pg 5 Whittaker/Watson

If $\frac{p^2}{q^2} = 2$

Then $\frac{(2q-p)^2}{(p-q)^2} = \frac{4q^2 - 4qp + p^2}{p^2 - 2pq + q^2} =: Q$

~~But $p^2 = 2q^2$~~ key But $p^2 = 2q^2$

$$Q = \frac{6q^2 - 4qp}{3q^2 - 2pq} = \frac{2(3q^2 - 2pq)}{3q^2 - 2pq} = 2.$$

$\frac{p}{q} > 1$ $\quad p > q$

$p - q > 0$

+ if $p - q = q$ $\quad p = 2q$

How get $\boxed{p - q < q}$?

$$y - x = \frac{x(x^2+6)}{3x^2+2} - \frac{(3x^3+2x)}{3x^2+2} = \frac{-2x^3+4x}{3x^2+2}$$

$$= \frac{2x(2-x^2)}{3x^2+2}$$

$$y^2 - 2 = \frac{x^2(x^2+6)^2}{(3x^2+2)^2} - 2\,\frac{(3x^2+2)^2}{(3x^2+2)^2}$$

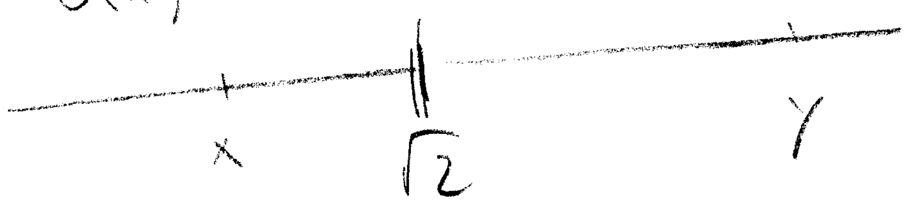
$$= \frac{x^6 + 12x^4 + 36x^2 - 18x^4 - 24x^2 - 8}{(3x^2+2)^2}$$

$$= \frac{x^6 - 6x^4 + 24x^2 - 8}{(3x^2+2)^2} < \frac{(x^2-2)^3}{(3x^2+2)^2}$$

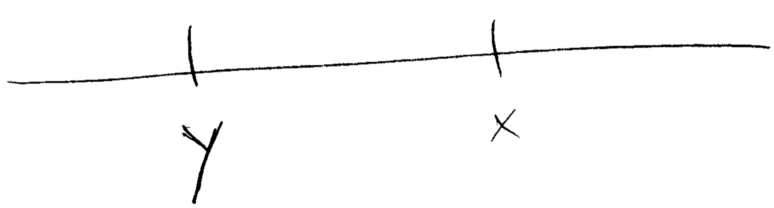
$\binom{3}{1} = 6 \qquad \binom{3}{2} = \frac{3\cdot 2\cdot 1}{1!}$

$y = O(x) \qquad y^2 - 2 = O(x^2)$

$y - x = O(x)$



$y =$ $\qquad$ $y >$



PS 8 w/w

$$\{|x+iy|+|u+iv|\}^2 =$$

$$= (x^2+y^2) + (u^2+v^2) + 2\sqrt{(x^2+y^2)(u^2+v^2)}$$

$$= \quad '' \quad + \quad '' \quad + 2\sqrt{\underline{x^2}u^2 + x^2v^2 + y^2u^2 + \underline{y^2}v^2}$$

$$= \quad '' \quad + \quad '' \quad + 2\sqrt{(xu+yv)^2 + (xv-yu)^2}$$

$$(xu)^2 + 2xuyv + (yv)^2 \qquad -2xyuv$$

$$\geq x^2+y^2 + u^2+v^2 + 2|xu+yv|$$

$$\sqrt{a^2+b^2} \geq \sqrt{a^2} = |a| \geq a.$$

$$\geq x^2+y^2+u^2+v^2 + 2(xu+yv) = |x+iy+u+iv|^2$$

$$= x^2 + 2xu + u^2 + \qquad \cancel{(x+u)^2 + (y+v)^2}$$

$$y^2 + 2yv + v^2 \qquad \cancel{x^2}$$

$$= (x+u)^2 + (y+v)^2 = |x+iy+u+iv|^2$$

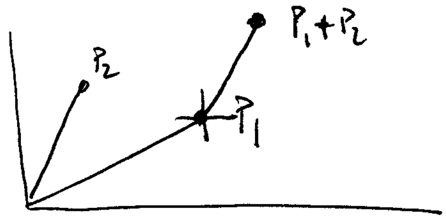
$$\therefore |x+iy|+|u+iv| \geq |x+iy+u+iv|$$

Pg 9 w/w

$$= \{ x^2u^2 - \cancel{2xy}uv + y^2v^2 + x^2v^2 + \cancel{2xy}vu + y^2u^2 \}^{1/2}$$

$$= \{ x^2(u^2+v^2) + y^2(v^2+u^2) \}^{1/2}$$

$$= \{ (x^2+y^2)(v^2+u^2) \}^{1/2}$$



$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$$
$$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

$$z_1z_2 = r_1r_2\big(\cos\theta_1\cos\theta_2 + i(\sin\theta_2\cos\theta_1 + \sin\theta_1\cos\theta_2)$$
$$- \sin\theta_1\sin\theta_2\big)$$

$$= r_1r_2\big(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\big)$$

pg 12 w/w

$$\frac{\frac{15n+19}{1+n^3}}{\frac{1}{n^2}} = \frac{15n^3+19n^2}{n^3+1} = \frac{15+19/n}{1+1/n^3}$$

Then $\left|\frac{\frac{15n+19}{1+n^3}}{\frac{1}{n^2}}\right| \leq 16 \quad \forall n$

$\therefore \quad \frac{15n+19}{1+n^3} = O\left(\frac{1}{n^2}\right)$

pg 12 ugw

$$x_n \geq \alpha - \tfrac{1}{2}\epsilon \qquad n \geq n_1$$

$$x_n < \alpha + \tfrac{1}{2}\epsilon \qquad n \geq n_2$$

$n = \mathrm{Max}(n_1, n_2)$

$$\begin{cases} x_n - \alpha \geq -\dfrac{\epsilon}{2} \Leftrightarrow \cancel{\alpha - x_n < \dfrac{\epsilon}{2}} \\ x_n - \alpha < \dfrac{\epsilon}{2} \end{cases}$$

$$\Rightarrow \quad |x_n - \alpha| < \frac{\epsilon}{2}$$

Example 2. P_v: $\lim (z_m - z_m') = l - l'$

given $\epsilon > 0$

let $n_1 \ni |z_m - l| < \frac{\epsilon}{2}$ $\quad m > n_1$

& $n_2 \ni |z_m' - l'| < \frac{\epsilon}{2}$ $\quad m > n_2$

Then $|z_m - z_m' - (l - l')| = |z_m - l - z_m' + l'|$

$\leq |z_m - l| + |z_m' - l'| < \epsilon$ if $m > \mathrm{Max}(n_1, n_2)$

P_x: $\lim z_m z_m' = l l'$

$|z_m z_m' - \ell\ell'|$

$\cancel{(z_m - \ell)(z_m' + \ell') = z_m z_m' +}$. Both ℓ + ℓ' finite

$|z_m z_m' - \ell z_m' + \ell z_m' - \ell\ell'|$

$|z_m'(z_m - \ell) + \ell(z_m' - \ell')|$

$\leq |z_m'||z_m - \ell| + |\ell||z_m' - \ell'|$

Then if $|z_m' - \ell'| < \frac{\epsilon}{2|\ell|}$ $\forall\ m > n_1$

$\Rightarrow$ $|z_m'| < \ell' + \frac{\epsilon}{2|\ell|}$

$\therefore \leq \left(\ell' + \frac{\epsilon}{2|\ell|}\right)|z_m - \ell| + \frac{\epsilon}{2}$ $\forall\ m > n_1$

pick $n_2 \ni |z_m - \ell| \leq \dfrac{\epsilon}{2\left(\ell' + \frac{\epsilon}{2|\ell|}\right)}$

Then Above is $\ni$

$\leq \epsilon$ $\forall\ m \geq \max(n_1, n_2)$

· if l or $l' = \infty$. Sufficient to consider one only, then

Say WOLOG $l = \infty$.
given N $\exists$ n_1 ∋
$\Rightarrow$ $|z_m| > N$ $\forall$ $m > n_1$

Then $|z_m z_m'| = |z_m||z_m'| > N|z_m'|$

if ~~$m > n_2$~~

$\cancel{|z_m' - l'| <}$ $\qquad$ $\cancel{|z_m' - l'|} < 1$

$\cancel{|z_m'|} <$

$\cancel{(-1+l') < z_m' < 1} + l'$

$\Rightarrow$ $\cancel{|z_m z_m'| > N|z_m'|} >$

let a ∋ $l' - a > 0$ Then $\exists$ n_2 ∋

$$|z_m' - l'| < a \quad \forall \quad m > n_2$$

$$\Rightarrow \quad a - l' < z_m' < a + l'$$

Then $|z_m z_m'| > N|z_m'| > N z_m' > N(a-l')$ can be made as large as needed.

Pr: $\lim\left(\frac{z_m}{z_m'}\right) = \frac{l}{l'}$

$$\left|\frac{z_m}{z_m'} - \frac{l}{l'}\right| = \left|\frac{l' z_m - z_m' l}{z_m' l'}\right| = \left|\frac{l' z_m - l' l - z_m' l + l' l}{z_m' l'}\right|$$

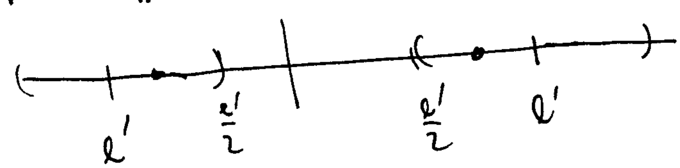
$$\frac{|l'||z_m - l| + |l||z_m' - l'|}{|z_m'||l'|} = \frac{|z_m - l|}{|z_m'|} + \frac{|l|}{|l'||z_m'|}|z_m' - l'|$$

Case I; l, l' both finite.

Now: $|z_m' - l'| < l'/2$ pick

$\cancel{-l' + l' < z_m' < l' + l'} \Rightarrow \cancel{0 < z_m' < 2l'}$

$\therefore$ $\cancel{|z_m'| < 2l'}$



$\Rightarrow$

$$-\frac{l'}{2} + l' < z_m' < l' + \frac{l'}{2}$$

$$\Rightarrow \quad \frac{l'}{2} < z_m' < \frac{3}{2} l' \Rightarrow |z_m'| > \frac{|l'|}{2} \text{ see picture above}$$

$$\frac{1}{|z_m'|} < \frac{1}{|l'|/2}$$

$\therefore$

$\therefore$

$$\therefore \left|\frac{z_m}{z_m'} - \frac{l}{l'}\right| < \frac{2}{|l'|}|z_m - l| + \frac{2|l|}{|l'|^2}|z_m' - l'|$$

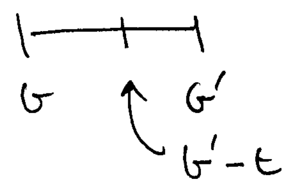
~~pick~~ $\forall\, m > n_1$

pick $n_2 \ni |z_m - l| < \frac{\epsilon}{2} \cdot \frac{|l'|}{2} \quad \forall\, m > n_2$

& $n_3 \ni |z_m' - l'| < \frac{\epsilon}{2} \frac{|l'|^2}{2|l|} \quad \forall\, m > n_3$

Then $\left|\frac{z_m}{z_m'} - \frac{l}{l'}\right| < \epsilon \quad \forall\, m > \mathrm{Max}(n_1, n_2, n_3)$.

$(1+a)^n > 1+na$.



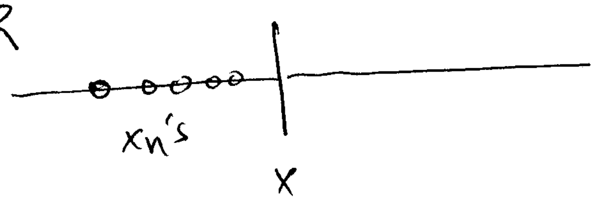
$$G' - \epsilon = \frac{G'}{2} + \frac{G}{2} = \frac{1}{2}(G'+G)$$

Given $x_{n+1} \geq x_n$ $\{x_n\}$ Pg 12 w/w

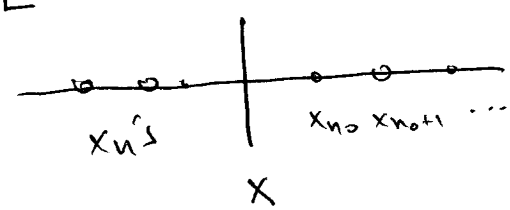
$L = \{x \mid x \in \mathbb{Q} \text{ \& } x_n \geq x$ ~~some~~ $\forall n > n_0(x)\}$

$R = \{x \mid x \in \mathbb{Q} \text{ \& } x_n < x \quad \forall n.\}$

$x \in R$



$x \in L$



$x_n - \alpha \geq \frac{-1}{2}\epsilon$

$\alpha - x_n \leq \frac{1}{2}\epsilon$ $\qquad x_n - \alpha < \frac{\epsilon}{2}$

Ex 2: if $l' \neq 0$ $\exists\, n_1 \ni |z_m' - l'| < \epsilon_1$ $m > n_1$

& $\therefore$ denominator is not close to zero.

$0 < x < 1$ $\qquad x = (1+a)^{-1}$ $\qquad a = 0 \Rightarrow x = 1$

$a = \infty \Rightarrow x = 0.$

$1 < \frac{1}{x} < \infty$ $\qquad \frac{1}{x} = 1 + a$

$0 < \frac{1}{x} - 1 < \infty$ $\qquad a = \frac{1}{x} - 1.$

$0 < a < \infty.$

$$\frac{1}{(1+a)^n} < \frac{1}{1+na}$$

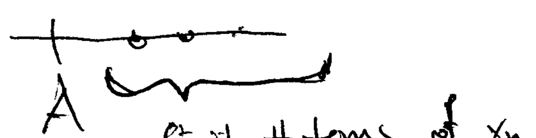
$$(1+a)^n > 1+na$$

$$(1+a)^n = \sum_{k=0}^{n} \binom{n}{k} a^k$$

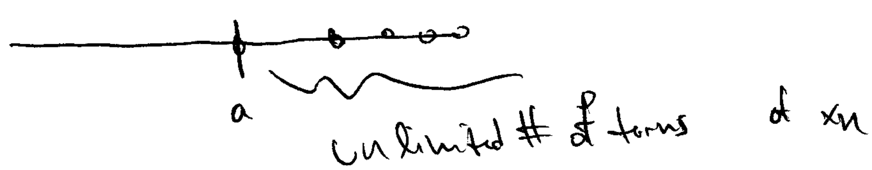
<u>pg 13 w/w</u>

in def of cluster point it does <u>not</u> say all terms after some point that is the difference

Bolzano's Thm: $A \in \mathbb{R}$ class

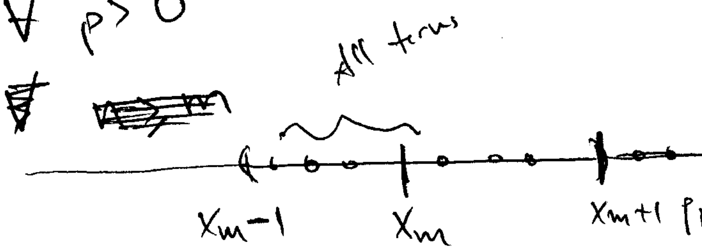


$a \in L$ class



Sups & Infs are for sequences w/ limit points & that don't have limits themselves

$|x_{m+p} - x_m| < 1 \qquad \forall p > 0$



$\lambda_1 = \min\{x_i \mid i = 1, \ldots, m\}$

$\rho_1 = \max\{x_i \mid i = 1, \ldots, m\}$

$$-1 + \lambda_1 < -1 + x_m < x_{m+p} < 1 + \underline{x_m} < 1 + \rho_1 \qquad \forall p > 0$$

$$-1 + \lambda_1 < x_{m+p} < 1 + \rho_1 \qquad \forall p > 0$$

$\forall n \quad \lambda_1 - 1 < x_n < 1 + \rho_1$

let $\epsilon < \frac{G-H}{4}$ $\quad \exists n \ni |x_{n+p} - x_n| < \epsilon \quad \forall p > 0$

G & H l.p. $\Rightarrow$ q + r +

$$|G - x_{n+q}| < \epsilon \qquad |H - x_{n+r}| < \epsilon.$$

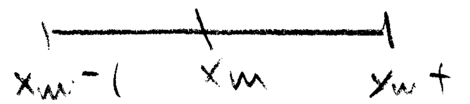
Th $\quad |G - H| < |G - x_{n+q}| + | \cdots | < 4\epsilon$

$\Rightarrow G - H < 4\epsilon$

Pg 14 w/w

$\therefore \ |x_{m+p} - x_m| < 1 \quad \forall \quad p \geq 1$ → All terms $x_{m+1}, \ldots$ are in this interval

$\{x_1, x_2, \ldots, x_m, x_{m+1}, \ldots\}$



$\alpha_1 = \min\{x_1, x_2, x_3, \ldots, x_m\}$

$\beta_1 = \max\{x_1, \ldots, x_m\}$

$\therefore \quad \beta_1 > x_1, x_2, \ldots, x_m$ in particular $\beta_1 > x_m$

& $\beta_1 + 1 >$ All $x_n \ \forall n$

Sim $\alpha_1 < x_1, \ldots, x_m$ in particular $\alpha_1 < x_m$

$\Rightarrow \alpha_1 - 1 <$ All $x_n \ \forall n$

By Bolzano - Weierstrass Theorem

$\{u_n\}$

$S\{u_n\} = \{s_n\} \quad s_n = \sum_{k=1}^{n} u_k$

$S\{s_n\} = S^2\{u_n\} = \{\sum_{k=1}^{n} S\{u_k\}_k\}$

$|S_n - S_{n+p}| < \epsilon$

$|S_{n+p} - S_n| < \epsilon$

$|S_{n,p}| < \epsilon. \quad \forall p > 0.$

$S_{n,n} = U_{n+1} + U_{n+2} + \cdots U_{n+n}$

$= \underbrace{\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}}_{n \text{ terms}} > \underbrace{\frac{1}{2n} + \frac{1}{2n} + \cdots}_{n\ t}$

$\frac{1}{2n} \cdot n = \frac{1}{2}.$

$S_{2^{n+1}} = \sum_{k=1}^{2^{n+1}} \frac{1}{k} = 1 + \sum_{k=2}^{2} + \sum_{k=3}^{8} + \cdots \sum_{k=2^n-1}^{2^{n+1}} \frac{1}{k}$

$\begin{array}{ccccc} S_{1,1} = \frac{1}{2} & S_{2,2} & S_{4,4} & S_{8,8} \cdots & S_{2^n,2^n} \\ 0 & 1 & 2 & 3 & n \end{array}$

$> 1 + \underbrace{\frac{1}{2} + \frac{1}{2} + \cdots \frac{1}{2}}_{n+1 \text{ term}}$

pg 15 w/w

$$\therefore \; = 1 + \frac{n+1}{2} = \frac{n+3}{2} \longrightarrow \infty.$$

pg 16 w/w

$$\sum_{k=1}^{\infty} z^k = \sum_{k=0}^{\infty} z^{k+1} = z \sum_{k=0}^{\infty} z^k = z \frac{1}{1-z} = \frac{z}{1-z}$$

$$\sum_{k=0}^{\infty} z^{-k} = \frac{1}{1-1/z} = \frac{z}{z-1} = -\frac{z}{1-z}.$$

pg 17 w/w

$$\sum_{n=1}^{m} a_n f_n = \sum_{n=1}^{m} (S_n - S_{n-1}) f_n = \cancel{\sum \Delta S_{n-1} f_n}$$

$$\leq \; = \sum_{n=1}^{m} S_n f_n - \sum_{n=1}^{m} S_{n-1} f_n$$

$$= \sum_{n=1}^{m} S_n f_n - \sum_{n=0}^{m-1} S_n f_{n+1} \qquad \text{but } S_0 \equiv 0$$

$$= \sum_{n=1}^{m-1} S_n (f_n - f_{n+1}) + S_m f_m$$

$$\leq \cancel{A\left[\sum_{n=1}^{m-1} \quad + f_m\right]} = A\left(f_{n-1}\Big|_1^m + f_m\right)$$

$$\cancel{\sum \Delta f_n}$$

$$\le A\left\{\sum_{n=1}^{m-1}(-\Delta f_n) + f_m\right\}$$

$$= A\left\{-f_n\Big|_1^m + f_m\right\} = A\left[\cancel{-f_m}+f_1+\cancel{f_m}\right]$$

$$= Af_1$$

$$\left|\sum_{n=1}^{m} a_n f_n\right| \le \sum_{n=1}^{m-1} |S_n||f_{n+1}-f_n| + |S_m||f_m|$$

$$\le A\left\{\sum_{n=1}^{m-1}|f_{n+1}-f_n| + |f_m|\right\}.$$

$$\left|\sum_{n=m+1}^{m+q}\right| = \left|\sum_{n=1}^{m+q} - \sum_{n=1}^{m}\right| \le \left|\sum_{n=1}^{m+q} a_n\right| + \left|\sum_{n=1}^{m} a_n\right|$$

2k.

$$\left|\sum_{n=m+1}^{m+q} a_n\right| \le A f_{m+1} \quad \text{By Abel's ineq}$$

$$A = \max_{\substack{n\\=0,1,2,\cdots}}\left\{\left|\sum_{k=m+1}^{m+1+n} a_k\right|\right\}$$

$$\left|\sum_{n=1}^{p} a_n\right| < k \qquad \forall\, p = 1, 2, 3, \dots$$

Thus $|a_1|, |a_1+a_2|, |a_1+a_2+a_3|, \dots$

are all bounded by k.

A in $\left|\displaystyle\sum_{n=m+1}^{m+p} a_n f_n\right| \le A f_{m+1}$.

$$A = \underbrace{|a_{m+1}|}_{q=1},\ \underbrace{|a_{m+1}+a_{m+2}|}_{q=2},\ \dots \qquad \le 2k.$$

$\vdots$

θ fixed

Pg 18 W/W

Ex 1: $\left|\sum_{n=1}^{p} \sin n\theta\right| = \left| \sin\frac{(p+1)\theta}{2} \sin\frac{p\theta}{2} \operatorname{cosec}\frac{\theta}{2} \right|$

$\leq \left|\operatorname{cosec}\frac{\theta}{2}\right| = k$ ind of p

By using summation techniques in Difference eqs.

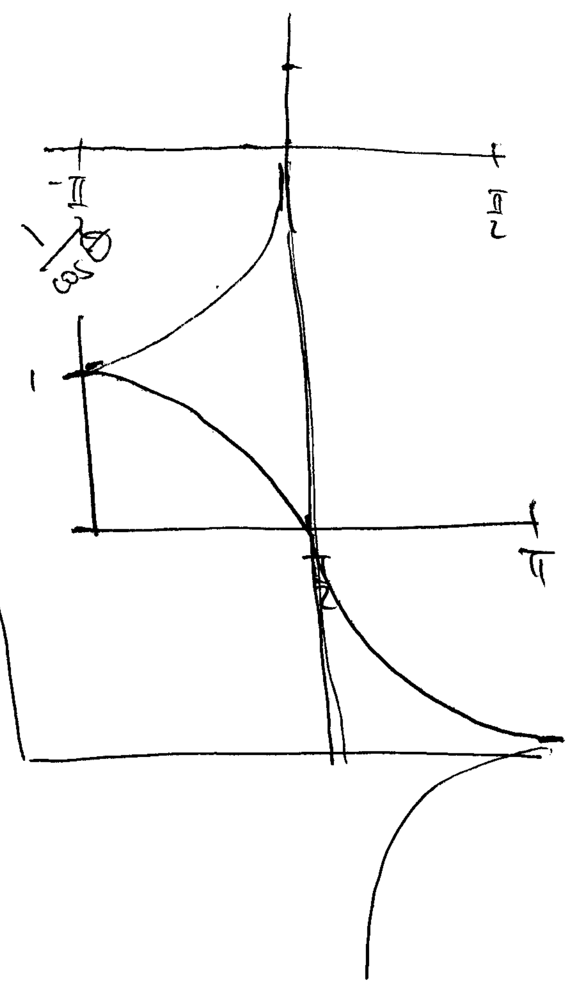
$0 < \theta < 2\pi$

$0 < \frac{\theta}{2} < \pi$ $\qquad \operatorname{cosec}\frac{\theta}{2} = \frac{1}{\cos(\frac{\theta}{2})}$

$\frac{\theta}{2} = \frac{\pi}{2}$

$\Rightarrow \theta = \pi.$

Thus $\underline{\theta \neq \pi}$.



$\Rightarrow$

$$\sum_{n=1}^{p} \sin n\theta = \frac{1}{2i}\sum_{n=1}^{p} e^{in\theta} - \frac{1}{2i}\sum_{n=1}^{p} e^{-in\theta}$$

$$= \frac{1}{2i}\left(\frac{e^{in\theta}}{i\theta}\right|_1^{p+1} - \frac{1}{2i}\left(\frac{e^{-in\theta}}{-i\theta}\right|_1^{p+1}$$

$=$

$$\sum_{n=1}^{p} r^n$$

$\Delta r^n = r^{n+1} - r^n = r^n(r-1)$

$\Rightarrow \quad r^n = \frac{\Delta r^n}{r-1}$

Pg 18 w/w

Show: $\left| \sum_{n=1}^{p} \sin n\theta \right| \leq \dfrac{1}{\cos \frac{\theta}{2}}$

$$\sum_{n=1}^{p} r^n = \frac{r^{p+1} - r^1}{r-1}$$

$$\sin n\theta = \frac{e^{in\theta} - e^{-in\theta}}{2i}$$

Now: $\dfrac{1}{2i} \displaystyle\sum_{n=1}^{P} e^{in\theta} = \dfrac{1}{2i} \dfrac{e^{ip\theta + i\theta} - e^{i\theta}}{e^{i\theta} - 1}$

$$\therefore \sum_{n=1}^{p} \sin n\theta = \frac{1}{2i}\left(\frac{e^{i\theta(p+1)} - e^{i\theta}}{e^{i\theta} - 1} \right) - \frac{1}{2i}\left(\frac{e^{-i\theta(p+1)} - e^{-i\theta}}{e^{-i\theta} - 1} \right)$$

$$= \frac{1}{2i} e^{i\theta}$$

$$\sum_{n=1}^{p} \cos n\theta = \frac{1}{2}\sum_{n=1}^{p} e^{in\theta} + \frac{1}{2}\sum_{n=1}^{p} e^{-in\theta}$$

$$= \frac{1}{2}\,\frac{e^{i\theta n}}{e^{i\theta}-1}\Bigg|_{1}^{p+1} + \frac{1}{2}\,\frac{e^{-in\theta}}{e^{-i\theta}-1}\Bigg|_{1}^{p+1}$$

$$= \frac{1}{2}\frac{1}{(e^{i\theta}-1)}\left[e^{i\theta(p+1)} - e^{i\theta}\right] + \frac{1}{2}\frac{1}{e^{-i\theta}-1}\left[e^{-i\theta(p+1)} - e^{-i\theta}\right.$$

$$= \frac{1}{2}\left(\frac{e^{i\theta}}{e^{i\theta}-1}\right)\left(e^{i\theta p}-1\right) + \frac{e^{-i\theta}}{2(e^{-i\theta}-1)}\left(e^{-i\theta p}-1\right)$$

$$= \frac{(1-e^{i\theta})(e^{i\theta p}-1)}{2(e^{i\theta}-1)(e^{-i\theta}-1)} + \frac{(1-e^{-i\theta})(e^{-i\theta p}-1)}{2(e^{-i\theta}-1)(e^{i\theta}-1)}$$

$$= \frac{e^{i\theta p}-1-e^{i\theta(p+1)}+e^{i\theta}+e^{-i\theta p}-1-e^{-i\theta(p+1)}+e^{-i\theta}}{2(1-2\cos\theta+1)}$$

$$= \frac{2\cos p\theta - 2 - 2\cos(p+1)\theta + 2\cos\theta}{4(1-\cos\theta)}$$

$$\frac{-2+2\cos\theta+2\cos p\theta-2\cos(p+1)\theta}{4(1-\cos\theta)}$$

$$=-\frac{1}{2}+\frac{\cos p\theta-\cos(p+1)\theta}{2(1-\cos\theta)}$$

$$=-\frac{1}{2}+\frac{1}{2}\cos p\theta-\cos p\theta\sin\theta+\sin$$

$$\frac{e^{ip\theta}+e^{-ip\theta}-e^{i(p+1)\theta}-e^{-i(p+1)\theta}}{2(\underbrace{2-e^{i\theta}-e^{-i\theta}})}$$

$$-(\underbrace{e^{i\theta}-2+e^{-i\theta}})$$

$$-\left(e^{i\theta/2}-e^{-i\theta/2}\right)^2$$

$$\to\frac{e^{ip\theta}+e^{-ip\theta}-e^{ip\theta}e^{i\theta}-e^{-ip\theta}e^{-i\theta}}{e^{ip\theta}(1-e^{i\theta})+e^{-ip\theta}(1-e^{-i\theta})}$$

$$= 2\,\mathrm{Re}\; e^{ip\theta}(1-e^{i\theta})$$

$$= 2\,(\cos p\theta + i\sin p\theta)(1-\cos\theta - i\sin\theta)$$

$$= 2\,\cos(p\theta)(1-\cos\theta) + \sin p\theta\,\sin\theta$$

Pg 18 w/w

Ex 1:

If $\left|\sum_{n=1}^{p} \sin n\theta\right| < \frac{1}{\cos\frac{\theta}{2}}$

Then a direct application of Dirchlet's test gives $\sum_{n=1}^{\infty} a_n \sin n\theta$ series converges trivially +

$\forall$ $0<\theta<2\pi$. if $\theta=0$ $\forall \theta \in \mathbb{R}$

Sin if $\theta = 2\pi$. As the

for $\sum_{n=1}^{\infty} f_n \cos n\theta$ consider $\sum_{n=1}^{p} \cos n\theta = ?$

using exponentials or difference eq technique ~~$2\cos n\theta = \cos(n+1)\theta - \cos n\theta$~~

one can get $\sum_{n=1}^{p} \cos n\theta = \frac{\sin\theta(n-\frac{1}{2})}{2\sin\frac{\theta}{2}}$

$\left|\sum_{n=1}^{p} \cos n\theta\right| \le \frac{1}{2\left|\sin\frac{\theta}{2}\right|}$

$\frac{\theta}{2} = \pm n\pi$

$\theta = \pm 2\pi n \quad n = 0,1,2,3.$

if $0<\theta<2\pi$

Again by Dirchlet's test $\sum_{n=1}^{\infty} f_n \cos n\theta$ converges. if $\theta \neq$ even multiple of π.

<u>Ex 2</u>: Show (a) $\sum_{n=1}^{\infty} (-1)^n f_n \cos n\theta$ if $\theta \in \mathbb{R}$ + not odd mult of π

(b) $\sum_{n=1}^{\infty} (-1)^n f_n \sin n\theta \quad \forall \theta$.

We consider (b) 1st from <u>Ex 1</u>: $\sum_{n=1} f_n \sin n\theta$ converges $\forall \theta$

$$\sum_{n=1}^{\infty} f_n \sin n(\theta + \pi) \quad \text{conv.} \ \forall \theta$$

$$= \sum_{n=1}^{\infty} f_n(\sin n\theta \cos n\pi + \underbrace{\cos n\theta \sin n\pi}_{=0}) = \sum_{n=1}^{\infty} f_n (-1)^n \sin n\theta .$$

consider (a) 1st. from Ex 1: $\sum_{n=1}^{\infty} f_n \cos n\theta$ converges $\forall \theta \neq 2n\pi \quad n \in \mathbb{Z}$

~~Note~~ Then $\sum_{n=1}^{\infty} f_n \cos n(\theta + \pi)$ converges $\forall \theta \neq -\pi + 2n\pi \quad n \in \mathbb{Z}$.

$$\Rightarrow \sum_{n=1}^{\infty} f_n \cos n\theta \cos n\pi \quad \text{conv} \ \forall \theta \neq (2n-1)\pi \quad n \in \mathbb{Z}.$$

$$\Rightarrow \sum_{n=1}^{\infty} f_n (-1) \cos n\theta \quad \text{conv} \ \forall \theta \neq (2n+1)\pi \quad \mathbb{Z} \in \mathbb{Z}.$$

Pg 19 w/w

$$\frac{1}{1^{s-1}} + \frac{1}{2^{s-1}} + \frac{1}{2^{2(s-1)}} + \frac{1}{2^{3(s-1)}} + \dots \frac{1}{2^{(p-1)(s-1)}}$$

$$\sum_{k=0}^{p-1}\left[\left(\frac{1}{2}\right)^{s-1}\right]^k < \frac{1}{1-(1/2)^{s-1}} = \frac{1}{1-2^{1-s}}$$

$p = 2$
$2^p - 1 = 3$

$s < 1 . \quad x > 1$

$s < 1 ; x > 1$

$x^s < x$

$\frac{1}{x^s} > \frac{1}{x}$

$\left(\frac{1}{x}\right)^s > \frac{1}{x}$

$\frac{1}{x} < 1$

$3^2 = 9$

$10^{.5} = 3.16.$

$10^{.5} < 10.$

Pg 20 w/w

$|u_n| < C|v_n|$

$\Rightarrow \frac{|u_n|}{|v_n|} < C$

Ex 2: $\sum_{k=1}^{\infty} \frac{1}{k^2(z - e^{ki})}$

let $|z| \neq 1$

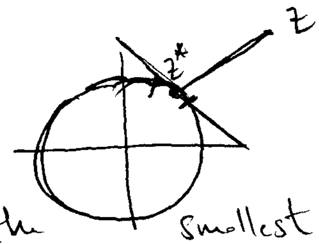
Then $|z - z_n| \geq \big|1 - |z|\big| = \epsilon^{-1}$

Pg 20 w/o

Ex 2: Show $\sum_{n=1}^{\infty} \frac{1}{n^2(z-z_n)}$ is conv.

$\forall z \not\ni |z| = 1$ $\qquad z_n = e^{in}$

given a $z \ni |z| \neq 1$



$\exists$ a $z_n \ni |z - z_n|$ is the smallest. say z_n^*

Then $\left| \frac{1}{n^2} \frac{1}{z - z_n} \right| < \frac{1}{n^2} \frac{1}{|z - z_n^*|} \quad \forall n$

As $\sum 1/n^2 < \infty$ the above

$$|z - z_n| \geq \big| |z| - |z_n| \big| \geq \big| |z| - 1 \big| = 1 - |z|$$

$\cancel{\big| |a| - |b| \big| \leq |a+b| \leq |a| + |b|}$

$$\frac{1}{k^2|z-e^{ik}|} \leq \frac{c}{k^2} \qquad \big|1-|z|\big| > c^{-1}$$

Ex 3: $\displaystyle\sum_{k=1}^{\infty} 2^k \sin\left(\frac{z}{3^k}\right)$ conv. Absolutly !! ??

$$\lim_{n\to\infty} \left(3^n \sin\left(z/3^n\right) \right) = \lim_{n\to\infty} \left(3^n \sum_{k=0}^{\infty} \frac{z^{2k+1}(-1)^k}{(2k+1)!\, 3^{n(2k+1)}} \right)$$

$$= z.$$

$\exists\, k.$ $\quad |3^n \sin(z/3^n)| < k$

$$\therefore \boxed{\left|2^n \sin\left(\frac{z}{3^n}\right)\right|} \leq k\left(\frac{2}{3}\right)^n$$

$|\sin x| \leq |x| \quad \forall\, x \in \mathbb{R}.$

$$\therefore \left|2^n \sin\frac{z}{3^n}\right| \leq \frac{2^n}{3^n}|z| \quad \forall\, x \in \mathbb{R}$$

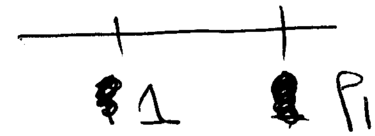
$\overline{\lim} |u_n|^{1/n} > 1$

$\Rightarrow$ ~~[illegible]~~

If $\overline{\lim} |u_n|^{1/n} = \infty$

How show $\lim |u_n| = \infty$ + divergence

If $\overline{\lim} |u_n|^{1/n} = \rho > 1$ + finite



$\Rightarrow \exists\, m \ni$ when $n \geq m$ $|u_n|^{1/n} \geq \rho > 1$

Then $|u_n| \geq \rho^n > 1^n$

As $\sum \rho^n$ div $\sum |u_n|$

$\overline{\lim} |u_n|^{1/n} > 1$ if $u_n \to 0$

$|u_n| \to 0.$

$|u_n|^{1/n} \not\to 0.$

Not true!!

Pr: If $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = l < 1$

$\exists\ n_1 \ni \forall n > n_1$

$$\left|\left|\frac{u_{n+1}}{u_n}\right| - l\right| < \epsilon$$

$$l - \epsilon < \left|\frac{u_{n+1}}{u_n}\right| < l + \epsilon \qquad \text{pick } \epsilon < \frac{1-l}{\cancel{4}}$$

Then

$$\left|\frac{u_{n+1}}{u_n}\right| < l + \left(\frac{1-l}{\cancel{4}}\right) = 1 \quad \forall n > n_1$$

If $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = l > 1$

$|z| < 1$

$(1+n^{-1})^n - |z^n| >$

$$\sum_{k=0}^{n} \binom{n}{k} n^{-(n-k)} = 1 + 1 + \frac{(n-1)}{2n} + \cdots$$

$$1n^{-n} + \cdots + \binom{n}{n-2} n^{-(n-n+2)} + \binom{n}{n-1} n^{-1} + 1$$

$$\binom{n}{n-2} = \binom{n}{2} \qquad \binom{n}{n-1} = \binom{n}{1}$$

$$\cdots\quad \frac{n(n-1)}{2} n^{-2} + \frac{n n^{-1}}{1} + 1$$

$$= 1 + 1 + \frac{n-1}{2n} + \cdots$$

$$> \cancel{1} + 1 + \frac{n-1}{2n} + \cdots - \cancel{1} > \underline{1}$$

<u>pg 23 w/w</u>

$$\left(1+\frac{1}{n}\right)^{-(1+\frac{1}{2}c)} = 1 - \left(1+\frac{c}{2}\right)\frac{1}{n} + O(1/n^2)$$

$$\therefore \underbrace{n\left\{\frac{v_{n+1}}{v_n} - 1\right\}}_{\substack{\parallel \\ B_n}} \longrightarrow -1-\frac{c}{2}$$

$-1-\frac{c}{2}$

$$A_n = n\left\{\frac{u_{n+1}}{u_n} - 1\right\}$$

$$\overline{\lim_{n\to\infty}}\, A_n = -1-\frac{c}{2}$$

$$\Rightarrow \forall \epsilon > 0 \ \exists n_1 \ni \forall n > n_1$$

$$\left|A_n + 1 + \frac{c}{2}\right| \le \epsilon$$

$$-1-c < -1$$

pg 21 w/w

$\overline{\lim_{n\to\infty}} |u_n|^{1/n} < 1$ As $|u_n|^{1/n} > 0 \ \forall n$

if $\sigma = \overline{\lim_{n\to\infty}} |u_n|^{1/n} < 1$

Then by def of of lim sup $\exists \ \infty$ # terms

$\sigma - \epsilon < |u_n|^{1/n} < \sigma + \epsilon$ pick $\epsilon \ni$

$\sigma + \epsilon < \rho < 1$

Then ∞ # terms $\Rightarrow \exists \ m \ni |u_n|^{1/n} < \rho \ \forall n > m.$

If $\exists$ only finite # of points greater than $\sigma + \epsilon$ we are o.k. Then run index up to that point..

If $\exists \ \infty$ # of points > then $\sigma + \epsilon$ then we get σ not the smallest limit pt $\therefore$

If $\overline{\lim} |u_n|^{1/n} > 1$ $\Rightarrow \exists \ \infty$ # of $n \ni 1 < \rho < \sigma - \epsilon < |u_n|^{1/n}$

" σ

$\forall n > m$

$\Rightarrow |u_n| > (\sigma - \epsilon)^n > \rho^n > 1^n$

$\therefore |u_n| \to \infty.$

pg 22 w/w:

$$\left|\frac{u_{n+1}}{u_n}\right| < \rho \qquad n > r.$$

$$|u_{r+1}| + |u_{r+2}| + \cdots + |u_{r+m}| = |u_{r+1}|\left(1 + \frac{|u_{r+2}|}{|u_{r+1}|}\right.$$

~~But~~ $\cancel{\frac{|u_{r+1}|}{|u_r|}} < \rho$; $\cancel{\frac{|u_{r+2}|}{|u_{r+1}|} = \frac{|u_{r+2}|}{|u_{r+1}|}}$

$$\left. + \frac{|u_{r+3}|}{|u_{r+1}|} + \frac{|u_{r+4}|}{|u_{r+1}|} + \cdots \right)$$

Now: $\frac{|u_{r+2}|}{|u_{r+1}|} < \rho$

$$\frac{|u_{r+3}|}{|u_{r+1}|} = \frac{|u_{r+3}|}{|u_{r+2}|}\,\frac{|u_{r+2}|}{|u_{r+1}|} < \rho^2$$

$$\vdots$$

$$\frac{|u_{r+n}|}{|u_{r+1}|} = \rho^{n-1}$$

pg 22 w/w

$1 - \ell > 0$

If $\left| \left[\left| \frac{u_{n+1}}{u_n} \right| - \ell \right] \right| < \frac{1}{2}(1-\ell)$

$$\left| \frac{u_{n+1}}{u_n} \right| < \frac{1}{2} - \frac{\ell}{2} + \ell = \frac{1}{2}(1+\ell) < 1 \quad \text{As } \ell < 1$$

If $\left| \frac{u_{n+1}}{u_n} \right| > 1 \Rightarrow 1 + \epsilon < \left| \frac{u_{n+1}}{u_n} \right|$ for ∞ many n

$$\Rightarrow |u_{n+1}| > r|u_n| > r^2 |u_{n-1}| \cdots > r^{n+1} |u_0|$$

w/ $r > 1 \Rightarrow |u_{n+1}| \to \infty$.

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pg 23 w/w

$$|z^n - (1+n^{-1})^n| \geq (1+n^{-1})^n - |z|^n \geq \sum_{k=}$$

$$(1+n^{-1})^n = \sum_{k=0}^{n} \binom{n}{k} (n^{-k}) \geq 1 + 1 + \frac{n-1}{2n} + \cdots - 1 = 1 + \frac{n-1}{2n}$$

> 1

$$\binom{n}{2} = \frac{n!}{(n-2)!\,2!} = \frac{n(n-1)}{2} n^{-2} = \frac{n-1}{2n}$$

pg 22 w/w

If $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = l$ w/ $l > 1$

Then given $\epsilon > 0$ $\exists$ n_0 ∋

$$\left|\left|\frac{u_{n+1}}{u_n}\right| - l\right| < \epsilon \quad \forall\ n > n_0$$

$$\Rightarrow \quad l - \epsilon < \left|\frac{u_{n+1}}{u_n}\right| < l + \epsilon$$

pick ϵ ∋ $l - \epsilon = \rho > 1$

Then $\left|\frac{u_{n+1}}{u_n}\right| > \rho \quad \forall\ n > n_0$

Then $\frac{|u_{n_0+1}|}{|u_{n_0}|} > \rho \Rightarrow |u_{n_0+1}| > \rho |u_{n_0}|$

$$\frac{|u_{n_0+2}|}{|u_{n_0+1}|} = \frac{|u_{n_0+2}|}{|u_{n_0}|} \cdot \frac{|u_{n_0}|}{|u_{n_0+1}|} > \rho$$

$$\frac{|u_{n_0+2}|}{|u_{n_0}|} > \rho \frac{|u_{n_0+1}|}{|u_{n_0}|} > \rho^2$$

In gen $|u_{n_0+p}| > \rho^p |u_{n_0}|$

$$\therefore \lim_{p \to \infty} |u_{n_0+p}| = \infty.$$

pg 23 w/w

If $\forall\, n > r$ $\quad |u_{n+1}| > |u_n| \quad \forall n$

Then $\lim\limits_{n \to \infty} \dfrac{|u_{n+1}|}{|u_n|} > 1$

$|u_{r+1}| > |u_r|$

$|u_m| = |u_{m+r-r}|$

As $\overline{\lim\limits_{n\to\infty}} \underbrace{n\left\{\left|\dfrac{u_{n+1}}{u_n}\right| - 1\right\}}_{\lim_1} = -1-c.$

& $\lim\limits_{n\to\infty} \underbrace{n\left\{\left|\dfrac{v_{n+1}}{v_n}\right| - 1\right\}}_{\lim_2} = -1-\frac{1}{2}c$

Then

$-1-c \qquad -1-\frac{1}{2}c \qquad -1$

$\therefore \lim_2 > \lim_1$

$\therefore \exists\, m \ni n > m$

$$n\left\{\left|\frac{u_{n+1}}{u_n}\right| - 1\right\} < n\left\{\left|\frac{v_{n+1}}{v_n}\right| - 1\right\}$$

$$\Rightarrow \left|\frac{u_{n+1}}{u_n}\right| < \left|\frac{v_{n+1}}{v_n}\right| \qquad \forall\, n > m$$

If ~~[illegible]~~ Both $\left|\frac{U_{n+1}}{U_n}\right| \longrightarrow 1$

$n > m.$

$+ \quad \frac{V_{n+1}}{V_n} \longrightarrow 1$

If $|U_n| > V_n \quad \forall \ n$

Then $|U_{n+1}| > V_{n+1}$ As Both are positive

$$\frac{|U_{n+1}|}{|U_n|} > \frac{V_{n+1}}{V_n} \quad \rightarrow\leftarrow$$

If $|U_n|$ + V_n osscilate in magnitude around each other ?

How get $|U_n| \leqslant V_n$

Consider $\sum_{n=1}^{\infty} n^r \exp\left(-k \sum_{m=1}^{n} \frac{1}{m}\right) \qquad k > 0 \quad r > 0$

$$\left|\frac{U_{n+1}}{U_n}\right| = \left|\frac{(n+1)^r}{n^r} \exp\left(-k\frac{1}{n}\right)\right| = \left|e^{-k/n}\left(1+\frac{1}{n}\right)^r\right| \xrightarrow{n\to\infty} 1$$

Raabe $n\left\{\left|\frac{U_{n+1}}{U_n}\right| - 1\right\} = n\left\{\left(1 - \frac{k}{n} + \frac{k^2}{n^2 2} - \cdots\right)\left(1 + \frac{r}{n} + \frac{r(r-1)}{2n} + \cdots\right) - 1\right\}$

$$= n\left\{ \cancel{1} + \frac{r}{n} + \frac{r(r-1)}{2n} - \frac{k}{n} - \frac{kr(r-1)}{2n^2} + \frac{k^2}{n^2 2} + \cdots - \cancel{1} \right\}$$

$$= \underbrace{\left(r + \frac{r(r-1)}{2} - k\right)}_{A_1} + O\left(\frac{1}{n}\right)$$

$$r + \frac{r^2}{2} - \frac{r}{2} - k = \frac{r}{2} + \frac{r^2}{2} - k < -1$$

Then $\frac{1}{2}(r+1)r < (k+1)$

So for absolute convergence req

$$\frac{r(r+1)}{k+1} < 2$$

For range $r > k$ let ~~$r = k+\eta$~~

$k = r - \eta$

$\Rightarrow$

$$r(r+1) < 2(r-\eta)$$

plot η as fn of r.

Pg 24 w/w

$$\left|\frac{u_{n+1}}{u_n}\right| = \frac{\dfrac{a(a+1)(a+2)\cdots(a+n)}{\quad}}{\dfrac{a(a+1)(a+2)\cdots(a+n-1)\, b(b+1)(b+2)\cdots(b+n-1)}{(n-1)!\, c(c+1)\cdots(c+n-1)}}$$

could finish.

$$|z| = 1$$

$$\left|\frac{u_{n+1}}{u_n}\right| = \frac{\left|1 - \frac{(1-a)}{n}\right| \left|1 + \frac{b-1}{n}\right|}{\left|1 + \frac{c-1}{n}\right|}$$

$$= \left|1 + \frac{a-1}{n}\right| \left|1 + \frac{b-1}{n}\right| \left|1 - \frac{c-1}{n} + O\left(\frac{1}{n^2}\right)\right|$$

$$= \left|\left|1 + \frac{a-1+b-1}{n} + O\left(\frac{1}{n^2}\right)\right|\right| \qquad \Big|$$

$$= \left|1 - \left(\frac{c-1}{n}\right) + \frac{a+b-2}{n} + O(n^{-2})\right|$$

$$= \left| 1 + \frac{a+b-c-1}{n} + O(n^{-2}) \right|$$

$$=$$

$$= \left| 1 + \frac{D'}{n} + O\left(\frac{1}{n^2}\right) + i\left(\frac{D''}{n} + O(n^{-2})\right) \right|$$

$$= \left\{ \left(1 + \frac{D'}{n} + O\left(\frac{1}{n^2}\right)\right)^2 + \left(\frac{D''}{n} + O(n^{-2})\right)^2 \right\}^{1/2}$$

$$= \left[\left(1 + \frac{D'}{n}\right)^2 + \left(\frac{D''}{n}\right)^2 + O(n^{-2}) \right]^{1/2}$$

$$= \left(1 + \frac{2D'}{n} + O(n^{-2}) \cdot \right)^{1/2}$$

$$= 1 + \frac{D'}{n} \Rightarrow$$

need $-1-c = D'$

$\Rightarrow -\underline{c} = a'+b'-c' < 0$

pg 25 w/w

consider $S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ & rearrange the terms.

Now in Σ ~~take~~ 2 positive terms then 1 negative $\therefore$

~~every set of 3 is~~

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots$$ (order indecated by the summation.

$\Sigma = \ldots$ Now in S take in order given one positive + one negative,

Then let $\sigma_n := 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$, then

$$\cancel{S_{2n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k} = \sum_{k=2,4,6,\cdots}^{2n} \frac{(-1)^{k+1}}{k} + \sum_{k=1,3,5,\cdots}^{2n-1} (-1)^{k+1}}$$

$$\cancel{= \sum_{k=1}^{n} \frac{(-1)^{2k+1}}{2k} + \sum}$$

$$\cancel{\sigma_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}}$$

$$\sigma_{2n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2n-1} + \frac{1}{2n}$$

$$\sigma_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

$\therefore\ \sigma_{2n} - \sigma_n = \cancel{1 + \left(\frac{1}{2} - 1\right) + \left(\frac{1}{3} - \frac{1}{2}\right)}$

$\cancel{1 - \frac{1}{2}}$

$$= \underline{1} + \left(\underline{\tfrac{1}{2}} - \underline{\underline{1}}\right) + \underline{\tfrac{1}{3}} + \left(\underline{\tfrac{1}{4}} - \underline{\underline{\tfrac{1}{2}}}\right) + \underline{\tfrac{1}{5}} + \left(\underline{\tfrac{1}{6}} - \underline{\underline{\tfrac{1}{3}}}\right) + \cdots + \underline{\tfrac{1}{2n-1}} + \left(\underline{\tfrac{1}{2n}} - \underline{\underline{\tfrac{1}{n}}}\right)$$

= is from $-\sigma_n$

▬ is from σ_{2n}

$$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots + \frac{1}{2n-1} - \frac{1}{2n} = S_{2n}$$

Now $\sum_{3n} = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots$

(take all positive terms $\frac{1}{4n-3}$, then all negative terms)

$$= 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \cdots + \frac{1}{4n-1} + \quad - \frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} \cdots - \frac{1}{2n}$$

~~Now~~ $\cancel{\sum_{3} = 1 + \frac{1}{3} - \frac{1}{2}}$ * (See pg 2b for explanation) *

$3 \cdot 1$

$\cancel{\sum_{6} = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4}}$

$= 3 \cdot 2$

Now Add in $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{4n}$ ~~something~~ & factor out a $-\frac{1}{2}$

$$= \underline{1} + \frac{1}{2} + \underline{\frac{1}{3}} + \frac{1}{4} + \cdots + \underline{\frac{1}{4n-2}} + \underline{\frac{1}{4n-1}} + \frac{1}{4n} - \quad -\frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)$$

▬ = ~~in 1st part of~~ $\sum_{3n}$ $\qquad \cancel{-\frac{1}{2}\sigma_n}$

=

$$= \sigma_{4n} - \frac{1}{2} + \frac{1}{4} - \frac{1}{6} - \frac{1}{8} \cdots - \frac{1}{4n} - \frac{1}{2}\sigma_n$$

$$= \sigma_{4n} - \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2n}\right) - \frac{1}{2}\sigma_n$$

$$= \sigma_{4n} - \frac{1}{2}\sigma_{2n} - \frac{1}{2}\sigma_n$$

$$= (\sigma_{4n} - \sigma_{2n}) + \frac{1}{2}(\sigma_{2n} - \sigma_n)$$

from $S_{2n} = \sigma_{2n} - \sigma_n$

$$= S_{4n} + \frac{1}{2}S_{2n}$$

$\therefore$ taking limits on Both sides

$$\Sigma = S + \frac{1}{2}S$$

$$\Rightarrow \Sigma = \frac{3}{2}S$$

picked 3n because we note

$$\sum_{\substack{3 \\ 3\cdot 1}} = 1 + \frac{1}{3} - \frac{1}{2}$$

$$\sum_{\substack{6 \\ 3\cdot 2}} = \left(1 + \frac{1}{3}\right) - \frac{1}{2} + \left(\frac{1}{5} + \frac{1}{7}\right) - \frac{1}{4}$$

$$= \left(1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7}\right) - \left(\frac{1}{2} + \frac{1}{4}\right)$$

$$\sum_{\substack{9 \\ 3\cdot 3}} = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6}\right)$$

$$\therefore \sum_{3n} = \left(1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{4n-3} + \frac{1}{4n-1}\right) - \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2n}\right)$$

~~$S'_k = S_n$ + terms of S w/ suffices greater than~~

Assum to get 1st n terms in S we must Add up m terms in S'

$\therefore$ If $k > m$

S'_k contains 1st n terms in S + poss. other terms not in S_n.

$\therefore$ $S'_k = S_n$ + terms in $\underline{\underline{S}}$ w/ indices greater than n.

$$S_n = \sum_{k=1}^{n} u_k$$

$S = S_1 + S_2 + S_3 + \cdots + S_n + S_{n+1} + \cdots$

$$S'_k = \sum_{p=1}^{k} \tilde{u}_p = \sum$$

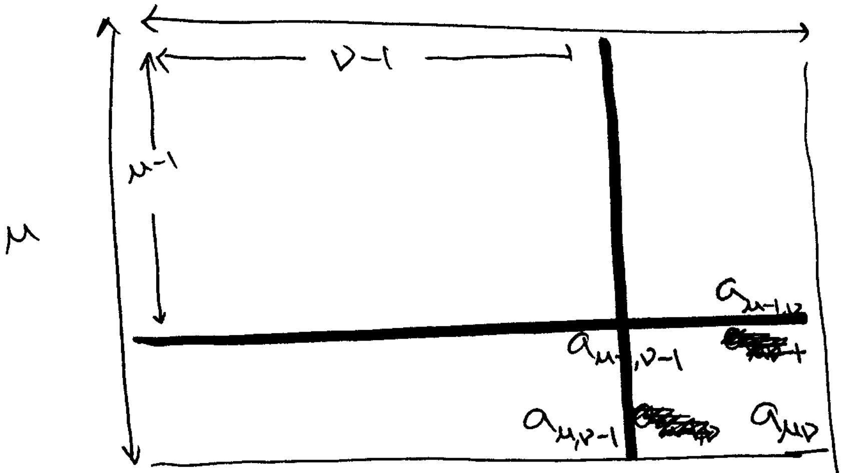
$\therefore$ $S'_k - S = S_n - S$ + terms in S w/ indices $> n$.

$$|S'_k - S| \le |S - S_n| + \sum_{\text{All } k > n} |u_k|$$

$$\le \underline{|S - S_n|} + \frac{\epsilon}{2}$$

$$\le \lim_{p \to \infty} \{ |u_{n+1}| + |u_{n+2}| + \cdots + |u_{n+p}| \}.$$

pg 27 w/w



$a_{\mu\nu} = (S_{\mu\nu} - S_{\mu,\nu-1}) - (S_{\mu-1,\nu} - S_{\mu-1,\nu-1})$ From drawing:

insert S everywhere

$$= \left((S_{\mu\nu} - S) - (S_{\mu,\nu-1} - S)\right)$$

$$- \left((S_{\mu-1,\nu} - S) - (S_{\mu-1,\nu-1} - S)\right)$$

$$\therefore |a_{\mu\nu}| \leq |S_{\mu\nu} - S| + |S_{\mu,\nu-1} - S|$$

$$+ |S_{\mu-1,\nu} - S| + |S_{\mu-1,\nu-1} - S| \longrightarrow 0$$

If $\left| S_{\mu+p,\nu+\sigma} - S_{\mu,\nu} \right| < \epsilon$ when $\mu > m$ & $\nu > n$

& $p, \sigma > 0$

Then if $\mu > m+n$ ~~keep~~ ~~Both~~ index

$$\left| S_{\mu+p,\mu+p} - S_{\mu,\mu} \right| < \epsilon$$

By taking $\nu = \mu$ & $\sigma = p$ in "Cauchy criterion".

~~But the first~~ Then sequence $S_{\mu,\mu}$ is cauchy by Before th & $\therefore$ $S_{\mu,\mu}$ has a limit say S

Then can show that $S_{\mu,\nu}$ gets "close to" this S. i.e

can be anything i.e. $\mu+p = \nu+\sigma$ works.

$$\left| S_{\mu,\nu} - S \right| \le \underbrace{\left| S_{\mu,\nu} - S_{\mu+p,\nu+\sigma} \right|}_{\frac{\epsilon}{2} \text{ (By Cauchy for double series)}} + \underbrace{\left| S_{\mu+p,\nu+\sigma} - S \right|}_{\text{cauchy sequence has a limit.}}$$

pick $\mu+p = \nu+\sigma$ large enough Then $\left| S_{\mu+p,\nu+\sigma} - S \right|$

Pg 27 w/w

~~Pf that~~ $\cancel{\left|S_{\mu+p,\nu+\beta} - S_{\mu,\nu}\right| < \epsilon}$ ~~when~~ $\cancel{\mu > m}$

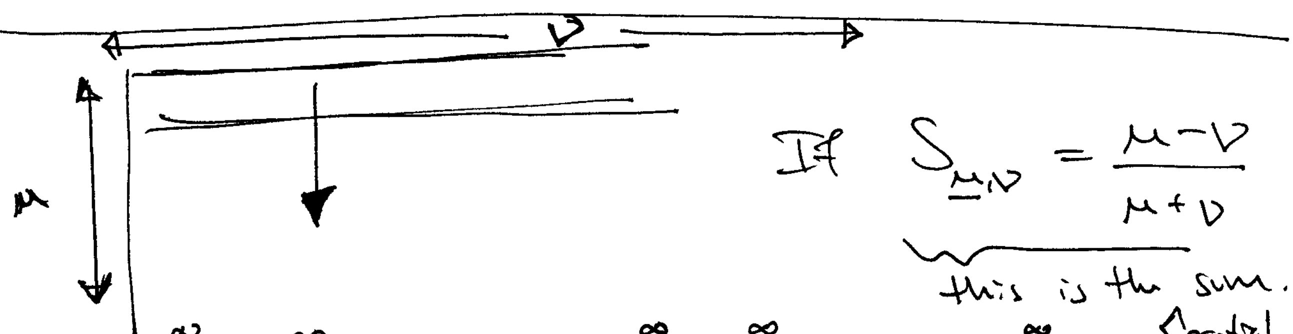
$\cancel{\nu > n}$.

If ~~$\mu > m+n$~~

$$\cancel{\left|S_{\mu+p,\mu+p} - S_{\mu,\mu}\right| < \epsilon}$$

$$\cancel{S_{\mu,\mu} \longrightarrow S.}$$

$\cancel{S_{\mu+p,\nu+\beta} \longrightarrow S}$ if ~~$\mu+p = \nu+\beta$~~



If $\underbrace{S_{\mu,\nu}}_{} = \frac{\mu-\nu}{\mu+\nu}$

this is the sum.

Then sum by columns is

$$\cancel{\sum_{\nu=1}^{\infty}\sum_{\mu=1}^{\infty}\frac{\mu-\nu}{\mu+\nu} = \sum_{\nu=1}^{\infty}\left(\sum_{\mu=1}^{\infty}\frac{\mu}{\mu+\nu} - \nu\sum_{\mu=1}^{\infty}\frac{1}{\mu+\nu}\right)}$$ partial sum

$\therefore$ $\cancel{S_\mu}$ $\nexists$

Sum by columns $:= \lim\limits_{\nu\to\infty}\lim\limits_{\mu\to\infty} S_{\mu,\nu}$

Sum by rows $:= \lim\limits_{\mu\to\infty}\lim\limits_{\nu\to\infty} S_{\mu,\nu}$

w/ $S_{\mu,\nu} = \sum_{j=1}^{\nu}\sum_{i=1}^{\mu} a_{ij}$

$\therefore$ Sum by columns $= 1$

Sum by rows $= -1$

If $S_{\mu,\nu}$ converges to S Pg 27 W/W

~~$|S_{\mu+p} - S - (S_{\mu,\nu} - S)| \leq |S_{\mu+}$~~

Then

$$|S_{\mu+p,\nu+6} - S - (S_{\mu,\nu} - S)| \leq$$

$$|S_{\mu+p,\nu+6} - S| + |S_{\mu,\nu} - S| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Pg 28 W/W

Let:

$$\lim_{\nu \to \infty} S_{\mu,\nu} = A_\mu$$

How Do?

Then

$$\left|\lim_{\nu \to \infty} S_{\mu,\nu} - S\right| \leq \left|\lim_{\nu \to \infty} S_{\mu,\nu} - S_{\mu,6}\right| + |S_{\mu,6} - S|$$

$$< \epsilon$$

---

pg 29 w/w

$$\lim_{\nu\to\infty} S_{\mu,\nu} = \text{Sum of 1st } \mu \text{ rows of the array.}$$

μ

Ex 1 $\sum |u|$

$$|b_1| + |b_2| + \cdots + |b_\mu| \le \lim_{\nu\to\infty} t_{\mu,\nu} \le t$$

Ex 2: $\displaystyle\sum_{k=0}^{\infty} \frac{z^k}{2^k}$ Then this conv Absoltly

$$\lim_{k\to\infty} \left| \frac{\frac{z^{k+1}}{2^{k+1}}}{\frac{z^k}{2^k}} \right| = \frac{|z|}{2} < 1$$

$$\Rightarrow |z| < 2$$

& $\displaystyle\sum_{k=0}^{\infty} \frac{1}{z^k}$ conv Absl $\left| \frac{\frac{1}{z^{k+1}}}{\frac{1}{z^k}} \right| = \frac{1}{|z|} < 1$

$$\Rightarrow |z| > 1$$

Then both conv. absolutly in the annulus. $1 < |z| < 2$

$\therefore$ The product, by Cauchy's thm. is absolutly convergent & converges.

Cauchy's test for Absolute conv $\overline{\lim}_{n\to\infty} |a_n|^{1/n} < 1$

$$\overline{\lim}_{n\to\infty} |a_n z^n|^{1/n} = \overline{\lim}_{n\to\infty} |a_n|^{1/n}|z| = |z|\,\overline{\lim}_{n\to\infty} |a_n|^{1/n}$$

$$= |z|\left(\underline{\lim}_{n\to\infty} |a_n|^{-1/n}\right)^{-1} = \frac{|z|}{r} < 1$$

$$\left(\overline{\lim}_{n\to\infty} |a_n|^{1/n}\right)^{-1} = \underline{\lim}_{n\to\infty} |a_n|^{-1/n}$$

0

~~Pg 31 w/w~~

$$\cancel{n|a_n||z|^{n-1} \le nM r_1^{-n}|z|^{n-1}}$$

$$\cancel{M|z|^{-1}\, n\frac{|z|^n}{r_1^n}}$$

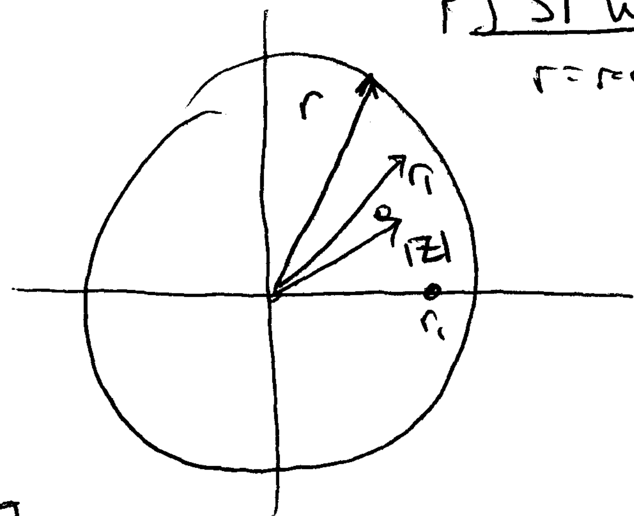
$$\cancel{= \frac{M}{|z|}\, n\frac{|z|^n}{r_1^n}}$$

Corollary: $\sum_{n=0}^{\infty} \frac{a_n z^{n+1}}{n+1}$ has the same circle of conv. as $\sum_{n=} a_n z^n$

for if $\cancel{|z| < r_1}$

__pg 31 w/w__

r = radius of conv.



$\exists M.$

$$|a_n r_1^n| < M \quad \forall n \geq 0.$$

$$\Rightarrow |a_n| < M r_1^{-n} \quad \forall n \geq 0$$

Then ~~$n|a_n||z|^{n-1}$~~

Then $n|z|^{n-1}|a_n| < n|z|^{n-1} M r_1^{-n}$

$$= \frac{M}{r_1} n\left(\frac{|z|}{r_1}\right)^{n-1}$$

But ratio $\lim_{n\to\infty} \left(\frac{(n+1)}{n} \frac{|z|^n}{r_1^n} \cdot \frac{r_1^{n-1}}{|z|^{n-1}}\right)$

$$= \frac{|z|}{r_1} \cdot 1 < 1$$

$$\Rightarrow |z| < r_1$$

Corollary

Assume $\sum_{n=0}^{\infty} \frac{a_n z^{n+1}}{n+1}$ had a larger or

smaller circle of convergence. Then it larger

= derived series $\sum_{n=0}^{\infty} a_n z^n$ has larger ~~cnt~~ radius of conv than

we are told it has $\rightarrow\leftarrow$

Pg 32 w/w

$\Pi_{n-1} = (1+a_1)(1+a_2)\cdots(1+a_{n-1})$ $\ast$

$\Pi_n = \Pi_{n-1}(1+a_n)$

$\therefore \lim_{n\to\infty} \Rightarrow \Pi = \Pi(1+\lim_{n\to\infty} a_n)$ & $\Pi \neq 0$ By def.

$\prod_{n=2}^{\infty}(1-n^{-1})$ $\qquad 1-\frac{1}{n} = \frac{n-1}{n}$

$\Pi_n = (1-\frac{1}{2})($

$= \frac{2-1}{\cancel{2}}\cdot\frac{\cancel{3}-1}{\cancel{3}}\cdot\frac{\cancel{4}-1}{\cancel{4}}\cdot\frac{\cancel{5}-1}{\cancel{5}}\cdots,\frac{\cancel{n}-1}{n} = \frac{1}{n}$

$\therefore \lim_{n\to\infty}\Pi_n = 0.$ ~~converges~~ diverge to zero.

Ex 1: $\prod_{n=1}^{\infty}(1+a_n)$ converges.

$\exists m \ni$

$\Rightarrow \left|\prod_{k=1}^{n}(1+a_k) - \Pi\right| < \epsilon \quad \forall n \geq m.$

$= \left|\exp\left\{\sum_{k=1}^{n}\log(1+a_k)\right\} - \Pi\right| < \epsilon \quad \forall n \geq m$

$$\left| \exp\left\{ \sum_{k=1}^{n} \log(1+a_k) \right\} - \exp\{\log \Pi\} \right| < \epsilon$$

$\Rightarrow \exists\, m_2 \ni$ if $n > m_2$

$$\left| \sum_{k=1}^{n} \log(1+a_k) - \log \Pi \right| < \epsilon \Rightarrow \sum \text{ converges}$$

<u>Ex 2</u> $\quad \Pi\left(\frac{\sin\left(\frac{z}{k}\right)}{\left(\frac{z}{k}\right)}\right) = \frac{z}{k} + \frac{\sin(z/k)''|_{?}}{2!\,(z/k)}$

$$\sum_{k=0}^{\infty} \frac{(-1)^k (z/k)^{2k+1}}{(2k+1)!\,(z/k)} = \sum_{k=0}^{\infty} \frac{(-1)^k (z/k)^{2k}}{(2k+1)!}$$

$$= 1 - \frac{(z/k)^2}{3!} + \frac{(z/k)^4}{5!} + \cdots$$

$$= 1 - \frac{\lambda_k}{k^2}$$

$f(k^{-1}) = \sin(z/k)$

~~$f'(k) = \cos(z/k)\left(\frac{-z}{k^2}\right)$~~

$f'(k^{-1}) = z\cos(z/k)$

$f''(k^{-1}) = -z^2 \sin(z/k)$

$f'''(k^{-1}) = -z^3 \cos(z/k)$

$\sin(z/k) = 0 + \frac{z}{k}$

$\quad - z^3\left(\frac{1}{k}\right)^3$

$$\frac{\sin z/k}{z/k} = 1 -$$

pg 33 w/w

1

Taylor series of $\log(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k}$ Thus

$$= x - \frac{1}{2}x^2 + \sum_{k=3}^{\infty} \frac{(-1)^{k+1} x^k}{k}$$

$$\left| \frac{\log(1+x) - x}{x} \right| = \left| -\frac{1}{2}x + \sum_{k=3}^{\infty} \frac{(-1)^{k+1} x^{k-1}}{k} \right|$$

$$\Rightarrow \left| x^{-1}\log(1+x) - 1 \right| \leq \frac{1}{2}|x| + \sum_{k=3}^{\infty} \frac{|x|^{k-1}}{k} \leq$$

As $x \leftrightarrow a_k$ & $|a_k| < 1/2$

$$\text{LHS} \leq \frac{1}{2^2} + \sum_{k=3}^{\infty} \frac{1}{2^{k}k} \leq \frac{1}{2^2} + \sum_{k=3}^{\infty} \frac{1}{2^{k-1}} =$$

$$\sum_{n=N_1}^{N_2} a^n = \frac{1}{(a-1)} a^n \Big|_{N_1}^{N_2+1} = \frac{1}{(1/2 - 1)} \left(\frac{1}{2}\right)^n \Big|_{2}^{\infty} =$$

$$\Delta a^n = a^{n+1} - a^n = a^n(a-1)$$

$$a^n = \frac{1}{a-1}\Delta a^n \Rightarrow \sum a^n = \frac{1}{(a-1)} a^n \Big|_{N_1}^{N_2+1}$$

$$\therefore -2\left(0-\frac{1}{4}\right)=\frac{1}{2}$$

Example I:

If $\prod_{n=1}^{\infty}(1+a_n)$ conv

$\Rightarrow \prod_{n=1}^{N}(1+a_n)=\Pi_N$ has a limit

$\Rightarrow \log \Pi_N=\sum_{n=1}^{N}\log(1+a_n)$

& if Π_N ~~does~~ converges $\Rightarrow \Pi_N \xrightarrow{N\to\infty} \Pi \neq 0$

$\therefore \log \Pi_N \longrightarrow \log \Pi$ As log is a continuous fn.

$\Rightarrow \sum_{n=1}^{N}\log(1+a_n) \xrightarrow{N\to\infty} \log \Pi$ & sum converges

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Example 2: (see other set of notes) pg 1

$$\frac{\sin(z/n)}{z/n}=\frac{1}{z/n}\sum_{k=0}^{\infty}\frac{(-1)^k (z/n)^{2k+1}}{(2k+1)!}=\sum_{k=0}^{\infty}\frac{(-1)^k (z/n)^{2k}}{(2k+1)!}=$$

$$= z/n-\frac{1}{6}(z/n)^3+$$

$$f(x) = \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + R_{N+1}(x)$$ only x real

$$+ \frac{f^{(N+1)}(\xi)(x-x_0)^{N+1}}{(N+1)!}$$ $\cancel{x<\xi<}$ ξ between x + x_0.

Reminder of taylor's thm only for $x \in \mathbb{R}$

$\therefore$ $\sin(x/n) = x/n + 0 - \cos(\xi(x))\frac{x^3}{6n^3}$ exact.

1st deri. cos, 2nd $=0.$ $-\sin$, 3rd der $-\cos$

$\therefore$ $$\boxed{\frac{\sin(x/n)}{x/n} = 1 - \frac{\cos(\xi(x))/6\, x^2}{n^2}}$$ exact. $\forall$

Then from power series which is valid $\forall z \in \mathbb{C}$

we get

$$\frac{\sin(z/n)}{z/n} = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k (z/n)^{2k}}{(2k+1)!} = 1 + \frac{1}{n^2} \sum_{k=1}^{\infty} \frac{(-1)^k z^{2k}}{(2k+1)!\, n^{2k-2}}$$

Now $\sum_{k=1}^{\infty} \frac{(-1)^k z^{2k}}{(2k+1)!\, n^{2(k-1)}}$ converges $\forall z$ $= \cancel{F_n(z)}$ ~~for n=1,2,...~~

$= \lambda_n(z)$ for $n=1,2,\ldots$

$\cancel{F_1(z) < F_2(z)}$

$=$ ~~$F_n(z)$~~ Now $\lambda_n(z)$ is a seq of decreasing fns

$\lambda_1(z) > \lambda_2(z) \ldots$

$\therefore \quad \frac{\sin(z/n)}{z/n} = 1 + \frac{\lambda_n}{n^2}$

if $\sum \frac{\lambda_n}{n^2}$ is Absolutly convergent we have that the product is absolutly convergent. $|\lambda_n|$ How?

~~(Pm~~~~-1)~~~~th~~ ~~terms~~ pg 34 W/W O = order not exceeding

$$\left(1 \mp \frac{z}{m\pi}\right)\left(e^{\pm \frac{z}{m\pi}}\right) - 1 = \left(1 \mp \frac{z}{m\pi}\right)\left(1 \pm \frac{z}{m\pi} + O\left(\frac{1}{m^2}\right)\right) - 1$$

$$= \left(1 \mp \frac{z}{m\pi}\right) \sum_{k=0}^{\infty} \frac{1}{k!} \frac{z^k}{m^k \pi^k} - 1 \Bigg] \quad 1 - \frac{z^2}{m^2\pi^2} + O\left(\frac{1}{m^2}\right) - 1$$

$=$

$= O\left(\frac{1}{m^2}\right)$ + $\therefore$ convergence

is absolut

Ex 1:

$$\prod_{n=1}^{\infty}\left[\left(1-\frac{z}{c+n}\right)e^{z/n}\right]$$

question of Absolute convergence of

consider $\displaystyle\sum_{n=1}^{\infty}\left(\left(1-\frac{z}{c+n}\right)e^{z/n}-1\right)$

This sum has general term

$$\left(1-\frac{z}{c+n}\right)\left(1+\frac{z}{n}+\frac{z^2}{2n^2}+O(n^{-3})\right)-1$$

$$=\cancel{1}+\frac{z}{n}+\frac{z^2}{2n^2}-\frac{z}{c+n}-\frac{z^2}{n(c+n)}+O(n^{-3})-\cancel{1}$$

$$=\frac{z}{n}-\frac{z}{c+n}+\frac{z^2}{2n^2}-\frac{z^2}{n(c+n)}+O(n^{-3})$$

$$=\frac{z}{n}-\frac{z}{n}\left(\frac{1}{1+c/n}\right)+$$

$$=\frac{z}{n}-\frac{z}{n}\left(1-\frac{c}{n}+O(n^{-2})\right)+O(n^{-2})$$

$$=+\frac{zc}{n^2}+O(n^{-2})=O(n^{-2}).$$ Absolutely conv

Ex 2:

$$\prod_{n=2}^{\infty}\left\{1-\left(1-\frac{1}{n}\right)^{-A}z^{-n}\right\}$$

Abs conv. $\Rightarrow$ $\sum_{n=2}^{\infty}\left(1-\frac{1}{n}\right)^{-n} z^{-n}$ is Abs. convergent.

Now ~~$\lim_{n\to\infty}\left(1-\frac{1}{n}\right)$~~ $\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{-n}$ type 1^{∞}

indeterminate

type $\frac{0}{0}$

$$e^{\lim_{n\to\infty}\left(-n\log\left(1-\frac{1}{n}\right)\right)} = e^{\lim_{n\to\infty}\frac{-\log\left(1-\frac{1}{n}\right)}{\frac{1}{n}}}$$

$$= e^{-\lim_{n\to\infty}\frac{\frac{-1}{1-\frac{1}{n}}\left(\cancel{\frac{+1}{n^2}}\right)}{\cancel{\frac{-1}{n^2}}}} = e^{1}.$$

$$\therefore \lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\frac{\left(1-\frac{1}{n+1}\right)^{-(n+1)}}{\left(1-\frac{1}{n}\right)^{-n}}\frac{|z|^{-n-1}}{|z|^{-n}}$$

$$= \frac{1}{|z|} < 1 \quad \Rightarrow \quad |z| > 1$$

<u>Ex 3:</u> $$\frac{1\cdot 2\cdot 3\cdot \cdots (m-1)}{(z+1)(z+2)\cdots(z+m-1)}\, m^z = a_m$$

z neg integer one term in the denom vanishes.

$$= \underbrace{\left(\frac{1}{z+1}\right)\left(\frac{2}{z+2}\right)\cdots\left(\frac{m-1}{z+m-1}\right)}_{m-1 \text{ factors}} \cdot \underbrace{\left(\frac{2}{1}\right)^z\left(\frac{3}{2}\right)^z\left(\frac{4}{3}\right)^z\cdots\left(\frac{m}{m-1}\right)^z}_{m-1 \text{ factors}}$$

$$= \underbrace{\left(\left(\frac{2}{1}\right)^z \frac{1}{z+1}\right)}_{\text{1st term}}\underbrace{\left(\left(\frac{3}{2}\right)^z\left(\frac{2}{z+2}\right)\right)}_{\text{2nd}}\underbrace{\left(\left(\frac{4}{3}\right)^z \frac{3}{z+3}\right)}_{\text{3rd}}\cdots\left(\left(\frac{m}{m-1}\right)^z \frac{m-1}{z+m-1}\right)$$

product w/ each term nth term is

$$\frac{n}{z+n}\cdot\left(\frac{n+1}{n}\right)^z = \frac{1}{(z/n+1)}\left(1+\frac{1}{n}\right)^z$$

$$= \left(1+\frac{1}{n}\right)^z\left(1+\frac{z}{n}\right)^{-1} = \left(1+\frac{z}{n}+\frac{z(z-1)}{2}\frac{1}{n^2}+O(n^{-3})\right)\cdot\left(1-\frac{z}{n}+O(n^{-2})\right)$$

$$(1+\alpha)^z = 1+\alpha z+\frac{z(z-1)}{2}\alpha^2+\cdots$$

$$= 1-\cancel{\frac{z}{n}}+O(n^{-2})+\cancel{\frac{z}{n}}-\frac{z^2}{n^2}+O(n^{-3})$$

$$= 1-O(n^{-2})$$ Absolutely conv.

Ex 4:

$$\underbrace{z\left(1-\frac{z}{\pi}\right)\left(1-\frac{z}{2\pi}\right)\left(1+\frac{z}{\pi}\right)}\underbrace{\left(1-\frac{z}{3\pi}\right)\left(1-\frac{z}{4\pi}\right)\left(1+\frac{z}{2\pi}\right)}\underbrace{\left(1-\frac{z}{4\pi}\right)\left(1-\frac{z}{5\pi}\right)\cdots}$$

group terms

$$\lim_{k\to\infty} z\left(1-\frac{z}{\pi}\right)\left(1-\frac{z}{2\pi}\right)\left(1+\frac{z}{\pi}\right)\cdots\left(1-\frac{z}{(2k-1)\pi}\right)\left(1-\frac{z}{2k\pi}\right)\left(1+\frac{z}{k\pi}\right)$$

$k=1$ $\quad z\left(1-\frac{z}{\pi}\right)\left(1-\frac{z}{2\pi}\right)\left(1+\frac{z}{\pi}\right)$

put in $e^{\pm z/n\pi}$'s everywhere.

$$= \lim_{k\to\infty}\Big[z\left(1-\frac{z}{\pi}\right)e^{z/\pi}\left(1-z/2\pi\right)e^{z/2\pi}\left(1+z/\pi\right)e^{-z/\pi}$$

$$\cdots\left(1-z/(2k-1)\pi\right)e^{z/(2k-1)\pi}\left(1-z/2k\pi\right)e^{z/2k\pi}\left(1+z/k\pi\right)e^{-z/k\pi}\Big]$$

$$= \cdot\, \underbrace{e^{-z/\pi}\, e^{-z/2\pi}\, e^{+z/\pi}} \cdots\; e^{-z/(2k-1)\pi}\; e^{-z/2k\pi} \cdot e^{+z/k\pi} \cdot$$

factor at $\frac{z}{\pi}(-1-\frac{1}{2}+1-\frac{1}{3}-\frac{1}{4}+\frac{1}{2}-\frac{1}{5}-\frac{1}{6}+\frac{1}{3}\cdots$

$$\lim_{k\to\infty}\Big[e^{\frac{z}{\pi}\left(\cancel{-1}\underbrace{-\frac{1}{2}+\cancel{1}}_{1/2}-\frac{1}{3}\underbrace{-\frac{1}{4}+\frac{1}{2}}_{\frac{1}{4}}-\cdots-\frac{1}{(2k-1)}\underbrace{-\frac{1}{2k}+\frac{1}{k}}_{+\frac{1}{2k}}\right)}$$

$$\cdot z\left(1-\frac{z}{\pi}\right)e^{z/\pi}\left(1-z/2\pi\right)e^{z/2\pi}\left(1+z/\pi\right)e^{-z/\pi}\cdots$$

$$\left(1-\frac{z}{(2k-1)\pi}\right)e^{z/(2k-1)\pi}\left(1-\frac{z}{2k\pi}\right)e^{z/2k\pi}\left(1+z/k\pi\right)e^{-z/k\pi}\Big]$$

. . . .

$$= \lim_{k\to\infty}\Big[e^{+z/\pi\left(-1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\frac{1}{6}\cdots-\frac{1}{(2k-1)}+\frac{1}{2k}\right)}$$

$$z\,\underline{(1-z/\pi)e^{z/\pi}}\;\underline{\underline{(1-z/2\pi)e^{z/2\pi}}}\;\underline{(1+z/\pi)}e^{-z/\pi}\cdots$$

$$(1-z/(2k-1)\pi)e^{z/(2k-1)\pi}\,(1-z/2k\pi)e^{z/2k\pi}\,(1+z/k\pi)e^{-z/k\pi}\Big]$$

$$= \lim_{k\to\infty}\Big[e^{-\frac{z}{\pi}\sum_{n=1}^{k}\frac{(-1)^{n+1}}{n}}\, z\,\underline{\left(1-\frac{z}{\pi}\right)e^{z/\pi}}\;\underline{\left(1+\frac{z}{\pi}\right)}e^{-z/\pi}\;\underline{\underline{\left(1-\frac{z}{2\pi}\right)e^{+z/2\pi}}}$$

$$(1+z/2\pi)e^{-z/2\pi}\cdots\Big]$$

$$\text{But} \Rightarrow \lim_{k\to\infty}\left(e^{-z/\pi\sum_{n=1}^{k}\frac{(-1)^{n+1}}{n}}\right)z$$

Thus product is made up of terms $\left(1\mp\frac{z}{r\pi}\right)e^{\pm r\pi}$ from section 2.71

Example: Pg 36 W/W

∞ Determinate

∞ Det converges ⟺ if product of diagonals converge Absolutly & + sum of non-diagonals converge absolutly.

$$\lim_{m\to\infty} \begin{vmatrix} 1 & \alpha_1 & 0 & 0 & \cdots & 0 \\ B_1 & 1 & \alpha_2 & 0 & \cdots & 0 \\ 0 & B_2 & 1 & \alpha_3 & \cdots & \\ 0 & \cdots & 0 & B_m & 1 & \end{vmatrix}$$

$$\lim_{n\to\infty} \begin{bmatrix} 1 & \alpha_1 & 0 & \cdots & & 0 \\ B_1 & 1 & \alpha_2 & 0 & \cdots & 0 \\ 0 & B_2 & 1 & \alpha_3 & \cdots & \\ & & & \ddots & \ddots & \\ & & & & B_n & 1 \end{bmatrix}$$

tridiagonal matrix that grows ∞

That diagonal converges Abs. is trivial

for The sum of the non diag to conver Abs we need

$$\sum_{j=1}^{\infty} |\alpha_j| + |B_j|$$ to converge

$\Rightarrow$ Both $\sum |\alpha_j|$ + $\sum |\beta_j|$ converge

$\Rightarrow$ $\sum |\alpha_j \beta_j|$ converge Absolutly

$\Rightarrow$ $\sum \alpha_j \beta_j$ converge. This is ∞ But converge $\Rightarrow \sum \alpha_j \beta_j < \infty$
Abs

Now if $\sum \alpha_j \beta_j < \infty$ converge Absolutly

$\Rightarrow \sum |\alpha_j \beta_j| < \infty$

want $\sum |\alpha_j| < \infty$ + $\sum |\beta_k| < \infty$

But $\frac{(-1)^n}{n}$ $\qquad \frac{(-1)^n}{n}$

How show $\sum \alpha_j \beta_j$ converge Abs $\Rightarrow$

$\sum |\alpha_j|$ + $\sum |\beta_k|$ con Abs ?

~~pg 38~~ WW

~~$\log(10) = \log\left(\frac{2n+1}{2n-1}\right)^n$~~

~~$\log 10$~~

Ex 1

$$\lim_{n\to\infty} e^{-na} n^b = \lim_{n\to\infty} \sum_{k=0}^{\infty} \frac{(-1)^k n^k a^k}{k!} n$$

$$n^b = e^{b\log n}$$

$$\lim_{n\to\infty} e^{b\log n - na} = 0 \qquad \text{As } b\log n < an$$

Som $n > n_0$

$$\lim_{n\to\infty} (n^{-a}\log n) = 0. \qquad n^a > \log n \quad n > n_0.$$

(2) <u>pg 38 w/w</u>

1st Check nth term test

$$\lim_{n\to\infty}\left(n\ln\left(\frac{2n+1}{2n-1}\right)\right) \stackrel{?}{=} 1$$

$$\|$$

$$= \lim_{n\to\infty} \frac{\ln\left(\frac{2n+1}{2n-1}\right)}{\frac{1}{n}} \quad \text{" type } \frac{0}{0}\text{"}$$

$$= \lim_{n\to\infty} \frac{\left(\frac{2n-1}{2n+1}\right)\frac{-1}{(n-\frac{1}{2})^2}}{\frac{-1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{(n-\frac{1}{2})^2}\left(\frac{2n-1}{2n+1}\right) = 1 \checkmark.$$

no information.

pg 39 w/D

(2) consider $\sum_{n=1}^{\infty} \left\{ 1 - n\log\left(\frac{2n+1}{2n-1}\right) \right\}$

$$= \sum_{n=1}^{\infty} \left\{ 1 - n\log\left(\frac{2n-1+2}{2n-1}\right) \right\}$$

$$= \sum_{n=1}^{\infty} \left\{ 1 - n\log\left(1 + \frac{2}{2n-1}\right) \right\}$$

$$= \sum_{n=1}^{\infty} \left\{ 1 - n\log\left(1 + \frac{1}{n-1/2}\right) \right\} \qquad \log(1+x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} x^{k+1}$$

$$= \sum_{n=1}^{\infty} \left\{ 1 - n\left(\frac{1}{n-1/2} + \frac{-1}{2}\frac{1}{(n-1/2)^2} + O\left(\frac{1}{n^3}\right)\right) \right\}$$

$$= \sum_{n=1}^{\infty} \left\{ 1 - \frac{n}{n-1/2} + \frac{n}{2(n-1/2)^2} + O\left(\frac{1}{n^2}\right) \right\}$$

$$= \sum_{n=1}^{\infty} \left\{ \cancel{1} - \frac{\cancel{n-1/2}}{\cancel{n-1/2}} - \frac{1}{2}\frac{1}{(n-1/2)} + \frac{n}{2(n-1/2)^2} + O(n^{-2}) \right\}$$

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} |1 - n\log$$

$$= \sum_{n=1}^{\infty} \left\{ -\frac{1}{2}\frac{1}{(n-1/2)} + \frac{n}{2(n-1/2)^2} + O(n^{-2}) \right\}$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} \left(\frac{-(n-1/2)}{(n-1/2)^2} + \frac{n}{(n-1/2)^2} \right) + O(n^{-2})$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} O(n^{-2}) \quad \therefore \text{ converges. One method}$$

Attempt new ratio test w/ $u_n = 1 - n\log\left(\frac{2n+1}{2n-1}\right)$

$$\lim_{n\to\infty} \left| \frac{1-(n+1)\log\left(\frac{2n+3}{2n+1}\right)}{1-n\log\left(\frac{2n+1}{2n-1}\right)} \right|$$ type $\frac{0}{0}$ by argument on 1st page

$$= \lim_{n\to\infty} \left| \frac{-\log\left(\frac{2n+3}{2n+1}\right) - (n+1)\left(\frac{2n+1}{2n+3}\right)\frac{-1}{(n+1/2)^2}}{-\log\left(\frac{2n+1}{2n-1}\right) - n\left(\frac{2n-1}{2n+1}\right)\frac{-1}{(n-1/2)^2}} \right|$$ type "$\frac{0}{0}$" $\frac{0-0}{0-0}$

$$= \cancel{\lim_{n\to\infty} \frac{\log\left(\frac{2n+3}{2n+1}\right)}{}} -$$

$$= \lim_{n\to\infty} \left| \frac{\log\left(\frac{2n+3}{2n+1}\right) - \left(\frac{2n+1}{2n+3}\right)\frac{n+1}{(n+1/2)^2}}{\log\left(\frac{2n+1}{2n-1}\right) - \left(\frac{2n-1}{2n+1}\right)\frac{n}{(n-1/2)^2}} \right|$$

L Hopitals

$$= \lim_{n\to\infty} \left| \left(\frac{2n+1}{2n+3}\right) \cdot \right.$$

If $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = 1$ No information, Apply to

consider $\left|\frac{U_{n+1}}{U_n}\right| = \left|\frac{1-(n+1)\log\left(1+\frac{1}{n+1/2}\right)}{1-n\log\left(1+\frac{1}{n-1/2}\right)}\right|$

$=$

pg 38 w/10

(3) $$\sum_{n=1}^{\infty}\left(\frac{1\cdot3\cdot5\cdots 2n-1}{2\cdot4\cdots 2n}\cdot\frac{4n+3}{2n+2}\right)^2$$

$$= \sum_{n=1}^{\infty}\left(\frac{1\cdot2\cdot3\cdot4\cdot5\cdots(2n-1)2n}{(2\cdot4\cdot6\cdots2n)^2}\ \frac{2n+3/2}{n+1}\right)^2$$

$$= \sum_{n=1}^{\infty}\left(\frac{(2n)!}{2^{2n}(n!)^2}\ \frac{2n+3/2}{n+1}\right)^2$$

$\frac{4}{2}+$

consider the Ratio test

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\frac{\left(\frac{(2(n+1))!}{2^{2n+2}(n+1)!^2}\ \frac{2n+7/2}{n+2}\right)^2}{\left(\frac{2n!}{2^{2n}(n!)^2}\ \frac{2n+3/2}{n+1}\right)^2}$$

$$= \lim_{n\to\infty}\left(\frac{2n+7/2}{2n+3/2}\right)^2\left(\frac{n+1}{n+2}\right)^2\left(\frac{n!}{(n+1)!}\right)^4\left(\frac{(2n+2)!}{2n!}\right)^2\frac{2^{4n}}{2^{4n+4}}$$

$$= \lim_{n\to\infty} 1\cdot 1\ \frac{1}{(n+1)^4}\ \frac{(2n+2)^2(2n+1)^2}{2^4} = 1.$$

pg 38 w/w

(7) If $1 < \alpha < \beta$

Then $\frac{1}{2^\beta} < \frac{1}{2^\alpha}$

$\frac{1}{4^\beta} < \frac{1}{2^\alpha}$

$\therefore$ Sum given $< \sum \frac{1}{n^\alpha}$ $\Leftarrow$ conv As $\alpha > 1$

But $\frac{U_{2n+1}}{U_{2n}} = \frac{\frac{1}{(2n+1)^\alpha}}{\frac{1}{(2n)^\beta}} = \frac{(2n)^\beta}{(2n+1)^\alpha} = \frac{(2n)^{\beta-\alpha}}{\cancel{(2n)^\alpha}} (1+\frac{1}{2n})^\alpha$

But $\beta - \alpha > 0$ $\therefore$ $(2n)^{\beta-\alpha} \xrightarrow[n\to\infty]{} \infty$.

pg 38 w/w

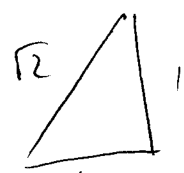
④ $\sum_{n=1}^{\infty} (-1)^{n+1} 2^n \sin^{2n} z$ $\quad z \in \mathbb{R}$. How do for complex z?

$$\lim_{n\to\infty} \left| \frac{2^{n+1}}{2^n} \frac{\sin^{2(n+1)} z}{\sin^{2n} z} \right| = 2 \lim_{n\to\infty} \left| \sin^2 z \right| = 2|\sin^2 z| < 1$$

$$\boxed{|x - n\pi| < \frac{\pi}{4}}$$

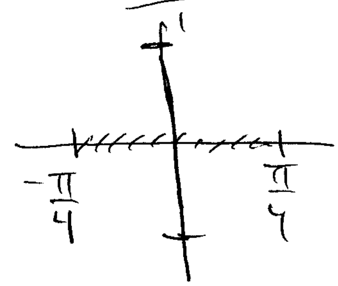
$\Rightarrow |\sin^2 z| < \frac{1}{2}$

$\Rightarrow |\sin z| < \frac{1}{\sqrt{2}}$ $\quad z \in \mathbb{R}$



$\Rightarrow \sin z < \frac{1}{\sqrt{2}}$

$\Rightarrow z < \frac{\pi}{4}$



Root test $\overline{\lim_{n\to\infty}} \left(|2^n \sin^{2n} z|^{1/n} \right)$

$$= 2 \overline{\lim_{n\to\infty}} |\sin^2 z| = 2|\sin^2 z| < 1$$

Pg 39 w/w

(9) $$\sum_{n=1}^{\infty} \frac{n z^{n-1}\{(1+n^{-1})^n - 1\}}{(z^n-1)\{z^n-(1+n^{-1})^n\}}$$

denom vanishes if $z^n = 1 \Rightarrow z = e^{\frac{2\pi i k}{n}}$ $k=0,1,2,\dots n-1$

(This is a=0
k = 0,1,...,n-1
n = 1,2,3, ...)

+ $z^n = (1+\frac{1}{n})^n = \cancel{(1+\frac{1}{n})} e^{\cancel{2\pi i k n}}$

$\cancel{z = (1+\frac{1}{n})^{1/n} e^{2\pi i k}}$

$z^n = (1+\frac{1}{n})^n e^{2\pi i k}$

$\Rightarrow z = (1+\frac{1}{n}) e^{2\pi i k/n}$

This is a=1
k = 0,1,2, ..., n-1
n = 1,2,3, ...)

Same conditions as in book. Now consider the summand. consider n a constant & try ~~an~~ a partial fraction expansion.

$\cancel{n z^{n-1}}$

$$\frac{n z^{n-1}\{(1+n^{-1})^n - 1\}}{(z^n-1)\{z^n-(1+n^{-1})^n\}} \overset{?}{=} \frac{A}{z^n-1} + \frac{B}{z^n-(1+n^{-1})^n}$$

$$\lim_{n\to\infty}\left|\underbrace{\frac{n+1}{n}}_{1}\underbrace{\frac{z^n}{z^{n+1}}}_{|z|}\frac{\{(1+\frac{1}{n+1})^{n+1}-1\}}{\{(1+\frac{1}{n})^n-1\}}\frac{(z^n-1)\{z^n-(1+\frac{1}{n})^n\}}{(z^{n+1}-1)\{z^{n+1}-(1+\frac{1}{n+1})^{n+1}\}}\right|$$

$$\frac{e-1}{e-1}$$

$$=\lim_{n\to\infty}\left|\frac{(z^n-1)}{(z^{n+1}-1)}\frac{\{z^n-(1+\frac{1}{n})^n\}}{\{z^{n+1}-(1+\frac{1}{n+1})^{n+1}\}}\right|$$

$$z^{2n}\ \cancel{z^{n+1}}-1=(z^n-1)(z+$$

$$z^{n+1}-1=(z-1)\sum$$

know

$$\sum_{k=0}^{n} z^k=\frac{1-z^{n+1}}{1-z}=\quad z^{n+1}-1=(z-1)\sum_{k=0}^{n} z^k$$

$$\therefore\ \cancel{\lim_{n\to\infty}}\ \cancel{\left|\sum_{k=}\right.}$$

$$=\lim_{n\to\infty}\left|\frac{(z^n-1)}{(z^{n+1}-1)}\frac{\left(\left(\frac{z}{(1+\frac{1}{n})}\right)^n-1\right)}{\left(\left(\frac{z}{1+\frac{1}{n+1}}\right)^{n+1}-1\right)}\underbrace{\frac{(1+\frac{1}{n})^n}{(1+\frac{1}{n+1})^{n+1}}}_{\frac{e}{e}}\right|$$

$$= \lim_{n\to\infty} \frac{\sum_{k=0}^{n-1} z^k}{\sum_{k=0}^{n} z^k} \frac{\sum_{k=0}^{n-1} \frac{z^k}{(1+\frac{1}{n})^k}}{\sum_{k=0}^{n} \frac{z^k}{\left(1+\frac{1}{n+1}\right)^k}}$$

where next? - - - -

- - - -

Back to partial fractions

$$A = \frac{n z^{n-1}\{(1+n^{-1})^n - 1\}}{z^n - (1+n^{-1})^n}\Bigg|_{z=e^{\frac{2\pi i k}{n}}}$$

$$= n e^{\frac{2\pi i k(n-1)}{n}} = n\exp\left\{2\pi i k\left(1-\frac{1}{n}\right)\right\}$$

$$= n e^{-2\pi i k/n}$$

$$B = \frac{n z^{n-1}\{(1+n^{-1})^n - 1\}}{z^n - 1}\Bigg|_{z=(1+\frac{1}{n})e^{2\pi i k/n}} = n\left(1+\frac{1}{n}\right)^{n-1} e^{-2\pi i k/n}$$

$$\Rightarrow \sum_{n=1}^{\infty} = \sum_{n=1}^{\infty} \frac{n e^{-2\pi i k/n}}{z^n - 1} + n$$

(10) Pg 390/W 1

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{s-1} + \sum_{n=1}^{\infty}\left[\frac{1}{n^s} + \frac{1}{(s-1)}\left\{\frac{1}{(n+1)^{s-1}} - \frac{1}{n^{s-1}}\right\}\right]$$

$$\text{"} \int n^{-s}\,dn \text{"} = \left.\frac{n^{-s+1}}{-s+1}\right| = \frac{-1}{s-1}\frac{1}{n^{1-s}}$$

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if $s>1$

$$S_n := \frac{1}{s-1} + \sum_{k=1}^{n}\left[\frac{1}{k^s} + \frac{1}{(s-1)}\left\{\frac{1}{(k+1)^{s-1}} - \frac{1}{k^{s-1}}\right]\right.$$

$$= \frac{1}{s-1} + \sum_{k=1}^{n}\frac{1}{k^s} + \frac{1}{(s-1)}\sum_{k=1}^{n}\Delta_k \frac{1}{(k)^{s-1}}$$

$$= \frac{1}{s-1} + \sum_{k=1}^{n}\frac{1}{k^s} + \frac{1}{(s-1)}\left.\frac{1}{k^{s-1}}\right|_1^{n+1}$$

$$= \frac{1}{s-1}\left[1 + \frac{1}{(n+1)^{s-1}} - 1\right] \quad (\text{if } s>1) + \sum_{k=1}^{n}\frac{1}{k^s}$$

$$= \frac{1}{(s-1)(n+1)^{s-1}} + \sum_{k=1}^{n}\frac{1}{k^s}$$

$$\lim_{n\to\infty} S_n =$$

(18)

pg 40 w/w

1

$$\sum_{n=0}^{\infty} u_n = ?$$

$$S_n = \sum_{j=0}^{n} u_j = \sum_{j=3}^{n} u_j =$$

$$S_3 = \frac{-1}{\sqrt{2}}$$

$$S_4 = \cancel{\frac{-1}{\sqrt{2}}} + \cancel{\frac{1}{\sqrt{2}}} + \frac{1}{2} + \frac{1}{2(\sqrt{2})} = \frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right)$$

$$S_5 = S_4 + \frac{-1}{\sqrt{3}} = \frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right) - \frac{1}{\sqrt{3}}$$

$$S_6 = \frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right) - \cancel{\frac{1}{\sqrt{3}}} + \cancel{\frac{1}{\sqrt{3}}} + \frac{1}{3} + \frac{1}{3\sqrt{3}}$$

$$= \frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right) + \frac{1}{3}\left(1 + \frac{1}{\sqrt{3}}\right)$$

$$S_7 = S_6 - \frac{1}{\sqrt{\frac{7+1}{2}}}$$

$$S_8 = \frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right) + \frac{1}{3}\left(1 + \frac{1}{\sqrt{3}}\right) + \frac{1}{4}\left(1 + \frac{1}{\sqrt{4}}\right) > \sum_{k=2}^{8/2=4} \frac{1}{k}$$

$$\therefore S_{2n} > \sum_{k=2}^{n} \frac{1}{k} \xrightarrow{n \to \infty} \infty \quad \therefore \text{ Sum diverges (Sequence of partial sums does not converge)}$$

$$S_{2n-1} = S_{2n} - \frac{1}{\sqrt{n}}$$

$$\therefore \; |S_{2n-1} - S_n| < \frac{1}{\sqrt{n}}$$

$$u_{2n-1}^2 = \frac{1}{n}$$

$$u_{2n}^2 = \left(\frac{1}{\sqrt{n}} + \frac{1}{n} + \frac{1}{n\sqrt{n}}\right)^2 > \left(\frac{1}{\sqrt{n}}\right)^2 = \frac{1}{n}$$

$$\therefore \; \sum_{n=2}^{\infty} u_n > \sum_{n=2}^{\infty} \frac{1}{n} \quad \therefore \text{ diverges.}$$

But $\displaystyle\prod_{n=0}^{\infty}(1+u_n) = \prod_{n=3}^{\infty}(1+u_n)$

$$\cancel{\prod_n = \left(1 - \frac{1}{\sqrt{2}}\right)\left(1 - \frac{1}{\sqrt{2}} + \frac{1}{2} + \frac{1}{2\sqrt{2}}\right)}$$

$$\prod_3 = 1 + u_3 = 1 - \frac{1}{\sqrt{2}}$$

$$\prod_4 = \left(1 - \frac{1}{\sqrt{2}}\right)(1 + u_4) = \left(1 - \frac{1}{\sqrt{2}}\right)\left(1 + \frac{1}{\sqrt{2}} + \frac{1}{2} + \frac{1}{2\sqrt{2}}\right)$$

$$= 1 + \cancel{\frac{1}{\sqrt{2}}} + \cancel{\frac{1}{2}} + \cancel{\frac{1}{2\sqrt{2}}} - \cancel{\frac{1}{\sqrt{2}}} - \cancel{\frac{1}{2}} - \cancel{\frac{1}{2\sqrt{2}}} - \frac{1}{4} = 1 - \frac{1}{4}$$

$$\Pi_5 = \Pi_4(1+u_5)$$
$$= \left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{\sqrt{3}}\right)$$

$$\Pi_6 = \left(1-\frac{1}{4}\right)\left(1-\frac{1}{\sqrt{3}}\right)\left(1+\frac{1}{\sqrt{3}}+\frac{1}{3}+\frac{1}{3\sqrt{3}}\right)$$
$$= \left(1-\frac{1}{4}\right)\left[1 + \cancel{\frac{1}{\sqrt{3}}} + \cancel{\frac{1}{3}} + \cancel{\frac{1}{3\sqrt{3}}} - \cancel{\frac{1}{\sqrt{3}}} - \cancel{\frac{1}{3}} - \cancel{\frac{1}{3\sqrt{3}}} - \frac{1}{9}\right]$$
$$= \left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)$$

$$\Pi_7 = \Pi_6\left(1 - \frac{1}{4}\right)$$

$$\Pi_8 = \left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\left(1-\frac{1}{16}\right)$$

$$\therefore \quad \Pi_{2n} = \prod_{k=2}^{n}\left(1 - \frac{1}{k^2}\right) \quad \text{converges Absolutely.}$$

[illegible]

$$\Pi_{2n-1} = \Pi_{2n-2}\left(1 - \frac{1}{\sqrt{n}}\right)$$

$$\Rightarrow \left|\Pi_{2n-1} - \Pi_{2(n-1)}\right| \le \frac{\Pi_{2(n-1)}}{\sqrt{n}} \xrightarrow{n\to\infty} 0$$

[illegible] [illegible]

get $\prod_{2n} = \prod_{k=2}^{n}\left(1-\frac{1}{k^2}\right)$ compute this product.

$\cancel{\left(1-\frac{1}{4}\right)}$ $\cancel{\sum_{k=2}^{2}} = \cancel{\prod_2 = \prod_{k=2}^{2}\left(1-\frac{1}{k^2}\right) = 1-\frac{1}{4}}$

$\cancel{= \frac{4-1}{4} = \frac{3}{4}}$

$$\prod_4 = \prod_{k=2}^{2}\left(1-\frac{1}{k^2}\right) = \cancel{\prod_2} \left(1-\frac{1}{4}\right) = \frac{4-1}{4} = \frac{3}{4}$$

$$\prod_6 = \prod_{k=2}^{3}\left(1-\frac{1}{k^2}\right) = \frac{3}{4}\left(1-\frac{1}{9}\right) = \frac{3}{4}\left(\frac{9-1}{9}\right) = \frac{\cancel{3}^{1}\cdot\cancel{8}^{2}}{\cancel{4}\cdot\cancel{9}_{3}} = \frac{2}{3}$$

$$\prod_8 = \prod_{k=2}^{4}\left(1-\frac{1}{k^2}\right) = \frac{2}{3}\left(1-\frac{1}{16}\right) = \frac{2}{\cancel{3}}\,\frac{\cancel{15}^{5}}{\cancel{16}_{8}} = \frac{5}{8}$$

$$\prod_{16} = \frac{5}{8}\left(1-\frac{1}{25}\right) = \frac{\cancel{8}}{\cancel{8}}\,\frac{\cancel{24}^{3}}{\cancel{25}_{5}} = \frac{3}{5}$$

$\vdots$

$$\prod_{2n} = \left(1-\frac{1}{2}\right)\left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{3}\right)\cdots \cancel{\left(1-\frac{1}{n}\right)\left(1+\frac{1}{n}\right)}$$

$$= \cancel{\left(\frac{2}{2}\right)\left(\frac{3}{2}\right)\left(\frac{2}{3}\right)\cdots} \quad \left(1-\frac{1}{4}\right)\left(1+\frac{1}{4}\right)\left(1-\frac{1}{5}\right)\left(1+\frac{1}{5}\right)\cdots$$

$$= \left(\frac{2-1}{2}\right)\left(\frac{3}{\cancel{2}}\right)\left(\frac{\cancel{2}}{\cancel{3}}\right)\left(\frac{4}{\cancel{3}}\right)\left(\frac{\cancel{3}}{\cancel{4}}\right)\left(\frac{\cancel{5}}{4}\right)\left(\frac{4}{\cancel{5}}\right)\cdots\left(1-\frac{1}{n-1}\right)\left(1+\frac{1}{n-1}\right)\left(1-\frac{1}{n}\right)\cdot\left(1+\frac{1}{n}\right)$$

$$\Pi_{2n} = \frac{1}{2} \cdots \left(\frac{\cancel{n-2}}{\cancel{n-1}}\right)\left(\frac{\cancel{n}}{\cancel{n-1}}\right)\left(\frac{\cancel{n+1}}{\cancel{n}}\right)\left(\frac{n+1}{n}\right)$$

$$= \frac{1}{2}\left(\frac{n+1}{n}\right) = \frac{1}{2}\left(1+\frac{1}{n}\right)$$

$$\therefore \Pi = \lim_{n\to\infty} \Pi_{2n} = \frac{1}{2}.$$

pg 42 w/w

Example:

$f(x)$ & $\phi(x)$ cont at x_1

Then given $\epsilon > 0$ $\exists$ δ_1 & δ_2 $\ni$ $|x_1 - x| < \delta_1$

$|f(x) - f(x_1)| < \epsilon/2$ & if $|x_1 - x| < \delta_2$ $|\phi(x) - \phi(x_1)| < \epsilon/2$

Then

$$\left| f(x) \pm \phi(x) - (f(x_1) \pm \phi(x_1)) \right| \le |f(x) - f(x_1)| + |\phi(x) - \phi(x_1)|$$

$$< \epsilon$$

if $\delta = \operatorname{Min}(\delta_1, \delta_2)$ & $|x - x_1| < \delta$.

~~pg 43 w/~~

pg 44 w/w

$$x^2 \sum_{k=0}^{\infty} \frac{1}{(1+x^2)^k}$$

$$\lim_{k\to\infty} \left| \frac{\frac{1}{(1+x^2)^{k+1}}}{\frac{1}{(1+x^2)^k}} \right| = \lim_{k\to+\infty} \left| \frac{1}{(1+x^2)} \right| = \frac{1}{1+x^2} < 1 \quad \forall\, x \in \mathbb{R}$$

$$S_n = x^2 \sum_{k=0}^{n-1} \frac{1}{(1+x^2)^k} = x^2 \left. \frac{\left(\frac{1}{1+x^2}\right)^k}{\left(\frac{1}{1+x^2} - 1\right)} \right|_0^{n}$$

$\sum a^n =$

$\Delta a^n = a^n(a-1)$

$\sum a^n = \left. \frac{a^n}{a-1} \right|_{\ell_2}^{\ell_1+1}$

$$= \frac{x^2(x^2+1)}{(1-(1+x^2))} \left(\frac{1}{(1+x^2)^{n}} - 1 \right)$$

$$= -(1+x^2)\left(\qquad \right) = 1+x^2 - \frac{1}{(1+x^2)^{n-1}}$$

pg 46 hw

1

Ex 1:

$$x \sum_{n=1}^{\infty} \frac{1}{((n-1)x+1)(nx+1)}$$

$$\frac{1}{((n-1)x+1)(nx+1)} = \frac{A}{(n-1)x+1} + \frac{B}{nx+1}$$

$$A = \frac{1}{nx+1}\Bigg|_{x=-\frac{1}{n-1}} \qquad B = \frac{1}{(n-1)x+1}\Bigg|_{-\frac{1}{n}}$$

$$= \frac{n-1}{-n+n-1} \qquad B = \frac{n}{-(n-1)+n}$$

$$= 1-n \qquad B = n$$

$$\frac{x}{(\;)(\;)} = \frac{-x(n-1)}{(n-1)x+1} + \frac{xn}{nx+1}$$

$$= x\Delta\left(\frac{n-1}{(n-1)x+1}\right)$$

$$\therefore x\sum_{k=1}^{n} \frac{1}{((k-1)x+1)(kx+1)} = x\left(\frac{k-1}{(k-1)x+1}\right)\Bigg|_{1}^{n+1} =$$

$$x\frac{n}{nx+1} - 0 = \frac{xn}{xn+1} = 1 - \frac{1}{xn+1}$$

$$S(x) = x\sum_{k=1}^{\infty}\frac{1}{((k-1)x+1)(kx+1)} = x\frac{k-1}{(k-1)x+1}\Bigg|_{1}^{\infty}$$

$$= 1$$

$$\therefore \quad R_n(x) = S(x) - S_n(x) = 1 - \left(1 - \frac{1}{nx+1}\right) = \frac{1}{nx+1}$$

See Next pg

Ex 2:

type of partial fractions irreducible quadratics

$$\frac{x(n(n+1)x^2-1)}{(1+n^2x^2)(1+(n+1)^2x^2)} = \frac{Ax+B}{1+n^2x^2} + \frac{Cx+D}{1+(n+1)^2x^2}$$

$$\Rightarrow \quad Ax+B = \frac{x(n(n+1)x^2-1)}{1+(n+1)^2x^2}$$

$$\text{Root}\left[\,x^2(n+1)^2+1=0\,\right]$$

To make series uniformly convergent $\Rightarrow \exists N$ (2b)

$\ni \; \forall \; x \in \mathbb{R} \quad |R_n(x)| < \epsilon$ eq to

$|R_{n,p}(z)| < \epsilon \; \forall \; n > N \; \forall z \in \mathbb{C}$

$\Rightarrow \left|\frac{1}{nx+1}\right| < \epsilon \quad \forall \; n > N$

$\Rightarrow \epsilon^{-1} < |nx+1| \Longrightarrow$ if I make $n \uparrow \infty$ take $x_n = \frac{1}{n}$

Then get

$\Rightarrow \cancel{-\epsilon^{-1} < nx+1 < \epsilon^{-1}}$

$\Rightarrow \cancel{-\frac{1}{x}(\epsilon^{-1}+1) < n < \frac{1}{x}(\epsilon^{-1}-1)} \qquad x > 0$

Say for $x > 0$ get

$nx+1 > \epsilon^{-1}$ or $nx+1 < -\epsilon^{-1}$

$n > \frac{1}{x}(\epsilon^{-1}-1)$

Does $\exists \; N \ni \; N > \frac{1}{x}(\epsilon^{-1}-1) \; \forall \; x \in \mathbb{R}$

No $x \to 0$

Pg 46 W/W

1

Ex 2: Type of partial fractions : irreducible quadratics

$$\frac{x(n(n+1)x^2-1)}{(1+n^2x^2)(1+(n+1)^2x^2)} = \frac{Ax+B}{1+n^2x^2} + \frac{Cx+D}{1+(n+1)^2x^2}$$

$$Ax+B\Big|_{x=\frac{\pm i}{n}} = \frac{x(n(n+1)x^2-1)}{1+(n+1)^2x^2}\Bigg|_{x=\frac{\pm i}{n}}$$

Take $+i$

$$\frac{Ai}{n}+B = \frac{+\frac{i}{n}\left(n(n+1)\left(\frac{-1}{n^2}\right)-1\right)}{1-\frac{(n+1)^2}{n^2}} \cdot \frac{n^2}{n^2}$$

$$= \frac{-ni\left(\frac{1}{n}(n+1)+1\right)}{n^2-(n^2+2n+1)} = \frac{-i(n+1+n)}{-2n-1}$$

$$= \frac{i(2n+1)}{2n+1} = i$$

$A = 1\cdot n$; $B = 0$

$$Cx+D\Big|_{x=\frac{\pm i}{n+1}} = \frac{x(n(n+1)x^2-1)}{1+n^2x^2}\Bigg|_{x=\frac{\pm i}{(n+1)}}$$

Take +

$$\frac{Ci}{n+1} + D = \frac{\frac{i}{(n+1)}\left(-\frac{n(n+1)}{(n+1)^2} - 1\right)}{1 - \frac{n^2}{(n+1)^2}} \cdot \frac{(n+1)^2}{(n+1)^2}$$

$$= \frac{i}{(n+1)} \frac{\left(-n(n+1) - (n+1)^2\right)}{\left((n+1)^2 - n^2\right)}$$

$$= -i \frac{(n + n+1)}{\cancel{n^2} + 2n+1 - \cancel{n^2}} = -i$$

$$\Rightarrow \quad C = -(n+1) \quad ; \quad D = 0$$

$$\therefore \quad \frac{x(n(n+1)x^2 - 1)}{(1+n^2x^2)(1+(n+1)^2x^2)} = \frac{nx}{1+n^2x^2} - \frac{(n+1)x}{1+(n+1)^2x^2}$$

$$= -x\Delta\left(\frac{n}{1+n^2x^2}\right)$$

If: $S(x) = \sum_{k=1}^{\infty} -x\,\Delta\left(\frac{n}{1+n^2x^2}\right)$

$$= -x \left. \frac{n}{1+n^2x^2} \right|_1^{\infty} = -x\left(0 - \frac{1}{1+x^2}\right)$$

$$= \frac{x}{1+x^2}$$

$$R_n(x) = S(x) - S_n(x) = \frac{x}{1+x^2} + x\sum_{k=1}^{n} \Delta_k\left(\frac{k}{1+k^2x^2}\right)$$

$$= \frac{x}{1+x^2} + x \left. \frac{k}{1+k^2x^2} \right|_1^{n+1} = \cancel{\frac{x}{1+x^2}} + \frac{x(n+1)}{1+(n+1)^2x^2} - \cancel{\frac{x}{1+x^2}}$$

$$= \frac{x(n+1)}{1+(n+1)^2x^2} \quad \checkmark$$

For Uniform convergence given $\epsilon > 0$ $\exists N \ni \forall x \in \mathbb{R}$

$|R_n(x)| < \epsilon$ if $n > N$

$$\Rightarrow |x| < (\underbrace{1+(n+1)^2x^2}_{(n+1)})\epsilon \Rightarrow \underbrace{(n+1)\epsilon^{-1}}|x| < 1 + (n+1)^2x^2$$

$$\Rightarrow 0 < 1 - (n+1)\epsilon^{-1}|x| + (n+1)^2x^2$$

$$0 < 1 - (n+1)\epsilon^{-1}|x| + (n+1)^2 x^2 \qquad (*)$$

Solve for n to see how far to sum given an x value equality at:

$$(n+1) = \frac{\epsilon^{-1}|x| \pm \sqrt{\epsilon^{-2}|x|^2 - 4(x^2)}}{2x^2}$$

$$= \frac{1}{2}|x|^{-1}\left\{ \epsilon^{-1} \pm \sqrt{\epsilon^{-2} - 4} \right\}$$

$$\epsilon^{-2} - 4 > 0$$

if $\frac{1}{\epsilon^2} > 4$

$$\epsilon^2 < 4^{-1}$$

$$\epsilon < 1/2$$

If $(n+1)$ larger than this # ineq. $(*)$ is set.

But As $|x| \to 0$ $\quad n+1 \to \infty$. $\therefore$ No N constant

ind. of x exists $\ni$ $n+1 \leq N$.

pg 46 w/w

$$S(z) - S_n(z) = \lim_{p\to\infty} R_{N,p}(z) - R_{N,n-N}(z)$$

$$- \left(U_{N+1}(z) + U_{N+2}(z) + \cdots + U_{n-N}(z)\right)$$

check

$$\sum_{k=n+1}^{\infty} U_k(z) = \sum_{k=1}^{\infty} U_{N+k}(z) - \sum_{k=1}^{n-N} U_{N+k}(z)$$

$N+\alpha = n$

$\alpha = n-N$

$$= \sum_{k=n-N+1}^{\infty} U_{N+k}(z)$$

$n+1,$

$$= \sum_{k=n+1}^{\infty} U_k(z)$$

same

pg 46 w/w

$$S(x) = \lim_{n\to\infty}\left(1 - \frac{1}{nx+1}\right) = \begin{cases} 0 & x = 0 \\ 1 & x \neq 0 \end{cases}$$

For convergence must have $|R_n(z)| < \epsilon$ $\forall$ n independent

& But $R_n = \frac{1}{nx+1}$

$\therefore$ $\left|\frac{1}{nx+1}\right| < \epsilon$ cannot be solved for n

w/o an x dependence

i.e. $x > 0$ get $n > \frac{1}{x}\left(\frac{1}{\epsilon} - 1\right)$

But we see, for a fixed N, all that is required is to take $|x| \le x^* < |x_2|$ & obtain a violation of $|P_n(x^*)| > \epsilon$ & thus

But if $D = [10, \infty)$ then

take $|x_2| = \dfrac{\epsilon^{-1} + \sqrt{\epsilon^{-2} - 4}}{2(n+1)} < 10$

$\Rightarrow n+1 > \epsilon^{-1}$ — independent of x^*!!

~~convergence is uniform~~, But not for

pg 47 w/w

~~Ex 1~~

$$S(x) = x + \sum_{n=2}^{\infty} \left(x^{\frac{1}{2n-1}} - x^{\frac{1}{2(n-1)-1}} \right) = x + \sum_{n=2}^{\infty} \Delta\, x^{\frac{1}{2(n-1)-1}}$$

$$= x + x^{\frac{1}{2(n-1)-1}} \Big|_{2}^{\infty} = x + (1 - x) = \underline{1}$$

$$\& \quad S_n(x) = x + x^{\frac{1}{2(n-1)-1}} \Big|_{2}^{n+1} = x + x^{\frac{1}{2n-1}} - x$$

$$= x^{\frac{1}{2n-1}}$$

$\therefore\ S(0) = 1$

$S_n(0) = 0$

$\&\ \lim_{n\to\infty} S_n(0) = 0$

Uniform conv. on def $\exists\ N$ independent of x $\ni$ $|R_n(x)| < \epsilon$ $\forall n > N$ & $\forall x \in \mathbb{R}$

find seq $x_n \ni |R_n(x_n)| > \epsilon$ $\forall n$ &

$x_n \to 0$ pts in domain

$S_n(x) = x^{\frac{1}{2n-1}}$

$u_n(x) > 0$

$x^{\frac{1}{2n-1} - \frac{1}{2n-3}} > 0$

$x^{\frac{-3+1}{(2n-1)(2n-3)}} \overset{?}{>} 0$

$x^{\frac{-2}{(2n-1)(2n-3)}} > 0$ yes $\forall$ $x \neq 0$

pg 48 w/w

Ex 2:

$$\sum_{n=1}^{\infty} \frac{-2z(1+z)^{n-1}}{\{1+(1+z)^{n-1}\}\{1+(1+z)^n\}}$$

Now: (My way...)

$$\frac{-2z(1+z)^{n-1}}{(1+(1+z)^{n-1})(1+(1+z)^n)} \overset{?}{=} \frac{A}{1+(1+z)^{n-1}} + \frac{B}{1+(1+z)^n}$$

$$= \frac{A(1+(1+z)^n) + B(1+(1+z)^{n-1})}{(1+(1+z)^n)(1+(1+z)^{n-1})} \quad *$$

$$= \frac{A+B+(A(1+z)+B)(1+z)^{n-1}}{(\quad)(\quad)}$$

$\Longleftrightarrow$ ~~A+B+(A(1+~~ or Back to $*$

$$-2z(1+z)^{n-1} = A(1+(1+z)^{n}) + B(1+(1+z)^{n-1})$$

$$\Leftrightarrow A(1+(1+z)^n) = B(1+(1+z)^{n-1}) + 2z(1+z)^{n-1}$$

If $A+B=0$

& $A(1+z)+B=-2z$

$$\Rightarrow A+B+Az=-2z \Rightarrow \begin{array}{l} A=-2 \\ B=2 \end{array}$$

Then check

$$\frac{-2}{1+(1+z)^{n-1}}+\frac{2}{1+(1+z)^{n}}$$

$$=\frac{\cancel{-2}-2(1+z)^{n}+\cancel{2}+2(1+z)^{n-1}}{(\quad)(\quad)}=\frac{(1+z)^{n-1}(-2(1+z)+2)}{(\quad)(\quad)}$$

$$=\frac{-2z(1+z)^{n-1}}{(\quad)(\quad)} \checkmark$$

Thus

$$2\sum_{k=1}^{n}\Delta\left(\frac{1}{1+(1+z)^{n-1}}\right)=2\,\frac{1}{1+(1+z)^{n-1}}\Bigg|_{1}^{n+1}$$

$$=2\left[\frac{1}{1+(1+z)^{n}}-\frac{1}{2}\right]$$

o/w way:

Consider $\dfrac{1-(1+z)^n}{1+(1+z)^n} - \dfrac{(1-(1+z)^{n-1})}{1+(1+z)^{n-1}} = {}^{+}\Delta\left(\dfrac{1-(1+z)^{n-1}}{1+(1+z)^{n-1}}\right)$

$$= \frac{(1+(1+z)^{n-1})(1-(1+z)^n) - (1+(1+z)^n)(1-(1+z)^{n-1})}{(1+(1+z)^n)(1+(1+z)^{n-1})}$$

$$= \frac{\cancel{1} \mp (1+z)^n + (1+z)^{n-1} - \cancel{(1+z)^{2n-1}} - [\cancel{1} - (1+z)^{n-1} + (1+z)^n - \cancel{(1+z)^{2n-1}}]}{(\quad)(\quad)}$$

$$= \frac{-2(1+z)^n + 2(1+z)^{n-1}}{(\quad)(\quad)} = \frac{2(1+z)^{n-1}\{-(1+z)+1\}}{(\quad)(\quad)}$$

$$= \frac{-2z(1+z)^{n-1}}{(\quad)(\quad)} \quad ✓$$

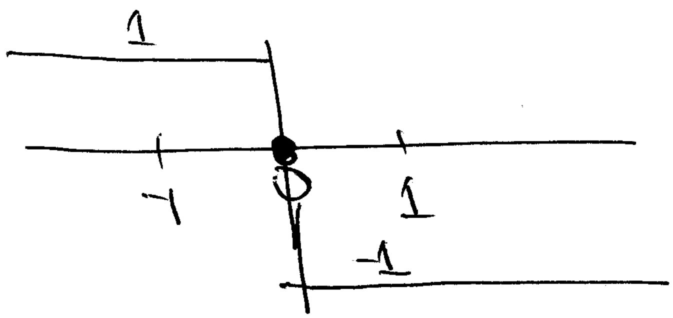
$$\therefore \sum_{k=1}^{n} \Delta u_k = u_k \Big|_1^{n+1} = \frac{1 \mp (1+z)^{k-1}}{1+(1+z)^{k-1}} \Bigg|_1^{n+1}$$

$$= \frac{1-(1+z)^n}{1+(1+z)^n} - \frac{1-1}{2} = \frac{1-(1+z)^n}{1+(1+z)^n}$$

if $z > -1$

$\Rightarrow \quad z+1 > 0$

If $(z+1) < 1$ i.e. $(z < 0)$ $\lim_{n\to\infty} S_n = 1$; If $z+1 = 1$ i.e. $(z=0)$ $\lim_{n\to\infty} S_n = 0$; If $z+1 > 1$ i.e. $(z > 0)$ $\lim_{n\to\infty} S_n = -1$



Series discont. $\therefore$ cannot be uniformly convergent.

Remainder for $z > 0$

$$\left| \frac{1-(1+z)^n}{1+(1+z)^n} + 1 \right| = \left| \frac{2}{1+(1+z)^n} \right|$$

$\therefore \quad |R_n(z)| < \epsilon$

$$\Rightarrow \quad \frac{2}{|1+(1+z)^n|} < \epsilon$$

Take

$$\Rightarrow \quad \cancel{2\epsilon^{-1} < 1+(1+z)^n}$$

$$\cancel{(2\epsilon^{-1}-1) < (1+z)^n}$$

$$\cancel{\frac{\log(2\epsilon^{-1}-1)}{\log}}$$

pg 51 W/W

$w_1 = \alpha_1 + i\beta_1 \quad w_2 = \alpha_2 + i\beta_2$

$|mw_1 + nw_2| = |m\alpha_1 + n\alpha_2 + i(m\beta_1 + n\beta_2)|$

$= ((\alpha_1 m + \alpha_2 n)^2 + (\beta_1 m + \beta_2 n)^2)^{1/2}$

consider series $\sum' \frac{1}{(m^2+n^2)^{\alpha/2}}$

quotent of corresponding terms is $\quad \underline{\mu = n/m}$

$$\frac{b_{mn}}{a_{mn}} = \frac{\frac{1}{(m^2+n^2)^{\alpha/2}}}{\frac{1}{((\alpha_1 m + \alpha_2 n)^2 + (\beta_1 m + \beta_2 n)^2)^{\alpha/2}}} = \left(\frac{(\alpha_1 m + \alpha_2 n)^2 + (\beta_1 m + \beta_2 n)^2}{m^2+n^2}\right)^{\alpha/2}$$

$$= \left\{\frac{(\alpha_1 + \alpha_2 \mu)^2 + (\beta_1 + \beta_2 \mu)^2}{1+\mu^2}\right\}^{\alpha/2} = F(\mu)$$

Then $F(\mu) \geq F^*$ w/ F^* minimum over all μ.

$\therefore$ let original series of terms be a_{mn} + new series be b_{mn} Then $\frac{b_{mn}}{a_{mn}} \geq F^*$

$\Rightarrow b_{mn} \geq F^* a_{mn}$ + $\therefore$ by

Comparison test If $\sum_{m,n} b_{mn}$ converges so must $\sum_{m,n} a_{mn}$.

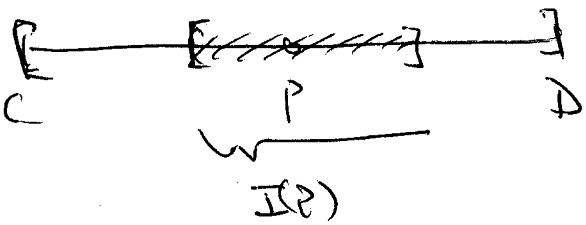
$$S_{p,q} := \sum_{m=-p}^{p}{}' \sum_{n=-q}^{q}{}' \frac{1}{(m^2+n^2)^{\alpha/2}} = 4 \sum_{m=1}^{p} \sum_{n=1}^{q} \frac{1}{(m^2+n^2)^{\alpha/2}}$$

$(1,\dots,p)$ $(1,\dots,q)$

$(-1,\dots,-p)$ $(-1,\dots,-q)$

$$\leq 4 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{(m^2+n^2)^{\alpha/2}} \leq 4 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty}{}' \frac{1}{(m^2+n^2)^{\alpha/2}}$$

Pg 53 w/w



have ∞ # of closed intervals covering the closed interval $[C,D]$

Heine-Borel $\Rightarrow$ $\exists$ a finite cover of the line segment $[C,D]$

Pg 57 w/w

Ex 1: If $\frac{f(x)-f(x')}{x-x'} > 0 \quad \forall \; x, x'$

Assume (increasing monotonic; decreasing monotonic is sim w/ $>$ $\Leftrightarrow$ $<$)

$$\Rightarrow f(x)-f(x') > 0 \qquad x-x' > 0 \quad (1)$$

$$f(x)-f(x') < 0 \qquad x-x' < 0 \quad (2)$$

(1) $\Rightarrow \; x > x' \Rightarrow f(x) > f(x')$

(2) $\Rightarrow \; x < x' \Rightarrow f(x) < f(x')$

Now $x_1, x_2, \dots, x_n$

$$F_a^b = |f(a)-f(x_1)| + |f(x_1)-f(x_2)| + \dots + |f(x_n)-f(b)|$$

$$= +f(\cancel{x_1}) - f(a) + f(\cancel{x_2}) - f(\cancel{x_1}) + \dots + f(b) - f(\cancel{x_n})$$

$$= f(b) - f(a) = |f(b)-f(a)|$$

Ex 2:

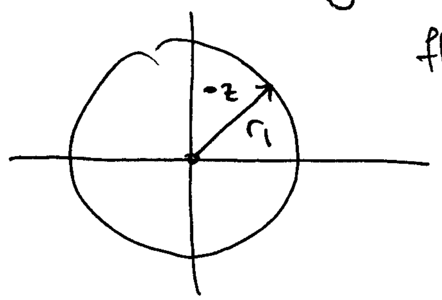
$$f(x) = \underbrace{\tfrac{1}{2}\{F_a^x + f(x)\}}_{g(x)} - \underbrace{\tfrac{1}{2}\{F_a^x - f(x)\}}_{h(x)} \qquad \text{yes}$$

Pr: $g(x)$ & $h(x)$ are increasing monotonic take $x' > x$

$$\frac{g(x)-g(x')}{x-x'} = \frac{1}{2}\left(\frac{F_a^x - F_a^{x'} + f(x) - f(x')}{x-x'}\right)$$

Pg 58 W/W

3.73



$f(z) = 0.$

$a_m + a_{m+1} z + \cdots$ = original series modified by ÷ by z + subtracting a finite # of terms.

⇒ radius of convergence is the same as the original series.

$$S = a_{m+1} + a_{m+2} z + \cdots = \sum_{n=1}^{\infty} a_{m+n} z^{n-1}$$

$$|S| \le \sum_{n=1}^{\infty} |a_{m+n}| |z|^{n-1} \le \sum_{n=1}^{\infty} |a_{m+n}| r^{n-1}$$

∴ $\cancel{\sum_{n=0}^{\infty} a_{m+n} z}$

$$\left| \sum_{n=1}^{\infty} a_{m+n} z^{n} \right| \le |z| \left| \sum_{n=1}^{\infty} a_{m+n} z^{n-1} \right| \le$$

& $L = a_m + zS$ + $|S| < \sum_{n=1}^{\infty} |a_{m+n}| r^{n-1}$

$$r_1 \quad \frac{\frac{1}{2}|a_m|}{\sum_{n=1}^{\infty} |a_{m+n}| r^{n-1}}$$

Pg 59 w/w

(1) $$\sum_{n=1}^{\infty} \frac{z^{n-1}}{(1-z^n)(1-z^{n+1})} = ?$$

Now:

$$\frac{z^{n-1}}{(1-z^n)(1-z^{n+1})} = \frac{\text{"}A\text{"}}{1-z^n} + \frac{\text{"}B\text{"}}{1-z^{n+1}}$$

$$= \frac{A(1-z^{n+1}) + B(1-z^n)}{(1-z^n)(1-z^{n+1})}$$

Factor out a z^{n-1}

$$= z^{n-1} \frac{\{A(z^{-n+1} - z^2) + B(z^{-n+1} - z)\}}{(1-z^n)(1-z^{n+1})}$$

$$= z^{n-1} \frac{\{(A+B)z^{-n+1} - Az^2 - Bz\}}{(1-z^n)(1-z^{n+1})}$$

$$= z^{n-1} \frac{\{(A+B)z^{-n+1} - z(Az+B)\}}{(1-z^n)(1-z^{n+1})}$$

A & B (Must be independent of n)

Pick $\quad -z(Az+B) = 1$

$\quad + \quad A+B = 0.$

$A = -B$

$-Bz + B = \frac{-1}{z}$

$B(1-z) = \frac{-1}{z} \rightarrow B = \frac{-1}{z(1-z)}$

$\quad$ & thus $A = \frac{1}{z(1-z)}$

$$S_N = \sum_{n=1}^{N} \frac{z^{n-1}}{(1-z^n)(1-z^{n+1})} = \sum_{n=1}^{N} \frac{1}{z(1-z)} \left[\frac{1}{1-z^n} - \frac{1}{1-z^{n+1}} \right]$$

$$= +\frac{1}{z(z-1)} \sum_{n=1}^{N} \Delta\left(\frac{1}{1-z^n}\right) = \frac{1}{z(z-1)} \left(\frac{1}{1-z^n}\right|_1^N$$

$$= \frac{1}{z(z-1)} \left[\frac{1}{1-z^N} - \frac{1}{1-z} \right]$$

Now

$$\lim_{N\to\infty} S_N(z) = \frac{1}{z(z-1)} \left[1 - \frac{1}{1-z} \right] \quad \text{if } |z|<1$$

$$\therefore \quad S(z) = \frac{\cancel{1}-z\cancel{-1}}{z(z-1)(1-z)} = \frac{1}{(1-z)^2}$$

$$\& \lim_{N\to\infty} S_N(z) = \frac{1}{z(z-1)}\left[-\frac{1}{1-z}\right] = \frac{1}{z(1-z)^2} \quad \text{when } |z|>1$$

$$\therefore \quad S(z) = \begin{cases} \dfrac{1}{(1-z)^2} & |z|<1 \\ \dfrac{1}{z(1-z)^2} & |z|>1 \end{cases}$$

Now for $|z|<1$

$$R_N(x) = \frac{1}{(1-z)^2} - \frac{1}{z(z-1)(1-z^N)} + \frac{-1}{z(1-z)^2}$$

& for $|z|>1$

$$R_N(z) = \frac{1}{z(1-z)^2} - \frac{1}{z(z-1)(1-z^N)} - \frac{1}{z(1-z)^2}$$

$$\therefore \quad |R_N(z)| = \frac{1}{|z||z-1||1-z^N|} < \epsilon$$

$$\underbrace{\big||z|^N-1\big|}_{\text{positive when } |z|>1} < |1-z^N| < \epsilon|z||z-1|$$

$$|z|^N-1 < \epsilon|z||z-1|$$

$$\Rightarrow \quad |z|^N < \epsilon|z||z-1|+1$$

$$\Rightarrow \quad N\log|z| < \log(1+\epsilon|z||z-1|)$$

$\Rightarrow N > \dfrac{\log(1+\epsilon|z||z-1|)}{\log|z|} =: n(z)$ $\quad |z| < 1$

$\Rightarrow \log|z| < 0$

As $|z| \to 1$ $\quad \cancel{n}(z) \to \infty$

$|z| = 1$

& thus the series is non-uniformly convergent on $|z| \geq 1$

$|z-1|^2 = (z-1)(\bar{z}-1)$

$= z\bar{z} - z - \bar{z} + 1$

$= 2 - z - \bar{z}$

$= 2 - 2\mathrm{Re}(z) = 2(1 - \mathrm{Re}(z))$

~~Thus~~ Thus we cannot expect necessarily the series to be the same when summed ~~from~~ w/ $|z| < 1$ & w/ $|z| > 1$

This is reflected in $S(z)$.

pg 59 w/w

$$\sum_{k=0}^{\infty} 2^k \sin\left(\frac{z^{-1}}{3^k}\right)$$

② From $|\sin z| \leq |z| \quad \forall z \in \mathbb{R}$

we get $\left|2^k \sin\left(\frac{1}{3^k z}\right)\right| \leq \frac{2^k |z|^{-1}}{3^k} = \left(\frac{2}{3}\right)^k |z|^{-1}$

As $\frac{2}{3} < 1$ The given series converges Absolutely.

Now to show we don't have uniform convergence

consider.

$$2^k \sin\left(\frac{1}{3^k z}\right) = \frac{2^k}{2i}\left[e^{\frac{i}{3^k z}} - e^{\frac{-i}{3^k z}}\right] = \frac{2^{k-1}}{i}\left[e^{''} - e^{''}\right]$$

Then $\Delta_k\, 2^{k-1} e^{i/3^k z} = 2^k e^{i/3^{k+1} z} - 2^{k-1}$

--- Let $\epsilon > 0$ Be given Then we would like to pick values

let $z_k = \frac{1}{3^k}$ Then $z_k \xrightarrow{k \to \infty} 0$ of $z \ni$

$|R_n(z_n)| \geq \epsilon \ \forall n$.

Then series is not uniformly convergent.

$$S_n(x) = \sum_{k=0}^{n} 2^k \sin\left(\frac{1}{3^k z}\right)$$

How do ??

Pg 59 w/w

(3) $u_n(x) = 2n^2 x e^{-n^2x^2} - 2(n-1)^2 x e^{-(n-1)^2x^2}$

$= 2x\left[n^2 e^{-n^2x^2} - (n-1)^2 e^{-(n-1)^2x^2}\right]$

$= 2x \, \Delta \, n^2 e^{-n^2x^2}$

$\therefore \quad S_n(x) = \sum_{k=1}^{n} 2x \Delta k^2 e^{-k^2x^2} = 2x\left(k^2 e^{-k^2x^2}\right\Big|_1^n$

$= 2x\left(n^2 e^{-n^2x^2} - e^{-x^2}\right)$

Then $\lim_{n\to\infty} S_n(x) = -2xe^{-x^2} \qquad x \neq 0$

$= 0 \qquad x = 0. \Big\} S(x)$

Then if $x \neq 0$

$R_n(x) = S(x) - S_n(x) = -2xn^2 e^{-n^2x^2}$

$\therefore \quad |R_n(x)| = 2|x| n^2 e^{-n^2x^2} \overset{\text{want}}{<} \epsilon.$

But $n^2 e^{-n^2x^2} \leq$ ~~...~~

$= e^{2\log n} e^{-n^2x^2}$

want something larger than $n^2 e^{-n^2x^2}$ Then

I can make $|R_n(x)| < \epsilon$

By making that ~~#~~ smaller than ϵ. Now.

$\log n^2 < n^2$ for $n \gg 1$ → wait work As $|x| < 1$.

$$\Rightarrow e^{2\log n} e^{-n^2x^2} < e^{n^2} e^{-n^2x^2} = e^{n^2(1-x^2)}$$

If $e^{n^2(1-x^2)} < \epsilon$ then.

$$\cancel{\log n^2(1-x^2)} \quad (1-x^2)n^2 < \log \epsilon$$

$$e^{2\log n - n^2x^2} < \epsilon \qquad \lim_{n\to\infty} (2\log n - n^2x^2) = -\infty \qquad x \neq 0.$$

$$\Rightarrow 2\log n - n^2x^2 < \log \epsilon$$

$$-x^2\left\{-\frac{2\log n}{x^2} + n^2\right\} < \log \epsilon$$

$$\Rightarrow n^2 - \frac{2\log n}{x^2} > -\frac{\log \epsilon}{x^2} > 0 \quad (\text{As } \epsilon < 1)$$

solving $n^2 - \frac{2\log n}{x^2} = -\frac{\log \epsilon}{x^2}$

Using perturbations we get to 1st order

$$n_0 = \sqrt{-\frac{\log\epsilon}{x^2}} = \frac{\sqrt{-\log\epsilon}}{|x|}$$

Then $n_1^2 - \frac{2\log n_0}{x^2} = -$

$$n^2 = -\frac{\log\epsilon}{x^2} + \frac{2\log n}{x^2}$$

$$n^2 \sim -\frac{\log\epsilon}{x^2} \qquad n\to\infty$$

$$\Rightarrow n \sim \sqrt{-\frac{\log\epsilon}{x^2}} \qquad n\to\infty$$

$$\Rightarrow n = \sqrt{-\frac{\log\epsilon}{x^2}}\,(1+\phi(n)) \qquad \cancel{\phi(n)\to 0} \quad n\to\infty$$

$\Rightarrow$ put in equality.

$$\Rightarrow -\frac{\log\epsilon}{x^2}(1+2\phi(n)+\phi^2(n)) = -\frac{\log\epsilon}{x^2} + \frac{2}{x^2}\cancel{\log(1+2\phi(n)+\phi^2(n))}$$
$$+\frac{2}{x^2}\log\left(\sqrt{-\frac{\log\epsilon}{x^2}}\,(1+\phi(n))\right)$$

$\Rightarrow$ dropping ϕ^2

$$-\frac{2\log\epsilon}{x^2}\phi(n) \sim \cancel{\frac{2\log}{x^2}} \quad \frac{2}{x^2}\log\left(\cancel{\sqrt{-\frac{\log\epsilon}{x^2}} + \sqrt{\frac{\log\epsilon}{x^2}}\,\phi(n)}\right)$$

pg 78 w/w

$$\int_{z_0}^{z} dz = \int_a^b \left(\frac{dx}{dt} + i\frac{dy}{dt}\right) dt = x(b) - x(a) + i(y(b) - y(a))$$

$$\overline{\int_{z_0}^{z} z\,dz} = \int_a^b \left(x\frac{dx}{dt} - y\frac{dy}{dt}\right) + i\left(x\frac{dy}{dt} + y\frac{dx}{dt}\right) dt$$

$$= \frac{x^2}{2}\Big|_a^b - \frac{y^2}{2}\Big|_a^b + i\int_a^b \underbrace{x\frac{dy}{dt} + y\frac{dx}{dt}}_{\frac{d}{dt}(xy)} dt$$

$$+ ixy\Big|_a^b$$

$$= \frac{1}{2}(x^2 - y^2 + 2ixy)\Big|_a^b = \frac{1}{2}(z^2 - z_0^2)$$

Pg 79 W/W

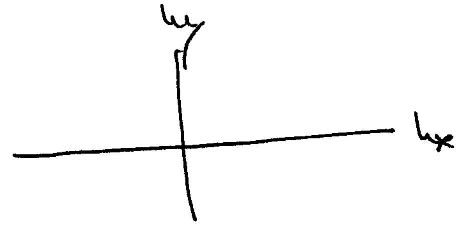
Corollary:

Show: $\frac{d}{dz}\sum_{n=0}^{\infty} u_n(z) = \sum_{n=0}^{\infty} \frac{d}{dz} u_n(z)$ if The RHS converges uniformly
& The LHS is convergent

Then on pg 78 W/W If sum is uniformly convergent

$$\cancel{\int_C \left(\sum_{n=0}^{\infty} \frac{d}{dz} u_n(z)\right) dz = \sum_{n=0}^{\infty} \int_C \frac{d}{dz} u_n(z)\, dz = \sum_{n=0}^{\infty} u_n(z)}$$

~~Comp 13 2000~~



consider $f(z) = \bar{z}$

Then $h \in \mathbb{C}$

$$\frac{1}{h}(\bar{z} + \bar{h} - \bar{z}) = \frac{\bar{h}}{h} = \frac{h_x - i h_y}{h_x + i h_y} \cancel{\frac{(h_x - i h_y)}{(h_x - i h_y)} = \frac{h_x^2 + h_y^2 - 2i h_x h_y}{h_x^2 + h_y^2}}$$

Take aproach w/ $h_y \equiv 0$ $h_x \to 0$.

$$\Rightarrow \frac{h_x}{h_x} = 1$$

Take aproach w/ $h_x \equiv 0$ & $h_y \to 0$

$$\frac{-i h_y}{i h_y} = -1$$

& $-1 \neq 1$. $\bar{z}$ is <u>not</u> analytic

$$\lim_{h \to 0} \frac{1}{h} \left\{ \frac{z - a - (z - a + h)}{(z-a)(z-a+h)} \right\} = \lim_{h \to 0} \frac{1}{\cancel{h}} \left\{ \frac{-\cancel{h}}{(z-a)(z-a+h)} \right\}$$

$$= -1 \frac{1}{(z-a)^2}$$

$z^{1/2}$ at $z = 0$ $\quad \lim_{h \to 0} \frac{h^{1/2}}{h} = \lim_{h \to 0} \frac{1}{h^{1/2}} = \infty$. ? Not analytic

$$\frac{f(z') - f(z)}{z' - z} - l = v(z, z')$$

Ex 1: Hard method

(i) $\lim_{h \to 0} \frac{(z+h)^2 - z^2}{z+h-z} = \lim_{h \to 0} \frac{+2zh + h^2}{h} = \lim_{h \to 0} (2z + h) = 2z$

All pts it is analytic

(ii) $\operatorname{cosec} z = \frac{1}{\cos z}$ if $\cos z = 0$. could not be analytic

$\Rightarrow z = n\pi$

Ex 2: $z = x + iy$ $f(z) = u + iv$. Show:

$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$ ind. of means $h \to 0$.

$$\Rightarrow \lim_{h \to 0} \frac{u(z+h) + iv(z+h) - u(z) - iv(z)}{h}$$

$$\Rightarrow \lim_{h \to 0} \frac{u(x+h_x, y+h_y) - u(x,y) + i(v(x+h_x, y+h_y) - v(x,y))}{h_x + ih_y} \cdot \frac{(h_x - ih_y)}{h_x - ih_y}$$

$$\Rightarrow \lim_{h \to 0} \frac{\qquad}{h_x^2 + h_y^2}$$

$$= \frac{(\Delta u\, h_x - \Delta v\, h_y)}{h_x^2 + h_y^2} + i\,\frac{(-\Delta u\, h_y + h_x \Delta v)}{h_x^2 + h_y^2}$$

consider path $h_y \equiv 0$

$$\Rightarrow \quad \frac{\Delta u}{h_x} = \frac{u(x+h_x, y) - u(x,y)}{h_x} \xrightarrow[\text{ind } 0]{\text{limit}} \Rightarrow$$

$$= \frac{\Delta u}{h_x} + i\,\frac{\Delta v}{h_x} \xrightarrow{\text{lim}} \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x}$$

path $h_x \equiv 0$

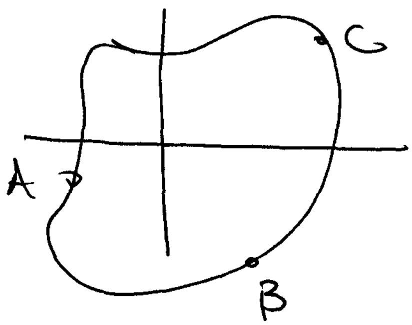
$$\Rightarrow \quad -\frac{\Delta v}{h_y} + i\,-\frac{\Delta u}{h_y} \xrightarrow{\text{limit}} -\frac{\partial v}{\partial y} - i\,\frac{\partial u}{\partial y}$$

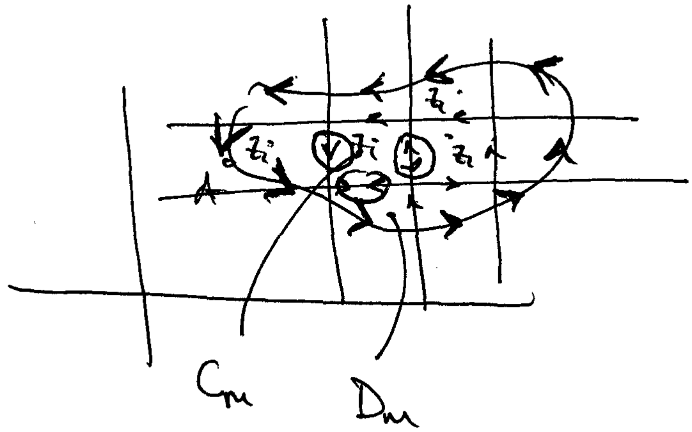
equating real & imag par

Check

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} \quad \& \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Pg 85 w/w





Note: All interior integrals cancel

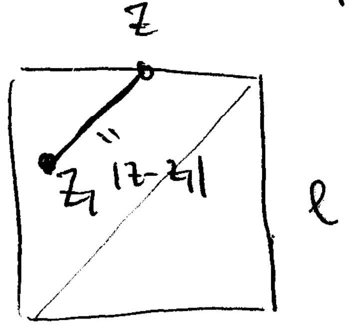
For (C_n)

$$\int_{(C_n)} |(z-z_1)\, v\, dz| \le \epsilon\, l\sqrt{2} \int_{C_n} |dz| = \epsilon\, l\sqrt{2}\, 4\, l$$

$$= l^2\, 4\sqrt{2}\,\epsilon$$

$$= 4 A_n \epsilon \sqrt{2}$$

magnitude of $|v| \le \epsilon$

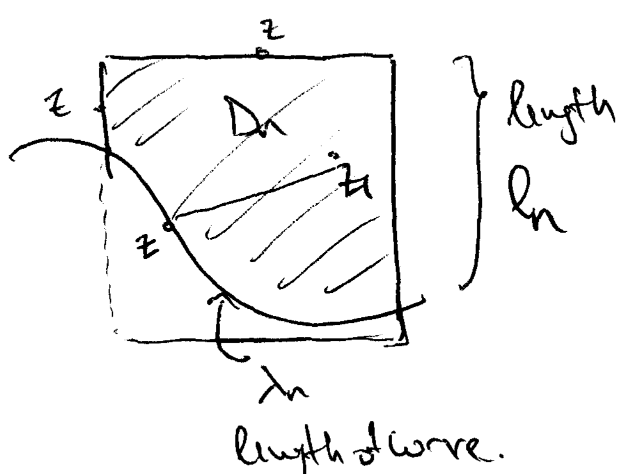


$|z - z_1| < \sqrt{2}\, l$

For D_n

$$\int_{D_n} |(z-z_1)\, v\, dz| \le \epsilon$$

mag. of $|v| \le \epsilon$



& $|z - z_1| \le \sqrt{2}\, l_n$ As before

$\cancel{\le l_n\sqrt{2}\,(4l_n + 4\lambda_n)}$ 4 sides & 4

Then

$$\int_{D_n} |(z-z_1)r\,dz| \le \epsilon\sqrt{2}\, l_n \int_{D_n} |dz|$$

But $\int_{D_n} |dz| \le (4l_n + 4\lambda_n)$ 4 sides & 4 curves

$$\therefore \int_{D_n} |(z-z_1)r\,dz| \le 4\epsilon\sqrt{2}\,(l_n^2 + l_n\lambda_n) = 4\epsilon(A_n' + l_n'\lambda_n)$$

$$\therefore \left|\int_C f(z)\,dz\right| \le \sum_{n=1}^{M}\left|\int_{C_n} f(z)\,dz\right| + \sum_{n=1}^{N}\left|\int_{D_n} f(z)\,dz\right|$$

$$< 4\epsilon\sqrt{2}\left(\sum_{n=1}^{M} A_n + \sum_{n=1}^{N} A_n' + \sum_{n=1}^{N} l_n' \lambda_n\right)$$

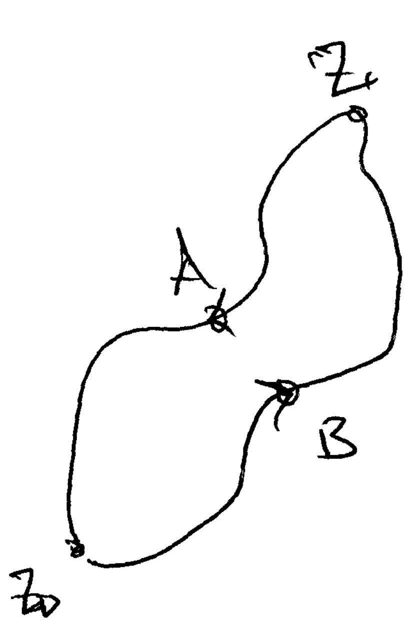
But let l sq

$$< 4\sqrt{2}\,\epsilon\left(\sum_{n=1}^{M} A_n + \sum_{n=1}^{N} A_n' + l\underbrace{\sum_{n=1}^{N} \lambda_n}_{\lambda \text{ total curve leng}}\right)$$

$$= 4\epsilon\sqrt{2}\left(l^2 + l\lambda\right)$$

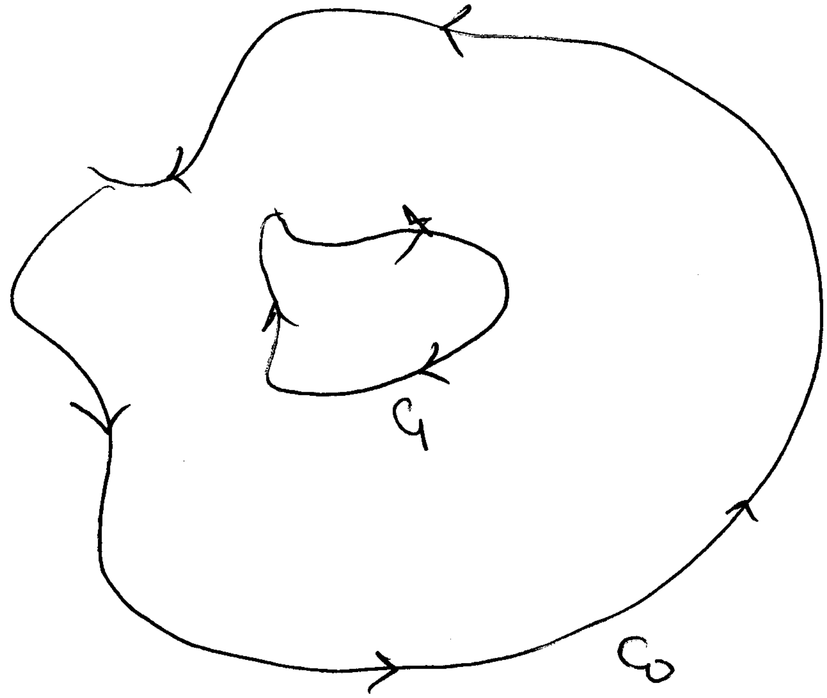
---

corollary 1:



The $\displaystyle\int_{z_0Az_1} f(z)\,dz = \int_{z_0Bz_1} f(z)\,dz$

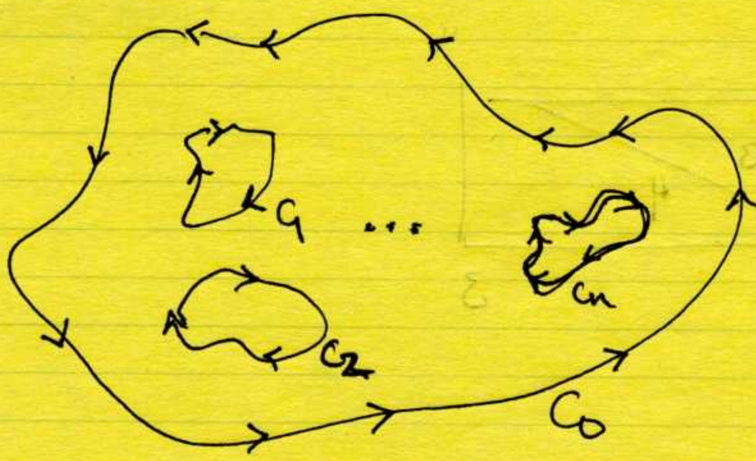
Cor 2



$$\int_{C'} f(z)\,dz = \int_{\underset{\circlearrowleft}{C_0}} f(z)\,dz + \int_{\underset{\circlearrowright}{C_1}} f(z)\,dz = 0$$

Cor 3:

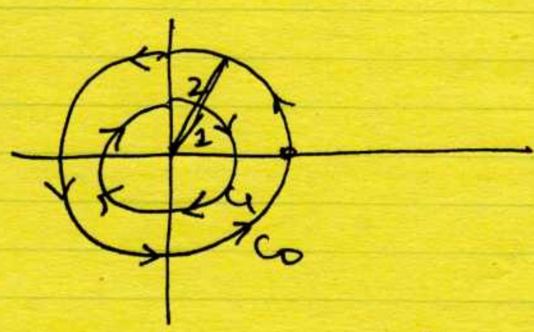
Corollary 3 pg 87 W/W



Boundary is $C_0, C_1, C_2, \ldots C_n$ Note that "walking along" each a person would have the region to their left.

Thus if All the Arrows were reversed then the integral would still be zero.

Example: Let $f(z) = \frac{1}{z}$

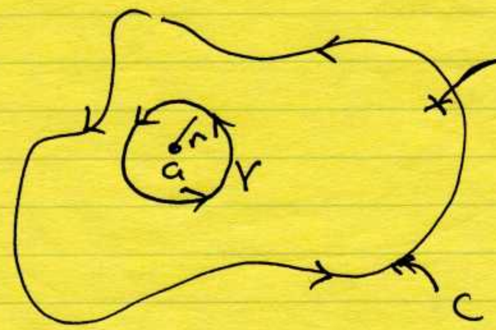


$$\int_C f(z)dz = \int_{C_0} + \int_{C_1} = \int_{\theta=0}^{2\pi} \cancel{2}^{-1}e^{-i\theta}\,\cancel{2}ie^{i\theta}\,d\theta + \int_{\theta=0}^{2\pi} e^{i\theta}(-ie^{-i\theta})d\theta$$

$z = 2e^{i\theta}$, $dz = 2ie^{i\theta}d\theta$ | $z = 1e^{-i\theta}$, $dz = -ie^{-i\theta}d\theta$

$$= i\int_0^{2\pi} d\theta - i\int_0^{2\pi} d\theta = 0 \quad ✓.$$

5.21)



w/ $r < \delta$
+ inside the Region.

$\frac{f(z)}{z-a}$ analytic at All pts except a.

let $\tilde{C} = C$ w/ γ

Then $\int_{\tilde{C}} = 0$

$$\Rightarrow \int_C + \int_\gamma = 0$$

$$\rightarrow \int_C \frac{f(z)}{z-a}\,dz = \int_{-\gamma} \frac{f(z)}{z-a}\,dz$$ w/ Both integrals being taken counterclockwise

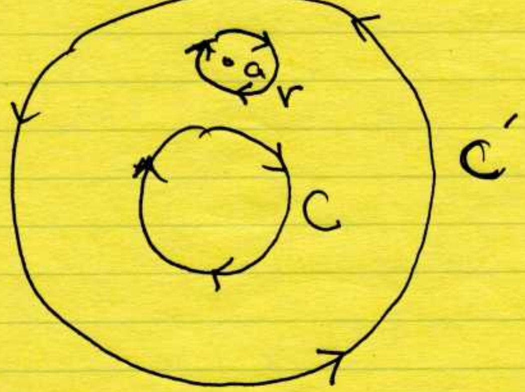
- - - - -

let $z-a = re^{i\theta}$ $\quad dz = rie^{i\theta}d\theta$

$$\int_\gamma \frac{dz}{z-a} = \int_{\theta=0}^{2\pi} \frac{rie^{i\theta}d\theta}{re^{i\theta}} = i2\pi$$

$$\Rightarrow \int_\gamma dz = \int_{\theta=0}^{2\pi} ire^{i\theta}\, d\theta = \frac{ir}{i}\left(e^{i\theta}\right|_0^{2\pi} = 0.$$

Corollary:



Then $\displaystyle \int_{\substack{C' \\ \circlearrowleft}} + \int_{\substack{C \\ \circlearrowright}} + \int_{\substack{\gamma \\ \circlearrowright}} = 0$

Do to $\frac{f(z)}{z-a}$. Then As Before we just had

$$\int_{\substack{C \\ \circlearrowleft}} + \int_\gamma = 0$$

$$\Rightarrow \int_{\substack{C \\ \circlearrowleft}} = -\int_\gamma \longrightarrow f(a)\, 2\pi i$$

get $\displaystyle f(a) = \int_{C' \circlearrowleft} + \int_{C \circlearrowright}$

Then $f(a) = \frac{1}{2\pi i}\int_{\substack{C \\ \circlearrowleft}} \frac{f(z)}{z-a}\,dz - \frac{1}{2\pi i}\int_{\substack{C' \\ \circlearrowleft}} \frac{f(z)}{z-a}\,dz$

$$\lim_{h\to 0} \frac{1}{2\pi i h}\left\{ \int_C \frac{f(z)\left(z-a-(z-a-h)\right)}{(z-a)(z-a-h)}\,dz \right.$$

$$= \lim_{h\to 0} \frac{1}{2\pi i \cancel{h}}\cancel{h} \int_C \frac{f(z)}{(z-a)(z-a-h)}\,dz = \lim_{h\to 0} \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-a)(z-a-h)}$$

Now

$$\frac{1}{(z-a)(z-a-h)} - \frac{1}{(z-a)^2} = \frac{(z-a)-(z-a-h)}{(z-a)^2(z-a-h)}$$

$$= \frac{h}{(z-a)^2(z-a-h)}$$

Thus $\frac{1}{(z-a)(z-a-h)} = \frac{1}{(z-a)^2} + \frac{h}{(z-a)^2(z-a-h)}$

$\frac{1}{|z-a-h|}$ But know

$|z-a|-|h| \le |z-a-h|$

$\vdots$

Want to show that if $|h| < \frac{|z-a|}{2}$

Then $\dfrac{1}{|z-a-h|}$ is Bounded.

$$\frac{1}{|z-a-h|} \overset{\leq}{\neq} \frac{1}{|z-a|-|h|} < \frac{1}{|z-a|-\frac{1}{2}|z-a|} = \frac{2}{|z-a|}$$

if $|h| < \frac{1}{2}|z-a|$

$$\therefore \left|\frac{f(z)}{(z-a)^2(z-a-h)}\right| \leq \frac{2|f(z)|}{|z-a|^3}\Bigg|_{\text{on } C} \leq K.$$

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z-a}$$

$$f'(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}(-1)^2 = \checkmark$$

Same as if one took the derivative under the integral sign

<u>pg 90 W&W</u>

$$\frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{h}\left\{\frac{(z-a)^2-(z-a-h)^2}{(z-a)^2(z-a-h)^2}\right\}$$

$$= \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{h}\left\{\frac{\cancel{(z-a)^2} - \cancel{(z-a)^2} + 2h(z-a) - h^2}{(z-a)^2(z-a-h)^2}\right\}$$

$$= \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-a)^2(z-a-h)^2}\, 2\left\{(z-a) - \frac{h}{2}\right)$$

Then

$$\cancel{\frac{1}{(z-a)^3}}\ T$$

$$\frac{2(z-a-h/2)}{(z-a-h)^2(z-a)^2} - \frac{2}{(z-a)^3} = \frac{(z-a-h/2)(z-a) - (z-a-h)^2}{(z-a)^3(z-a-h)^2}$$

÷ by 2

$$= \frac{\cancel{(z-a)^2} - \frac{h}{2}(z-a) - \cancel{(z-a)^2} + 2h(z-a) - h^2}{(\quad)^3(\quad)^2}$$

$$= \frac{\frac{3}{2}h(z-a) - h^2}{(\quad)^3(\quad)^2}$$

$$\therefore \quad \frac{2(z-a-h/2)}{(z-a-h)^2(z-a)^2} = \frac{2}{(z-a)^3} + \frac{3h(z-a) - 2h^2}{(z-a)^3(z-a-h)^2}$$

$$\therefore \quad \frac{1}{2\pi i}\int_C f(z)\,dz\ \frac{2(z-a-h/2)}{(z-a-h)^2(z-a)^2} = \frac{1}{2\pi i}\int_C f(z)\frac{2}{(z-a)^3}\,dz$$

$$+ \underbrace{\frac{1}{2\pi i}\int_C \frac{f(z)(3h(z-a) - 2h^2)}{(z-a)^3(z-a-h)^2}\,dz}_{h\left[\frac{1}{2\pi i}\int_C \frac{f(z)(3(z-a)-2h)}{(z-a)^3(\quad)^2}\,dz\right.}$$

pg 91 w/w

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}} dz$$

Then if C is a circle radius r.

$$|f^{(n)}(a)| \le \cancel{\frac{n!}{2\pi} \int_C \frac{|f(z)|}{r^{n+1}} r \, |d\theta|}$$

$$|f^{(n)}(a)| \le \frac{n!}{2\pi} \int_C \frac{|f(z)|}{|z-a|^{n+1}} |dz|$$

on C $|f(z)| \le M$

& if $z-a = re^{i\theta}$

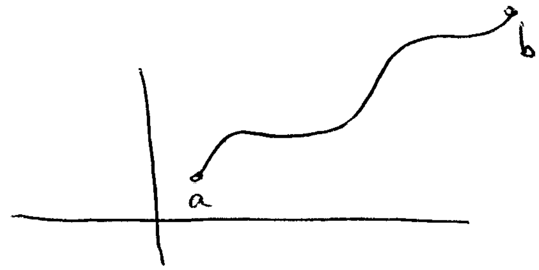
$|z-a| = |r|$

$$= \frac{n!}{2\pi} \frac{M}{r^{n+1}} \int_C |dz| = \frac{M n!}{\cancel{2\pi} r^{n+1}} \cancel{2\pi} r$$

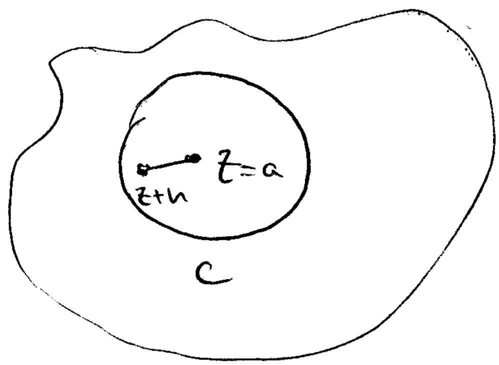
$$= \frac{M}{r^n} n!$$

pg 92 w/w

$$\int_a^b f(t,z)\,dt$$



5.4



$$f(a+h) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a-h}\,dz$$

But $$\frac{1}{z-a-h} = \frac{1}{z-a}\,\frac{1}{1-h/(z-a)}$$

But $$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = \sum_{k=0}^{N} x^k + R_{N+1}(x)$$

$$\Rightarrow R_{N+1}(x) = \frac{1}{1-x} - \sum_{k=0}^{N} x^k = \frac{1}{1-x} - \frac{(x^{N+1}-1)}{x-1}$$

From 1st Day of Class

$$\Delta x^k = x^k(x-1)$$

$$\Rightarrow \sum x^k = \frac{1}{x-1}\sum \Delta x^k = \frac{1}{x-1} x^k \Big|_{N_1}^{N_2+1}$$

$$\therefore \quad R_{N+1}(x) = \frac{1}{1-x} + \frac{x^{N+1}-1}{1-x} = \frac{x^{N+1}}{1-x}$$

$$\therefore \quad \frac{1}{1-x} = \sum_{k=0}^{N} x^k + \frac{x^{N+1}}{1-x}$$

$$\frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z-a-h} = \frac{1}{2\pi i}\int_C \frac{1}{(z-a)} \frac{f(z)\,dz}{1-\left(\frac{h}{z-a}\right)}$$

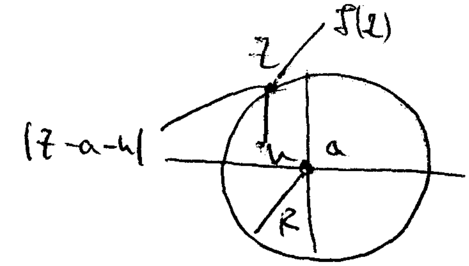
$$= \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\left(\sum_{k=0}^{n} \frac{h^k}{(z-a)^k} + \frac{h^{n+1}}{(z-a)^{n+1}} \frac{1}{1-\left(\frac{h}{z-a}\right)}\right) dz$$

$$= \frac{1}{2\pi i}\int_C f(z)\left(\sum_{k=0}^{n} \frac{h^k}{(z-a)^{k+1}} + \frac{h^{n+1}}{(z-a)^{n+1}(z-a-h)}\right) dz$$

$$= \frac{1}{2\pi i}\int_C f(z)\left(\frac{1}{z-a} + \frac{h}{(z-a)^2} + \frac{h^2}{(z-a)^3} + \cdots + \frac{h^n}{(z-a)^{n+1}} + \frac{h^{n+1}}{(z-a)^{n+1}(z-a-h)}\right) dz$$

$$= f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^n}{n!}f^{(n)}(a)$$

$$+ \frac{1}{2\pi i}\int_C \frac{f(z)h^{n+1}}{(z-a)^{n+1}(z-a-h)}\,dz$$



$\dfrac{|f(z)|}{|z-a-h|}$ is cont.

$$\therefore \left| \frac{1}{2\pi i}\int_C \frac{f(z)\,h^{n+1}}{(z-a)^{n+1}(z-a-h)}\,dz \right| \le \frac{M}{2\pi}\left|\int_C \frac{|dz|\,|h|^{n+1}}{|z-a|^{n+1}}\right|$$

$$\le \frac{M}{\cancel{2\pi}}\left(\frac{h^{n+1}}{R^{n+1}}\right)\cancel{2\pi} R = \frac{Mh^{n+1}}{R^n} \qquad \text{let } n \to \infty$$

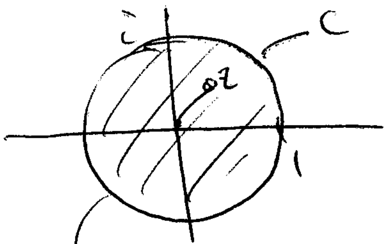
$$\left(\frac{h}{R}\right)^n \to 0.$$

$$z = a + h \to h = z - a$$

$$f(z) = f(a) + (z-a)f'(a) + \cdots$$

Ex 1

pg 94 W/W



$$f(z) = \frac{a - \cos\theta}{a^2 - 2a\cos\theta + 1} + i\,\frac{\sin\theta}{a^2 - 2a\cos\theta + 1}$$ $a > 1$ at points on circle

$$f^{(n)}(0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{z^{n+1}}\,dz$$ let $z = e^{i\theta}$, $dz = ie^{i\theta}\,d\theta$

$$= \frac{n!}{2\pi \cancel{i}}\int_0^{2\pi} \frac{e^{-i(n+1)\theta}\,\cancel{i}\,e^{i\theta}\left((a-\cos\theta) + i\sin\theta\right)}{a^2 - 2a\cos\theta + 1}\,d\theta$$

$$= \frac{n!}{2\pi}\int_0^{2\pi} \frac{e^{-in\theta}\left(a - \frac{1}{2}\cancel{e^{i\theta}} - \frac{1}{2}e^{-i\theta} + \frac{\cancel{i}}{2\cancel{i}}\left(\cancel{e^{i\theta}} - e^{-i\theta}\right)\right)}{a^2 - a(e^{i\theta} + e^{-i\theta}) + 1}\,d\theta$$

w/ $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$ $\sin\theta = \ldots$

$$= \frac{n!}{2\pi}\int_0^{2\pi} \frac{e^{-in\theta}\left(a - e^{-i\theta}\right)}{\underbrace{a^2 - a(e^{i\theta} + e^{-i\theta}) + 1}_{(a - e^{i\theta})(a - e^{-i\theta})}}\,d\theta$$ factor:

$$= \frac{n!}{2\pi} \int_0^{2\pi} \frac{e^{-in\theta}}{a - e^{i\theta}} \, d\theta$$

$$= \frac{n!}{2\pi} \int_0^{2\pi} \frac{e^{-in\theta}}{a - e^{i\theta}} \, d\theta$$

Johns Dents class method

$$= \frac{n!}{2\pi} \frac{1}{a} \int_0^{2\pi} \frac{e^{-in\theta}}{1 - \frac{e^{i\theta}}{a}} \, d\theta$$

$$= \frac{n!}{2\pi} \frac{1}{a} \int_0^{2\pi} e^{-in\theta} \sum_{k=0}^{\infty} \frac{e^{ik\theta}}{a^k} \, d\theta$$

series is uniformly convergent wrt to θ.

By Weierstrass M test

$$= \frac{n!}{2\pi} \frac{1}{a} \sum_{k=0}^{\infty} \frac{1}{a^k} \int_0^{2\pi} e^{i(k-n)\theta} \, d\theta$$

$$\left| \frac{e^{ik\theta}}{a^k} \right| \le \frac{1}{a^k}. \quad \forall \theta.$$

$\|$

0 unless $k = n$ get 2π.

$$= \frac{n!}{\cancel{2\pi}} \frac{\cancel{2\pi}}{a} \frac{1}{a^n} = \frac{n!}{a^{n+1}}$$

Same !!

W/W method

$$= \frac{n!}{2\pi} \int_0^{2\pi} \frac{e^{-in\theta}}{a - e^{i\theta}}\, d\theta = \frac{n!}{2\pi i} \oint_C \frac{z^{-(n+1)}}{a - z}\, dz$$

w/ $z = e^{i\theta}$

$dz = i e^{i\theta}\, d\theta$

$$e^{-i(n+1)\theta} \; e^{i\theta}\, d\theta$$

$$= \frac{n!}{2\pi i} \int_C \frac{\frac{1}{a-z}}{z^{(n+1)}}\, dz$$

$=$

$$= \frac{d^n}{dz^n}\left(\frac{1}{a-z}\right)\Big|_{z=0} = \frac{n!}{a^{n+1}}$$

check

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n = \sum_{n=0}^{\infty} \frac{z^n}{a^{n+1}} = \frac{1}{a}\,\frac{1}{1 - z/a} = \frac{1}{a - z}$$

pg 94 w/w

Ex 2:

$$\sum_{\nu=0}^{\infty} a_\nu z^\nu = f(z) \Rightarrow a_\nu = \frac{f^{(\nu)}(0)}{n!}$$

$$\cancel{a_\nu = \frac{1}{2\pi i}\int_C \frac{f(z)}{z^{n+1}}\,dz} \qquad \cancel{\frac{dz}{z^{n+1}}} \longleftrightarrow$$

Then Arithmetic mean of $z^{-n}f(z)$ is

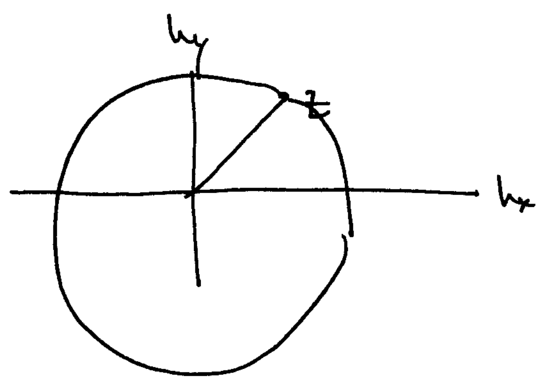
$$\frac{1}{2\pi}\int_0^{2\pi} \frac{f(z)}{z^n}\,d\theta = \underbrace{\frac{1}{2\pi i}\int_C \frac{f(z)}{z^{n+1}}\,dz}_{\frac{f^{(n)}(0)}{n!} \text{ By Cauchy integral Formula.}} = \frac{f^{(n)}(0)}{n!} = a_n$$

Ex 3: Pg 95 w/w

$f(z) = z^r$ Branch pts at $z=0$ & $z=\infty$.

Then $f(z+h) = (z+h)^r$ for fixed z. As a fn of h point of non analyticity is $h=-z$. Thus if $|h|<|z|$

$(z+h)^r$ is analytic



Then by Taylors thm.

$$f(z+h) = (z+h)^r = \sum_{k=0}^{\infty} \frac{h^k}{k!} f^{(k)}(z)$$

Now
$$f^{(1)}(z) = r(z)^{r-1}$$
$$f^{(2)}(z) = r(r-1)z^{r-2}$$
$$f^{(3)}(z) = r(r-1)(r-2)z^{r-3}$$
$$\vdots$$
$$f^{(k)}(z) = r(r-1)(r-2)\cdots(r-k+1)z^{r-k}$$

$$= \sum_{k=0}^{\infty} \frac{h^k}{k!} r(r-1)(r-2)\cdots(r-k+1)z^{r-k}$$

$$= z^r + rz^{r-1}h + \frac{r(r-1)z^{r-2}h^2}{2} + \frac{r(r-1)(r-2)z^{r-3}h^3}{6} + \cdots$$

pg 95 w/w

$$\frac{d}{dt}\left\{\sum_{m=1}^{n-1}\frac{h^m}{m!}(1-t)^m f^{(m)}(a+th)\right\} = \sum_{m=1}^{n-1}\frac{h^m}{m!}\Big(m(1-t)^{m-1}(-1)f^{(m)}(a+th)$$

$$+ (1-t)^m f^{(m+1)}(a+th)h\Big)$$

$$= \sum_{m=1}^{n-1} -\frac{h^m(1-t)^{m-1}}{(m-1)!}f^{(m)}(a+th) + \frac{h^{m+1}(1-t)^m f^{(m+1)}(a+th)}{m!}$$

$$= \sum_{m=1}^{n-1}\left\{\frac{h^{m+1}(1-t)^m f^{(m+1)}(a+th)}{(m!)} - \frac{h^m(1-t)^{m-1}f^{(m)}(a+th)}{(m-1)!}\right\}$$

$$= \sum_{m=1}^{n-1}\Delta_m\left(\frac{h^m(1-t)^{m-1}f^{(m)}(a+th)}{(m-1)!}\right) = \left.\frac{h^m(1-t)^{m-1}f^{(m)}(a+th)}{(m-1)!}\right|_1^n$$

$$= \frac{h^n(1-t)^{n-1}f^{(n)}(a+th)}{(n-1)!} - \cancel{h(1-t)}\,\frac{h(1-t)f^{(1)}(a+th)}{0!}$$

$$= \frac{h^n(1-t)^{n-1}f^{(n)}(a+th)}{(n-1)!} - h(1-t)f'(a+th)$$

$$-\int_0^1 \longrightarrow$$

$$\sum_{m=1}^{n-1} \frac{h^m}{m!}(1-t)^m f^{(m)}(a+th)\Bigg|_0^1 = \frac{h^n}{(n-1)!}\int_0^1 (1-t)^{n-1} f^{(n)}(a+th)\,dt$$

$$-\frac{h}{h} f(a+th)\Big|_0^1$$

$$\Rightarrow -\sum_{m=1}^{n-1} \frac{h^m}{m!} f^{(m)}(a) = \frac{h^n}{(n-1)!}\int_0^1 (1-t)^{n-1} f^{(n)}(a+th)\,dt \quad -\left(f(a+h)-f(a)\right)$$

$$\Rightarrow \quad f(a+h) = f(a) + \sum_{m=1}^{n-1} \frac{h^m}{m!} f^{(m)}(a) + \underbrace{\frac{h^n}{(n-1)!}\int_0^1 (1-t)^{n-1} f^{(n)}(a+th)\,dt}_{R_n}$$

pick $p \le n$

Then

$$R_n = \frac{h^n}{(n-1)!}\int_0^1 (1-t)^{p-1}(1-t)^{n-p} f^{(n)}(a+th)\,dt$$

let

$$L \le \underbrace{\left|(1-t)^{n-p} f^{(n)}(a+th)\right|}_{\text{no absolute value}} \le U \qquad 0 \le t \le 1$$

Then

$$\Rightarrow \quad L\int_0^1 (1-t)^{p-1}\,dt \le \int_0^1 (1-t)^{p-1}(1-t)^{n-p} f^{(n)}(a+th)\,dt \le \int_0^1 U(1-t)^{p-1}\,dt$$

As $(1-t)^n$

It $\exists\ \theta \in [0,1] \ni$

$$\int_0^1 (1-t)^{n-1} f^{(n)}(a+th)\,dt = p^{-1}(1-\theta)^{n-p} f^{(n)}(a+\theta h)$$

$$\therefore \quad R_n = \frac{h^n}{(n-1)!\,p}(1-\theta)^{n-p} f^{(n)}(a+\theta h) \qquad p = \begin{cases} 1 \\ n \end{cases}$$

$$\Rightarrow p = n \qquad R_n = \frac{h^n}{n!} f^{(n)}(a+\theta h)$$

$$p = 1 \qquad R_n = \frac{h^n}{(n-1)!}(1-\theta)^{n-1} f^{(n)}(a+\theta h)$$

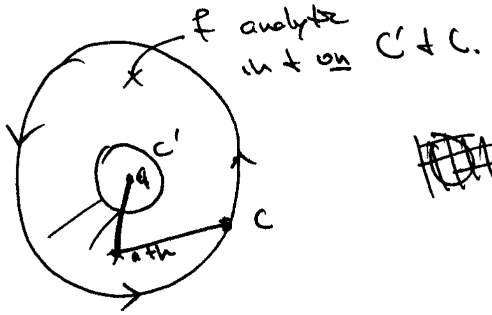
$n = 1$

$$\Rightarrow f(a+h) = f(a) + \begin{cases} h f'(a+\theta h) & \text{Lagrange} \\ h f'(a+\theta h) & \text{Cauchy.} \end{cases}$$

$$\Rightarrow f(a+h) = f(a) + h f'(a+\theta h) \qquad 0 \le \theta \le 1$$

$$\Rightarrow f(x) = f(x_0) + (x-x_0) f'(\xi)$$

ξ between x_0 & x.



$$f(a+h) = \frac{1}{2\pi i}\int_{C} \frac{f(z)}{z-a-h}\,dz - \frac{1}{2\pi i}\int_{C'} \frac{f(z)}{z-a-h}\,dz$$

Now As in Pf of Taylors thm pg ... we know

$$\frac{1}{1-x} = \sum_{k=0}^{N} x^k + \frac{x^{N+1}}{1-x}.$$ pg 92 Notes pg 2 $\forall x$

Thus

$$\frac{1}{z-a-h} = \frac{1}{(z-a)}\frac{1}{\left(1-\frac{h}{z-a}\right)} = \frac{1}{z-a}\left(\sum_{k=0}^{n}\frac{h^k}{(z-a)^k} + \frac{\left(\frac{h}{z-a}\right)^{n+1}}{1-\left(\frac{h}{z-a}\right)}\right)$$

on C Then Done in this way As

$$\left|\frac{h}{z-a}\right| < 1 \Leftrightarrow |h| \le \underbrace{|z-a|}_{\text{distance from center to outer circle}}$$

Must be large enough to put a+h into annulus

How know $\left|\frac{h}{z-a}\right| \overset{?}{<} 1$

$$\Rightarrow \frac{1}{z-a-h} = \sum_{k=0}^{n} \frac{h^k}{(z-a)^{k+1}} + \frac{1}{(z-a-h)}\,\frac{h^{n+1}}{(z-a)^{n+1}}$$

For

$$\frac{1}{z-a-h} = -\frac{1}{h}\,\frac{1}{\left(1-\frac{(z-a)}{h}\right)}$$

on C'

on C' $\quad \frac{|z-a|}{|h|} < 1$

$$|z-a| < |h|$$

Then

$$= -\frac{1}{h}\left(\sum_{k=0}^{n} \frac{(z-a)^k}{h^k} + \frac{\frac{(z-a)^{n+1}}{h^{n+1}}}{1-\frac{(z-a)}{h}}\right)$$

$$= -\left(\sum_{k=0}^{n} \frac{(z-a)^k}{h^{k+1}} + \frac{(z-a)^{n+1}}{(h-(z-a))h^{n+1}}\right)$$

Then

$$f(a+h) = \frac{1}{2\pi i}\int_C f(z)\left\{\frac{1}{(z-a)} + \frac{h}{(z-a)^2} + \frac{h^2}{(z-a)^3} + \cdots + \frac{h^n}{(z-a)^{n+1}}\right\} + \frac{h^{n+1}}{(z-a)^{n+1}(z-a-h)}\Bigg\}\,dz$$

note sign

$$+ \frac{1}{2\pi i}\int_{C'} f(z)\left\{\frac{1}{h} + \frac{z-a}{h^2} + \frac{(z-a)^2}{h^3} + \cdots + \frac{(z-a)^n}{h^{n+1}} \overset{-}{+} \frac{(z-a)^{n+1}}{h^{n+1}(z-a-h)}\right\}dz$$

Now:

$$h^{n+1}\left|\int_C \frac{f(z)\,dz}{(z-a)^{n+1}(z-a-h)}\right| \le \frac{h^{n+1}M}{R^{n+1}}\,2\pi R \xrightarrow{n\to\infty} 0$$

Now $\left|\dfrac{f(z)}{z-a-h}\right|$ is cont on C so $\le M$

& $\left|\dfrac{f(z)}{z-a+h}\right| \le M$

$$\Rightarrow \quad f(a+h) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a} + \frac{h}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}\,dz + \cdots + \frac{h^n}{2\pi i}\int_C \frac{f(z)}{(z-a)^n} + \cdots$$

$$\frac{1}{h}\frac{1}{2\pi i}\int_{C'} f(z)\,dz + \frac{1}{h^2}\frac{1}{2\pi i}\int_{C'} f(z)(z-a)\,dz + \cdots$$

$$= a_0 + a_1 h + a_2 h^2 + \cdots$$

$$+ \frac{b_1}{h} + \frac{b_2}{h^2} + \cdots$$

w/ $a_n = \dfrac{1}{2\pi i}\displaystyle\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz$ & $b_n = \dfrac{1}{2\pi i}\displaystyle\int_{C'} f(z)(z-a)^{n-1}\,dz$

WTS: $J_n(x) = \frac{1}{2\pi}\int_0^{2\pi} \cos(n\theta - x\sin\theta)\,d\theta$ pg 100 w/w

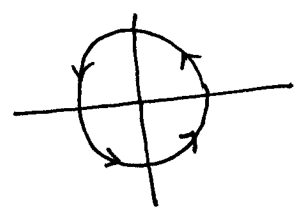
1

$e^{\frac{x}{2}(z - 1/z)}$ is analytic $\forall\, z \neq 0$ & $z \neq \infty$

Then in this annulus Laurent's theorem gives

$$e^{x/2(z - 1/z)} = \sum_{k=0}^{\infty} a_k z^k + \sum_{k=1}^{\infty} \frac{b_k}{z^k} = \sum_{k=-\infty}^{\infty} a_k z^k$$

w/ $a_n = \frac{1}{2\pi i}\int_{C_t} \frac{f(t)}{(t-0)^{n+1}}\,dt$ ✓



Then taking C to be the unit circle $t = e^{i\theta}$ $dt = ie^{i\theta}d\theta$

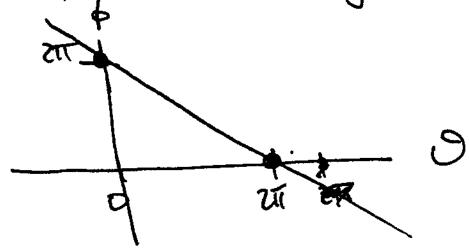
gives

$$a_n = \frac{1}{2\pi i}\int_0^{2\pi} \frac{e^{\frac{x}{2}(e^{i\theta} - e^{-i\theta})}}{e^{i\theta(n+1)}}\, i e^{i\theta}\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta} e^{\frac{x}{2}(2i\sin\theta)}\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} e^{i(x\sin\theta - n\theta)}\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cos(x\sin\theta - n\theta)\, d\theta + \frac{i}{2\pi}\int_0^{2\pi} \sin(x\sin\theta - n\theta)\, d\theta$$

following hint let $\theta = 2\pi - \phi$, $d\theta = -d\phi$. Then 2nd integral becomes



$$= -\int_{\phi=2\pi}^{0} \sin(x\sin(2\pi-\phi) - n(2\pi-\phi))\, d\phi$$

$$= -\int_{2\pi}^{0} \sin(x\sin(2\pi-\phi) - 2\pi n + n\phi)\, d\phi$$

$\sin(2\pi-\phi) = -\sin\phi$

$\hookrightarrow \sin 2\pi\cos\phi - \cos 2\pi \sin\phi = -\sin\phi$ ✓.

$$= -\int_0^{2\pi} \sin(x\sin\phi - n\phi)\, d\phi \qquad \left(\therefore\ I = \int_0^{2\pi} \sin(x\sin\theta - n\theta)\, d\theta \right.$$

$$= -\int_0^{2\pi} \sin(x\sin\phi - n\phi)\, d\phi = -I \Big)$$

$\rightarrow 2I = 0 \rightarrow I = 0$ ✓.)

$$f(z) = e^{\frac{x}{2}(z - z^{-1})} \longrightarrow e^{\frac{x}{2}(-z^{-1} + z)} = f(z)$$

$\therefore \quad f(z) = f(-z^{-1})$

$\quad\quad \parallel$

$a_0 + a_1 z + \cdots +$

$$\sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \frac{b_n}{z^n} = f(z) \quad \text{Then}$$

$$f(-z^{-1}) = \sum_{n=0}^{\infty} \frac{a_n(-1)^n}{z^n} + \sum_{n=1}^{\infty} b_n(-1)^n z^n = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^{-n} = f(z)$$

$$\Rightarrow \quad a_0 + \sum_{n=1}^{\infty}$$

$\Rightarrow$ Match powers of z.

$$a_n(-1)^n = b_n \quad \Big| \quad b_n(-1)^n = a_n$$

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pg 103

~~**Ex 2:**~~ $\cancel{f(z) = \frac{e^{\frac{c}{z-a}}}{(e^{z/a}-1)} =: \frac{h(z)}{g(z)}}$

$\cancel{g(z) = e^{z/a} - 1}$ ~~But~~ $\cancel{g(z) = 0}$

$\cancel{g'(z) = \frac{e^{z/a}}{a} \neq 0 \quad \forall z \in \mathbb{C}}$ $\cancel{z = 2\pi n i a \quad n = 0, \pm 1, \pm 2, \cdots}$

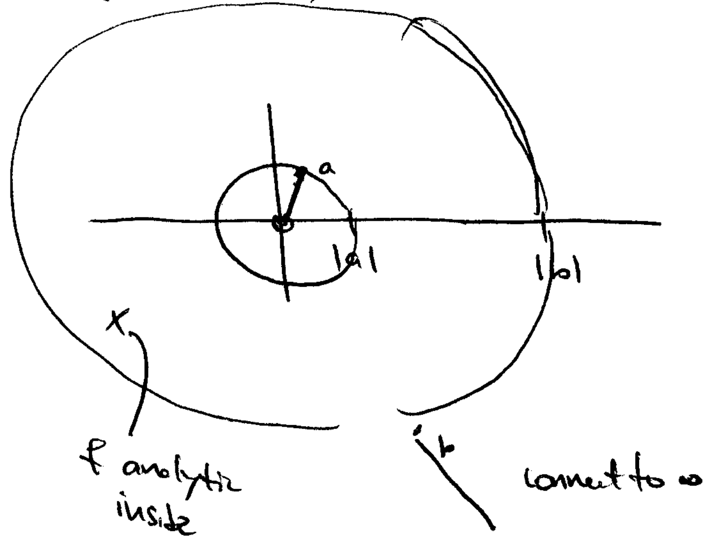
~~Then these are all simple poles.~~

pg 101 w/w

Ex 2: branch pts for $\left(\frac{bz}{(z-a)(b-z)}\right)^{1/2}$ are

$z=0,\ z=a,\ z=b$

$z=\infty.$



$$\frac{1}{2\pi i}\int_C \frac{dz}{z^{n+1}}\left\{\frac{bz}{(z-a)(b-z)}\right\}^{1/2}$$

$z = re^{i\theta}$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{re^{i\theta}\,d\theta}{r^{n+1}e^{i(n+1)\theta}}\left\{\frac{bre^{i\theta}}{(re^{i\theta}-a)(b-re^{i\theta})}\right\}^{1/2}$$

inside factor out a $bre^{i\theta}$

$$= \frac{1}{2\pi}\int_0^{2\pi} r^{-n}e^{-ni\theta}\,d\theta\ \Bigg\{\underbrace{\frac{\cancel{bre^{i\theta}}}{(re^{i\theta}-a)\cancel{bre^{i\theta}}(r^{-1}e^{-i\theta}-b^{-1})}}_{\displaystyle \underbrace{\frac{1}{re^{i\theta}\left(1-\frac{a}{r}e^{-i\theta}\right)\left(r^{-1}e^{-i\theta}-b^{-1}\right)}}_{\displaystyle \frac{1}{\left(1-\frac{a}{r}e^{-i\theta}\right)\left(1-\frac{r}{b}e^{i\theta}\right)}}}\Bigg\}^{1/2}$$

mult $re^{i\theta}$ through 2nd term

$$= \frac{1}{2\pi} \int_0^{2\pi} r^{-n} e^{-ni\theta} \underbrace{\left(1 - \frac{r}{b} e^{i\theta}\right)^{-1/2}} \underbrace{\left(1 - \frac{a}{r} e^{-i\theta}\right)^{-1/2}} \, d\theta$$

From taylor series

Ex 3 Show

pg 102 w/w

$$f(z) = e^{uz + v/z} = a_0 + a_1 z + a_2 z^2 + \cdots + \frac{b_1}{z} + \frac{b_2}{z^2} + \cdots \qquad u, v, \text{ parameters,}$$

$$= \sum_{k=0}^{\infty} a_k z^k + \sum_{k=1}^{\infty} \frac{b_k}{z^k}$$

w/ $a_n = \frac{1}{2\pi}\int_0^{2\pi} e^{(u+v)\cos\theta} \cos\{(u-v)\sin\theta - n\theta\} d\theta$

$b_n = \frac{1}{2\pi}\int_0^{2\pi} e^{(u+v)\cos\theta} \cos\{(v-u)\sin\theta - n\theta\} d\theta$

Then this is a Laurent expansion.

$f(z)$ is analytic $\forall\, z \neq 0, \infty$ $\therefore$ Then by Laurent's thm

$$a_n = \frac{1}{2\pi i}\int_C e^{(uz + v/z)} \frac{dz}{z^{n+1}} \qquad b_n = \frac{1}{2\pi i}\int_C e^{(uz + v/z)} z^{n-1}\, dz$$

pick $C =$ unit circle. $z = e^{i\theta}$ $\quad 0 \leq \theta < 2\pi$

$dz = ie^{i\theta} d\theta$

$$a_n = \frac{1}{2\pi \cancel{i}} \int_0^{2\pi} e^{(ue^{i\theta} + ve^{-i\theta})} e^{-i(n+1)\theta} \cancel{ie^{i\theta}}\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{(ue^{i\theta} + ve^{-i\theta})} e^{-in\theta}\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \exp\{u\cos\theta + iu\sin\theta + v\cos\theta - iv\sin\theta - in\theta\}\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \exp\{(u+v)\cos\theta + i(u\sin\theta - v\sin\theta - n\theta)\}\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{(u+v)\cos\theta}\left[\cos\left((u-v)\sin\theta - n\theta\right) + i\sin\left((u-v)\sin\theta - n\theta\right)\right] d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{(u+v)\cos\theta} \cos\left\{(u-v)\sin\theta - n\theta\right\} d\theta$$

$$+ \frac{i}{2\pi}\int_0^{2\pi} e^{(u+v)\cos\theta} \sin\left\{(u-v)\sin\theta - n\theta\right\} d\theta$$

(If 2nd integral is zero then we will have shown the result for a_n)

consider the second integral let $I = \int_0^{2\pi} e^{(u+v)\cos\theta} \sin\left\{(u-v)\sin\theta - n\theta\right\} d\theta$

let $\phi = 2\pi - \theta \qquad \theta = 2\pi - \phi$

$$\cos(\theta) = \cos\phi$$
$$\sin\theta = \sin(2\pi - \phi) = \sin 2\pi\cos\phi - \sin\phi\cos 2\pi = -\sin\phi$$

$$I = \int_{2\pi}^{0} e^{(u+v)\cos\phi} \sin\left\{(u-v)\sin\phi - n2\pi + n\phi\right\} d\phi$$

$$= -\int_0^{2\pi} e^{(u+v)\cos\phi} \sin\left\{(v-u)\sin\phi - n\phi\right\} d\phi = -I$$

$\Rightarrow I = 0$ thus we have the expression!

For b_n, Use a trick suggested by Ex 1.

$z \to \dfrac{V}{Uz}$ Then $Uz + \dfrac{V}{z} = \dfrac{V}{z} + \dfrac{V}{\frac{V}{Uz}} = \dfrac{V}{z} + Uz$ ✓.

This is the fn is unchang if we let $z \to \frac{v}{uz}$ thus

$$f(z) = f\left(\frac{v}{uz}\right)$$

$$\underbrace{\sum_{k=0}^{\infty} a_k z^k}_{} + \underbrace{\sum_{k=1}^{\infty} \frac{b_k}{z^k}}_{} = \underbrace{\sum_{k=0}^{\infty} a_k \frac{v^k}{u^k z^k}}_{} + \underbrace{\sum_{k=1}^{\infty} \frac{b_k u^k z^k}{v^k}}_{}$$

Thus

$$a_k = \frac{u^k}{v^k} b_k \quad + \quad b_k = \frac{v^k}{u^k} a_k$$

$$\therefore \quad b_k = \frac{v^k}{u^k} \frac{1}{2\pi} \int_0^{2\pi}$$

doesn't look like expression given

$$b_n = \frac{1}{2\pi \cancel{i}} \int_{C,\, 0 \le \theta \le 2\pi} e^{(uz + v/z)} e^{i(n-1)\theta} \cancel{i e^{i\theta}}\, d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \exp\left\{ u e^{i\theta} + v e^{-i\theta} + in\theta \right\} d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \exp\left\{ (u+v)\cos\theta + i \left\{ (u-v)\sin\theta \underset{\text{plus}}{+} n\theta \right\} \right\} d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} e^{(u+v)\cos\theta} \cos\left\{ (u-v)\sin\theta + n\theta \right\} + \frac{i}{2\pi} \int_0^{2\pi} e^{(u+v)\cos\theta} \sin\left\{ (u-v)\sin\theta + n\theta \right\} d\theta$$

Following the same ~~xxx~~ procedure as above we get the imag. part to also vanish.

pg 103 w/w

Ex 2: $f(z) = \dfrac{e^{1/z-a}}{(e^{z/a}-1)} =: \dfrac{h(z)}{g(z)}$

in $g(z) = e^{z/a} - 1$ But $g(z) = 0 \rightarrow z = 2\pi n i a$

$g'(z) = \dfrac{e^{z/a}}{a} \neq 0 \;\forall\; z \in \mathbb{C}$. $n = 0, \pm 1, \pm 2, \ldots$

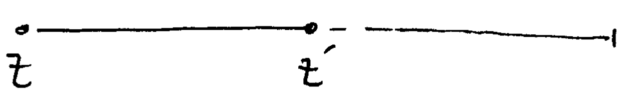
Then these are all single poles.

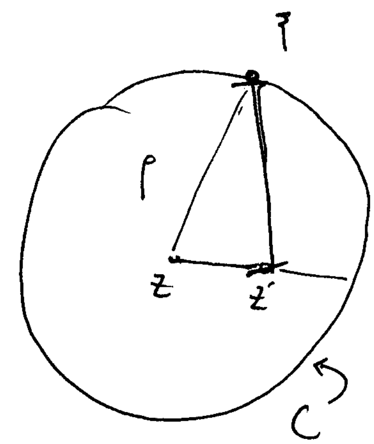
pg 105 w/w

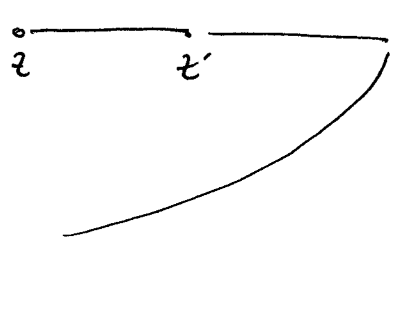
$$\cancel{f(z) - f(z') = \frac{1}{2\pi i}\int_C \left(\frac{f(\xi)}{\xi - z} - \frac{f(\xi)}{\xi - z'}\right) d\xi}$$

$$\cancel{= \frac{1}{2\pi i}\int_C \left\{\frac{1}{\xi - 1}\right.}$$

$$f(z') - f(z) = \frac{1}{2\pi i}\int_C \left(\frac{f(\xi)}{\xi - z'} - \frac{f(\xi)}{\xi - z}\right) d\xi = \frac{1}{2\pi i}\int_C \left\{\frac{1}{\xi - z'} - \frac{1}{\xi - z}\right\} f(\xi)\, d\xi$$







$$\xi = z + \rho e^{i\theta}$$

$$|\xi - z'| \le |z - z'| + \underbrace{|z - \xi|}_{\rho}$$

$$\Rightarrow \quad |\xi - z'| \ge \big|\,|z - \xi| - |z - z'|\,\big|$$

$$\ge |\rho - \rho/2| = \rho/2.$$

$$|f(z')-f(z)| = \frac{1}{2\pi}\left|\int_C \frac{\xi - z - \xi + z'}{(\xi-z')(\xi-z)} f(\xi)\, d\xi\right|$$

$$= \frac{1}{2\pi}\left|\int_C \frac{z'-z}{(\xi-z')(\xi-z)} f(\xi)\, d\xi\right|$$

$$|\xi - z| = \rho$$

$$\leq \frac{1}{2\pi} \frac{M\,|z'-z|}{(\rho/2)\rho}\int_C |d\xi|$$

w/ $\boxed{\left|\frac{f(\xi)}{\cancel{(\xi-z)}}\right| \leq M \quad \xi \text{ on } C}$

$$\leq \frac{1}{\cancel{2\pi}} \frac{M\,|z'-z|}{\rho^2/2}\, \cancel{2\pi}\cancel{\rho} = 2M\,|z'-z|\,\rho^{-1}$$

$$|f(z)-f(z')| \leq 2M\,|z-z'|\,\rho^{-1} \longrightarrow 0$$

Pg105 W/W

1

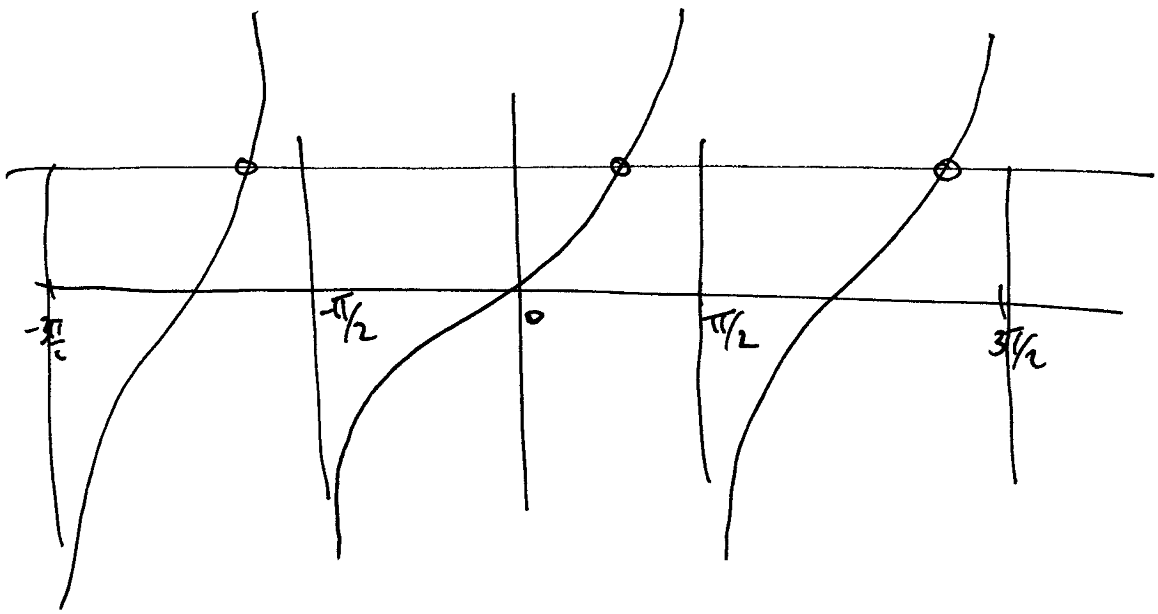
Bounded Because we have removed the part that grows

$a_1 z + a_2 z^2 + \dots + a_n z^n$ or the

Pg 106 w/w

$$\cos\left(\frac{\theta+2\pi}{2}\right) = \cos\left(\frac{\theta}{2}+\pi\right) = -\cos\frac{\theta}{2}$$

$$\sin\frac{1}{2}(\theta+2\pi) = \sin\left(\frac{\theta}{2}+\pi\right) = -\sin\frac{\theta}{2}$$



<u>pg 108 w/w</u>

(1) obtain

$$f(z) = f(a) + 2\left\{ \frac{z-a}{2} f'\left(\frac{z+a}{2}\right) + \frac{(z-a)^3}{2^3\, 3!} f'''\left(\frac{z+a}{2}\right) + \frac{(z-a)^5}{2^5\, 5!} f^{(v)}\left(\frac{z+a}{2}\right) + \dots \right\}$$

$$= f(a) + 2 \sum_{k=0}^{\infty} \frac{(z-a)^{2k+1}}{2^{2k+1}\,(2k+1)!} f^{(2k+1)}\left(\frac{z-a}{2}\right)$$

consider ~~Exacts~~ $\Leftrightarrow$ $$\frac{f(z) - f(a)}{2} = \sum_{k=0}^{\infty} \frac{(z-a)^{2k+1}}{(2k+1)!\; 2^{2k+1}} \underbrace{f^{(2k+1)}\left(\frac{z+a}{2}\right)}$$

$$\frac{f(z) - f(a)}{2} = \frac{1}{4\pi i} \int_C$$

Thoughts

In problem 2 let $m=1$ to get

$$f(z) = f(a) + (z-a)\left\{ f'\left(a + \underbrace{\frac{z-a}{2}}_{\frac{z+a}{2}}\right) + f'\left(\right.\right.$$

Note sum on the Right is odd in z.

+ The sum has no integral remainder term thus

This term must go to zero.

~~$\frac{z-a}{2} = \eta \Rightarrow z = a + 2\eta$~~

let $h = \frac{z-a}{2} \Rightarrow z = a + 2h.$

$$f(a+2h) = f(a) + 2\sum_{k=0}^{\infty} h^{2k+1} f^{(2k+1)}\cancel{(h)}$$

To ~~attempt this I will proceed as in the pf of taylors thm.~~

~~(Although an could prob. get this~~

$$\frac{z+a}{2} = \frac{2a+2h}{2} = \underline{a+h}.$$

$\therefore$

$$f(a+2h) = f(a) + 2\sum_{k=0}^{\infty} h^{2k+1} f^{(2k+1)}(a+h)$$

Thus I'll attempt to get this As in the pf of Taylor

$$f(a+2h) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a-2h}\,dz = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z-a-h-h} =$$

$$\frac{1}{z-(a+h)-h}$$

want powers of $\frac{1}{(z-(a+h))^i}$

Now

$$\frac{1}{z-(a+h)-h} = \frac{1}{z-(a+h)}\,\frac{1}{1-\frac{h}{z-(a+h)}}$$

$$= \frac{1}{z-(a+h)}$$

Now: From before. $\frac{1}{1-x} = \sum_{k=0}^{N} x^k + \frac{x^{N+1}}{1-x}$. Exact.

$$\therefore \frac{1}{1-\left(\frac{h}{z-(a+h)}\right)} = \sum_{k=0}^{N} \frac{h^k}{(z-(a+h))^k} + \frac{\frac{h^{N+1}}{(z-(a+h))^{N+1}}}{\left(1-\frac{h}{z-(a+h)}\right)}$$

$$= \sum_{k=0}^{N} \frac{h^k}{(z-(a+h))^k} + \cancel{\frac{h^{N+1}}{(z-(a+h))^{N+1}}\frac{1}{(z-(a+h))(z-(a+h)-h)}}$$

Thus:

$$f(a+2h) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-(a+h))}\left(\sum_{k=0}^{N}\frac{h^k}{(z-(a+h))^k} + \frac{h^{N+1}}{1-\frac{h}{z-(a+h)}}\right)dz$$

$$= \frac{1}{2\pi i}\int_C f(z)\left(\sum_{k=0}^{N}\frac{h^k}{(z-(a+h))^{k+1}} + \frac{h^{N+1}}{z-(a+h)-h}\right)dz$$

$$= \sum_{k=0}^{N} h^k \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-(a+h))^{k+1}}\,dz + h^{N+1}\frac{1}{2\pi i}\int \frac{f(z)\,dz}{z-(a+2h)}$$

But $\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-(a+h))^{k+1}}\,dz = \frac{1}{k!}f^{(k)}(a+h)$

Pg 108 w/w

(5)

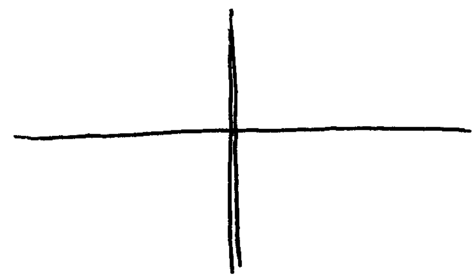
$$(1-z^2)^{-1/2} = 1 + \frac{1}{2}z^2 + \frac{1\cdot 3}{2\cdot 4}z^4 + \cdots + \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}z^{2n}$$

$$+ \frac{3\cdot 5\cdots(2n+1)}{2\cdot 4\cdots(2n)}(1-z^2)^{-1/2}\int_0^z t^{2n+1}(1-t^2)^{-1/2}\,dt$$

This looks like the taylor series of $\frac{1}{\sqrt{1-z^2}}$

z^2-1

$1-z^2<$ About the point $z=0$.



Pg 105 W/W

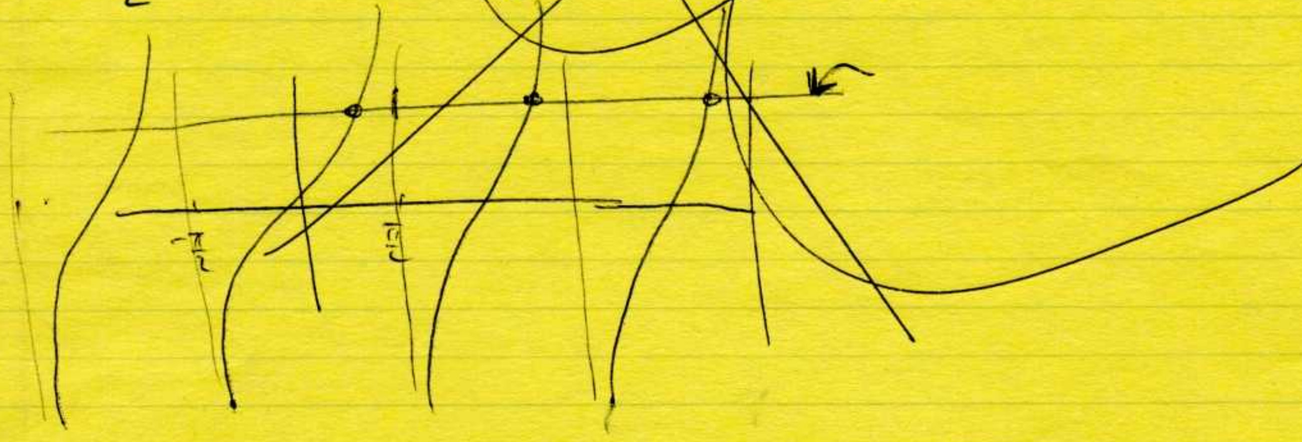
bounded because we have removed the part that grows i.e.

$a_1 z + a_2 z^2 + \dots + a_n z^n$ or the

Pg 106 W/W

$$\cos \tfrac{1}{2}(\theta + 2\pi) = \cos\left(\frac{\theta}{2} + \pi\right) = -\cos\frac{\theta}{2}$$

$$\sin \tfrac{1}{2}(\theta + 2\pi) = \sin\left(\frac{\theta}{2} + \pi\right) = -\sin\frac{\theta}{2}$$



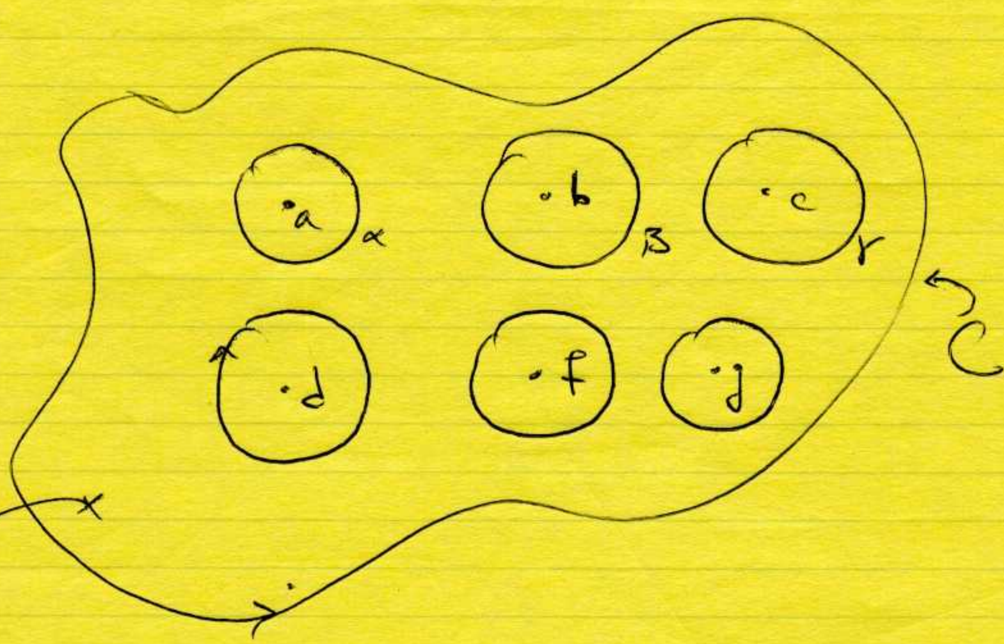
Pg 111 W/W

$$\int_\alpha \frac{dz}{(z-a)^r} \qquad z - a = \rho e^{i\theta} \qquad dz = \rho i e^{i\theta}\, d\theta$$

$$= \int_0^{2\pi} \frac{\rho i e^{i\theta} d\theta}{\rho^r e^{ri\theta}} = \rho^{-r+1} i \int_0^{2\pi} e^{i\theta(1-r)} d\theta = \rho^{1-r} \frac{e^{i\theta(1-r)}}{(1-r)} \Big|_0^{2\pi}$$

$r \neq 1 \Rightarrow = 0.$

$$\int_{\alpha}^{r=1} \frac{dz}{z-a} = i\int_0^{2\pi} d\theta = 2\pi i$$



etc

f analytic

Then

$$\int_{C\,\circlearrowleft} + \sum_{\text{circles}} \int_{\circlearrowright} = 0$$

$$\Rightarrow \int_{C\,\circlearrowleft} - \sum_{\text{circles}} \int_{\circlearrowleft} = 0 \Rightarrow \int_C = \sum_{\text{circles}} \int$$

<u>Ex 1</u>:

$$\int_0^{2\pi} \frac{d\theta}{1-2p\cos\theta+p^2} = \oint_C$$

let $e^{i\theta} = z$

$ie^{i\theta}d\theta = dz \Rightarrow d\theta = -ie^{-i\theta}dz$

$\Rightarrow d\theta = -iz^{-1}dz$

$$\int_C \frac{-iz^{-1}\,dz}{1-\frac{2p}{2}(z+z^{-1})+p^2} = -i\int_C \frac{dz}{z-p(z^2+1)+zp^2}$$

$$= \frac{-1}{i}\int_C \frac{dz}{pz^2+p-z-zp^2} = \frac{-1}{i}\int_C \frac{dz}{pz^2-(1+p^2)z+p}$$

$$= \frac{-1}{i}\int_C \frac{dz}{(pz-1)(z-p)}$$

Points where $\frac{1}{(pz-1)(z-p)}$ is not analytic

$z = 1/p \qquad z = p \qquad$ let $0<p<1$

Then the only one in the unit circle is $z=p$.

+ this is a simple pole

$$\therefore \int_C \frac{dz}{(\ )(\ )} = 2\pi i \sum \text{Residue}$$

$$= 2\pi i \lim_{z\to p} \cancel{(z-p)} \frac{1}{(pz-1)\cancel{(z-p)}} = \frac{2\pi i}{p^2-1}$$

$\Rightarrow$ integral is $\frac{2\pi}{1-p^2}$

Ex 2: $0<p<1$

$$\int_0^{2\pi} \frac{\cos^2 3\theta}{1-2p\cos 2\theta+p^2}\,d\theta$$

Thought: This looks different but remember $\cos n\theta$ can always be written as powers of $\cos\theta$, $\sin\theta$

Again let $z=e^{i\theta}$

Now:

$$\cos 3\theta = \frac{e^{3i\theta}+e^{-3i\theta}}{2} = \frac{z^3+z^{-3}}{2}$$ Sim for $\cos 2\theta$

Then Also $d\theta = -i\frac{dz}{z}$

$$= \int_C \frac{1}{2^2}\frac{(z^3+z^{-3})^2}{1-\frac{2p}{2}(z^2+z^{-2})+p^2}\left(-i\frac{dz}{z}\right) =$$

$$= \frac{-i}{4}\int_C \frac{z^{-6}(z^6+1)^2}{z^{-2}\left(z^2(1+p^2)-p(z^4+1)\right)}\frac{dz}{z} = \frac{+i}{4}\int_C \frac{(1+z^6)^2\,dz}{z^5(pz^4-(1+p^2)z^2+p)}$$

$$= \frac{i}{4}\int_C \frac{(1+z^6)^2\,dz}{z^5(pz^2-1)(z^2-p)}$$

~~Residues~~ Singularities of f in unit circle are $z=\pm\sqrt{p}$ & 0.
Both in

& $z^2 = 1/p$ (No)) ~~But 0 is~~

~~Result~~ $= \frac{i}{4} 2\pi i \sum$ Res in unit $\bigcirc$

$$\lim_{z\to-\sqrt{p}} \frac{\cancel{(z+\sqrt{p})}(1+z^6)^2}{z^5(pz^2-1)(z-\sqrt{p})\cancel{(z+\sqrt{p})}} = \frac{(1+p^3)^2}{-p^{5/2}(p^2-1)(-2\sqrt{p})}$$

$$= \frac{+(p^3+1)^2}{2(1-p^2)p^3} = \frac{(p^3+1)^2}{2p^3(1-p^2)}$$

For $\lim_{z\to+\sqrt{p}} \frac{\cancel{(z-\sqrt{p})}(1+z^6)^2}{z^5(pz^2-1)\cancel{(z-\sqrt{p})}(z+\sqrt{p})} = \frac{(p^3+1)^2}{p^{5/2}(2\sqrt{p})(p^2-1)}$

$$= \frac{-(1+p^3)^2}{2p^3(1-p^2)}$$

Initially one would think that $z=0$ is a pole of order 5 + thus not important on ~~eq~~ evaluating the integral. but

expanding the top gives.

$1 + 2z^6 + z^{12}$ $\Rightarrow$ 0 is has a pole of order 1.

$\Rightarrow$ ~~$\lim_{z\to0} \frac{z(1+z^6)^2}{z^5(pz^2-1)(z^2-p)} = \frac{1}{(-1)(-p)} \lim_{z\to0} \frac{(1+z^6)^2}{z^4}$~~

$=$ Now

$$\frac{(1+z^6)^2}{z^5(pz^2-1)(z^2-p)} = \frac{1+2z^6+z^{12}}{z^5(pz^2-1)(z^2-p)} = \underbrace{\frac{1}{z^5(pz^2-1)(z^2-p)}}_{\text{integral is zero.}} + \underbrace{\frac{2}{z(\ )(\ )}}_{\text{Has non zero pole}} + \underbrace{\frac{z^7}{(\ )(\ )}}_{\text{int is zero}}$$

pg 112 W/w

∴ Thus Residue of O is

$$\lim_{z\to 0} \frac{\cancel{z}\,\cancel{2}}{\cancel{z}\cancel{(pz^2-1)(z^2-p)}} = \cancel{\frac{2}{(-1)(-p)}} = \frac{2}{p}. \quad \text{(wrong)}$$

$$\frac{(1+z^6)^2}{z^5(pz^2-1)(z^2-p)} \sim \text{~~single pole?~~} \quad \text{No.} \quad \underline{1/0.}$$

How tell?

$$\lim_{z\to 0} \frac{z(1+z^6)^2}{\cancel{z}^5(pz^2-1)(z^2-p)} = \lim_{z\to 0} \boxed{} + \boxed{\tfrac{1}{z}} + \boxed{}$$

Single pole ∃ $f(z) = \dfrac{\phi(z)}{z}$ w/ ϕ analytic

~~⇒ $z=0$ single zero of denominator $z^5(pz^2-1)(z^2-p)$~~

This is a pole of order $\underline{5}$ in the expression

$$\frac{(1+z^6)^2}{z^5(pz^2-1)(z^2-p)}$$ The residue is found from pg 1003 Onel

to be

$$\frac{1}{4!}\lim_{z\to 0}\frac{d^4}{dz^4}\,\cancel{z^5}\frac{(1+z^6)^2}{\cancel{z^5}(pz^2-1)(z^2-p)}$$

$$= \frac{1}{24}\lim_{z\to 0}\frac{d^4}{dz^4}\frac{(1+z^6)^2}{(pz^2-1)(z^2-p)}$$

~~Assuming b/w correct~~

Ex 3:

$$\int_0^{2\pi} e^{\cos\theta}\cos(n\theta - \sin\theta)\, d\theta$$

let $e^{i\theta} = z$

1st consider $\displaystyle\int_0^{2\pi} e^{\cos\theta}\, e^{i(n\theta - \sin\theta)}\, d\theta$ Then real part is 1st integral & imag pt is 2nd integral.

$$= \int_0^{2\pi} e^{in\theta} e^{\cos\theta - i\sin\theta}\, d\theta = \int_0^{2\pi} (e^{i\theta})^n e^{e^{-i\theta}}\, d\theta$$

let $z = e^{i\theta}$ $\quad dz = ie^{i\theta} d\theta = iz\, d\theta$

$$\Rightarrow d\theta = \frac{dz}{iz}$$

$$= \int_C z^n e^{z^{-1}} \frac{dz}{iz} = \frac{1}{i}\int_C z^{n-1} e^{z^{-1}}\, dz = 2\pi i \sum_{\text{in unit }\odot} \text{Res}\left(\frac{z^{n-1} e^{z^{-1}}}{i}\right)$$

pg 113 w/w

1

$$\int_{-\rho}^{\rho} Q(z)\,dz + \int_{\Gamma} Q(z)\,dz = 2\pi i \sum \text{Res}$$

$$\rightarrow \left| \int_{-\rho}^{\rho} Q(z)\,dz - 2\pi i \sum R \right| = \left| \int_{\Gamma} Q(z)\,dz \right|$$

$z = \rho e^{i\theta}$

$$\left| \int_{\Gamma} Q(z)\,dz \right| = \left| \int_0^{\pi} Q(\rho e^{i\theta}) \rho e^{i\theta} i\,d\theta \right| \le \int_0^{\pi} |Q(\rho e^{i\theta}) \rho| \, |d\theta|$$

$$< \frac{\epsilon}{\pi} \int_0^{\pi} d\theta = \epsilon$$

$$\therefore \lim_{\rho \to \infty} \int_{-\rho}^{\rho} Q(z)\,dz = 2\pi i \sum R$$

Ex 1:

$$\frac{1}{(z^2+1)^3} = \frac{1}{(z-i)^3(z+i)^3}$$ residue of order 3.

$$\frac{1}{2!}\lim_{z\to i}\frac{d^2}{dz^2}\frac{1}{(z+i)^3}\Bigg|_{z=i} = \frac{1}{2}\lim_{z\to i}\frac{d}{dz}\frac{-3}{(z+i)^4}$$

$$= -\frac{3}{2}(-4)\frac{1}{(z+i)^5}\Bigg|_{z=i} = \frac{6}{2^5 \underbrace{i^5}_{i}} = \frac{3}{2^4 i} = \frac{3}{16i}$$

$$\therefore \int_{-\infty}^{\infty}\frac{dx}{(x^2+1)^3} = 2\pi i \sum \text{Res} = 2\pi\left(\frac{3}{16}\right) = \frac{3\pi}{8}$$

~~Ex 2:~~ $\cancel{\int_{-\infty}^{\infty}\frac{x^4}{(a+bx^2)^4}}$

Check conditions for the theorem:

1) ✓ 3) ✓
2) ✓ 4) ✓

Ex 2: $\displaystyle\int_{-\infty}^{\infty}\frac{x^4\,dx}{(a+bx^2)^4}$ check conditions 1) ✓ 3) ✓ 2) ✓ 4) ✓

$$(\sqrt{a}-\sqrt{b}\,ix)(\sqrt{a}+\sqrt{b}\,ix)$$

poles at $x = \pm\frac{\sqrt{a}}{\sqrt{b}\,i} = \mp\frac{\sqrt{a}\,i}{\sqrt{b}}$ take 2nd one.

$$\int_{-\infty}^{\infty} \frac{x^4\,dx}{(a+bx^2)^4} = 2\pi i \sum \text{Res}$$

$x = +\frac{\sqrt{a}}{\sqrt{b}}\,i$ is a pole of order $\underline{4.}$

$$\frac{x^4}{(a+bx^2)^4} = \frac{x^4}{a^4(\underbrace{1+b/a\,x^2}))}$$

$$\hookrightarrow = \frac{x^4}{b^4(a/b+x^2)^4} = \frac{x^4}{b^4\left(i\sqrt{a/b}+x\right)^4\left(-i\sqrt{a/b}+x\right)^4}$$

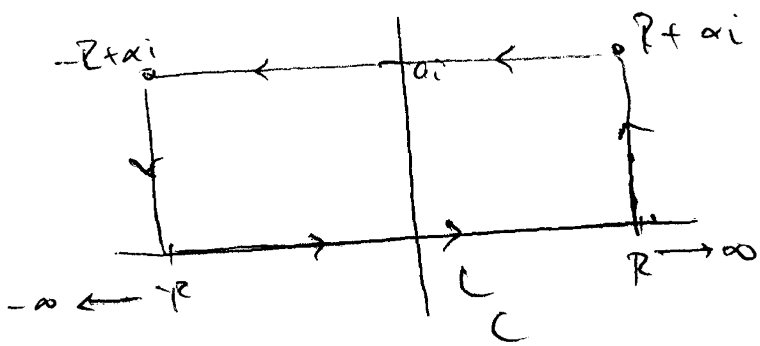
$\therefore$ Residue is

$$\frac{1}{3!}\frac{d^3}{dz^3}\frac{\cancel{(z-i\sqrt{a/b})^4}\; z^4}{\cancel{(z-i\sqrt{a/b})^4}\,(z+i\sqrt{a/b})^4\, b^4}$$

$$= \frac{1}{6b^4}\frac{d^3}{dz^3}\frac{z^4}{(z+i\sqrt{a/b})^4}\Bigg|_{z=+i\sqrt{a/b}}$$

finish on MMA.

Ex 3:



$\int_C e^{-\lambda z^2} dz$ is analytic $\forall z$

$\therefore \int_C e^{-\lambda z^2} dz = 0$

$||$

$$\int_{-R}^{R} e^{-\lambda x^2} dx + \int_{y=0}^{\alpha} e^{-\lambda(R+yi)^2} i\,dy + \int_{R}^{-R} e^{-\lambda(x+\alpha i)^2} dx + \int_{y=\alpha}^{0} e^{-\lambda(-R+iy)^2} i\,dy = 0$$

$z=x$, $dz=dx$ $z=R+iy$, $dz=i\,dy$ $z=x+\alpha i$, $dz=dx$ $z=-R+iy$, $dz=i\,dy$

group ⌴ terms.

$$i\int_{y=0}^{\alpha} e^{-\lambda(R^2+2Ryi-y^2)} dy + -i\int_{y=0}^{\alpha} e^{-\lambda(R^2-2Ryi-y^2)} dy + '' + '' = 0$$

$$ie^{-\lambda R^2}\int_{y=0}^{\alpha} e^{-\lambda(2Ryi-y^2)}$$

$$\int_{-R}^{R} e^{-\lambda x^2}\,dx \quad - \int_{-R}^{R} e^{-\lambda(x^2+2\alpha x i-\alpha^2)}\,dx$$

$$= \int_{-R}^{R} e^{-\lambda x^2}\Big\{1-\underbrace{e^{-\lambda(2\alpha x i-\alpha^2)}}_{e^{\lambda\alpha^2}e^{-2\lambda\alpha x i}}\Big\}\,dx$$

$$e^{\lambda\alpha^2}\cos(2\lambda\alpha x)+ie^{\lambda x^2}\sin(2\lambda\alpha x)$$

Thus At this time my suggestion is to write it as follows.

$$* \quad \underbrace{\int_{-R}^{R} e^{-\lambda(x^2+2x\alpha i-\alpha^2)}\,dx}_{} = \underbrace{\int_{-R}^{R} e^{-\lambda x^2}\,dx}_{\text{known as } \Gamma \text{ fn.}} + i\int_{y=0}^{\alpha} e^{-\lambda(R+iy)^2}\,dy + i\int_{y=\alpha}^{0} e^{-\lambda(-R+iy)^2}\,dy$$

$$e^{\lambda\alpha^2}\int_{-R}^{R} e^{-\lambda x^2}\big(\cos(2\lambda\alpha x)-i\sin(2\lambda\alpha x)\big)\,dx$$

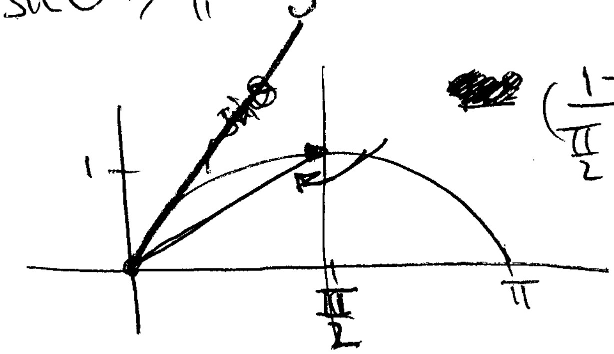
$$= \dots$$

$$= \int_{-R}^{R} e^{-\lambda x^2}\left(\cos(2\lambda\alpha x) - i\sin(2\lambda\alpha x)\right)dx$$

$$= e^{-\lambda\alpha^2}\Big[\underbrace{\qquad\qquad\qquad}_{3 \text{ terms}}$$

$$\left|\int_\Gamma e^{miz} Q(z)\,dz\right| < \int_0^{\pi} \frac{\epsilon}{\pi}\rho e^{-m\rho\sin\theta}\,d\theta = \frac{2\epsilon}{\pi}\int_0^{\pi/2}\rho e^{-m\rho\sin\theta}\,d\theta$$

$$\sin\theta \geq \frac{2}{\pi}\theta, \qquad 0 \le \theta \le \frac{\pi}{2}$$



$$\left(\frac{1-0}{\frac{\pi}{2}-0}\right)(x-0) = y$$

$$\Rightarrow y = \frac{2}{\pi}x$$

$$e^{-m\rho\sin\theta} < e^{-m\rho\frac{2}{\pi}\theta}$$

$$\therefore \left|\int_\Gamma e^{imz} Q(z)\,dz\right| \le \underbrace{\frac{2\epsilon}{\pi}\int_0^{\pi/2}\rho e^{-\frac{2m\rho}{\pi}\theta}\,d\theta}_{\text{I can now do the integral}}$$

$$= \frac{2\epsilon}{\cancel{\pi}}\cancel{\rho}\left(\frac{1}{-\frac{2m\cancel{\rho}}{\cancel{\pi}}}\right)e^{-\frac{2m\rho}{\pi}\theta}\Big|_0^{\pi/2}$$

$$= -\frac{\epsilon}{m}\left[e^{-m\rho}-1\right] = \frac{\epsilon}{m}\left[1-e^{-m\rho}\right] < \frac{\epsilon}{m}$$

$f(\theta)$ $\frac{\sin\theta}{\theta}$

$$f(0) = 1$$

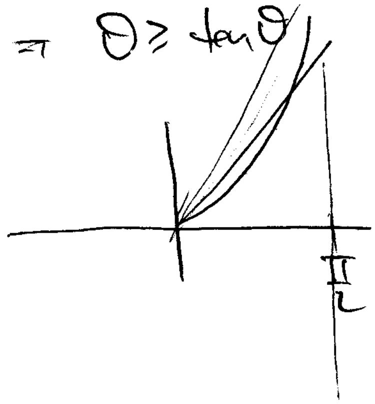
$$f(\pi/2) = \frac{2}{\pi}$$

$$f'(\theta) = \frac{\cos\theta}{\theta} - \frac{\sin\theta}{\theta^2} = \frac{\theta\cos\theta - \sin\theta}{\theta^2}$$

But $0 < \theta < \frac{\pi}{2}$ $\quad$ $\underbrace{\theta \geqslant \frac{\sin\theta}{\cos\theta}}$ $\quad$ $1 \leq \frac{\sin\theta}{\theta} \leq \frac{2}{\pi}\theta$

$\theta \leq \sin\theta \leq \frac{2}{\pi}\theta$

Something wrong here... $\Rightarrow \theta \geq \tan\theta$



Pg 116 w/w

Ex 1: $a > 0$

look at $\dfrac{e^{ix}}{x^2+a^2}$

Show $\displaystyle\int_0^\infty \underbrace{\frac{\cos x}{x^2+a^2}}_{Q(x)} dx = \frac{\pi}{2a} e^{-a}$

Ask questions about $\dfrac{e^{ix}}{x^2+a^2}$ ...

Check $Q(x)$ sat conditions in 6.22. then (only poles at $\pm ia$)

i) ✓ iii) $x \dfrac{1}{x^2+a^2}$ uniform, yes.

ii) ✓

iv) Don't know about this one ?

Thus $\displaystyle\int_0^\infty Q(x)\cos x \, dx = \pi i \sum R'$ w/ R' sum of Residue

in upper ½ plane of $e^{iz}/(z^2+a^2)$

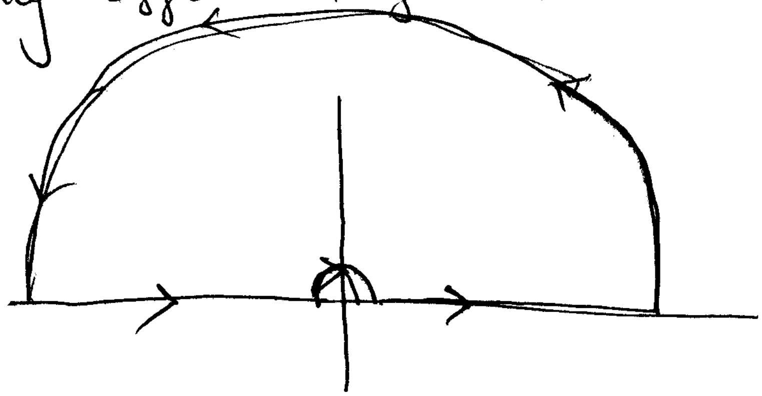
$$\lim_{z\to ia} \cancel{(z-ia)} \frac{e^{iz}}{\cancel{(z-ia)}(z+ia)} = \frac{e^{i(ia)}}{2ia} = \frac{e^{-a}}{2ia}$$

$$\therefore \int_0^\infty \frac{\cos x}{x^2+a^2} dx = \frac{\pi}{2a}\left(e^{-a}\right).$$

Ex 2: $\displaystyle\int_0^\infty \frac{\cos 2ax - \cos 2bx}{x^2} dx \overset{?}{=} \int_0^\infty \frac{\cos 2ax}{x^2} dx - \int_0^\infty \frac{\cos 2bx}{x^2} dx$

Then consider

Following suggestion integration Then is



Then $f(z) =$

Ex 3: $b > 0 \quad m \geq 0$

$$\int_0^\infty \frac{3x^2 - a}{(x^2+b^2)^2} \cos mx \, dx = \ldots$$

consider $Q(z) = \dfrac{3z^2 - a}{(z^2+b^2)^2} e^{imz}$ Check conditions

i) ✓ iii) ✓

ii) ✓ iv) Not sure?

From pg 115 (i) then is

prime prob. denotes That the residues are of diff

$$\int_0^\infty Q(x) \cos mx \, dx = \pi i \sum R' \quad \text{w/ } R' \text{ residues of } Q(x) e^{imx} \text{ fn}$$

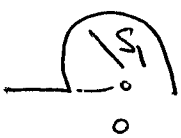
then the original.

$$\frac{3z^2 - a^2}{(z-ib)^2 (z+ib)^2}$$ Then ib is pole order 2

$$\to \frac{1}{1!} \lim_{z \to ib} \frac{d}{dz} \frac{\cancel{(z-ib)^2}(3z^2 - a^2)}{\cancel{(z-ib)^2}(z+ib)^2} = \frac{1}{1} \frac{d}{dz} \frac{3z^2 - a^2}{(z+ib)^2}$$

$$\int_{\gamma_1} Q(z)\,dz = \int_{\pi}^{0} Q(a+\delta_1 e^{i\theta})\,\delta_1 e^{i\theta}\, i\, d\theta$$

But $Q(a+\delta_1 e^{i\theta})$



$\quad\quad\quad\quad \shortparallel$

$Q(a+z)$ has $z=0$ a simple pole

& by hypothesis $Q(a+z)z \longrightarrow a'$ uniformly as $z \to 0$

$\Longleftrightarrow \quad Q(a+\delta_1 e^{i\theta})\,\delta_1 e^{i\theta} \longrightarrow a'$ $\therefore$

$$\lim_{\delta_1 \to 0} \int_{\gamma_1} Q(z)\,dz = \lim_{\delta_1 \to 0} a' i(-\pi) = -\pi i a'$$

$$\Rightarrow \quad P\int_{-\rho}^{\rho} Q(z)\,dz + \int_{\Gamma_\rho} Q(z)\,dz = 2\pi i \sum R + \pi i \sum R_0$$

$$= 2\pi i\left(\sum R + \frac{1}{2}\sum R_0\right)$$

For $x \to \infty$

$$x^a Q(x) \longrightarrow 0 \quad \Rightarrow \quad x^a Q(x) \sim \frac{1}{x^r} \quad x \to \infty \quad \text{w/ } r>0$$

Thus $x^{a-1} Q(x) \sim \dfrac{1}{x^{r+1}}$ $r>0$ $\Rightarrow$ convergence

For $x \to 0$

$x^a Q(x) \to 0$ $\qquad x^{a-1} Q$

$\Rightarrow \quad x^a Q(x) \sim x^r \qquad r > 0 \qquad x \to 0.$

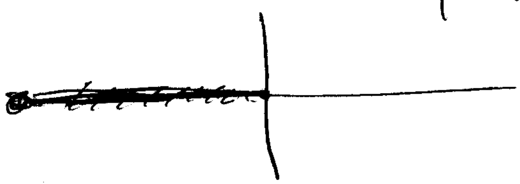
$\therefore \quad x^{a-1} Q(x) \sim x^{r-1} = \frac{1}{x^{1-r}}$ $\therefore$ have convergence at $x = 0$

For $a \in \mathbb{C}$ $\qquad z^a = e^{a \log z}$.

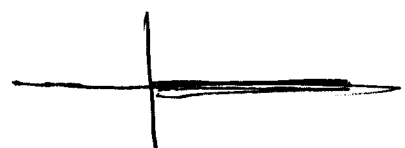
+ The ~~principle~~ branch of $z^a \iff$ principle branch of $\log z$.

principle branch $\log z = \log|z| + \operatorname{Arg} z \, i$

$-\pi < \operatorname{Arg} z \leq \pi$



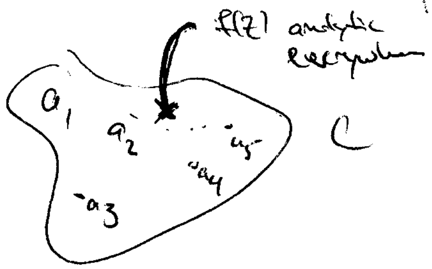
So $-z$ has principle branch



Thus if we consider just branch pts of $\log z$, $0, \infty$

Then we must have our contour flow around the line segment from $(0, \infty)$

pg 119 WW



$\phi(z)$ analytic except at some poles.

$$\frac{1}{2\pi i}\int_C f(z)\,\frac{\phi'(z)}{\phi(z)}\,dz = \sum \mathrm{Res}\ \frac{f(z)\,\phi'(z)}{\phi(z)}$$

$$\phi(z) = A(z-a_1)^{r_1} + B(z-a_1)^{r_1+1} + \cdots$$

$$\phi'(z) = A r_1 (z-a_1)^{r_1-1} + B(r_1+1)(z-a_1)^{r_1} + \cdots$$

$$f(z) = f(a_1) + (z-a_1)f'(a_1) + \cdots$$

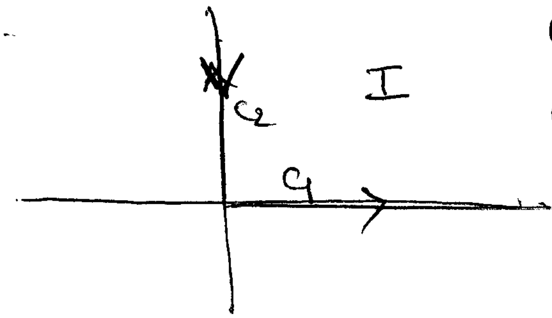
$$\therefore\quad \frac{f(z)\,\phi'(z)}{\phi(z)} - \frac{r_1 f(a_1)}{z-a_1}$$

P) 120 W/W

Ex 3: 1st check $f(z) = z^6 + 6z + 10 = 0$

must not have any poles ✓ & zeros on the contour

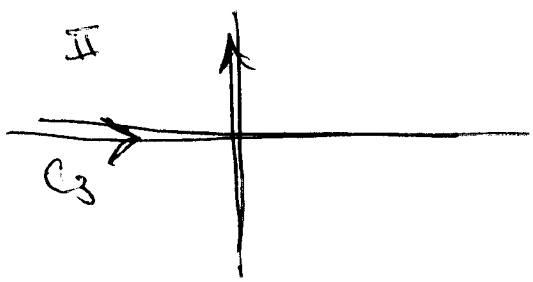
A



C_1 $z > 0$ $\therefore$ LHS > 0 no zeros

C_2 $z = iy$

$-y^6 + 6iy + 10 = 0$ has no real # solutions



C_3 [illegible] $z = x < 0$

let $z = -x$ $(x > 0)$

$x^6 - 6x + 10 = 0$