4-4} & \\ & & \end{bmatrix}$$

since the elt r_{22} cannot be calculated (since it is negative) the matrix C is Not SPD.

$$D = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ -1 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & \sqrt{-1} \end{bmatrix}$$

since the elt r_{33} cannot be calculated (since it is negative) the matrix D is Not SPD.

pg 40 Watkins

Ex 1.4.24 From the inner product formulation ~~given~~ of Cholesky's algorithm we have the following flops counts. Working from inside to outside

The for $k = 1, \cdots, i-1$ executes 2 flops on each iteration giving

$$\sum_{k=1}^{i-1} 2 = 2(i-1) \text{ flops}$$

The division by a_{ii} requires one more flop giving a total of $2i-1$ flops this loop is executed for $j = i+1, i+2, \cdots, n$ or ~~n-i~~ $n-(i+1)+1 = n-i$ iterations giving $(n-i)(2i-1)$ flops (assuming the square root ~~squareroot~~ executes in 1 flop (which is prob not true (more likely $O(100)$ flops)) we have $(n-i)(2i-1)+1$. The $a_{ii} \leftarrow a_{ii} - a_{ki}^2$ requires 2 flops + is executed $i-1$ times resulting in $2(i-1)$ flops. In total the entire inner loop requires $(n-i)(2i-1)+1+2(i-1)$ flops

executing this for $i = 1, 2, \cdots, n$ gives

$$\sum_{i=1}^{n} \left((n-i)(2i-1) + 1 + 2(i-1) \right) = n + 2\sum_{i=1}^{n-1} i + \sum_{i=1}^{n} (n-i)(2i-1)$$

$$= n + \frac{2(n)(n-1)}{2} + \cancel{n\sum_{i=1}^{n} 2i - 1 - \sum_{i=1}^{n} 2i} \sum_{i=1}^{n} (2ni - n - 2i^2 + i)$$

$$= n + n(n-1) + 2n\sum_{i=1}^{n} i - n^2 - 2\sum_{i=1}^{n} i^2 + \frac{n(n+1)}{2}$$

$$= \cancel{n} + \cancel{n^2} - \cancel{n} + 2n\frac{n(n+1)}{2} - \cancel{n^2} - 2\left(\frac{n^3}{3}\right) + \frac{n(n+1)}{2}$$

$$= n^2(n+1) - \frac{2}{3}n^3 + O(n^2) = \frac{n^3}{3} + O(n^2)$$

pg 42 Watkins

Ex 1.4.29 From Example 1.4.18

$$A = \begin{bmatrix} 4 & -2 & 4 & 2 \\ -2 & 10 & -2 & -7 \\ 4 & -2 & 8 & 4 \\ 2 & -7 & 4 & 7 \end{bmatrix} = \left[\begin{array}{c|ccc} 4 & -2 & 4 & 2 \\ \hline -2 & 10 & -2 & -7 \\ 4 & -2 & 8 & 4 \\ 2 & -7 & 4 & 7 \end{array}\right] \equiv \left[\begin{array}{c|c} a_{11} & b^T \\ \hline b & \hat{A} \end{array}\right]$$

So $r_{11} = +\sqrt{a_{11}} = \sqrt{4} = 2$

$$s^T = \frac{1}{2}(-2, 4, 2) = (-1, 2, 1)$$

$$\tilde{A} = \hat{A} - ss^T = \begin{bmatrix} 10 & -2 & -7 \\ -2 & 8 & 4 \\ -7 & 4 & 7 \end{bmatrix} - \begin{bmatrix} 1 & -2 & -1 \\ -2 & 4 & 2 \\ -1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 0 & -6 \\ 0 & 4 & 2 \\ -6 & 2 & 6 \end{bmatrix}$$

Then $r_{22} = \sqrt{9} = 3$

$$s^T = \frac{1}{3}(0, -6) = (0, -2)$$

$$\tilde{A} = \begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}$$

Then $r_{33} = \sqrt{4} = 2$

$$s^T = \frac{1}{2}(2) = 1$$

$$\tilde{A} = 2 - 1^2 = 1 \quad \therefore \quad r_{44} = 1$$

Reassembling the pieces we have

$$R = \begin{bmatrix} 2 & -1 & 2 & 1 \\ 0 & 3 & 0 & -2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Checking $R^T R \stackrel{?}{=} A$

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ -1 & 3 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ 1 & -2 & 1 & 1 \end{bmatrix} \cancel{\begin{bmatrix} 2 & 0 & 0 & 0 \\ -1 & 3 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ 1 & -2 & 1 & 1 \end{bmatrix}} \begin{bmatrix} 2 & -1 & 2 & 1 \\ 0 & 3 & 0 & -2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$-1-6$

$$= \begin{bmatrix} 4 & -2 & 4 & 2 \\ -2 & 10 & -2 & -7 \\ 4 & -2 & 8 & 4 \\ 2 & -7 & 4 & 7 \end{bmatrix}$$

yes ✓.

Ex 1.4.30

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 10 \\ 3 & 10 & 16 \end{bmatrix}$$

$r_{11} = \sqrt{1} = 1$

$s^T = \frac{1}{1}(2\ 3) = (2\ 3)$

$$\tilde{A} = \begin{bmatrix} 5 & 10 \\ 10 & 16 \end{bmatrix} - \begin{bmatrix} 4 & 6 \\ 6 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 4 & 7 \end{bmatrix}$$

$r_{22} = \sqrt{1} = 1$

$s^T = \frac{1}{1}(4) = 4$

$\tilde{A} = 7 - 16 = -9$

But $r_{33} = \sqrt{-9}$ which is imaginary showing that No Cholesky factor exists

+ A is <u>Not</u> positive definite.

Ex 1.4.31

Ex 1.4.21 is to show that A given by

$$A = \begin{bmatrix} 16 & 4 & 8 & 4 \\ 4 & 10 & 8 & 4 \\ 8 & 8 & 12 & 10 \\ 4 & 4 & 10 & 12 \end{bmatrix}$$ is positive definite. To show this

it is sufficient to ~~str~~ compute its Cholesky factor $A = R^T R$.

Using the outer product formulation for this problem we partition A as

$$A = \begin{bmatrix} a_{11} & b^T \\ b & \hat{A} \end{bmatrix} \quad \& \quad R = \begin{bmatrix} r_{11} & s^T \\ 0 & \hat{R} \end{bmatrix}$$

Giving when we equate coefficients of the product

$$r_{11} = + \sqrt{a_{11}}$$

$$s^T = r_{11}^{-1} b^T$$

$$\cancel{\hat{A} = \hat{R}^T\hat{R} + ss^T} \quad \hat{A} - ss^T = \hat{R}^T\hat{R}$$

We begin with $r_{11} = \sqrt{16} = 4$

Then $s^T = \frac{1}{4}(4\ 8\ 4) = (1\ 2\ 1)$

$$\hat{A} = \begin{bmatrix} 10 & 8 & 4 \\ 8 & 12 & 10 \\ 4 & 10 & 12 \end{bmatrix}$$

So $\hat{R}$ satisfies $\hat{R}^T\hat{R} = \hat{A} + \cancel{\frac{1}{16}}\begin{bmatrix} \cancel{16} & \cancel{32} & \cancel{16} \\ \cancel{32} & \cancel{64} & \cancel{32} \\ \cancel{16} & \cancel{32} & \cancel{16} \end{bmatrix}$

$$= \hat{A} - \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 6 & 3 \\ 6 & 8 & 8 \\ 3 & 8 & 11 \end{bmatrix}$$

continuing we have $r_{22} = +\sqrt{9} = 3$

$$s^T = \frac{1}{3}(6 \;\; 3) = (2 \;\; 1)$$

$$\hat{R}^T\hat{R} = \begin{bmatrix} 8 & 8 \\ 8 & 11 \end{bmatrix} - \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 6 & 10 \end{bmatrix}$$

so $r_{33} = 2$

& $s^T = \frac{1}{2}(6) = 3$

$$R_{44}^2 = 10 - 9 = 1 \Rightarrow r_{44} = 1$$

Since all components exist & are real the Cholesky factor of A exists & A is symmetric positive definite.

pg 42 Watkins

Ex 1.4.32

Ex 1.4.22 seeks to determine whether A, B, C, or D are positive definite. We'll do so by computing the Cholesky factor of each if it exists the matrix is positive definite

For A (since it is not symmetric) we will skip it since all the algorithms assume A must be symmetric

For $B = \begin{bmatrix} 4 & 2 & 6 \\ 2 & 2 & 5 \\ 6 & 5 & 29 \end{bmatrix}$ the outer product formulation proceeds as follows

$$r_{11} = 2$$

$$s^T = \frac{1}{2}(2\ 6) = (1\ 3)$$

Then $$\hat{R}^T\hat{R} = \begin{bmatrix} 2 & 5 \\ 5 & 29 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \end{bmatrix}[1\ 3] = \begin{bmatrix} 2 & 5 \\ 5 & 29 \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 2 & 20 \end{bmatrix}$$

$r_{22} = 1$

$$s^T = \frac{1}{1}(2) = 2$$

Then $r_{33}^2 = 20 - 4 = 16$ so $r_{33} = 4$

This matrix is positive definite. Checking

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 2 & 4 \end{bmatrix}\begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 6 \\ 2 & 2 & 5 \\ 6 & 5 & 29 \end{bmatrix}$$ yes

For $C = \begin{bmatrix} 4 & 4 & 8 \\ 4 & -4 & 1 \\ 8 & 1 & 6 \end{bmatrix}$

we have $r_{11} = 2$, then $s^T = \frac{1}{2}(4\ 8) = (2\ 4)$

$\hat{R}^T \hat{R} = \begin{bmatrix} -4 & 1 \\ 1 & 6 \end{bmatrix} - \begin{bmatrix} 4 & 8 \\ 8 & 16 \end{bmatrix} = \begin{bmatrix} -8 & -7 \\ -7 & -10 \end{bmatrix}$ then

$r_{22} = \sqrt{-8}$ which is imaginary & ∴ No cholesky factor exists &

∴ the matrix is Not ~~post~~ positive definite

For $D = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 1 \end{bmatrix}$

$r_{11} = 1$ & $s^T = \frac{1}{1}(1\ 1) = (1\ 1)$

$\hat{R}^T \hat{R} = \begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

$r_{22} = 1$ & $s^T = 1$

$r_{33}^2 = 0 - 1 = -1 < 0$ & ∴ No cholesky factorization exist &

∴ D is Not positive definite,

Ex 1.4.33

~~for i=1:n~~

~~if (a_{ii} < 0) error No cholesky factorization~~

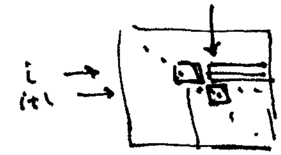
~~$a_{ii} = +\sqrt{a_{ii}}$~~

~~for j=i+1:n~~ ~~for k=i+1:n~~

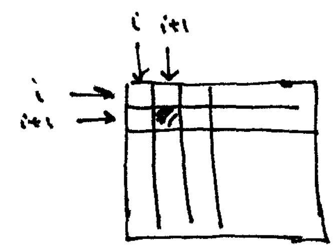
~~$a_{ij} = a_{ij}/a_{ii}$~~ ~~$a_{ik} = a_{ik}/a_{ii}$~~

~~end~~ ~~end~~

~~for k~~



```
for i=1:n        /* row loop */
  if (a_ii < 0) error No cholesky factorization your matrix is Not P.D
  a_ii = +sqrt(a_ii)     /* compute r_ii */
  /* compute s^T matrix */
  for k=i+1:n      /* column loop */
     a_ik = a_ik/a_ii
  end
  /* adjust Â submatrix by ss^T */
  for j=i+1:n    /* row loop */
    for k=j:n       /* column loop */
       a_jk = a_jk - a_ij a_ik      /* outer product update */
    end
  end
end
```



Ex 1.4.34

(a) computing s^T requires $n-(i+1)+1 = n-i$ flops for each index i.

The adjusting of $\hat{A}$ by $s s^T$ requires

$$\sum_{j=i+1}^{n} \sum_{k=j}^{n} 2 \text{ flops}$$

$$= \cancel{\sum_{j=i+1}^{n} 2(n-j)} = \cancel{2(n-i)^2} \quad \sum_{j=i+1}^{n} 2(n-j+1) = 2\sum_{j=1}^{n-i} j = 2\frac{(n-i)(n-i+1)}{2}$$

$$= (n-i)(n-i+1)$$

Thus each loop in i requires

$$1 + (n-i) + \cancel{2(n-i)^2 + 1} \; (n-i)^2 + (n-i) = 1 + 2(n-i) + (n-i)^2$$
$$= (n-i+1)^2$$

Summing for $i=1$ to n we have for the total # of flops

$$\cancel{\sum_{i=1}^{n} (1 + 2(n-i)^2 + (n-i)) = n + 2\sum_{i=1}^{n-1} i^2 + \sum_{i=1}^{n-1} i}$$

$$\cong \cancel{n + 2\left[\frac{(n-1)^3}{3}\right] + \frac{n(n+1)}{2}}$$

$$\cong \cancel{\tfrac{2}{3}n^3} \quad \sum_{i=1}^{n} (n-i+1)^2 = \sum_{i=1}^{n} i^2 \cong \frac{n^3}{3}$$

(b) Based on the pseudocode just given + the one on 1.4.17 one can see that exactly the same opperations are performed but the ordering of the for loops is different.

Ex 1.4.35 let $R = \left[\begin{array}{c|c} R_j & T_{n-j} \\ \hline 0 & V \end{array}\right]$

+ $A = \left[\begin{array}{c|c} A_j & A_{12} \\ \hline A_{12}^T & A_{n-j} \end{array}\right]$

Then $A = R^T R$

$$\Rightarrow \left[\begin{array}{c|c} A_j & A_{12} \\ \hline A_{12}^T & A_{n-j} \end{array}\right] = \left[\begin{array}{c|c} R_j & T_{n-j} \\ \hline 0 & V \end{array}\right]^T \left[\begin{array}{c|c} R_j & T_{n-j} \\ \hline 0 & V \end{array}\right]$$

$$= \left[\begin{array}{c|c} R_j^T & 0 \\ \hline T_{n-j}^T & V^T \end{array}\right] \left[\begin{array}{c|c} R_j & T_{n-j} \\ \hline 0 & V \end{array}\right]$$

$$= \left[\begin{array}{c|c} R_j^T R_j & R_j^T T_{n-j} \\ \hline T_{n-j}^T R_j & T_{n-j}^T T_{n-j} + V^T V \end{array}\right]$$

Then $A_j = R_j^T R_j \quad A_{12} = R_j^T T_{n-j}$

$+ \quad A_{n-j} = T_{n-j}^T T_{n-j} + v^T v$

Thus we see from the 1st eq ~~[illegible]~~ $A_j = R_j^T R_j$ that R_j is the cholesky factorization of A_j

... looks like the partition in ~~[illegible]~~. Eq 1.4.36 is better
j should be the principal sub matrix's dimension

Ex 1.4.38

In Example 1.4.18 we have

$$A = \begin{bmatrix} 4 & -2 & 4 & 2 \\ -2 & 10 & -2 & -7 \\ 4 & -2 & 8 & 4 \\ 2 & -7 & 4 & 7 \end{bmatrix}$$ Using the bordered form of Cholesky's method

~~we have~~ $\cancel{R_1 = 4}$ ~~so~~ $r_{11} = \sqrt{4} = 2$ we have $R_1 = [r_{11}] = [2]$

~~now since~~ Now partition the leading principal submatrices of $\underline{\underline{A_2}}$ as

~~Then by partitioning R + A into~~

~~let j=2 then eqs 1.4.37 become~~

$\cancel{A_1 = R_1^T R_1}$

$\cancel{c = R_1^T h}$

$$\left[\begin{array}{c|c} 4 & -2 \\ \hline -2 & 10 \end{array}\right] = \left[\begin{array}{c|c} 4 & 0 \\ \hline h & r_{22} \end{array}\right]\left[\begin{array}{c|c} 4 & h \\ \hline 0 & r_{22} \end{array}\right]$$

We 1st calculate h from c via

$$c = R_{j-1}^T h \quad \text{then} \quad r_{jj}^2 = a_{jj} - h^T h$$

In this case $(j=2)$ these become

$$\Rightarrow \; -2 = 2h \Rightarrow h = -1 \quad \text{then}$$

$$r_{22}^2 = 10 - (1)^2 = 9 \Rightarrow r_{22} = +3 \qquad \text{so} \quad R_2 = \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}$$

Letting $j=3$ + partitioning A_3 into ~~3x3 matrix~~ 4 pieces we have

$$\left[\begin{array}{cc|c} 4 & -2 & 4 \\ -2 & 10 & -2 \\ \hline 4 & -2 & 8 \end{array}\right] = \left[\begin{array}{cc|c} 2 & 0 & \\ -1 & 3 & h \\ \hline \multicolumn{2}{c|}{h^T} & r_{33} \end{array}\right]\left[\begin{array}{cc|c} 2 & -1 & \\ 0 & 3 & h \\ \hline \multicolumn{2}{c|}{0} & r_{33} \end{array}\right]$$

So $\quad c = R_2^T h$

$$\Rightarrow \begin{pmatrix} 4 \\ -2 \end{pmatrix} = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \Rightarrow \begin{array}{l} h_1 = 2 \\ h_2 = 0 \end{array}$$

Thus $\quad r_{33} = +\sqrt{a_{33} - h^T h} = \sqrt{8-4} = 2$

Then

$$R_3 = \begin{bmatrix} 2 & -1 & 2 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$\Big\{$ check

$$R_3^T R_3 \stackrel{?}{=} A_3$$

$$\begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 0 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 2 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & -2 & 4 \\ -2 & 10 & -2 \\ 4 & -2 & 8 \end{bmatrix} \text{ yes.}$$ $\Big\}$

Then construct $R_4 = \left[\begin{array}{c|c} R_3 & h \\ \hline 0 & r_{44} \end{array}\right]$ from A_4

$$R_3^T h = c = \begin{bmatrix} 2 \\ -7 \\ 4 \end{bmatrix}$$

Then

$$\begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 0 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -7 \\ 4 \end{bmatrix}$$

$\Rightarrow \quad h_1 = 1, \quad -1 + 3h_2 = -7 \Rightarrow h_2 = -2$

$$2 + 2h_3 = 4 \Rightarrow h_3 = 1$$

~~Then~~ ~~+~~

Then $r_{44} = +\sqrt{7-1^2-4-1^2} = 1$

Giving in total

$$R_4 = \begin{bmatrix} 2 & -1 & 2 & 1 \\ 0 & 3 & 0 & -2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Ex 1.4.39 Example 1.4.20 attempts to find the Cholesky factorization of the following matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 10 \\ 3 & 10 & 16 \end{bmatrix}$$ we will use the bordered form of

Cholesky factorization. This starts with the element $r_{11} = +\sqrt{a_{11}} = 1$

& defines the 1st ~~principal~~ leading principal submatrices as $R_1 = [r_{11}] = [1]$

Partitioning the 2nd leading principal submatrices as

$$R_2 = \begin{bmatrix} R_1 & h \\ 0 & r_{22} \end{bmatrix} \quad \& \quad A_2 = \begin{bmatrix} A_1 & c \\ c^T & a_{22} \end{bmatrix}$$

we have $R_1^T h = c$ or $1(h) = 2 \Rightarrow h = 2$

Then $r_{22} = +\sqrt{5-2^2} = 1$

Continuing set $R_3 = \begin{bmatrix} R_2 & h \\ 0 & r_{33} \end{bmatrix}$ & $A_3 = \begin{bmatrix} A_2 & c \\ c^T & a_{33} \end{bmatrix}$

$c = R_2^T h$

$$\Rightarrow \begin{bmatrix} 3 \\ 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$

$\Rightarrow h_1 = 3$ & $10 = 2h_1 + h_2$

$\Rightarrow 10 = 6 + h_2 \Rightarrow h_2 = 4$

$\therefore \quad r_{22} = +\sqrt{a_{22} - h^T h} = \sqrt{16 - 9 - 16} = \sqrt{-9}$ which is imaginary,

& thus No Cholesky factor exists.

Ex 1.4.40 Ex 1.4.21 asks for the Cholesky decomposition of the matrix

$$A = \begin{bmatrix} 16 & 4 & 8 & 4 \\ 4 & 10 & 8 & 4 \\ 8 & 8 & 12 & 10 \\ 4 & 4 & 10 & 12 \end{bmatrix}$$

. We will use the bordered form of Cholesky's algorithm

$R_1 = [r_{11}] = [4]$

$$\left[\begin{array}{c|c} 16 & 4 \\ \hline 4 & 10 \end{array}\right] = \begin{bmatrix} 4 & 0 \\ h & r_{22} \end{bmatrix} \begin{bmatrix} 4 & h \\ 0 & r_{22} \end{bmatrix}$$

$\Rightarrow \quad 4 = 4h \Rightarrow h = 1$

then $r_{22} = +\sqrt{10 - 1^2} = 3$

~~Re.~~ Partitioning A_3 + R_3 we have

$$\left[\begin{array}{cc|c} 16 & 4 & 8 \\ 4 & 10 & 8 \\ \hline 8 & 8 & 12 \end{array}\right] = \left[\begin{array}{c|c} R_2^T & 0 \\ \hline h^T & r_{33} \end{array}\right]\left[\begin{array}{c|c} R_2 & h \\ \hline 0 & r_{33} \end{array}\right]$$

$$\begin{pmatrix} 8 \\ 8 \end{pmatrix} = R_2^T h$$

$$= \begin{bmatrix} 4 & 0 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \Rightarrow \quad h_1 = 2$$

$$2 + 3h_2 = 8 \Rightarrow h_2 = 2$$

Then $r_{33} = +\sqrt{12 - h^T h} = \sqrt{12-4-4} = 2$

Thus $R_3 = \begin{bmatrix} 4 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{bmatrix}$

Partitioning $A_4 = A$ + R_4 we have

$$\begin{bmatrix} 4 \\ 4 \\ 10 \end{bmatrix} = R_3^T \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 1 & 3 & 0 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}$$

$$h_1 = 1 \Rightarrow \begin{pmatrix} 4 \\ 10 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 8 \end{pmatrix} = \begin{bmatrix} 3 & 0 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} h_2 \\ h_3 \end{bmatrix}$$

$$\Rightarrow h_2 = 1 \Rightarrow h_3 = 3$$

Then $r_{44} = +\sqrt{12 - 1^2 - 1^2 - 9} = \sqrt{12-11} = 1$

Thus $R = \begin{bmatrix} 4 & 1 & 2 & 1 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

pg 43 Watkins

Ex ~~1.4.1~~ 1.4.41

Ex 1.4.22 is to determine which of the following matrices in positive definite. Since A is Not symmetric we will not consider this matrix For $B = \begin{bmatrix} 4 & 2 & 6 \\ 2 & 2 & 5 \\ 6 & 5 & 29 \end{bmatrix}$ Using the bordered form of the Cholesky factorization 1st $r_{11} = 2$, then we solve

~~Let~~ $2 = 2h \Rightarrow h = 1$

Then $r_{22} = +\sqrt{2 - 1^2} = 1$

Then $\begin{bmatrix} 6 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \Rightarrow \begin{matrix} h_1 = 3 \\ h_2 = 5 - 3 = 2 \end{matrix}$

The $r_{33} = \sqrt{29 - 9 - 4} = 4$

So $R = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$

So this matrix is P.D.

For the Matrix $C = \begin{bmatrix} 4 & 4 & 8 \\ 4 & -4 & 1 \\ 8 & 1 & 6 \end{bmatrix}$

$r_{11} = 2$ Then ~~2~~ $4 = 2h \Rightarrow h = 2$

The $r_{22} = +\sqrt{-4 - 2^2} = \sqrt{-8}$ which is complex & this matrix does not have a Cholesky factor

If $D = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 1 \end{bmatrix}$

Then $r_{11} = \sqrt{1} = 1$

$1 = 1 \cdot h \Rightarrow h = 1$

So $r_{22} = \sqrt{2 - 1^2} = 1$

$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = h_1 = 1 \Rightarrow h_2 = 1$

$r_{33} = \sqrt{1 - 1^2 - 1^2} = \sqrt{-1}$ which can not be computed & thus this matrix is not P.D.

Pg 43 Watkins

Ex 1.4.42

(a) In the bordered form of Cholesky's method

~~for i=1,2,3,...,n we must perform~~

For $i=1$ $r_{11} = +\sqrt{a_{11}}$ requires "1" flop

Then to compute R_2 from R_1 we must

(1) calculate h from

$$c = R_1^T h$$

(2) calculate r_{jj} from

$$r_{jj} = +\sqrt{a_{jj} - h^T h}$$

Thus the algorithm is basically

$i=1$ $r_{11} = +\sqrt{a_{11}}$

~~i=2~~

for $i = 2:n$

/* calculate h from */

$c = R_i^T h$

/* calculate ~~r_{jj}~~ ~~from~~ r_{ii} from */

$r_{ii} = \sqrt{a_{ii} - h^T h}$

end

Counting flops we have 1 flop for r_{11}

Then performing step i ~~we calculate~~ when calculating

$$c = R_i^T h$$

we require $\cong$ ~~flops~~ i^2 flops.

Then in performing $r_{ii} = \sqrt{a_{ii} - h^T h}$ since h is of size i $h^T h$ requires $O(i)$ flops + in total $O(i)$ flops

Thus our total flop count is given by

$$\sum_{i=1}^{n} i^2 \cong \frac{n^3}{3}$$

~~exc~~ exactly the same as the other versions of this algorithm.

$\frac{n^3}{3}$

Ex 1.4.42

(a) In the bordered form of Cholesky's decomposition, doing each stage ~~j we assume we have R_{j-1} (the principle submatrix of size~~ $i \times i$)

~~we solve for h in $c = R_{j-1}^T$~~

$j = 2, 3, \ldots, n$ we assume we have R_{j-1} ~~of size $= j-1$~~ ~~R_{j-1}~~

the $j-1$st principle submatrix of R. Then to compute R_j we must

calculate h from $c = R_{j-1}^T h$ using forward substitution

& then $r_{jj} = +\sqrt{a_{jj} - h^T h}$.

Now since the solution for h in the above requires

$(j-1)^2$ flops, the cost to compute r_{jj} requires 1 flop.

Since this is done for $j = 2, 3, \ldots, n$, the total # of flops

required is given by

$$\sum_{j=2}^{n} (j-1)^2 + 1 = n-1 + \sum_{j=2}^{n} (j-1)^2 = n-1 + \sum_{j=1}^{n-1} j^2$$

$$\approx n-1 + \frac{(n-1)^2}{3} = O\left(\frac{n^3}{3}\right)$$ the same flop count as

before.

(b) In the inner product formulation we 1st perform the operation $r_{jj} = \sqrt{a_{jj} - h^T h}$

& then the back substitution

In the outer product formulation ~~a~~ we divide by the ~~elimination entry~~ ~~subtract from every element & reduce the size of~~ pivot in the 1st row then do a rank 1 update to elliminate the elements ~~below~~ below the pivot ~~effectively~~ performing ~~forward~~ back substitution.

Ex 1.4.44

Consider $I = C(C^{-1}) \cancel{=} I$

& take the transpose of both sides, then

$$\cancel{I} = I = (C^{-1})^T C^T$$

so $\cancel{(C^T)^{-1}} =$ since $(C^{-1})^T$ opperating on the left of of C^T produces the identity is is the inverse of C^T so

$$(C^T)^{-1} = (C^{-1})^T$$

Ex 1.4.45

for $k = 1, 2, \cdots, s$

$A_{kk} \leftarrow \text{cholesky}(A_{kk})$ (if cholesky fails set error flg)

for ~~k~~ $j = k, k+1, \cdots, s$

$A_{kj} \leftarrow A_{kk}^{-T} \cdot A_{kj}$

end

~~for $i, j = k+1, k+2, \cdots, s$~~

~~for~~

for $i = k+1, k+2, \cdots, s$

for $j = i, i+1, \cdots, s$

$A_{ij} \leftarrow A_{ij} - A_{ki}^{T} A_{kj}$

end

end

end

pg 45 Watkins

Ex 1.4.47 The Block version of the inner product form of Cholesky's method will proceed "row" by "row". Thus consider ~~the~~ ~~into~~ a block partition of A as given in the text

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1s} \\ & A_{22} & \cdots & A_{2s} \\ & & A_{33} \cdots & A_{3s} \\ & & & \vdots \\ & & & A_{ss} \end{bmatrix} = \begin{bmatrix} R_{11}^T & & & \\ R_{12}^T & R_{22}^T & & \\ R_{13}^T & & & \\ \vdots & & & \\ R_{1s}^T & \cdots & \cdots & R_{ss}^T \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} & R_{13} & \cdots & R_{1s} \\ & R_{22} & R_{23} & \cdots & R_{2s} \\ & & & & \vdots \\ & & & & R_{ss} \end{bmatrix}$$

~~For~~ ~~i = 1, 2, ..., s~~ ~~/* for each block row */~~

The 1st row is equal to

$A_{1j} = R_{11}^T R_{1j}$ ~~Also~~ for $j = 1, 2, \ldots, s$

when $j=1$ we have $A_{11} = R_{11}^T R_{11}$

then $R_{1j} = R_{11}^{-T} A_{1j}$ $j = 2, 3, 4, \ldots, s$

Then 2nd row is equal to

$$A_{2j} = R_{12}^T R_{1j} + R_{22}^T R_{2j} \qquad j = 2, 3, \ldots, s$$

so for $j=2$ we have

$$R_{22}^T R_{22} = A_{22} - R_{12}^T R_{12}$$

& $R_{2j} = R_{22}^{-T}(A_{2j} - R_{12}^T R_{1j})$ $j = 3, 4, \ldots, s$

The pattern has become clear ~~&~~ the algorithm is

~~for k = 1, 2, ...;~~

for $i = 1, 2, 3, \cdots, s$ /* for each block row */

  for $k = 1, 2, \cdots, i-1$ /* adjust the ~~...~~ ~~...~~ diagonal et */

    $A_{ii} \leftarrow A_{ii} - A_{ki}^T A_{ki}$

  end

  $A_{ii} \leftarrow \text{cholesky}(A_{ii})$ /* error if this does not exist */

  for $j = i+1, \cdots, s$

    for $k = 1, 2, \cdots, i-1$

      $A_{ij} \leftarrow A_{ij} - A_{ki}^T A_{kj}$

    $A_{ij} \leftarrow A_{ii}^{-T} A_{ij}$

Ex 1.4.48 ~~...~~ ~~The algorithms~~ Each algorithm steps are identical ~~...~~ just grouped into blocks or not $\therefore$ the same ~~...~~ calculations must be done in the two versions

---

$$2d^3 / 3d^2 = \frac{2}{3} d$$

Ex 1.4.49 The operation count for

$A_{kj} \leftarrow A_{kk}^{-T} A_{kj}$ can ~~be~~ be derived as follows

since A_{kk} is replaced by its Cholesky factor the matrix A_{kk} at this stage of the algorithm is upper triangular & therefore ~~the~~ the transpose is lower triangular. ~~The~~ The application of A_{kk}^{-T} on ~~the~~ ~~vector~~ a column vector will therefore require $d^2 - d = d^2 + O(d)$ flops since the dimension of A_{kk} is assumed to be $d \times d$. Since this must be performed d times (once for each column of A_{kj}) the total flop count will be

$$d(d^2 + O(d)) = d^3 + O(d^2).$$

Ex 1.4.50 The total work required for a Cholesky factorization is given by $O(n^3/3)$. In the Cholesky decomposition of the main diagonal blocks we have $s \cdot O(d^3/3)$ flops ~~for~~ for each of the s diagonals. The calculation of $A_{kj} \leftarrow A_{kk}^{-T} A_{kj}$ requires d^3 flops ~~&~~ each time it is performed. The corresponding fraction of flops ~~is~~ ~~given by~~ for the Cholesky decomposition of the diagonal blocks is given by

$$\frac{s \cdot d^3/3}{n^3/3} = \frac{s d^3}{n^3}$$

Since the reduction step $A_{ij} \leftarrow A_{ij} - A_{ki}^T A_{kj}$ is performed at most about $s^2/2$ times the ratio of the total work in this case is

$$\frac{\frac{s^2}{2}(d^3)}{n^3/3} = \frac{s^2 d^3}{6n^3}$$

Since $n = ds$ we have the two fractions of

$$\frac{s\cancel{d^3}}{s^3\cancel{d^3}} = \frac{1}{s^2} \quad \& \quad \frac{s^2 \cancel{d^3}}{6(\cancel{d^3}s^3)} = \frac{s^2}{6s^3} = \frac{1}{6s}$$

When $s = 10$ we have

$$\frac{1}{10^2} = \frac{1}{100} \quad \& \quad \frac{1}{60}$$

When $s = 20$ we have

$$\frac{1}{20^2} = \frac{1}{400} \quad \& \quad \frac{1}{6(20)} = \frac{1}{120}$$

Showing the dominant computation required by the reduction step.

~~Ex 1.4.52 Let $x = e_i$ vector of all zeros but with a one in the ith spot. Then $e_i^T A$ ... $x^T A x = e_i^T A e_i = a_{ii} > 0$~~

Ex 1.4.54 ~~Let x to be a vector of with~~ If A_{11} is of size $d_1 \times d_1$ then let x be a vector with its 1st d_1 elements nonzero & its remaining elements zero. Then from the partition given we have here

$$\begin{bmatrix} x_1 \\ \vdots \\ x_{d_1} \\ 0 \\ \vdots \\ 0 \end{bmatrix}^T \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_{d_1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_{d_1} \\ 0 \\ \vdots \\ 0 \end{bmatrix}^T \begin{bmatrix} A_{11}\begin{bmatrix} x_1 \\ \vdots \\ x_{d_1} \end{bmatrix} \\ A_{21}\begin{bmatrix} x_1 \\ \vdots \\ x_{d_1} \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} x_1 \\ \vdots \\ x_{d_1} \end{bmatrix}^T A_{11} \begin{bmatrix} x_1 \\ \vdots \\ x_{d_1} \end{bmatrix} > 0 \quad \forall \; x_1 \cdots x_{d_1} \quad \& \therefore A_{11} \text{ is positive definite}$$

Specifying our vector x to consist initially of d_1 zeros & the remaining nonzero elements, ~~we~~ in the same way we can show A_{22} is positive definite

~~Ex 1.7.36 Consider a nonzero vector v & consider~~

~~$v^T X^T A X v = (Xv)^T A (Xv) = y^T A y > 0$~~

~~where $y = Xv$. Thus $X^T A X$ is positive definite.~~

~~Note if X were singular this result is not true since there would exist a nonzero v such that $0 = Xv$ & therefore $v^T X^T A X v = 0$ in contradiction to the requirement of positive definite~~

pg 48 watkins

Ex 1.4.59 Assume A is symmetric positive definite, ~~then~~ then we will show by induction on the dimension of A that A can be factored into a lower triangular matrix R^T + an upper triangular matrix R. If $m=1$ then defining $r = +\sqrt{a}$ is the explicit construction of the Cholesky factors for this scalar case + thus the factorization exists $m=1$. Assume the cholesky ~~Re~~ factorization exists for $m \leq k$, we desire to show that a factorization exists for $m=k+1$. To show this case, consider the 1-1 principal leading submatrix partitioning of A as in the book i.e. consider A as given by

$$A = \begin{bmatrix} a_{11} & b^T \\ b & \hat{A} \end{bmatrix}$$ Then defining $r_{11} = +\sqrt{a_{11}}$ + $s = \frac{1}{r_{11}} b$ we have that

$$\begin{bmatrix} r_{11} & 0 \\ s & I \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \tilde{A} \end{bmatrix} \begin{bmatrix} r_{11} & s^T \\ 0 & I \end{bmatrix}$$ will equal A if

$\tilde{A} = \hat{A} - ss^T$. ~~From~~ the or the schur complement of a_{11} in A. From the fact that the schur complement of a_{11} in A

is positive definite & is of size $m=k$ we can use the induction hypothesis to assume the existence of a Cholesky factor $\tilde{R}$. Namely $\begin{bmatrix} 1 & 0 \\ 0 & \tilde{A} \end{bmatrix}$ can be written

$$= \begin{bmatrix} 1 & 0 \\ 0 & \tilde{R}^T\tilde{R} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \tilde{R}^T \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \tilde{R} \end{bmatrix}$$ & A then

has the following factorization

$$A = \begin{bmatrix} r_{11} & 0 \\ s & I \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \tilde{R}^T \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \tilde{R} \end{bmatrix}\begin{bmatrix} r_{11} & s^T \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} r_{11} & 0 \\ s & \tilde{R}^T \end{bmatrix}\begin{bmatrix} r_{11} & s^T \\ 0 & \tilde{R} \end{bmatrix}$$

Since these two matrices satisfy the requirements of a Cholesky factor we have shown the existence of a Cholesky factor for ~~matrix~~ $m=k$ & hence ∴ by induction proved the existence of a Cholesky factor for all m.

pg 49 Watkins

Ex 1.4.61 Let the sequence of j's as stated in the problem be given & consider a vector $\underline{x}$ with the only non-zero elements corresponding to the components $j_1, j_2, \cdots, j_k$ & the others zero

Then $\underline{x}^T A x > 0 \quad \forall \; \underline{x}$ & $\therefore$ for the $\underline{x}$ specified above

But in addition $\underline{x}^T A x = \tilde{x}^T \tilde{A} \tilde{x}$ where

$\tilde{x}$ is a vector of k components the numerical values of which are given from the $j_1, j_2, \cdots, j_k$ components of $\underline{x}$. & $\tilde{A}$ is the $k \times k$ matrix obtained by intersecting rows $j_1, j_2, \cdots, j_k$. Since

$\tilde{x}^T \hat{A} \tilde{x} > 0 \quad \forall \; \tilde{x}$ & $\tilde{A}$ is symmetric we have that

$\tilde{A}$ is symmetric positive definite

Ex 1.4.63

(a) To prove that $(AB)^* = B^* A^*$ consider the (i,j)th element of $((AB)^*)_{ij} = \overline{(AB)}_{ji}$

$$= \overline{\sum_{k=1}^{n} A_{jk} B_{ki}} = \sum_{k=1}^{n} \bar{A}_{jk} \bar{B}_{ki} = \sum_{k=1}^{n} (A^*)_{kj} (B^*)_{ik}$$

$$= \sum_{k=1}^{n} (B^*)_{ik} (A^*)_{kj} = \text{the } (i,j)\text{th element of } B^* \cdot A^*$$

(b) If A is hermitian, then x^*Ax is real $\forall x \in \mathbb{C}^n$.

let $\alpha = x^*Ax$ then if α is considered a 1×1 matrix we have that $\alpha^* = (x^*Ax)^* = x^*A^*x = x^*Ax$

since A is hermitian, thus $\alpha^* = \bar{\alpha} = \alpha$ & we have shown that

α is real

Ex 1.4.64 If M is $n\times n$ nonsingular matrix then M^*M is positive definite. Let $\underline{x} \in \mathbb{C}^n$ then

$$x^*M^*Mx = (Mx)^*(Mx) = y^*y = \sum_{i=1}^n \bar{y}_i y_i = \sum_{i=1}^n |y_i|^2 > 0$$

$\forall\ y \neq 0$. Since $y \neq 0 \Leftarrow x \neq 0$ (Because M is nonsingular)

we have that $x^*(M^*M)x > 0 \quad \forall\ x \neq 0$ & $\therefore$

M^*M is positive definite

Ex 1.4.65 Following exactly the proof of uniqueness given ~~for~~ the real version of the Cholesky Factorization we can show using say the inner product formulation that the decomposition is unique if it exists. In exactly the same way as existence was shown for the real version of the Cholesky factorization the existence of the complex factorization must hold

Ex 1.4.66

(a) $P =$ ~~What~~ $E \cdot I$ where E is the voltage drop + I is the current

Then since the voltage drop accross a resistor ~~equils~~ equals $E = IR$ we have that the the power dissipated by the resistor is given by $P = (IR) \cdot I = I^2 R$ or replacing the current by the voltage drop $P = E\left(\frac{E}{R}\right) = \frac{E^2}{R}$

(b) From the figure drawn in 1.9 the current through the "R" resistor is $x_i - x_j$ + thus this resistor draws power

$$P = (x_i - x_j)^2 \cdot R$$

Thus $P > 0$ unless $x_i = x_j$ in which case it draws no power. Notice that the matrix expression

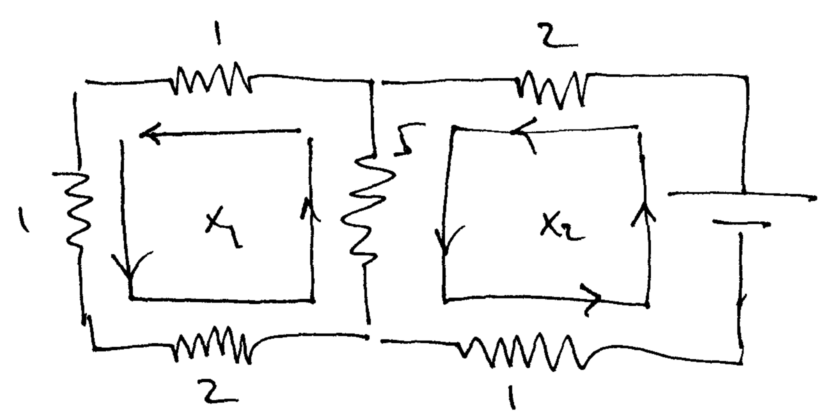
$$\begin{bmatrix} x_i & x_j \end{bmatrix} \begin{bmatrix} R & -R \\ -R & R \end{bmatrix} \begin{bmatrix} x_i \\ x_j \end{bmatrix} = \begin{bmatrix} x_i & x_j \end{bmatrix} \begin{bmatrix} Rx_i - Rx_j \\ -Rx_i + Rx_j \end{bmatrix}$$

$$= Rx_i^2 - Rx_ix_j - Rx_ix_j + Rx_j^2$$

$$= \cancel{R(x_i^2 - x_ix_j}$$

$= R(x_i^2 - 2x_ix_j + x_j^2) = R(x_i - x_j)^2$ the same expression for the power drawn by this resistor

(c) Considering the circuit from Example 1.2.8



Then the total power dissipated by all the resistors is given by

$$1x_1^2 + 1x_1^2 + 2x_1^2 + 5(x_1 - x_2)^2 + 2x_2^2 + 1x_2^2$$

$$\Rightarrow \quad \cancel{5x_1^2 + 2x_2^2}$$

$$4x_1^2 + 5(x_1^2 - 2x_1x_2 + x_2^2) + 3x_2^2$$

$$\Rightarrow \quad 9x_1^2 - 10x_1x_2 + 8x_2^2$$

or $9x_1^2 - 5x_1x_2 - 5x_1x_2 + 8x_2^2$

Which can be written in matrix form as

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 9 & -5 \\ -5 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

For A ~~defined as given to~~ will be positive definite iff it has a Cholesky factorization which we now compute

$$\begin{bmatrix} 9 & -5 \\ -5 & 8 \end{bmatrix} = \begin{bmatrix} r_{11} & 0 \\ r_{12} & r_{22} \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix} = \begin{bmatrix} r_{11}^2 & r_{11}r_{12} \\ r_{11}r_{12} & r_{12}^2 + r_{22}^2 \end{bmatrix}$$

$\Rightarrow \quad r_{11}^2 = 9 \quad \Rightarrow \quad r_{11} = +3$

Then $\quad r_{12} = \dfrac{-5}{r_{11}} = -\dfrac{5}{3}$

& $\quad r_{12}^2 + r_{22}^2 = 8 \quad \Rightarrow \quad r_{22}^2 = 8 - \left(\dfrac{5}{3}\right)^2 = 8 - \dfrac{25}{9}$

$$= \frac{72}{9} - \frac{25}{9} = \frac{47}{9}$$

$\therefore \quad r_{22} = +\dfrac{\sqrt{47}}{3}$

so
$$A = \begin{bmatrix} 9 & -5 \\ -5 & 8 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ -\frac{5}{3} & \frac{\sqrt{47}}{3} \end{bmatrix} \begin{bmatrix} 3 & -5/3 \\ 0 & \frac{\sqrt{47}}{3} \end{bmatrix}$$

we can ~~~~ check our expansion on the RHS. by multiplying $R^T R$ together to get

$$= \begin{bmatrix} 9 & -5 \\ -5 & \frac{25}{9} + \frac{47}{9} \end{bmatrix} = \begin{bmatrix} 9 & -5 \\ -5 & \frac{72}{9} \end{bmatrix}$$

$$\begin{array}{r} \cancel{25} \\ 47 \\ \hline 72 \end{array}$$

$$= \begin{bmatrix} 9 & -5 \\ -5 & 8 \end{bmatrix}$$

Thus since A has a Cholesky factor it <u>is</u> positive definite.

(d) . . .

Ex 1.4.67

(a)

Then the voltage drop across this resistor is given by $x_i - x_j$ so

By Ex 1.4.66 part (a) an expression for the power dissipated by this resistor is given by $\frac{E^2}{R} = \frac{(x_i - x_j)^2}{R} = C(x_i - x_j)^2$. Expanding this expression we have

$C[x_i^2 - x_i x_j - x_i x_j + x_j^2]$ which as a quadratic form can be written as $[x_i \;\; x_j]\begin{bmatrix} C & -C \\ -C & C \end{bmatrix}\begin{bmatrix} x_i \\ x_j \end{bmatrix}$

(b) The power drawn from the circuit in Figure 1.1 is given by the following expression

$$P = \left(\frac{1}{1}\right)(x_1 - x_2)^2 + \frac{1}{1}(x_1 - x_3)^2 + \frac{1}{2}(x_2 - x_4)^2 + \frac{1}{5}(x_3 - x_4)^2$$

$$+\left(\frac{1}{2}\right)(x_5 - x_3)^2 + \frac{1}{1}(x_5 - x_4)^2$$

Show $P \geq 0$ unless $x_i = x_j \;\; \forall \; i \neq j$

(c) ~~Expanding~~ ~~Consider the matrix H given by~~

$\cancel{H = \begin{bmatrix} \frac{1}{1} & -\frac{1}{1} \\ -\frac{1}{1} & \end{bmatrix}}$

Expanding our expression for P we have

$$P = x_1^2 - 2x_1x_2 + x_2^2 + x_1^2 - 2x_1x_3 + x_3^2$$
$$+ \tfrac{1}{2}(x_2^2 - 2x_2x_4 + x_4^2) + \tfrac{1}{5}(x_3^2 - 2x_3x_4 + x_4^2)$$
$$+ \tfrac{1}{2}(x_5^2 - 2x_3x_5 + x_3^2) + x_6^2 - 2x_4x_6 + x_4^2$$

$$= (1+1)x_1^2 - 2x_1x_2 + (1+\tfrac{1}{2})x_2^2$$
$$- 2x_1x_3 + (1+\tfrac{1}{5}+\tfrac{1}{2})x_3^2 + (\tfrac{1}{2}+\tfrac{1}{5}+1)x_4^2$$
$$+ \tfrac{1}{2}x_5^2 - \tfrac{1}{2}(2x_3x_5) + x_6^2 - 2x_4x_6$$
$$- \tfrac{1}{2}(2x_2x_4) - \tfrac{2}{5}x_4x_3$$

Now consider H defined as

$$H = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & \frac{3}{2} & 0 & -\frac{1}{2} & 0 & 0 \\ -1 & 0 & 1+\frac{1}{5}+\frac{1}{2} & -\frac{1}{5} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{5} & \frac{1}{2}+\frac{1}{5}+1 & 0 & -1 \\ 0 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$

$$\therefore\ H = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & \frac{3}{2} & 0 & -\frac{1}{2} & 0 & 0 \\ -1 & 0 & \frac{17}{10} & -\frac{1}{5} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{5} & \frac{17}{10} & 0 & -1 \\ 0 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$

Then $x^T H x = P$

(d) From Example 1.2.6 we obtain the matrix

$$A = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 3/2 & 0 & -1/2 \\ -1 & 0 & \frac{17}{10} & -\frac{1}{5} \\ 0 & -1/2 & -\frac{1}{5} & \frac{17}{10} \end{bmatrix}$$

which is the 1st principal submatrix in H. & since H is positive definite so is A. We know A is positive definite from the jth leading submatrices of a positive definite matrix (Proposition 1.4.53)

(e) One could construct a matrix H, ~~later~~ ~~earlier~~ show that it is positive definite by showing its equivalence to a positive definite quadratic form i.e. the power dissipated & then show that A is a principal submatrix of H & invoking the principle sub matrix theorem on H we have that A is then positive definite. Or one could just ~~compute the~~ ~~Cholesky factor~~ show a Cholesky factorization exists for A & ∴ it is positive definite. Since this latter

Approach seems easier it is what we will do.

pg 81 Watkins

Ex 1.4.68

(a) For a ~~Hooke~~ Hookian spring the restoring force is proportional to the stretched distance $F(s) = +ks$ thus the strain energy is given by

$$\int_0^x F(s)\,ds = \int_0^x +ks\,ds = +k\frac{s^2}{2}\Big|_0^x = +\frac{kx^2}{2}$$ is the strain energy

(b) Since the total stretch in the spring is given by $x_1 - x_2$ the residual strain energy is given by $\frac{k(x_1-x_2)^2}{2}$. which when expanded gives $\frac{k}{2}\left[x_1^2 - x_1x_2 - x_1x_2 + x_2^2\right]$ which can be written as given in the text.

(c) The total strain energy in the system of Example 1.2.10 is given by

$$\frac{4x_1^2}{2} + \frac{4}{2}(x_1-x_2)^2 + \frac{4}{2}(x_2-x_3)^2 + \frac{4}{2}x_3^2 > 0 \text{ unless } x_i = 0 \ \forall i$$

(d) Expanding the above we have

$$2x_1^2 + 2x_1^2 - 4x_1x_2 + 2x_2^2 + 2x_2^2 - 4x_2x_3 + 2x_3^2 + 2x_3^2$$

$$= 4x_1^2 - 4x_1x_2 + 4x_2^2 - 4x_2x_3 + 4x_3^2$$

which can be written as

$$\frac{1}{2}\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}\begin{bmatrix} 8 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Since this expression is always positive $\forall\ \underline{x}$ the matrix A is positive definite

(e) Finish...

Ex 1.4.69 $C_{ij} = v_j^T v_i$ so $C_{ji} = v_i^T v_j = v_j^T v_i = C_{ij}$ so

C is symmetric + $C_{ii} = v_i^T v_i = 1$ since each vector is assumed to have unit variance. Let $V = [v_1 \, v_2 \cdots, v_k]$ be a matrix with columns given by the k vector samples then

$C = V^T V$. + If the vectors v_i are linearly independent

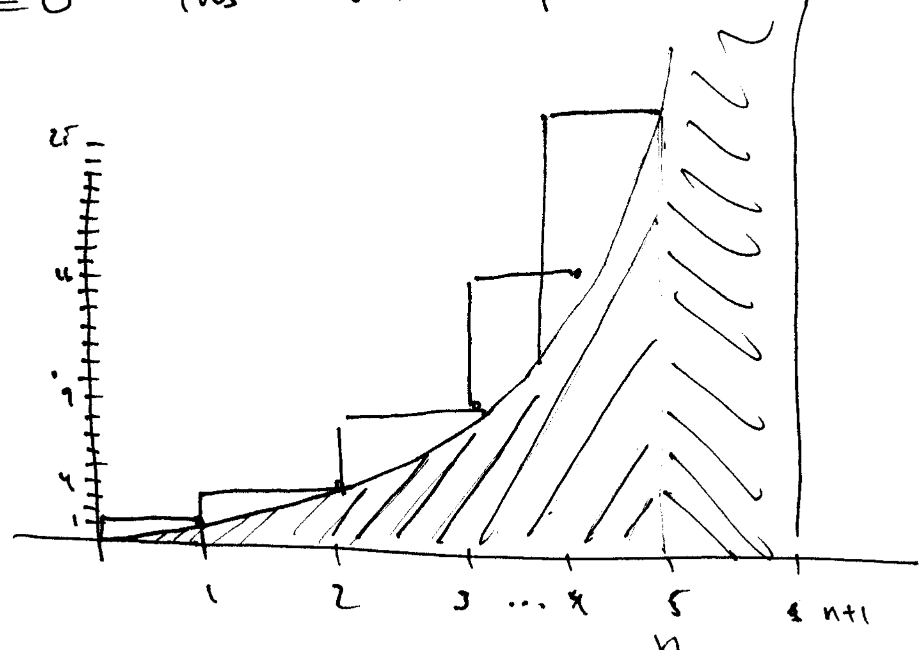
V has a trivial null space + ∴ $V^T V$ is positive definite. If the v_i's are linearly dependent then there exists a

$\underline{c} = (c_1, c_2, \cdots, c_k)$ not all zero

nontrivial vector c_i such that $V\underline{c} = 0$ + then

$\underline{c}^T V^T V \underline{c} = 0$ thus $V^T V$ is positive semi definite

Ex 1.4.70



So $\displaystyle\int_0^n i^2 \, di \le \sum_{i=0}^{n} i^2 \le \int_0^{n+1} i^2 \, di$

evaluating the integrals on each side of the summation

$$\frac{n^3}{3} \le \sum_{i=0}^{n} i^2 \le \frac{(n+1)^3}{3} \qquad n$$

$$\therefore \quad \sum_{i=1}^{n} i^2 = \frac{n^3}{3} + O(n^2)$$

Ex 1.4.71

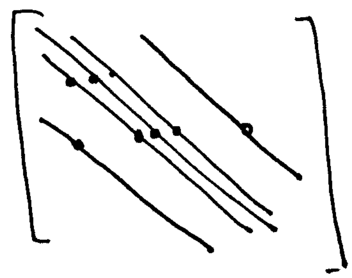
Skipped.

Ex 1.5.3 Ignoring B.C. for the fixed beam

pg 56 Watkins

$A =$

So $A =$



Ex 1.5.4 $m \times m$ network then we have m^2 equations each row has 5 non zeros still. The bandwidth of the system is given by m since we must connect with the ~~element~~ unknowns below & above the unknown the equation is derived from.

So if $m = 100$ we have 10^4 equations w/ 5 non zeros per row & bandwidth 100.

if $m = 1000 = 10^3$ we have 10^6 equations w/ 5 nonzeros per row & bandwidth 1000.

Ex 1.5.6 The envelope of the matrix in 1.5.5 is given by the element "-1" above the diagonal entry "2". in every column but also including ~~the~~ all elements above 2 in the last column of A, i.e. since the element $(1,n)$ is nonzero our prescription for the envelope is that all elements below this element to the diagonal must be included in the envelope

If A is $n \times n$ the upper triangl occupies $O(\frac{n^2}{2})$ storage while the envelope itself occupies $O(2n+n) = O(3n)$ storage. We thus have the envelope occupying a fraction $O\left(\frac{3n}{\frac{n^2}{2}}\right) = O\left(\frac{6}{n}\right)$ of the storage of the entire upper triangular matrix.

pg 58 Watkins Notes

If $n=100$ then with utilizing the envelope structure our Cholesky factorization requires about $\frac{1}{3}n^3 \cong 3.3 \cdot 10^5$ flops.

Exploiting the banded structure we require $ns^2 = 100(10)^2 = 10^4$ flops

Without exploiting the banded structure the forward & back solve requires

$2 \cdot n^2 = 2 \cdot 10^4$ flops. while when we exploit this structure we ~~...~~

require $4n \cdot s = 4 \cdot 10^2 \cdot 10 = 4 \cdot 10^3$

Total cost to solve $Ax=b$ when A is a fully dence symmetric positive definite matrix is given

by $\frac{1}{3}n^3 + 2n^2$

V.S. the total cost to solve for x $Ax=b$ when A is a banded (with ~~...~~ semi-bandwidth s) & A is symmetric positive definite is given by

$n \cdot s^2 + 4n \cdot s$

Ex 1.5.11

(a) If $m = 100$, $n = 10^4$ & we compute a dense Cholesky decomposition followed by the forward + backward substitutions then the total flop count is given by

$\cancel{\frac{1}{3}n^4}$

$$\frac{1}{3}n^3 + 2n^2 = \frac{1}{3}(10^{12}) + 2(10^8)$$

With a bandwidth of $2m$ or a semi-band width of $s = m = \cancel{n} = 100$ we have

$$ns^2 + 4n\cdot s = 10^4(10^2)^2 + 4(10^4)(10^2) = 10^8 + 4\cdot 10^6$$

Thus the flop count drops by ~~to~~ roughly $10^4 = 10{,}000$

Storing only the semi band requires at most $(s+1)\cdot n = \cancel{(101)(1000)}$ $(101)(10^4) \cong 10^6$ where as storing the entire matrix in dense format requires ~~a~~ $n^2 = (10^4)^2 = 10^8$ elements. The sparse format ~~is~~ requires 1% the storage of the dense matrix.

(b) If $m = 1000$, $n = (10^3)^2 = 10^6$. Then ~~the~~ the dense solution method requires $\frac{1}{3}(10^6)^3 + 2(10^6)^2 =$

while the sparse storage format requires $ns^2 + 4ns = 10^6(10^3)^2 + 4(10^6)(10^3)$

$= \cancel{10^7 + 4\cdot 1}$

$= 10^6\cdot 10^6 + 4\cdot 10^9$

pg 70 Watkins

Ex 1.7.2 To show that $\hat{A}\hat{x} = \hat{b}$ + $Ax = b$ have the same solutions when $\hat{A}\hat{x} = \hat{b}$ is obtained from $Ax = b$ by elementary operations we have to show that any solution to $Ax=b$ is also a solution to $\hat{A}\hat{x} = \hat{b}$ + then the converse. Since any solution to $Ax = b$ ~~stays~~ ~~the~~ is unaffected by elementary operations it will also be a solution to $\hat{A}\hat{x} = \hat{b}$. ~~Any solution to $\hat{A}\hat{x}=\hat{b}$~~

By recognizing that every elementary operation has a elementary operation that is its inverse, the ~~solutions~~ ~~+~~ system $\hat{A}\hat{x} = \hat{b}$ can be converted to $Ax = b$ by using elementary operations. Thus any solutions to $\hat{A}\hat{x} = \hat{b}$ are also solutions to $Ax = b$, and we have shown that the solutions to $Ax = b$ + $\hat{A}\hat{x} = \hat{b}$ are the same.

Ex 1.7.6

Defining $m_{i1} = \dfrac{a_{i1}}{a_{11}}$

Then m_{i1} times the 1st row gives a row that looks like

$$\left[m_{i1}a_{11} \quad m_{i1}a_{12} \quad m_{i1}a_{13} \quad \cdots \quad m_{i1}a_{1n} \;\middle|\; m_{i1}b_1 \right]$$

$$= \left[a_{i1} \quad m_{i1}a_{12} \quad m_{i1}a_{13} \quad \cdots \quad m_{i1}a_{1n} \;\middle|\; m_{i1}b_1 \right]$$

This row subtracted from the ith row will give the following

$$\left[a_{i1} - a_{i1} \quad a_{i2} - m_{i1}a_{12} \quad a_{i3} - m_{i1}a_{13} \quad \cdots \quad a_{in} - m_{i1}a_{1n} \,\middle|\, b_i - m_{i1}b_1 \right]$$

$$= \left[0 \quad a_{i2} - m_{i1}a_{12} \quad a_{i3} - m_{i1}a_{13} \quad \cdots \quad a_{in} - m_{i1}a_{1n} \,\middle|\, b_i - m_{i1}b_1 \right]$$

Giving a zero in the $(i,1)$ position.

Ex 1.7.10

(a) To guarantee that no row exchanges will be required it is sufficient to guarantee that the k-th leading principal submatrix of A is nonsingular for $k = 1, 2, 3, \ldots, n$

For the matrix A given, the invertability of the 4 leading principal submatrices of A can be determined by considering the determinants of A_k.

Computing we have

$$|A_1| = 2 \neq 0$$

$$|A_2| = \begin{vmatrix} 2 & 1 \\ -2 & 0 \end{vmatrix} = 0+2 = 2 \neq 0$$

$$|A_3| = \begin{vmatrix} 2 & 1 & -1 \\ -2 & 0 & 0 \\ 4 & 1 & -2 \end{vmatrix} = +2\begin{vmatrix} 1 & -1 \\ 1 & -2 \end{vmatrix} = 2(-2+1) = -2 \neq 0$$

$$|A_4| = \begin{vmatrix} 2 & 1 & -1 & 3 \\ -2 & 0 & 0 & 0 \\ 4 & 1 & -2 & 6 \\ -6 & -1 & 2 & -3 \end{vmatrix} = +2\begin{vmatrix} 1 & -1 & 3 \\ 1 & -2 & 6 \\ -1 & 2 & -3 \end{vmatrix}$$

$$= 2\left[1\begin{vmatrix} -2 & 6 \\ 2 & -3 \end{vmatrix} - 1\begin{vmatrix} -1 & 3 \\ 2 & -3 \end{vmatrix} + 1\begin{vmatrix} -1 & 3 \\ -2 & 6 \end{vmatrix}\right]$$

$$= 2\left[(6-12) - (3-6) + \overset{0}{(-6+6)}\right]$$

$$= 2\left[-6+3\right] = 2(-3) = -6 \neq 0$$

Since $|A_k| \neq 0 \quad \forall\ k = 1, 2, 3, 4$ we know that A can be transformed into upper-triangular form by operations of type 1 only.

(b) To transform $Ax = b$ into $Ux = y$ we 1st form the Augmented system

$$\left[\begin{array}{cccc|c} 2 & 1 & -1 & 3 & 13 \\ -2 & 0 & 0 & 0 & -2 \\ 4 & 1 & -2 & 6 & 24 \\ -6 & -1 & 2 & -3 & -14 \end{array}\right]$$

Then $m_{21} = \frac{-2}{2} = -1$ Thus the 1st transformation becomes

$m_{31} = \frac{4}{2} = 2$

$m_{41} = -\frac{6}{2} = -3$

$$\begin{array}{r} 39 \\ -14 \\ \hline 25 \end{array}$$

$$\left[\begin{array}{c|ccc|c} 2 & 1 & -1 & 3 & 13 \\ \hline -1 & 1 & -1 & 3 & 11 \\ 2 & -1 & 0 & 0 & -2 \\ -3 & 2 & -1 & 6 & 25 \end{array}\right]$$

From which we see that $m_{32} = \frac{-1}{1} = -1$

$m_{42} = \frac{2}{1} = 2$

-22

& we obtain

$$\left[\begin{array}{c|c|cc|c} 2 & 1 & -1 & 3 & 13 \\ \hline -1 & 1 & -1 & 3 & 11 \\ \hline 2 & -1 & -1 & 3 & 9 \\ -3 & 2 & 1 & 0 & 13 \end{array}\right]$$

Then $m_{43} = \frac{1}{-1} = -1$ so we obtain

$$\left[\begin{array}{cccc|c} 2 & 1 & -1 & 3 & 13 \\ -1 & 1 & -1 & 3 & 11 \\ 2 & -1 & -1 & 3 & 9 \\ -3 & 2 & -1 & 3 & 12 \end{array}\right]$$

(c) Back substitution would give

$$x_4 = 4$$

$$-x_3 = 9 - 3(4) = -3 \Rightarrow x_3 = 3$$

$$x_2 = 11 + x_3 - 3x_4 = 11 + 3 - 12 = 2$$

$$2x_1 = 13 - 2 + 3 - 12 = 16 - 14 = 2 \Rightarrow x_1 = 1$$

Then checking our solution $x = [1\ 2\ 3\ 4]^T$ we have

$$\begin{bmatrix} 2 & 1 & -1 & 3 \\ -2 & 0 & 0 & 0 \\ 4 & 1 & -2 & 6 \\ -6 & -1 & 2 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2+2-3+12 \\ -2 \\ 4+2-6+24 \\ -6-2+6-12 \end{bmatrix} = \begin{bmatrix} 13 \\ -2 \\ 24 \\ -14 \end{bmatrix} \checkmark$$

Text ~~Exercises~~ PJ 81 Wetkin

The 2nd row of U will be given by multiply the 2nd row of L into each column of U. This product gives the elements of a_{2j}. Thus we have

$$a_{2j} = l_{21} u_{1j} + 1 u_{2j} \qquad j \geq 2$$

but since we know the 1st row of U the elements u_{1j} are all known, so is the value l_{21}. Thus the only unknown in the above is u_{2j} so we have

$$u_{2j} = a_{2j} - l_{21} u_{1j} \qquad j \geq 2$$

which is how we will determine u_{2j} $(j \geq 2)$.

To Determine the second column of L, multiply the ith row of L into the fixed 2nd column of U. This product equals the element

$$a_{i2} = l_{i1} u_{12} + l_{i2} u_{22} \qquad i \geq 3$$

Since we now know the 1st, 2nd rows of U, and the 1st column of L we know everything in the above except l_{i2}. Solving for l_{i2} we have

$$\therefore \quad l_{i2} = \left(\frac{1}{u_{22}}\right)(a_{i2} - l_{i1} u_{12})$$

PS 82 Wetterie

Gaussian elimination ~~s~~ with ~~non~~ the inner product formulation.

For the example given

$$A = \begin{bmatrix} 2 & 4 & 2 & 3 \\ -2 & -5 & -3 & -2 \\ 4 & 7 & 6 & 8 \\ 6 & 10 & 1 & 12 \end{bmatrix}$$

compute rows of U according to $u_{kj} = a_{kj} - \sum_{m=1}^{k-1} l_{km} u_{mj} \quad j = k, k+1, \ldots, n$

& columns of L according to $l_{ik} = u_{kk}^{-1}\left(a_{ik} - \sum_{m=1}^{k-1} l_{im} u_{mk}\right) \quad i = k+1, \ldots, n$

So $A = \left[\begin{array}{c|ccc} 2 & 4 & 2 & 3 \\ \hline -1 & -5 & -3 & -2 \\ 2 & 7 & 6 & 8 \\ 3 & 10 & 1 & 12 \end{array}\right]$ in Step 1 we computed $U(1,:)$ & $L(:,1)$

Then in Step 2 we compute $U(2,:)$ & $L(:,2)$ so

$A = \left[\begin{array}{cc|cc} 2 & 4 & 2 & 3 \\ \hline -1 & -1 & -1 & 1 \\ \hline 2 & 1 & 6 & 8 \\ 3 & 2 & 1 & 12 \end{array}\right]$

$u_{22} = a_{22} - 4(-1) = -5 + 4 = -1$

$u_{23} = -3 - (-1)(2) = -3 + 2 = -1$

$u_{24} = -2 - (-1)(3) = -2 + 3 = 1$

$l_{32} = \frac{1}{(-1)}\cancel{(7 - (-1)(2))} = \cancel{-1(7+2) = -9}$

$$l_{32} = \frac{1}{(-1)}(7 - 2(4)) = -(-1) = 1$$

$$l_{42} = \frac{1}{(-1)}(10 - 3 \cdot 4) = 2$$

Then in step 3 we compute $U(3,:)$ + $L(:,3)$ so

$$A = \begin{bmatrix} 2 & 4 & 2 & 3 \\ -1 & -1 & -1 & 1 \\ 2 & 1 & 3 & 1 \\ 3 & 2 & -1 & 12 \end{bmatrix}$$

$$U_{33} = 6 - (-1)(1) - 2(2) = 6 + 1 - 4 = 3$$

$$U_{34} = 8 - (1)(1) - 2 \cdot 3 = 8 - 1 - 6 = 1$$

$$l_{43} = \frac{1}{3}(1 - 2(-1) - 2(3)) = \frac{1}{3}(1 + 2 - 6) = -1$$

Finally we compute the U_{44} element

$$U_{44} = 12 - (-1)(1) - 2 - 9 = 12 + 1 - 11 = 2$$

Ex 1.7.26 In Ex 1.7.10 our matrix was $A = \begin{bmatrix} 2 & 1 & -1 & 3 \\ -2 & 0 & 0 & 0 \\ 4 & 1 & -2 & 6 \\ -6 & -1 & 2 & -3 \end{bmatrix}$

Then using the inner product formulation to decompose A into $L \cdot U$ we have using $u_{kj} = a_{kj} - \sum_{m=1}^{k-1} l_{km} u_{mj}$ &

Step 1 compute $U(1,:)$ & $L(:,1)$ $\qquad l_{ik} = u_{kk}^{-1}\left(a_{ik} - \sum_{m=1}^{k-1} l_{im} u_{mk}\right)$

$$\begin{bmatrix} 2 & 1 & -1 & 3 \\ -1 & 0 & 0 & 0 \\ 2 & 1 & -2 & 6 \\ -3 & -1 & 2 & -3 \end{bmatrix}$$

Step 2 compute $U(2,:)$ & $L(:,2)$

$$\begin{bmatrix} 2 & 1 & -1 & 3 \\ -1 & 1 & -1 & 3 \\ 2 & -1 & -2 & 6 \\ -3 & 2 & 2 & -3 \end{bmatrix}$$

$u_{22} = 0 - (-1)(1) = 1$

$u_{23} = 0 - (-1)(-1) = -1$

$u_{24} = 0 - 3(-1) = 3$

$l_{32} = \frac{1}{1}(1 - 2(1)) = -1$

$l_{42} = \frac{1}{1}(-1 - (-3)(1))$

$= -1 + 3 = 2$

Step 3 compute $U(3,:)$ & $L(:,3)$

$$\begin{bmatrix} 2 & 1 & -1 & 3 \\ -1 & 1 & -1 & 3 \\ 2 & -1 & -1 & 3 \\ -3 & 2 & -1 & -3 \end{bmatrix}$$

$u_{33} = -2 - (-1)^2 - (-1)(2)$

$= -2 - 1 + 2 = -1$

$u_{34} = 6 - (-1)(3) - 2(3) = 3$

$l_{43} = \frac{1}{(-1)}(2 - (-1)(2) - (-3)(-1))$

$= -(2 + 2 - 3) = -1$

Eg: Step 4 compute $U(4,4)$

$$u_{44} = -3 - (-1)(3) - 2(3) - (-3)(3)$$

$$= -3 + 3 - 6 + 9 = 3 \quad \text{giving}$$

$$\begin{bmatrix} 2 & 1 & -1 & 3 \\ -1 & 1 & -1 & 3 \\ 2 & -1 & -1 & 3 \\ -3 & 2 & -1 & 3 \end{bmatrix}$$

pg 83 Watkins

Ex 1.7.27

To derive an outer-product formulation of the LU decomposition we decompose $A = LU$ into the following blocks

$$\begin{bmatrix} a_{11} & b^T \\ c & \hat{A} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ l & \hat{L} \end{bmatrix} \begin{bmatrix} u_{11} & s^T \\ 0 & \hat{U} \end{bmatrix}$$

Where if A is $n \times n$, then b is $(n-1)\times 1$, c is $(n-1)\times 1$, $\hat{A}$ is $(n-1)\times(n-1)$, l is $(n-1)\times 1$, $\hat{L}$ is $(n-1)\times(n-1)$, s is $(n-1)\times 1$, $\hat{U}$ is $(n-1)\times(n-1)$.

By multiplying the blocks in the right hand side we have

$$\begin{bmatrix} a_{11} & b^T \\ c & \hat{A} \end{bmatrix} = \begin{bmatrix} u_{11} & s^T \\ l u_{11} & l s^T + \hat{L}\hat{U} \end{bmatrix}$$

So we have

$$u_{11} = a_{11}$$
$$s^T = b^T$$
$$l u_{11} = c$$
$$\hat{L}\hat{U} + l s^T = \hat{A}$$

Thus the ~~algorithm~~ outer-product formulation of the LU decomposition involves solving for the following

$$u_{11} = a_{11}$$
$$s^T = b^T$$
$$l = \frac{1}{a_{11}} c$$

Solve for $\hat{L}$ + $\hat{U}$ in $\hat{L}\hat{U} = \hat{A} - l s^T = \hat{A} - (\frac{1}{a_{11}} c) b^T = \hat{A} - \frac{cb^T}{a_{11}}$

This algorithm suits itself very well into a recursive ~~formulation~~ implementation.

Note that the ~~step~~ calculation $\cancel{\hat{A} - \frac{cb^T}{a_{11}}}$ $\hat{A} - (\frac{c}{a_{11}}) b^T$ is exactly

the results of performing row operations of type 1 to produce

zeros in the 1st row of A.

Pg 85 Watkins

Ex 1.7.32 Theorem 1.7.31 states that if A is positive then A can be expressed in exactly one way as a product $A = LDL^T$, with L lower triangular, D a diagonal ~~Ext~~ matrix with positive elements. From Theorem 1.7.30 since our A ~~#~~ ~~[illegible]~~ ~~#~~ being positive definite is symmetric and has leading principal submatrices that are nonsingular it has a unique decomposition given by $A = LDL^T$ with L unit lower triangular & D is a diagonal matrix. Since L is unit lower triangular it is invertible so we have

$$D = (L^{-1})A(L^{-1})^T$$

But this matrix $(L^{-1})A(L^{-1})^T$ must be positive definite by Proposition 1.4.55 from the book. Since D must be positive definite $x^T D x > 0$ for $x \neq 0$. If $x = e_i$ the standard basis vector with a one in the ith element, then $x^T D x = e_i^T D e_i = d_{ii} > 0$ so D has positive elements on the diagonal.

Ex 1.7.37 If A is symmetric & it has been reduced to the form given Then from the outer product formulation given in exercise 1.7.27 gives

$$A^{(1)} = \hat{A} - \frac{1}{a_{11}} c b^T$$

with $\hat{A} = A(2:n, 2:n)$, $c = A(2:n,1)$, $b = A(1,2:n).'$;

in ~~matlab~~ ~~matlb~~ matlab notation. From this explicit expression we have

$$A^{(1)T} = \hat{A}^T - \frac{1}{a_{11}} b c^T = \hat{A} - \frac{1}{a_{11}} c b^T$$

since $b = c$ by the symmetry of A.

— — — —

Another way?

The given reduction can be obtained as a sequence of elementary matrix operations as

$$\left[\begin{array}{c|ccc} a_{11} & a_{12} & \cdots & a_{1n} \\ \hline 0 & & & \\ \vdots & & A^{(1)} & \\ 0 & & & \end{array}\right] = M_{1n} \cdots M_{13} M_{12} A$$

Then transposing both sides we have

$$\left[\begin{array}{c|ccc} a_{11} & 0 & \cdots & 0 \\ \hline a_{12} & & & \\ \vdots & & A^{(1)T} & \\ a_{1n} & & & \end{array}\right] = A^T M_{12}^T M_{13}^T \cdots M_{1n}^T$$

$$= A M_{12}^T M_{13}^T \cdots M_{1n}^T .$$

The multiplication of M_{12}^T on the left combines the <u>columns</u> of A in such a way as to produce 0's in the 1st row of the product. Because A is symmetric these operations working on the columns of A will produce the same elliminations as M_{12} operating on the ~~right~~ right of A but in the ~~rows~~ columns rather than the rows. Thus the lower right matrix of $A M_{12}^T M_{13}^T \cdots M_{1n}^T$ will give the same result as $A^{(1)}$

$A^{(1)}$ & we have $A^{(1)} = A^{(1)T}$ so $A^{(1)}$ is symmetric

Ex 1.7.38

(a) A bordered form for the LU decomposition can be derived by considering the following factorizations of $A = LU$ derived by considering the ~~principal~~ leading principal submatrix of A. We start by considering a 1×1 matrix

$a_{1,1} = 1 \cdot a_{1,1}$

so $l_{11} = 1$ + $u_{11} = a_{11}$

Considering a recursive formulation if we have L_j + U_j can we calculate how to calculate L_{j+1} + U_{j+1} when L_j + U_j are the leading principal submatrices of size $j \times j$ of L + U respectively

By partitioning $A_j = L_j U_j$ as

$$\begin{bmatrix} A_{j-1} & b \\ c^T & a_{jj} \end{bmatrix} = \begin{bmatrix} L_{j-1} & 0 \\ h^T & 1 \end{bmatrix} \begin{bmatrix} U_{j-1} & g \\ 0 & u_{jj} \end{bmatrix}$$

giving the equations

$$A_{j-1} = L_{j-1} U_{j-1}$$

$$b = L_{j-1} g$$

$$c^T = h^T U_{j-1} \iff U_{j-1}^T h = c$$

$$a_{jj} = h^T g + u_{jj}$$

~~Solving~~ Thus since we know L_{j-1} + U_{j-1} we can solve for the unknowns that will enable us to build L_j + U_j from the components L_{j-1} + U_{j-1}. These

~~elements~~ ~~are~~ unknowns are g, h, & u_{jj}. Solving we have

$$g = L_{j-1}^{-1} b$$

$$h = (U_{j-1}^{T})^{-1} c$$

& finally ~~&~~ $u_{jj} = a_{jj} - h^T g$. This gives the bordered form of the LU decomposition (we let $j = 1, 2, \dots, n$)

(b) Prove that the lower envelope of L (by rows) equals the lower envelope of A (by rows) and that the upper envelope of U by columns equals the upper envelope of A (by columns)

From the bordered decomposition just discussed lets consider the envelope of A, let the vector b be given by

$$b = \begin{bmatrix} 0 \\ \hat{b} \end{bmatrix}$$ where $\hat{b} \in \mathbb{R}^{s_j}$ i.e. the jth column has an envelope that only has s_j elements. When we solve for g since L_{j-1} is lower triangular the vector g will have only s_j possibly non zero elements i.e. $g = \begin{bmatrix} 0 \\ \hat{g} \end{bmatrix}$

with $\hat{g} \in \mathbb{R}^{s_j}$.

$$\cancel{\begin{bmatrix} g \end{bmatrix} = \begin{bmatrix} 0 \\ \hat{b} \end{bmatrix}}$$

$$\cancel{g = \begin{bmatrix} 0 \\ \hat{g} \end{bmatrix}}$$

~~Thus the envelope of~~ since $\hat{b}(1) \neq 0$

When we solve for $\hat{g}(1)$ it cannot be equal to zero

$$1\,\hat{g}(1) = \hat{b}(1) \Rightarrow \hat{g}(1) = \hat{b}(1) \neq 0.$$

Thus the envelope of U & A are the same. To show that the envelope of L & the row envelope of A are the same consider the row c^T. Assume that

$$c = \begin{bmatrix} 0 \\ \hat{c} \end{bmatrix} \text{ w/ } \hat{c} \in \mathbb{R}^{s_j}$$

so that the jth row has s_j elements in its envelope.

~~Thus~~ Now when we solve for h with

$U_{j-1}^T h = c$ since c has $n - s_j$ leading zeros h will have $n - s_j$ leading zeros thus $h = \begin{bmatrix} 0 \\ \hat{h} \end{bmatrix}$ w/ $\hat{h} \in \mathbb{R}^{s_j}$. Thus the lower row envelope of A equals the lower row envelope of L. Again if $\hat{c}(1) \neq 0$ $\hat{h}(1) \neq 0$ so the two envelopes are the same.

Ex 1.7.39

If A is non singular & A has an LU decomposition, then we can factor L & U to obtain the LU factorization of the principal submatrices of A, i.e.

$$A_k = L_k U_k$$

But L_k is not singular since it is unit lower triangular.
& U_k is not singular because if it was ~~there would have to be a~~ then U_k must have a zero on an element of U_k. But then U would be singular making A singular. Thus the submatrix A_k is non singular for each $k=1,2,\cdots,n$

Ex 1.7.40

I'll follow the factorizations from the book for performing an ~~factorization from A. Then an LDL^T factorization~~ Cholesky factorization of $A = R^T R$. Then an LDL^T factorization comes easily from the Cholesky factorization. We ~~divide the diagonal~~ ~~assign~~ to divide each row of R by the element on its diagonal, so let $R = D_1 \hat{R}$ w/ D_1 a diagonal matrix with the elements on the diagonal of R then $\hat{R} = D_1^{-1} R$.
With this transformation

$$A = (D_1\hat{R})^T (D_1\hat{R})$$
$$= \hat{R}^T D_1^T D_1 \hat{R} = \hat{R}^T D_1^2 \hat{R}$$

Then let $L = \hat{R}^T$ + $D = R_1^2$ + we have a decomposition given by

$A = LDL^T$. So to compute this decomposition we compute the R^TR ~~EQ~~ decomposition + then modify it to compute LDL^T.

To compute R^TR ~~requires~~ using any of the methods suggested inner-product, outer-product, + bordered formulation all require $O(\frac{n^3}{3})$ flops to compute. Then to perform the row scaling to compute $\hat{R} = R_1^{-1}R$ will require $1 + 2 + 3 + \cdots + (n-1)$ flops $= O(n^2)$ flops to compute $\hat{R}$ from R. Since this is ~~of~~ of sub order to ~~o~~ $n^3/3$ the time to ~~facto~~ factor LDL^T will require $O(n^3/3)$.

Obviously by not enforcing the symmetry of A we could perform a straightforward LU decomposition w/ $O(\frac{2}{3}n^3)$ flops. Given $A = LU$ factoring out the diagonal from D would give ~~DG~~ $U = D\hat{U}$ + $A = LD\hat{U}$ But since A is symmetric $\hat{U} = L^T$ + we have the desired factorization. The ~~direct~~ 3 methods then all require $O(\frac{2}{3}n^3)$ and the formulation given in Ex 1.7.37 is the inner product formulation

pg 87 Watkins

Ex 1.7.41

Theorem 1.7.33 states that if A is ~~symmetric~~ positive definite then it can be expressed in one way as $MD^{-1}M^T$ w/ M lower triangular, D diagonal w/ positive elements + the main diagonal elements of M the same as those of D. To compute this decomposition we will 1st compute the Cholesky decomposition + further modify the algorithm to return the matrices M + D. We can perform the Cholesky decomposition in $O(n^3/3)$ flops using any of the given methods inner, outer, or bordered factorizations. Once we have R in $A = RR^T$ we can define $D = \text{diag}(R)$ to be the diagonal matrix with elements taken from the main diagonal of R. Then the following manipulations explain what to do given that I will factor R as $R = LD$

$$
\begin{aligned}
A = RR^T &= (LD)(LD)^T \\
&= LD^2L^T \\
&= LD^2 \cdot D^{-2} D^2 L^T \\
&= (LD^2) D^{-2} (LD^2)^T \\
&= (RD)(D^2)^{-1}(RD)^T
\end{aligned}
$$

From the above if we define $M = RD$ then have the required factorization $A = M(D^2)^{-1}M^T$.

Thus from the R matrix we can compute M by multiplying each column of R with the element in its diagonal. Since R is upper triangular this only requires $O(n^2)$ flops. In the same way the diagonal matrix requested can be obtained by ~~one~~ computing $1/r_{ii}^2$.

Ex 1.7.42

~~In problem exercise 1.7.41 we gave one explicit formulas for how to calculate~~

Lets assume to reach a contradiction that there existed two such matrices that satisfied the conditions of Theorem 1.7.33 this means that

$$A = M_1 D_1^{-1} M_1^T \quad + \quad A = M_2 D_2^{-1} M_2^T$$

so we have that

$$M_1 D_1^{-1} M_1^T = M_2 D_2^{-1} M_2^T$$

$$\Rightarrow \quad V_1 M_1^T = V_2 M_2^T \qquad \text{w/} \quad V_i = M_i D_i^{-1}$$

$$\Rightarrow \quad V_2^{-1} V_1 = M_2^T M_1^{-T}$$

which can be done since the M_i's is lower triangular + has positive elements on its diagonal (+ is therefore invertible)

M_2 is lower triangular so M_2^T is upper triangular

M_1^{-T} will also be upper triangular so $M_2^T M_1^{-T}$ will be upper triangular

On the other hand V_i is lower triangular so

$V_2^{-1} V_1$ will be lower triangular. Thus we have a

lower triangular matrix equal to an upper triangular matrix which

cannot be true ~~as~~ unless each is actually a diagonal matrix.

~~That~~ which must be the common diagonal between them.

But since M_i + D_i have the same diagonal elements V_i has

diagonal elements given by 1's + thus $V_2^{-1} V_1$ ~~must~~ ~~equal~~

has ~~one~~ ones on the diagonal + is diagonal so $V_2^{-1} V_1 = I$

$\Rightarrow V_1 = V_2$ + $M_2^T M_1^{-T} = I \Rightarrow M_1 = M_2$

which together give $D_1 = D_2$. ~~Combining all of these~~ thus the

uniqueness result is proved.

Ex 1.7.43

(a) If A has the properties given then A ~~can~~ ~~be~~ ~~decomposed~~ ~~as~~ has an ~~A=~~ LU decomposition ~~as~~ without requiring pivoting, i.e.

$$A = LU = L D \hat{U}$$ when we have factored out the diagonal element from each row of U producing a unit upper triangular matrix $\hat{U}$.

$$= L D_1 (D_1^{-1} D D_1^{-1}) D_1 \hat{U}$$ we then reinsert unknown diagonal matrices D_1 to multiply L + $\hat{U}$.

$$= (L D_1) [D_1^{-1} D D_1^{-1}] (D_1 \hat{U})$$

For all matrices to have the same diagonal elements we require that

~~this~~ requires that $D_1^{-1} D D_1^{-1} = D_1 \Rightarrow$ ~~$D_1^3 = D \Rightarrow D_1 = \sqrt[3]{D}$~~

Since the left hand ~~side~~ side is the diagonal elements of the middle

$\therefore$ ~~$A = (LD^{1/3})(D^{1/3})(D^{1/3}\hat{U})$~~

matrix and the Right hand side D_1 is the value of the diagonal elements of both the products LD_1 + $D_1\hat{U}$. Solving this last equation for the unknown matrix D_1 we obtain

$$D_1 = D^{1/3}$$

requiring us to compute the 3rd roots of D. Given these values our decomposition then becomes $A = (LD^{1/3})(D^{1/3})(D^{1/3}\hat{U})$

Defining $M = LD^{1/3}$ $\tilde{D} = D^{-1/3}$ + $U = D^{1/3}\hat{U}$ we have our requested $M\tilde{D}^{-1}U$ decomposition

$D^{-2/3}U$

(b) The algorithm begins by computing the L^*U^* decomposition of A using any of the methods discussed thus far. Then ~~we divide each row of U by~~ because the U required by our $MD^{-1}U$ decomposition is $D^{1/3}\hat{U} = D^{-2/3}U^*$ w/ U^* the U produced by the LU decomposition we can compute the cube root of the diagonal elements of U^*. These elements go on the diagonal of U^* ~~+ will compose the elements~~. Then divide the ith row of U^* (excluding the diagonal) by $d_{ii}^{-2/3}$ then multiply ~~divide~~ the jth column of L^* by $d_{jj}^{1/3}$ + we have computed M.

Pg 88 Watkins

Ex 1.7.47

(a) We can show the given ~~M_1^{-1}~~ ~~inverse~~ matrix is the inverse of M by just multiplying them & observing that their product is the identity we have.

$$M_1 M_1^{-1} = \begin{bmatrix} 1 & 0 \\ m_1 & I_{n-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -m_1 & I_{n-1} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ m_1 - m_1 & I_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & I_{n-1} \end{bmatrix}$$

(b) Consider $M_1^{-1}A = \cancel{\begin{bmatrix} 1 & 0 \\ -m_1 & I_{n-1} \end{bmatrix} A \left[\begin{array}{c|cccc} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ \hline a_{21} & a_{22} & \cdots & & a_{2n} \\ \vdots & \vdots & & & \\ a_{n1} & a_{n2} & \cdots & & a_{nn} \end{array}\right]}$

$$= \cancel{\left[\begin{array}{c|ccc} a_{11} \quad a_{12} & a_{12} & a_{13} \cdots & a_{1n} \\ \hline -a_{11} m_1 + \begin{bmatrix} a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} & & & \end{array}\right]}$$

$$= \begin{bmatrix} 1 & 0 \\ -m_1 & I_{n-1} \end{bmatrix} \begin{bmatrix} a_{11} & b^T \\ c & \hat{A} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & b^T \\ c - a_{11} m_1 & \hat{A} - m_1 b^T \end{bmatrix}$$

then to make the elements $(2,1), (3,1), (4,1), \dots, (n,1)$

of $M_1^{-1}A$ zero we must have

$$c - a_{11}m_1 = 0 \qquad \text{so} \qquad m_1 = \frac{c}{a_{11}}$$

with c the $A(2{:}n,1)$ elements from A

(c) The matrix just derived is $A^{(1)}$ & ~~to~~ ~~the~~ ~~form~~ to transform $A^{(1)}$ into $A^{(2)}$ we will multiply by M_2^{-1}

this matrix will have the form

$$M_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \\ 0 & m_{32} & \\ 0 & m_{42} & I_{n-2} \\ \vdots & \vdots & \\ 0 & m_{n2} & \end{bmatrix}$$

~~then~~ this can be done to create a "column" of zeros below the ~~main~~ main diagonal in the 2nd column, in the same way as in part (b)

(d) By performing the $\overset{n-1}{\wedge}$ Gauss transforms one at a time to A we have

$$M_{n-1}^{-1} \cdots M_3^{-1} M_2^{-1} M_1^{-1} A = U$$

~~then~~ & multiplying by the M's on the left we obtain

$$A = M_1 M_2 \cdots M_{n-2} M_{n-1} U$$

defining L as $L = M_1 M_2 \cdots M_{n-1}$, we have $A = LU$

since each M_i is unit lower triangular the product will be unit lower triangular.

$\therefore$ L is unit lower triangular by Ex 1.7.45

(e)

Ex 1.7.48

pg 89 Watkins

The existence of an LU decomposition for all matrices A that are (nonsingular) & with leading principal submatrices that are nonsingular can be seen by applying Gauss matrices M_k one at a time to create zeros below the diagonal in column k of A. Then after n-1 application of Gauss matrices we will have reduced A to an upper triangular matrix U with $M_{n-1} M_{n-2} \cdots M_2 M_1 A = U$

so since each Gauss matrix is invertible we have

$$A = M_1^{-1} M_2^{-1} \cdots M_{n-2}^{-1} M_{n-1}^{-1} U$$

Since the product $M_1^{-1} M_2^{-1} \cdots M_{n-2}^{-1} M_{n-1}^{-1}$ equals a unit lower triangular matrix thus proving the existence of the LU decomposition.

pg 90 Watkins

Ex 1.7.49

The ~~...~~ Gauss matrices are given by

$$M_k = \begin{bmatrix} I_{k-1} & & \\ & 1 & 0 \\ & m_k & I_{n-k} \end{bmatrix}$$

the vector m_k is in the k-th column of the matrix M.

which can be written as the following ~~...~~ ~~product~~ form

$$M_k = I - vw^T$$

w/ $v = \begin{bmatrix} 0 \\ 1 \\ m_k \end{bmatrix}$ w/ the 0 representing the 0 ~~...~~ column matrix of size $n-1-(n-k) = \cancel{n}-1-\cancel{n}+k = k-1$ & m_k a column of $n-k$ numbers.

& $w = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ w/ the 1st block of 0's of size $n-k$, a 1 and then a 2nd block of 0's of size $n-1-(n-k) = k-1$.

Ex 1.7.50

(a) By multiply the block matrices on the left together we get

$$\begin{bmatrix} I_k & 0 \\ -M & I_{n-k} \end{bmatrix}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ -MA_{11}+A_{21} & -MA_{12}+A_{22} \end{bmatrix}$$

Then to make the block matrix of the ~~two~~ (2,1) position 0

$$-MA_{11}+A_{21}=0$$

$$\Rightarrow \quad MA_{11}=A_{21}$$

$$\Rightarrow \quad M = A_{21}A_{11}^{-1}$$ which since we are assuming that A_{11} is invertible ~~is~~ ~~complete~~ the M is unique. The (2,2) position will then become

$$-A_{21}A_{11}^{-1}A_{12}+A_{22},$$

which we define to be $\tilde{A}_{22}$.

(b) We can now block invert the matrix $\begin{bmatrix} I_k & 0 \\ -M & I_{n-k} \end{bmatrix}$

to complete the block LU ~~&~~ decomposition.

The block inverse of this matrix is given by $\begin{bmatrix} I_k & 0 \\ M^* & I_{n-k} \end{bmatrix}$

Which can be shown by simply multiply the two block matrices & showing they produce the identity.

$$\begin{bmatrix} I_k & 0 \\ -M & I_{n-k} \end{bmatrix}\begin{bmatrix} I_k & 0 \\ M & I_{n-k} \end{bmatrix} = \begin{bmatrix} I_k & 0 \\ -M+M & I_{n-k} \end{bmatrix} = \begin{bmatrix} I_k & 0 \\ 0 & I_{n-k} \end{bmatrix} = I$$

Thus by left multiply by this inverse gives

$$A = \begin{bmatrix} I_k & 0 \\ M & I_{n-k} \end{bmatrix}\begin{bmatrix} A_{11} & A_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix}$$

(C) Assuming that the leading principal submatrices of A_{11} are non singular