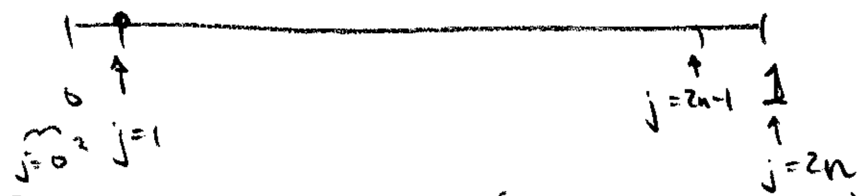


ODE:

$$-\frac{d^2u}{dx^2} = f(x) \Rightarrow -\frac{(u_{j-1} - 2u_j + u_{j+1})}{h^2} = f_j$$

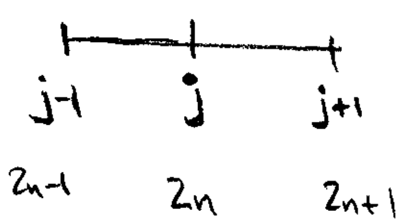
$$= \frac{-u_{j-1} + 2u_j - u_{j+1}}{h^2} = f_j \qquad j = 2, \dots, 2n-1$$

evaluating at $j = 1$ gives

$$\frac{-\cancel{u_0}^{\,0} + 2u_1 - u_2}{h^2} = f_1 \Rightarrow h^{-2}(2u_1 - u_2) = f_1$$

Then to discretize $\frac{du(1)}{dx} = 0$ write central differences of $\frac{d^2u}{dx^2}$ as

$$-\frac{\left(\frac{(u_{j+1} - u_j)}{h} - \frac{(u_j - u_{j-1})}{h}\right)}{h} = f_j$$



evaluat at $j = 2n-1$ gives

$$\cancel{-\frac{\left[\frac{(u_{2n} - u_{2n-1})}{h} - \frac{(u_{2n-1} - u_{2n-2})}{h}\right]}{h}} = f_{2n-1}$$

Evaluate at $j = 2n$ gives

As forward Approx to $\frac{du(1)}{dx} = 0$

$$-\frac{\left[\frac{(\cancel{U_{2n+1} - U_{2n}})}{h} - \frac{(U_{2n} - U_{2n-1})}{h}\right]}{h} = f_{2n}$$

$$\Rightarrow \frac{U_{2n} - U_{2n-1}}{h^2} = f_{2n}$$

Book has a ½ here? How get?

Gauss Seidel $\quad u_1^m = U_2^{(m-1)} + h^2 f_1$

$$-u_{j-1}^m + 2u_j^m = u_{j+1}^{(m-1)} + h^2 f_j \qquad j = 2, 3, \ldots, 2n-1 \quad \text{eq } 2.2.5$$

$$-U_{2n-1}^m + U_{2n}^m = \frac{1}{2} h f_{2n}$$

- - - Assuming periodicity $U(1) = U(0)$ is this enough of a BC to entirely determine the problem sol.?

let $e^m = U^m - U^\infty$ since U^∞ satisfys the above equations exactly i.e.

$$\cancel{U_1^\infty = U_2^\infty + h^2 f_1}$$

$$-U_{j-1}^\infty + 2U_j^\infty = U_{j+1}^\infty + h^2 f_j$$

$$\cancel{-U_{2n-1}^\infty + U_{2n}^\infty = \frac{1}{2} h f_{2n}}$$

subtracting these from the ones above gives

~~$U^m_1 = U_1$~~

One would expect the continuous solution to be periodic (why?)

Then Assuming U^m & U^∞ are also periodic subtraction

$$-e^m_{j-1} + 2e^m_j = e^{m-1}_{j+1} \qquad e^m_j = e^m_{j+2n} \qquad \text{eq } 2.2.7$$

$$e^m_j = \sum_{\alpha=0}^{2n-1} c^m_\alpha e^{ij\theta_\alpha} \qquad \theta_\alpha = \frac{\pi\alpha}{n} \qquad \alpha \text{ th Fourier mode}$$

let $e^m_{j-1} \rightarrow c^m_\alpha e^{i(j-1)\theta_\alpha}$ into Above

$$-c^m_\alpha e^{i(j-1)\theta_\alpha} + 2c^m_\alpha e^{i(j)\theta_\alpha} = c^{m-1}_\alpha e^{i(j+1)\theta_\alpha}$$

$$c^m_\alpha \left[-e^{-i\theta_\alpha} + 2 \right] = c^{m-1}_\alpha e^{i\theta_\alpha}$$

$$c^m_\alpha = \underbrace{\frac{e^{i\theta_\alpha}}{2 - e^{-i\theta_\alpha}}}_{g(\theta_\alpha)} c^{m-1}_\alpha \qquad \text{eq } 2.2.9$$

$$|g(\theta_\alpha)| = \left(g(\theta_\alpha)^* g(\theta_\alpha) \right)^{1/2}$$

⋮

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$$\left(g(\theta_\alpha)^* g(\theta_\alpha)\right)^{1/2} = \left(\frac{e^{-i\theta_\alpha}}{2-e^{i\theta_\alpha}} \cdot \frac{e^{+i\theta_\alpha}}{2-e^{-i\theta_\alpha}}\right)^{1/2}$$

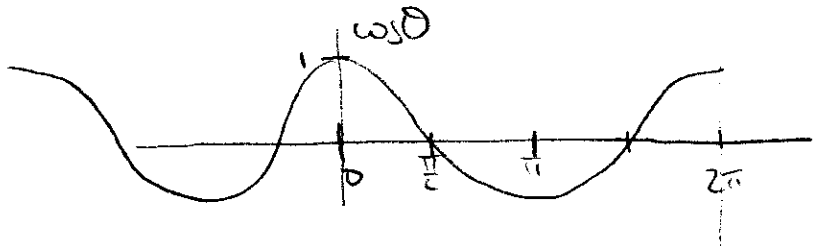
$$= \frac{1}{\left(4 - 2e^{-i\theta_\alpha} - 2e^{i\theta_\alpha} + 1\right)^{1/2}}$$

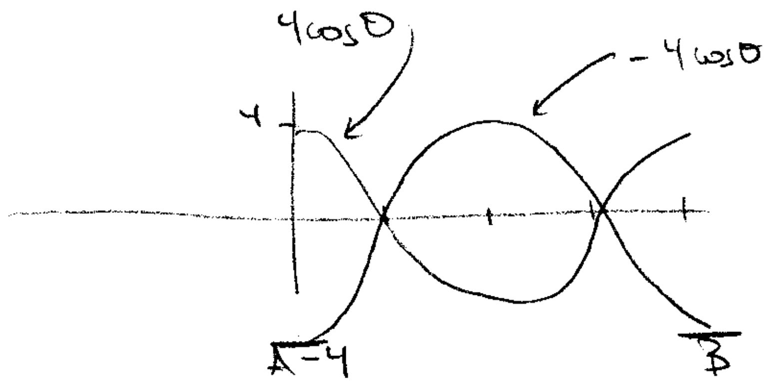
$$|g(\theta_\alpha)| = \frac{1}{\left(5 - 2^2\cos\theta_\alpha\right)^{1/2}} = \frac{1}{\left(5 - 4\cos\theta_\alpha\right)^{1/2}} \qquad \text{eq } 2.2.10$$

$\theta_\alpha = \dfrac{2\pi\alpha}{2n} = \dfrac{\pi\alpha}{n}$ $\quad \alpha = 0, 1, 2, \ldots, 2n-1 \Rightarrow \theta_\alpha$ from 0 to $\dfrac{(2n-1)\pi}{n}$

$= 2\pi - \dfrac{\pi}{n} \approx 2\pi$ for large n.

$\max\limits_\alpha |g(\theta_\alpha)| =$ occurs when $5 - 4\cos\theta_\alpha$ is the smallest





When I shifts the graph up by 5 these two Points will be the lowest

$5 - 4\cos\theta$

$$\max_\alpha |g(\theta_\alpha)| = |g(0)| = |g(2\pi)|$$
$$= \underline{1} \quad \text{eq } 2.2.11$$

$g(\theta_1) =$

$\theta_1 \ll 1$ for large n

For some reason we don't need decay of the Fourier mode $\alpha = 0$. ?

$$\max_\alpha |g(\theta_\alpha)| = |g(\theta_1)|$$
$$= \frac{1}{\left(5 - 4\left(1 - \frac{\theta_1^2}{2} + O(\theta_1^4)\right)\right)^{1/2}} = \frac{1}{\left(1 + 2\theta_1^2 + O(\theta_1^4)\right)^{1/2}}$$
$$= 1 - \theta_1^2 + O(\theta_1^4) = 1 - (2h\pi)^2 + O(h^4)$$

$\theta_1 = \frac{\pi(1)}{n} = 2h\pi$ $\qquad = 1 - 4\pi^2 h^2 + O(h^4) \quad$ eq 2.2.12

$h = \frac{1}{2n} \Rightarrow n = \frac{1}{2h}$

Define smoother $\rho = \max\{|g(\theta_\alpha)| \quad \theta_\alpha \in \Theta_r\}$

$$\rho = \frac{1}{(5 - 4\cos\theta_{cn})^{1/2}} = \frac{1}{(5 - 4\cos(\frac{c\pi}{n}))^{1/2}} = \frac{1}{(5 - 4\cos(c\pi))^{1/2}}$$

$c = 1$

$$\Rightarrow \rho = \frac{1}{(5 - 4\cos(\pi))^{1/2}} = \frac{1}{9^{1/2}} = \frac{1}{3} \quad \text{don't get } 2.2.17\,?$$

Ex 2.2.1 Point Jacobi gives $M = \mathrm{diag}(A) \equiv D$

Then $Ay = b$

$$A = D - N$$

$$Dy^{m+1} = Ny^m + b \Rightarrow y^{m+1} = D^{-1}Ny^m + D^{-1}b$$

to make the damped update y^{m+1} as

$$y^* = D^{-1}Ny^m + D^{-1}b$$

$$y^{m+1} = \omega y^* + (1-\omega)y^m$$

$$= \omega(D^{-1}Ny^m + D^{-1}b) + (1-\omega)y^m$$

$$= \omega D^{-1}Ny^m + (1-\omega)y^m + \omega D^{-1}b$$

$$Dy^{m+1} = \omega Ny^m + (1-\omega)Dy^m + \omega b$$

Applied to problem 2.2.5 which is really 2.2.3

$$-u_{j-1} + 2u_j - u_{j+1} = h^2 f_j$$

$$\Rightarrow 2u_j^{m+1} = \omega(u_{j-1}^m + u_{j+1}^m) + 2(1-\omega)u_j^m + \omega h^2 f_j \quad \Big] \text{ Point Jacobi iteration}$$

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This is a linear expression & to be consistent we can Decompose into a discrete fourier series as in eq ??.3

$$e_j^m = \sum_{\alpha=0}^{2n-1} c_\alpha^m e^{j\theta_\alpha} \qquad \theta_\alpha = \frac{2\pi\alpha}{2n} = \frac{\pi\alpha}{n} \quad \text{Then}$$

putting one Fourier component in gives $c_\alpha^m e^{j\theta_\alpha}$

In rest[?] the error terms $h^2 f_j$ drops away

$$2c_\alpha^{m+1} = \omega\left(e^{-i\theta_\alpha} + e^{+i\theta_\alpha}\right)c_\alpha^m + 2(1-\omega)c_\alpha^m$$

$$= \left(2(1-\omega) + 2\omega\cos\theta_\alpha\right)c_\alpha^m$$

$$c_\alpha^{m+1} = \underbrace{\left((1-\omega) + \omega\cos\theta_\alpha\right)}_{g(\theta_\alpha)} c_\alpha^m$$

Don't see where Boundary conditions come into play?

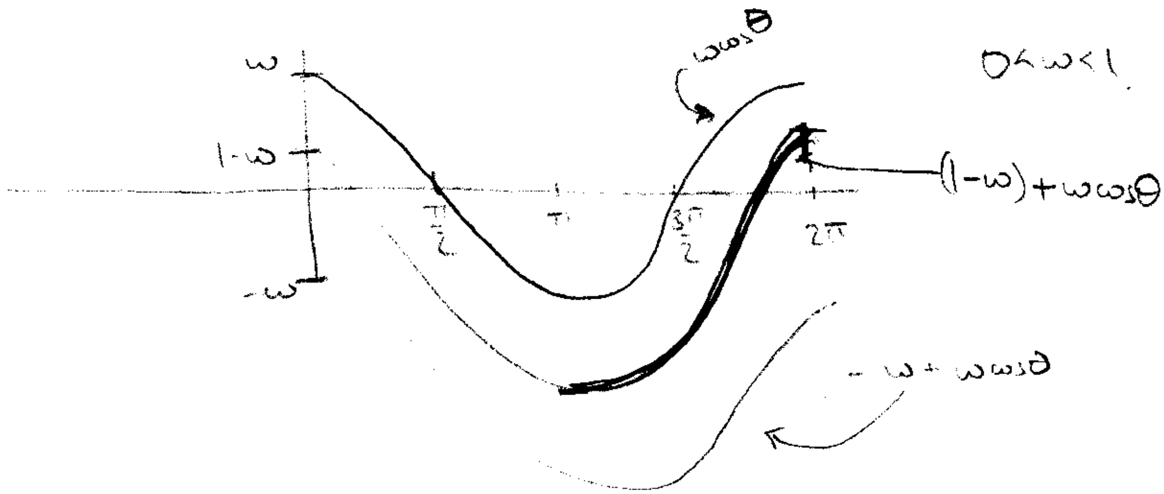
$$|g(\theta_\alpha)| = |(1-\omega) + \omega\cos\theta_\alpha|$$

Smoothing factor is defined as $\rho = \max\{|g(\theta_\alpha)| : \theta_\alpha \in \theta r\}$

$$\theta r = \left\{\theta_\alpha = \frac{\pi\alpha}{n},\ \alpha > n,\ \alpha = 1, 2, 3, \ldots, 2n-1\right\}$$

ρ is maximized over all coarse or rough wave #'s.

plot $(1-\omega) + \omega\cos\theta \qquad \theta \in \pi, 2\pi$



ON $(\pi, 2\pi)$ largest values of $(1-\omega)+\omega\cos\theta$ occur at $\theta \simeq 2\pi$.

Then $\rho = (1-\omega)+\omega\cos\theta_{\alpha}\Big|_{\alpha=2n-1} = (1-\omega)+\omega\cos\left(\frac{(2n-1)\pi}{n}\right)$

$$= (1-\omega)+\underbrace{\omega\cos\left(2\pi - \frac{\pi}{n}\right)}$$

$$= \qquad \omega\left(1-\frac{\pi^2}{n^2}+O\left(\frac{1}{n^4}\right)\right)$$

$$= 1-\omega+\omega-\frac{\omega\pi^2}{n^2}+O\left(\frac{1}{n^4}\right) = 1-\frac{\omega\pi^2}{n^2}+$$

$h = \frac{1}{2n} \Rightarrow n = \frac{1}{2h}$

$$\Rightarrow \rho = 1-\omega 4h^2\pi^2+O(h^4)$$

$$= 1-4\pi^2\omega h^2+O(h^4)$$

I would guess $\omega=0$ is not a smoother.

PS 8 Wessling 07-23-01

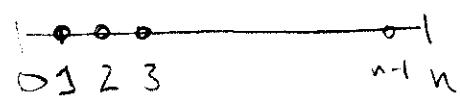
Ex 2.2.2 w/ Drichlet B.C. $U(0) = U(1) = 0$ the discrete representation should also hold this condition also $\Rightarrow$

Discrete Fourier sin transform:

$$U_j = \sum_{k=1}^{n-1} c_k \sin(j\theta_k) \qquad \theta_k = \frac{\pi k}{n} \qquad w/ \qquad I = \{1, 2, \cdots, n-1\}$$

$$c_k = \frac{2}{n} \sum_{j=1}^{n-1} U_j \sin(j\theta_k)$$

Then problem 2.2.5 w/ Drichlet B.C. becomes



Writting discretized version of $-\frac{d^2U}{dx^2}$ at node 1 & node n-1

we get $j = 1$ in eq 2.2.3

$$h^{-2}(0 + 2U_1 - U_2) = f_1 \rightarrow h^{-2}(2U_1 - U_2) = f_1$$

$j = n-1$ in eq 2.2.3

$$h^{-2}(-U_{n-2} + 2U_{n-1} - 0) = f_{n-1}$$

Thus for Drichlet B.C get following 3 eq

$$h^{-2}(2U_1 - U_2) = f_1$$

$$h^{-2}(-U_{i-1} + 2U_i - U_{i+1}) = f_i$$

$$h^{-2}(-U_{n-2} + 2U_{n-1}) = f_{n-1}$$

Then from section Chapter 7 U_i can be approximated as

a Fourier sin series

$$U_i = \sum_{k=1}^{n-1} C_k \sin(i\theta_k)$$

since our problem is linear we can put one term in k work w/ it,

Questions: 1) what do I do w/ the f_i's? Decompose them in a sin series?

2) What about Boundary conditions how do I handle them?

To ignore both of these problems I'll just work w/ the Gauss-Seidel version of

$$-U_{i-1}^{n} + 2U_i^{n} - U_{i+1}^{n-1} = 0$$

$$-C_k^{n}\sin((i-1)\theta_k) + 2C_k^{n}\sin(i\theta_k) - C_k^{n-1}\sin((i+1)\theta_k) = 0$$

$$= -C_k^{n}\sin(i\theta_k - \theta_k) + 2C_k^{n}\sin(i\theta_k) - C_k^{n-1}\sin(i\theta_k + \theta_k) = 0$$

$$C_k^{n} = \frac{C_k^{n-1}\sin((i+1)\theta_k)}{2\sin(i\theta_k) - \sin((i-1)\theta_k)}$$

$$\Rightarrow C_k^{n} = \underbrace{\frac{\sin((i+1)\theta_k)}{2\sin(i\theta_k) - \sin((i-1)\theta_k)}}_{g} C_k^{n-1}$$

How do I get rid of grid point index i

$$= \frac{\sin(i\theta_k)\cos\theta_k + \sin\theta_k\cos(i\theta_k)}{2\sin(i\theta_k) - [\sin(i\theta_k)\cos\theta_k - \sin\theta_k\cos(i\theta_k)]}$$

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$$= \frac{\cos\theta_k \ \sin(i\theta_k) + \sin\theta_k \cos(i\theta_k)}{(2-\cos\theta_k)\sin(i\theta_k) + \sin\theta_k \cos(i\theta_k)}$$

Ex 2.2.3

$$-\epsilon \frac{d^2U}{dx^2} + c\frac{dU}{dx} = f$$

Since I am not given B.C. I'll just descretize the PDE.

$$-\frac{\epsilon}{\Delta x^2}(U_{i+1} - 2U_i + U_{i-1}) + \frac{c}{2\Delta x}(U_{i+1} - U_{i-1}) = f_i$$

$\Delta x = h$

$$\Rightarrow \left(-\frac{\epsilon}{h^2} + \frac{c}{2h}\right)U_{i+1} + \left(\frac{2\epsilon}{h^2}\right)U_i + \left(-\frac{\epsilon}{h^2} - \frac{c}{2h}\right)U_{i-1} = f_i$$

Then Gauss-Seidel Applied to this eq becomes

$$\left(-\frac{\epsilon}{h^2} + \frac{c}{2h}\right)U^{n-1}_{i+1} + \frac{2\epsilon}{h^2}U^n_i + \left(-\frac{\epsilon}{h^2} - \frac{c}{2h}\right)U^n_{i-1} = f_i \overset{=0}{} \text{ For time being.}$$

Then Assuming periodic B.C.'s U^n_i satisfys a discrete fourier series Because problem is linear $U^n_i = C^n_\alpha e^{iI\theta_\alpha}$ $\theta_\alpha = \frac{\pi\alpha}{n}$

$$\Rightarrow \left(-\frac{\epsilon}{h^2} + \frac{c}{2h}\right)C^{n-1}_\alpha e^{I\theta_\alpha} + \frac{2\epsilon}{h^2}C^n_\alpha + \left(-\frac{\epsilon}{h^2} - \frac{c}{2h}\right)C^n_\alpha e^{-I\theta_\alpha} = 0$$

$$C^n_\alpha = \frac{\left(-\frac{\epsilon}{h^2} + \frac{c}{2h}\right)e^{I\theta_\alpha}}{\frac{2\epsilon}{h^2} + \left(-\frac{\epsilon}{h^2} - \frac{c}{2h}\right)e^{-I\theta_\alpha}} C^{n-1}_\alpha$$

$$C_\alpha^n = \frac{\frac{1}{h}\left(-\frac{\epsilon}{h} + \frac{c}{2}\right)e^{I\theta_\alpha}}{\frac{2\epsilon}{h^2} - \frac{1}{h}\left(+\frac{\epsilon}{h} + \frac{c}{2}\right)e^{-I\theta_\alpha}} C_\alpha^{n-1}$$

$$= \frac{\left(-\frac{\epsilon}{h} + \frac{c}{2}\right)e^{I\theta_\alpha}}{\left(\frac{2\epsilon}{h} - \left(\frac{\epsilon}{h} + \frac{c}{2}\right)e^{-I\theta_\alpha}\right)} C_\alpha^{n-1}$$

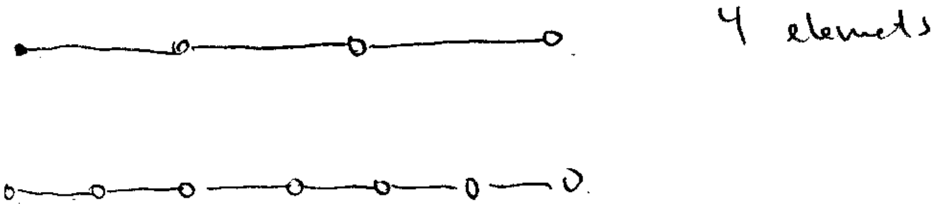
$$= \frac{\left(-\frac{\epsilon}{h} + \frac{c}{2}\right)e^{I\theta_\alpha}\left(\frac{2\epsilon}{h} - \left(\frac{\epsilon}{h} + \frac{c}{2}\right)e^{I\theta_\alpha}\right)}{\left(\frac{2\epsilon}{h} - \left(\frac{\epsilon}{h} + \frac{c}{2}\right)e^{-I\theta_\alpha}\right)\left(\frac{2\epsilon}{h} - \left(\frac{\epsilon}{h} + \frac{c}{2}\right)e^{I\theta_\alpha}\right)} C_\alpha^{n-1}$$

$$= \frac{\text{''}}{\frac{4\epsilon^2}{h^2} - 2\left(\frac{\epsilon}{h} + \frac{c}{2}\right)\cos\theta_\alpha + \left(\frac{\epsilon}{h} + \frac{c}{2}\right)^2} C_\alpha^{n-1}$$

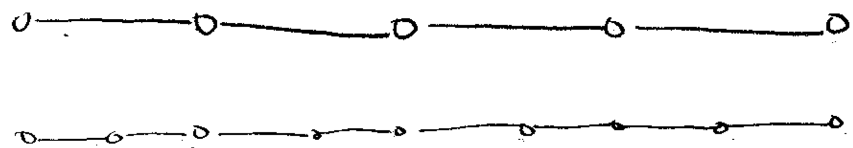
Compute $|g(\theta)|$ & check that

$$\rho = \max\{|g(\theta_\alpha)| : \theta_\alpha \in \Theta_r\} > 1$$

An example of 4 & 8 elements will help explain



Have to have even # of cells



Doesn't seem to be working. Assume simply that however the discretization is done the fine grid has twice as many elements as the coarse grid.

Take 4 & 8 as an example

$$\frac{1}{2}\begin{pmatrix} 1 & 0 & & \\ 2 & 0 & & \\ 1 & 1 & & \\ & 2 & & \\ & 1 & 1 & \\ & & 2 & \\ & & 1 & 1 \\ & & & 2 \\ & & & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(u_1+u_0) \\ u_1 \\ \frac{1}{2}(u_1+u_2) \\ u_2 \\ \frac{1}{2}(u_2+u_3) \\ u_3 \\ \frac{1}{2}(u_3+u_4) \\ u_4 \\ \frac{1}{2}(u_4 \cancel{+} u_5) \end{pmatrix}$$

influence of u_1

influence of u_2

of u_3

Something incorrect about boundaries

$$P = \frac{1}{2}\begin{pmatrix} 1 & & \\ 2 & & \\ 1 & 1 & 0 \\ & 2 & \\ & 1 & 1 \\ 0 & & 2 \\ & & 1 \quad 1 \end{pmatrix}$$

In our $4 \to 8$ example

$$P = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

Then Restriction

$$R\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{pmatrix} = \begin{pmatrix} \frac{1}{4}(u_1 + 2u_2 + u_3) \\ \frac{1}{4}(u_3 + 2u_4 + u_5) \\ \frac{1}{4}(u_5 + 2u_6 + u_7) \\ \frac{1}{4}(u_7 + 2u_8 + ?) \end{pmatrix}$$

$$R = \frac{1}{4}\begin{pmatrix} 1 & 2 & 1 & & & & & \\ & & 1 & 2 & 1 & & & \\ & & & & 1 & 2 & 1 & \\ & & & & & & 1 & 2 \end{pmatrix}$$

Now $$P^T = \frac{1}{2}\begin{pmatrix} 1 & 2 & 1 & & & & & \\ . & . & 1 & 2 & 1 & & & \\ . & . & . & . & 1 & 2 & 1 & . \\ . & . & . & . & . & . & 1 & 2 \end{pmatrix} = 2R$$

$e = \hat{v} - v$ $\hat{v}$ is an approximation to $Av = f$

Note: this definition is backwards from the commonly referenced one.

No $Ae = A\hat{v} - Av = A\hat{v} - f = -r$ eq 2.3.11

$\Rightarrow$ $A(-e) = r$

Let $\bar{v}$ be the coarse grid approx of the above eq

$\Rightarrow$ $\bar{v}$ solves $\bar{A}\bar{v} = \bar{r}$ exactly

Then to update $\hat{v}$ to obtain the next guess we can use.

$$\hat{v} = \hat{v} + P\bar{v} \cong \hat{v} - e = \hat{v} - (\hat{v} - v) = v$$ motivation for eq 2.3.13

$\uparrow$ to $-e$

since $\bar{v}$ is the coarse grid approx. when I project to the fine grid I get a fine grid approximation of $-e$

Ex 2.3.1

Au_i can be written in compact form as

$$\Omega = [0,1] \quad u(0) = \frac{du(1)}{dx} = 0$$

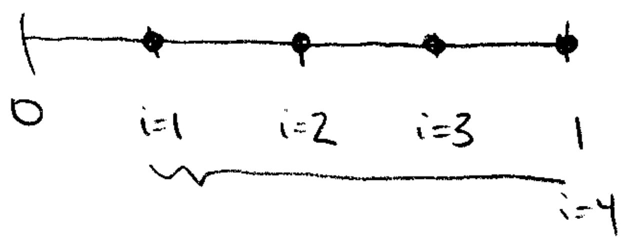
$$G = \{x = x_j = jh,\ j = 1, 2, \dots, 2n,\ h = 1/2n\}$$

excludes 0 (Dirichlet condition like in Finite Elements)

but includes end point of Neumann

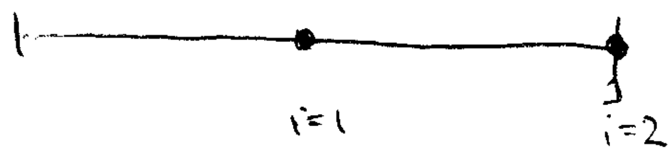
For ease of understanding lets put down an example grid. take $n = 2$

$$G = \{x_j = jh,\ j = 1, 2, 3, 4,\ h = 1/4\}$$



if $v \in G \Rightarrow \vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}$

Then $\bar{G} = \{\bar{x}_j = j\bar{h},\ j = 1, 2,\ h = 1/2\}$



$\vec{\bar{v}} \in G \Rightarrow \bar{v} = \begin{pmatrix} \bar{v}_1 \\ \bar{v}_2 \end{pmatrix}$

Dirichlet condition.

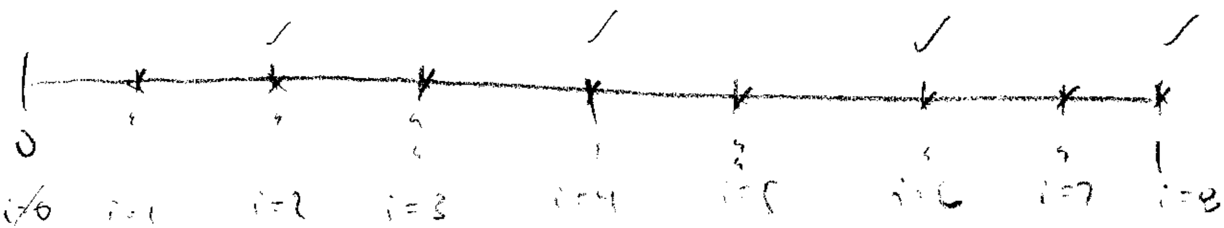
$$\& \quad P\begin{pmatrix} \bar{v}_1 \\ \bar{v}_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(\bar{v}_1 + 0) \\ \bar{v}_1 \\ \frac{1}{2}(\bar{v}_1 + \bar{v}_2) \\ \bar{v}_2 \end{pmatrix} \quad \therefore \quad P = \begin{pmatrix} \frac{1}{2} & 0 \\ 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 2 & 0 \\ 1 & 1 \\ 0 & 2 \end{pmatrix}$$

$\therefore R =$

Note: there are really No internal nodes in this case. Do case w/ $n = 4$. & see this example.

$n = 4$

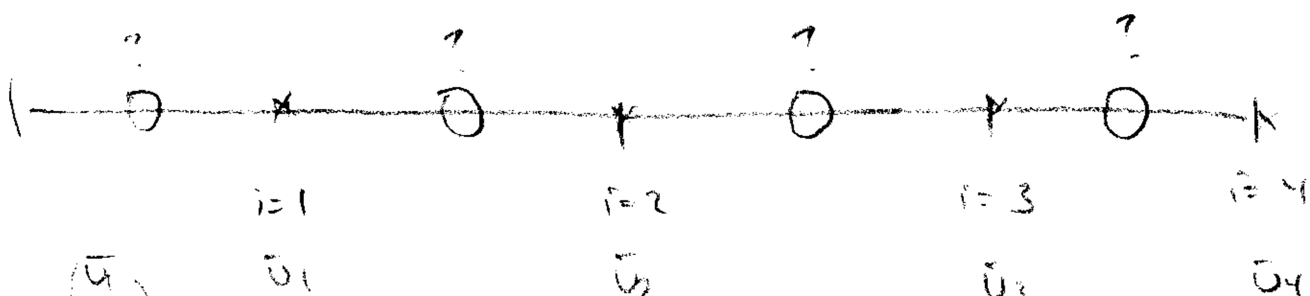
$$G = \{ x_j = jh,\ j = 1,2,3,4,5,6,7,8,\ h = 1/8 \}$$



$\bar{U} \in$

$$\bar{U} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{pmatrix}$$

$$\bar{G} = \{ x_j = j\bar{h},\ j = 1,2,3,4\ \ \bar{h} = 1/4 \}$$



$$\bar{U} = \begin{pmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \bar{u}_3 \\ \bar{u}_4 \end{pmatrix}$$

$$P\begin{pmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \bar{u}_3 \\ \bar{u}_4 \end{pmatrix} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \begin{pmatrix} \frac{1}{2}(0 + \bar{u}_1) \\ \bar{u}_1 \\ \frac{1}{2}(\bar{u}_1 + \bar{u}_2) \\ \bar{u}_2 \\ \frac{1}{2}(\bar{u}_2 + \bar{u}_3) \\ \bar{u}_3 \\ \frac{1}{2}(\bar{u}_3 + \bar{u}_4) \\ \bar{u}_4 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 1 & & & \\ \frac{1}{2} & \frac{1}{2} & & \\ & 1 & & \\ & \frac{1}{2} & \frac{1}{2} & \\ & & 1 & \end{pmatrix}$$

Thus

$$P = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{pmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & & & & & & \\ & & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & & & & \\ & & & & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & & \\ & & & & & & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} \frac{1}{2} & 1 & \frac{1}{2} & & & & & \\ & & \frac{1}{2} & 1 & \frac{1}{2} & & & \\ & & & & \frac{1}{2} & 1 & \frac{1}{2} & \\ & & & & & & \frac{1}{2} & 1 \end{bmatrix}$$

Different [illegible] to end conditions of prob not these [illegible] because they don't Add to 1 + the value of $u_{n+1} \stackrel{?}{=} 0$

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$$R A \vec{v}_i = R \begin{cases} h^{-2}(2v_1 - v_2) & i=1 \\ h^{-2}(-v_{i-1} + 2v_i - v_{i+1}) & i = 2,3,\dots,2n-1 \\ h^{-2}(-v_{2n-1} + v_{2n}) & i = 2n \end{cases}$$

↑ Restriction operator acting on these equations

$$\tfrac{1}{4} h^{-2}(2v_1 - v_2) + \tfrac{1}{2} h^{-2}(-v_1 + 2v_2 - v_3) + \tfrac{1}{4} h^{-2}(-v_2 + 2v_3 - v_4)$$

$$= \tfrac{1}{4} h^{-2}(-v_{i-2} + 2v_{i-1} - v_i) + \tfrac{1}{2} h^{-2}(-v_{i-1} + 2v_i - v_{i+1}) + \tfrac{1}{4} h^{-2}(-v_i + 2v_{i+1} - v_{i+2})$$

$$\tfrac{1}{4}\left(h^{-2}(-v_{2n-2} + 2v_{2n-1} - v_{2n})\right) + \tfrac{1}{2} h^{-2}(-v_{2n-1} + v_{2n})$$

$$= h^{-2}\left[\left(\tfrac{1}{2} - \tfrac{1}{2}\right)v_1 + \left(-\tfrac{1}{4} + 1 - \tfrac{1}{4}\right)v_2 + \left(-\tfrac{1}{2} + \tfrac{1}{2}\right)v_3 - \tfrac{1}{4} v_4\right] \quad \bar{i} = 1$$

$$h^{-2}\left[-\tfrac{1}{4} v_{i-2} + \left(\tfrac{1}{2} - \tfrac{1}{2}\right)v_{i-1} + \left(-\tfrac{1}{4} + 1 - \tfrac{1}{4}\right)v_i + \left(-\tfrac{1}{2} + \tfrac{1}{2}\right)v_{i+1} + \left(-\tfrac{1}{4}\right)v_{i+2}\right] \quad \bar{i} = 2,3,\dots,n- \quad i = 2\bar{i}$$

$$h^{-2}\left[-\tfrac{1}{4} v_{2n-2} + \left(\tfrac{1}{2} - \tfrac{1}{2}\right)v_{2n-1} + \left(-\tfrac{1}{4} + \tfrac{1}{2}\right)v_{2n}\right] \quad \bar{i} = n$$

$$= h^{-2}\left[\tfrac{1}{2}u_2 - \tfrac{1}{4}u_4\right] \qquad \bar{i} = 1$$

$$h^{-2}\left[-\tfrac{1}{4}u_{i-2} + \tfrac{1}{2}u_i - \tfrac{1}{4}u_{i+2}\right] \qquad \bar{i} = 2, 3, \cdots, n-1 \quad i = 2\bar{i}$$

$$h^{-2}\left[-\tfrac{1}{4}u_{2n-2} + \tfrac{1}{4}u_{2n}\right] \qquad \bar{i} = n$$

$\bar{A} = RAP$ replace u_i by $P\bar{u}_j$

$$P\bar{u}_j = \begin{array}{cc} \begin{pmatrix} \tfrac{1}{2}\bar{u}_1 \\ \bar{u}_1 \\ \tfrac{1}{2}(\bar{u}_1 + \bar{u}_2) \\ \bar{u}_2 \\ \vdots \\ \tfrac{1}{2}(\bar{u}_{p-1} + \bar{u}_p) \\ \bar{u}_p \\ \tfrac{1}{2}(\bar{u}_p + \bar{u}_{p+1}) \\ \vdots \\ \tfrac{1}{2}(\bar{u}_n + \bar{u}_{n-1}) \\ \bar{u}_n \end{pmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ \vdots \\ 2p-1 \\ 2p \\ 2p+1 \\ \vdots \\ 2n-1 \\ 2n \end{matrix} \end{array}$$

Thus As hint suggests replace u_i by $P\bar{u}_j$

$$= \bar{A}\bar{u}_j = \begin{cases} h^{-2}\left[\tfrac{1}{2}(\bar{u}_1) - \tfrac{1}{4}\bar{u}_2\right] \\ h^{-2}\left[-\tfrac{1}{4}\bar{u}_{2p-1} + \tfrac{1}{2}\bar{u}_{2p} - \tfrac{1}{4}\bar{u}_{2p+2}\right] \\ h^{-2}\left[-\tfrac{1}{4}(\bar{u}_{n-1}) + \tfrac{1}{4}\bar{u}_n\right] \end{cases}$$

Thus as $\bar{h} = 2h$

$$\bar{A}\bar{u}_j = \bar{h}^{-2}\left[2\bar{u}_1 - \bar{u}_2\right]$$

$$\bar{h}^{-2}\left[-\bar{u}_{2p-1} + \bar{u}_{2p} - \bar{u}_{2p+2}\right] \qquad \text{eq } 2.3.10$$

$$\bar{h}^{-2}\left[-\bar{u}_{n-1} + \bar{u}_n\right]$$

$$e^{2/3} = u^{2/3} - u$$

$$= u^{1/3} + P\bar{u} - u = u^{1/3} - u + P\bar{u}$$

$$= e^{1/3} + P\bar{u}$$

$$= e^{1/3} + P\bar{A}^{-1}Rr \qquad *$$

$$= e^{1/3} + P\bar{A}^{-1}R(f - Au^{1/3})$$

What is exact def of $\bar{u}^{1/3}$?

$$= e^{1/3} + P\bar{A}^{-1}Rf - P\bar{A}^{-1}RAu^{1/3} \qquad **$$

sol to $\bar{A}\bar{u}^{1/3} = Rr = R(f - Au^{1/3})$

exact sol to

$\Rightarrow \bar{u}^{1/3} = \bar{A}^{-1}Rr$ put in line eq $*$ & we get

$$e^{2/3} = e^{1/3} + P\bar{u}^{1/3}$$

but from $**$ we get eq 2.4.1

$$e^{2/3} = e^{1/3} + P\bar{A}^{-1}Rf - P\bar{A}^{-1}RA(e^{1/3} + u)$$

$e^{1/3} = u^{1/3} - u$

$u^{1/3} = e^{1/3} + u$

$$= e^{1/3} - P\bar{A}^{-1}RAe^{1/3} + \underbrace{P\bar{A}^{-1}Rf - P\bar{A}^{-1}RAu}_{=0}$$

$$e^{2/3} = (\underbrace{I - P\bar{A}^{-1}RA}_{E})e^{1/3} \qquad \text{eq } 2.4.1$$

$\bar{e} = \cdots$ & $\bar{e}_j = e^{1/3}_{2j}$

let $e^{1/3} = d + P\bar{e}$ d is vector that makes up the difference between $e^{1/3}$ & $P\bar{e}$.

Then $e^{2/3} = Ee^{1/3} = Ed + EP\bar{e}$

$$EP = P - P\bar{A}^{-1}RAP$$

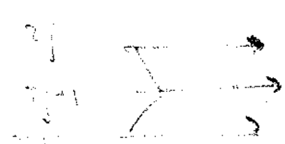
$$= P(I - \bar{A}^{-1}RAP) \equiv 0$$

w/ $\bar{A} = RAP$ $\bar{A}^{-1}RAP = \bar{A}^{-1}\bar{A} = I$

$e^{2/3} = Ed$ eq 2.4.4

From 2.4.3 $d = e^{1/3} - P\bar{e}$

$$= \begin{cases} e^{1/3}_1 - \frac{1}{2}e^{1/3}_2 & 1 \\ e^{1/3}_2 - \bar{e}^{1/3}_2 & 2 \\ e^{1/3}_3 - \frac{1}{2}(e^{1/3} & 3 \\ \vdots & \\ e^{1/3}_{2j} - e^{1/3}_{2j} = 0 & 2j \\ e^{1/3}_{2j+1} - \frac{1}{2}(e^{1/3}_{2j} + e^{1/3}_{2j+2}) = -\frac{1}{2}e^{1/3}_{2j} + e^{1/3}_{2j+1} - \frac{1}{2}e^{1/3}_{2j+2} & \end{cases}$$



eq 2.4.5

$RAd = ?$

RAd was computed in the Exercise 2.3.1

$$RAd = h^{-2}\left[\tfrac{1}{2} d_2 - \tfrac{1}{4} d_4\right] \qquad \bar{\imath} = 1$$

$$h^{-2}\left[-\tfrac{1}{4} d_{i-2} + \tfrac{1}{2} d_i - \tfrac{1}{4} d_{i+2}\right] \qquad \bar{\imath} = 2, 3, \ldots, n-1, \quad i = 2\bar{\imath}$$

$$h^{-2}\left[-\tfrac{1}{4} d_{2n-2} + \tfrac{1}{4} d_{2n}\right] \qquad \bar{\imath} = n$$

$$= h^{-2}\left[\tfrac{1}{2}\cdot 0 \quad \right] = 0$$

0 as each comp 2,4,6

0

Since $e^{2/3} = Ed = (I - P\bar{A}^{-1}RA)d$

$d \quad RAd = 0 \Rightarrow e^{2/3} = d$ as 2,4,7

Pg 16 Wessling

Ex 3.2.1 Given Uniform ellipticity, we get

$$a_{\alpha\beta} V_\alpha V_\beta \geq C V_\alpha V_\alpha$$

let $C = 0$ In $\mathbb{R}^2$

$$a_{11} V_1 V_1 + a_{12} V_1 V_2 + a_{21} V_2 V_1 + a_{22} V_2^2 \geq 0$$

$$= \quad a_{11} V_1^2 + 2a_{12} V_1 V_2 + a_{22} V_2^2 \geq 0$$

$\Rightarrow$

$e^1 = U^1 - U$ $\qquad 2U_1^m = U_2^{m-1} + h^2 f_1$

Doing 1 step on $U^{2/3}$ produces U^1

$$\Rightarrow \quad 2U_1^1 = U_2^{2/3} + h^2 f_1$$

$$(1) \qquad -U_{j-1}^1 + 2U_j^1 = U_{j+1}^{2/3} + h^2 f_j \qquad j = 2, 3, \dots, 2n-1$$

$$-U_{2n-1}^1 + U_{2n}^1 = \tfrac{1}{2} h^2 f_{2n}$$

Then since U satisfies these equations exactly $\Rightarrow$

$$2U_1 = U_2 + h^2 f_1$$

$$(2) \qquad -U_{j-1} + 2U_j = U_{j+1} + h^2 f_j$$

$$-U_{2n-1} + U_{2n} = \tfrac{1}{2} h^2 f_{2n}$$

Subtracting eqs (2) from (1) we get

$$2e_1^1 = e_2^{2/3}$$

$$-e_{j-1}^1 + 2e_j^1 = e_{j+1}^{2/3} \qquad j = 2, 3, \dots, 2n-1 \qquad \text{eq } 2.4.8$$

$$-e_{2n-1}^1 + e_{2n}^1 = 0$$

Using 2.4.7 to replace $e^{2/3}$ w/ d we get

$$2e_1^1 = d_2$$

$$-e_{j-1}^1 + 2e_j^1 = d_{j+1} \qquad j = 2, 3, 4, \dots, 2n-1$$

$$-e'_{2n-1} + e'_{2n} = 0 \quad (2)$$

Then using 2.4.5 we get

$$e'_1 = 0 \qquad \text{eq } 2.4.9.1$$

looking at $j = 2, 4, 6, \ldots, 2n-2$ get $-e'_{j-1} + 2e'_j = d_{j+1}$

or $-e'_{2p-1} + 2e'_{2p} = d_{2p+1}$

Since j is even

$$e'_{2p} = \frac{1}{2} d_{2p+1} + \frac{1}{2} e'_{2p-1} \qquad (1)$$

looking at $j = 3, 5, 7, \ldots, 2n-1$ get $\qquad j = 2p+1$

since j is odd

$$-e'_{2p} + 2e'_{2p+1} = \cancelto{0}{d_{2p+2}}$$

$$e'_{2p+1} = \frac{1}{2} e'_{2p}$$ using this relationship in eq(1)

eq 2.4.9.2

gives

$$e'_{2p} = \frac{1}{2} d_{2p+1} + \frac{1}{4} e'_{2p-2} \qquad \text{eq } 2.4.9.2$$

Then equation (2) becomes

$$e'_{2n} = e'_{2n-1}$$

Thus we get in summary

$$e_1^1 = 0$$

$$e_{2j}^1 = \frac{1}{2} d_{2j+1} + \frac{1}{4} e_{2j-2}^1 \qquad e_{2j+1}^1 = \frac{1}{2} e_{2j}^1 \quad j = 1, 2, \cdots, n-1$$

$$e_{2n}^1 = e_{2n-1}^1 \qquad \text{or eqs } 2.4.9$$

Then $e_2^1 = \frac{1}{2} d_3 + 0 = \frac{1}{2} d_3$

$$e_3^1 = \frac{1}{2} e_2^1 = \frac{1}{2^2} d_3$$

$$e_4^1 = \frac{1}{2} d_5 + \frac{1}{4} e_2^1 = \frac{1}{2} d_5 + \frac{1}{2} d_3$$

$$e_5^1 = \frac{1}{2} e_4^1 = \frac{1}{2}\left(\frac{1}{2}\right)(d_3 + d_5)$$

Don't see how to get 2.4.10.

$$|e_{2j}^1| \leq \frac{2}{3} \|d\|_\infty$$

$$|e_{2j}^1| \leq \frac{2}{3} \|e^{2/3}\|_\infty \qquad \text{How?}$$

$$\Rightarrow \|e^1\|_\infty \leq \frac{2}{3} \|e^{2/3}\|_\infty \leq \frac{2}{3} \|e^0\|_\infty$$

Ex 24.1 Assume Neumann condition holds at $x = 0$

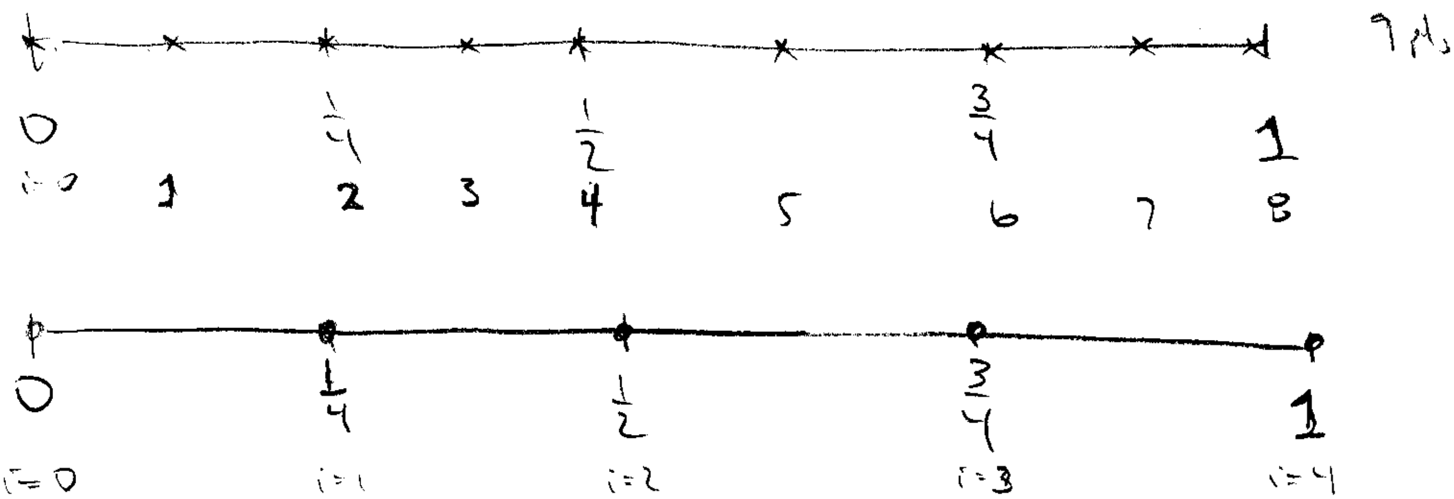
$$G = \{ jh,\ j = 0,1,2,3,\dots,2n \}$$
$$\bar{G} = \{ j\bar{h},\ j = 0,1,2,\dots,n \}$$

Now both grid definitions include the point 0.

$$h = \frac{1}{2n}$$
$$\bar{h} = \frac{1}{n}$$

Lets Do simple example w/ $n = 4$.

$$G = \{ jh,\ j = 0,1,2,3,4,5,6,7,8 \}$$
$$\bar{G} = \{ j\bar{h},\ j = 0,1,2,3,4 \}$$



At $x = 0$ we must discretize $\frac{du(0)}{dx} = 0$

At $x = 1$ " " " $\frac{du(1)}{dx} = 0$

Assume these are the neumann conditions to satisfy.

To get the 2 grid algorithm to work w/ these BC's I need to write down A, smoothing operator (gauss-seidel), $\bar{A}$ operator Prolongation operator P, restriction operator R + done.

· Discretizing ODE: $-\frac{d^2u}{dx^2} = f$ gives

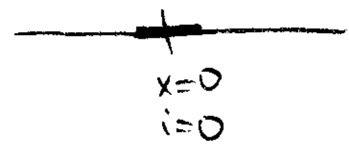
$$+h^{-2}(-u_{j-1} + 2u_j - u_{j+1}) = f_j$$

evaluating this eq at $j=0$ gives

$\underline{\frac{du}{dx}(0) = 0}$

$$h^{-2}(-u_{-1} + 2u_0 - u_1) = f_0$$

$$\Leftrightarrow h^{-2}(-u_{-1} + u_0 + u_0 - u_1) = f_0$$



Since $u_0 - u_{-1} \cong f'(0)h = 0$

We get the following (I think)

$$h^{-2}(u_0 - u_1) = f_0$$

Evaluating this eq at $j=2n$ gives

$$h^{-2}(-u_{2n-1} + 2u_{2n} - u_{2n+1}) = f_{2n}$$

$$= \quad h^{-2}(-u_{2n-1} + u_{2n} + u_{2n} - u_{2n+1}) = f_{2n}$$

But $u_{2n} - u_{2n+1} \cong -f'(x_{2n})h = 0$

$$\Rightarrow \quad h^{-2}(-u_{2n-1} + u_{2n}) = f_{2n}$$

Thus matrix A operating on $\bar{U}$ gives the following system of eqs

$$h^{-2}(u_0 - u_1) = f_0$$

$$h^{-2}(-u_{j-1} + 2u_j - u_{j+1}) = f_j \qquad 2 \le j \le 2n-1$$

$$h^{-2}(-u_{2n-1} + u_{2n}) = f_{2n}$$

· Then the smoothing operator can be represented (Gauss-Seidel) as

$$h^{-2}(u_0^m - u_1^{m-1}) = f_0$$

$$h^{-2}(-u_{j-1}^m + 2u_j^m - u_{j-1}^{m-1}) = f_j \qquad 1 \le j \le 2n-1$$

$$h^{-2}(-u_{2n-1}^m - u_{2n}^m) = f_{2n}$$

$$\Rightarrow \quad u_0^m = u_1^{m-1} + h^2 f_0$$

$$-u_{j-1}^m + 2u_j^m = u_{j-1}^{m-1} + h^2 f_j \qquad 1 \le j \le 2n-1$$

$$-u_{2n-1}^m - u_{2n}^m = h^2 f_{2n}$$

$$\underbrace{\begin{bmatrix} 1 & 0 & & & & & \\ -1 & 2 & 0 & & & & \\ & -1 & 2 & 0 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -1 & 2 & 0 & \\ & & & & -1 & 2 & 0 \\ & & & & & -1 & -1 \end{bmatrix}}_{W} \begin{bmatrix} u_0^m \\ u_1^m \\ \vdots \\ \\ u_{2n}^m \end{bmatrix} = \begin{bmatrix} 0 & 1 & & & & \\ 1 & 0 & & & & \\ & 1 & 0 & & & \\ & & 1 & 0 & & \\ & & & \ddots & \ddots & \\ & & & & 1 & 0 \\ & & & & 0 & 0 \end{bmatrix} \begin{bmatrix} u_0^{m-1} \\ u_0^{m-1} \\ \vdots \\ \\ u_{2n}^{m-1} \end{bmatrix} + h^2 \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ \\ f_{2n} \end{bmatrix}$$

Non-symmetric is actually lower diagonal matrix

∴ To write down $\bar{A}$ operator I must define P & R

$$\Rightarrow \quad \bar{A} = RAP$$

$$P\begin{pmatrix}\bar{U}_0\\ \bar{U}_1\\ \bar{U}_2\\ \bar{U}_3\\ \bar{U}_4\end{pmatrix} = \begin{pmatrix}\bar{U}_0\\ \frac{1}{2}(\bar{U}_0+\bar{U}_1)\\ \bar{U}_1\\ \frac{1}{2}(\bar{U}_1+\bar{U}_2)\\ \bar{U}_2\\ \frac{1}{2}(\bar{U}_2+\bar{U}_3)\\ \bar{U}_3\\ \frac{1}{2}(\bar{U}_3+\bar{U}_4)\\ \bar{U}_4\end{pmatrix} = \frac{1}{2}\begin{bmatrix}2 & & & & \\ 1 & 1 & & & \\ & 2 & & & \\ & 1 & 1 & & \\ & & 2 & & \\ & & 1 & 1 & \\ & & & 2 & \\ & & & 1 & 1\\ & & & & 2\end{bmatrix}\begin{bmatrix}\bar{U}_0\\ \bar{U}_1\\ \bar{U}_2\\ \bar{U}_3\\ \bar{U}_4\end{bmatrix}$$

Then restriction operator is given by:

$$R\begin{bmatrix}U_0\\ U_1\\ U_2\\ U_3\\ U_4\\ U_5\\ U_6\\ U_7\\ U_8\\ U_9\end{bmatrix} = \frac{1}{4}\begin{bmatrix}4 & & & & & & & & \\ 0 & 1 & 2 & 1 & & & & & \\ & & & 1 & 2 & 1 & & & \\ & & & & & 1 & 2 & 1 & \\ & & & & & & & & 4\end{bmatrix}\begin{bmatrix}U_0\\ U_1\\ U_2\\ U_3\\ U_4\\ U_5\\ U_6\\ U_7\\ U_8\\ U_9\end{bmatrix}$$

Here I have used maybe a Non-standard restriction opperator at the boundaries

Next following Exercise 2.3.1 derive the $\bar{A}$ w/ these definitions of prolongation & restriction.

$\left\{ \int v\,du = vu\Big| - \int u\,dv \right\}$

$(Lu, v) = (s, v) \quad \forall\, v \in H$

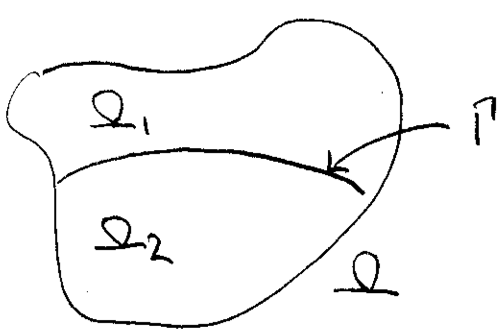
$\|$

$$-\int (a_{\alpha\beta} u_{,\alpha})_{,\beta}\, v\, d\Omega + \int (b_\alpha u)_{,\alpha}\, v\, d\Omega + \int c u v\, d\Omega = (s, v)$$

$\|$

$$\underbrace{-\int_{\partial\Omega} v (a_{\alpha\beta} u_{,\alpha}) n_\beta\, d\Gamma + \int_{\Omega} (a_{\alpha\beta} u_{,\alpha}) v_{,\beta}\, d\Omega}\; + b(u,v) + (cu, v)$$

$$\int_\Omega a_{\alpha\beta} u_{,\alpha} v_{,\beta}\, d\Omega - \int_{\partial\Omega} a_{\alpha\beta} u_{,\alpha} n_\beta v\, d\Gamma \qquad \text{eq } 3.2.4$$



consider the domain to be ∞
$\ni \partial\Omega$ can be taken at ∞.

Then operating on the weak form of the equations & ignoring for the time being "external" B.C

$$a(u,v) = \int_\Omega a_{\alpha\beta} u_{,\alpha} v_{,\beta}\, d\Omega$$

$$= \int_{\Omega_1} a_{\alpha\beta} u_{,\alpha} v_{,\beta}\, d\Omega + \int_{\Omega_2} a_{\alpha\beta} u_{,\alpha} v_{,\beta}\, d\Omega$$

This is required because $a_{\alpha\beta}(x)$ is two different fns on Ω^1 & Ω^2.

Then apply the Gauss Divergence theorem to each sub-domain in Ω_1 we get:

$$\int_{\Omega_1} a_{\alpha\beta} u_{,\alpha} v_{,\beta}\, d\Omega = -\int_{\Omega_1} (a_{\alpha\beta} u_{,\alpha})_{,\beta} v\, d\Omega + \int_{\partial\Omega_1} a_{\alpha\beta} u_{,\alpha} v\, n_\beta\, d\Gamma$$

n_β outward normal of domain Ω_1

$$\boxed{\int u\,dv = uv\Big| - \int v\,du}$$

Then
$$\int_{\Omega_2} a_{\alpha\beta} u_{,\alpha} v_{,\beta}\, d\Omega = -\int_{\Omega_2} (a_{\alpha\beta} u_{,\alpha})_{,\beta} v\, d\Omega + \int_{\partial\Omega_2} a_{\alpha\beta} u_{,\alpha} v\, \hat{n}_\beta\, d\Gamma$$

$\hat{n}_\beta$ = outward normal of domain Ω_2.

Then

$$a(u,v) = \int_{\Omega_1} + \int_{\Omega_2} = -\int_{\Omega_1} - \int_{\Omega_2} + \int_{\partial\Omega_1} a^1_{\alpha\beta} u^1_{,\alpha} n^1_\beta v\, d\Gamma$$

$$+ \int_{\partial\Omega_2} a^2_{\alpha\beta} u^2_{,\alpha} n^2_\beta v\, d\Gamma$$

$$= -\int_{\Omega} a_{\alpha\beta} u_{,\alpha} v_{,\beta}\, d\Omega + \int_{\Gamma} (a^1_{\alpha\beta} u^1_{,\alpha} - a^2_{\alpha\beta} u^2_{,\alpha}) n^1_\beta v\, d\Gamma \qquad \text{eq 3.27}$$

n^1_β outward normal of domain Ω_1

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Argument for 3.2.8 is the following.

Given that we must satisfy 3.2.5 A simplified version of $b = c = 0$

gives $a(u,v) = 0 \quad \forall v \in H$.

$$\| $$

But $-\int_\Omega (a_{\alpha\beta} u_{,\alpha})_{,\beta} v \, d\Omega + \int_\Gamma (a'_{\alpha\beta} u'_{,\alpha} - a^2_{\alpha\beta} u^2_{,\alpha}) n'_\beta v \, d\Gamma = 0 \quad \forall v \in H$

A suitable choice of v (i.e. delta fns where needed) gives a necessary requirement of

3.2.8

Ex 3.2.1 $\quad a_{\alpha\beta} v_\alpha v_\beta \geq C v_\alpha v_\alpha$ in 2D

$$a_{11} v_1^2 + a_{22} v_2^2 + 2 a_{12} v_1 v_2 \geq C(v_1^2 + v_2^2) \quad \forall v \in \mathbb{R}^2$$

let $v_1 = 1, \; v_2 = 0 \qquad v_1 = 0, \; v_2 = 1$

$$\Rightarrow \quad a_{11} \geq C > 0 \qquad a_{22} \geq C > 0$$

How show just elements on coeff a's from condition on all v's?

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$$-\frac{1}{2}\left\{\nabla\left(a\left(\frac{u_{j+1}-u_j}{h}\right)\right)+\Delta\left(a\left(\frac{u_j-u_{j-1}}{h}\right)\right)\right\}=s_j$$

$$=-\frac{1}{2}\left\{\frac{\frac{a_j(u_{j+1}-u_j)}{h}-\frac{a_{j-1}(u_j-u_{j-1})}{h}}{h}+\frac{\frac{a_{j+1}(u_{j+1}-u_j)}{h}-\frac{a_j(u_j-u_{j-1})}{h}}{h}\right\}$$

$$=s_j$$

$$=-\frac{1}{2h^2}\left\{a_ju_{j+1}-a_ju_j-a_{j-1}u_j+a_{j-1}u_{j-1}+a_{j+1}u_{j+1}-a_{j+1}u_j-a_ju_j+a_ju_{j-1}\right\}$$

$$=s_j$$

$$=\frac{-1}{2h^2}\left\{(a_{j-1}+a_j)u_{j-1}+(-a_{j-1}-2a_j-a_{j+1})u_j+(a_j+a_{j+1})u_{j+1}\right\}=s_j$$

$$=\frac{+1}{2h^2}\left\{-(a_{j-1}+a_j)u_{j-1}+(a_{j-1}+2a_j+a_{j+1})u_j-(a_j+a_{j+1})u_{j+1}\right\}=s_j$$

eq 3.3.5

$j=1,2,\cdots,n-1$

$a(0)u_1(0)=f$ at $j=0$

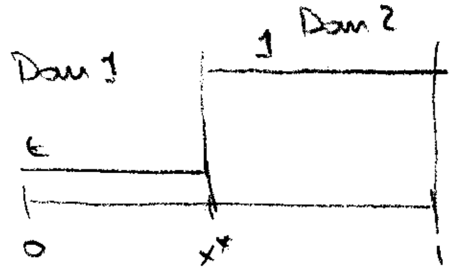
$$\frac{1}{2h^2}\left\{\underbrace{-(a_{-1}+a_0)u_{-1}+(a_{-1}}+2a_0+a_1)u_0-(a_0+a_1)u_1\right\}=s_0$$

take

$$-(a_{-1}+a_0)u_{-1}+(a_{-1}+a_0)u_0=2f$$

why 2?

At $j=0$



$$-(a(x)u_{,1})_{,1} = 0$$

Jump cond 3.2.3 at $x = x^*$

$$\epsilon \lim_{x \to x^{*-}} u_{,1} = 1 \lim_{x \to x^{*+}} u_{,1} \qquad \text{eq } 3.3.7$$

Assume $u(x) = u_0 + u_1 x$ in both domains

Domain 1 : $u(0) = 0$

$$-\epsilon u_{xx} = 0$$

$$u^1 = u_0^1 + u_1^1 x$$

$$u^1(0) = 0 \Rightarrow u_0^1 = 0$$

$$u^1(x) = u_1^1 x$$

Dom 2 $u(1) = 1$

$$-u_{xx} = 0$$

$$u^2 = u_0^2 + u_1^2 x$$

$$u^2(1) = u_0^2 + u_1^2 = 1 \Rightarrow u_0^2 = 1 - u_1^2$$

$$u^2(x) = 1 - u_1^2 + u_1^2 x$$

$$= 1 + (x-1)u_1^2 = 1 + u_1^2(x-1)$$

w/ matching condition (jump condition) on $\underbrace{u_x}$

$$\epsilon u_1^1 = u_1^2$$

But fn must be continuous at u

$$u^1(x^*) = u_1^1 x^* = 1 + u_1^2(x^* - 1) = u^2(x^*)$$

$$u_1^1 x^* = 1 + \epsilon u_1^1 (x^* - 1)$$

$$-1 = (\epsilon x^* - \epsilon - x^*) u_1^1$$

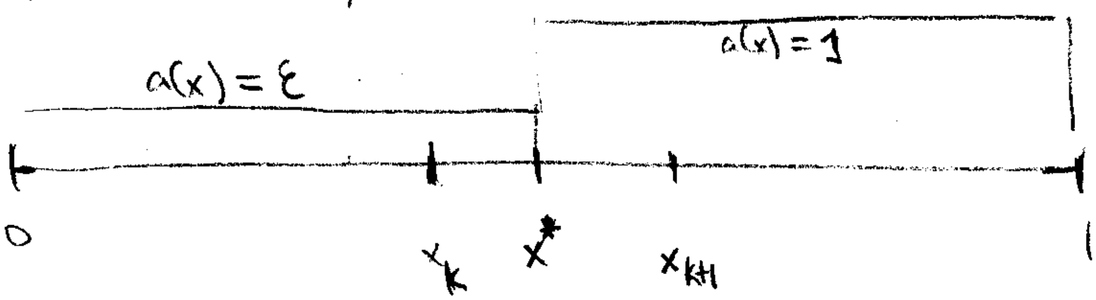
$$u_1^1 = \frac{-1}{-x^* + \epsilon x^* - \epsilon} = \frac{1}{x^* - \epsilon x^* + \epsilon} \equiv \alpha$$

Then $u_1^2 = \epsilon\alpha$

So $u^1(x) = \alpha x \qquad 0 < x < x^*$

$u^2(x) = 1 + \epsilon\alpha(x-1) \qquad x^* < x < 1$] eq 3.3.8

--- Assume discontinuity is in the middle of a cell.



Then writting eq 3.3.5 down for $j = 1, \dots, k-1$ we get

$$(-2\epsilon u_{j-1} + 4\epsilon u_j - 2\epsilon u_{j+1})/2h^2 = 0 \quad ✓$$

$$\Rightarrow \quad -u_{j-1} + 2u_j - u_{j+1} = 0 \quad ✓$$

For $j = k$ we get

$$-2\epsilon u_{k-1} + (3\epsilon+1)u_k - (\epsilon+1)u_{k+1} = 0 \quad ✓$$

For $j = k+1$

$$-(\epsilon+1)u_k + (\epsilon+3)u_{k+1} - 2u_{k+2} = 0 \quad ✓$$

For $k+2 \leq j \leq n-1$

$$-2u_{j-1} + 4u_j - 2u_{j+1} = 0 \Leftrightarrow -u_{j-1} + 2u_j - u_{j+1} = 0$$

Postulating $u_j = \alpha j \quad 0 \le j \le k \qquad u_j = \beta j - \beta n + 1 \quad k+1 \le j \le n$

w/ BC $u_0 = 0$ & $u_n = 1$

$u_0 = 0$ ✓ $\qquad u_n = 1$ ✓

Now for $1 \le j \le k-1$ & $k+2 \le j \le n-1$ we require

$$-u_{j-1} + 2u_j - u_{j+1} = 0 \qquad -u_{j-1} + 2u_j - u_{j+1} = 0$$

$\Vert$ $\qquad$ Is satisfied also.

$$-\alpha(j-1) + 2\alpha j - \alpha(j+1) \stackrel{?}{=} 0$$
$$= 0 \checkmark$$

Now eq for $j = k$ is w/ assumed solution:

$$-2\epsilon u_{k-1} + (1+3\epsilon)u_k - (\epsilon+1)u_{k+1} = 0 \quad \checkmark$$

$$-2\epsilon\alpha(k-1) + (1+3\epsilon)\alpha k - (\epsilon+1)(\beta(k+1) - \beta n + 1) = 0 \quad \checkmark$$

$$\Rightarrow \left(-2\epsilon(k-1) + (1+3\epsilon)k\right)\alpha - (1+\epsilon)(k+1-n)\beta = \epsilon + 1 \quad \checkmark$$

Now eq for $j = k+1$ is

$$-(\epsilon+1)\alpha k + (3+\epsilon)\left(\beta(k+1) - \beta n + 1\right) - 2\left(\beta(k+2) - \beta n + 1\right) = 0 \quad \checkmark$$

$$\Rightarrow -(1+\epsilon)k\alpha + \left[(3+\epsilon)(k+1-n) - 2(k+2-n)\right]\beta = -(3+\epsilon) + 2 \quad \checkmark$$

Then 2 eqs become for α & β

$$(-2\epsilon k + 2\epsilon + k + 3\epsilon k)\alpha - (1+\epsilon)(k+1-n)\beta = \epsilon+1 \quad ✓$$

$$-(1+\epsilon)k\alpha + \left[(3+\epsilon)(k+1-n) - \underbrace{2(k+1-n+1)}\right]\beta = -(1+\epsilon) \quad ✓$$

$$\left[(1+\epsilon)(k+1-n) \overset{-2(k+1-n)-2}{-2}\right]\beta = \ldots \quad ✓$$

$$\Rightarrow \quad (\epsilon k + k + 2\epsilon)\alpha - (1+\epsilon)(k+1-n)\beta = 1+\epsilon \quad ✓$$

$$-(1+\epsilon)k\alpha + \left[(1+\epsilon)(k+1-n) - 2\right]\beta = -(1+\epsilon) \quad ✓$$

Solving w/ MMA to simplify gives:

$$\alpha = \frac{1+\epsilon}{k(1-\epsilon^2)+\epsilon(1+n-\epsilon+n\epsilon)} = \frac{1+\epsilon}{k(1-\epsilon^2)+\epsilon(1-\epsilon+n(1+\epsilon))}$$

$$\left.\begin{aligned}\beta &= \frac{\epsilon(1+\epsilon)}{k(1-\epsilon^2)+\epsilon(1+n-\epsilon+n\epsilon)} = \epsilon\alpha\end{aligned}\right\} \text{eq } 3.3.10$$

$$\therefore \quad \alpha = \frac{1}{k(1-\epsilon)+\epsilon\left(\frac{1-\epsilon}{1+\epsilon}+n\right)} = \frac{1}{\epsilon\left(\frac{1-\epsilon}{1+\epsilon}\right)+\epsilon(n-k)+k}$$

Then $$U_k = \alpha k = \frac{k}{\epsilon\left(\frac{1-\epsilon}{1+\epsilon}\right)+\epsilon(n-k)+k} = \frac{kh}{\epsilon h\left(\frac{1-\epsilon}{1+\epsilon}\right)+\epsilon hn+(1-\epsilon)hk}$$

$$\therefore\ y = \frac{x_k}{\epsilon h\left(\frac{1-\epsilon}{1+\epsilon}\right) + \epsilon + (1-\epsilon)x_k} \quad \text{eq } 3,3,11$$

If $x^* = x_{k+1}$ then exact solution is $U = \alpha x = \dfrac{x}{x_{k+1} - \epsilon x_{k+1} + \epsilon}$

so that $U(x_k) = \dfrac{x_k}{(1-\epsilon)x_{k+1} + \epsilon}$ eq 3,3,12

Then the error

$$U(x_k) - y = \frac{x_k}{(1-\epsilon)x_{k+1} + \epsilon} - \frac{x_k}{\epsilon h\left(\frac{1-\epsilon}{1+\epsilon}\right) + \epsilon + (1-\epsilon)x_k}$$

taylor expanding in terms of h gives

$$= \frac{x_k}{(1-\epsilon)x_{k+1} + \epsilon} - \frac{x_k}{(1-\epsilon)\left[x_k + \frac{\epsilon h}{1+\epsilon}\right] + \epsilon}$$

$$= \frac{x_k}{(1-\epsilon)x_{k+1} + \epsilon} - \frac{x_k}{(1-\epsilon)\left(x_k + h + \frac{\epsilon h - h - \epsilon h}{1+\epsilon}\right) + \epsilon}$$

$$= \frac{x_k}{(1-\epsilon)x_{k+1} + \epsilon} - \frac{x_k}{(1-\epsilon)\left(x_{k+1} - \frac{h}{1+\epsilon}\right) + \epsilon}$$

$$= O\left(\frac{h(1-\epsilon)}{1+\epsilon}\right)$$

Book got $O\left(\frac{\epsilon h(1-\epsilon)}{1+\epsilon}\right)$ I didnt get the ϵ

If $x^* = x_k + \frac{h}{2}$

Exact solution now is $U(x_k) = \dfrac{x_k}{\left(x_k + \frac{h}{2} - \epsilon\left(x_k + \frac{h}{2}\right) + \epsilon\right)} = \dfrac{x_k}{(1-\epsilon)x_k + \epsilon + \frac{h}{2}(1-\epsilon)}$

Then the error satisfies

$$U(x_k) - u_k = \frac{x_k}{(1-\epsilon)x_k + \epsilon + \frac{h}{2}(1-\epsilon)} - \frac{x_k}{(1-\epsilon)x_k + \epsilon + \frac{h}{2}(1-\epsilon)\left[\frac{2\epsilon}{1+\epsilon}\right]}$$

$$= O\left(\frac{\epsilon(1-\epsilon)h}{1+\epsilon}\right)$$ Book got different result,

Both terms have $O(h)$ error around the discontinuity.

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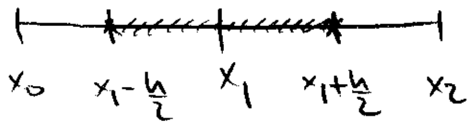
$$a(u,v) = -\int_{\Omega} (a_{\alpha\beta} u_{,\alpha})_{,\beta}\, v \, d\Omega$$



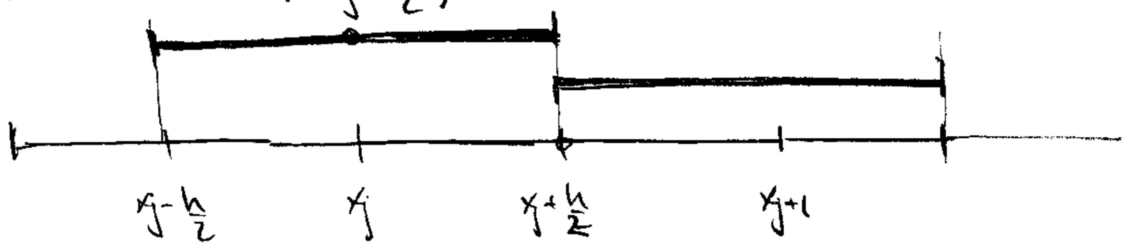
$$= -\int_{\Omega_j} (a_1 u_{,1})_{,1}\, d\Omega = -\int_{x_j-\frac{h}{2}}^{x_j+\frac{h}{2}} (a_1 u_{,1})_{,1}\, dx$$

a_1 is now a constant due to averaging over cells

$$= -(a\, u_{,1} \Big|_{x_j-\frac{h}{2}}^{x_j+\frac{h}{2}}$$

$1 \le j \le n-1$ eq 3.3.18

Now to approximate $u_{,1}(x_j+\frac{h}{2})$



Given by our discretization $\exists$ a jump in $a(x)$ at $x_j+\frac{h}{2}$

$$a^1 u^1_{,1} \cong a_j \frac{(u^-_{j+1/2} - u_j)}{h/2} = \frac{2}{h} a_j (u^-_{j+1/2} - u_j)$$

$$a^2 u^2_{,1} \cong a_{j+1} \frac{(u_{j+1} - u^+_{j+1/2})}{h/2} = \frac{2}{h} a_{j+1} (u_{j+1} - u^+_{j+1/2})$$

] eqs 3.3.19

Then the jump condition at $x_j+\frac{h}{2}$ gives

$$a^1 u^1_{,1} = a^2 u^2_{,1}$$ setting the two eqs above equal gives

$$\frac{a_j}{a_{j+1}}\left(u^-_{j+1/2} - u_j\right) = u_{j+1} - u^+_{j+1/2}$$

Now $a(x)$ can be discontinuous but in the weak formulation $u(x)$ must be continuous $\Rightarrow$ $u^-_{j+1/2} = u^+_{j+1/2}$: Above becomes

$$\left(\frac{a_j}{a_{j+1}} + 1\right) u_{j+1/2} = \frac{a_j}{a_{j+1}} u_j + u_{j+1}$$

$$u_{j+1/2} = \frac{a_j u_j + a_{j+1} u_{j+1}}{a_j + a_{j+1}} \qquad \text{eq } 3.3.20$$

Then we can approximate $a u'(x_j + \frac{h}{2}) \cong \frac{2}{h} a_j \left(\frac{a_j u_j + a_{j+1} u_{j+1}}{a_j + a_{j+1}} - u_j \right)$

$$= \frac{2}{h} a_j \left(\frac{\cancel{a_j u_j} + a_{j+1} u_{j+1} - \cancel{a_j u_j} - a_{j+1} u_j}{a_j + a_{j+1}} \right)$$

$$= \frac{2}{h} a_j \left(\frac{a_{j+1}}{a_j + a_{j+1}} \right) (u_{j+1} - u_j) = \underbrace{\left(\frac{2 a_j a_{j+1}}{a_j + a_{j+1}} \right)}_{w_j} \left(\frac{u_{j+1} - u_j}{h} \right)$$

eq 3.3.21

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$$-\left(w_j\frac{(u_{j+1}-u_j)}{h} - w_{j-1}\frac{(u_j-u_{j-1})}{h}\right) = s_j h \quad j = 1,2,\cdots,n-1$$

eq 3,3,23.

a(x) smooth $w_j = \dfrac{2a_j a_{j+1}}{(a_j + a_{j+1})}$ ~~taylor expanded gives~~ want give a_{j+1} terms

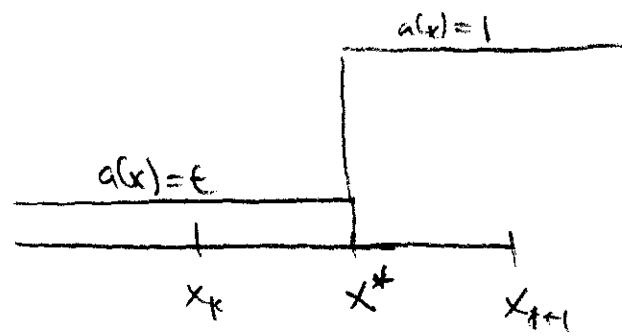
Since $\sqrt{ab} \le \frac{1}{2}(a+b) \quad a,b > 0$

$$ab \le \frac{1}{4}(a+b)^2$$

$$\therefore \quad w_j \le \frac{2(\frac{1}{4})(a_j+a_{j+1})^2}{a_j+a_{j+1}} = \frac{a_j+a_{j+1}}{2}$$

eq 3,3,23 is

$$w_{j-1}\frac{(u_j-u_{j-1})}{h} - w_j\frac{(u_{j+1}-u_j)}{h} = 0$$



Then $\omega_{k-1} = \dfrac{2\epsilon^2}{2\epsilon} = \epsilon \qquad \omega_k = \dfrac{2\epsilon}{\epsilon+1}$

$$\omega_{k+1} = 1$$

$\therefore \quad w_j = \epsilon$ for $1 \le j \le k-1$; $w_k = \dfrac{2\epsilon}{1+\epsilon}$; $w_j = 1$ for $k+1 \le j \le n-1$

eq 3,3,24

To solve this discontinuous difference equation

Assume $u_j = \alpha j \quad 0 \le j \le k$ &

$u_j = \beta j - \beta n + 1 \quad k+1 \le j \le n$

Note: BC's are met!!

Then eq 3,3,23 for $j = 1, 2, \ldots, k-1$ gives

$$\frac{\epsilon}{h}(\alpha j - \alpha(j-1)) - \frac{\epsilon}{h}(\alpha(j+1) - \alpha j) \stackrel{?}{=} 0 \quad \checkmark$$

$$\frac{\epsilon}{h}(\alpha) - \frac{\epsilon}{h}(\alpha) = 0 \quad \checkmark \checkmark$$

Then eq 3,3,23 for $j = k+2, k+3, \ldots, n$ gives

$$1\left(\frac{\beta j - \beta(j-1)}{h}\right) - 1\left(\frac{\beta(j+1) - \beta j}{h}\right) = 0 \quad \checkmark \checkmark$$

Eq 3,3,23 for $j = k$ is

$$\frac{\epsilon}{h}(\alpha k - \alpha(k-1)) - \frac{2\epsilon}{1+\epsilon}\left(\frac{\beta(k+1) - \beta n + 1 - \alpha k}{h}\right) = 0 \quad \checkmark \qquad (1)$$

Eq 3,3,23 for $j = k+1$

$$\frac{2\epsilon}{1+\epsilon}\left(\frac{\beta(k+1) - \beta n + 1 - \alpha k}{h}\right) - \frac{1}{h}\left(\beta(\cancel{k}+2) - \cancel{\beta n + 1} - \beta(\cancel{k}+1) + \cancel{\beta n - 1}\right) = 0 \quad (2)$$

Eqs (1) & (2) become

$$\alpha - \frac{2}{1+\epsilon}(\beta(k+1-n) + 1 - k\alpha) = 0$$

$$\frac{2\epsilon}{1+\epsilon}(\beta(k+1-n) - \alpha k + 1) - \beta = 0$$

$$\left(1+\frac{2k}{1+t}\right)\alpha - \frac{2}{1+t}(k+1-n)\beta = \frac{2}{1+t}$$

$$-k\frac{2t}{1+t}\alpha + \left[\frac{2t}{1+t}(k+1-n) - 1\right]\beta = -\frac{2t}{1+t}$$

$$\Rightarrow \quad \alpha = \frac{-2}{-1+2k(-1+t)+t-2nt} \qquad \& \qquad \beta = t\alpha$$

$$= \frac{2}{1-t+2k-\underbrace{2kt+2nt}_{2t(n-k)}} = \frac{1}{\frac{1-t}{2}+k+t(n-k)}$$

What is w in eq 3.3.25? $\quad x_k = hk \quad 1 = nh \Rightarrow n = \frac{1}{h}$

$\Rightarrow k = \frac{x_k}{h}$

using Above

$$\therefore \alpha = \frac{1}{\left(\frac{1-t}{2}\right)+\frac{2k(1-t)}{2}+\frac{2nt}{2}} = \frac{1}{\frac{1-t}{2}+k(1-t)+nt} = \frac{1}{\frac{1-t}{2}+\frac{x_k}{h}(1-t)+\frac{1}{h}t}$$

$$= \frac{h}{(1-t)\left[x_k+\frac{h}{2}\right]+t} \quad \text{eq } 3.3.26$$

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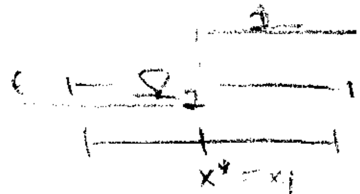


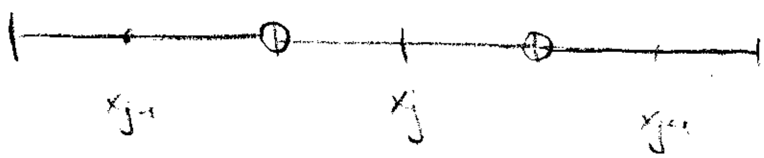
x_j is the volume center.

(3.2.7)

$$a(u,v) = -\int_\Omega (a u_{,1})_{,1} v \, d\Omega + \lim_{x \to x^{*-}} a(x) u_{,1} - \lim_{x \to x^{*+}} a(x) u_{,1}$$

$$= -a u_{,1} \Big|_{x_j - h/2}^{x_j + h/2} + \lim_{x \to x^{*-}} a(x) u_{,1} - \cdots \qquad \text{eq } 3.3.27$$

Discontinuity in $a(x)$ gives rise to discontinuity in U_x. ∃ two terms cancel.



$$u_{,1}(x_{j+h/2}) \approx \frac{1}{h}(U_j - U_{j-1})$$

Then

$$a(u,v) \approx -\frac{a_{j+1/2}}{h} (\qquad \text{gives eq } 3.3.28$$

$$\Rightarrow \quad -a_{j-1/2} U_{j-1} + (a_{j-1/2} + a_{j+1/2}) U_j - a_{j+1/2} a_{j+1} \qquad \text{eq } 3.3.29$$

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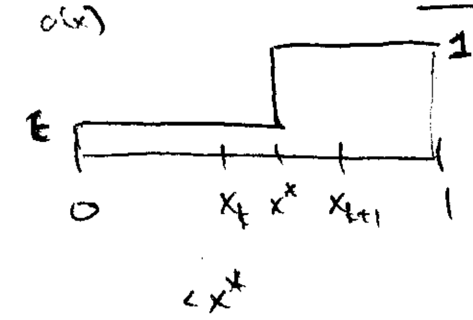


Ex 3.3.1 Now eq 3.3.9 is

$$U_j = \alpha j \qquad 0 \le j \le k$$

$$U_j = \beta(j-n) + 1 \qquad k+1 \le j \le n$$

depending on the neumeral discretization α & β can be different values (say for finite volume or finite differences)

The exact solution is however constant & is

$$U = \alpha x \qquad 0 \le x \le x^*$$

$$= \epsilon\alpha(x-1) + 1 \qquad x^* \le x \le 1$$

w/ $\alpha = \dfrac{1}{x^* - \epsilon x^* + \epsilon}$ given by eq 3.3.8

To do this problem we will use a finite difference discretization w/ $x^* = x_{k+1}$

Then the approximate solution is (exact α & β factors are). Note other examples would be worked similarly.

$$\beta = \epsilon\alpha \qquad \alpha = \frac{1}{\left(\epsilon\frac{(1-\epsilon)}{1+\epsilon} + \epsilon(n-k) + k\right)}$$

So that

$$U_j = \begin{cases} \dfrac{j}{\epsilon\frac{(1-\epsilon)}{1+\epsilon} + \epsilon(n-k) + k} & 0 \le j \le k \\[2ex] \dfrac{\epsilon(j-n)}{\epsilon\left(\frac{1-\epsilon}{1+\epsilon}\right) + \epsilon(n-k) + k} + 1 & k+1 \le j \le n \end{cases}$$

Now $|U|_\infty = \text{Max}\{|U_j| : 0 \le j \le n\}$

So $|\mathcal{E}|_\infty = |u(x_j) - U_j| = \text{Max}\{|u(x_j) - U_j| : 0 \le j \le n\}$ So that we

need to evaluate $u(x_j) - U_j$

writing U_j in terms of x_j gives $\left\{ j = \frac{x_j}{h} \right\}$

$$U_j = \begin{cases} \dfrac{x_j}{\epsilon h \frac{(1-\epsilon)}{1+\epsilon} + \epsilon(1-x_k) + x_k} & 0 \le j \le k \\[2ex] \dfrac{\epsilon(x_j - 1)}{\epsilon h \frac{(1-\epsilon)}{1+\epsilon} + \epsilon(1-x_k) + x_k} + 1 & k+1 \le j \le n \end{cases}$$

Thus for $0 \le j \le k$

$$u(x_j) - U_j = \frac{x_j}{x_{k+1} - \epsilon x_{k+1} + \epsilon} - \frac{x_j}{\epsilon h \frac{(1-\epsilon)}{1+\epsilon} + x_k - \epsilon x_k + \epsilon}$$

$$= \frac{x_j}{x_{k+1} - \epsilon x_{k+1} + \epsilon} - \frac{x_j}{\epsilon h\left(\frac{1-\epsilon}{1+\epsilon}\right) - h + x_{k+1} - \epsilon x_{k+1} + \epsilon h + \epsilon}$$

$$= \quad '' \quad - \frac{x_j}{x_{k+1} - \epsilon x_{k+1} + \epsilon + \boxed{}}$$

w/ $\boxed{} = \dfrac{\epsilon h(1-\epsilon)}{1+\epsilon} - h + \epsilon h$

So $= \frac{\epsilon h(1-\epsilon)}{1+\epsilon} - (1-\epsilon)h = (1-\epsilon)h\left[\frac{\epsilon}{1+\epsilon} - 1\right]$

$$= (1-\epsilon)h\left(\frac{\cancel{\epsilon} - 1 - \cancel{\epsilon}}{1+\epsilon}\right) = -\frac{(1-\epsilon)h}{1+\epsilon}$$

Thus

$$U(x_j) - U_j = \frac{x_j}{x_{k+1} - \epsilon x_{k+1} + \epsilon} - \frac{x_j}{x_{k+1} - \epsilon x_{k+1} + \epsilon - h\frac{(1-\epsilon)}{1+\epsilon}} \qquad 0 \le j \le k$$

$$= \frac{x_j}{x_{k+1} - \epsilon x_{k+1} + \epsilon}\Bigg[1 - \underbrace{\frac{x_{k+1} - \epsilon x_{k+1} + \epsilon}{x_{k+1} - \epsilon x_{k+1} + \epsilon - h\frac{(1-\epsilon)}{1+\epsilon}}}_{\text{constant independent of } j}\Bigg]$$

Thus

$\max_{0 \le j \le k} |U(x_j) - U_j| = \underline{\underline{|U(x_k) - U_k|}}$ or term from below

For $k+1 \le j \le n$

$$U(x_j) - U_j = \frac{\epsilon(x_j - 1)}{x_{k+1} - \epsilon x_{k+1} + \epsilon} + \cancel{X} - \left[\frac{\epsilon(x_j - 1)}{\epsilon h\frac{(1-\epsilon)}{1+\epsilon} + \epsilon(1-x_k) + x_k} + \cancel{X}\right]$$

$$= \frac{\epsilon(x_j - 1)}{x_{k+1} - \epsilon x_{k+1} + \epsilon} - \frac{\epsilon(x_j - 1)}{\epsilon h\frac{(1-\epsilon)}{1+\epsilon} + \epsilon(1 - x_{k+1} + h) + x_{k+1} - h}$$

$$U(x_j) - u_j = \frac{\epsilon(x_j-1)}{x_{k+1}-\epsilon x_{k+1}+\epsilon} - \frac{\epsilon(x_j-1)}{x_{k+1}-\epsilon x_{k+1}+\epsilon+\underbrace{\frac{\epsilon h(1-\epsilon)}{1+\epsilon}+\epsilon h-h}_{*}}$$

$*$ Becomes

$$\frac{\epsilon h(1-\epsilon)}{1+\epsilon} - h(1-\epsilon) = h(1-\epsilon)\left[\frac{\epsilon}{1+\epsilon}+\frac{-1-\epsilon}{1+\epsilon}\right] = \frac{-h(1-\epsilon)}{1+\epsilon}$$

Thus

$$U(x_j) - u_j = \frac{\epsilon(x_j-1)}{x_{k+1}-\epsilon x_{k+1}+\epsilon}\underbrace{\left[1-\frac{x_{k+1}-\epsilon x_{k+1}+\epsilon}{x_{k+1}-\epsilon x_{k+1}+\epsilon-\frac{h(1-\epsilon)}{1+\epsilon}}\right]}_{\text{independent of } j} \qquad k+1\le j\le n$$

Now: This difference is largest when $j=k+1$ so

$$U(x_{k+1}) - u_{k+1} = \frac{\epsilon(x_{k+1}-1)}{x_{k+1}-\epsilon x_{k+1}+\epsilon}\left[1-\frac{x_{k+1}-\epsilon x_{k+1}+\epsilon}{x_{k+1}-\epsilon x_{k+1}+\epsilon-\frac{h(1-\epsilon)}{1+\epsilon}}\right]$$

Thus

$$\max_{0\le j\le n} |U(x_j)-u_j| = \max\left\{\frac{x_k}{x_{k+1}-\epsilon x_{k+1}+\epsilon}\left[\frac{x_{k+1}-\epsilon x_{k+1}+\epsilon}{x_{k+1}-\epsilon x_{k+1}+\epsilon-\frac{h(1-\epsilon)}{1+\epsilon}}-1\right],\right.$$

$$\left.\frac{\epsilon(1-x_{k+1})}{x_{k+1}-\epsilon x_{k+1}+\epsilon}\left[\frac{x_{k+1}-\epsilon x_{k+1}+\epsilon}{x_{k+1}-\epsilon x_{k+1}+\epsilon-\frac{h(1-\epsilon)}{1+\epsilon}}-1\right]\right\}.$$

This max depends soley on which is larger x_t

$\epsilon(1-x_{t+1})$ or

x_t v.s $\epsilon(1-x_t-h)$ As $h\to 0$ this becomes

x_t v.s. $\epsilon(1-x_t)$ & this depends on the problem at hand.

One should next Taylor expand the $\dfrac{x_{t+1}-\epsilon x_{t+1}+\epsilon}{x_{t+1}-\epsilon x_{t+1}+\epsilon-h\frac{(1-\epsilon)}{1+\epsilon}}-1$

v.s. h to discover the order of convergence of the l_∞ norm.

Now l_2 norm

$$|u|_2 = h\left\{\sum_{j=0}^{n} u_j^2\right\}^{1/2}$$

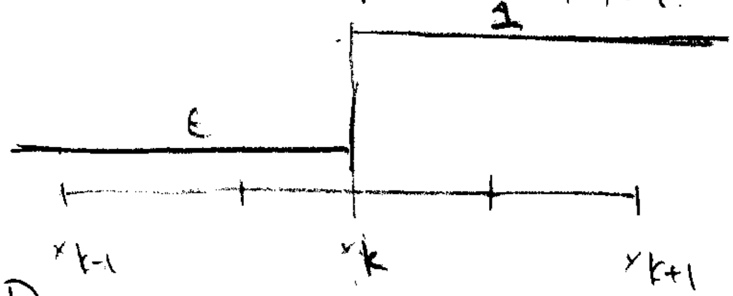
$$\Rightarrow \frac{|u|_2}{h} = \left\{\sum_{j=0}^{k} u_j^2 + \sum_{j=k+1}^{n} u_j^2\right\}^{1/2}$$

$$= \left\{\sum_{j=0}^{k} \frac{x_j^2}{(x_{k+1} - \epsilon x_{k+1} + \epsilon)^2}\left(1 - \frac{x_{k+1} - \epsilon x_{k+1} + \epsilon}{x_{k+1} - \epsilon x_{k+1} + \epsilon - \frac{h(1-\epsilon)}{1+\epsilon}}\right)^2\right.$$

$$\left. + \sum_{j=k+1}^{n} \frac{\epsilon^2 (x_j - 1)^2}{(x_{k+1} - \epsilon x_{k+1} + \epsilon)^2}\left(1 - \frac{x_k - \epsilon x_{k+1} + \epsilon}{x_{k+1} - \epsilon x_{k+1} + \epsilon - \frac{h(1-\epsilon)}{1+\epsilon}}\right)^2\right\}^{1/2}.$$

Ex 3.3.2 Assume a solution of the form 3.3.9 into 3.3.29.

for $j \le k-2$



$$-\cancel{\epsilon}(\cancel{\alpha}(j-1)) + (2\cancel{\epsilon})\cancel{\alpha}j - \cancel{\epsilon}\,\cancel{\alpha}(j+1) \stackrel{?}{=} 0$$

$0 = 0$ yes

for $j \ge k+2$

$$-1(\beta(\cancel{j}-\cancel{1}) - \beta n + 1) + 2(\beta\cancel{j} - \beta n + 1) - 1(\beta(\cancel{j}+\cancel{1}) - \beta n + 1) \stackrel{?}{=} 0$$ yes

$j = k-1$

$-\epsilon\alpha(k-2)$

Based on eq 3.4.3 & expanding we get for a 2D grid $\alpha = 1,2$, $\beta = 1,2$

$$-\frac{1}{2}\left[\nabla_\beta(a_{1\beta}\Delta_1 + a_{2\beta}\Delta_2) + \Delta_\beta(a_{1\beta}\nabla_1 + a_{2\beta}\nabla_2)\right]U$$

$$+ \frac{1}{2}(\nabla_1 + \Delta_1)(b_1 U) + \frac{1}{2}(\nabla_2 + \Delta_2)(b_2 U) + cU = S$$

$\Rightarrow$

$$-\frac{1}{2}\left[\nabla_1(a_{11}\Delta_1 + a_{21}\Delta_2) + \nabla_2(a_{12}\Delta_1 + a_{22}\Delta_2)\right.$$

$$\left. + \Delta_1(a_{11}\nabla_1 + a_{21}\nabla_2) + \Delta_2(a_{12}\nabla_1 + a_{22}\nabla_2)\right]U$$

$$+ \frac{1}{2}(\nabla_1 + \Delta_1)(b_1 U) + \frac{1}{2}(\nabla_2 + \Delta_2)(b_2 U) + cU = S$$

Now: to determine the stencil of this discretization it suffices to look at the 2nd order terms as expanded in 3.4.5. to do This we get

$$-\frac{1}{2}\left[\nabla_1\left(a_{11}\left(\frac{U_{j+e_1} - U_j}{h}\right) + a_{21}\left(\frac{U_{j+e_2} - U_j}{h}\right)\right)\right.$$

$$+ \nabla_2\left(a_{12}\left(\frac{U_{j+e_1} - U_j}{h}\right) + a_{22}\left(\frac{U_{j+e_2} - U_j}{h}\right)\right)$$

$$+ \Delta_1\left(a_{11}\left(\frac{U_j - U_{j-e_1}}{h}\right) + a_{21}\left(\frac{U_j - U_{j-e_2}}{h}\right)\right)$$

$$+ \Delta_2\left(a_{12}\left(\frac{u_j - u_{j-e_1}}{h}\right) + a_{22}\left(\frac{u_j - u_{j-e_2}}{h}\right)\right)\Bigg]$$

$$\Rightarrow -\frac{1}{2h^2}\Big[a_{11}(u_{j+e_1} - u_j) - a_{11,j-e_1}(u_j - u_{j-e_1})$$

$$+ a_{21}$$