pg 125 W/W

$$\frac{d}{dt}\sum_{m=1}^{n}(-1)^m(z-a)^m\phi^{(n-m)}(t)f^{(m)}(a+t(z-a))$$

$$=\sum_{m=1}^{n}(-1)^m(z-a)^m\phi^{(n-m+1)}(t)f^{(m)}(a+t(z-a))$$

$$+\sum_{m=1}^{n}(-1)^m(z-a)^{m+1}\phi^{(n-m)}(t)f^{(m+1)}(a+t(z-a))$$

To get powers of $(z-a)$ the same

decrement 2nd sum

$$=\sum_{m=1}^{n}\cdots\qquad+\sum_{m=2}^{n+1}(-1)^{m-1}(z-a)^m\phi^{(n-m+1)}(t)f^{(m)}(\ldots)$$

$$=\qquad''\qquad-\sum_{m=2}^{n+1}(-1)^m(\quad)^m\phi^{(n-m+1)}(t)f^{(m)}(\ldots)$$

loose $m=2,\ldots,n$ terms

$$= (-1)^1 (z-a)^1 \phi^{n-1}(t) f^{(1)}(a+t(z-a))$$

$$-(-1)^{n+1}(z-a)^{n+1} \phi^{(\cancel{n}-\cancel{n}-\cancel{1}+1)}(t) f^{(n+1)}(a+t(z-a))$$

$$= -(z-a)\,\phi^{(n)}(t) f^{(1)}(a+t(z-a)) + (-1)^n (z-a)^{n+1} \phi(t) f^{(n+1)}(a+t$$

$\phi^{(n)}(t)$ const $= \phi^{(n)}(0)$

$\int_0^1 \to dt$ on RHS gives.

$$= -(z-a)\,\phi^{(n)}(0) \int_0^1 f^{(1)}(a+t(z-a))\,dt + (-1)^n (z-a)^{n+1} \int_0^1 \phi(t) f^{(n+1)}(\cdots)$$

$$\eta = a + t(z-a)$$

$$d\eta = (z-a)\,dt$$

$$= -\phi^{(n)}(0) \int_{\eta=a}^{z} f^{(1)}(\eta)\,d\eta + (-1)^n (z-a)^{n+1} \int_0^1 \phi(t) f^{(n+1)}(a+t(z-a))\,dt$$

$$= -\phi^{(n)}(0)\big(f(z)-f(a)\big) + \quad \cdots$$

∴ LHS becomes:

$$\sum_{m=1}^{n} (-1)^m (z-a)^m \phi^{(n-m)}(t) f^{(m)}(a+t(z-a)) \Bigg|_0^1$$

$$\Rightarrow \phi^{(n)}(0)\big(f(z)-f(a)\big) =$$ ~~$\sum_{m=1}^{n} (-1)^{m+1} (z-a)^m \phi^{(n-m)}(t) \Big[f^{(m)}(z) - f^{(m)}(a) \Big]$~~

$$= \sum_{m=1}^{n} (-1)^{m-1} (z-a)^m \Big[\phi^{(n-m)}(1) f^{(m)}(z) - \phi^{(n-m)}(0) f^{(m)}(a) \Big]$$

$$+ (-1)^n (z-a)^{n+1} \int_0^1 \phi(t) f^{(m+1)}(a+t(z-a))\, dt.$$

let $\phi(t) = (t-1)^n$ $\qquad \phi' = n(t-1)^{n-1}$

$\phi^2 = n(n-1)(t-1)^{n-2}$

$\vdots$

$\phi^{(k)} = n(n-1)\cdots(n-k+1)(t-1)^{n-k}$

$$\phi^{(k)} = \binom{n}{k}(t-1)^{n-k}$$

$$\binom{n}{1} = \binom{n}{n-1} \qquad \binom{n}{n-m} = \binom{n}{m}$$

$$\binom{n}{2} = \binom{n}{n-2}$$

$$\Rightarrow \quad \phi^{(n)} = \binom{n}{n} = \underline{1} \qquad\qquad (t-1)^m \Big|_{t=1} = \underline{0}$$

$$\Rightarrow \quad f(z) - f(a) = \sum_{m=1}^{n} (-1)^{m-1}(z-a)^m \Big[\binom{n}{n-m}(t-1)^{n-n+m}\Big|_{t=1} f^{(m)}(z)$$

$$- \binom{n}{n-m}(t-1)^{n-n+m}\Big|_{t=0} f^{(m)}(a) \Big] + (-1)^n (z-a)^{n+1} \int_0^1 (t-1)^n f^{(n+1)}(a+t(z- \; dt.$$

$$\Rightarrow \quad f(z) - f(a) = \sum_{m=1}^{n} (-1)^{m-1}(z-a)^m \cancel{(-1)} \binom{n}{n-m} \cdot \cancel{(-1)}^m f^{(m)}(a)$$

$$+ (-1)^n (z-a)^{n+1} \int_0^1 (t-1)^n f^{(n+1)}(a+t(z-a))\, dt$$

$$f(z) - f(a) = \sum_{m=1}^{n} \binom{n}{n-m} f^{(m)}(a)\,(z-a)^m + R(x)$$

$$\binom{n}{n-m} = \frac{n!}{(n-m)!\,(n-n+m)!} = \frac{n!}{(n-m)!\,m!} =$$

$$\underset{\substack{\uparrow \\ m=1}}{\binom{n}{n-1}} = \binom{n}{1} = \frac{n!}{(n-1)!\,1!} = \boxed{n}$$

$$\left(\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x-x_0)^k \right.$$

$$\phi(t) = \frac{(t-1)^n}{n!} \quad ; \quad \phi^{(r)} = \frac{(t-1)^{n-k}}{(n-k)!}$$

$$\phi' = \frac{(t-1)^{n-1}}{(n-1)!} \qquad \phi^{(} \ \frac{1}{(n-(n-m))!} = \frac{1}{m!}$$

Everything the same [illegible]

$$f(z) - f(0) = \sum (-1)^{m+1} (z-a)^m \frac{\cancel{(-1)\,(-1)^m}}{m!} f^{(m)}(0) + \checkmark$$

$$\phi^{(n)}(0) = \frac{(t-1)^0}{0!} = 1 \ \checkmark.$$

$n \to 2n$ $$\phi(t) = \underbrace{t^n(t-1)^n}$$

$$= [t^2 - t]^n .$$

$$\phi' = n t^{n-1} (t-1)^n$$

$$\phi' = D[t^n(t-1)^n] = Dt^n \cdot (t-1)^n \longleftrightarrow$$

$$+ \sum ($$

Riemann product Rule . . .

$$\frac{\cos z/2}{\sin z/2} \qquad z/2 = n\pi \quad n = \overset{0}{\pm 1}, \pm 2, 3, \ldots$$

$$z = 2n\pi \quad n = \pm 1,$$

$z = 0$ Removable sing. w/ $\frac{z}{2} \cot z/2$.

$$= 1 - B_1 \frac{z^2}{2!} - B_2 \frac{z^4}{4!} - B_3 \frac{z^6}{6!} - \ldots$$

$$\left[\left(\frac{z}{2}\right)\cot(z/2) - 1\right]$$ Show All even $D^{2n}\Big|_{z=0} < 0$

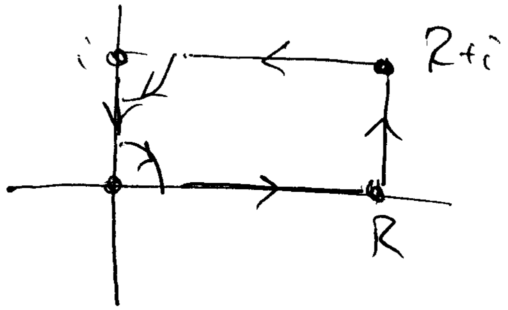
$$= -\sum_{k=0}^{\infty} |B_k| z^k/k!$$ has no zeros $\forall\, z \in \mathbb{R}$

plot & look at $\left(\frac{z}{2}\right)\cot(z/2) - \underline{1}$

$B_1 = 1/6$; $B_2 = 1/30$;

$$\int_0^\infty \frac{\sin px \, dx}{e^{\pi x} - 1} = ?$$

pg <u>122</u> <u>w/w</u> <u>prob</u> 2



$R \to +\infty$.

$$\frac{d}{dz}\left\{\frac{z}{2}\cot\left(\frac{z}{2}\right) - 1\right\} = \frac{1}{4}\frac{1}{\sin\left(\frac{z}{2}\right)^2}\left(-z + \sin z\right)$$

$\sin z \le z \quad \forall \; z \geq 0.$

$\therefore \quad \sin z - z \le 0 \quad \forall \; z \geq 0.$

$\therefore \quad \frac{d}{dz}\left\{ \qquad \right\} < 0 \quad \forall \; z \geq 0$

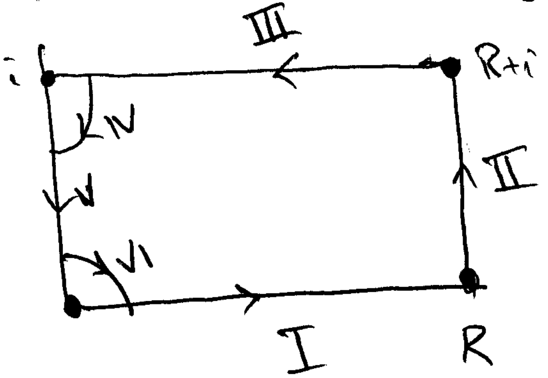
$\frac{d}{dz}\left\{ \qquad \right\}\Big|_{z=0} = 0 \qquad \therefore$

pg 122 W/W

Show

② $$\int_0^\infty \frac{\sin ax}{e^{2\pi x}-1}\,dx = \frac{1}{4}\frac{e^a+1}{e^a-1} - \frac{1}{2a}$$

Integrate $$\int \frac{e^{\pm aiz}}{e^{2\pi z}-1}\,dz$$



$$e^{2\pi z} = 1$$

$$2\pi z = \log 1 + 2\pi n i$$

$z = ni$ zeros of this fn $n = 0, \pm 1, \pm 2, \ldots$

$$\therefore \oint = 0$$

! $$\Rightarrow \int_{I} + \int_{II} + \int_{III} + \int_{IV} + \int_{VI} + \text{[illegible]}_{VII} = 0.$$

Show $\int_{VI} + \int_{IV} \longrightarrow 0$

$$\int_{VI} = \int$$

pg 155 w/w

$$S_n = \sum_{k=1}^{n} a_k$$

$$\mathcal{S} = \lim_{n\to\infty} \left(\frac{S_1 + S_2 + \dots \; S_n}{n} \right)$$

let $\sum_{m=1}^{\infty} a_m = S$ $\qquad \sum_{m=1}^{n} S_m = n\mathcal{S}_n$

pr $\mathcal{S}_n \longrightarrow S.$

pick n ∋ $\left| \sum_{m=n+1}^{n+p} a_m \right| < \epsilon. \quad \forall p > 0$

∋ $|S - S_n| < \epsilon$

$\nu > n$

$$\mathcal{S}_\nu = a_1 + a_2\left(1 - \frac{1}{\nu}\right) + a_3\left(1 - \frac{2}{\nu}\right)$$

$$+ \dots + a_n\left(1 - \cancel{np}\, \frac{n-1}{\nu}\right) + a_{n+1}\left(1 - \frac{n}{\nu}\right)$$

$$+ \dots + a_\nu\left(1 - \frac{\nu-1}{\nu}\right)$$

$$= \sum_{k=1}^{\nu} a_k + - a_2 \frac{1}{\nu} - a_3 \frac{2}{\nu} + \dots + - a_n \frac{(n-1)}{\nu} - a_{n+1} \frac{n}{\nu}$$

$$+ \dots + \; - a_\nu \frac{(\nu-1)}{\nu} \qquad \left(\cancel{-k+\frac{1}{\nu}}\right)$$

$$= -\frac{1}{\nu}\sum_{k=2}^{\nu}(k-1)a_k$$

But $\tilde{S}_\nu \equiv \frac{1}{\nu}\sum_{m=1}^{\nu} S_m$

Now $\tilde{S}_\nu$ from above $= \sum_{k=1}^{\nu}\left(1-\frac{k-1}{\nu}\right)a_k$

$$= \frac{1}{\nu}\sum_{k=1}^{\nu}(\nu-k+1)a_k$$

$$= \frac{1}{\nu}\Big(S_\nu + \underbrace{\sum_{k=1}^{\nu}(\nu-k)a_k}\Big)$$

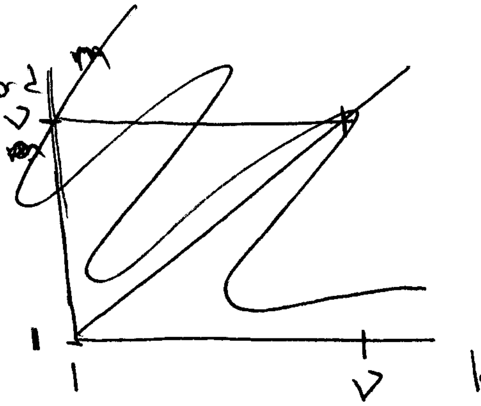
$$\sum_{k=1}^{\nu-1}(\nu-k)a_k$$

$$\sum_{k=1}^{\nu-1}(\nu-1-k+1)a_k$$

$$= S_{\nu-1} + \sum_{k=1}^{\nu-1}(\nu-1-k)a_k$$

drop last term again

$$S_\nu = \frac{1}{\nu}\sum_{m=1}^{\nu} S_m = \frac{1}{\nu}\underbrace{\sum_{m=1}^{\nu}\sum_{k=1}^{m}}_{\text{chang ord}} a_k$$

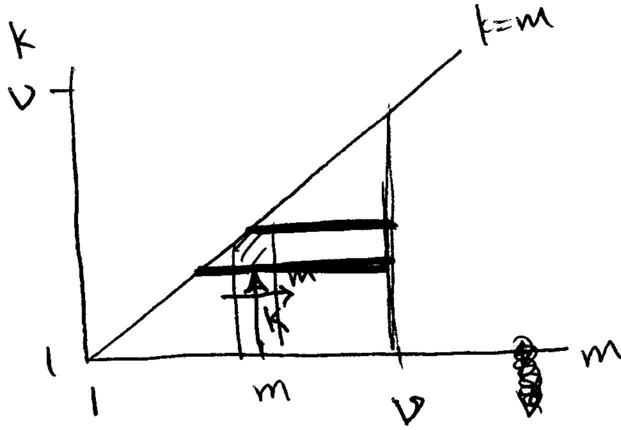


$1 < m < \nu$ out.

$1 < k < m$ in

$0 < m-1 < \nu-1$

$0 < k-1 < m-1$



$1 < k < \nu$ outer

$k < m < \nu$ inner loop

$$\therefore\ S_\nu \overset{?}{=} \frac{1}{\nu}\sum_{k=1}^{\nu}\sum_{m=k}^{\nu} a_k$$

$$= \frac{1}{\nu}\sum_{k=1}^{\nu} a_k \sum_{m=k}^{\nu} 1 = \frac{1}{\nu}\sum_{k=1}^{\nu} a_k(\nu+k-1)$$

$\nu + k - 1$

$$= \sum_{k=1}^{\nu} a_k\left(1 + \frac{k-1}{\nu}\right)$$ As in Book!!

Abel's ineq:

$$f_n \geq f_{n+1} > 0 \quad \forall n. \quad \Big|\sum_{n=1}^{m} a_n f_n\Big| \leq A f_1$$

$$w/ \quad A = \max\{|a_1|, |a_1+a_2|, |a_1+a_2+a_3|, \ldots, |a_1+a_2+\cdots+a_m|\}$$

$$\sum_{k=1}^{0} a_k = 0 \ ; \quad \sum_{k=1}^{-1} a_k = -\sum_{k=-1}^{1} a_k = -(a_{-1}+a_0+a_1)$$

$$\int_1^1 = 0 \qquad -1, 0, 1$$

$$\int_1^{-1} = -\int_{-1}^{1}$$

$$f_n = (1 - \frac{n-1}{\nu}) \qquad f_n \geq f_{n+1} > 0.$$

$\therefore$ ~~$\sum_{k=1}^{\nu}(1-\frac{k-1}{\nu})a_k \leq A \cdot 1$~~

$$A_n = \Big|\sum_{k=n+1}^{\nu}(1-\frac{k-1}{\nu})a_k\Big| \leq (1-\frac{n}{\nu})A = (1-\frac{n}{\nu})t.$$

$$A = \max\{|a_{n+1}|, |a_{n+1}+a_{n+2}|, |a_{n+1}+a_{n+2}+a_{n+3}|, \ldots$$

$$|a_{n+1}+\cdots+a_\nu|\} < \epsilon.$$

$\forall$ ∴

$$\left| S'_\nu - \sum_{k=1}^{n}\left(a_k - \frac{(k-1)}{\nu}\right)\right| < \left(1 - \frac{n}{\nu}\right)\epsilon$$

$\nu \to \infty$

S' limit pt of S_ν ?

$$\left| S' - \sum_{k=1}^{n} a_k \right| \leq \epsilon$$

Don't follow why not just limit?

$$|S' - s_n| \leq \epsilon$$

$$|S' - s| \leq |S' - s_n| + |s_n - s| \leq 2\epsilon.$$

How know S' has unique limit?

"Check"

$$S' = \lim_{n\to\infty} \frac{1}{n}\left(\sum_{k=1}^{n} s_k\right)$$

$$\sum_{k=0}^{\infty} z^k = \frac{1}{1-z} \qquad s_n = \sum_{k=1}^{n} z^k = \frac{z^{n+1}-1}{z-1}$$

$$\therefore \quad S' = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{z^{k+1}-1}{z-1} =$$

$$= \frac{1}{z-1} \lim_{n\to\infty} \frac{1}{n}\left(\sum_{k=1}^{n} z^{k+1} - \sum_{k=1}^{n} 1\right)$$

. . . .

$$\lim_{n\to\infty} \frac{z}{n}\sum_{k=1}^{n} z^{k} - \frac{n}{n}$$

$$\frac{1}{z-1} \lim_{n\to\infty} \left(\frac{z}{n}\left(\frac{z^{n+1}-1}{z-1} \right) - 1 \right) \qquad |z|<1$$

$$\frac{1}{z-1} \lim_{n\to\infty} \left(\frac{1}{n}\frac{z}{1-z} - 1 \right) = \frac{1}{1-z}$$ same sum !!

What if $S = \sum_{C_1} k$ still diverges.

What is $S = \sum_{k=1\ C_1}^{\infty} \frac{1}{k} = ?$

$S_{C_1} = \sum_{k=1}^{\infty} \frac{1}{k} = ?$

$\sum_{C_1} k = \lim_{n \to \infty} \frac{(n+1)(n+2)}{6} = \infty.$

pg 159 w/w

⑤ $1 + 0 + 0 - 1 + 0 + 1 + 0 + 0 + -1 + 0 + 1$

$S_1 = 1$

$S_2 = 1$

$S_3 = 1$

$S_4 = 0$

$S_5 = 0$

$S_6 = 1$

position of 1's

x_1

2 zeros, 1 (-1), 1 zero = 4

x_5

Pg 161 w/w

8 $e^{it} = \xi$

$$\Rightarrow \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\left(\frac{\xi^n + \xi^{-n}}{2}\right) + b_n\left(\frac{\xi^n - \xi^{-n}}{2i}\right)$$

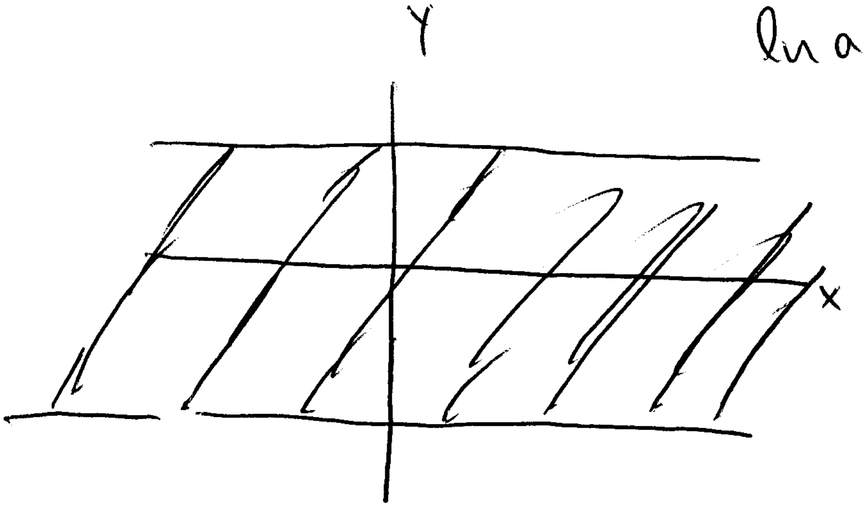
$$\Rightarrow \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \frac{1}{2}(a_n - i b_n)\xi^n + \frac{1}{2}(a_n + i b_n)\xi^{-n}$$

$$+ \frac{1}{2}\sum_{n=1}^{\infty}(a_n - i b_n)\xi^n + \frac{1}{2}\sum_{n=1}^{\infty}(a_n + i b_n)\xi^{-n}$$

$$+ \frac{1}{2}\sum_{n=1}^{a} c_n \xi^n + \frac{1}{2}\sum_{n=1}^{\infty} \bar{c}_n \xi^{-n}$$

$|\xi| = e^{-Y}$ ∴ $a \le e^{-Y} \le b$

$\ln a \le -y \le \ln b$



a = b = 1
Mapping of unit circle to ℝ.

$$f(z) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \sin kz}{k}$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \left(\frac{e^{ikz} - e^{-ikz}}{2i} \right.$$

$$= -\frac{1}{2i} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} (e^{iz})^k + \frac{1}{2i} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} (e^{-iz})^k$$

conv. of 1st $|e^{iz}| \leq 1$ sec $|e^{-iz}| \leq 1$

$\Rightarrow e^{-y} \leq 1$ $e^{y} \leq 1$

$-y \leq 0$ $y \leq 0$

$y \geq 0$ $y \leq 0$

Abel's thm: Told that $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} e^{\pm ixk}$ How conv. ??

$x \in \mathbb{R}$

then

$\lim_{r \to 1}$

pg 162 W/W

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \frac{e^{inx} + e^{-inx}}{2}$$

2 serie $\cancel{|e^{inx}|} <$

Rato test $\left| \frac{e^{ix(n+1)}}{e^{ix(n)}} \right| < 1$

$\Rightarrow \quad |e^{ix}| < 1 \qquad x = x + iy$

$\Rightarrow \quad e^{-y} < 1$

$-y < 0 \qquad y > 0$

other

Show $\sum \frac{(-1)^{k+1}}{k} \sin kx$ conv. uniformly

$\left| \sum_{k=1}^{?} \sin kx \right| = |\text{Trig terms}| \leq 2$ say

$f_n(z) = (-1$

$$\int \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx = \sum_{n=1}^{\infty} \frac{(-1)^{n} \cos nx}{n^2} \Bigg|_0^x + C_1$$

$$\frac{1}{4} x^2 = \sum_{n=1}^{\infty} \frac{(-1)^{\overset{-1}{n}}}{n^2} (1 - \cos nx) = -\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cos nx}{n^2}$$

$$+ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$$

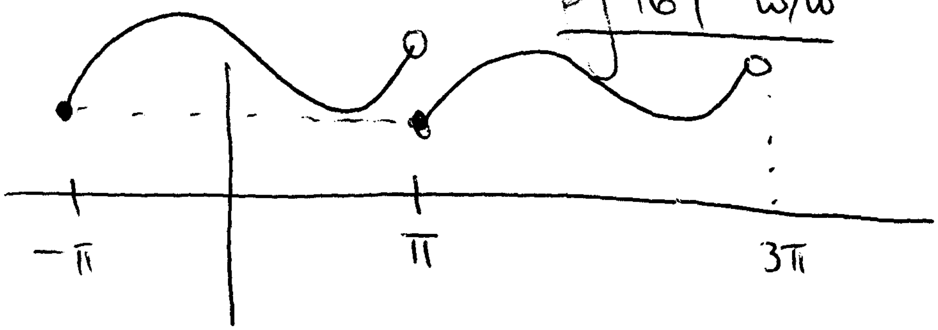
$$\Rightarrow \quad 2\pi C - \frac{1}{12} x^3 \Bigg|_{-\pi}^{\pi} = \sum_{n=1} \frac{(-1)^{n-1}}{n^2} \frac{\sin nx}{n} \Bigg|_{-\pi}^{\pi} \Bigg| \; C$$

$$= \sum 0 = 0$$

$$2\pi C - \frac{1}{12} (\pi^3 + \pi^3) = 0$$

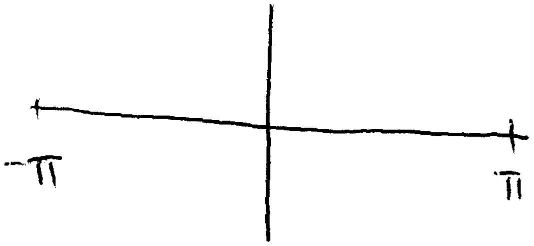
$$-\frac{\cancel{}}{6} \pi^3 = 0$$

$$C = \frac{\pi^2}{12}$$


pg 164 w/w

Dirichlet ~~[illegible]~~ req. that f(t) bounded on (-π, π)

but $\log|2\cos\frac{1}{2}x|$ is not as $\log 0 \to -\infty$

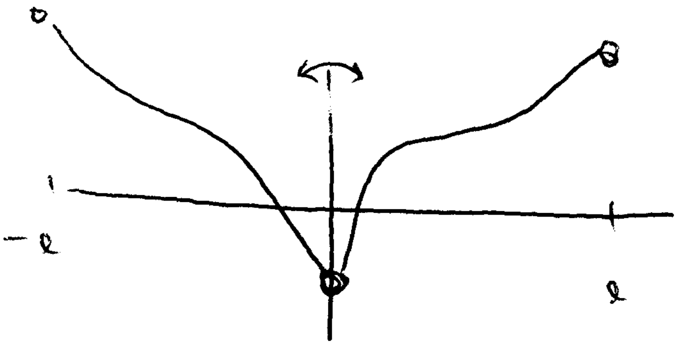


$\pi \longrightarrow b$

$-\pi \longrightarrow a$

$x(-\pi) = a$

$x(\pi) = b$ ✓



$a = -l$

$b = l$

$$\frac{1}{2}\{f(x+0)+f(x-0)\} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi}{2l}2x+\cancel{l}l\right) + b_n\sin\left(\frac{n\pi}{l}x\right)$$

$$\frac{1}{2}(2\ell)\,a_n = \int_{-\ell}^{\ell} f(t)\cos\left(\frac{n\pi t}{\ell}\right)dt$$

$$\ell b_n = \int_{-\ell}^{\ell} f(t)\sin\left(\frac{n\pi t}{\ell}\right)dt = \int_{-\ell}^{0} + \int_{0}^{\ell}$$

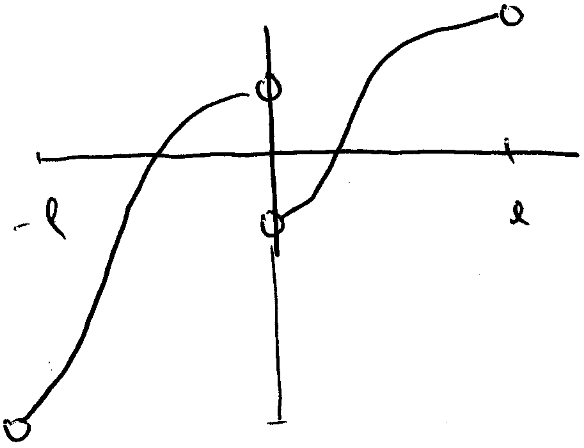
$$-\int_{0}^{\ell} f(-t)\,\underline{\qquad} +$$

$$= 0.$$

$f(t)$

$$\ell a_n = \int_0^{\ell} f(-t)\cos(\ )\,dt + \int = 2\int_0^{\ell}$$

$f(-x) = -f(-x)$



$$\ell a_n = \int \qquad + \int_0^{\ell}$$

$$\int_0^{\ell} f(-t)\cos(\ )\,dt$$

$$-\int f(t)$$

$$= 0.$$

Pg 166 W/W

7

$$\ell b_n = -\int_0^\ell f(l-t)\sin(\ )\,dt + \int_0^\ell \quad = 2\int_0^\ell$$

$$=$$

$$\therefore -\ell \leq x \leq 0$$

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{\ell}\right) + \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{\ell}\right) = 0$$

$$\pi a_n = 2\int_0^\pi \frac{1}{2}(\pi - x)\sin x\cos nx\,dx$$

sin cos nx + sin nx cos x

sin x cos nx − cos x sin nx

$$= \int_0^\pi (\pi - x)\left[\frac{1}{2}\left(\sin(x+nx)\right) + \sin(x-nx)\right)\right]dx \qquad n \neq \underline{1}$$

$$= \frac{1}{2}\left[\int_0^\pi (\pi - x)\left(\sin((n+1)x) + \sin((n-1)x)\right)dx\right]$$

$$= -\frac{1}{2}(\pi - x)\left(\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right)\Bigg|_0^\pi$$

$$+ \frac{1}{2}\int_0^\pi (-1)\left[\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]dx$$

$$= +\frac{1}{2}\left[\frac{1}{n+1} + \frac{1}{n-1}\right] - \frac{1}{2}\left(\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right|_0^\pi$$

$$\frac{1}{2}\left[\frac{2}{\ell}\int_0^\ell f(t)\cos\left(\frac{n\pi t}{\ell}\right)dt\right]$$

$$+\sum_{n=1}^{\infty}\frac{2}{\ell}\int_0^\ell f(t)\cos\left(\frac{n\pi t}{\ell}\right)dt\cdot\cos\left(\frac{n\pi}{\ell}x\right)+$$

$$\sum_{n=1}^{\infty}\frac{2}{\ell}\int_0^\ell f(t)\sin\left(\frac{n\pi t}{\ell}\right)$$

$1 = u$ [illegible] $=$

Pg 167 w/w

$f = LHS$

$$f' = \frac{\pi}{96}\left[-2(\qquad) + (\pi-2x)(2\pi-4x)\right]\Big|_0$$

$$= \frac{\pi}{96}\left[-2\pi^2 + \pi(2\pi)\right] = 0 \checkmark$$

$$f'(\pi) = \frac{\pi}{96}\left[-2(\pi^2) + (-\pi)(-2\pi)\right] = 0 \checkmark$$

$$f(0) = \frac{\pi}{96}\left[\pi(\pi^2)\right]$$

~~[illegible]~~

$$f(\pi) = \frac{\pi}{96}\left[(-\pi)(-\pi^2)\right]$$

Then $f(\pi)\sin\pi n - f(0)\cdot 0 = 0.$

$$f'' = \frac{\pi}{96}\left[-2(2\pi-4x) + (-2)(2\pi-4x) + (\pi-2x)(-4)\right]$$

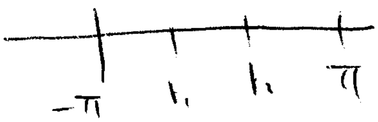
$$f''' = \frac{\pi}{96}\left[8+8+8\right] = \frac{3\cdot 8\pi}{96} = \frac{\cancel{3}\cdot\overset{\cancel{2}}{\cancel{4}}\cdot\cancel{2}\,\pi}{\cancel{2}(45+3)} = \frac{\pi}{4}.$$

$48 = \cancel{24}\cdot\cancel{2}\ 8$

$=$

~~Pg~~ 167 W/W

$$-\frac{1}{n^4}\frac{\pi}{4}\left(\cos n\pi - 1\right) =$$



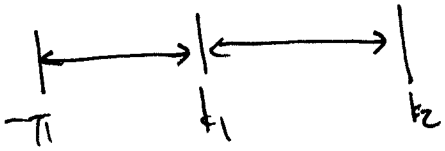
b)

$$\pi a_m = \left. f(t)\frac{\sin mt}{m}\right|_{-\pi}^{t_1} + \left. m^{-1} f(t)\sin mt\right|_{t_1}^{t_2}$$

$$+ \cdots \Big|_{t_{n-1}}^{t_n} + \left. m^{-1} f(t)\sin mt\right|_{t_n}^{\pi}$$

$$- m^{-1}\int_{-\pi}^{t_1} f'(t)\sin mt\,dt - m^{-1}\int_{t_1}^{t_2} f'(t)\sin mt\,dt \cdots$$

$$- m^{-1}\int_{t_n}^{\pi} f'(t)\sin mt\,dt$$



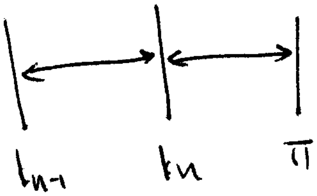
$\pi a_m =$ ~~$f(t) - f(-\pi^+)$~~

$$\frac{f(t_1^-)\sin m t_1^- - \cancelto{0}{f(-\pi^+)\sin(-\pi^+)}}{m}$$

$$+ \frac{f(t_2^-)\sin m t_2^- - f(t_1^+)\sin m t_1^-}{m}$$

$$+ \frac{f(k_3^-)\sin mk_3^- - f(k_2^+)\sin mk_2^+}{m} + \cdots$$

$$+ \frac{f(t_n^-)\sin mt_n^- - f(t_{n-1}^+)\sin mt_{n-1}^+}{m}$$



$$+ \frac{f(\pi)\cancelto{0}{\sin m\pi} - f(t_n^+)\sin mt_n^+}{m}$$

$$\Rightarrow \; -m^{-1}\left\{ \underbrace{\int_{-\pi}^{k_1} f' \sin mt\,dt + \int_{k_1}^{k_2} + \cdots + \int_{t_n}^{\pi}}_{\pi b_m'} \right\}.$$

$$\Rightarrow \quad \pi a_m = \frac{\pi A_m}{m} - \frac{\pi b_m'}{m}$$

$$\Rightarrow \quad a_m = \frac{A_m}{m} - \frac{b_m'}{m} \quad w/$$

What we are doing is writing coeff a_m in 2 parts one depends on the coeff of f' + another that depends on the "connectivity data" of f at the points $k_1, k_2, \ldots, k_n$

$$\pi A_m = \cancel{f(k_{r+1}^-)\sin mt_{r+1}^- - f(t_r^+)\sin mt_r^+}$$

$$= f(k_1^-)\sin mk_1 - f(k_1^+)\sin mk_1$$

$$+ f(k_2^-)\sin mk_2 - f(k_2^+)\sin mk_2 . +$$

$$+ \cdots \left(f(k_n^-) - f(k_n^+)\right) \sin m k_n$$

$$\therefore \quad \pi A_m = \sum_{r=1}^{n} \left(f(k_r^-) - f(k_r^+)\right) \sin m k_r$$

- - - - - - - -

$$\pi b_m = f(t) \underline{\cos m} t \qquad \pi b_m = \int_{-\pi}^{\pi} f(t) \sin mt \, dt$$

$$\vdots \quad \sin mt \to -\cos mt.$$

$$\pm m^{-1} \int_{k_n}^{\pi}$$

$$\pi b_m = \frac{- \sum_{r=1}^{n} \left(f(k_r^-) - f(k_r^+)\right) \cos m k_r + f(-\pi^+) \cos m\pi - f(\pi^-) \cos m}{m}$$

$$+ \; m^{-1} \left\{ \int_{-\pi}^{k_1} f' \cos mt + \int_{k_1}^{k_2} + \cdots \int_{k_n}^{\pi} \right\}.$$

$$= \cancel{m^{-1}} \frac{\pi B_m}{m} + \frac{\pi a_m'}{m}$$

$$\therefore \quad b_m = \frac{B_m}{m} + \frac{a_m'}{m}$$

w/ $$\pi B_m = -\sum_{r=1}^{n} \left(f(k_r^-) - f(k_r^+)\right) \cos m k_r - \cos m\pi \left(f(\pi^-) - f(\pi^+)\right)$$

Sin:

$$a_m' = \frac{A_m'}{m} - \frac{b_m''}{m} \qquad b_m' = \frac{B_m'}{m} + \frac{a_m''}{m}$$

w/ . a_m'' & b_m'' fo... coe of f''.

& $$\pi A_m' = \sum_{r=1}^{n} \sin m k_r \{ f'(k_r$$

$\therefore$

$$a_m = \frac{A_m}{m} - \frac{1}{m}\left(\frac{B_m'}{m} + \frac{a_m''}{m}\right) \qquad b_m = \frac{B_m}{m} + \frac{A_m'}{m^2} - \frac{b_m''}{m^2}$$

$$= \frac{A_m}{m} - \frac{B_m'}{m^2} - \frac{a_m''}{m^2}$$

Now we have a_m & b_m in terms of discontinuity data of f, f' & coeff of $f''(x)$.

if $A_m' = O(1)$ $B_m' = O(1)$

+ As integrals of $a_m'' = O(1)$ $b_m'' = O(1)$

if $A_m = 0$ + $B_m = 0$

$A_m = 0$ if $f(k_r^-) = f(k_r^+)$

+ $f(\pi -) = f(-\pi)$

pg 169 W/W

if $a_m' = b_m' = 0$ Then

$$a_m = \frac{A_m}{m} \qquad b_m = \frac{B_m}{m}$$

$$\begin{array}{ccc} \| & & \| \\ 0 & \Rightarrow A_m = 0 \,\Big|\, & \frac{1}{2m}(1-\cos m\pi) = \frac{B_m}{\cancel{m}} \end{array}$$

$$\Rightarrow B_m = \frac{1}{2}(1-\cos m\pi)$$

$$0 = \pi A_m = \sum_{r=1}^{n} \sin m k_r \left(f(k_r^-) - f(k_r^+) \right)$$

$$k_1 = 0$$

$$\pi\left(\frac{1}{2} - \frac{1}{2}\cos m\pi\right) = -\Big[\cos m\pi \left(f(\pi^-) - f(\pi^+)\right)$$
$$+ f(0^-) \cancel{+} - f(0^+)\Big]$$

$\forall$ m $\Rightarrow$

$$\frac{\pi}{2} = f(\pi^-) - f(\pi^+)$$

$$\frac{\pi}{2} = f(0^-) - f(0^+)$$

9.4

$$f(\pi) = f(-\pi) \quad \checkmark.$$

pg 170 w/w

My understanding.

pr:

$$\frac{1}{2}(f(x^+)+f(x^-)) = \lim_{m\to\infty}\frac{1}{m}\{S_1+S_2+\cdots S_m\}.$$

if $\sum_{n=1}^{\infty} A_n(x)$. w/ $S_k = \sum_{n=1}^{k} A_n$

$$\frac{A_0}{\cancel{2}} + \sum_{n=1}^{\infty} A_n(x)$$

$S_1 = A_0$

$S_2 = A_0 + A_1$

$S_3 = A_0 + A_1 + A_3$

$\vdots$

$$\lim_{m\to\infty}\frac{1}{m}\{mA_0 + \underline{A_1} + (\underline{A_1}+A_2) + (\underline{A_1}+A_2+A_3)+\cdots$$

$$+ (A_1+\underline{A_2}+A_3+\cdots+A_{m-1})\}$$

$$\lim_{m\to\infty}\frac{1}{m}\{mA_0 + (m-1)A_1 + (m-2)A_2 + \cdots + A_{m-1}(x)\}$$

Pr: $\lim_{m\to\infty}\frac{1}{m}\{mA_0 + (m-1)A_1 + (m-2)A_2 + \cdots + A_{m-1}(x)\}$

$$= \frac{1}{2}(f(x^+)+f(x^-)).$$

Now

$$\underline{\lim_{m\to\infty} \frac{1}{m}\Big\{}$$

$$A_0 + \sum_{n=1}^{m-1} S_n(x) = mA_0 + (m-1)A_1 + (m-2)A_2 + \cdots + A_{m-1}(x).$$

$$= \cancel{\frac{1}{\pi}\Big[m\int_{-\pi}^{\pi}\Big\{}$$

$$A_0 = \frac{1}{2}a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt$$

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\big(f(t)\cos nt\cos nx + f(t)\sin nt\sin nx\big)\,dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos n(t-x)\,dt$$

∴ Sum above is.

$$\cancel{\frac{m}{2\pi}} \quad \frac{1}{\pi}\int_{-\pi}^{\pi}\Big\{\frac{m}{2} + (m-1)\cos(x-t) + (m-2)\overset{\cos 2(x-t)}{\cancel{\cos(x-t)}} + \cdots$$

$$+ \cos\big((m-1)(x-t)\big)\Big\} f(t)\,dt$$

Now evaluate

$$\frac{m}{2} + (m-1)\cos(x-t) + (m-2)\cos 2(x-t) + \cdots + \cos(m-1)(x-t)$$

$\theta = x - t$

$$= \frac{m}{2} + \frac{(m-1)}{2}(e^{i\theta}) + \frac{(m-2)}{2}e^{2i\theta} + \cdots \frac{e^{(m-1)i\theta}}{2}$$

$$+ \sum_{k=0}^{m-1} \frac{(m-k)}{2}(e^{-i\theta})^k - \frac{m}{2}$$

$$= \frac{1}{2}\sum_{k=0}^{m-1} m(e^{i\theta})^k - \frac{1}{2}\sum_{k=0}^{m-1} k(e^{i\theta})^k + \frac{m}{2}\sum_{k=0}^{m-1} (e^{-i\theta})^k - \frac{1}{2}\sum_{k=0}^{m-1} k(e^{-i\theta})^k$$

$$-\frac{m}{2}$$

$$= \frac{m}{2}\left(\frac{e^{i\theta m}-1}{e^{i\theta}-1}\right) + \frac{m}{2}\left(\frac{e^{-i\theta m}-1}{e^{-i\theta}-1}\right) - \frac{m}{2}$$

$$-\frac{1}{2}\left[\frac{-(m-1)e^{i\theta m}}{1-e^{i\theta}} + \frac{e^{i\theta}(1-e^{i\theta(m-1)})}{(1-e^{i\theta})^2}\right]$$

$$-\frac{1}{2}\left[\frac{-(m-1)e^{-i\theta m}}{1-e^{-i\theta}} + \frac{e^{-i\theta}(1-e^{-i\theta(m-1)})}{(1-e^{-i\theta})^2}\right]$$

Now:

$$\sum_{k=0}^{n-1} k q^k = \frac{-(n-1)q^n}{1-q} + \frac{q(1-q^{n-1})}{(1-q)^2}$$

$$= \frac{m}{2}\left[\frac{(e^{-i\theta}-1)(e^{i\theta m}-1) + (e^{-i\theta m}-1)(e^{i\theta}-1)}{(e^{i\theta}-1)(e^{-i\theta}-1)}\right] - \frac{m}{2}$$

$$+ \frac{(m-1)}{2}\left[\frac{e^{i\theta m}}{1-e^{i\theta}} + \frac{e^{-i\theta m}}{1-e^{-i\theta}}\right] - \frac{1}{2}\left[\frac{e^{i\theta}(1-e^{i\theta(m-1)})}{(1-e^{i\theta})^2} + \frac{e^{-i\theta}(1-e^{-i\theta(m}}{(1-e^{-i\theta})^2}\right.$$

$$= \frac{m}{2}\left[\frac{e^{i\theta(m-1)} - e^{-i\theta} - e^{+i\theta m} + 1 + e^{-i\theta(m-1)} - e^{-i\theta m} - e^{i\theta} + 1}{1 - e^{i\theta} - e^{-i\theta} + 1}\right]^4$$

~~P~~g 170 W/W

Using footnote to save time.

$$\delta = \frac{m}{2} + \frac{m-1}{2}(\lambda + \lambda^{-1}) + \frac{(m-2)}{2}(\lambda^2 + \lambda^{-2}) + \cdots \frac{(\lambda^{m-1} + \lambda^{-(m-1)})}{2}$$

~~Mult Above δ by $(1-\lambda)$ gives~~

~~$\frac{m}{2}(1-\lambda) + \frac{m-1}{2}($~~

$$= \frac{m}{2} + \frac{m-1}{2}\lambda + \frac{m-2}{2}\lambda^2 + \cdots\cdots \frac{1}{2}\lambda^{m-1}$$

$$+ \frac{m-1}{2}\lambda^{-1} + \frac{m-2}{2}\lambda^{-2} + \cdots + \frac{1}{2}\lambda^{-(m-1)}$$

$-m+1$

$-(m-1)+1$

Mult above by $1-\lambda$ gives

$$\underset{\sim\sim}{\frac{m}{2}} + \underline{\frac{m-1}{2}}\lambda + \underline{\frac{m-2}{2}}\lambda^2 + \cdots + \underline{\frac{1}{2}}\lambda^{m-1}$$

$$-\underline{\frac{m}{2}}\lambda - \underline{\frac{(m-1)}{2}}\lambda^2 + \cdots \cancel{+} - \underline{\frac{2}{2}}\lambda^{m-1} - \frac{1}{2}\lambda^m$$

$$\underline{\frac{m-\cancel{1}}{2}}\lambda^{-1} + \underline{\frac{m-2}{2}}\lambda^{-2} + \underline{\frac{1(\cancel{m}\ 2)\lambda^{-(m-2)}}{2\cdots}} + \frac{1}{2}\lambda^{-(m-1)}$$

$$\underset{\sim\sim}{-\frac{m-1}{2}} - \underline{\frac{m-2}{2}}\lambda^{-1} \ \underline{\ \ }\ \cdots \cancel{+} \underline{\frac{1}{2}\lambda^{-(m-1)+1}} \cancel{+}$$

$$-\left(\frac{m-3}{2}\right)\lambda^{-2}$$

$$= \underset{\text{min}}{\frac{1}{2}} - \frac{\lambda}{2} - \frac{\lambda^2}{2} - \cdots \frac{1}{2}\lambda^{m-1} - \frac{1}{2}\lambda^{m}.$$

$$+ \frac{1}{2}\lambda^{-1} + \frac{1}{2}\lambda^{-2} + \cdots \frac{1}{2}\lambda^{-(m-2)} + \frac{1}{2}\lambda^{-(m-1)}$$

$$\circledast = \frac{1}{2}\Big\{ \lambda^{1-m} + \lambda^{2-m} + \cdots + \lambda^{-1} + 1$$

$$- \lambda - \lambda^2 \cdots - \lambda^{m-1} - \lambda^{m} \Big\}$$

$\therefore$

$$\frac{1}{2}\Big\{ m + (m-1)(\lambda+\lambda^{-1}) + (m-2)(\lambda^2+\lambda^{-2}) + \cdots + (\lambda^{m-1} + \lambda^{-(m-1)}) \Big\}$$

$$= \frac{(1-\lambda)^{-1}}{2}\Big\{ \lambda^{1-m} + 2^{2-m} + \cdots + \lambda^{-1} + 1 - \lambda - \lambda^2 - \cdots - \lambda^{m-1}$$

$$- \lambda^{m} \Big\}$$

· Cheating + going to Grad/Rysz

~~$\frac{m}{2} + m\cos(x-t)$~~

$$\frac{m}{2} + \sum_{k=1}^{m-1}(m-k)\cos k(x-t)$$

$$= \frac{m}{2} + m\sum_{k=1}^{m-1}\cos k(x-t) - \sum_{k=1}^{m-1}k\cos k(x-t)$$

= ~~$\frac{m}{2} + \frac{m}{2}\Big[1 + \sin($~~

pg 36 G/R

$$\sum_{k=0}^{n}\cos kx = \frac{1}{2}\left[1 + \frac{\sin(n+\frac{1}{2})x}{\sin\frac{x}{2}}\right]$$

$$\therefore \qquad \sum_{k=1}^{n-1}\cos kx = \frac{1}{2}\left[1 + \frac{\sin(n-\frac{1}{2})x}{\sin x/2}\right] - 1$$

$$= \frac{1}{2}\left[-1 + \frac{\sin(m-\frac{1}{2})x}{\sin x/2}\right]$$

= ~~$\frac{m}{2} + \frac{m}{2}\Big[$~~ $$= \frac{m}{2}\frac{\sin(m-\frac{1}{2})(x-t)}{\sin(\frac{x-t}{2})} - \sum_{k=1}^{m-1}k\cos k(x-t)$$

pg 170 w/w

pg 38 G/R:

$$= \cancel{\frac{m}{2}\frac{\sin(m-\frac{1}{2})(x-t)}{\sin\frac{x-t}{2}}} - \cancel{m\frac{\sin(\frac{2m-1}{2})(x-t)}{2\sin\frac{x-t}{2}}} + \frac{1-\cos m(x-t)}{4\sin^2\frac{(x-t)}{2}}$$

$$= \frac{1-\cos m(x-t)}{4\sin^2(\frac{x-t}{2})}$$

$$\sin^2\theta = \frac{1-\cos^2 2\theta}{2}$$

$$2\sin^2\theta = 1-\cos 2\theta$$

$$\therefore \quad \frac{1}{4\pi}\int_{-\pi}^{\pi}\frac{2\sin^2(\frac{m}{2}(x-t))}{\sin^2(\frac{x-t}{2})}f(t)\,dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\sin^2(\frac{m}{2}(x-t))f(t)}{\sin^2(\frac{x-t}{2})}\,dt$$

$u \overset{\text{let}}{=} t+x$ ~~[illegible]~~

~~[illegible]~~

$-t = -u+x$

so $x-t = -u+2x$.

$$= \cancel{\frac{1}{2\pi}\int_{-\pi+x}^{\pi+x}\frac{\sin^2(\frac{m}{2}(2x-u))}{\sin^2(\frac{2x-u}{2})}f(u-x)\,du}$$

$$f(u-x) \overset{?}{=} f(u)$$

$$= \cancel{\frac{1}{2\pi}\int_{-\pi+x}^{\pi+x}\sin^2(\tfrac{m}{2}(}$$

Is $\frac{\sin^2(\frac{m}{2}(x-t))}{\sin^2(\frac{x-t}{2})}$ periodic of period 2π? yes.

$$\frac{\sin^2\left(\frac{m}{2}(x-t)-\pi m\right)}{\sin^2\left(\frac{x-t}{2}-\pi\right)} = \frac{[}{\qquad\qquad\qquad} \quad \checkmark$$

Thus from Ashbaugh / Asmar

$$\int_0^T f(x)\,dx = \int_a^{a+T} f \qquad \text{if } f \text{ period } T$$

So $$\int_{-\frac{T}{2}}^{T/2} = \int_{0-\frac{T}{2}}^{T-\frac{T}{2}} = \int_0^T = \int_{-\frac{T}{2}+x}^{\frac{T}{2}+x}$$

$a = -\frac{T}{2}$ $\qquad$ $a = -\frac{T}{2}+x$

$$\therefore \int_{-\pi}^{\pi} \Leftarrow \int_{-\pi+x}^{\pi+x}$$

$$\therefore \int_{-\pi}^{\pi} = \int_{-\pi+x}^{\pi+x} \quad \checkmark$$

$$t = x \mp 2\theta$$

$$x + 2\theta$$

$$-\pi + x = x + 2\theta \Rightarrow \theta = -\frac{\pi}{2}$$

$$I = \frac{1}{2\pi}\int_{-\pi+x}^{x} + \frac{1}{2\pi}\int_{x}^{\pi+x}$$

$t = x - 2\theta$ | $t = x + 2\theta$

$\theta = 0$ upper lim | $x = x + 2\theta \Rightarrow \theta = 0$ low limit

$-\pi + \cancel{x} = \cancel{x} \mp 2\theta \Rightarrow \theta = \frac{\pi}{2}$ lower lim | $\pi + \cancel{x} = \cancel{x} + 2\theta \Rightarrow \theta = \frac{\pi}{2}$ upper limit

$$= \frac{1}{2\pi}\int_0^{\frac{\pi}{2}} \frac{\sin^2 \frac{m}{2}(2\theta)}{\sin^2\theta} f(x-2\theta)\, d\theta \cdot 2 + \frac{1}{2\pi}\int_0^{\frac{\pi}{2}} \frac{\sin^2 m\theta}{\sin^2\theta} f(x+2\theta)\, d\theta \cdot 2$$

$$= \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \frac{\sin^2 m\theta}{\sin^2\theta} f(x+2\theta)\, d\theta + \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \frac{\sin^2 m\theta}{\sin^2\theta} f(x-2\theta)\, d\theta$$

$$\therefore \lim_{m\to\infty} \frac{1}{m}\left\{A_0 + S_1 + S_2 + \cdots + S_{m-1}\right\}$$

$$= \lim_{m\to\infty}\left(\frac{1}{m\pi}\int_0^{\frac{\pi}{2}} \frac{\sin^2 m\theta}{\sin^2\theta} f(x+2\theta)\, d\theta + \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \frac{\sin^2 m\theta}{m\sin^2\theta} f(x-2\theta)\, d\theta\right)$$

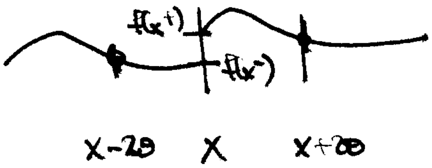
want

$$= \frac{1}{2}\left\{f(x+) + f(x-)\right\}$$

Pg 171 w/w

$\phi_1 = f(x+2\theta) - f(x+)$

$\phi_2 = f(x-2\theta) - f(x-)$



As f is cont from Right + Left.

$\exists\, \delta_1 \ni \quad |\phi_1(\theta)| < \epsilon \qquad 0 < \theta \le \frac{\delta_1}{2}$

$+ \exists\, \delta_2 \ni \quad |\phi_2(\theta)| < \epsilon \qquad 0 < \theta \le \frac{\delta_2}{2}$

put $\delta = \min\{\delta_1, \delta_2\}$

Then $|\phi(\theta)| < \epsilon \qquad 0 < \theta \le \frac{\delta}{2}.$

Then

$$\left|\frac{1}{m}\int_0^{\frac{\pi}{2}} \frac{\sin^2 m\theta}{\sin^2\theta}\,\phi(\theta)\,d\theta\right| \le \frac{1}{m}\int_0^{\frac{1}{2}\delta} \frac{\sin^2 m\theta}{\sin^2\theta}\,|\phi(\theta)|\,d\theta$$

$$+ \frac{1}{m}\int_{\delta/2}^{\pi/2} \frac{\sin^2 m\theta}{\sin^2\theta}\,|\phi(\theta)|\,d\theta$$

$$< \frac{\epsilon}{m}\int_0^{\delta/2} \frac{\sin^2 m\theta}{\sin^2\theta}\,d\theta + \frac{1}{m\sin^2 \delta/2}\int_{\delta/2}^{\pi/2} |\phi(\theta)|\,d\theta.$$

$\because \sin^2 m\theta \le 1$

$\because \frac{1}{\sin^2\theta} \le \frac{1}{\sin^2 \delta/2}$

$$< \frac{\epsilon}{m}\int_0^{\pi/2}\frac{\sin^2 m\theta}{\sin^2\theta}\,d\theta + \frac{1}{m\sin^2\delta/2}\int_0^{\pi/2}|\phi(\theta)|\,d\theta$$

$$\frac{\epsilon}{\cancel{m}}\cdot\frac{1}{2}\pi\cancel{m} + \frac{1}{m\sin^2\delta/2}\int_0^{\pi/2}|\phi(\theta)|\,d\theta$$

Now $$\int_0^{\pi/2}|\phi|\,d\theta \le \int_0^{\pi/2}\left(|f(\cancel{x}\pm 2\theta)| + |f(x\pm)|\right)d\theta$$

$$= \int_0^{\pi/2}|f(\cancel{x}\pm 2\theta)|\,d\theta + |f(x\pm)|\frac{\pi}{2}$$

$$u = x \pm 2\theta$$

$$du = \pm 2\,d\theta$$

$$= \int_x^{x\pm\pi}|f(u)|\left(\frac{\pm du}{2}\right) + |f(x\pm)|\frac{\pi}{2}$$

$$+ \quad \frac{1}{2}\int_x^{x+\pi}|f(u)|\,du \; +$$

$\cancel{\theta}$

$x = 0$

~~max(A, π)~~

f 2π period

$$\cancel{\frac{1}{2}\int_x^{x-\pi}}$$

$$\overset{+}{=} \frac{1}{2}\int_0^{\pi}|f(u)|\,du + |f(x^+)|\frac{\pi}{2}$$

$$- \quad -\frac{1}{2}\int_{x}^{x-\pi} |f(u)|\,du + \frac{\pi}{2}|f(x^-)|$$

$$= -\frac{1}{2}\int_{0}^{-\pi} |f(u)|\,du$$

$$= +\frac{1}{2}\int_{-\pi}^{0} |f(u)|\,du.$$

given ϵ. $\exists$ $\therefore$ δ

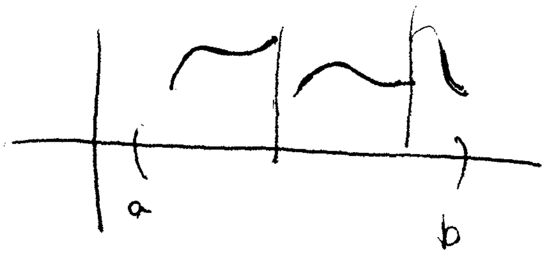
$$m\left(\frac{\pi}{2}\epsilon \sin^2 \delta/2\right) > \int_0^{\pi/2} |\phi(\theta)|\,d\theta$$

$$\therefore \left|\frac{1}{m}\int_0^{\pi/2} \frac{\sin^2 m\theta}{\sin^2\theta}\phi(\theta)\,d\theta\right| \leq \frac{\pi}{2}\epsilon + \frac{1}{\cancel{m}\cancel{\sin^2\delta/2}} \cdot \frac{\cancel{m}\pi\epsilon \cancel{\sin^2\delta/2}}{2}$$

$$= \pi\epsilon.$$

$$\therefore \lim_{m\to\infty} \frac{1}{m}\int_0^{\pi/2} \frac{\sin^2 m\theta}{\sin\theta}\phi(\theta)\,d\theta = 0.$$

Pg 172 W/W



$$\int_{-\pi}^{\pi} |f|\,dt = \pi . A.$$

$$a+\eta \le x \le b-\eta \qquad \eta > 0$$

$$U - \frac{1}{m}\left\{A_0 + \sum_{n=1}^{m-1} S_n(x)\right\} = U - \frac{1}{2\pi m}\int_{-\pi+x}^{\pi+x} \frac{\sin^2\frac{m}{2}(x-t)}{\sin^2\frac{1}{2}(x-t)} f(t)\,dt$$

Now $$\int_{-\pi+x}^{\pi+x} \frac{\sin^2\frac{m}{2}(x-t)}{\sin^2(\frac{1}{2}(x-t))}\,dt = \int_{\theta=\frac{x}{2}-\frac{(-\pi+x)}{2}}^{\theta=\frac{x}{2}-\frac{(\pi+x)}{2}} \frac{\sin^2 m\theta}{\sin^2\theta}\left(-\frac{dt}{2}\right)$$

$$\theta = \frac{x-t}{2}$$

$$d\theta = -\frac{dt}{2}$$

$$= \frac{2}{\cancel{1}}\int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \cdots\cdots \quad dt = 2\cancel{1}\int_{-\frac{\pi}{2}}^{\pi/2} \cdots\cdots \quad dt = 4\int_0^{\pi/2} \cdots\,dt$$

$$\|$$

$$4\frac{\pi m}{2}. = 2\pi m$$

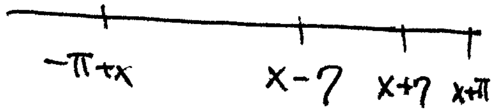
$\therefore$

$$U = \frac{2}{2\pi m}\int_{-\pi+x}^{\pi+x} U\,\frac{\sin^2\frac{m}{2}(x-t)}{\sin^2\frac{x-t}{2}}\,dt$$

$\therefore$

$$= \frac{1}{2\pi m}\left[\int_{-\pi+x}^{\pi+x}\left[\sigma \overset{\frac{\sin^2\frac{m}{2}(x-t)}{\sin^2\frac{x-t}{2}}}{\diagup} - \frac{\sin^2\frac{m}{2}(x-t)}{\sin^2\frac{x-t}{2}} f(t)\right]dt\right]$$

$$= \frac{1}{2\pi m}\left[\int_{-\pi+x}^{x-\eta} + \int_{x-\eta}^{x+\eta} + \int_{x+\eta}^{x+\pi}\right]\left(\qquad\right)dt.$$



$-\pi < -\eta < \underset{?}{\eta} < \pi$

$b-a < 2\pi$

As $a+\eta < b-\eta$

$b-a > 2\eta$

$\eta < \frac{b-a}{2} < \pi$ ✓.

$$= \frac{1}{2\pi m}\left[\int + \int + \int \frac{\sin^2}{\sin^2}\left(\sigma - f(t)\right)dt\right]$$

$$\geq \frac{1}{2\pi m}\left[\int_{-\pi+x}^{x-\eta} + \int_{x+\eta}^{x+\pi}\right]\frac{\sin^2}{\sin^2}\left(\sigma - f\right)dt.$$

$$\frac{1}{2\pi m} \int_{x-\pi}^{x-\eta} \frac{\sin^2 \frac{m}{2}(x-t)}{\sin^2 \frac{x-t}{2}} \left(U - f(t)\right) dt$$

$$\sin^2 \frac{x-t}{2} \geq \cancel{\sin^2 \frac{x-t}{2}}\Big| \quad ? \quad \frac{\sin^2 \eta/2}{}$$

$\geq$

$$\sin^2 \frac{x-t}{2}\Big|_{t=x-\pi} = \sin^2 \frac{\pi}{2} = 1$$

$$\sin^2 \frac{x-t}{2}\Big|_{t=x-\eta} = \sin^2 \eta/2.$$

$$\sin^2 \frac{m}{2}(x-t) \leq \sin^2 \frac{m}{2}(x-t)\Big|_{t=}$$

$$\Big|_{x-\eta} = \sin^2 \frac{m}{2}\eta$$

$$\Big|_{x-\pi} \quad \sin^2 \frac{m\pi}{2} = +1$$

$$0.$$

pg 172 w/w

1

I had from before

$$U - \frac{1}{m}\left\{A_0 + \sum_{n=1}^{m-1} S_n(x)\right\} \geq \frac{1}{2m\pi}\left\{\int_{-\pi+x}^{x-\eta} + \int_{x+\eta}^{x+\pi}\right\} \frac{\sin^2\frac{m}{2}(x-t)}{\sin^2\frac{1}{2}(x-t)}$$

$$\{U - f(t)\}dt$$

$$\geq \frac{-1}{2m\pi}\left\{\int_{-\pi+x}^{x-\eta} + \int_{x+\eta}^{x+\pi}\right.$$

is

$* \quad U - f(t) \geq -(|U|+|f|$

$$U - f > 0$$

or $|U| + U + |f| - f \geq 0$

$$|U| + U = \begin{cases} 2U & U > 0 \\ 0 & U < 0 \end{cases}$$

$$|f| - f = \begin{cases} -2f & f < 0 \\ 0 & f > 0 \end{cases}$$

$|U| + U \geq 0 \quad \forall\, U$

$|f| - f \geq 0 \quad \forall\, f$

$\therefore$

$* \Rightarrow |f| + |U| \geq f(t) - U$

~~$\sin^2\frac{m}{2}(x-t)$~~

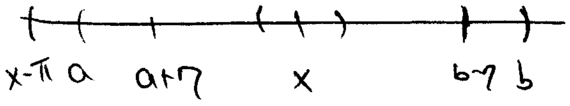
1

~~$t = -\pi + x = \sin^2\frac{m\pi}{2} = 0, 1$~~

~~$t = x - \eta = \sin^2\frac{m}{2}\eta$~~

~~$t = x + \eta =$~~

~~$t = x + \pi = 0, 1$~~



$$\geqslant -\frac{|U|}{2\pi m}\left\{ \int_{-\pi+x}^{x-\eta} + \int_{x+\eta}^{\pi+x} \right] dt$$

$$\cancel{x}-\eta+\pi-\cancel{x} \quad + \quad \pi+\cancel{x}-\cancel{x}-\eta$$

$$\pi-\eta+\pi-\eta \quad = \quad 2\pi-2\eta = 2(\pi-\eta)$$

---

plot $\dfrac{\sin^2\frac{m}{2}(x-t)}{\sin^2\frac{x-t}{2}}$ on $t=-\pi+x$

to $t=-\eta+x$

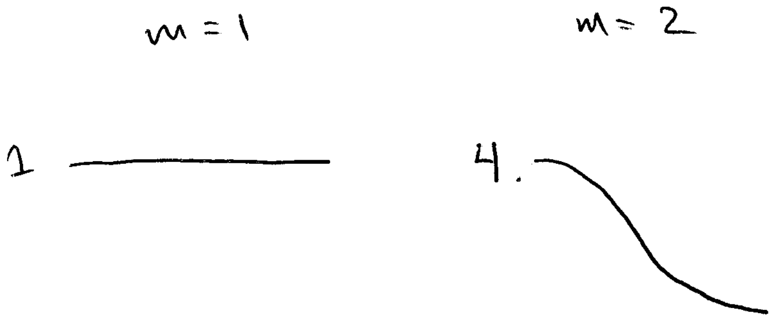
Same as graph from $t=x+\eta$

$t=x+\pi$

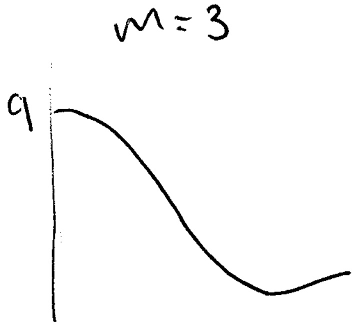
or $\xi=x-t=\pi$ $\qquad \eta\leq\xi\leq\pi$

$\xi=x-t=\eta$

$$\frac{\sin^2\left(\frac{m}{2}\xi\right)}{\sin^2\left(\frac{\xi}{2}\right)}$$



Now $\sin^2\left(\frac{x-t}{2}\right.$

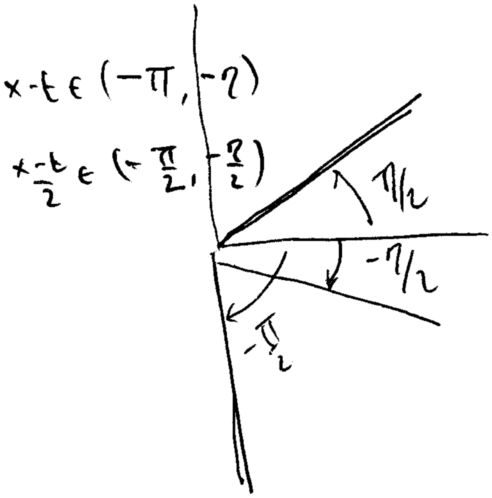


Has Max at $x = 0$?

$\underline{m=4}$

But $\sin^2\frac{x-t}{2} \geq \sin^2 \eta/2$ for

∴ $\frac{1}{\sin^2\frac{x-t}{2}} \leq \frac{1}{\sin^2 \eta/2}.$



$x-t \in (-\pi, -\eta)$

$\frac{x-t}{2} \in (-\frac{\pi}{2}, -\frac{\eta}{2})$

look at $\sin^2\frac{m}{2}(x-t)$

Can get want to show

$$\frac{\sin^2\frac{m}{2}(x-t)}{\sin^2\frac{x-t}{2}}\{U - f(t)\} \geq -\frac{|U| + |f(t)|}{\sin^2\eta/2}$$

on $x-t \in (\eta, \pi)$

iff $$\frac{\sin^2\frac{m}{2}(x-t)}{\text{[illegible]}}\left(\frac{\sin^2\eta/2}{\sin^2\frac{x-t}{2}}\right)\frac{(f(t)-U)}{|f|+U} \leq 1$$

.7

~~$\theta - \frac{1}{m}\sum$~~

$$U - \frac{1}{m}\left\{A_0 + \sum_{n=1}^{m-1} S_n(x)\right\} \geqslant -\frac{1}{2\pi m}\left\{\int_{x-\pi}^{x-\eta} + \int_{x+\eta}^{x+\pi}\right\}\frac{|U|+|f|}{\sin^2 \eta/2}\, d$$

$$U + \frac{1}{2\pi m}\left\{\int_{x-\pi}^{x-\eta} + \int_{x+\eta}^{x+\pi}\right\}\frac{|U|+|f|}{\sin^2 \eta/2}\, dt \geqslant \underline{\frac{1}{m}\left\{A_0 + \sum_{n=1}^{m-1} S_n(x)\right\}}.$$

$$\therefore \quad * \leq U + \frac{1}{2\pi m}\left\{\int_{x-\pi}^{x-\eta} + \int_{x-\eta}^{x+\eta} + \int_{x+\eta}^{x+\pi\,*}\right\}\frac{|U|+|f|}{\sin^2 \eta/2}$$

1st integral $|U|$ $\quad \cancel{x}-\cancel{\eta}-\cancel{x}+\pi+\cancel{x}+\cancel{\eta}-\cancel{x}+\cancel{\eta}+\cancel{x}+\pi-\cancel{x}-\cancel{\eta}$
$= 2\pi.$

$$\therefore \quad \frac{1}{m}\left\{A_0 + \sum_{n=1}^{m-1} S_n(x)\right\} \leq U + \frac{|U|}{\cancel{\pi} m \sin^2 \eta/2} + \frac{1}{2\cancel{\pi} m \sin^2 \eta/2}\int_{x-\pi}^{x+\pi}|f(t)|\,dt$$

$$\|$$

$$U + \left\{|U| + \tfrac{1}{2}A\right\}\Big/ m\sin^2 \eta/2$$

Should hv define

$$\int_{-\pi}^{\pi}|f|\,dt = 2\pi A. \quad \therefore \quad A = \frac{1}{2\pi}\int_{-\pi}^{\pi}|f|\,dt$$

To get 2nd to last step. ~~$-1 < -\frac{(U-f)}{|f|+|U|} < 0$~~

~~Must show~~

$$\geq \frac{|U|+|f|}{\sin^2 \eta/2}$$

$$-\sin^2\left(\frac{m}{2}\cdot x-t\right)\frac{\sin^2 \eta/2}{\sin^2 \frac{x-t}{2}}\,\frac{U-f}{|f|+|U|} \overset{?}{\leq} 1$$

yes because LHS $< 0 \leq 1$ trivially.

pg 172 W/W

consider

$$\frac{1}{m}\left\{A_0+\sum_{n=1}^{m-1}S_n\right\}-L = \frac{1}{2\pi m}\left\{\int_{-\pi+x}^{x-\eta}+\int_{x-\eta}^{x+\eta}+\int_{x+\eta}^{x+\pi}\right\}\frac{\sin^2\frac{m}{2}(x-t)}{\sin^2\frac{x-t}{2}}\Big)$$

$$\{f(t)-L\}\,dt$$

$$\geq \frac{1}{2\pi m}\left\{\int_{-\pi+x}^{x-\eta}+\int_{x+\eta}^{x+\pi}\right\}\frac{\sin^2\frac{m}{2}\cdot x-t}{\sin^2\frac{x-t}{2}}(f-L)\,dt$$

$$\frac{\sin^2\frac{m}{2}\cdot x-t\,(f-L)}{\sin^2\frac{x-t}{2}} \le \frac{|f|+|L|}{\sin^2\eta/2}$$

$$\Downarrow$$

$$\geq -\frac{(|f|+|L|)}{\sin^2\eta/2}$$

By fact that LHS is positive & RHS is negative..!!?

$$\therefore \text{LHS} \geq \frac{-1}{2\pi m}\left\{\int\quad\int\right\}\frac{|f|+|L|}{\sin^2\eta/2}\,dt$$

$$\geq -\frac{1}{2\pi m}\left\{\int\int_{x-\eta}^{x+\eta}\int\right\}\frac{|\;|+|\;|}{\sin^2\eta/2}\,dt$$

$$\frac{1}{m}\left\{A_0 + \sum_{n=1}^{m-1} S_n\right\} \geq L - \frac{1}{2\cancel{\pi} m}\left[\frac{\cancel{\pi} A}{\sin^2 \eta/2} + \frac{|L| 2\cancel{\pi}}{\sin^2 \eta/2}\right]$$

$$= L - \left[|L| + \frac{A}{2}\right] \Big/ m \sin^2 \eta/2 .$$

Pr

Pg 190 w/w

① (w) $$\frac{1-r\cos z}{1-2r\cos z+r^2} = 1 + r\cos z + r^2\cos 2z + \cdots$$

$$= 1 + \sum_{k=1}^{\infty} r^k\cos kz .$$

This is a cosine fourier expansion of LHS

$$\Rightarrow \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt = \frac{2}{\pi}\int_0^{\pi} f(t)\cos nt\,dt$$

$$= \frac{2}{\pi}\int_0^{\pi}\left(\frac{1-r\cos t}{1-2r\cos t+r^2}\right)\cos nt\,dt$$

$$= \frac{1}{\pi}\int_0^{2\pi}\frac{1-r\cos t}{1-2r\cos t+r^2}\cos nt\,dt$$

$$\cos t = \frac{e^{it}+e^{-it}}{2} \qquad z = e^{it}$$

$$dz = ie^{it}dt \Rightarrow dt = \frac{dz}{iz}$$

$$= \frac{1}{\pi}\oint_{|z|=1}\frac{1-\frac{r}{2}(z+\frac{1}{z})}{1-r(z+\frac{1}{z})+r^2}\left(z^n+\left(\tfrac{1}{z}\right)^n\right)\frac{dz}{iz}$$

$$= \frac{1}{i\pi}\oint \frac{1-\frac{r}{2}(z+1/z)}{z-r(z^2+1)+r^2z}\left(z^n+\left(\frac{1}{z}\right)^n\right) dz$$

$$= \frac{1}{\pi i}\oint \frac{z^n+(1/z)^n}{(z-r(z^2+1)+r^2z)} - \frac{r}{2\pi i}\oint \frac{(z+1/z)(z^n+(1/z)^n)}{-r+(1+r^2)z-rz}$$

$$= -\frac{1}{\pi i}\oint \frac{z^n+(1/z)^n}{rz^2-(1+r^2)z+r}\,dz + \frac{r}{2\pi i}\oint \frac{(z+1/z)(z^n+1/z}{rz^2+(1+r^2)z+r}$$

$$\frac{z^{2n}+1}{z^n(\underbrace{rz^2-(\quad)z+r}_{=0})}$$

$$z = \frac{1+r^2\pm\sqrt{(1+r^2)^2-4r^2}}{2r} = \frac{1+r^2\pm\sqrt{1+2r^2+r^4-4r^2}}{2r}$$

$$= \frac{1+r^2\pm\sqrt{(1-r^2)^2}}{2r} = \frac{1+r^2\pm(1-r^2)}{2r}$$

$+$

pg 193 W/W

㉘

$$S_n(x) = 2\sum_{r=1}^{n} (-1)^{r-1} \frac{\sin rx}{r}$$

$$S_n'(x) = 2\sum_{r=1}^{n} (-1)^{r-1} \cos rx$$

By Table or given on test

$$= 2\left[1 - \frac{\cos\left(\frac{n}{2}(x+\pi)\right)\cos\left(\frac{1}{2}\left[n(x+\pi)+x\right]\right)}{\cos\left(\frac{x}{2}\right)}\right]$$

$=$ – But product of cosigns (As when we integrate) becomes

$$\cos(A)\cos(B) = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$$

$\therefore$

$$\cos\left(\frac{n}{2}(x+\pi)\right)\cos\left(\frac{1}{2}(n(x+\pi)+x)\right)$$

$$= \frac{1}{2}\left[\cos\left(\frac{x}{2}+n(\pi+x)\right) + \cos\left(-\frac{x}{2}\right)\right]$$

$\therefore \mu t$

$$= 2\left[1 - \frac{1}{2}\frac{\cos\left(\frac{x}{2} + n(\pi + x)\right)}{\cos(x/2)} - \frac{1}{2}\right]$$

$$= 1 - \frac{\cos\left(\frac{x}{2} + n(\pi + x)\right)}{\cos(x/2)} = 0$$

Set to find Max/Min

$$\Rightarrow \quad \cos\left(\frac{x}{2} + n(\pi + x)\right) = \cos(x/2)$$

$$(1) \Rightarrow \quad \frac{x}{2} = \frac{x}{2} + n(\pi + x) + 2\pi m \qquad m = 0, \pm 1, \pm 2 \ldots$$

or $\quad \frac{x}{2} = -\left(\frac{x}{2} + n(\pi + x)\right) + 2\pi m$ ״

1st eq solving for x gives

$$x_m = -\frac{2m\pi + n\pi}{n} = -\frac{\pi}{n}(2m + n)$$

2nd eq gives

$$x_m = \frac{(2m - n)\pi}{1 + n}$$

As we wont x_m as close to π

but not equl π

$$x_m \underset{\neq}{<} \pi$$

for 1st eq $\Rightarrow$

$$-\frac{\cancel{\pi}}{n}(2m+n) < \cancel{\pi}$$

$$2m+n > -n$$

$$2m > -2n$$

$$m > -n$$

Take $m = -n+1$

Then $x_m = -\frac{\pi}{n}(-2n+2+n) = -\frac{\pi}{n}(-n+2)$

$$= \frac{\pi}{n}(2-n)$$

for 2nd eq

$$\frac{(2m-n)\cancel{\pi}}{1+n} \underset{\neq}{<} \cancel{\pi}$$

$$2m < n+1+n = 2n+1$$

$m < n+\frac{1}{2}$ take $m = \underline{n}$

Then $x_m = \frac{n}{n+1}\pi$

which is closer to π $\frac{n}{n+1}\pi$ or $\frac{2-n}{n}\pi$?

Consider $\frac{n}{n+1} - \frac{2-n}{n} = \frac{n^2}{(n+1)n} - \frac{(2-n)(n+1)}{n(n+1)}$

$$= \frac{n^2-(2n+2-n^2-n)}{n(n+1)}$$

$$= \frac{2n^2-n-2}{n(n+1)} > 0$$ for n large enough

$\therefore$ $\frac{n}{n+1}\pi$ is closer to π for large n.

<u>Now:</u>

$$S_n\left(\frac{n\pi}{n+1}\right) = 2\sum_{r=1}^{n} \frac{(-1)^{r-1}}{r} \sin\left(\frac{n\pi}{n+1} r\right)$$

$$= 2\sum_{r=1}^{n} \frac{(-1)^{r-1}}{r} \sin\left(\frac{(n+1)\pi r - \pi r}{(n+1)}\right)$$

$$= 2\sum_{r=1}^{n} \frac{(-1)^{r-1}}{r} \sin\left(\pi r - \frac{\pi}{n+1} r\right)$$

$$= 2\sum_{r=1}^{n} \frac{(-1)^{r-1}(-1)(-1)^{r}}{r} \sin\left(\frac{\pi}{n+1} r\right)$$

$$= 2\sum_{r=1}^{n} \frac{\sin\left(\frac{\pi r}{n+1}\right)}{r}$$

let

$$t_r = \frac{\pi r}{n+1} \to r = \frac{n+1}{\pi}$$

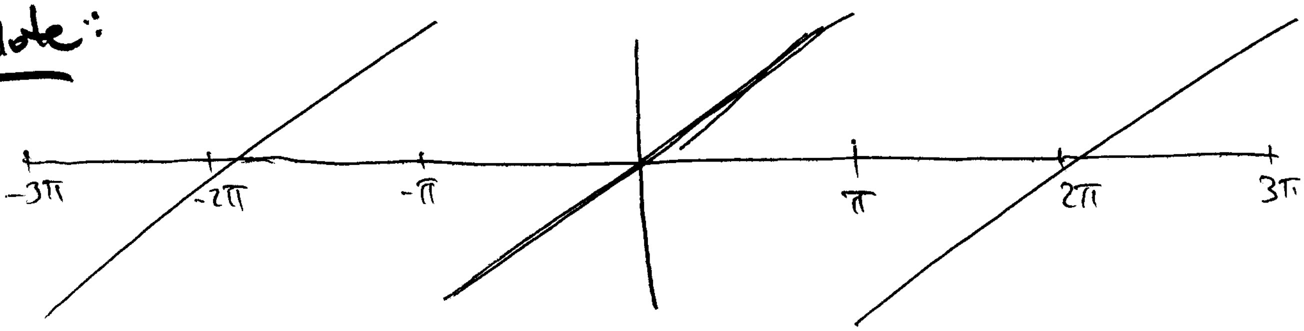
$$dt_r = \frac{\pi}{n+1}$$

$$= 2\sum_{t_r=\frac{\pi}{n+1}}^{t_r=\frac{\pi n}{n+1}} \frac{\sin t_r}{t_r}\underbrace{\left(\frac{\pi}{n+1}\right)}_{dt_r}$$

$$\xrightarrow{\lim n\to\infty} = 2\int_{t=0}^{\pi} \frac{\sin t}{t}\,dt$$

✓

Note:



on (-π, π) fourier series of x is

$$x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx) + b_n\sin(nx)$$

w/

$$a_n = \frac{1}{l}\int_{-l}^{l} x\cos(nx)\,dx = 0$$

As x is odd over (-π, π)

&

$$b_n = \frac{1}{l}\int_{-l}^{l} x\sin(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)\,dx = \frac{1}{\pi}\left[-x\frac{\cos(nx)}{n}\Big|_{-\pi}^{\pi} + \frac{1}{n}\int_{-\pi}^{\pi}\cos(nx)\,dx\right] = \frac{1}{\pi}\left[\frac{-1}{n}\left(\pi(-1)^n + \pi(-1)^n\right) + \frac{1}{n}\sin(nx)\Big|_{-\pi}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{2(-1)^{n+1}\pi}{n} + \cancelto{0}{\frac{1}{n}\sin(n\pi)}\right] = \frac{2(-1)^{n+1}}{n}$$

$$\therefore \quad x = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx)$$

The fourier series will converge to x $\forall x \in (-\pi, \pi)$ & to 0 At $x = \pm\pi$. & to periodic extensions of period 2π $\forall$ other x.

$$S_n = 2\sum_{r=1}^{n} \frac{(-1)^{r+1}}{n} \sin(rx)$$ has a maximum at

$\frac{n\pi}{n+1} < \pi$ that moves towards π as $n \to +\infty$.

But the height of this maximum $S_n\left(\frac{n\pi}{n+1}\right) \to 2\int_0^{\pi} \frac{\sin t}{t}\,dt$ As $n \to \infty$ & we can numerically calculate the value of this integral ~~by using taylor series~~ For example....

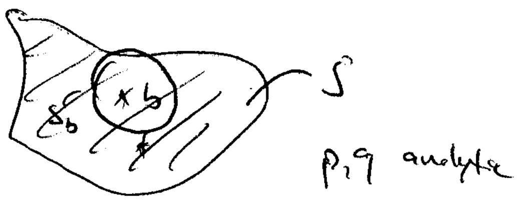
$$\frac{\sin t}{t} = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{(2k+1)!}$$

$$\int_0^{\pi} \frac{\sin t}{t}\,dt = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \frac{t^{2k+1}}{(2k+1)} \Bigg|_0^{\pi} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \frac{\pi^{2k+1}}{(2k+1)}$$

if one can get $G \cong 3{,}704 \ldots > \pi$ which over shoots the value that it should have of π.

pg 194 w/w

$$\frac{d^2u}{dz^2} + p(z)\frac{du}{dz} + q(z)u = 0$$



let $u = v\exp\left\{-\frac{1}{2}\int_b^z p(\xi)\,d\xi\right\}$

$$u' = v'\exp\{\ \} + v\left(\frac{-1}{2}p(z)\right)\exp\{\ \}$$

$$u'' = v''\exp\{\ \} + 2v'\left(-\frac{1}{2}\right)p(z)\exp\{\ \} + v\left(-\frac{1}{2}p(z)\exp\{\ \}\right)'$$

$$= v''e^{\square} - v'pe^{\square} + v\left(-\frac{p'}{2}e^{\square} - \frac{p}{2}\left(-\frac{p}{2}\right)e^{\square}\right) -$$

$$= \left(v'' - v'p - p'\frac{v}{2} + \frac{p^2v}{4}\right)e^{\square}$$ put into diff eq

$$\Rightarrow v'' - \cancel{v'p} - p'\frac{v}{2} + \overset{keep}{\cancel{\frac{vp^2}{4}}} + \cancel{pv'} - \overset{keep}{\cancel{\frac{vp^2}{2}}} + qv = 0$$

$$\Rightarrow v'' + \left(q - \frac{p'}{2} - \frac{p^2}{4}\right)v = 0$$

$$\Rightarrow v'' + J(z)v = 0$$

$(D^2 + J(z))v = 0$ $D^2 + J(z)$ is an easy op to invert.

$$V_0 = a_0 + a_1(z-b)$$

$$V_n = -\iint J(\xi)\, V_{n-1}(\xi)\, d\xi$$

$$\Rightarrow \quad V_n'' + J(z)\, V_{n-1} = 0$$

$$V_n'' = -J(z)\, V_{n-1}(z)$$

$$V_n' = - \int^z$$

Don't ~~see~~ $J(z)$ has no sing except those of p & q.

f How get $V_n(z) = ?$

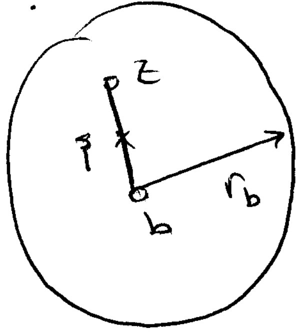
$\underline{n=0}$

$|a + a_1(z-b)| \le \mu$ ✓ By def of μ

$$|V_m(z)| = \left| \int_b^z (\xi - z)\, J(\xi)\, V_{m-1}(\xi)\, d\xi \right|$$

$$\le \frac{1}{(m-1)!} \int_b^z |\xi - z|\, |J(\xi)|\, \underline{\mu M^{m-1}}\, |\xi - b|^{2m-2}\, |d\xi|$$

$$\le \frac{\mu M^m}{(m-1)!} \int_b^z |\xi - z|\, |\xi - b|^{2m-2}\, |d\xi|$$



Now: $|\xi - z| \le |z-b|$ As
in integrand
integration is along a st. line w/ end points b & z.

Then $$|V_m(z)| \le \frac{\mu M^m |z-b|}{(m-1)!} \int_b^z |\xi - b|^{2m-2} \, |d\xi|$$

Change $\int_b^z |\xi - b|^{2m-2} \, |d\xi|$ to a real integral.

let $t = |\xi - b|$ $\quad t : 0 \to \mathbb{R}^+$ Along $b \to z$
$t : 0 \to |z-b|$

Thus on the t axis $$\underbrace{\int_0^{|z-b|} t^{2m-2} \, dt}_{} = \left. \frac{t^{2m-1}}{(2m-1)} \right|_0^{|z-b|}$$

$d\xi = \underset{\xi^{n+1}}{|} \quad \underset{\xi^n}{|} \rightarrow$

As Thus $|d\xi| = dt$ <u>only</u> if path

$[|d\xi| = \xi^{n+1} - \xi^n$ difference taken in$]$ is taken along a st. line

$$\therefore \; |V_m(z)| \le \frac{\mu M^m |z-b|}{(m-1)!} \left(\frac{|z-b|^{2m-1}}{(2m-1)} \right) \le \frac{\mu M^m |z-b|^{2m}}{m!}$$

But $\frac{1}{2m-1} < \frac{1}{m}$

4

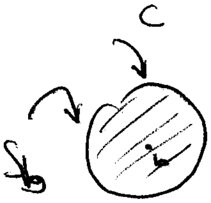
$$\sum \frac{u M^n r_b^{2n}}{n!} \; ; \quad \lim_{n\to\infty} \frac{\frac{M^{n+1} r_b^{2n+2}}{(n+1)!}}{\frac{M^n r_b^{2n}}{n!}} = \lim_{n\to\infty} \frac{M r_b^2}{(n+1)} = 0 < 1$$

$$\frac{d}{dz} v_n(z) = 0 + -\int_b^z J(\xi) v_{n-1}(\xi)\, d\xi$$

$$\frac{d^2 v_n}{dz^2} = -J(z) v_{n-1}(z)$$

5,3

Here the contour is $\int_b$



convergence is uniform then
+ each V_n is analytic there

$$\frac{d^2 v}{dz^2} = \frac{d^2 v_0}{dz^2} + \sum_{n=1}^{\infty} \frac{d^2 v_n}{dz^2}$$

$$= 0 + -J(z) \sum_{n=1}^{\infty} v_{n-1}(z) = -J(z) v(z)$$

$$v(b) = v_0(b) + \sum_{n=1}^{\infty} v_{n-1}(b) = a_0 + 0 = a_0$$

~~$\therefore \; |V_m(z)| \leq \frac{u M^m}{m!} |z-b|^{2m+1}$~~

~~$= \frac{|z-b|^{2m}}{2m}$~~

4

$$v'(b) = \frac{dv_0}{dz} + \sum_{n=1}^{\infty} \frac{dv_n}{dz} = a_1 + \sum_{n=1}^{\infty} \cancel{-J(z)v_n(z)}\Big|_{z=b} \frac{dv_n}{dz}\Big|_{z=b}$$

but $\frac{dv_n}{dz}\Big|_{z=b} = -\int_b^b \quad = 0.$

$$\frac{d^{n-2}}{dz^{n-2}} \frac{d^2}{dz^2} w + \frac{d^{(n-2)}}{dz^{n-2}} Jw = 0 \xleftarrow{\qquad} \text{question?}$$

$$w^n + \sum_{k=0}^{n-2} \binom{n-2}{k} J^{(k)} w^{(n-2-k)} = 0$$

$n-2-(n-1) = -1$

$$\cancel{w^{(n)}} + \quad w^{(n)} + Jw^{(n-2)} + \binom{n-2}{1} J' w^{(n-3)} + \binom{n-2}{2} J'' w^{n-4}$$

$$+ \cdots + \binom{n-2}{n-3} \underline{J}^{(n-3)} w' + \binom{n-2}{n-2} \underline{J^{(n-2)}} w = 0$$

evaluate at b.. w/ n = 2, 3, 4....

1st eq $\frac{d^2 w}{dz^2} = -Jw\Big|_{z=b}$ if $w(b) = 0$.

$\Rightarrow \frac{d^2 w}{dz^2}(b) = 0.$ $\quad \frac{d^3 w}{dz^3}\Big|_b = Aw + Bw'\Big|_b = 0.$

$\Rightarrow U(z) = v(z) e^{\square}$

$$u(z) = v(z)\exp\left\{ -\frac{1}{2}\int_b^z p(\xi)\,d\xi \right\}$$

$$u(b) = A_0 = a_0$$

$$u'(z) = v'(z)\exp\{\quad\} - \frac{1}{2}p(z)v(z)\exp\{\quad\}$$

$$u'(b) = A_1 = a_1 - \frac{1}{2}p(b)a_0$$

Ex 1: $(1-z^2)u'' - 2zu' + \frac{3}{4}u = 0$ $\frac{3}{4} = \frac{1}{2}(1+\frac{1}{2})$ (Legendre's eq order $\frac{1}{2}$)

ordinary pts $\forall\ z \neq \pm 1$ ∴ At $z=0$ a unique, analytic eq exists consider $u(z) = \sum_{k=0}^{\infty} a_k z^k$, $u' = \sum a_k k z^{k-1}$,

~~$\sum_{k=0}^{\infty} a_k z^k - \sum_{k=0}^{\infty} a_k z^{k+2} - z$~~

$$(1-z^2)\sum_{k=0}^{\infty} a_k k(k-1) z^{k-2} - 2\sum_{k=0}^{\infty} a_k k z^k + \frac{3}{4}\sum_{k=0}^{\infty} a_k z^k = 0$$

Pg 197 w/w

regular singular points.

$$\frac{d^2u}{dz^2} + p(z)\frac{du}{dz} + q(z)u = 0$$

$$(z-c)^2\frac{d^2u}{dz^2} + (z-c)\underbrace{(z-c)p(z)}_{P(z-c)}\frac{du}{dz} + \underbrace{(z-c)^2 q(z)}_{Q(z-c)}u = 0$$

$P(z-c)$ Analytic at $c = \sum_{n=0}^{\infty} p_n(z-c)^n$

$Q(z-c)$ Analytic at $c = \sum_{n=0}^{\infty} q_n(z-c)^n$

Assume. $u = (z-c)^{\alpha}\left[\sum_{n=0}^{\infty} a_n(z-c)^n\right]$ $\quad a_0 = 1$

$$u' = \alpha(z-c)^{\alpha-1}\sum_{n=0}^{\infty} a_n(z-c)^n + (z-c)^{\alpha}\sum_{n=1}^{\infty} a_n n(z-c)^{n-1}$$

$$u'' = \alpha(\alpha-1)(z-c)^{\alpha-2}\sum_{n=0}^{\infty} a_n(z-c)^n + 2\alpha(z-c)^{\alpha-1}\sum_{n=1}^{\infty} a_n n(z-c)^{n-1} + (z-c)^{\alpha}\sum_{n=2}^{\infty} a_n n(n-1)(z-c)^{n-}$$

put in diff eq.

pg 281 W/W

$$F(a,b;c;z) = \sum_{k=0}^{\infty} \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)\,k!} z^k$$

$\therefore$

$$F(1,B;1;z/B) = \sum_{k=0}^{\infty} \frac{\Gamma(1)\cancel{\Gamma(k+1)}\,\Gamma(B+k)}{\Gamma(1)\Gamma(B)\,\Gamma(k+1)\cancel{k!}} \frac{z^k}{B^k}$$

$$= \sum_{k=0}^{\infty} \frac{\Gamma(B+k)}{\Gamma(B)\,B^k} \frac{z^k}{k!}$$

$$\lim_{B\to\infty} F(1,B;1;z/B) = \lim_{B\to\infty} \sum_{k=0}^{\infty} \cdots = \sum_{k=0}^{\infty} \lim_{B\to\infty} \frac{\Gamma(B+k)}{\Gamma(B)\,B^k} \frac{z^k}{k!}$$

$$\frac{\Gamma(B+k)}{\Gamma(B)B^k} = \frac{(B+k)(B+k-1)\cdots(B+1)\cancel{\Gamma(B)}}{B \quad B \cdots\cdots B\,\cancel{\Gamma(B)}}$$

$$= \left(1+\frac{k}{B}\right)\left(1+\frac{k-1}{B}\right)\cdots\left(1+\frac{1}{B}\right) =$$

$$\therefore \lim_{B\to\infty} = 1.$$

$$\therefore \lim_{B\to\infty} F(1,B;1;z/B) = e^z \quad ✓.$$

pg 281 Whittaker/Watson

$$F(a,b;c;z) = 1 + \frac{a\cdot b}{1\cdot c} z + \frac{a(a+1)b(b+1)}{1\cdot 2\, c(c+1)} z^2$$

$$+ \frac{a(a+1)(a+2)b(b+1)(b+2)}{1\cdot 2\cdot 3\; c(c+1)(c+2)} z^3 + \cdots$$

$$\cancel{= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)k!} \frac{\Gamma(a+k+1)\Gamma(b+k+1)}{\Gamma(c+k+1)} z^k}$$

$$\doteq \Gamma(c+k+1) = \text{[illegible]} (c+k)\Gamma(c+k)$$

$$= (c+k)(c+k-1)\Gamma(c+k-1)$$

$$= (c+k)(c+k-1)\cdots(c+1)\, c\, \Gamma(c)$$

$$(c+k-1)(c+k-2)\cdots (c+1)c\Gamma(c)$$

$$= \Gamma(c+k)$$

$$= \sum_{k=0}^{\infty} \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)k!} z^k = F(a,b;c;z)$$

$$\frac{d}{dz}\Gamma(a,b;c;z) = \sum_{k=1}^{\infty} \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)\,(k-1)!} z^{k-1}$$

$$\sum_{k=a}^{\infty} \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \frac{\Gamma(a+k+1)\,\Gamma(b+1+1)}{\Gamma(c+k+1)\;k!} z^k$$

$$= \left(\sum_{k=0}^{\infty} \frac{\Gamma(c+1)}{\Gamma(a+1)\,\Gamma(b+1)} \frac{\Gamma(a+1+k)\,\Gamma(b+1+k)}{\Gamma(c+1+k)\,k!} z^k \right) \cdot \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \cdot \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(c+1)}$$

$$= \frac{a\,\cancel{\Gamma(a)}\,b\,\cancel{\Gamma(b)}}{c\,\cancel{\Gamma(c)}} \cdot \frac{\cancel{\Gamma(c)}}{\cancel{\Gamma(a)}\,\cancel{\Gamma(b)}} F(a+1, b+1; c+1; z)$$

$$= \frac{ab}{c} F(a+1, b+1; c+1; z)$$

pg 281 W/W

$$F(-n,\beta;\beta;-z)=\sum_{k=0}^{\infty}\frac{\Gamma\cancel{(\beta)}}{\Gamma(-n)\cancel{\Gamma(\beta)}}\,\frac{\Gamma(-n+k)\,\cancel{\Gamma(\beta+k)}}{\cancel{\Gamma(\beta+k)}}\,z^{k}$$

$$=\sum_{k=0}^{\infty}\frac{\Gamma(-n+k)}{\Gamma(-n)}z^{k}\;=\sum_{k=0}^{\infty}$$

Legendre Polynomials : pg 302 w/W

$$\frac{1}{\sqrt{1-2zh+h^2}} = P_0(z) + hP_1(z) + h^2P_2(z) + h^3P_3(z) + \cdots$$

expanding in powers of h

w/ $P_0 = 1$, $P_1 = z$, $P_2 = \frac{1}{2}(3z^2-1)$, . . .

or more generally

$$P_n(z) = \sum_{k=0}^{m} \frac{(-1)^k (2n-2k)!}{2^k k!(n-k)!(n-2k)!} z^{n-2k}$$

$P_n(1)$ let $z=1$ in

$$\frac{1}{\sqrt{1-2h+h^2}} = \frac{1}{\sqrt{(1-h)^2}} = \frac{1}{1-h} = \sum_{k=0}^{\infty} h^k$$

$$\Rightarrow \quad P_n(1) = \underline{1}$$

$h = -1$

$$\frac{1}{\sqrt{(1+h)^2}} = \sum_{k=0}^{\infty} (-1)^k h^k = \sum_{n=0}^{\infty} P_n(-1) h^n$$

$$\Rightarrow \quad P_n(-1) = (-1)^n$$

$$\frac{1}{\sqrt{1-h^2}} = \sum_{k=0}^{\infty} P_k(0) h^k$$

$\Big\downarrow$ is even fn

$\Rightarrow$ ~~[illegible]~~ ~~[illegible]~~ . $\Rightarrow P_{2k+1}(0) = 0$ As LHS.

pg 302 Whittaker / Watson

$$\frac{1}{\sqrt{1-2zh+h^2}} = f(h)$$

$$f'(h) = \frac{-(-2z+2h)}{2(1-2zh+h^2)^{3/2}} = \frac{+z-h}{(1-2zh+h^2)^{3/2}}$$

$$= (z-h)(1-2zh+h^2)^{-3/2}$$

~~$f'' =$~~

$$f'' = -(1-2zh+h^2)^{-3/2}$$

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$$

$$\frac{1}{1-(2zh-h^2)} = \sum_{k=0}^{\infty} (2zh-h^2)^k$$

Expand:

$$\frac{1}{\sqrt{1-x}} = (1-x)^{-1/2} = f(x)$$

$$f'(x) = -\tfrac{1}{2}(1-x)^{-3/2}(-1)$$

$$= \frac{(1-x)^{-3/2}}{2}$$

$$f'' = -\frac{3}{4}(1-x)^{-5/2}$$

$$f''' = \left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)(1-x)^{-7/2}$$

$\vdots$

$2k-1$

$2(k-1)+1$

$k-1$ $k-2$ 1

$$f^k(x) = \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{(2k-1)}{2}\right)(1-x)^{-\frac{(2k+1)}{2}}$$

$$= \frac{(-1)^k}{2^k}(1\cdot 3\cdot 5 \quad (2k-1))(1-x)^{-(2k+1)/2}$$

$$\therefore \quad \frac{1}{\sqrt{1-x}} = \sum_{k=0}^{\infty} \frac{(-1)^k(1\cdot 3\cdot 5 \cdots (2k-1))}{2^k\, k!}\, x^k$$

$$\downarrow$$

$$\frac{1\cdot 2\cdot 3\cdot 4\cdot 5 \cdots (2k-1)\cdot 2k}{2\cdot 4\cdot 6 \cdots 2k}$$

$$= \frac{(2k)!}{2^k\, k!}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k (2k)!}{4^k (k!)^2}\, x^k$$

Now $$\frac{1}{\sqrt{1-(2zh-h^2)}} = \sum_{k=0}^{\infty} \frac{(-1)^k\, 2k!}{4^k (k!)^2} (2zh-h^2)^k$$

$$\| $$

$$h^k (2z-h)^k$$

$$\| $$

$$h^k \sum_{j=0}^{k} \binom{k}{j} 2^{k-j} z^{k-j} (-1)^j h^j$$

$$\sum_{k=0}^{\infty} \left(h^{k+1},\, h^{k+2} \ldots\, h^{2k} \right)$$

$$h,\, h^2,$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k\, 2k!}{4^k (k!)^2}\, h^k \sum_{j=0}^{k} \frac{k!}{(k-j)!\,j!} (-1)^j z^{k-j} (-1)^j h^j$$

$$\left(\sum_{k=0}^{\infty} A_k h^k \sum_{j=0}^{k} B_j h^j \right)$$

Thus [illegible] power of h

=

$$= \sum_{k=0}^{\infty}\Big(\underbrace{\sum_{j=0}^{k}(A_k B_j)}_{A_k h^{k+j}}\Big)h^{k+j}$$

$$\sum_{k=0}^{\infty} h^k \sum_{j=0}^{k} \cdots \qquad k$$

$$= \Big[\, h^0 \sum_{j=0}^{0} \Big]_{k=0} + \sum_{k=1}^{\infty} h^k \sum_{j=0}^{k}$$

$$\underbrace{\qquad\qquad}_{(h^k,\, h^{k+1},\, h^{k+2},\, \ldots,\, h^{2k})}$$

$k = 1$

$h, h^2, h^3, \ldots, h^2 \quad = h, h^2$

$k = 2$

$h^2, h^3, h^4 \qquad h^2 \quad h^3 \quad h^4$

$k = 3$

$h^3, h^4, h^5, h^6. \qquad h^3 \quad h^4 \quad h^5 \quad h^6$

$$\therefore \sum_{k=0}^{\infty} A_k h^k \sum_{j=0}^{k} B_j(k) h^j$$

$$= A_0 + A_1 h\left(B_0^{(1)} + B_1^{(1)}h\right) + A_2 h^2\left(B_0^{(2)} + B_1^{(2)}h + B_2^{(2)}h^2\right)$$

$$+ A_3 h^3\left(B_0^{(3)} + B_1^{(3)}h + B_2^{(3)}h^2 + B_3^{(3)}h^3\right) + A_4 h^4\left(B_0^{(4)} + B_1^{(4)}h + B_2^{(4)}h^2\right.$$

$$\left. + B_3^{(4)}h^3 + B_4^{(4)}h^4\right) + \cdots$$

$$= A_0 + A_1 B_0^{(1)}h + A_1 B_1^{(1)}h^2 + A_2 B_0^{(2)}h^2 + A_2 B_1^{(2)}h^3 + A_2 B_2^{(2)}h^4$$

$$+ A_3 B_0^{(3)}h^3 + A_3 B_1^{(3)}h^4 + \underline{A_3 B_2^{(3)}h}^5 + A_3 B_3^{(3)}h^6 + A_4 B_0^{(4)}h^4 + A_4 B_1^{(4)}h^5$$

$$+ A_4 B_2^{(4)}h^6 + A_4 B^{(4)}$$

$$= A_0 + A_1 B_0^{(1)}h^1 + \left(A_{\underline{1}} B_1^{(1)} + A_{\underline{2}} B_0^{(2)}\right)h^2 + \left(A_2 B_1^{(2)} + A_{\underline{3}} B_0^{(3)}\right)h^3$$

$$+ \left(A_{\underline{2}} B_2 + A_{\underline{3}} B_1 + A_{\underline{4}} B_0\right)h^4 + \left(A_3 B_2^{(3)} + \right. \qquad \frac{3}{2}$$

$$= \sum_{k=0}^{\infty}\left(\sum_{j=k}^{\lceil \frac{k}{2}\rceil} A_j B_{(k-j)}^{(j)}\right)h^k \qquad B_\square = \frac{k!}{(k-\square)!\,\square!}(-1)^{\square}$$

$$\Big\|$$

$$\sum_{j=\lceil \frac{k}{2}\rceil}^{k} A_j B_{k-j} = \sum_{j=\lceil \frac{k}{2}\rceil}^{k} \frac{(-1)^j (2j)!}{4^j j!^2}\,\frac{k!\,(-1)^{k-j}}{(k-(k-j))!\,(k-j)!}\, z^{k-(k-j)}$$

Ex 2: pg 303 w/w

Show

$$\left(1-2h\cos\theta+h^2\right)^{-1/2} = \left(1+\tfrac{1}{2}he^{i\theta}+\frac{1\cdot3}{2\cdot4}h^2e^{2i\theta}+\cdots\right)$$
$$\left(1+\frac{h}{2}e^{-i\theta}+\frac{1\cdot3}{2\cdot4}h^2e^{-2i\theta}+\cdots\right)$$

LHS

$$\Rightarrow \left(1-h(e^{i\theta}+e^{-i\theta})+h^2\right)^{-1/2}$$

$$= \left(1-he^{i\theta}-he^{-i\theta}+h^2\right)^{-1/2}$$

$$= \left(1-e^{i\theta}h\right)^{-1/2}\left(1-e^{-i\theta}h\right)^{-1/2}$$

$$= \left[\sum_{k=0}^{\infty}\binom{-1/2}{k}(-1)^k e^{ik\theta}h^k\right]\left[\sum_{k=0}^{\infty}\binom{-1/2}{k}(-1)^k e^{-ik\theta}h^k\right]$$

~~$= \left(1-\tfrac{1}{2}e^{i\theta}h+\frac{(1/2)(1/2-1)}{2!}e^{2i\theta}h^2-\cdots\right)$~~

~~$\left(1-\tfrac{1}{2}e^{-i\theta}h+1/2(\right.$~~

$$= \sum_{k=0}^{\infty} \frac{(-\frac{1}{2})(-\frac{1}{2}-1)(-\frac{1}{2}-2)\cdots(-\frac{1}{2}-k+1)}{k!} (-1)^k e^{ik\theta} h^k$$

$$\sum_{k=0}^{\infty} \frac{(-\frac{1}{2})(-\frac{1}{2}-1)(-\frac{1}{2}-2)\cdots(-\frac{1}{2}-k+1)}{k!} (-1)^k e^{-ik\theta} h^k$$

$$= \sum_{k=0}^{\infty} \frac{(\frac{1}{2})^k (1)(1+2)(1+4)\cdots(1+2k-2)}{k!} e^{ik\theta} h^k$$

$$\sum_{k=0}^{\infty} \frac{(\frac{1}{2})^k (2k-1)!!\, e^{-ik\theta}}{k!} h^k \qquad (-1)^{ii} = 1$$

∴ $P_n(\cos\theta)$ nth term in the taylor series of $(1-2h\cos\theta+h^2)^{-1/2}$

Pg 306 wxw

Ex 2:

$$I = \int_0^1 P_m P_n \, dz$$

$m = n$

$$= \int_0^1 P_m^2 \, dx \propto \int_0^1 \frac{d^m}{dx^m}(1-x^2) \cdot \frac{d^m(1-x^2)}{dx^m} \, dx$$

$$= \frac{d^m(1-x^2)}{dx^m} \frac{d^{m-1}}{dx^{m-1}}(1-x^2) \Big|_0^1$$

$$- \int_0^1 \frac{d^{m-1}}{dx^{m-1}}(1-x^2) \cdot \frac{d^{m+1}}{dx^{m+1}}(1-x^2) \, dx$$

Ex. Credit

given $(n+1)P_{n+1} - (2n+1)zP_n + nP_{n-1} = 0$ (a)

& $P'_{n+1} - P'_{n-1} = (2n+1)P_n$ (b)

get $zP'_n - P'_{n-1} = nP_n$

consider:

$$\frac{d}{dz}(a) \rightarrow (n+1)P'_{n+1} - (2n+1)P_n - (2n+1)zP'_n + nP'_{n-1} = 0$$

put in (b) $\Rightarrow$ $(n+1)(b)$

~~[illegible]~~

$$(n+1)P'_{n-1} + (2n+1)(n\cancel{+1})P_n - (2\cancel{n+1})P_n - (2n+1)zP'_n + nP'_{n-1} = 0$$

$$\Rightarrow \underbrace{(2n+1)}P'_{n-1} + (\cancel{2n}+1)nP_n - (\cancel{2n}+1)zP'_n = 0$$

$2\cancel{n}+1$

~~[illegible]~~

$$\cancel{(2n+1) z P_n' - (n+2) P_{n-1}' = (2n+1)} n P_n$$

$$\cancel{z P_n' - \underset{\sim 1}{\left(\frac{n+2}{2n+1}\right)} P_{n-1}' = n P_n}$$

$$\Rightarrow \quad P_{n-1}' + n P_n - z P_n' = 0$$

$$\text{or} \quad z P_n' - P_{n-1}' = n P_n \qquad \text{(C).}$$

pg 309 w/w

Pr:

$$\frac{d}{dz}\{z(P_n^2+P_{n+1}^2)-2P_nP_{n+1}\} = (2n+3)P_{n+1}^2-(2n+1)P_n^2$$

$$\Vert$$

$$P_n^2+P_{n+1}^2+2zP_nP_n'+2zP_{n+1}P_{n+1}'$$

$$-2P_nP_{n+1}'-2P_n'P_{n+1}$$

$$= P_{n+1}\{P_{n+1}+2zP_{n+1}'-2P_n'\}$$

$$+P_n\{P_n+2zP_n'-2P_{n+1}'\}$$

$$= P_{n+1}\{P_{n+1}+2(zP_{n+1}'-P_n')\}$$

$$+P_n\{P_n+2zP_n'-2P_{n+1}'\}$$

$$= P_{n+1}\{P_{n+1}+2((n+1)P_{n+1})\}$$

$$+P_n\{P_n+2zP_n'-2(P_{n-1}'+(2n+1)P_n)\}.$$

— put in $P_{n+1}'-P_{n-1}'=(2n+1)P_n$

$$= P_{n+1}^2(2n+3)+\;\ldots$$

$$= (2n+3)P_{n+1}^2 + P_n \{-(2n+1)P_n - (2n+1)P_n - 2P_{n-1}' + P_n + 2zP_n' \}$$

$$= (2n+3)P_{n+1}^2 - (2n+1)P_n^2$$

$$+ P_n \{ \cancel{P_n} + 2zP_n' - 2nP_n - \cancel{P_n} - 2P_{n-1}' \}$$

$$= (2n+3)P_{n+1}^2 - (2n+1)P_n^2$$

$$+ P_n \{ 2(\underbrace{zP_n' - P_{n-1}' - nP_n}_{=0 \quad \text{by (C)}}) \}$$

$$= (2n+3)P_{n+1}^2 - (2n+1)P_n^2$$

㉕ <u>pg 333 W/W</u>

$$f(x) = \sum_{n=0}^{\infty} A_n P_n(x)$$

Then $$\int_1^x f(t)\,dt = \sum_{n=0}^{\infty} A_n \int_1^x P_n(t)\,dt$$

$$P_n(t) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (t^2-1)^n$$

$$= \cancel{\sum A_n} \quad A_0 \int_1^x dt + A_1 \int_1^x t\,dt$$

$$+ \sum_{n=2}^{\infty} A_n \frac{1}{2^n n!} \int_1^x \frac{d^n}{dx^n} (t^2-1)^n\,dt$$

$$= A_0(x-1) + A_1\left(\frac{x^2}{2} - \frac{1}{2}\right) + \sum_{n=2}^{\infty} \frac{A_n}{2^n n!} \frac{d^{n-1}}{dx^{n-1}} (t^2-1)^n \Big|_1^x$$

Pg 337 W/W

$$v'' + v' + \left(\frac{k}{z} + \frac{1/4 - m^2}{z^2}\right)v = 0.$$

$$v = e^{-z/2} W_{k,m}(z)$$

$$v' = -\frac{1}{2} e^{-z/2} W_{k,m}(z) + e^{-z/2} W_{k,m}'(z)$$

$$v'' = \frac{1}{4} e^{-z/2} W_{k,m}(z) - \frac{2}{2} e^{-z/2} W_{k,m}'(z) + e^{-z/2} W_{k,m}''(z)$$

$$v'' + v' + (\qquad)v =$$

$$\cancel{e}^{-z/2}\Big[W_{k,m}'' - \cancel{W'} + \frac{1}{4}W + \cancel{W'} - \frac{1}{2}W.$$

$$+ \frac{k}{z}W + \left(\frac{1/4}{z^2} - \frac{m^2}{z^2}\right)W\Big] = 0$$

$$= W'' + \text{[illegible]} + \left\{-\frac{1}{4} + \frac{k}{z} + \frac{1/4 - m^2}{z^2}\right\}W = 0.$$

$t = 1/z \qquad \frac{dt}{dz} = -\frac{1}{z^2} = -t^2$

$\frac{d}{dt} = \frac{dz}{dt}\frac{d}{dz} = -\frac{1}{t^2}\frac{d}{dz}$

$\frac{d}{dz} = -t^2 \frac{d}{dt}.$

$$\frac{d^2}{dz^2} = \frac{dt}{dz}\cdot\frac{d}{dt}\left(-t^2\frac{d}{dt}\right) = -t^2\left(-2t\frac{d}{dt} - t^2\frac{d^2}{dt^2}\right)$$

$$= t^4\frac{d^2}{dt^2} + 2t^3\frac{d}{dt}.$$

$$\Rightarrow \quad t^4\frac{d^2w}{dt^2} + 2t^3\frac{d}{dt}w + \left\{-\frac{1}{4} + kt + (1/4 - m^2)t^2\right]w = 0$$

$t = 0$ is $\lim_{t \to 0} t\left(\frac{2t^3}{t^4}\right) = 2$

$$\lim_{t \to 0} t^2\left(\frac{-1/4 + kt + (\ )t^2}{t^4}\right) = \infty$$

irreg sing pt.

let $W = \sum_{k=0}^{\infty} A_k z^{k+\alpha}$

$$W' = \sum_{k=0}^{\infty} A_k (k+\alpha) z^{k+\alpha-1}$$

$$W'' = \sum_{k=0}^{\infty} A_k (k+\alpha)(k+\alpha-1) z^{k+\alpha-2}$$

put into $z^2 W'' + \left\{ -\frac{1}{4} z^2 + kz + \frac{1}{4} - m^2 \right] W = 0$

$$\Rightarrow \sum_{k=0}^{\infty} A_k (k+\alpha)(k+\alpha-1) z^{k+\alpha} + \sum_{k=0}^{\infty} -\frac{1}{4} A_k z^{k+\alpha+2} + k \sum_{k=0}^{\infty} A_k z^{k+\alpha+1}$$

$$+ \frac{1}{4} \sum_{k=0}^{\infty} A_k z^{k+\alpha} - m^2 \sum_{k=0}^{\infty} A_k z^{k+\alpha} = 0.$$

All k's to n's

$$\Rightarrow \sum_{n=0}^{\infty} A_n (n+\alpha)(n+\alpha-1) z^{n+\alpha} - \frac{1}{4}\sum_{n=2}^{\infty} A_{n-2} z^{n+\alpha}$$

$$+ k \sum_{n=1}^{\infty} A_{n-1} z^{n+\alpha} + \frac{1}{4}\sum_{n=0}^{\infty} A_n z^{n+\alpha} - m^2 \sum_{n=0}^{\infty} A_n z^{n+\alpha} = 0$$

$$= A_0 \alpha(\alpha-1) z^{\alpha} + A_1(1+\alpha)(\alpha) z^{1+\alpha} + \sum_{n=2}^{\infty} A_n (n+\alpha)(n+\alpha-1) z^{n+\alpha}$$

$$- \frac{1}{4}\sum_{n=2}^{\infty} A_{n-2} z^{n+\alpha} + k A_0 z^{\alpha} + k \sum_{n=2}^{\infty} A_{n-1} z^{n+\alpha}$$

$$+ \frac{1}{4} A_0 z^{\alpha} + \frac{1}{4} A_1 z^{\alpha+1} + \frac{1}{4}\sum_{n=2}^{\infty} A_n z^{n+\alpha} - m^2 A_0 z^{\alpha} - m^2 A_1 z^{\alpha+1}$$

$$- m^2 \sum_{n=2}^{\infty} A_n z^{n+\alpha} = 0.$$

$$\Rightarrow A_0\left[\alpha^2 - \alpha + k + \frac{1}{4} - m^2\right] x^{\alpha} + A_1\left[\alpha + \alpha^2 + \frac{1}{4} - m^2\right] x^{\alpha+1}$$

$$\alpha^2 - \alpha - m^2 + k + 1/4 = 0$$

$$\alpha = \frac{+1 \pm \sqrt{1+4(m^2-k-1/4)}}{2}$$

$$= \frac{1}{2} \pm \frac{1}{2}\sqrt{4(m^2-k^2)-1+1}$$

[illegible]

$$k = -1/4$$

$$= \frac{1}{2} \pm \frac{\sqrt{4m^2-4k^2}}{2} = \frac{1}{2} \pm (m^2-k^2)^{1/2}$$

$$= \cancel{\frac{1}{2} \pm |m| \frac{\ }{2}\sqrt{1-\ }}$$

$$= \frac{1}{2} \pm m(1-(k/m)^2)^{1/2}$$

$$\therefore \; y_1 = z^{\frac{1}{2}+m(1-(k/m))^{1/2}}$$

eq A $y = \sum A_n x^{n+\alpha}$ put in.

$$= \sum_{n=0}^{\infty} A_n(n+\alpha)(n+\alpha-1)x^{n+\alpha} + \sum_{n=0}^{\infty} A_n(n+\alpha)x^{n+\alpha}$$

$$+ k\sum_{n=0}^{\infty} A_n x^{n+\alpha+1} + \frac{1}{4}\sum_{n=0}^{\infty} A_n x^{n+\alpha} - m^2\sum_{n=0}^{\infty} A_n x^{n+\alpha} = 0$$

$$\Rightarrow \quad '' \quad + \quad '' \quad + k\sum_{n=1}^{\infty} A_{n-1} x^{n+\alpha} + \cdots = 0$$

$\Rightarrow$

Pg 337 w/o

$$M_{\kappa m} = z^{\frac{1}{2}+m} e^{-\frac{1}{2}z} f(z)$$

$$M' = (m+\tfrac{1}{2}) z^{m-\frac{1}{2}} e^{-\frac{z}{2}} f$$

$$+ z^{m+\frac{1}{2}}\left(-\tfrac{1}{2}e^{-\frac{z}{2}}\right) f + z^{m+\frac{1}{2}}\left(e^{-\frac{z}{2}}\right) f'$$

$$= (m+\tfrac{1}{2})$$

$$M'' = (m+\tfrac{1}{2})(m-\tfrac{1}{2}) z^{m-\frac{3}{2}} e^{-\frac{z}{2}} f + (m+\tfrac{1}{2})(-\tfrac{1}{2}) z^{m-\frac{1}{2}} e^{-\frac{z}{2}} f$$

$$+ (m+\tfrac{1}{2}) z^{m-\frac{1}{2}} e^{-\frac{z}{2}} f' + (m+\tfrac{1}{2}) z^{m-\frac{1}{2}}\left(-\tfrac{1}{2}e^{-\frac{z}{2}}\right) f$$

$$+ z^{m+\frac{1}{2}} \tfrac{1}{4} e^{-\frac{z}{2}} f + z^{m+\frac{1}{2}}\left(-\tfrac{1}{2}e^{-\frac{z}{2}}\right) f'$$

$$+ (m+\tfrac{1}{2}) z^{m-\frac{1}{2}} e^{-\frac{z}{2}} f' + z^{m+\frac{1}{2}}(-\tfrac{1}{2}) e^{-\frac{z}{2}} f'$$

$$+ z^{m+\frac{1}{2}} e^{-\frac{z}{2}} f''$$

$$= (m+\tfrac{1}{2})(m-\tfrac{1}{2}) z^{m-\frac{3}{2}} e^{-\frac{z}{2}} f$$

putting in DE.

$$\Rightarrow -\frac{1}{4}e^{-z/2} z^{m-1/2} \left\{ f(z-4k) - 4(1+2m)f' \right.$$

$$\left. + 4f'' \right] = 0$$

$\Rightarrow f \Rightarrow$

$$-4f'' - 4(1+2m)f' + (z-4k)f = 0.$$

f''

$$M_{k,m} = z^{m+1/2} e^{-z/2} \left\{ 1 + \frac{(1/2 + m - k)}{1!(2m+1)} z + \frac{(1/2 + m - k)(1/2 + 1 + m - k)}{2!(2m+1)(2m+2)} z^2 + \cdots \right\}$$

$$= \sum_{n=0}^{\infty} \frac{z^n (1/2 + m - k)(1/2 + 1 + m - k) \cdots (1/2 + (n-1) + m - k)}{n!(2m+1)(2m+2) \cdots (2m+n)}$$

$$1 + \sum_{n=0}^{\infty} \frac{z^n}{n!\,\Gamma(2m+n+1)}$$

Pg 338 W/W

$$z^{-1/2-m} M_{k,m}(z) = (-z)^{-1/2-m} M_{-k,m}(-z)$$

or.

$$e^{-z/2} \left\{ 1 + \frac{1/2 + m - k}{1!(2m+1)} z + \frac{(1/2 + m - k)(3/2 + m - k)}{2!(2m+1)(2m+2)} z^2 + \cdots \right\}$$

$$=$$

PS ~~38.5~~ W/W

$$\int^{0^+} t^{-n-r-1} e^{+t}\, dt = \int^{0^+} t^{-n-r-1} \sum_{k=0}^{\infty} \frac{t^k}{k!}\, dt$$

$$= \int^{0^+} \left(\sum_{k=0}^{\infty} \frac{t^{k-n-r-1}}{k!} \right) dt$$

pick out term when $t^{k-n-r-1} = t^{-1}$

$\Rightarrow \quad k-n-r = 0 \qquad k \geq 0 \Rightarrow \quad n+r \geq 0$

$\Rightarrow k = n+r$

+ Residue is $\dfrac{1}{k!} = \dfrac{1}{(n+r)!}$

If $n+r < 0$ Residu is 0.

$\neq$

pg 355 Whittaker / Watson

$$J_n(z) = \frac{1}{2\pi i}\oint^{(0+)} u^{-n-1} e^{\frac{1}{2}z(u-1/u)}\, du \qquad n \in \mathbb{Z}$$

$$u = \frac{2t}{z} \qquad du = \frac{2}{z}dt$$

$$J_n(z) = \frac{1}{2\pi i}\left(\frac{z}{2}\right)^{n+1}\frac{\cancel{2}}{\cancel{z}}\oint t^{-n-1} e^{\frac{1}{2}z(\frac{2}{z}t - z/2t)}\, dt$$

$$= \frac{1}{2\pi i}\left(\frac{z}{2}\right)^{n}\oint t^{-n-1}\exp\left\{t - \frac{z^2}{4t}\right\} dt$$

$$e^t e^{-z^2/4t}$$

$$\sum_{r=0}^{\infty}\frac{(-1)^r (z/2)^{2r}}{r!\, t^r}$$

$$\frac{1}{t^2}$$

$n = 2$

$r = -2, -3, -2, -1, 0, 1, 2, 3$

$$= \frac{1}{2\pi i}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{z}{2}\right)^{n+2r}\oint t^{-n-r-1} e^t\, dt$$

Residue of t^{-n-r-1} if & ~~[illegible]~~ ~~[illegible]~~

$-n-r=0 \Rightarrow \underline{r=-n}$ or $\underline{n=-r}$

any

Pg 356 W/W

$$e^{\frac{1}{2}z(v-\frac{1}{v})} \quad \frac{1}{v^2}\left[\frac{2b\left(1+\frac{1}{v^2}\right)}{\left(1-\frac{2a}{v}-\frac{k}{v^2}\right)^2+\frac{4b^2}{v^2}}\right]$$

$$= \frac{1}{v^2}\sum_{n=1}^{\infty} A_n \theta^{n-1} e^{\frac{1}{2}z(v-\frac{1}{v})} = \sum_{n=1}^{\infty} A_n e^{\frac{1}{2}z(v-\frac{1}{v})} \quad v^{-n-1}$$

$$\theta = \frac{1}{v} \qquad \cancel{v^{-n-1}}\,2$$

$$\therefore \frac{1}{2\pi i}\oint^{(0+)} \longrightarrow$$

$$= \frac{1}{2\pi i}\int^{(0+)} e^{\frac{1}{2}z(v-\frac{1}{v})} \frac{2b\left(\frac{1}{v^2}+\frac{1}{v^4}\right)}{\left(1-\frac{2a}{v}-\frac{1}{v^2}\right)^2+\frac{4b^2}{v^2}} \, dv$$

$$= \sum_{n=1}^{\infty} A_n J_n(z)$$

let $\frac{1}{2}(v-v^{-1})-\alpha = t$

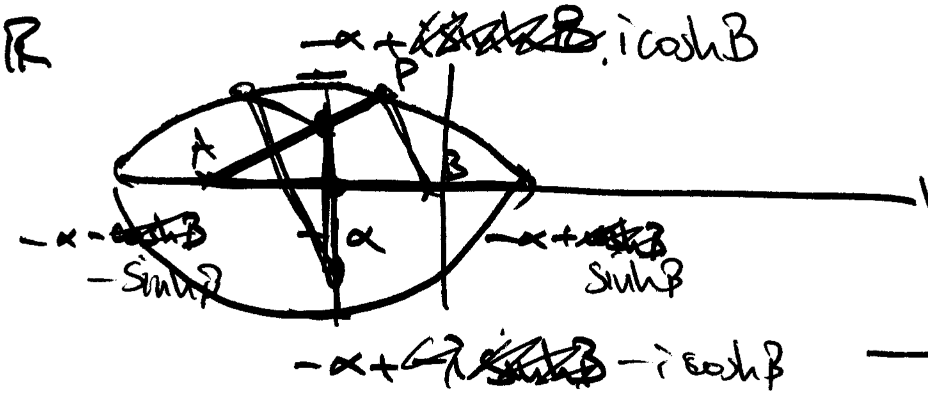
Then if $v = e^{B}e^{i\theta}$ $\quad 0 \leq \theta < 2\pi$

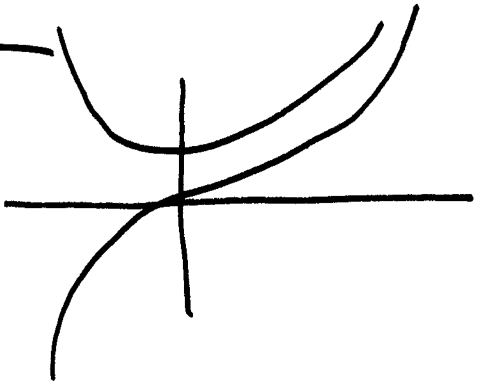
$$\frac{1}{2}\left(\underbrace{e^{B}e^{i\theta} - e^{-B}e^{-i\theta}}_{\cancel{e^{i\theta+B}} \; \cancel{-e^{-(i\theta+B)}}}\right) - \alpha = t$$

sinh

(over)

$\alpha \in \mathbb{R}$





foci of ellipse . . .

$|AP| + |BP| = \text{const} = 2\cosh B$ $\cosh B \geq \sinh B \;\; \forall B \in \mathbb{R}$

Using Shams at him series . . .

Distance from center $(-\alpha, 0)$ to focus F or F′

is $\sqrt{\cosh^2 B - \sinh^2 B} = 1$

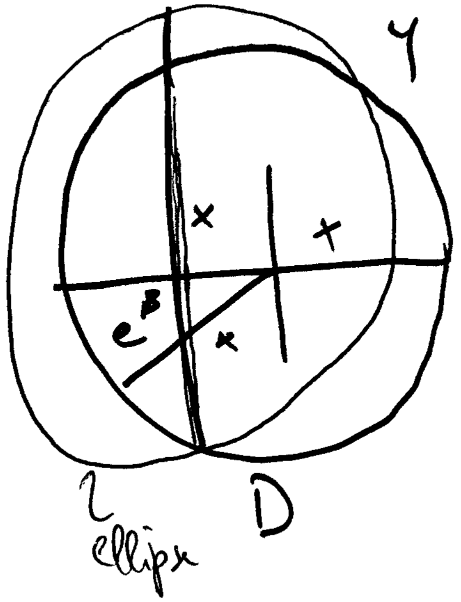
∴... foci are $(-\alpha, i)$ & $(-\alpha, -i)$.

Then $\displaystyle\sum_{n=1}^{\infty} A_n J_n(z) = \frac{1}{2\pi i}\int_D e^{z(t+\alpha)}$

$$t^2 = \frac{1}{2^2}(u-u^{-1})^2 - \frac{2\alpha}{2}(u-u^{-1}) + \alpha^2$$

$$= \frac{1}{4}(u^2 - 2 + u^{-2}) - \alpha u + u^{-1} + \alpha^2$$

$$\frac{u^2}{4} - \frac{1}{2} + \frac{1}{4u^2} - \alpha u + \frac{1}{u} + \alpha^2$$



$$\therefore t^2 + b^2$$

$$= ?$$

Now:

$$dt = \frac{1}{2}(du + u^{-2} du)$$

$$= \frac{1}{2}(1 + u^{-2}) du.$$

$$\therefore \frac{2b}{u^2}\underbrace{(1 + u^{-2}) du}_{2dt} = \frac{2b \cdot 2dt}{u^2}$$

Now

$$2(t+\alpha) = u - u^{-1} \quad *$$

$\times u \rightarrow$

$$u^2 - 1 = 2u(t+\alpha)$$

$$u^2 - 2u(t+\alpha) - 1 = 0$$

$$u = \frac{2(t+\alpha) \pm \sqrt{4(t+\alpha)^2 + 4}}{2}$$

$\Rightarrow \quad v = t+\alpha \pm \sqrt{(t+\alpha)^2+1}$

Take * & mult by v^{-1}

$\Rightarrow \quad 1 - v^{-2} = 2(t+\alpha)v^{-1}$

$\Rightarrow \quad v^{-2} + 2(t+\alpha)v^{-1} - 1 = 0$

$$v^{-1} = \frac{-2(t+\alpha) \pm \sqrt{4(t+\alpha)^2+4}}{2}$$

$$= -t-\alpha \pm \sqrt{(t+\alpha)+1}$$

Pg 356 W/W

$n = -m$

$$J_{-m}(z) = \cancel{\sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{1}{2}z\right)^{-m+2r}}{r!\,(-m+r)}} \quad \frac{1}{2\pi i}\left(\frac{z}{2}\right)^{n} \int^{(0^+)} t^{m-1} \cdot e^{t} e^{-z^2/4t}\, dt$$

$$= \frac{1}{2\pi i}\left(\frac{z}{2}\right)^{-m} \int^{0^+} t^{+m-1} e^{t} \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(4t)^k k!}$$

$$= \frac{1}{2\pi i}\left(\frac{z}{2}\right)^{-m} \sum_{k=0}^{\infty} \frac{z^{2k}(-1)^k}{k!\,4^k} \int^{0^+} t^{m-1+2k} e^{t}\, dt$$

$$\int^{0^+} t^{m-r-1} e^{t}\, dt = \int^{0^+} \sum_{k=0}^{\infty} \frac{t^{k+m-r-1}}{k!} = \cancel{\sum_{k=0}^{\infty} \frac{1}{k!} \int^{0^+} t^{k+m-r-1}}\, dt$$

pick of $\quad k+m-r-1 = -1 \qquad \cancel{m \geq 1}$

$\cancel{k \geq 0}$

$k+m = r. \geq 1$

$k = r-m. \qquad \cancel{k \geq 0}$

$m \geq 1$

$$= \frac{1}{(r-m)!}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \left(\frac{z}{2}\right)^{-m+2r} \underbrace{\frac{1}{2\pi i} \int^{0^+} t^{+m-r-1} e^t \, dt}_{\frac{1}{(r-m)!}}$$

$r-m \geq 0$

$\Rightarrow r \geq m.$ 0 else

$$= \sum_{r=m}^{\infty}$$

$k = r-m$ * req: $r \geq 0$

$m \geq 1$

+ * to be true $k \geq 0.$

Then As $k \geq 0 \Rightarrow r-m \geq 0$

$\Rightarrow r \geq m.$

$$\Rightarrow \sum_{r=m}^{\infty} \frac{(-1)^r (\frac{1}{2}z)^{2r-m}}{r!} \frac{1}{(r-m)!} = \sum_{r=0}^{\infty} \frac{(-1)^{m+r} (\frac{1}{2}z)^{2r+m}}{(m+r)!\, r!}$$

$$= (-1)^m J_m(z)$$

$$\therefore \; J_{-m}(z) = (-1)^m J_m(z).$$

PS 357 W/W

Shew: $J_n(y+z) = \sum_{m=-\infty}^{\infty} J_m(y)\, J_{n-m}(z)$

$$\exp\left\{\tfrac{1}{2}(y+z)\left(t-\tfrac{1}{t}\right)\right\} = \exp\left\{\tfrac{1}{2}y\left(t-1/t\right)\right\} \cancel{\exp}$$

$$\cdot \exp\left\{\tfrac{1}{2}z\left(t-\tfrac{1}{t}\right)\right\}$$

Mult both sides by t^{-n-1} & intg $\frac{1}{2\pi i}\int^{(0+)} \quad dt$

$$\Rightarrow J_n(y+z) = \frac{1}{2\pi i}\int^{(0+)} t^{-n-1}\exp\left\{\tfrac{1}{2}y\left(t-\tfrac{1}{t}\right)\right\}\exp\left\{\tfrac{1}{2}z\left(t-\tfrac{1}{t}\right)\right\} dt$$

$$= \frac{1}{2\pi i}\int^{(0+)} t^{-m-1}\exp\{\quad\}\, t^{-n+m}\exp\{\quad\}\, dt$$

$$= \frac{1}{2\pi i}\int^{(0+)} t^{-m-1}\exp\{\quad\}\, t^{-(n-m)}$$

---

integral not best way to do this

use Laurent thm to expand L.H.S in powers of t

$$\Rightarrow \sum_{n=-\infty}^{\infty} t^n J_n(y+z) = \underbrace{\sum_{m=-\infty}^{\infty} t^m J_m(y) \cdot \sum_{l=-\infty}^{\infty} t^l J_l(z)}$$

$$= \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} J_m(y) J_l(z) t^{m+l}$$

Setting coeff of $t^{m+l} = t^n$

$$\Rightarrow m+l = \underline{n} \rightarrow l = n-m$$

$$\Rightarrow \text{RHS} \Rightarrow \sum_{m=-\infty}^{\infty} \left(\sum_{m=-\infty}^{\infty} J_m(y) J_{n-m}(z) \right) t^n$$

$$\therefore J_n(y+z) = \sum_{m=-\infty}^{\infty} J_m(y) J_{n-m}(z).$$

PS 357 W/N

1

Ex 3:

$$e^{iz\cos\phi} = J_0(z) + 2i\cos\phi\, J_1(z) + 2i^2\cos 2\phi\, J_2(z) + \dots$$

$$i\cos\phi = \frac{i(e^{i\phi}+e^{-i\phi})}{2}$$

let $v = ie^{i\phi}$ $\quad v^{-1} = \frac{e^{-i\phi}}{i} = -ie^{-i\phi} \leftarrow$

$\therefore$ $i\cos\phi$ $\quad \Rightarrow \quad ie^{-i\phi} = -v^{-1}$

$$= \frac{v - v^{-1}}{2}$$

$$\therefore e^{iz\cos\phi} = e^{\frac{z}{2}(v-v^{-1})} = \sum_{n=-\infty}^{\infty} v^n J_n(z)$$

~~$\int_n$~~ $J_{-n}(z) = (-1)^n J_n(z)$

$$\Rightarrow \sum_{n=-\infty}^{-1} v^n J_n + J_0 + \sum_{n=1}^{\infty} v^n J_n$$

$$= \sum_{n=+\infty}^{1} u^{-n} J_{-n} + \cdots$$

$$= \sum_{n=1}^{\infty} (-1)^n u^{-n} J_n + J_0 + \sum_{1}^{\infty} u^n J_n$$

$$= J_0(z) + \sum_{1}^{\infty} \left(u^n + (-1)^n u^{-n}\right) J_n$$

$$= J_0(z) + \sum_{n=1}^{\infty} \left(i^n e^{in\phi} + \underset{i^n}{(-1)^{2n}} e^{-in\phi}\right) J_n$$

$$+ \sum_{n=1}^{\infty} i^n \left(\underbrace{e^{in\phi} + e^{-in\phi}}_{2 i^n \cos n\phi}\right)$$

$$= J_0(z) + 2\sum_{n=1}^{\infty} i^n \cos n\phi \, J_n(z) \qquad \checkmark$$

Ex 4

Pg 357 WW

Show: $r^2 = x^2+y^2$

$x = r\cos\theta$
$y = r\sin\theta$ } use this maybe?

$$J_0(r) = J_0(x)J_0(y) - 2J_2(x)J_2(y) + 2J_4(x)J_4(y)$$

$- \cdots$

$$J_0(r) \equiv \frac{1}{2\pi i}\int^{0+} t^{-1}\exp\left\{t - \frac{x^2}{4t} - \frac{y^2}{4t}\right\} dt$$

$$= \frac{1}{2\pi i}\int^{0+} t^{-1}\exp\{t -$$

$e^t e^{-x^2/4t} e^{-y^2/4t}$

$$\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{4^k t^k k!}\sum_{l=0}^{\wedge}\frac{(-1)^l y^{2l}}{4^l t^l l!}$$

$$\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{k+l} x^{2k} y^{2l}}{4^k t^k k! \cdot 4^l t^l l!}$$

$$e^{\frac{1}{2}r\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} t^n J_n(r)$$

$\exp\left\{\frac{1}{2}r^2\right.$

$$J_n(x^2+y^2) = \sum_{m=-\infty}^{\infty} J_m(x^2)J_{n-m}(y^2)$$

$$J_0(x^2+y^2) = \sum_{m=-\infty}^{\infty} J_m(x^2)\, J_{-m}(y^2)$$

$$= \sum_{m=-\infty}^{-1} + J_0(x^2) J_0(y^2)$$

$$+ \sum_{m=1}^{\infty}$$

$$= \sum_{m=1}^{\infty} J_{-m}(x^2) J_m(y^2) + J_0(x^2) J_0(y^2)$$

$$+ \sum_{m=1}^{\infty} J_m(x^2)\, J_{-m}(y^2)$$

$$J_{-m} \doteq (-1)^m J_m$$

$$\therefore = 2 \sum_{m=1}^{\infty} (-1)^m J_m(x^2)\, J_m(y^2) + J_0(x^2)\, J_0(y^2)$$

$$J_0(x^2) \overset{?}{=} J_0(x)$$

$$J_m(x^2) =$$

$$\therefore \; J_0(\underline{r}^2) = J_0(x^2)J_0(y^2) + 2\sum_{m=+1}^{\infty}(-1)^m J_m(x^2)J_m(y^2)$$

Thus from Above

$$\frac{1}{t^{k+l}} = t^{-(k+l)}$$

let $k+l = n$

$l = n-k$

~~$r = \sqrt{x^2+y^2}$~~

~~$J_0(r) = \frac{1}{2\pi i}$~~

$$\int$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^n x^{2k} y^{2n-2k}}{\underbrace{\cancel{4^k}\cancel{t^k}\, k!\, \cancel{4^{n-k}}\cancel{t^{n-k}}}_{4^n}(n-k)!}$$

$$(x^2)^k(y^2)^{n-k} \to (x^2+y^2)^n$$

$$\binom{n}{k} \cdot \frac{n!}{k!(n-k)!}$$

$$\therefore \; J_0(r) = \frac{1}{2\pi i}\int^{0^+} t^{-1}e^t \sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,t^n 4^n}\sum_{k=0}^{n}\frac{n!\,x^{2k}y^{2n-2k}}{k!(n-k)!}$$

pg 357 W/W

$r^2 = x^2 + y^2$

$$J_0(r) = J_0(x)J_0(y)$$

$$- 2J_2(x)J_2(y) + 2J_4(x)J_4(y) -$$

$$= J_0(x)J_0(y) + 2\sum_{k=1}^{\infty}(-1)^k J_{2k}(x)J_{2k}(y)$$

$$J_n(z) = \frac{1}{2\pi i}\left(\frac{1}{2}z\right)^n \int^{(0^+)} t^{-n-1} \exp\left\{t - \frac{z^2}{4t}\right\} dt$$

$$J_0(r) = \frac{1}{2\pi i}\int^{0^+} t^{-1} \exp\left\{t - \frac{x^2}{4t} - \frac{y^2}{4t}\right\} dt$$

$$\|$$

$$= e^t e^{-x^2/4t} e^{-y^2/4t}$$

$$= e^t \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^{i+j} x^{2i} y^{2j}}{4^{i+j} t^{i+j} i! j!}$$

Now

$$\int^{0^+} e^t \exp \cdots dt = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^{i+j} x^{2i} y^{2j}}{4^{i+j} i! j!} \int^{0^+} e^t t^{-1-i-j} dt$$

$$\|$$

$$\sum \frac{t^k}{k!}$$

$k - i - j = 0$

$0 \leq k = i+j$ Resdu is $\frac{1}{(i+j)!}$

$$\therefore J_0(r) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{i+j} x^{2i} y^{2j}}{4^{i+j} \, i! \, j! \, (i+j)!}$$

pg 357 WYW

$$\frac{d}{dz}J_n = \frac{1}{2\pi i}\,\frac{n}{2}\left(\tfrac{1}{2}z\right)^{n-1}\int^{0+} t^{-n-1}\exp\left(t-z^2/4t\right)dt$$

$$+\frac{1}{2\pi i}\left(\tfrac{1}{2}z\right)^{n}\int^{0+} t^{-n-1}\exp\left(t-z^2/4t\right)dt\,\Big(\underbrace{-\frac{2z}{4t}}_{-\frac{z}{2t}}\Big)$$

$$\frac{d^2}{dz^2}J_n = \frac{1}{2\pi i}\,\frac{n}{2}\left(\frac{n-1}{2}\right)\left(\tfrac{1}{2}z\right)^{n-2}\int^{0+} t^{-n-1}\cdots$$

$$+\frac{1}{2\pi i}\,2\left(\frac{n}{2}\right)\left(\tfrac{1}{2}z\right)^{n-1}\int^{0+} t^{-n-1}\exp\left\{t-z^2/4t\right\}dt\left(\frac{-z}{2t}\right)$$

$$+\frac{1}{2\pi i}\left(\tfrac{1}{2}z\right)^{n}\int^{0+} t^{-n-1}\,\cancel{\text{ex}}\,\exp\left\{t-z^2/4t\right\}\left[\frac{-1}{2t}+\frac{z^2}{4t^2}\right]$$

$$\therefore\ \frac{d}{d\cancel{z}}J_n = \frac{1}{2\pi i}\left(\tfrac{1}{2}z\right)^{n}\int^{0+} t^{-n-1}\exp\{\cdots\}\left[\frac{n}{\cancel{2}}\left(\frac{2}{z}\right)-\frac{z}{2t}\right]dt$$

$$\&\ \frac{d^2J_n}{dz^2} = \frac{1}{2\pi i}\left(\tfrac{1}{2}z\right)^{n}\int^{0+} t^{-n-1}\exp\{\cdots\}\Big[\frac{n}{\cancel{2}}\left(\frac{n-1}{\cancel{z}}\right)\frac{\overset{2}{\cancel{z}}}{z^2}+n\frac{2}{z}\left(\frac{-z}{2t}\right)$$
$$-\frac{1}{2t}+\frac{z^2}{4t^2}\Big]dt$$

Now $\frac{d^2J_n}{dz^2}+\frac{1}{z}\frac{dJ_n}{dz}+\left(1-\frac{n^2}{z^2}\right)J_n =$

$$\frac{n}{z^2} - \frac{1}{2t}$$

$$= \frac{1}{2\pi i}\left(\tfrac{1}{2}z\right)^n \int^{0+} t^{-n-1} \left\{ \frac{n(n-1)}{z^2} - \frac{n}{t} - \frac{1}{2t} + \frac{z^2}{4t^2} \cancel{\frac{z}{8tz}} \cancel{\frac{z}{4t^2}}\right.$$

$$\left. + 1 - \frac{n^2}{z^2} \right\} \cancel{\exp} \left\{ \cdots \right\} dt$$

$$+\cancel{\frac{n}{z^2}} - \underline{\frac{1}{2t}}$$

$$\cancel{\frac{n^2}{z^2}} - \cancel{\frac{n}{z^2}} - \underline{\frac{n}{t}} - \underline{\frac{1}{2t}} + \frac{z^2}{4t^2} \cancel{-\frac{1}{4t}} + \cancel{\frac{z}{4t}} + 1 - \cancel{\frac{n^2}{z^2}}$$

$$t^{-n-1} \left\{ 1 - \frac{(n+1)}{t} + \frac{z^2}{4t^2} \right\} \exp \left\{ \cdots \right\}$$

$$= t^{-(n+1)} \left\{ \cdots \right\} \exp \left\{ \cdots \right\}$$

$$\overset{?}{=} \frac{d}{dt}\, t^{-n-1} \exp\{\ \} = (-n-1)t^{-n-2} \exp\{\ \}$$

$$+ t^{-n-1}\left(1 + \frac{z^2}{4t^2}\right) \exp$$

$$= \left(\frac{-n-1}{t^{n+2}} + \frac{1}{t^{n+1}} + \frac{z^2/4t^2}{t^{n+1}} \right) \exp\{\cdots\}$$

$$= \frac{\left(1 - \frac{(n+1)}{t} + z^2/4t^2\right)}{t^{n+1}} \exp\{\cdots\} \quad ✓.$$

PS 357 w/w.

$$\lim_{z\to 0} z\left(\frac{1}{z}\right) = 1 \quad \text{Anal}$$

$$\lim_{z\to 0} z^2\left(1-\frac{n^2}{z^2}\right) \quad \text{Anal}$$

$$t = \frac{1}{z}$$

$$dt = -\frac{1}{z^2}dz = -t^2\,dz$$

$$\frac{d}{dz} = \cancel{\frac{dt}{dz}}\frac{d}{dt} = -t^2\frac{d}{dt}$$

$$\frac{d^2}{dz^2} = \cancel{\frac{dt}{dz}}\frac{d}{dt}\left(-t^2\frac{d}{dt}\right) = -t^2\left(-2t\frac{d}{dt} + -t^2\frac{d^2}{dt^2}\right)$$

$$= t^4\frac{d^2}{dt^2} + 2t^3\frac{d}{dt}$$

$$\Rightarrow \quad t^4\frac{d^2y}{dt^2} + 2t^3\frac{dy}{dt} + t\frac{dy}{dt}\left(-t^2\right) + \left(1-n^2t^2\right)y = 0$$

$$y'' + \left(\frac{2}{t}-\frac{1}{t}\right)y' + \frac{1-n^2t^2}{t^4}\,y = 0$$

$t=0 \Rightarrow z=\infty$ irreg sing pt.

pg 358 w/w

Ex 1: (0)

$$e^{\frac{1}{2}z(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty} t^n J_n(z)$$

(2) $\partial_z \longrightarrow$

$$\frac{1}{2}(t-t^{-1})e^{(\ )} = \sum_{n=-\infty}^{\infty} t^n J_n'$$

(1) $\partial_z^2 \longrightarrow$

$$\frac{1}{4}(t-t^{-1})^2 e^{(\ )} = \sum_{n=-\infty}^{\infty} t^n J_n''$$

$\partial_t \rightarrow$ of (0)

~~$\frac{1}{2}(1+t^{-2})e^{\frac{1}{2}z(t-t^{-1})} + \frac{1}{2}(t-t^{-1})e^{\cdots} \cdot \frac{1}{2}z(1+t^{-2}) = \sum_{n=-\infty}^{\infty} n t^{n-1} J_n$~~

[illegible]

$\Rightarrow$ [illegible] ~~$\left[\frac{1}{2}(1+t^{-2}) + \frac{1}{4}z\underbrace{(t-t^{-1})(1+t^{-2})}_{t-t^{-3}}\right]e^{(\ )}$~~

~~Mult by t~~

$\Rightarrow$ ~~$\left(\frac{1}{2}(t+t^{-1}) + \underbrace{\frac{1}{4}z(t-t^{-1})(t+t^{-1})}_{\frac{1}{4}z(t^2-t^{-2})}\right]e^{(\ )} = \sum n t^n J_n$~~

$$\tfrac{1}{2}z(1+t^{-2})\,e^{\frac{1}{2}z(\ \ )} = \sum n t^{n-1} J_n$$

mult by t

$$\tfrac{1}{2}z(t+t^{-1})e^{\frac{1}{2}(\ \ )} = \sum n t^{n} J_n$$

$\partial_t \to$

$$\left\{\tfrac{1}{2}z(1-t^{-2}) + \tfrac{1}{2}z(t+t^{-1})\,\tfrac{1}{2}z(1+t^{-2})\right]e^{(\ )} = \sum n^2 t^{n-1} J_n$$

mult by t

$$\Rightarrow \left\{\tfrac{1}{2}z(t-t^{-1}) + \tfrac{1}{4}z^2(t+t^{-1})(t+t^{-1})\right]e^{(\ )} = \sum n^2 t^{n} J_n$$

Then

$$\sum t^n\left\{z^2 J_n + z J_n + (z^2-n^2)J_n\right]$$

$$= \left\{\frac{z^2}{4}(t-t^{-1})^2 + \frac{z}{2}(\cancel{t}-t^{-1}) + z^2\cdot 1\right.$$

$$-\left[\frac{1}{2}z\cancel{(t}-t^{-1}) + \frac{1}{4}z^2(t+t^{-1})^2\right]\Big\}e^{(\;)}$$

$$= \Big\{\frac{z^2}{4}(\cancel{t^2}-2+\cancel{t^{-2}}) + z^2$$

$$-\frac{1}{4}z^2(\cancel{t^2}+2+\cancel{t^{-2}})\Big\}e^{(\;)}$$

$$= \Big\{\frac{z^2}{4}(-2) + z^2 - \frac{2z^2}{4}\Big\}e^{(\;)}$$

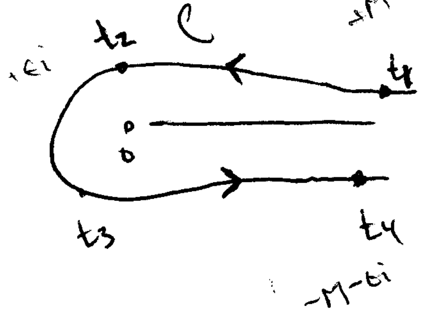
$$= 0 \quad \checkmark.$$

Pg 358 W/W

§12.22 gives:

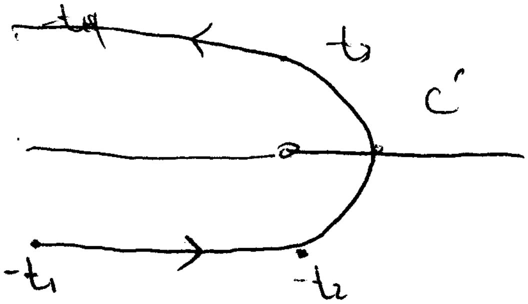
$$\Gamma(z) = \frac{-1}{2i\sin\pi z}\int_C (-t)^{z-1}e^{-t}\,dt$$

$$\Longleftrightarrow \quad \frac{1}{\Gamma(z)} = \frac{i}{2\pi}\int_C (-t)^{-z}e^{-t}\,dt.$$



$t \to -t$

$$\frac{1}{\Gamma(z)} = \frac{i}{2\pi}\int_{C'} t^{-z}e^{t}\,(-dt)$$



Thus C' is $\int_{-\infty}^{(0^+)}$

$$\Rightarrow \quad \frac{-2\pi}{i\Gamma(z)} = \int_{-\infty}^{(0^+)} t^{-z}e^{t}\,dt$$

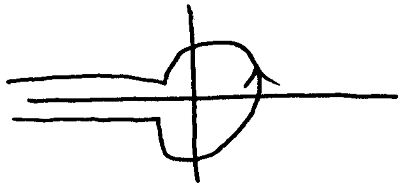
$$\therefore \quad J_n(z) = \frac{z^n}{2^{n+1}\cancel{\pi}\,\textcircled{i}}\sum_{r=0}^{\infty}\frac{(-1)^r z^{2r}}{2^{2r}\,r!}\left(\frac{\cancel{-2\pi}}{\textcircled{i}\,\Gamma(n+r+1)}\right)$$

$$= \sum_{r=0}^{\infty}\frac{(\underline{-1})^r z^{2r+n}}{2^{n+2r}\,r!\,\Gamma(n+r+1)}$$

p. 359 W/W

1

$$0 = \int_{-\infty}^{0^+} \frac{d}{dt}\left(t^{-n}\exp\left(t-\frac{z^2}{4t}\right)dt\right.$$



$$= \int_{-\infty}^{0^+}\left(-nt^{-n-1}+t^{-n}\left(1+\frac{z^2}{4t^2}\right)\right)\exp(\quad)\,dt$$

$$\int_{-\infty}^{0^+}\left(t^{-n}+\frac{z^2}{4}t^{-n-2}-nt^{-n-1}\right)\exp(\quad)\,dt$$

$$= \cancel{2\pi i}\left\{(2z^{-1})^{n-1}J_{n-1}(z)+\frac{z^2}{4}(2z^{-1})^{n+1}J_{n+1}\right.$$

$$\left.-n(2z^{-1})^{n}J_n\right\} = 0$$

$$\Rightarrow \quad \underset{\frac{1}{2}zJ_{n-1}}{(2z^{-1})^{-1}J_{n-1}} + \frac{z^2}{\cancel{4}}\underset{\frac{z}{2}}{(\cancel{2}z^{-1})^{1}}J_{n+1} - nJ_n = 0$$

$$J_{n-1}+J_{n+1} = \frac{2n}{z}J_n.$$

$$\int_{-\infty}^{0^+} -\frac{z}{2t}\, t^{-n-1} \exp$$

$$= -\frac{z}{2} \int t^{-n-2} \exp$$

$$= \frac{-z}{2^{n+2}\pi i} \underbrace{\int t^{-n-2} \exp\{\cdots\}}$$

$$= \frac{-\cancel{z}\; 2\cancel{\pi i}\,(\cancel{2}z^{-1})^{n\cancel{+1}}}{\cancel{2^{n+2}}\,\cancel{\pi i}} J_{n+1}$$

$$- z^{-n} J_{n+1} \qquad \checkmark$$

$$-nz^{-n-1} J_n + z^{-n} J_n' = -z^{-n} J_{n+1}$$

$$-nz^{-1} J_n + J_n' = -J_{n+1}$$

$$J_n' = \frac{n}{z} J_n - J_{n+1}$$

pg 359 Wht/Watson

$$J_{n-1} + J_{n+1} = \frac{2n}{z} J_n \quad * .$$

$J_n' = \frac{n}{z} \underline{J_n} - J_{n+1}$ put in J_n from *.

$$\therefore \quad J_n' = \frac{1}{2}(J_{n-1} + J_{n+1}) - J_{n+1} = \frac{1}{2}(J_{n-1} - J_{n+1})$$

(C) ✓

put in J_{n+1} from *

$$J_n' = \frac{n}{z} J_n - \left(\frac{2n}{z} J_n - J_{n-1}\right)$$

$$= -\frac{n}{z} J_n + J_{n-1} = J_{n-1} - \frac{n}{z} J_n \qquad \text{(D)} ✓$$

PS 360: W/W

Rc: $J_{n+1} + J_{n-1} = \frac{2n}{z} J_n$

consider

$$J_{n-1} = \sum_{r=0}^{\infty} \frac{(-1)^r (\tfrac{1}{2}z)^{n-1+2r}}{r!\,(n-1+r)!} = \sum_{r=0}^{\infty} \frac{(-1)^r (\tfrac{1}{2}z)^{n-1+2r}}{r!\,(n+r)!}(n+r)$$

Break $n+r \Rightarrow n + r$

$$= n\sum_{r=0}^{\infty} \frac{(-1)^r (\tfrac{1}{2}z)^{n-1+2r}}{r!\,(n+r)!} + \sum_{r=\cancel{0}\,1}^{\infty} \frac{(-1)^r\, r \;\cdots}{r!}$$

$$+ \sum_{r=1}^{\infty} \frac{(-1)^r}{(r-1)!\,(n+r)!}$$

$$+ \sum_{r=0}^{\infty} \frac{(-1)^{r+1} (\tfrac{1}{2}z)^{n+1+2r}}{r!\,(n+1+r)!}$$

$$n\left(\tfrac{1}{2}z\right)^{-1} \sum_{r=0}^{\infty} \qquad - J_{n+1}$$

$$= \frac{2n}{z} J_n - J_{n+1}$$

PV:

$$J_n' = \frac{n}{z} J_n - J_{n+1} \qquad J_n = \sum_{r=0}^{\infty} \frac{(-1)^r (\frac{1}{2} z)^{n+2r}}{r!\,(n+r)!}$$

$$J_n' = \sum_{r=0}^{\infty} \frac{(-1)^r (n+2r)\, \frac{1}{2}\, (\frac{1}{2} z)^{n+2r-1}}{r!\,(n+r)!}$$

~~$= \sum_{r=0}^{\infty} \frac{(-1)^r (n+r)\, \frac{1}{2} (\frac{1}{2} z)^{-1} (\frac{1}{2} z)^{n+2r}}{r!\,(n+r)!}$~~

~~$+ \sum_{r=0}^{\infty} \frac{(-1)^r\, r\, \frac{1}{2} (\frac{1}{2} z)^{-1} (\frac{1}{2} z)^{n+2r}}{r!\,(n+r)!}$~~

~~$= \sum_{r=0}^{\infty} \frac{(-1)^r (\frac{1}{z}) (\frac{1}{2} z)^{n+2r}}{r!\,(n+r-1)!} + \sum_{r=1}^{\infty} \frac{(-1)^r \frac{1}{z} (\frac{1}{2} z)^{n+2r}}{(r-1)!\,(n+r)!}$~~

~~$+ \sum_{r=0}^{\infty} \frac{(-1)^{r+1} \frac{1}{z} (\frac{1}{2} z)^{n+2r+2}}{r!\,(n+1+r)!}$~~

~~$+ (\frac{1}{2} z)^{1} \frac{(-1)}{z} \sum_{r=0}^{\infty} \frac{(-1)^r (\frac{1}{2} z)^{n+1+2r}}{r!\,(n+1+r)!}$~~

$\frac{d}{z}$

Break up $n+2r \Rightarrow n + 2r$

$$\Rightarrow n \sum_{r=0}^{\infty} \frac{(-1)^r \frac{1}{2}(\frac{1}{2}z)^{-1}(\frac{1}{2}z)^{n+2r}}{r!\,(n+r)!} + 2\sum_{r=0}^{\infty} \frac{(-1)^r r\, z^{-1}(\frac{1}{2}z)^{n+2r}}{r!\,(n+r)!}$$

$$= \frac{n}{z}\sum_{r=0}^{\infty} \frac{(-1)^r(\frac{1}{2}z)^{n+2r}}{r!\,(n+r)!} + \frac{2}{z}\sum_{r=1}^{\infty} \frac{(-1)^r r\,(\frac{1}{2}z)^{\cdots}}{\cdots}$$

$$= \frac{n}{z}J_n \qquad + \frac{2}{z}\sum_{r=}^{\infty} \frac{(-1)^r(\frac{1}{2}z)^{n+2r}}{!(r-1)!(n+r)!}$$

$$+ \frac{2}{z}\sum_{r=0}^{\infty} \frac{(-1)^{r+1}(\frac{1}{2}z)^{n+2r+2}}{r!\,(n+1+r)}$$

$$+ \frac{\cancel{2}}{\cancel{z}}\left(\frac{1}{\cancel{2}}z\right)^{(-1)} \sum_{r=0}^{\infty} \frac{(-1)^r(\frac{1}{2}z)^{n+2r+1}}{r!\,(n+1+r)!}$$

$$- J_{n+1}(z)$$

$$\therefore \quad J_n' = \frac{n}{z}J_n(z) - J_{n+1}(z) \qquad *$$

Pr :

$$J_n' = \frac{1}{2}\left[J_{n-1} - J_{n+1}\right]$$

Break $n+2r$ into $(n+r) + (r)$

$$\sum_{r=0}^{\infty} \frac{(-1)^r \, \frac{1}{2} \left(\frac{1}{2}z\right)^{n+2r-1}}{r!\,(n+r-1)!}$$

$$+ \sum_{r=\cancel{0}\,1}^{\infty} \frac{(-1)^r \, r \, \frac{1}{2} \left(\frac{1}{2}z\right)^{n+2r-1}}{r!\,(n+r)!}$$

↑
1st term vanishes

$$\frac{r}{r!} = \frac{1}{(r-1)!}$$

$$= \underbrace{\frac{1}{2} \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{1}{2}z\right)^{n-1+2r}}{r!\,(n-1+r)!}}_{\frac{1}{2} J_{n-1}(z)} + \sum_{r=0}^{\infty} \frac{(-1)^{r+1} \, \frac{1}{2} \left(\frac{1}{2}z\right)^{n+2r+1}}{r!\,(n+r+1)!}$$

$$+ -\frac{1}{2} \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{1}{2}z\right)^{n+2r}}{r!\,(n+r+1)!}$$

$$-\frac{1}{2} J_{n+1}(z)$$

$$= \frac{1}{2}\left(J_{n-1} - J_{n+1}\right)$$ ✓

Pf:

$$J_n' = J_{n-1} - \frac{n}{z} J_n$$

Break $n+2r$ use $\Gamma(n+r+1)$.

$$J_n' = \sum_{r=0}^{\infty} (-1)^r$$

~~$n+r$~~ ~~$(n+r)$~~ $\Gamma(n+r)$

$$\frac{n}{z} J_n - \underline{J_{n+1}} = \frac{1}{2} J_{n-1} - \underline{\frac{1}{2} J_{n+1}}$$

$$\frac{n}{z} J_n = \frac{1}{2} J_{n-1} + \frac{1}{2} J_{n+1}$$

Should be able to get this from A, B, + C.

$$J_n = \frac{z}{2n}(J_{n-1} + J_{n+1})$$

$$J_n' = \frac{1}{2}\left[J_{n-1} - J_{n+1}\right]$$

use A $\left(-\frac{1}{2} J_{n+1} = -\frac{n}{z} J_n + \frac{1}{2} J_{n-1} \right)$

$$\Rightarrow J_n' = \frac{1}{2} J_{n-1} + -\frac{n}{z} J_n + \frac{1}{2} J_{n-1} = J_{n-1} - \frac{n}{z} J_n$$ ✓

24 $$J_n' = J_{n-1} - \frac{n}{z} J_n$$

$$2J_n' = \sum_{r=0}^{\infty} \frac{(-1)^r (n+2r)(\frac{1}{2}z)^{n+2r-1}}{r!(n+r)!}$$

$$= \sum \frac{(-1)^r (2n+2r-n)(\frac{1}{2}z)^{n+2r-1}}{r!(n+r)!}$$

$$= 2\sum \frac{(-1)(\frac{1}{2}z)^{n+2r-1}}{r!(n+r-1)!} - n \sum_{r=0}^{\infty} \frac{(-1)^r (\frac{1}{2}z)^{n+2r-1}}{r!(n+r)!}$$

$$2J_{n-1} \qquad - n\left(\frac{z}{2}\right)^{-1} J_n$$

$$\therefore J_n' = J_{n-1} - \frac{n}{z} J_n \qquad \checkmark$$

PS 360 W/W

Ex 2: $\frac{d}{dz}(z^n J_n) = z^n J_{n-1}(z)$

$$\frac{d}{dz}\left(\frac{1}{2\pi i} \frac{z^{2n}}{2^n} \int_{-\infty}^{0^+} t^{-n-1} \exp\left(t - \frac{z^2}{4t}\right) dt \right)$$

$$= \frac{1}{2\pi i} \underbrace{\frac{2n z^{2n-1}}{2^n} \int_{-\infty}^{0^+}} + \frac{1}{2\pi i} \frac{z^{2n}}{2^n} \int_{-\infty}^{0^+} t^{-n-1} \left(-\frac{z}{2t}\right) \exp(\ ) dt$$

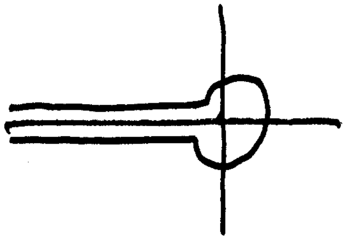
$$+ \frac{1}{2\pi i} \frac{-z^{2n+1}}{2^{n+1}} \underbrace{\int_{-\infty}^{0^+} t^{-n-2} \exp(\ )}_{2\pi i (2z^{-1})^{n+1} J_{n+1}(z)}$$

$$n z^{n-1} J_n + z^n J_n' = \cancel{n z^{n-1} J_n} + z^n \left(J_{n-1} - \cancel{\frac{n}{z} J_n} \right)$$

$$= z^n J_{n-1}$$

Pg 360 Wht/Watson

Ex 3: Show $J_0' = -J_1$



$$J_0 = \frac{1}{2\pi i}\int_{-\infty}^{(0^+)} t^{-1}\exp\left(t - \frac{z^2}{4t}\right)dt$$

$$J_0' = \frac{1}{2\pi i}\int_{-\infty}^{(0^+)} t^{-1}(-)\left(\frac{2z}{4t}\right)\exp\left(t - \frac{z^2}{4t}\right)dt$$

$$= \frac{-z}{2^2\pi i}\int_{-\infty}^{(0^+)} t^{-1-1}\exp\left(t - \frac{z^2}{4t}\right)dt$$

$$= -J_1(z) \qquad \checkmark$$

<u>pg 360 WW</u>

<u>Ex 4</u>: Show

$$\underset{\underset{2^4}{\parallel}}{16} J^{IV} = J_{n-4} - 4J_{n-2} + 6J_n - 4J_{n+2} + J_{n+4}$$

Use $J_n' = \frac{1}{2}(J_{n-1} - J_{n+1})$

$$J_n'' = \frac{1}{2}(J_{n-1}' - J_{n+1}')$$

$$= \frac{1}{2^2}(J_{n-2} - J_n - J_n + J_{n+2})$$

$$= \frac{1}{4}(J_{n-2} - 2J_n + J_{n+2})$$

etc...

PS 360 W/W

Ex 4: Show

$$\underset{\underset{2^4}{\|}}{16} J_n^{iv} = J_{n-4} - 4J_{n-2} + 6J_n - 4J_{n+2} + J_{n+4}$$

~~$J_n' = \frac{n}{2} J_n - J_{n+1}$~~

~~$J_n'' = -\frac{n}{2^2} J_n + \frac{n}{2} J_n$~~

Use $J_n' = \frac{1}{2}(J_{n-1} - J_{n+1})$

mistake

$$J_n'' = \frac{1}{2}(J_{n-1}' - J_{n+1}') = \frac{1}{2^2}[J_{n-2} - J_n]$$

$$J_n''' = \frac{1}{2}(J_{n-1}'' - J_{n+1}'') = \frac{1}{2^3}[J_{n-3} - J_{n-1} - J_{n-1} + J_{n+1}]$$

$$= \frac{1}{8}[J_{n-3} - 2J_{n-1} + J_{n+1}]$$

$$J_n^{iv} = \frac{1}{2}(J_{n-1}''' - J_{n+1}''')$$

$$= \frac{1}{16}(J_{n-4} - 2J_{n-2} + J_n - J_{n-2} + 2J_n - J_{n+2})$$

$$\Rightarrow 16 J_n^{iv} = J_{n-4} -$$

Pg 360 W/W

Ex 5

Shew $J_2 - J_0 = 2J_0''$

from $\frac{d}{dz}\{z^{-n}J_n(z)\}$ $\qquad J_0' = -J_1$

$= -z^{-n}J_{n+1}(z)$ $\qquad J_0'' = -J_1'$

C

put $n=0$

$\Rightarrow J_0' = -J_1$

Now $J_0'' = -J_1'$ but (C) w/ $n=1$

$\Rightarrow J_1' = \frac{1}{2}\{J_0 - J_2\}$

$\Rightarrow 2J_0'' = -\{J_0 - J_2\} = J_2 - J_0$

Pj 360 W3/b

~~$\frac{d}{dz}(z^{-n}J_n) = -z^{-n}J_{n+1}$~~

$$J_n' = \frac{n}{z}J_n - J_{n+1}$$

$$z^{-n}J_n' = nz^{-n-1}J_n - J_{n+1}z^{-n}$$

$$\underbrace{z^{-n}J_n' - nz^{-n-1}J_n} = -J_{n+1}z^{-n}$$

$$\frac{d}{dz}(z^{-n}J_n) = -J_{n+1}z^{-n}$$

$$-\frac{1}{z}\frac{d}{dz}(z^{-n}J_n) = J_{n+1}z^{-n-1}$$

True for $r = \underline{1}$ Assum for k

$$(-1)^k\frac{d^k}{(z\,dz)^k}(z^{-n}J_n) = z^{-n-k}J_{n+k}$$

$$-1\,\frac{d}{z\,dz}\longrightarrow \quad (-1)^{k+1}\frac{d^{k+1}}{(z\,dz)^{k+1}}(z^{-n}J_n) = \frac{-1}{z}\Big[(-n-k)z^{-n-k-1}J_{n+k}$$

$$z^{-n-k}J_{n+k}'\Big]$$

$$= (n+k)z^{-n-k-2}J_{n+k} - z^{-n-k-1}J_{n+k}'$$

$$= z^{-n-k-1}\left[(n+k)z^{-1}J_{n+k} - \cancel{z}J_{n+k}'\right]$$

$$J_{n+k}' = \frac{(n+k)}{z} J_{n+k} - J_{n+k+1} \quad \text{by B.}$$

$$\therefore \quad \frac{n+k}{z} J_{n+k} - J_{n+k}' = J_{n+k+1}.$$

$+$ IHS $\Rightarrow$ $z^{-n-k-1} J_{n+k+1}$ true for $k+1$ ✓.

$\underline{17.21}^2$ $y = z^{-1/2} v$ ~~$x = y(x)$~~

$z = x/2i$ ~~$v = z^{1/2} y(x) = z^{1/2} y(2iz)$~~

Do $z = x/2i$ 1st

~~$\frac{d}{dx} = \frac{1}{2i}\frac{d}{dz}$~~ ~~$\frac{d^2}{dx^2} = -\frac{1}{4}\frac{d^2}{dz^2}$~~

$\therefore$ B.E. $\Rightarrow$

~~$-\frac{1}{4}\frac{d^2y}{dz^2} + \frac{\cancel{2i}}{x}\frac{dy}{\cancel{2i}\,dz} + \left(1 - \frac{n^2}{x^2}(-4)\right)y = 0$~~

~~$\Rightarrow y'' - \frac{3}{x}y' - \left(1 + \frac{4n^2}{x^2}\right)y = 0$~~

let ~~$Y =$~~

$$\cancel{-\frac{1}{4}\frac{d^{2}}{dz^{2}}+\frac{1}{}}$$

$$\frac{d}{dz} = 2i\frac{d}{dx}$$

$$\frac{d^2}{dz^2} = -4\frac{d^2}{dx^2}$$

$\therefore$ BE $\Rightarrow$

$$\cancel{-4}\frac{d^2}{dx^2}y + \frac{\cancel{2i}(\cancel{2i})}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}(-4))\right)y = 0$$

$$\underset{\frac{d^2y}{dx^2}}{\underset{\shortparallel}{y''}} + \frac{1}{x}y' + \left(-\frac{1}{4} - \frac{n^2}{x^2}\right)y = 0$$

let $y = z^{-1/2}v = \dfrac{x^{-1/2}}{(2i)^{-1/2}}v$

$$= \sqrt{2i}\,x^{-1/2}v$$

$$\frac{dy}{dx} = \sqrt{2i}\left[-\frac{1}{2}x^{-3/2}v + x^{-1/2}v'\right]$$

$$\frac{d^2y}{dx^2} = \sqrt{2i}\left(-\frac{1}{2}\left(-\frac{3}{2}\right)x^{-5/2}v - x^{-3/2}v' + x^{-1/2}v''\right]$$

pd in

$\Rightarrow$ $\cancel{\phi}$ $\frac{3}{4}x^{-5/2}v - x^{-3/2}v' + x^{-1/2}v'' + \cancel{x^{-3/2}v'}$

$$+ -\frac{1}{2}x^{-5/2}v + x^{-3/2}v'$$

$-\left(\frac{1}{4}+\frac{n^2}{x^2}\right)x^{-1/2}v = 0$ mult by $x^{+5/2}$

$\Rightarrow \frac{3}{4}v - \cancel{xv'} + \underline{x^2 v''} - \frac{1}{2}v + \cancel{xv'} - \left(\frac{1}{4}+\frac{n^2}{x^2}\right)x^2 v$

$= 0$

$\div\, x^2$

$v'' + \left(\frac{3}{4x^2} - \frac{1}{2x^2} - \frac{1}{4} - \frac{n^2}{x^2}\right)v = 0$

$-\frac{1}{4} + \frac{1}{4x^2} - \frac{n^2}{x^2}$

$v'' + \left(-\frac{1}{4} + \frac{1/4 - n^2}{x^2}\right)v = 0$

$y = y(z) \longrightarrow$

$= \left(\frac{x}{2i}\right)^{-1/2} v(x)$

But dn still $\frac{d}{dz} \longrightarrow \frac{d}{dx}$.

pg 360 W/W

17.212

$W_{k,m}(z)$ sol.

$$W'' + \left\{ -\frac{1}{4} + \frac{k}{z} + \frac{1/4 - m^2}{z^2} \right\} W = 0 .$$

$\therefore$ sol of $\quad V'' + \left\{ -\frac{1}{4} + \frac{1/4 - n^2}{x^2} \right\} V = 0$

is $V = W_{0,m}$.

---

pg 361 W/W

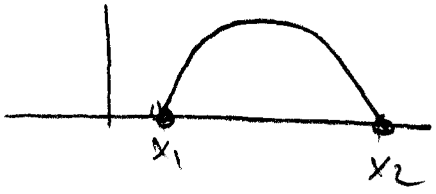
pg 361 W/W

(B) $\Rightarrow$ $$z^{-n} J_{n+1} = -\frac{d}{dz}\{z^{-n} J_n\}$$

if z_1 & z_2 are zeros of $z^{-n} J_n$

$f(x) = \frac{d}{dz} g(x)$

~~$f(x_2) - f(x_2) = g(x_2)$~~



$$\int_{x_1}^{x_2} f = g(x_2) - g(x_1) = 0$$

$\therefore$ $\int_{x_1}^{x_2} f\,dx = 0$ If $f(x)$ did not change sign at least once during x_1 to x_2 $\int_{x_1}^{x_2} f \neq 0$ Thus $\exists$ at least one zero of f between x_1 & x_2.

$\Rightarrow$ $\exists$ at least one zero of $z^{-n} J_{n+1}$ between two zeros of $-z^{n} J_n$

~~[illegible]~~

⑧ $J_n' = J_{n-1} - \frac{n}{z} J_n$

$z^{+n} \rightarrow$

$$z^{+n} J_n' = z^{+n} J_{n-1} - n z^{+n-1} J_n$$

$$\Rightarrow \quad z^{+n} J_n' + n z^{+n-1} J_n = z^{+n} J_{n-1}$$

$$\Rightarrow \quad \frac{d}{dz}\left(z^n J_n\right) = z^n J_{n-1}$$

or $n \rightarrow n+1$

$$z^{n+1} J_n = \frac{d}{dz}\left(z^{n+1} J_{n+1}\right)$$

PS 361 W/W

$z^{-n}J_n$ + $\frac{d}{dz}\{z^{-n}J_n\}$ have no comm

zeros if $y = z^{-n}J_n$

for $\quad y' = -nz^{-n-1}J_n + z^{-n}J_n'$

$$y'' = -n(-n-1)z^{-n-2}J_n - 2nz^{-n-1}J_n' + z^{-n}J_n''$$

Then $zy'' = -n(-n-1)z^{-n-1}J_n - 2nz^{-n}J_n' + z^{-n+1}J_n''$

$+(2n+1)y' = -n(2n+1)z^{-n-1}J_n + (2n+1)z^{-n}J_n'$

$zy = z^{-n+1}J_n$

+

$$\Rightarrow -n(-n-1)z^{-n-1}J_n - 2nz^{-n}J_n' + z^{-n-1}\left(-zJ_n' - (z^2-n^2)J_n\right)$$

$$-n(2n+1)z^{-n-1}J_n + (2n+1)z^{-n}J_n' + z^{-n+1}J_n$$

$$= n^2\cancel{z^{-n-1}}J_n + n\cancel{z^{-n-1}}J_n - \cancel{2nz^{-n}J_n'} - \cancel{z^{-n}J_n'}$$

$$-\cancel{z^{-n+1}J_n} + n^2\cancel{z^{-n-1}}J_n - 2n^2\cancel{z^{-n-1}}J_n - n\cancel{z^{-n-1}}J_n$$

$$+2n\cancel{z^{-n}J_n'} + \cancel{z^{-n}J_n} + \cancel{z^{-n+1}J_n}$$

$= 0$!!

if $y'(z) = y(z) = y''(z) = 0$

Thus All zeros of $z^{-n}J_n$ are simple

if ξ

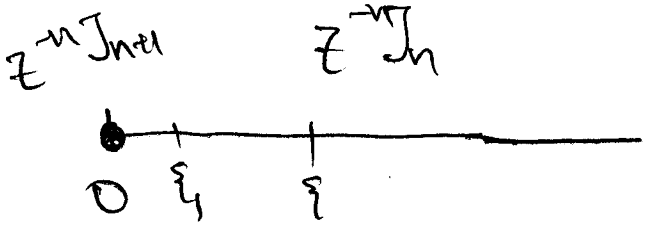
$$z^{-n}J_n(z)$$

$$(-z)^{-n}J_n(-z) \overset{?}{=} z^{-n}J_n(z) \text{ if } n \text{ even}$$

$$= +z^{-n}J_n(z)$$

$J_n(-z) = J_n(z)$ n even

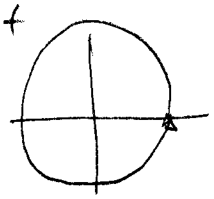
$J_n(-z) = -J_n(z)$ n odd



Pg 362 W/W

$$J_n = \frac{1}{2\pi i}\int^{(0^+)} u^{-n-1} e^{\frac{1}{2}z\left(u-\frac{1}{u}\right)}\, du$$

$n \in \mathbb{Z}^+$



$$u = e^{i\theta}$$

$$J_n = \frac{1}{2\pi \cancel{i}}\int_0^{\cancel{0}\,2\pi} e^{-(n+\cancel{1})i\theta} e^{\frac{1}{2}z\left(e^{i\theta}-e^{-i\theta}\right)} \cancel{ie^{i\theta}}\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{-ni\theta} e^{iz\sin\theta}\, d\theta \quad * \quad \begin{matrix} 0 \longrightarrow -\pi \\ 2\pi \to \pi \end{matrix}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-n(u+\pi)} e^{iz\sin(u+\pi)}\, du \;\Bigg|\; \begin{matrix} u = \theta - \pi \\ du = d\theta \end{matrix} \qquad \theta = (u+\pi)$$

$$\sin(u+\pi) = \sin u\cos\pi + \cancel{\sin\pi}^{\,0}$$

$$= -\sin u$$

$e^{-inu} e^{-in\pi}$??

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-niu} e^{-in\pi} e^{-iz\sin u}\, du$$

why is this wrong

If just ~~go~~ go from $-\pi$ to π

$$J_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ni\theta} e^{+iz\sin\theta}\, d\theta$$

$$J_n = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - z\sin\theta)\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cos(n\theta - z\sin\theta)\,d\theta .$$

show equivalent

$$* \quad J_n = \frac{1}{2\pi}\int_0^{2\pi} e^{-(n\theta - z\sin\theta)i}\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cos(n\theta - z\sin\theta)\,d\theta - \frac{1}{2\pi} i\int_0^{2\pi} \sin(n\theta - z\sin\theta)\,d\theta$$

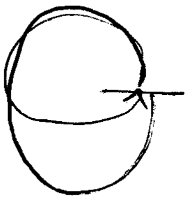
$$\frac{1}{2\pi}\left[\int_{-\pi}^{0} \quad + \quad \int_{0}^{\pi}\right]$$

$$\frac{1}{2\pi}\left[\int_{0}^{\pi} e^{in\theta}e^{-iz\sin\theta}\,d\theta + \int_{0}^{\pi} e^{-in\theta}e^{+iz\sin\theta}\,d\theta\right]$$

$$= \frac{1}{2\pi}\int_{0}^{\pi} e^{i(n\theta - z\sin\theta)}\,d\theta + = \frac{1}{\pi}\int_{0}^{\pi}\cos(n\theta - z\sin\theta)\,d\theta$$

Pg 362 w/w

$$J_n = \frac{1}{2\pi \cancel{i}} \int_0^{2\pi} e^{-i\theta(n\cancel{+1})} e^{\frac{z}{2}(e^{i\theta}-e^{-i\theta})} \cancel{ie^{i\theta}}\, d\theta$$



$$= \frac{1}{2\pi} \int_{``0"}^{``2\pi"} e^{-i\theta n} e^{iz\sin\theta}\, d\theta$$

$$= \cancel{\frac{1}{2\pi} \int_0^{2\pi} e^{i(n\theta}}$$

$$= \frac{1}{2\pi} \int_0^{\pi} e^{-ni\theta + iz\sin\theta}\, d\theta + \frac{1}{2\pi} \int_{\pi}^{2\pi} e^{-i\theta n + iz\sin\theta}\, d\theta.$$

let $2\pi - \phi = \theta$

$\theta = \pi \quad \phi = \pi$

$\theta = 2\pi \quad \phi = 0.$

$\sin(2\pi - \phi)$

$= \underline{0} - \sin\phi \cos 2\pi$

$= -\sin\phi$

$$= \frac{1}{2\pi} \int_0^{\pi} e^{-i(2\pi - \phi)n + iz\sin(2\pi-\phi)} (\cancel{-}d\phi)$$ 1st part of J_n

$$= \frac{1}{2\pi} e^{-i2\pi n} \int_0^{\pi} e^{i\phi n - iz\sin\phi}\, d\phi$$

$\cancel{\phi = z\phi}$

Pg 362 w/w

Ex 1:

$$|J_n| \le \frac{1}{\pi}\int_0^\pi |\cos(\quad)|\,d\theta \le \frac{1}{\pi}\int_0^\pi 1\,d\theta = \underline{1}$$

Ex 2

$$y = \frac{1}{\pi}\int_0^\pi \cos(n\theta - z\sin\theta)\,d\theta$$

$$\frac{dy}{dz} = -\frac{1}{\pi}\int_0^\pi \sin(n\theta - z\sin\theta)\sin\theta(-1)\,d\theta$$

$$= \frac{1}{\pi}\int_0^\pi \sin(n\theta - z\sin\theta)\sin\theta\,d\theta$$

$$\frac{d^2y}{dz^2} = \frac{1}{\pi}\int_0^\pi \cos(n\theta - z\sin\theta)\,{-\sin^2\theta}\,d\theta$$

Then

$$\frac{d^2y}{dz^2} + \frac{1}{z}\frac{dy}{dz} + \left(1 - \frac{n^2}{z^2}\right)y = \frac{1}{\pi}\int_0^\pi \Big[-\cos(n\theta - z\sin\theta)\overset{\cancel{1} - \cos^2\theta}{\sin^2\theta}$$

$$+ \frac{\sin\theta}{z}\sin(n\theta - z\sin\theta) + \cos(\cancel{n\theta} - z\sin\theta) - \frac{n^2}{z^2}\cos(n\theta - z\sin\theta)\Big]\,d\theta$$

$$= \frac{1}{\pi}\int_0^\pi \left(\underline{\cos^2\theta}\cos(\quad) + \frac{\sin\theta}{z}\sin(\quad) - \frac{n^2}{z^2}\cos(\quad)\right)d\theta$$

pg 364 W/W

$k \in \mathbb{Z}^+.$

$$z^{-n-r} J_{n+r} = (-1)^r \frac{d^r}{(z\,dz)^r} \{ z^{-n} J_n(z) \}$$

$$d(z^2/2) = z\,dz$$

$\nu = 1/2$

$$z^{-1/2-r} J_{r+1/2} = (-1)^r \frac{d^r}{d(z^2/2)^r} \{ z^{-1/2} J_{1/2} \}$$

$$= (-1)^r 2^r \frac{d^r}{d(z^2)^r} \left\{ \frac{\sqrt{2} \sin z}{\sqrt{\pi}\, z} \right\}$$

$$J_{r+1/2} = z^{1/2+r} (-1)^r 2^r \frac{\sqrt{2}}{\sqrt{\pi}} \frac{d^r}{d(z^2)^r} \left(\frac{\sin z}{z} \right)$$

$$(2z)^{k+1/2}$$

$r \to k$

$$J_{k+1/2} = \frac{(-1)^k (2z)^{k+1/2}}{\sqrt{\pi}} \frac{d^k}{d(z^2)^k} \left(\frac{\sin z}{z} \right)$$

$$\|$$

$$\frac{d^k}{d(z^2)^k} \left(\frac{\sin (z^2)^{1/2}}{(z^2)^{1/2}} \right)$$

$$\frac{d^k\left(\sin(z^2)^{1/2}\cdot(z^2)^{1/2}\right)}{d(z^2)^k}$$

Use Libnitz Rule.

Ex 1: $J_{-1/2} = \left(\frac{2}{\pi x}\right)^{1/2}\cos x$. Shew.

~~See Hildebrant.~~

~~[illegible]~~ 7

$$J_{-1/2} = \cancel{\sum_{r=0}^{\infty}\frac{(-1)^r z^{-1/2+2r}}{2^{-1/2+2r}\,\Gamma(r+1/2)}}$$

See pg 175 Hildebrant prob 31.

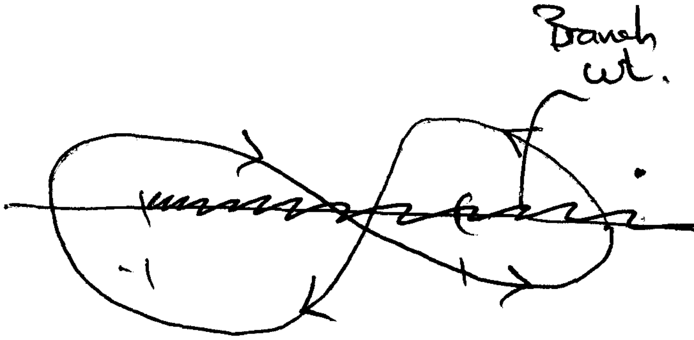
Pg 364 W/W

Ex 1 using (A) $\quad J_{-1/2} = \left(\frac{2}{\pi z}\right)^{1/2} \cos z .$

$$J_{1/2} = \frac{1}{\pi}\int_0^{\pi} \cos\left(-\frac{\theta}{2} - z\sin\theta\right) d\theta - \frac{1}{\pi}\int_0^{\infty} e^{-\theta/2 - z\sinh\theta}\, d\theta \quad ?$$

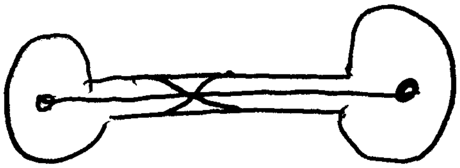
$$y = z^n \int_A^{(+1+,-1-)} (t^2-1)^{n-1/2} \cos(zt)\, dt$$

<u>pg 365 WW</u>



for $t-1$ $\qquad e^{2\pi i(n-1/2)}$

for $t+1$ $\qquad e^{-2\pi i(n-1/2)}$ $\qquad$ mult tog give 1



for convergence

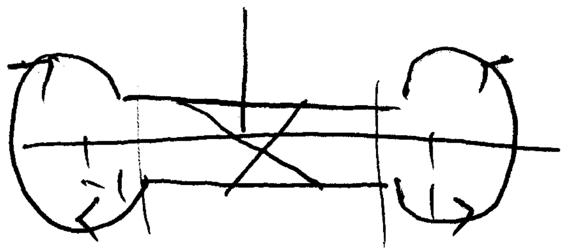
power of $t-1$ + $t+1$

~~must be~~ is $-n+1/2$

$\therefore \ Re(-n+1/2) < 1$

$Re(n-1/2) > -1$

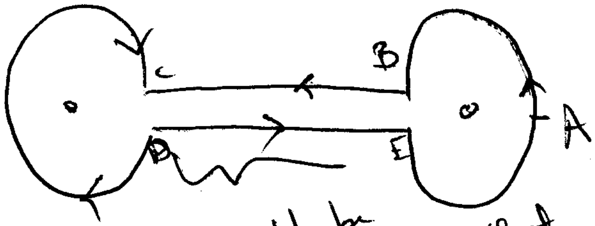
$Re(n+1/2) > 0$



Claim $\displaystyle\int_{0,\delta}^{-1+\delta} \xrightarrow{\delta \to 0} 0$

$$\int_0^{2\pi} \delta^{2r} e^{2ri\theta} (\delta^2 e^{2i\theta} - 1)^{n-1/2} \delta i e^{i\theta} \, d\theta$$

$$= \delta^{2r+1} i \int_0^{2\pi} e^{(2r+1)\theta} (\delta^2 e^{2i\theta} - 1)^{n-1/2} \, d\theta \xrightarrow{\delta \to 0} 0$$



Should be crossover point (How affect Branch cuts?)

$$\int_A^{(1+,-1-)} = \cancelto{0}{\int_A^B} + \int_{\underline{B}}^C + \cancelto{0}{\int_C^D} + \int_{\underline{D}}^E + \cancelto{0}{\int_E^A}$$

$$\int_B^C = \int_1^{-1} = e^{i\pi(n-1/2)} \int_1^{-1} t^{2r} (t^2-1)^{n-1/2} \, dt$$

Arg $t-1$ is $e^{i\pi}$

Arg $t+1$ is 1

$$\int_D^E = e^{-\pi i(n-\frac{1}{2})}\int_{-1}^{1} t^{2r}(t^2-1)^{n-\frac{1}{2}}\,dt$$

Arg $t-1$ is $e^{-\pi i}$

Arg $t+1$ is ~~$e^{\pi i}$~~ $\underline{1}$

$$\int_A^{(+1+,\,-1-)} t^{2r}(t^2-1)^{n-\frac{1}{2}}\,dt = e^{(n-\frac{1}{2})\pi i}\int_1^{-1} t^{2r}(1-t^2)^{n-\frac{1}{2}}\,dt$$

$$+\, e^{-(n-\frac{1}{2})\pi i}\int_{-1}^{1} t^{2r}(1-t^2)^{n-\frac{1}{2}}\,dt$$

$$= e^{(n-\frac{1}{2})\pi i}\ \cancel{2\int_D^{E}} - 2\int_{-1}^{1} t^{2r}(1-t^2)^{n-\frac{1}{2}}\,dt\ )$$

$$+\, e^{-(n-\frac{1}{2})\pi i}\int_{-1}^{1} t^{2r}(1-t^2)^{n-\frac{1}{2}}\,dt$$

$$= -2e^{(n-1/2)\pi i}\int_0^1 t^{2r}(1-t^2)^{n-1/2}\,dt + 2e^{-(n-1/2)\pi i}\int_0^1 t^{2r}(1-t^2)^{n-1/2}d$$

$$= -2\left[e^{(n-1/2)\pi i} - e^{-(n-1/2)\pi i}\right]\int_0^1 t^{2r}(1-t^2)^{n-1/2}\,dt$$

$$= -2\cdot 2i\sin((n-1/2)\pi)\int_0^1 t^{2r}(1-t^2)^{n-1/2}\,dt$$

$$u = t^2$$

$$du = 2t\,dt = 2\sqrt{u}\,dt \Rightarrow dt = \frac{du}{2\sqrt{u}}$$

$$= -4i\sin((n-\tfrac{1}{2})\pi)\,\frac{1}{2}\int_0^1 u^r(1-u)^{n-1/2}\,\frac{du}{\sqrt{u}}$$

$$= -2i\sin((n-1/2)\pi)\underbrace{\int_0^1 u^{r-1/2}(1-u)^{n-1/2}\,du.}_{\dfrac{\Gamma(r+1/2)\,\Gamma(n+1/2)}{\Gamma(n+r+1)}}$$

<u>Beta fn</u>

$$Y = \sum_{r=0}^{\infty} \frac{(-1)^r z^{n+2r}}{(2r)!} \frac{2i \sin\left(\left(n+\frac{1}{2}\right)\pi\right)\Gamma\left(r+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{\underbrace{\Gamma(n+r+1)}_{\text{keep}}}$$

$$= 2i\sin\left(\left(n+\frac{1}{2}\right)\pi\right) \sum_{r=0}^{\infty} \frac{(-1)^r z^{n+2r}}{2r(2r-1)(2r-2)\cdots 4\cdot 3\cdot 2\cdot 1} \Gamma\left(r+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)$$

$\underline{\text{Ex } 3}$ $\underline{\text{Pg } 365 \ W/W}$

Show $z^{k+1/2} \frac{d^k}{d(z^2)^k}\left(\frac{\cos z}{z}\right)$ $J_{k+1/2}(z)$

$$\Rightarrow \quad \frac{d^2y}{dz^2} + \frac{1}{z}\frac{dy}{dz} + \left(1 - \frac{(k+1/2)^2}{z^2}\right)y = 0$$

$\frac{d}{d(z^2)} =$ [illegible]

$$\frac{df}{d(z^2)} = \frac{df}{d\eta}\frac{d\eta}{dz}$$

$\eta = z^2$

$\frac{d\eta}{dz} = 2z$

$$= 2z \frac{df}{d(z^2)}$$

$$\Rightarrow \quad \frac{df}{d(z^2)} = \frac{1}{2z}\frac{df}{d(z^2)}$$

Pg 366 W/W

Pg 193 Rudin

$$\int_0^1 t^{x-1}(1-t)^{y-1}\,dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

Then $\displaystyle\int_0^1 u^{r-1/2}(1-t)^{n-1/2}\,du$

$$= \int_0^1 u^{r+1/2-1}(1-u)^{n+1/2-1}\,du = \frac{\Gamma(r+1/2)\Gamma(n+1/2)}{\Gamma(n+r+1)}$$

Pg 360 w/W

17.6

consider

$$Y_n(z) = 2\pi e^{n\pi i}\frac{(J_n(z)\cos n\pi - J_{-n}(z))}{\sin 2n\pi}$$

2n not an integer. n integer → 2n integer.

if n integer

$$Y_n = \lim_{\epsilon\to0} 2\pi e^{(n+\epsilon)\pi i}\frac{J_{n+\epsilon}(z)\cos(n+\epsilon)\pi - J_{-n-\epsilon}(z)}{\sin 2(n+\epsilon)\pi}$$

$$\cos(n+\epsilon)\pi = \cos(n\pi)\cos\epsilon\pi - \cancel{\sin n\pi}^{\,0}$$

$$= (-1)^n\cos\epsilon\pi$$

~~$Y_n = \lim_{\epsilon\to0}$~~ $\sin(2n\pi + 2\pi\epsilon) = 0 + \sin 2\pi\epsilon\cos 2\pi n$

$\sin 2\pi\epsilon.$

$$\therefore Y_n = \lim_{\epsilon\to0}\frac{2\pi e^{n\pi i}(J_{n+\epsilon}(z)(-1)^n - J_{n-\epsilon}(z))}{\sin 2\pi\epsilon}$$

$$= \lim_{\epsilon\to0}\left(\frac{2\pi}{\sin 2\pi\epsilon}\right)\left(J_{n+\epsilon}(z) - (-1)^n J_{n-\epsilon}(z)\right)$$

$$= \cancel{\lim_{\epsilon\to0}\left(\frac{J_{n+\epsilon}(z) - (-1)^n J_{n-\epsilon}(z)}{\epsilon}\right)}$$

$$= \lim_{\epsilon \to 0} \epsilon \left(\frac{2\pi\epsilon}{\sin 2\pi\epsilon} \right) \left(J_{n+\epsilon}(z) - (-1)^n J_{n-\epsilon}(z) \right)$$

$$= \lim_{\epsilon \to 0} \overset{1}{\overset{\shortparallel}{\left(\frac{2\pi\epsilon}{\sin 2\pi\epsilon} \right)}} \lim_{\epsilon \to 0} \left(\frac{J_{n+\epsilon}(z) - (-1)^n J_{n-\epsilon}(z)}{\epsilon} \right)$$

$$= \lim_{\epsilon \to 0} \frac{J_{n+\epsilon}(z) - (-1)^n J_{n-\epsilon}(z)}{\epsilon}$$

Pg 360 W/W

$$Y_n(z) = \frac{J_n(z)\cos(nz) - J_{-n}(z)}{\sin n\pi}$$

$$\sin 2n\pi = \text{[illegible]}$$

$$2\sin(n\pi)\cos(n\pi)$$

$\therefore$

$$Y_n = \pi e^{n\pi i} \frac{(J_n(z)\cos n\pi - J_{-n}(z))}{\sin(n\pi)\cos(n\pi)}$$

$$= \frac{\pi e^{n\pi i}}{\cos(n\pi)} Y_n(z)$$

$$\therefore \; Y_n(z) = \frac{\cos(n\pi) Y_n}{\pi e^{n\pi i}}.$$

pg 374 W/W

$$\frac{1}{t-z} = O_0(t)J_0(z) + 2O_1(t)J_1(z) + 2$$ ~~$= \sum_{n=0}^{\infty} O_n(t)J_n(z)$~~

$$= O_0(t)J_0(z) + 2\sum_{n=0}^{\infty} O_n(t)J_n(z)$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial z}\right)\frac{1}{t-z} = \frac{1}{(t-z)^2} - \frac{1}{(t-z)^2} = 0.$$

$\Rightarrow$ ~~$\sum_{n=0}^{\infty} O_n'(t)J_n(z) + \sum_{n=0}^{\infty} O_n(t)J_n'(z)$~~ $= 0$

As uniformly convergent we can

$$O_0'(t)J_0(z) + 2\sum_{n=0}^{\infty} O_n'(t)J_n(z)$$

$$+ O_0(t)J_0'(z) + 2\sum_{n=0}^{\infty} O_n(t)J_n'(z) = 0$$

But $2J_n'(z) = J_{n-1} - J_{n+1}$

$\Rightarrow$ ~~$O_0'(t)J_0(z) + \sum_{n=0}^{\infty} O_n'(t)(J_{n-1} - J_{n+1})$~~

Ex 5:

Pg 379 w/w

$v \leftrightarrow M$

$$M = E - e\sin E \qquad \cos v = \frac{\cos E - e}{1 - e\cos E} \qquad |e|<1$$

2 eqs Assume e = eccentricity is given.

other variables are E, M, v

Goal:

elliminate E + write v in terms of M

Ans:

$$v = M + 2(1-e^2)^{1/2} \sum_{m=1}^{\infty} \sum_{k=0}^{\infty} \left(\tfrac{1}{2}e\right)^k J_m^k(me)\, \frac{1}{m} \sin(mM).$$

F.H.

3 second pb   pg 379 W/W

If

$$J_0^2 + 2\sum_{k=1}^{\infty} J_k^2 = 1$$

Then $|J_0^2| \le 1$ if not we have an immediate $\rightarrow\!\!\times\!\!\leftarrow$

$+$ [illegible] $+$ $|2J_k^2| \le 1$

$\forall k$

[illegible] $\Rightarrow$ $J_k^2 \le \frac{1}{2}$

$\Rightarrow$ [illegible] [illegible] $\Rightarrow$ $|J_k| \le 2^{-1/2}$

$\sqrt{x^2} = |x|$ ✓

PS 379 W/W

② $\sin(z\sin\theta) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!} \sin^{2k+1}\theta$

gives only odd powers of z^p!

sum of coeff is

$\lfloor 1$

$\sin\theta = \sin\theta$

$\sin 3\theta = \sin 2\theta + \theta = \sin 2\theta\cos\theta + \sin\theta\cos 2\theta$

$= 2\sin\theta\cos^2\theta + \sin\theta(\cos^2\theta - \sin^2\theta)$

$= 2\sin\theta\cos^2\theta + \sin\theta\cos^2\theta - \sin^3\theta$

$= 2\sin\theta(1-\sin^2\theta) + \sin\theta(1-\sin^2\theta)$

$- \sin^3\theta$

$= 2\sin\theta - 2\sin^3\theta + \sin\theta - \sin^3\theta$

$- \sin^3\theta$

sum of coeff is

$= 3\sin\theta - 4\sin^3\theta$

$\lfloor 1$

$\sin 5\theta =$ ~~$\sin 2\theta + 3\theta$~~

$= \frac{1}{2i}(e^{5i\theta} - e^{-i5\theta})$

$= \sin(2\theta + 3\theta)$

$= \sin 3\theta\cos 2\theta + \sin 2\theta\cos 3\theta$

?

1b

$$\cos 3\theta = \cos 2\theta + \theta$$

$$= \cos 2\theta \cos\theta - \sin 2\theta \sin\theta$$

$$= \cos^3\theta - \sin^2\theta \cos\theta - 2\sin^2\theta \cos\theta$$

$$= \cos^3\theta - 3\sin^2\theta \cos\theta$$

$$= \cos\theta(1 - \sin^2\theta - 3\sin^2\theta) = \cos\theta(1-4\sin^2\theta)$$

sum of coeff is $\boxed{-1}$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$= 1 - \sin^2\theta - \sin^2\theta = 1 - 2\sin^2\theta$$

Claim

$$\cos 2n\theta = \sum \text{even powers of } \sin\theta$$

$$\cos (2n+1)\theta = \cos\theta \sum \text{even powers } \sin\theta.$$

$$\cos 4\theta = \cos 2\theta + 2\theta$$

$$= 1 - 2\sin^2 2\theta$$

$$= 1 - 2(1 - \cos^2 2\theta)$$

$$= 1 - 2 + 2\cos^2 2\theta = -1 + 2\cos^2 2\theta$$

$$= -1 + 2(1 - 2\sin^2\theta)^2 =$$ ~~$-1+2(1-4+4$~~

$$= -1 + 2(1 - 4\sin^2\theta + 4\sin^4\theta)$$

sum of coeff is

$$= 1 - 8\sin^2\theta + 8\sin^4\theta$$

$\boxed{1}$

$$= \sin 3\theta\,(\cos^2\theta - \sin^2\theta) + 2\sin\theta\cos\theta\cos 3\theta$$

$=$ ~~[illegible]~~

$$= (3\sin\theta - 4\sin^3\theta)(1 - 2\sin^2\theta)$$

$$+ 2\sin\theta\cos^2\theta(1 - 4\sin^2\theta)$$

$$= 3\sin\theta - 6\sin^3\theta - 4\sin^3\theta + 8\sin^5\theta$$

$$+ (2\sin\theta - 8\sin^3\theta)(1 - \sin^2\theta)$$

$$= 3\sin\theta - \underline{10\sin^3\theta} + \underline{8\sin^5\theta}$$

$$+ 2\sin\theta - \underline{2\sin^3\theta} - \underline{8\sin^3\theta} + \underline{8\sin^5\theta}$$

$\Rightarrow$ ~~[illegible]~~

$$\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$$

sum of coeff is $\underline{1}$

Need to get general formula for $\sin p\theta$ in terms of powers of $\sin\theta$

Same w/ $\cos p\theta$ in terms of $\sin\theta$.

perhaps use fourier expansions of

$$\sin^{p_1}\theta = \sum A_n^{p_1} \sin n\theta$$

$$\sin^{p_2}\theta = \sum A_n^{p_2} \sin n\theta$$

$$\sin^{p_3}\theta = \sum A_n^{p_3} \sin n\theta$$

As each sum $\sum A_n^{p_i} \sin n\theta$ is finite

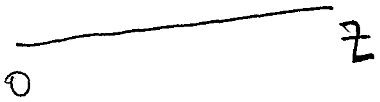
& each $A_n^{p_i}$ is known

$$C_1 \sin^{p_1}\theta + C_2 \sin^{p_2}\theta + C_3 \sin^{p_3}\theta = \sum \left(C_1 A_n^{p_1} + C_2 A_n^{p_2} + C_3 A_n^{p_3}\right) \sin n\theta$$

pick $C_1, \dots, C_n \ni$

$$C_1 A_n^{p_1} + C_2 A_n^{p_2} + C_3 A_n^{p_3} = \delta_{mn}$$

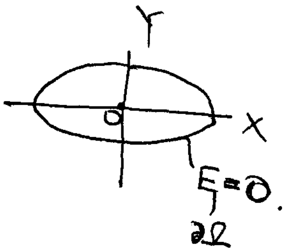
Pg 404 Whittaker/Watson



$$\frac{1}{c^2}\frac{\partial E}{\partial t} = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}$$

$$\frac{\partial H_x}{\partial t} = -\frac{\partial E}{\partial y}$$

$$\frac{\partial H_y}{\partial t} = \frac{\partial E}{\partial x}$$



$$\Rightarrow \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = \frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2}$$

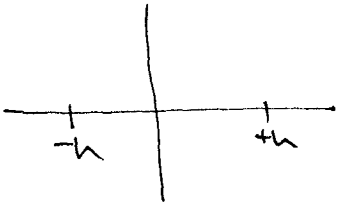
let $V = v(x,y)\cos(pt+\epsilon)$

put into $*$

$$\Rightarrow -\frac{1}{c^2} v(x,y)\cos(\cancel{pt+\epsilon})\, p^2 = \cos(\cancel{pt+\epsilon})\frac{\partial^2 v}{\partial x^2} + \cos(\cancel{pt+\epsilon})\frac{\partial^2 v}{\partial y^2}$$

$$\Rightarrow \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{p^2}{c^2} v = 0$$

let $x + iy = h\cosh(\xi + i\eta)$



$$h(\cosh(\xi)\cosh(i\eta) + \sinh(\xi)\sinh(i\eta))$$

$$\left\{ \cosh(\alpha + B) \overset{?}{=} \cosh(\alpha)\cosh(B) + \sinh(\alpha)\sinh(B) \right.$$

$$\frac{e^{\alpha}+e^{-\alpha}}{2}\cdot\frac{e^{\beta}+e^{-\beta}}{2} + \frac{e^{\alpha}-e^{-\alpha}}{2}\cdot\frac{e^{\beta}-e^{-\beta}}{2}$$

opposite than for trigs

$$\frac{1}{4}\Big(e^{\alpha+B} + \cancel{e^{\alpha-B}} + \cancel{e^{-\alpha+B}} + e^{-(\alpha+B)}$$

$$+\big(e^{\alpha+B} - \cancel{e^{\alpha-B}} - \cancel{e^{-\alpha+\beta}} + e^{-(\alpha+B)}\big)\Big)$$

$$= \frac{1}{4}\left(2e^{\alpha+\beta}) + 2e^{-(\alpha+B)}\right) = \cosh(\alpha+\beta)$$

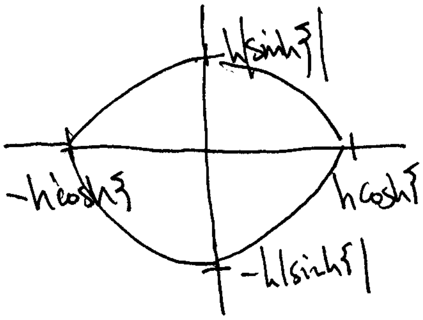
$$\therefore \quad x+iy = h\cosh(\xi)\cos(\eta) + ih\sinh(\xi)\sin(\eta)$$

$$\Rightarrow \quad x = h\cosh(\xi)\cos(\eta)$$ Then

$$y = h\sinh(\xi)\sin(\eta)$$

ξ constant

$$\Rightarrow \quad \frac{x^2}{h^2\cosh^2(\xi)} + \frac{y^2}{h^2\sinh^2(\xi)} = 1$$



$\xi > 0$ $\begin{cases} \xi = 0 \Rightarrow \text{line from } -h \text{ to } h \\ \xi > 0 \Rightarrow \text{ellipse forms.—} \end{cases}$

η constant

$$\Rightarrow \quad \frac{x^2}{h^2\cos^2(\eta)} - \frac{y^2}{h^2\sin^2(\eta)} = 1$$ hyperbolas

Pg 408 Whit/Watson

$$G(\eta) = \int_{-\pi}^{\pi} e^{mh\cos\eta\cos\theta}\phi(\theta)\,d\theta$$

G even ✓

G periodic ✓

$$\frac{dG}{d\eta} = \int_{-\pi}^{\pi} e^{mh\cos\eta\cos\theta}\phi(\theta)(-mh\sin\eta\cos\theta)\,d\theta$$

$$\frac{d^2G}{d\eta^2} = \int_{-\pi}^{\pi}\Big(\phi(\theta)e^{mh\cos\eta\cos\theta}(-mh\sin\eta\cos\theta)^2 + \phi(\theta)e^{mh\cos\eta\cos\theta}(-mh\cos\eta\cos\theta)\Big)d\theta$$

$$= \int_{-\pi}^{\pi}\phi(\theta)e^{mh\cos\eta\cos\theta}\left[m^2h^2\sin^2\eta\cos^2\theta - mh\cos\eta\cos\theta\right]d\theta$$

Then $\frac{d^2G}{d\eta^2} + (A + m^2h^2\cos^2\eta)G$

$$= \int_{-\pi}^{\pi}\Big\{m^2h^2(\sin^2\eta\cos^2\theta$$