

$$F_{n+1} = F_n + F_{n-1}$$

$$n \geq 1$$

$$F_0 = 0; F_1 = 1$$

$$F_{n+1}x^n = F_nx^n + F_{n-1}x^n$$

$$\sum_{n=1}^{\infty} F_{n+1}x^n = \sum_{n=1}^{\infty} F_nx^n + \sum_{n=1}^{\infty} F_{n-1}x^n$$

sum must be over range of values for which the difference equation is valid. Diff eq is not valid for $n=0$.

$$\sum_{n=1}^{\infty} F_{n+1}x^n = \sum_{n=1}^{\infty} F_nx^n + \sum_{n=1}^{\infty} F_{n-1}x^n$$

$$\text{define } F(x) = \sum_{n=0}^{\infty} F_nx^n$$

$$\sum_{n=2}^{\infty} F_nx^{n-1} = \sum_{n=0}^{\infty} F_nx^n + \sum_{n=0}^{\infty} F_nx^{n+1}$$

$$\{F_0=0\}$$

$$\sum_{n=0}^{\infty} F_nx^{n-1} - F_1x^0 - F_0x^{-1} = F(x) + xF(x)$$

$$\Rightarrow x^{-1}F(x) - 1 - 0 = (1+x)F(x)$$

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$$\frac{F(x) - x}{x} = (1+x) F(x)$$

$$F - x = (x+x^2)F$$

$$(x^2+x-1)F = -x$$

$$F = \frac{-x}{x^2+x-1} = \frac{x}{1-x-x^2}$$

$$y_i = 2^i$$

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$$(1.4.9) \quad D(x) = \frac{6}{12} \sum_{i=1}^4 (2^{i+1} - 2 \cdot 2^i + 2^{i-1}) x^i$$

$$= 6 \sum_{i=1}^4 (\cancel{2^{i+1}} - \cancel{2^{i+1}} + 2^{i-1}) x^i$$

$$= 6 \sum_{i=1}^4 2^{i-1} x^i = 3 \sum_{i=1}^4 2^i x^i$$

$$\{2^4 = 8 \cdot 2\}$$

$$= 3 [2x + 4x^2 + 8x^3 + 16x^4]$$

$$= 6x [1 + 2x + 4x^2 + 8x^3]$$

Then (~~1.4.10~~) (1.4.11) gives :

$$\cancel{Z_1} = \cancel{x \cancel{(\sqrt{3}-2)^5 Z_4}}$$

$$Z_1 + (\sqrt{3}-2)^5 Z_4 = -D(\sqrt{3}-2)$$

$$Z_1 + (-\sqrt{3}-2)^5 Z_4 = -D(-\sqrt{3}-2)$$

Now solving this eq gives :

$$\cancel{(\sqrt{3}-2)}$$

$$Z_1 = \frac{\begin{vmatrix} -D(\sqrt{3}-2) & 1 \\ -D(\sqrt{3}-2) & 1 \end{vmatrix}}{\begin{vmatrix} 1 & (\sqrt{3}-2)^5 \\ 1 & (\sqrt{3}-2)^5 \end{vmatrix}}$$

$$\cancel{Z_4} Z_4 = \frac{01-17-03}{2}$$

$$D(\sqrt{3}-2) = 6(\sqrt{3}-2)(1+2(\sqrt{3}-2)) \dots$$

$$f(n, k) = f(n-1, k) + f(n-1, k-1) \text{ w/ } f(n, 0) = 1$$

valid for $n = 1, 2, 3, \dots$

+ $k = 1, 2, 3, \dots \infty$

Assume $f(n, k) = 0$ if $k > n$

$f(n, k)$ = # of subsets of a set w/ n elts that contain k elements.

$$f(n, 0) = 1 \quad f(n, k) = 0 \text{ if } k > n$$

Mult by x^k + sum $k \geq 1$

$$\sum_{k=1}^{\infty} f(n, k) x^k = \sum_{k=1}^{\infty} f(n-1, k) x^k + \sum_{k=1}^{\infty} f(n-1, k-1) x^k$$

||

||

||

$$\sum_{k=0}^{\infty} f(n, k) x^k - f(n, 0) = \sum_{k=0}^{\infty} f(n-1, k) x^k - f(n-1, 0) + \sum_{k=0}^{\infty} f(n-1, k) x^{k+1}$$

$$B_n(x) - 1 = B_{n-1}(x) - 1 + x B_{n-1}(x)$$

$$B_0(x) = \sum_{k=0}^{\infty} f(0, k) x^k$$

$$B_n(x) = (1+x) B_{n-1}(x)$$

$$= 1$$

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$$B_1(x) = (1+x) \cdot 1$$

$$B_2(x) = (1+x)^2$$

$\vdots$

$$B_n(x) = (1+x)^n$$

$$B_n(x) = (1+x)^n \quad n=0,1,2,\dots$$

$$\sum_{n \geq 0} B_n(x) \gamma^n = \sum_{n \geq 0} (1+x)^n \gamma^n$$

$$= \sum_{n \geq 0} \left(\sum_{k=0}^n \binom{n}{k} x^k \right) \gamma^n$$

Following convention:

$$= \sum_{n \geq 0} \left(\sum_k \binom{n}{k} x^k \right) \gamma^n$$

$$= \sum_{n \geq 0} \sum_k \binom{n}{k} x^k \gamma^n$$

using:

$$= \sum_{n \geq 0} ((1+x)\gamma)^n = \frac{1}{1 - (1+x)\gamma} = \text{2D generating function for } \binom{n}{k}$$

$$\binom{n}{k} = [x^k \gamma^n] (1 - \gamma(1+x))^{-1}$$

$$\sum_n \binom{n}{k} \gamma^n = [x^k] \sum_{n \geq 0} \left(\sum_k \binom{n}{k} x^k \right) \gamma^n$$



$$\sum_{n \geq 0} \binom{n}{k} Y^n = ?$$

From:

$$\frac{1}{1-y(1+x)} = \sum_{n \geq 0} \sum_k \binom{n}{k} x^k Y^n = \sum_k \left(\sum_{n \geq 0} \binom{n}{k} Y^n \right) x^k = \text{[scribbled out]}$$

$$= [x^k] \left(\frac{1}{1-y(1+x)} \right) = [x^k] \left(\frac{1}{1-y-yx} \right) = \frac{1}{1-y} [x^k] \left(\frac{1}{1-\frac{y}{1-y}x} \right)$$

$$= \frac{1}{1-y} [x^k] \left(\frac{1}{1-(\frac{y}{1-y})x} \right) = \frac{1}{1-y} \cdot \left(\frac{y}{1-y} \right)^k = \frac{Y^k}{(1-Y)^{k+1}}$$

$$\sum_{n \geq 0} \binom{n}{k} Y^n = \frac{Y^k}{(1-Y)^{k+1}}$$

$$k=0 \Rightarrow \sum_{n \geq 0} \binom{n}{0} Y^n = \frac{1}{1-Y} \quad \checkmark$$

$$k=1 \Rightarrow \sum_{n \geq 0} \binom{n}{1} Y^n = \frac{Y}{(1-Y)^2} = \sum_{n \geq 0} n Y^n = \frac{Y}{(1-Y)^2}.$$

$$\sum_{n \geq 0} \binom{n}{k} \gamma^n = \frac{\gamma^k}{(1-\gamma)^{k+1}}$$

$$\Rightarrow \frac{1}{(1-\gamma)^{k+1}} = \sum_{n \geq 0} \binom{n}{k} \gamma^{n+k} = \sum_{n \geq -k} \binom{n+k}{k} \gamma^n = \sum_{n \geq 0} \binom{n+k}{k} \gamma^n$$

$$\left(\frac{1}{1-\gamma}\right)^k = \sum_{n \geq 0} \binom{n+k-1}{k-1} \gamma^n = \sum_{n \geq 0} \binom{n+k-1}{n+k-1-(k-1)} \gamma^n = \sum_{n \geq 0} \binom{n+k-1}{n} \gamma^n$$

Now $\binom{n+k}{k} = \binom{n+k}{n+k-k} = \binom{n+k}{n}$

So

P₄:

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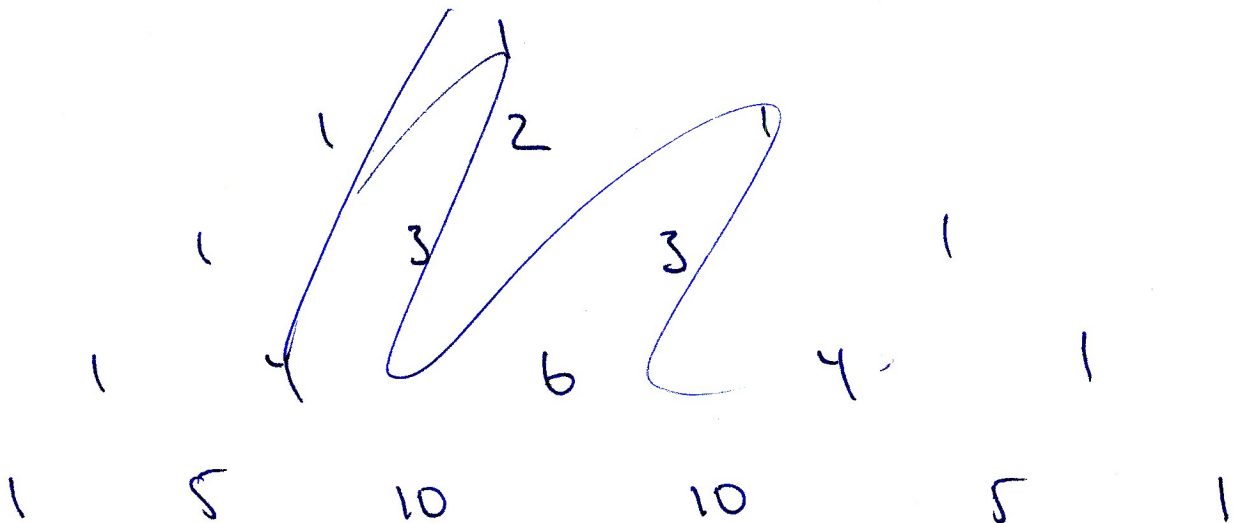
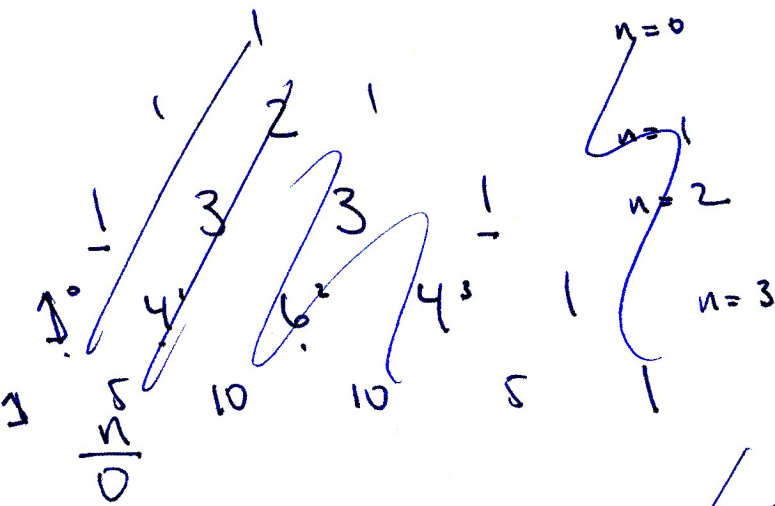
$$\left\{ \begin{matrix} n \\ 2 \end{matrix} \right\} = 2^{n-1} - 1$$

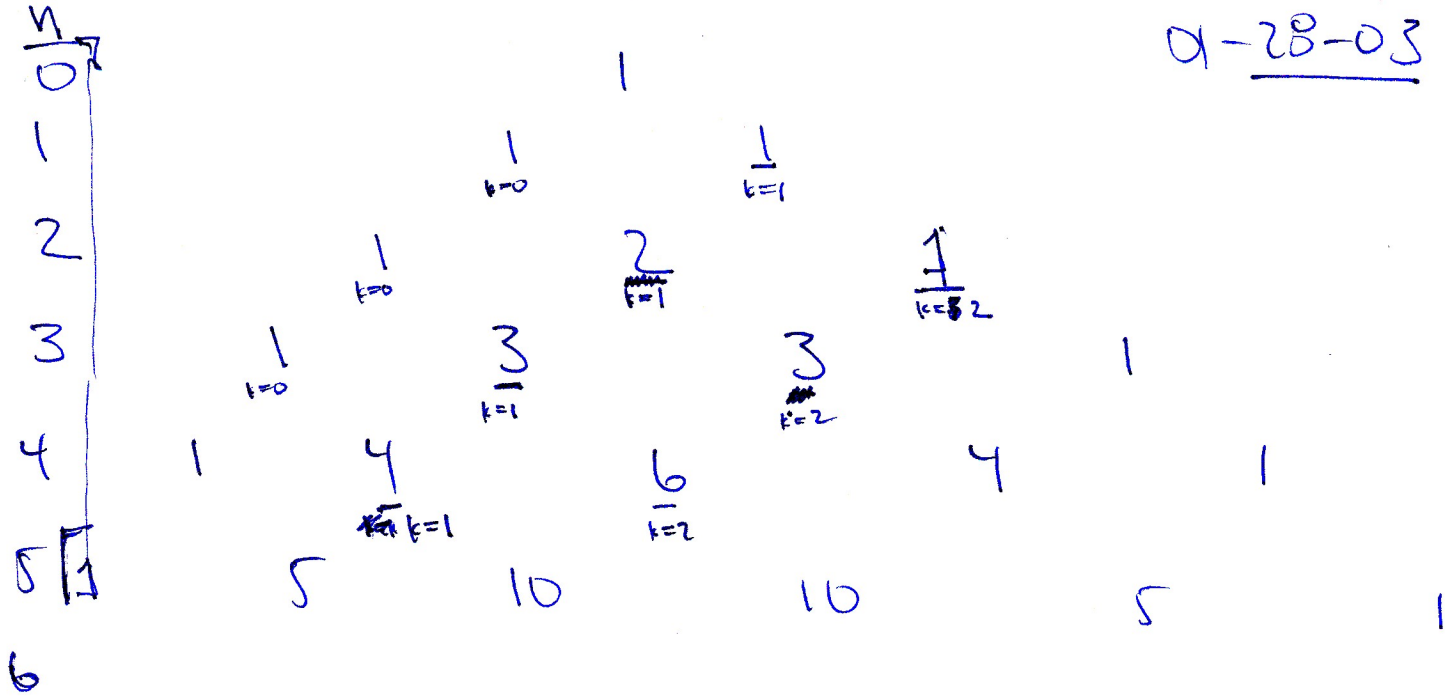
$\left\{ \begin{matrix} n \\ 2 \end{matrix} \right\} = \# \text{ of ways to split a set w/ } n \text{ elts into 2 sets.}$

$$= \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \dots + \binom{n}{n}$$

↑ ↑

selecting singleton sets + set w/ n-1 elts selecting sets w/ 2 elts + n-2 elts





- | | | | |
|----------------------------------|----------------------------------|-------------------------------|---------------------|
| $\{a, b\}$ | $\{a, b, c\}$ | $\{a, b, c, d\}$ | $\{a, b, c, d, e\}$ |
| $\{a, b\}$ | $\{a, b\}$ | $\binom{4}{1}$ singleton sets | $\binom{5}{1}$ |
| $\{a, b\}$ | $\{a\}, \{b, c\}$ | $\binom{4}{2}$ | $\binom{5}{2}$ |
| $\{a\}, \{b\}$ | $\{b\}, \{a, c\}$ | | $\binom{5}{3}$ |
| | $\{c\}, \{a, b\}$ | | |
- $\{4\}^? = 10$
 $= 2^3 - 1$
 $= 7$

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$$\begin{Bmatrix} n \\ k \end{Bmatrix} = \begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix} + k \begin{Bmatrix} n-1 \\ k \end{Bmatrix}$$

$$A_n(y) = \sum_k \begin{Bmatrix} n \\ k \end{Bmatrix} y^k$$

$$\begin{aligned} A_n(y) &= \sum_k \begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix} y^k + \sum_k k \begin{Bmatrix} n-1 \\ k \end{Bmatrix} y^k \\ &= y \sum_k \begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix} y^{k-1} + \cancel{\sum_k \begin{Bmatrix} n-1 \\ k \end{Bmatrix} \frac{D}{Dy} y^k} y \end{aligned}$$

$$= y A_{n-1}(y) + y \sum_k k \begin{Bmatrix} n-1 \\ k \end{Bmatrix} y^{k-1}$$

$$= y A_{n-1}(y) + y \sum_k \begin{Bmatrix} n-1 \\ k \end{Bmatrix} \frac{D}{Dy} y^k$$

$$= y A_{n-1}(y) + y \frac{d}{dy} A_{n-1}(y)$$

$$= \{y(1 + D_y)\} A_{n-1}(y)$$

$$A_0(y) = \sum_k \begin{Bmatrix} 0 \\ k \end{Bmatrix} y^k$$

- 1

$$A_1 = y(1 + \frac{1}{2y})(1)$$

$$A_1 = \cancel{y(1 + \frac{1}{2y})} =$$

$$A_1 = y(1)$$

$$A_2 = y(1 + \frac{1}{2y})(y) = y(y+1) = y+y^2$$

$$A_3 = \cancel{y(y+y^2+2y)} = \cancel{y+y^2+2y}$$

$$= y(1 + \frac{1}{2y})(y+y^2)$$

$$= y[y+y^2+1+2y] = y[3y+1+y^2]$$

$$= y+3y^2+y^3 \quad \text{etc...}$$

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$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \sum_{r=0}^k (-1)^{k-r} \frac{r^n}{r!(k-r)!}$$

check ~~$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \sum_{r=0}^k (-1)^{k-r} \frac{r^n}{r!(k-r)!}$~~

$$\left\{ \begin{matrix} n \\ 1 \end{matrix} \right\} = 1$$

$$\left\{ \begin{matrix} n \\ 1 \end{matrix} \right\} = \sum_{r=0}^1 (-1)^{1-r} \frac{r^n}{r!(1-r)!} = \frac{1^n}{1!0!} = 1 \quad \checkmark$$

$$\left\{ \begin{matrix} 13 \\ 19 \end{matrix} \right\} = \sum_{r=0}^{19} (-1)^{19-r} \frac{r^{13}}{r!(19-r)!}$$

$$= (-1)^{19} \sum_{r=0}^{19} (-1)^r \frac{r^{13}}{r!(19-r)!}$$

$$= (-1)^{19} \left[\cancel{r} \frac{-1}{18!} + \frac{2^{13}}{2(17!)} - \dots \right]$$

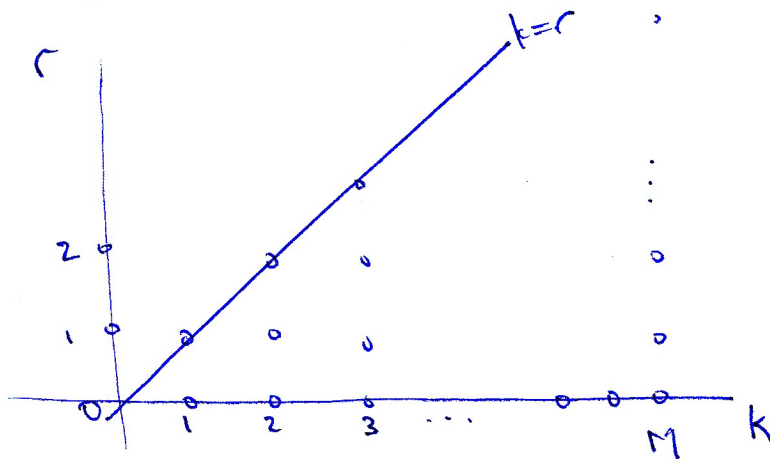
How show sums to zero?

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$$b(n) = \sum_{k=1}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \sum_{k=1}^M \sum_{r=1}^k \frac{(-1)^{k-r} r^n}{r! (k-r)!}$$

$$M \geq n$$

≠ Change order of 2D summation.



Change summation into sum over r 1st then k

For given r k must be: $r \leq k \leq M$

$$+ \quad 1 \leq r \leq M$$

$$\therefore b(n) = \sum_{r=1}^M \sum_{k=r}^M \frac{(-1)^{k-r} r^n}{r! (k-r)!}$$

$$= \sum_{r=1}^M \frac{r^n}{r!} \sum_{k=r}^M \frac{(-1)^{k-r}}{(k-r)!}$$

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$$\Rightarrow b(n) = \sum_{r=1}^M \frac{r^{n-1}}{(r-1)!} \sum_{k=r}^M \frac{(-1)^{k-r}}{(k-r)!}$$

$$\approx \sum_{r=1}^M \frac{r^{n-1}}{(r-1)!} \underbrace{\sum_{k=0}^{M-r} \frac{(-1)^k}{k!}}_{e^{-1} \quad M \rightarrow \infty}$$

~~Subtract r from each term~~
 Add r to each term in summand

$$b(n) = \frac{1}{e} \sum_{r \geq 0} \frac{r^n}{r!}$$

$$b(0) = \frac{1}{e} \sum_{r \geq 0} \frac{1}{r!} = 1 \quad \checkmark$$

$$b(1) = \frac{1}{e} \sum_{r \geq 0} \frac{r}{r!} = \frac{1}{e} \sum_{r \geq 1} \frac{1}{(r-1)!} = \frac{1}{e} \sum_{r \geq 0} \frac{1}{r!} = 1 \quad \checkmark$$

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(1)
(a)

$$p(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} n x^n = x \sum_{n=0}^{\infty} n x^{n-1}$$

$$= x \frac{d}{dx} \sum_{n=0}^{\infty} x^n = x \frac{d}{dx} \left(\frac{1}{1-x} \right) = \frac{x(-1)}{(1-x)^2} = \frac{-x}{(1-x)^2}$$

(b)

$$p(x) = \sum_{n=0}^{\infty} a_n x^n = \alpha \sum_{n=0}^{\infty} n x^n + \beta \sum_{n=0}^{\infty} x^n$$

$$= \frac{-\alpha x}{(1-x)^2} + \frac{\beta}{(1-x)}$$

$$(c) \quad p(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} n^2 x^n = \sum_{n=0}^{\infty} n(n-1+1) x^n$$

$$= \sum_{n=0}^{\infty} n(n-1) x^n + \sum_{n=0}^{\infty} n x^n = x^2 \sum_{n=0}^{\infty} \frac{d^2}{dx^2} x^n + \frac{-x}{(1-x)^2}$$

$$= x^2 \frac{d^2}{dx^2} \left(\frac{1}{1-x} \right) - \frac{x}{(1-x)^2}$$

$$= x^2 \frac{d}{dx} \left(\frac{-1}{(1-x)^2} \right) - \frac{x}{(1-x)^2}$$

$$(d) a_n = \dots$$

use results from pt (a) & (c).

$$(e) G\{n^P\} = x^P \frac{d^P}{dx^P} \left(\frac{1}{1-x} \right) - G\{n^{P-1}\}. \quad \exists \text{ a recurrence relationship like the class...}$$

$$n^P = n^{P-1} (n -$$

$$n^2 = \cancel{n(n-1)} n(n-1+1) = n(n-1) + n$$

$$n^3 = n(n-2+2)n = n(n-2+2)(n-1+1)$$

$$= n(n-1+1)(n-2+2)$$

$$= n(n-1)(n-2+2) + n(n-2+2)$$

$$(f) P(x) = \sum_{n \geq 0} 3^n x^n = \frac{1}{1-3x}$$

$$(g) P(x) = \frac{5}{1-7x} - \frac{3}{1-4x}$$

(2)

$$\begin{aligned}
 (a) \quad B(x) &= \sum_{n \geq 0} \frac{a_n}{n!} x^n = \sum_{n \geq 0} \frac{n}{n!} x^n = \sum_{n \geq 1} \frac{x^n}{(n-1)!} \\
 &\quad (n=0 \text{ term is } 0) \\
 &= \sum_{n \geq 0} \frac{x^{n+1}}{(n+1)!} = x e^x.
 \end{aligned}$$

$$(b) \quad \text{B} \quad \text{EG}(x) = \alpha x e^x + \beta e^x.$$

$$\begin{aligned}
 (c) \quad \text{EG}(x) &= \sum_{n \geq 0} \frac{n^2}{n!} x^n = \sum_{n \geq 1} \frac{n-1+1}{(n-1)!} x^n \\
 &= \sum_{n \geq 2} \frac{x^n}{(n-2)!} + \sum_{n \geq 1} \frac{x^n}{(n-1)!} \\
 &= \sum_{n \geq 0} \frac{x^{n+2}}{(n+2)!} + \sum_{n \geq 0} \frac{x^{n+1}}{(n+1)!} \\
 &= x^2 e^x + x e^x
 \end{aligned}$$

②

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$$(d) \quad G(x) = \sum_{n \geq 0} \frac{a_n}{n!} x^n$$

$$= \sum_{n \geq 0} \frac{(\alpha n^2 + \beta n + \gamma)}{n!} x^n = \alpha \sum_{n \geq 0} \frac{n^2 x^n}{n!} + \beta \sum_{n \geq 0} \frac{n x^n}{n!} + \gamma e^x$$

$$= \alpha G\{n^2\} + \beta G\{n\} + \gamma e^x$$

$$= \alpha \sum_{n \geq 1} \frac{n x^n}{(n-1)!} + \beta \sum_{n \geq 1} \frac{x^n}{(n-1)!} + \gamma e^x$$

$$= \alpha \sum_{n \geq 0} \frac{(n+1) x^{n+1}}{n!} + \beta \sum_{n \geq 0} \frac{x^{n+1}}{n!} + \gamma e^x$$

$$= \alpha \sum_{n \geq 0} \frac{n}{n!} x^{n+1} + \alpha \sum_{n \geq 0} \frac{x^{n+1}}{n!} + \beta x e^x + \gamma e^x$$

$$= \alpha \sum_{n \geq 1} \frac{1}{(n-1)!} x^{n+1} + \alpha x e^x + \beta x e^x + \gamma e^x$$

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$$= \alpha \sum_{n \geq 0} \frac{1}{n!} x^{n+2} + (\alpha x + \beta x + r) e^x$$

$$= \alpha x^2 e^x + \dots$$

(e) Do each power of x & All linearly

$$(f) \sum_{n \geq 0} \frac{3^n x^n}{n!} = e^{3x}$$

$$(g) 5e^{7x} - 3e^{4x}$$

③

$$(a) \sum_{n \geq 0} (a_n + c) x^n = \sum_{n \geq 0} a_n x^n + c \sum_{n \geq 0} x^n$$

$$= f(x) + \frac{c}{1-x}$$

$$(b) x f(x) + \frac{c}{1-x}$$

$$(c) \sum_{n \geq 0} n a_n x^n = x \sum_{n \geq 0} n a_n x^{n-1} = x \sum_{n \geq 0} \frac{d}{dx} (a_n x^n)$$

$$= x \frac{d}{dx} f(x) = x f'(x)$$

$$(d) \sum_{n \geq 0} p(n) a_n x^n \quad \text{convert each power of } n \text{ into derivatives}$$

$$(e) \sum_{n \geq 0} b_n x^n = \cancel{\sum_{n \geq 0} a_n x^n} = 0 + a_1 x + a_2 x^2 + \dots = f(x) - a_0$$

(f) ~~0-1~~

$$f(x) = 0x^0 + 0x^1 + x^2 + \underbrace{a_3x^3 + a_4x^4 + \dots}$$

$$f(x) = a_0 + a_1x + a_2x^2$$

$$g(x) = \frac{2}{x - a_0}$$

$$-a_0 - a_1x - (a_2 - 1)x^2 + f(x)$$

$$(g) f(x) = a_0 + 0x + a_2x^2 + 0x^3 + a_4x^4 + \dots$$

$$= f(x) - a_1x - a_3x^3 - a_5x^5 - \dots$$

$$= f(x) - x [a_1 + a_3x^2 + a_5x^4 + \dots] \quad \text{How sum?}$$

(h)

$$g(x) = a_1 + a_2x + a_3x^2 + \dots$$

$$= \frac{1}{x} (a_1x + a_2x^2 + a_3x^3 + \dots) = \frac{1}{x} (f - a_0)$$

$$(c) g(x) = a_h + a_{h+1}x + a_{h+2}x^2 + \dots$$

$$\frac{1}{x^h} (x^h a_h + a_{h+1}x^{h+1} + a_{h+2}x^{h+2} + \dots)$$

$$= \frac{1}{x^h} (f(x) - \sum_{k=0}^{h-1} a_k x^k)$$

$$(j) \frac{1}{x^2} (f(x) - a_0 - a_1 x)$$

$$+ \frac{3}{x} (f(x) - a_0)$$

$$+ f(x) = \left(\frac{1}{x^2} + \frac{3}{x} + 1 \right) f(x)$$

$$= \left(\frac{1}{x^2} + \frac{3}{x} \right) a_0 - \frac{1}{x^2} a_1$$

$$(k) \dots$$

a) ④

$$g(x) = \sum_{n \geq 0} \frac{(a_n + c)}{n!} x^n = f(x) + ce^x$$

b) $af(x) + ce^x$

c) $g = \sum_{n \geq 0} \frac{nx^n}{n!} = \sum_{n \geq 1} \frac{x^n}{(n-1)!} = x \sum_{n \geq 0} \frac{x^n}{n!} = xe^x$

d) Do each power of n .

e) $g = 0 + \frac{a_1 x}{1!} + \frac{a_2 x^2}{2!} + \dots = f(x) - a_0$

f) $g = f - a_0 - \frac{a_1 x}{1!} - \frac{a_2 x^2}{2!} + \frac{x^2}{2}$

g) $g = a_0 + \frac{0x}{1} + \frac{a_2 x^2}{2!} + \frac{0x^3}{3!} + \frac{a_4 x^4}{4!}$

? how to?

$$f(x) = a_1 + \frac{a_2 x}{1!} + \frac{a_3 x^2}{2!} + \frac{a_4 x^3}{3!}$$

~~$$\int_0^x g(\xi) d\xi = a_1 x + \frac{a_2 x^2}{2!} + \frac{a_3 x^3}{3!} + \dots$$~~

$$f(x) - a_0$$

$$\therefore \int_0^x g(\xi) d\xi = f(x) - a_0$$

$$\Rightarrow g(x) = \frac{1}{x} f(x)$$

$$(i) \quad a_n + a_{n+1}x + a_{n+2}x^2 + a_{n+3}x^3 + \dots = g(x)$$

integrate $g(x)$ n times to get $f(x)$

$$\underbrace{\int \int \int \dots \int_0^x}_{n \text{ integrals}} g(\xi) d\xi = f(x)$$

$$\Rightarrow g(x) = \frac{1}{x^n} f(x)$$

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(j) $\frac{1^2}{x^2} + 3\frac{1}{x} + 7$

(k) $\frac{1^2}{x^2} = -\frac{1}{x} - 7$

⑧

$$(a) [x^n] e^{2x} = [x^n] \sum_{n \geq 0} \frac{(2x)^n}{n!} = [x^n] \sum_{n \geq 0} \frac{2^n}{n!} x^n = \frac{2^n}{n!}$$

$$(b) \left[\frac{x^n}{n!} \right] \sum_{n \geq 0} \frac{x^n}{n!} = x^n$$

$$(c) \left[\frac{x^n}{n!} \right] \sum_{n \geq 0} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \left[\frac{x^n}{n!} \right] \left(x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \frac{1}{7!} x^7 + \dots \right)$$

$$\Rightarrow \left[\frac{x^n}{n!} \right] \sin x = 0 \quad \left\{ \begin{array}{l} \text{sign: } +1, -1, +1, -1, \dots \\ n: 1, 3, 5, 7, \dots \\ p: n=2p+1 \quad 0, 1, 2, 3, \dots \end{array} \right\}$$

$$\left[\frac{x^n}{n!} \right] \sin x = 0 \quad n \text{ even}$$

$$\left[\frac{x^n}{n!} \right] \sin x = (-1)^{\frac{n-1}{2}} \quad n \text{ odd}$$

$$= (-1)^{\frac{n-1}{2}} \quad n \text{ odd: } \{1, 3, 5, 7, \dots\}$$

$$(d) [x^n] \left\{ \frac{1}{(1-ax)(1-bx)} \right\}$$

$$= [x^n] \left\{ \frac{A}{1-ax} + \frac{B}{1-bx} \right\}$$

$$= [x^n] \left\{ A \sum_{n \geq 0} a^n x^n + B \sum_{n \geq 0} b^n x^n \right\} = A a^n + B b^n$$

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$$\frac{1}{(1-ax)(1-bx)} = \frac{A}{1-ax} + \frac{B}{1-bx}$$

$$A = \frac{1}{1-bx} - \frac{B(1-ax)}{1-bx}$$

$$\Big|_{x=1/a}$$

$$A = \frac{1}{1-\frac{1}{a}} = \frac{a}{a-b}$$

$$B = \frac{1}{1-ax} - \frac{A(1-bx)}{1-ax} = \frac{1}{1-\frac{a}{b}} = \frac{b}{b-a}$$

$$\Big|_{x=1/b}$$

(e) $[x^n](1+x^2)^m$

$$[x^n] \sum_{k=0}^m \binom{m}{k} x^{2k} = [x^n] \sum_{k=0}^m \binom{m}{k} 1^k (x^2)^{m-k}$$

$$= \sum_{n=0}^{\infty} \binom{m}{n/2} x^n$$

$$[x^n] \left(\binom{m}{0} x^0 + \binom{m}{1} x^2 + \binom{m}{2} x^4 + \dots + \binom{m}{m} x^{2m} \right)$$

$$= 0$$

$$n \geq 2m+1 \text{ or } n \text{ odd}$$

$$= \binom{m}{n/2}$$

$$n \leq 2m \text{ \& } n \text{ even}$$

⑥ (a) $a_{n+1} = 3a_n + 2 \quad n \geq 0 \quad a_0 = 0$

mult by x^n & sum $\sum_{n \geq 0}$

$$\sum_{n \geq 0} a_{n+1} x^n = 3 \sum_{n \geq 0} a_n x^n + 2 \sum_{n \geq 0} x^n$$

$$\sum_{n \geq 1} a_n x^{n-1} = 3f(x) + \frac{2}{1-x}$$

~~$$x \sum_{n \geq 0} a_n x^n =$$~~

$$x^{-1} \sum_{n \geq 1} a_n x^n = 3f + \frac{2}{1-x}$$

$$\Rightarrow x^{-1} \left[\sum_{n \geq 0} a_n x^n - a_0 \right] = 3f + \frac{2}{1-x}$$

$$\Rightarrow x^{-1} (f - \cancel{a_0}^0) = 3f + \frac{2}{1-x}$$

$$\frac{f}{x} - 3f = \frac{2}{1-x}$$

$$f \left(\frac{1}{x} - 3 \right) = \frac{2}{1-x}$$

$$f(x) = \frac{2}{1-x} \cdot \frac{1}{\frac{1}{x} - 3}$$

$$= \frac{2}{1-x} \left(\frac{x}{1-3x} \right) = \frac{2x}{(1-x)(1-3x)}$$

$$= 2x \left[\frac{A}{1-x} + \frac{B}{1-3x} \right]$$

$$\text{w/ } \frac{1}{(1-x)(1-3x)} = \frac{A}{1-x} + \frac{B}{1-3x}$$

$$(1) \quad \frac{1}{1-x} = \frac{A(1-3x)}{1-x} + B \quad \Big|_{x=1/3} \Rightarrow B = \frac{1}{1-1/3} = \frac{3}{2}$$

$$(2) \quad \frac{1}{1-3x} = A + \frac{B(1-x)}{1-3x} \quad \Big|_{x=1} \Rightarrow A = \frac{1}{1-3} = -\frac{1}{2}$$

$$\therefore f(x) = 2x \left[-\frac{1}{2} \frac{1}{1-x} + \frac{3}{2} \frac{1}{1-3x} \right]$$

$$\text{Check: } \frac{1}{(1-x)(1-3x)} = x \left[\frac{-1(1-3x) + 3(1-x)}{(1-x)(1-3x)} \right] = x \left[\frac{-1 + \cancel{3x} + 3 - \cancel{3x}}{(1-x)(1-3x)} \right] \checkmark$$

So

$$f(x) = 2x \left[-\frac{1}{2} \sum_{n \geq 0} x^n + \frac{3}{2} \sum_{n \geq 0} 3^n x^n \right]$$

$$= -x \sum_{n \geq 0} x^n + 3x \sum_{n \geq 0} 3^n x^n$$

$$= - \sum_{n \geq 0} x^{n+1} + 3 \sum_{n \geq 0} 3^n x^{n+1}$$

$$= \sum_{n \geq 0} (-1 + 3^{n+1}) x^{n+1}$$

$$= \sum_{n \geq 1} (-1 + 3^n) x^n$$

$$\therefore a_0 = 0$$

$$a_n = -1 + 3^n \quad n \geq 1$$

Check

$$o_{n+1} = -1 + 3^{n+1}$$

$$= 3^n 3(3^n) - 1$$

$$= 3(o_n + 1) - 1 = 3o_n + 2 \quad \checkmark$$

$$(b) \quad c_{n+1} = \alpha c_n + \beta \quad n \geq 0 \quad a_0 = 0$$

$$\sum_{n \geq 0} c_{n+1} x^n = \alpha \sum_{n \geq 0} c_n x^n + \frac{\beta}{1-x}$$

$$\sum_{n \geq 1} c_n x^{n-1} = \alpha \frac{1}{1-x} + \frac{\beta}{1-x}$$

$$x^{-1} \left[\sum_{n \geq 0} c_n x^n - a_0 \right] = \alpha \frac{1}{1-x} + \frac{\beta}{1-x}$$

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$$(c) \quad a_{n+2} = 2a_{n+1} - a_n \quad n \geq 0 \quad a_0 = 0, a_1 = 1$$

$$\sum_{n \geq 0} a_{n+2} x^n = 2 \sum_{n \geq 0} a_{n+1} x^n - \sum_{n \geq 0} a_n x^n$$

$$\sum_{n \geq 2} a_n x^{n-2} = 2 \sum_{n \geq 1} a_n x^{n-1} - f(x)$$

$$x^{-2} \left[\sum_{n \geq 0} a_n x^n - a_1 x - a_0 \right] = \frac{2}{x} \left[\sum_{n \geq 0} a_n x^n - a_0 \right] - f$$

$$x^{-2} [f - x] = \frac{2}{x} [f] - f$$

$$\frac{1}{x^2} f - \frac{2}{x} f + f = \frac{1}{x}$$

$$f = \frac{\frac{1}{x}}{\left(\frac{1}{x^2} - \frac{2}{x} + 1\right)} = \frac{\frac{1}{x}}{\left(\frac{1}{x} - 1\right)^2} \cdot \frac{x^2}{x^2}$$

$$= \frac{x}{(x-1)^2} = x \frac{1}{\frac{d}{dx} \frac{(-1)}{(x-1)}} = -x \frac{1}{\frac{d}{dx} (x-1)}$$

$$= x \left[\frac{A}{x-1} + \frac{Bx+C}{(x-1)^2} \right] \quad \times \text{ Not correct. Don't need to do}$$

$$7 = x \left(\sum_{n \geq 0} x^n \right)^2 = -x \frac{1}{1-x} \sum_{n \geq 0} (-1)^n x^n$$

$$= x \frac{1}{1-x} \sum_{n \geq 0} x^n = x \sum_{n \geq 0} n x^{n-1} = \sum_{n \geq 0} n x^n$$

$$\therefore a_n = n \quad n \geq 0$$

Check

$$\begin{aligned} a_{n+2} &= n+2 = \boxed{n+1} + 1 = 2n+2 - n \\ &= 2a_{n+1} - a_n \quad \checkmark. \end{aligned}$$

$$(d) \quad a_{n+1} = \frac{a_n}{3} + 1$$

$$\sum_{n \geq 0} a_{n+1} x^n = \frac{1}{3} \sum_{n \geq 0} a_n x^n + \sum_{n \geq 0} x^n$$

$$= \frac{1}{3} 7 + \frac{1}{1-x}$$

$$\sum_{n \geq 1} a_n x^{n-1} = \quad \quad \quad "$$

$$x^{-1} \left[\sum_{n \geq 0} a_n x^n - a_0 \right] = \frac{1}{3} 7 + \frac{1}{1-x}$$

$$\frac{7}{x} - \frac{7}{3} = \frac{1}{1-x}$$

$$f = \frac{1}{1-x} \cdot \frac{1}{\left(\frac{1}{x} - \frac{1}{3}\right)} \cdot \frac{3x}{3x}$$

$$= \frac{3x}{(1-x)(3-x)}$$

$$= 3x \left[\sum_{n=0}^{\infty} x^n \right] + \frac{x}{(1-x)(1-x/3)}$$

$$= \cancel{3x} \left[\cancel{\sum_{n=0}^{\infty} x^n} \right] + \frac{x}{(1-x)(1-x/3)}$$

$$= x \left[\frac{A}{1-x} + \frac{B}{1-x/3} \right]$$

$$= x \left[A \sum_{n=0}^{\infty} x^n + B \sum_{n=0}^{\infty} \frac{x^n}{3^n} \right]$$

$$= A \sum_{n=0}^{\infty} x^{n+1} + B \sum_{n=0}^{\infty} \frac{x^{n+1}}{3^n}$$

$$= \sum_{n=0}^{\infty} \left(A + \frac{B}{3^n} \right) x^{n+1}$$

$$= \sum_{n=1}^{\infty} \left(A + \frac{B}{3^{n-1}} \right) x^n$$

w/ A & B

$$\frac{1}{(1-x)(1-x/3)} = \frac{A}{1-x} + \frac{B}{1-x/3}$$

$$\frac{1}{1-x} = \frac{A(1-x/3)}{1-x} + \cancel{B} \quad \Big| \quad \Rightarrow B = -\frac{1}{2}$$

$x=3$

$$\frac{1}{1-x/3} = \cancel{A} + \frac{B(1-x)}{1-x/3} \quad \Big| \quad \Rightarrow A = \frac{1}{1-1/3} = \frac{1}{2/3} = \frac{3}{2}$$

$x=1$

Then $a_n = \frac{3}{2} + -\frac{1}{2}(3)^{-n+1} \quad n \geq 1 \quad \text{or } a_n = \frac{3}{2} - \frac{1}{2}3^{n-1}$

$$a_0 = \frac{3}{2} - \frac{1}{2}3 = 0 \quad \checkmark$$

o check

$$\begin{aligned} a_{n+1} &= \frac{3}{2} - \frac{1}{2} \frac{1}{3^n} = \frac{3}{2} - \frac{1}{2 \cdot 3} \frac{1}{3^{n-1}} \\ &= \cancel{\frac{3}{2}} \frac{1}{2 \cdot 3} \left[\right] = \frac{3}{2} - \frac{1}{3} \left[\frac{1}{2 \cdot 3^{n-1}} \right] \\ &= \frac{3}{2} - \frac{1}{3} \left[-a_n + \frac{3}{2} \right] \\ &= \frac{a_n}{3} + \frac{3}{2} - \frac{1}{2} = \frac{a_n}{3} + 1 \quad \checkmark \end{aligned}$$

⑦ 1.6.3 is

$$\{n\}_k = \{n-1\}_{k-1} + k \{n-1\}_k$$

$$(n, k) \neq (0, 0)$$

$$\{0\}_0 = 1$$

From the $b(n)$ of all possible

$$b(n) = \sum_{k=1}^n \{n\}_k$$

$k=0$

$k=1$

$k=2$

...

$k=n-1$

All partitions of $[n]$ in which the class that contains "n" contains 0 other letters

All partitions of $[n]$ in which the class that contains "n" contains 1 other letter

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(a) $a_{n+1} = 3a_n + 2 \quad n \geq 0 \quad a_0 = 0$

$$\sum_{n \geq 0} a_{n+1} \frac{x^n}{n!} = 3 \sum_{n \geq 0} \frac{a_n x^n}{n!} + 2 \sum_{n \geq 0} \frac{x^n}{n!}$$

$$\sum_{n \geq 1} \frac{a_n x^{n-1}}{(n-1)!} = 3f(x) + 2e^x$$

~~$$\sum_{n \geq 1} \frac{a_n x^{n-1}}{(n-1)!} = 3f(x) + 2e^x$$~~

~~$$\sum_{n \geq 1} \frac{n a_n x^{n-1}}{n!} = 3f(x) + 2e^x$$~~

$$\Rightarrow \sum_{n \geq 1} \frac{a_n}{n!} \frac{d}{dx} (x^n) = 3f(x) + 2e^x$$

$$\Rightarrow \frac{d}{dx} \sum_{n \geq 1} \frac{a_n x^n}{n!} = 3f + 2e^x$$

~~$$\frac{d}{dx} \left[\sum_{n \geq 0} \frac{a_n x^n}{n!} - a_0 \right] = 3f + 2e^x$$~~

$$f' - 3f = 2e^x$$

To solve $f' - 3f = 2e^x$

Mult by e^{-3x}

$$\underbrace{e^{-3x} f' - 3e^{-3x} f}_{\frac{d}{dx}(e^{-3x} f)} = 2e^{-2x}$$

$$\frac{d}{dx}(e^{-3x} f) = 2e^{-2x}$$

$$e^{-3x} f = -e^{-2x} + C$$

$$f = -e^x + Ce^{3x}$$

$$f(0) = -1 + C = a_0 = 0 \Rightarrow C = +1$$

$$\therefore f(x) = e^{3x} - e^x$$

$$= \sum_{n \geq 0} \frac{3^n x^n}{n!} - \sum_{n \geq 0} \frac{x^n}{n!} = \sum_{n \geq 0} \frac{x^n}{n!} (3^n - 1)$$

$$\therefore a_n = 3^n - 1$$

$$a_{n+1} = 3(3^n) - 1 = 3(a_n + 1) - 1$$

$$= 3a_n + 2 \quad \checkmark$$

$$(b) a_{n+1} = \alpha a_n + \beta$$

same as (a) but with different initial.

$$(c) a_{n+2} = 2a_{n+1} - a_n \quad n \geq 0 \quad a_0 = 0; a_1 = 1$$

$$\Rightarrow \sum_{n \geq 0} a_{n+2} \frac{x^n}{n!} = 2 \sum_{n \geq 0} a_{n+1} \frac{x^n}{n!} - \underbrace{\sum_{n \geq 0} a_n \frac{x^n}{n!}}_{f(x)}$$

$$\Rightarrow \sum_{n \geq 2} a_n \frac{x^{n-2}}{(n-2)!} = 2 \sum_{n \geq 1} a_n \frac{x^{n-1}}{(n-1)!} - f(x)$$

$$\Rightarrow \sum_{n \geq 2} a_n \frac{(n-1)x^{n-2}}{(n-1)!} = 2 \sum_{n \geq 1} a_n \frac{n x^{n-1}}{n!} - f(x)$$

$$\sum_{n \geq 2} a_n \frac{1}{dx} \left(\frac{x^{n-1}}{(n-1)!} \right) = 2 \sum_{n \geq 1} a_n \frac{1}{dx} \left(\frac{x^n}{n!} \right) - f(x)$$

$$\Rightarrow \sum_{n \geq 2} a_n \frac{1}{dx} \left(\frac{n x^{n-1}}{n!} \right)$$

$$\sum_{n \geq 2} a_n \frac{1}{dx} \cdot \frac{1}{dx} \left(\frac{x^n}{n!} \right) = 2 \frac{1}{dx} \left(\sum_{n \geq 1} a_n \frac{x^n}{n!} \right) - f(x)$$

$$\int_{-x}^x (f(x) - a_0 x^0 - \frac{a_1 x^1}{1!}) = 2 \int_{-x}^x (f(x) - f_0) - f$$

$$\Rightarrow \int_{-x}^x (f(x) - x) = 2f'(x) - f$$

$$f''(x) - 2f'(x) + f = 0$$

$$\text{let } f = x^1$$

$$f' = 1x^{1-1} \quad \times \text{ Not correct ans.}$$

$$f$$

$$\text{let } f = e^{\lambda x}$$

$$f' = \lambda e^{\lambda x}$$

$$f'' = \lambda^2 e^{\lambda x}$$

$$\lambda^2 - 2\lambda + 1 = 0$$

$$(\lambda - 1)^2 = 0 \quad \lambda = 1 \quad \text{one solution } f = e^x$$

$$\text{let } f = v(x)e^x$$

$$f' = v'e^x + ve^x$$

$$f'' = v''e^x + 2v'e^x + ve^x \quad \text{put in D.E.}$$

$$v''e^x + 2v'e^x + ve^x - 2v'e^x - 2v'e^x + ve^x = 0$$

$$v'' = 0 \quad v = Ax + B. \quad \Rightarrow \text{2nd solution } f = xe^x$$

So

~~Let~~ $f(x) = Ae^x + Bxe^x$

$$a_n = A \sum_{n \geq 0} \frac{x^n}{n!} + B \sum_{n \geq 0} \frac{x^{n+1}}{n!}$$

$$= A \sum_{n \geq 0} \frac{x^n}{n!} + B \sum_{n \geq 1} \frac{x^n}{(n-1)!}$$

$$= A + \sum_{n \geq 1} (A + Bn) \frac{x^n}{n!}$$

Since $A = a_0$, we get $A = 0$.

∴ $B = a_1$ " " $B = 1$

∴ ~~f(x)~~ =

$$\therefore f(x) = \sum_{n \geq 1} n \frac{x^n}{n!} \quad \text{etc} \therefore a_n = n$$

Check $a_{n+2} = n+2 = 2n - n + 2$
 $= \cancel{2n} \quad 2(n+1) - n$

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$$f(2n) = f(n)$$

$$f(2n+1) = f(n) + f(n+1)$$

$$F(x) = \sum_{n \geq 1} f(n) x^{n-1}$$

$$F(x^2) = \sum_{n \geq 1} f(n) x^{2n-2}$$

$$x^2 \overline{F(x^2)} = \sum_{n \geq 1} \overline{f(n)} x^{2n} = \sum_{n \geq 1} \overline{f(2n)} x^{2n}$$

to

$$(1+x+x^2) F(x^2) = \sum_{n \geq 1} f(n) (1+x+x^2) x^{2n-2}$$

$$= \sum_{n \geq 1} f(n) x^{2n-2} + \sum_{n \geq 1} f(n) x^{2n-1} + \sum_{n \geq 1} f(n) x^{2n}$$

$$\sum_{n \geq 1} f(2n) x^{2n-2}$$

$$F(x) = \sum_{n \geq 1} f(n) x^{n-1}$$

$$= \sum_{n=1,3,5,7,\dots} f(n) x^{n-1} + \sum_{n=2,4,6,8,\dots} f(n) x^{n-1}$$

$$= \sum_{n \geq 0} f(2n+1) x^{2n+1-1} + \sum_{n \geq 1} f(2n) x^{2n-1}$$

$$= \sum_{n \geq 0} (f(n) + f(n+1)) x^{2n} + \sum_{n \geq 1} f(n) x^{2n-1}$$

$n \neq 0$ must absorb term ...

$$= f(0) + f(1) + \sum_{n \geq 1} (f(n) + f(n+1)) x^{2n} + \sum_{n \geq 1} f(n) x^{2n-2+1}$$

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$$F(x) = f(0) + f(1) + x f(x^2) + x^2 f(x^2)$$

$$+ \sum_{n \geq 1} f(n+1) x^{2n}$$

$$= f(0) + f(1) + (x + x^2) f(x^2) + \sum_{n \geq 2} f(n) x^{2n-2}$$

$$= f(0) + \sum_{n \geq 1} f(n) x^{2n-2} + (x + x^2) f(x^2)$$

$$= f(0) + (1 + x + x^2) f(x^2)$$

$$F(x) = (1 + x + x^2) f(x^2)$$

$$= (1 + x + x^2) (1 + x^2 + x^4) f(x^4)$$

$$= (\quad) (\quad) (1 + x^4 + x^8) f(x^8)$$

$$= \prod_{j \geq 0} (1 + x^{2^j} + x^{2^{j+1}})$$

(10)

(c)

$$\mu = \sum_{n \geq 0} n p_n$$

$$P(x) = \sum_{n \geq 0} p_n x^n$$

 $\hat{=}$

$$P'(x) = \sum_{n \geq 0} n p_n x^{n-1}$$

$$\therefore P'(1) = \mu$$

$$\sigma^2 = \langle (X - \mu)^2 \rangle = \langle X^2 - 2\mu X + \mu^2 \rangle$$

$$= \langle X^2 \rangle - 2\mu^2 + \mu^2 = \langle X^2 \rangle - \mu^2$$

$$\sigma = \sqrt{\langle X^2 \rangle - \mu^2} =$$

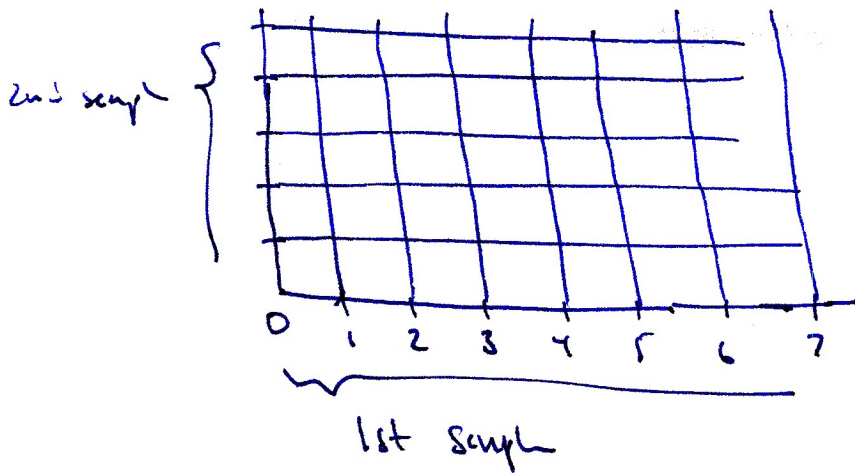
$$\langle X^2 \rangle = \sum_{n \geq 0} n^2 p_n = \frac{1}{x} \left(x \sum_{n \geq 0} n p_n x^{n-1} \right) \Big|_{x=1} = \frac{1}{x} (x P'(x)) \Big|_{x=1}$$

$$= \cancel{P'(1)} \mu P'(1) + x P''(1) \Big|_{x=1}$$

$$= P'(1) + P''(1)$$

$$\therefore \sigma^2 = P'(1) + P''(1) - P'(1)^2 = \mu + P''(1) - \mu^2$$

(b) $P_n^{(2)}$ = can get by getting pairs $0, n$
 $1, n-1$
 $2, n-2$
 $\vdots$



$$P_n^{(2)} = P_0 P_n + P_1 P_{n-1} + P_2 P_{n-2} + \cdots + P_{n-1} P_1 + P_n P_0$$

$$= \sum_{k=0}^n P_k P_{n-k}$$

$$P_2(x) = \sum_{n \geq 0} \left(\sum_{k=0}^n P_k P_{n-k} \right) x^n$$

$$= P(x)^2$$

(c) $P_n^{(L)}$ = ~~$\sum_{k=0}^n P_k P_{n-k}$~~
 ~~$\sum_{k=0}^n P_k P_{n-k}$~~
 ~~$\sum_{k=0}^n P_k P_{n-k}$~~

$$\sum_{\text{All } \vec{v} \rightarrow} P_{v_1} P_{v_2} \cdots P_{v_k}$$

$$v_1 + v_2 + \cdots + v_k = n$$

$$\begin{aligned} \therefore P_k(x) &= \sum_{n \geq 0} P_n^{(k)} x^n \\ &= P(x)^k \end{aligned}$$

(d) ~~we~~ =

$$\mu^{(k)} = \frac{1}{\int x} P_k(x) \Big|_{x=1} = \frac{1}{\int x} k P(x)^{k-1} P'(x) \Big|_{x=1}$$

$$= k P(1)^{k-1} P'(1) = k\mu$$

$$B^{(k)} = \sqrt{P'(1) + P''(1) - P'(1)^2}$$

$$= \sqrt{k P(x)^{k-1} + k(k-1) P(x)^{k-2} - k^2 P(x)^{k-2}}$$

$$= \sqrt{k P(x)^{k-2} (P(x) + (k-1) P(x) - k P(x))} \Big|_{x=1}$$

$$= \sqrt{k\mu^2}$$

$$P' = k P(x)^{k-1} \cdot P'(x) \rightarrow P'(1) = k\mu$$

$$P'' = k(k-1) P(x)^{k-2} (P'(x))^2 + k P(x)^{k-1} P''(x)$$

$$\Rightarrow P''(1) = k(k-1)\mu^2 + k P''(1)$$

Then

$$B^{(k)} = \sqrt{k u + k(k-1)u^2 + k p''(1) - k^2 u^2}$$

$$= \sqrt{k u - k u^2 + k(B^2 - p'(1) + u^2)}$$

$$= \sqrt{\cancel{k u} - \cancel{k u^2} + k B^2 - \cancel{k u} + \cancel{k u^2}}$$

$$= \sqrt{k} B$$

$$(c) \quad A(x) = A(0) + \sum_{n \geq 1} A(n) x^n = 1 + \sum_{n \geq 1} A(n) x^n$$

$$B(x) = A(x)^k$$

$$= \sum_{n \geq 0} 1 + \sum_{n \geq 1} b_n x^n$$

log $\Rightarrow$

$$\log B(x) = k \log A(x)$$

$\frac{1}{x} \Rightarrow$

$$\frac{1}{B(x)} B'(x) = \frac{k}{A(x)} A'(x)$$

$$A(x) B'(x) = k B(x) A'(x) \quad \text{mult by } x$$

$$A(x) \times B'(x) = k B(x) \times A'(x)$$

~~*~~

$$\left(1 + \sum_{n \geq 1} a_n x^n\right) \left(x \sum_{n \geq 1} n b_n x^{n-1}\right) = k \left(1 + \sum_{n \geq 1} b_n x^n\right) \left(\sum_{n \geq 1} n a_n x^n\right)$$

$$\Rightarrow \left(1 + \sum_{n \geq 1} a_n x^n\right) \left(\sum_{n \geq 1} n b_n x^n\right) = k \left(1 + \sum_{n \geq 1} b_n x^n\right) \left(\sum_{n \geq 1} n a_n x^n\right)$$

$$\Rightarrow \sum_{n \geq 1} n b_n x^n + \sum_{n_1 \geq 1} \sum_{n_2 \geq 1} a_{n_1} x^{n_1} n_2 b_{n_2} x^{n_2}$$

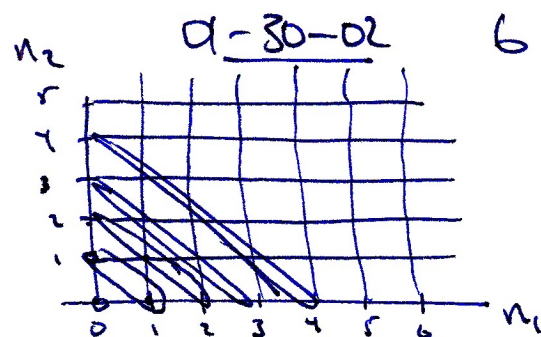
$$= k \left(\sum_{n \geq 1} n a_n x^n + \sum_{n_1 \geq 1} \sum_{n_2 \geq 1} b_{n_1} x^{n_1} n_2 a_{n_2} x^{n_2} \right)$$

$$\Rightarrow \sum_{n \geq 1} n b_n x^n + \sum_{n_1 \geq 1} \sum_{n_2 \geq 1} n_2 a_{n_1} b_{n_2} x^{n_1+n_2}$$

$$\left\{ \begin{array}{l} n_1 + n_2 = n \\ n_2 = n - n_1 \end{array} \right\}$$

$$= k \left(\sum_{n \geq 1} n a_n x^n + \sum_{n_1 \geq 1} \sum_{n_2 \geq 1} n_2 a_{n_2} b_{n_1} x^{n_1+n_2} \right)$$

$$\Rightarrow \sum_{n \geq 1} n b_n x^n + \sum_{n \geq 1} \sum_{n_1=1}^n (n-n_1) a_{n_1} b_{n-n_1} x^n$$



$$= k \left(\sum_{n \geq 1} n a_n x^n + \sum_{n \geq 1} \sum_{n_1=1}^n (n-n_1) a_{n-n_1} b_{n_1} x^n \right)$$

$$\Rightarrow \sum_{n \geq 1} \left(\cancel{n b_n} + \sum_{k=1}^n (n-k) a_k b_{n-k} - k n a_n - \sum_{k=1}^n (n-k) a_{n-k} b_k \right) x^n = 0$$

$$\Rightarrow n b_n - k n a_n + \sum_{k=1}^n (n-k) (a_k b_{n-k} - a_{n-k} b_k) = 0$$

$\{ k=n \text{ term vanishes} \}$

$$\Rightarrow n b_n = k n a_n + \sum_{k=1}^{n-1} (n-k) (a_k b_{n-k} - a_{n-k} b_k)$$

$$\therefore b_n = k a_n + \sum_{k=1}^{n-1} \frac{(n-k)}{n} (a_k b_{n-k} - a_{n-k} b_k)$$

$$\begin{aligned}
 (f) \quad P_1 &= .1 \\
 P_2 &= .2 \\
 P_3 &=
 \end{aligned}$$

p^* = total # of spots on all 100 throws is ≤ 300

Don't see what for this is?

$$(g) \quad P_k = \frac{1}{m} \quad 1 \leq k \leq m$$

Prob $\{S_n \leq j\}$

Don't see how?

⑪ $f(n) = \# \text{ subsets of } [n] \text{ that contain no 2 consecutive elements}$
for integer n .

For example

$$f(1) = 2 ; f(2) = 3 ; f(3) = 5 ; f(4) = 8$$

$$\begin{array}{cccc} \emptyset, \{1\} & \emptyset, \{1\}, \{2\} & \emptyset, \{1\}, \{2\}, \{3\} & \emptyset, \{1\}, \{2\}, \{3\}, \{4\} \\ & & \{1,3\} & \{1,3\}, \{1,4\}, \{2,4\}, \end{array}$$

from the sets ~~of~~ of length n split them into 2 piles

Left pile:

~~sets~~ subsets of $[n]$
that do not contain n

claim there are $f(n-1)$
of them.

Right pile:

subsets of $[n]$ that
do contain n .

claim there are ? many of them

can put set " n " into

$$\begin{array}{ccc} 1 + & n-2 & + \\ \uparrow & \uparrow & \\ \text{singleton} & \text{pair} & \text{triplets} \\ & \{1, n\} & \\ & \{2, n\} & \\ & \{3, n\} & \\ & \vdots & \\ & \{n-2, n\} & \end{array}$$

$$f(5) = 12$$

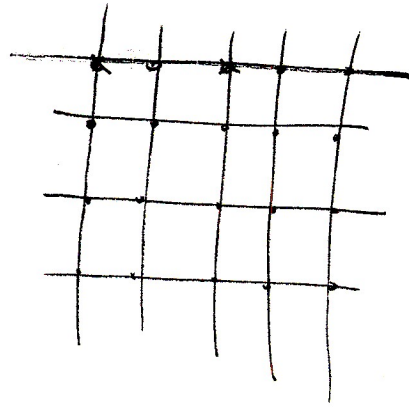
$\phi, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}$

$\{1,3\}, \{1,4\}, \{1,5\}$

$\{2,4\}, \{2,5\}$

$\{3,5\}$

sets of 3? $\{1,3,5\}$



How compute the # in Rght file?

19/26 w17

02-03-03 (

(12) $f(n, k) = \text{pile 1} + \text{pile 2}$

pile 1 = All ~~k~~ subsets
that contain n + don't
contain two consecutive elements

pile 2 = All subsets of length k
that don't contain n + don't
contain 2 consecutive elements

in pile 1 = $f(n-1, k-1)$

$\{ -, -, -, \dots, -, - \}$

+ $f(n-1, k)$

$f(n, k) = f(n-1, k-1) + f(n-1, k)$

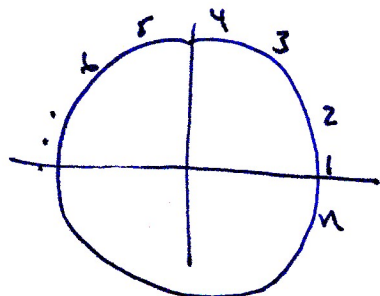
⋮

Don't think correct since ~~not every~~ not every
element counted by $f(n, k)$ (by adding k)
would not yield an element of

$f(n, k)$

(13) ?

(14)



$\{1, 2\}$

$g(n) = \#$ of ways of choosing a subset
w/ no 2 elts consecutive

$g(1) =$
 $g(2)$

$g(n) = g(n-1)$

(15) ?

$$(16) \quad \frac{1}{(1-x^2)^2} = \frac{1}{(1-x)^2(1+x)^2} = \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{1+x} + \frac{D}{(1+x)^2}$$

$$\frac{A}{1-x} + B \frac{1}{1-x} + \frac{C}{1+x} + \frac{D}{(1+x)^2}$$

$$\frac{1}{1-x^2} = \sum_{k=0}^{\infty} x^{2k}$$

$$\frac{1}{(1-x^2)^2} = \left(\sum_{k=0}^{\infty} x^{2k} \right) \left(\sum_{k=0}^{\infty} x^{2k} \right)$$

(17) ?

⋮

$$(21) (a) \quad e^{ex} = \int$$

$$\frac{1}{(1-x)^k} = ?$$

$$\frac{1}{\frac{d}{dx} \frac{1}{1-x}} = \frac{1}{(1-x)^2}$$

$$\frac{1^2}{\frac{d^2}{dx^2} \frac{1}{1-x}} = \frac{2}{(1-x)^3}$$

$$\vdots$$

$$\frac{1^p}{\frac{d^p}{dx^p} \frac{1}{1-x}} = \frac{p!}{(1-x)^{p+1}}$$

$$\therefore \frac{1}{1-x} = \sum_{n \geq 0} x^n$$

$$\left(\frac{1}{1-x}\right)^k = \frac{1}{(k-1)!} \frac{d^{k-1}}{dx^{k-1}} \frac{1}{1-x} = \frac{1}{(k-1)!} \frac{d^{k-1}}{dx^{k-1}} \sum_{n \geq 0} x^n$$

$$= \frac{1}{(k-1)!} \sum_{n \geq 0} n(n-1)(n-2) \cdots (n-(k-1)+1) x^{n-k+1}$$

$$= \sum_{n \geq 0} \frac{n(n-1)(n-2) \cdots (n-k+2)}{(k-1)!} x^{n-k+1}$$

Ass $k-1 \rightarrow n$

$$= \sum_{n \geq -k+1} \frac{(n+k-1)(n+k-2)(n+k-3) \cdots (n+k-1-k+2)}{(k-1)!} x^n$$

$$= \sum_{n \geq 0} \frac{(n+k-1)(n+k-2)(n+k-3) \cdots (n+1)}{(k-1)!} x^n$$

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$$\frac{(n+k-1)(n+k-2)(n+k-3) \dots (n+2)(n+1)n!}{(k-1)! n!}$$

$$= \binom{n+k-1}{n}$$

$$Z^k \xleftrightarrow{\text{op. 1}} \left\{ \sum_{n_1+n_2+\dots+n_k=n} a_{n_1} a_{n_2} \dots a_{n_k} \right\}_0^\infty$$

given n & k how many in order ~~sum~~ terms are in the sum above?

$$n_k = n - \sum_{p=1}^{k-1} n_p \quad \text{so specifying } k-1 \text{ terms}$$

$$\begin{aligned} 0 \leq n_1 \leq n \\ 0 \leq n_2 \leq n \\ \vdots \\ 0 \leq n_k \leq n \end{aligned} \Rightarrow \sum n_p = n \quad \text{How solve?}$$

pg 37 wilf

$$\begin{aligned} \frac{1}{(1-x)^k} &= \frac{x}{(k-1)!} \frac{d^{k-1}}{dx^{k-1}} \frac{1}{(1-x)} \quad \text{---} \quad \frac{1}{(k-1)!} \frac{d^k}{dx^k} \frac{1}{(1-x)} \\ &= \frac{1}{(k-1)!} \frac{d^k}{dx^k} \sum_{n \geq 0} x^n \\ &= \frac{1}{(k-1)!} \end{aligned}$$

Thus $\frac{1}{(1-x)^k} \xleftrightarrow{\text{coeff}} \left\{ \binom{n+k-1}{n} \right\}$

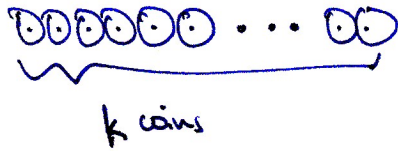
so that

$$[x^n] \left(\frac{1}{(1-x)^4} \right) = \binom{n+3}{n} = \binom{n+3}{3}$$

$$\begin{aligned} [x^n] \frac{x(1+x)}{(1-x)^4} &= [x^n] \left[\frac{x}{(1-x)^4} + \frac{x^2}{(1-x)^4} \right] \\ &= \cancel{2} \binom{n+2}{3} + \binom{n+1}{3} \quad \checkmark \end{aligned}$$

$$= \frac{(n+2)(n+1)n(n-1)!}{3!(n-1)!} + \frac{(n+1)(n)(n-1)}{3!}$$

$$= \frac{(n+1)n}{6} [n+2 + n-1] = \frac{(2n+1)(n+1)n}{6}$$



3 coins 2 spaces to begin

$$f(k) = \sum_{j=1}^k (k-j)f(j) + 1$$

$$= - \sum_{j=1}^k (j-k)f(j) + 1$$

$$= \sum_{j=1-k}^0 j f(j+k)$$

Product of $\{j\}_1^\infty$ and $\{f(j)\}_1^\infty$

$\uparrow$ $\frac{x}{(1-x)^2}$ $\uparrow$ $F(x)-1$

$$F(x) - 1 = \frac{x}{(1-x)^2} (F(x) - 1) + \underbrace{\left[\frac{1}{1-x} - 1 \right]}_{\frac{1-(1-x)}{1-x}}$$

$$\Rightarrow F-1 = \frac{x}{(1-x)^2} (F-1) + \frac{x}{1-x}$$

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2

$$\sum_{n \geq 0} \frac{a_{n+1} x^n}{n!}$$

$$f' \longleftrightarrow \{a_{n+1}\}_0^\infty$$

$$f(x) = \sum_{n \geq 0} \frac{a_n x^n}{n!} \quad \begin{array}{l} F(0) = a_0 = 0 \\ F' \end{array}$$

pg 41 with

$$\sum_{r,s \geq 0} \frac{a_r b_s}{r! s!} x^{r+s} = \sum_{n \geq 0} x^n \sum_{r+s=n} \frac{a_r b_s}{r! s!}$$

let $n = r+s$

$$\text{Then } \left[\frac{x^n}{n!} \right] (fg) = \sum_{r+s=n} \frac{n! a_r b_s}{r! s!} = \sum_{r+s=n} \frac{n!}{r! (n-r)!} a_r b_{n-r}$$

$$= \sum_{r=0}^n \binom{n}{r} a_r b_{n-r} \quad \text{* } \text{~~(n-r)~~$$

$$b_{n+1} = \sum_k \binom{n}{k} b(k) \cdot 1 \quad n \geq 0$$

$\downarrow$ eqt
 $\therefore$

$$\begin{aligned}
 B'(x) &= B(x) \left(\sum_{n \geq 0} \frac{x^n \cdot 1}{n!} \right) \\
 &= e^x B(x)
 \end{aligned}$$

$$\frac{B'}{B} = e^x$$

$$\ln B = e^x + C$$

$$B = C e^{e^x}$$

$$B(0) = 1 = C e^1 \Rightarrow C = e^{-1}$$

$$B(x) = e^{e^x - 1}$$

$$f(1) = \sum_{k=1} f(k-1)f(1-k) = f(0) \cdot f(0) = 1$$

$$f(2) = \sum_{k=1}^{n=2} f(k-1)f(2-k) = f(0)f(1) + f(1)f(0) = 2$$

$$f(3) =$$

$$f(n) = \sum_{k \geq 1} f(k-1)f(n-k)$$

$$= \sum_{k \geq 0}^{n-1} f(k)f(n-(k+1)) = \sum_{k \geq 0}^{n-1} f(k)f(n-k-1)$$

①

$$Q1) \quad \cos x = \sum_{k \geq 0} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}$$

By eq 2.13 let $\sum_{n \geq 0} b_n x^n$ be the inverse of $\cos x$

$$\cos b_0 = 1 \Rightarrow 1 \cdot b_0 = 1 \Rightarrow b_0 = 1$$

$$C_n = 0 = \sum_{k=0}^n a_k b_{n-k} = a_0 b_n + \sum_{k=1}^n a_k b_{n-k}$$

$$\Rightarrow b_n = -\frac{1}{a_0} \sum_{k=1}^n a_k b_{n-k} \quad n = 1, 2, 3, \dots$$

$$\text{Thus } b_1 = -\frac{1}{a_0} \sum_{k=1}^1 a_k b_{1-k} = -(a_1 b_0) = -0 = 0$$

$$b_2 = -\frac{1}{a_0} \sum_{k=1}^2 a_k b_{2-k} = -\left[\cancel{a_1 b_1} + a_2 b_0 \right] = +\frac{1}{2}$$

$$b_3 = -\frac{1}{a_0} \sum_{k=1}^3 a_k b_{3-k} = \cancel{\frac{1}{2} a_1 b_2} + \cancel{\frac{1}{2} a_2 b_1} - \sum_{k=1}^3 a_k b_{3-k}$$

$$= -\left[\cancel{a_1 b_2} + \cancel{a_2 b_1} + a_3 b_0 \right] = 0$$

$$b_4 = -\frac{1}{a_0} \sum_{k=1}^4 a_k b_{4-k} = -\left[\cancel{a_1 b_3} + a_2 b_2 + \cancel{a_3 b_1} + a_4 b_0 \right]$$

$$b_4 = - \left[-\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{4!} (4) \right]$$

$$\underline{0.2 - 0.8 - 0.3}$$

2

$$= - \left[-\frac{1}{4} + \frac{1}{4 \cdot 3 \cdot 2} \right]$$

$$= \frac{1}{4} \left[1 - \frac{1}{6} \right] = \frac{1}{4} \left[\frac{5}{6} \right] = \frac{5}{24}$$

$$\therefore \frac{1}{\cos x} \approx 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + O(x^6)$$

$$1 \stackrel{?}{=} \left(1 + \frac{x^2}{2} + \frac{5}{24}x^4 + O(x^6) \right) \left(1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6) \right)$$

$$= 1 - \cancel{\frac{x^2}{2}} + \frac{x^4}{24} + \cancel{\frac{x^2}{2}} - \frac{x^4}{4} + \frac{5}{24}x^4 + O(x^6)$$

$$= 1 + \left(\frac{1}{24} - \cancel{\frac{1}{4}} + \frac{5}{24} \right) x^4 + O(x^6)$$

$$(b) \quad (1+x)^m = \sum_{k=0}^m \binom{m}{k} x^k = 1 + mx + \frac{m(m-1)}{2} x^2 + \frac{m(m-1)(m-2)}{3 \cdot 2} x^3 + O(x^4)$$

$$1 = (1+x)^m \cdot \sum_{k=0}^{\infty} b_k x^k$$

$$a_0 b_0 = 1 \Rightarrow b_0 = 1$$

+ By multiplication of 2 power series

$$b_n = -\frac{1}{a_0} \sum_{k=1}^n a_k b_{n-k}$$

$$\Rightarrow b_1 = -1 \left[\sum_{k=1}^1 a_k b_{1-k} \right] = -a_1 b_0 = -m$$

$$b_2 = -\frac{1}{1} \sum_{k=1}^2 a_k b_{2-k} = -[a_1 b_1 + a_2 b_0] = -\left[m(-m) + \frac{m(m-1)}{2} \right]$$

$$= -\left[m^2 + \frac{m^2}{2} - \frac{m}{2} \right] = -\left[\frac{3m^2}{2} - \frac{m}{2} \right]$$

$$= -\frac{m}{2}(3m-1)$$

$$b_3 = -\frac{1}{1} \sum_{k=1}^3 a_k b_{3-k} = -[a_1 b_2 + a_2 b_1 + a_3 b_0]$$

$$= -\left[m(-1)\frac{m}{2}(3m-1) + \frac{m(m-1)}{2}(-m) + \frac{m(m-1)(m-2)}{6} \cdot 1 \right]$$

$$(c) \quad f(t) = 1 + t^2 + t^3 + t^5 + t^7 + t^{11} + O(t^{13})$$

$$a_0 b_0 = 1 \Rightarrow b_0 = 1$$

$$a b_n = -\frac{1}{a_0} \sum_{k=1}^n a_k b_{n-k} = -1 \sum_{k=1}^n a_k b_{n-k}$$

$$b_1 = \cancel{a_1} - \sum_{k=1}^1 a_k b_{1-k} = -[a_1 b_0] = 0$$

$$b_2 = - \sum_{k=1}^2 a_k b_{2-k}$$

$$= - [a_1 b_1 + a_2 b_0] = - [0 \cdot b_1 + 1 \cdot 1] = -1$$

$$b_3 = - \sum_{k=1}^3 a_k b_{3-k} = - [a_1 b_2 + a_2 b_1 + a_3 b_0]$$

$$\left\{ 1 = \sum_n a_n x^n \sum_n b_n x^n \right\} = - [0(-1) + (1)(0) + 1 \cdot 1] = -1$$

$$\therefore 1 + t^2 + \dots$$

(2)

(a) let $g(x)$ be the inverse of $f(x) = \sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$

Then ~~$f(g(x)) = x$~~

+ $g(f(x)) = x$

$$= x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{7!} + O(x^9)$$

~~$f(g(x)) = x$~~ Then w/ $g(x) = \sum_{n=0}^{\infty} c_n x^n$ requiring $g_0 = 0$

$$f(g(x)) = \sum_{n=0}^{\infty} c_n x^n - \frac{1}{6} \left(\sum_{n=0}^{\infty} c_n x^n \right)^3 + \dots = x$$

$$= \cancel{a_0} + \cancel{a_1 x} + \cancel{a_2 x^2} + \dots$$

Gives: $g(x) = \sum_{n=1}^{\infty} c_n x^n$

Then

$$f(g(x)) = \sum_{n=1}^{\infty} c_n x^n - \frac{1}{6} \left(\sum_{n=1}^{\infty} c_n x^n \right)^3 + \frac{1}{120} \left(\sum_{n=1}^{\infty} c_n x^n \right)^5 + O(x^7)$$

$$= a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots = x$$

$$- \frac{1}{6} \left[(a_1 x + a_2 x^2 + a_3 x^3 + \dots)^3 \right]$$

$$= a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots$$

$$- \frac{1}{6} [a_1^3 x^3 + \dots]$$

$$f(g(x)) = a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots$$

$$- \frac{1}{6} \left[(a_1x + a_2x^2 + \dots)(a_1x + a_2x^2 + \dots)(a_1x + a_2x^2 + \dots) \right]$$

$$= a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots$$

$$- \frac{1}{6} \left[(a_1x + a_2x^2 + \dots)(a_1^2x^2 + 2a_1a_2x^3 + a_2^2x^4 + \dots) \right]$$

$$= a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots$$

$$- \frac{1}{6} \left[a_1^3x^3 + 2a_1^2a_2x^4 + a_2a_1^2x^4 + \dots \right]$$

$$= a_1x + a_2x^2 + \left(a_3 - \frac{a_1^3}{6}\right)x^3 + O(x^4) = x$$

$$\Rightarrow a_1 = 1 ; a_2 = 0 ; a_3 = \frac{1}{6}$$

$$(b) y = \tan x$$

$$x = \tan^{-1}(y)$$

$$(c) y = x + x^2 \sqrt{1+x} = \sum_{n \geq 1} a_n x^n \quad \text{w/ } a_n \text{ known}$$

$$x = y - x(y)^2 \sqrt{1+x(y)}$$

~~letting~~ letting be coefficients of inverse series

$$(d) \quad x + x^3 = f(x)$$

$$\text{Then } g(x) = \sum_{n \geq 0} c_n x^n$$

$$f(g(x)) = \sum_{n \geq 0} c_n x^n + \left(\sum_{n \geq 0} c_n x^n \right)^3 = x$$

$$a_0 + a_0^3 = 0 \Rightarrow a_0 = 0 \text{ or } (1+a_0^2) = 0 \Rightarrow a_0 = \pm i$$

$$a_1 +$$

$$(3) \quad f'' + f = 0$$

$$f' = \sum_{n \geq 0} n a_n x^{n-1} = x^{-1} \sum_{n \geq 0} n a_n x^n$$

$$f'' = \sum_{n \geq 0} n(n-1) a_n x^{n-2}$$

$$f'' = -x^{-2} \sum_{n \geq 0} n a_n x^n + x^{-1} \sum_{n \geq 0} n^2 a_n x^{n-1}$$

$$= - \sum_{n \geq 0} n a_n x^{n-2} + \sum_{n \geq 0} n^2 a_n x^{n-2}$$

$$= \sum_{n \geq 0} n a_n (n-1) x^{n-2}$$

$$\sum_{n \geq 0} n(n-1) a_n x^{n-2}$$

$$\sum_{n \geq 0} n(n-1) a_n x^{n-2} + \sum_{n \geq 0} a_n x^n = 0$$

$$\sum_{n \geq 0} (n+2)(n+1)a_{n+2}x^n + \sum_{n \geq 0} a_n x^n = 0$$

$$\sum_{n \geq 0} ((n+2)(n+1)a_{n+2} + a_n)x^n = 0$$

$$\Rightarrow a_{n+2} = \frac{a_n}{(n+2)(n+1)} \quad n=0,1,2,\dots$$

④ (a) $\{n+7\}_0^\infty \rightarrow \frac{a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6}{x^7}$

~~(b)~~ ~~$\{n\}_0^\infty$~~ ~~$\rightarrow$~~

$$\{n\}_0^\infty + \{7\}_0^\infty \rightarrow xDf(x) + \frac{7}{1-x}$$

$$x f'(x) + \frac{7}{1-x}$$

$$\text{or } \{1\}_0^\infty \rightarrow f = \frac{1}{1-x}$$

$$(b) \{1\}_4^\infty = \{a_{n+4}\}_0^\infty =$$

$$\text{or } a_n \equiv 1$$

$$\Rightarrow \{n+7\}_0^\infty \rightarrow$$

$$f' = \frac{1}{(1-x)^2}$$

$$\frac{x}{(1-x)^2} + \frac{7}{1-x}$$

$$\left\{ a_{n+1} \right\}_0^\infty = \frac{1 - a_0 - a_1 x - a_2 x^2 - a_3 x^3}{x^4}$$

$$= \frac{\frac{1}{1-x} - 1 - x - x^2 - x^3}{x^4}$$

(c) $\{1, 0, 1, 0, 1, \dots\}$

let $f(x) = \sum_{n \geq 0} 1x^n = \frac{1}{1-x}$

$$\frac{f(x) + f(-x)}{2} = \sum_{n \geq 0} \left(\frac{1 + (-1)^n}{2} \right) x^n = \cancel{\frac{1}{1-x}} \frac{1}{2} \left(\frac{1}{1-x} + \frac{1}{1+x} \right)$$

(d) $\left\{ \frac{1}{n+1} \right\}_0^\infty \neq \frac{1}{1-x}$

If $f = \sum_{n \geq 0} x^n = \frac{1}{1-x}$

$$\int_0^x \frac{df}{1-f} = \int_0^x \frac{1}{1-f} = \sum_{n \geq 0} \frac{x^{n+1}}{n+1} = x \sum_{n \geq 0} \frac{x^n}{n+1}$$

$$\therefore x \sum_{n \geq 0} \frac{x^n}{n+1} = -\ln(1-x)$$

$$\left\{ \frac{1}{n+1} \right\}_0^\infty \xrightarrow{\text{qsst}} \frac{-\ln(1-x)}{x}$$

$$\left\{ \frac{1}{n+1} \right\}_2^\infty = \frac{f(x) - 0 - a_1 x}{x^2}$$

$$a_n = \frac{1}{n+1}$$

$$= \frac{f(x) - 1 - \frac{x}{2}}{x^2}$$

$$= \frac{-\frac{\ln(1-x)}{x} - 1 - \frac{x}{2}}{x^2}$$

$$(e) \left\{ \frac{1}{(n+5)!} \right\}_0^\infty = \left\{ \frac{1/n!}{(n+5)(n+4)(n+3)(n+2)(n+1)} \right\}_0^\infty$$

$$= \left\{ \frac{a_n}{(n+5)(n+4)(n+3)(n+2)(n+1)} \right\}_0^\infty \quad a_n = \frac{1}{n!}$$

Note: if $\{a_n\}_0^\infty \xrightarrow{\text{ops}} f(x)$ then

$$f(x) = \sum_{n \geq 0} a_n x^n$$

$$\int_0^x f(t) dt = \sum_{n \geq 0} \frac{a_n}{n+1} x^{n+1} = x \sum_{n \geq 0} \frac{a_n}{n+1} x^n$$

$$\Rightarrow \left\{ \frac{a_n}{n+1} \right\}_0^\infty \rightarrow \frac{1}{x} \int_0^x f(t) dt \equiv x^{-1} I(\cdot)$$

$$\left\{ \frac{a_n}{(n+2)(n+1)} \right\}_0^\infty \rightarrow x^{-1} I(x^{-1} I(\cdot))$$

$$\therefore \int_0^{\infty} \frac{c_n}{(n+5)(n+4)(n+3)(n+2)(n+1)} = x^{-1} I(x^{-1} I(x^{-1} I(x^{-1} I(x^{-1} I(x^{-1} I(f))))))$$

Since $f = e^x$

$$I(f) = e^x - 1 \dots$$

$$(f)_{(x)} = \sum_{n \geq 0} n F_n x^n$$

$$\sum_{n \geq 0} F_n x^n = \frac{x}{1-x-x^2}$$

$$F_0 = 0, F_1 = 1$$

$$\{n F_n\}_0^{\infty} \rightarrow x D F(x)$$

$$(g) \int_1^{\infty} \frac{n^2 + n + 1}{n!} = \int_1^{\infty} P(n) a_n \quad \text{w/ } a_n = \frac{1}{n!}$$

$$= P(xD)F = P(xD)e^x = \frac{x}{x^2} ((xD)^2 + (xD) + 1)e^x$$

$$\begin{aligned} \text{w/ } (xD)^2 &= xD(xD) \\ &= x[D + xD^2] \end{aligned}$$

$$\textcircled{5} \quad \sum_{k=0}^n \binom{n}{k} = 2^n$$

$$n=0,1,2,\dots$$

take the ~~E.G.F.~~ of both sides.

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^n \binom{n}{k} x^n \right) \neq \sum_{n=0}^{\infty} \binom{n}{k} x^n$$

Plot.

$$\text{If } f \xrightarrow{\text{egt}} \{a_n\}_0^{\infty} \quad g \xrightarrow{\text{egt}} \{b_n\}_0^{\infty}$$

$$fg \xrightarrow{\text{egt}} \left\{ \sum_{r=0}^n \binom{n}{r} a_r b_{n-r} \right\}$$

$$\text{taking } a_r = b_{n-r} = 1$$

$$\therefore \left\{ \sum_{r=0}^n \binom{n}{r} \right\} \xrightarrow{\text{egt}} e^x \cdot e^x = e^{2x} = \sum \xrightarrow{\text{egt}} \{2^n\} \quad \checkmark$$

⑥

$$n = \underbrace{\square + \square + \square + \square + \dots + \square + \square}_{k \text{ terms}}$$

~~k terms~~ non negative integers

$$n = n_1 + n_2 + \dots + n_k$$

$$S = \prod_{i=1}^k n_i$$

$$n=3 \quad k=2 \quad \text{0+3, 1+2, 2+1, 3+0}$$

$$S: \quad 0 \quad 2 \quad 2 \quad 0$$

$$\Sigma = 4$$

$$n = \square + \square + \square + \square + \dots + \square + \square$$

$$S = \prod_{i=1}^k n_i$$

x

x

x

⋮

$$\begin{aligned} f(n, k) &= \sum_{p=1}^{g(n, k)} S_p \\ &= \sum_{p=1}^{g(n, k)} S(n_1, n_2, n_3, \dots, n_k) \\ &= \sum_{i=1}^k \prod n_i \end{aligned}$$

of
ways
of
summing
to
n
non
neg integers

$$f(n, k) = \sum_{n_1+n_2+\dots+n_k=n} \left(\prod_{i=1}^k n_i \right)$$

$$= \sum_{n_1+n_2+\dots+n_k=n} (n_1 n_2 \dots n_k) = \sum_{n_1+n_2+\dots+n_k=n} a_{n_1} a_{n_2} \dots a_{n_k} \quad \text{if } a_n = n$$

Then $\left\{ \sum_{n_1+n_2+\dots+n_k=n} a_{n_1} a_{n_2} \dots a_{n_k} \right\} \xrightarrow{\text{ops}} f^k$

w/ $f(x) = \sum_{n \geq 0} a_n x^n$ in this case $= \sum_{n \geq 0} n x^n$

$$= x D \sum_{n \geq 0} x^n = x D \left(\frac{1}{1-x} \right) = \frac{x}{(1-x)^2}$$

$$\therefore f^k = \frac{x^k}{(1-x)^{2k}} = x^k \sum_{n \geq 0} \binom{n+k-1}{n} x^n$$

$$= \sum_{n \geq 0} \binom{n+k-1}{n} x^{n+k} = \sum_{n \geq k} \binom{n-1}{n-k} x^n$$

$$\therefore f(n, k) = \binom{n-1}{n-k}$$

$$\textcircled{7} \quad \left\{ \sum_{n_1+n_2+n_3+\dots+n_k=n} a_{n_1} a_{n_2} a_{n_3} \dots a_{n_k} \right\} \xrightarrow{\text{o.p.s.j.t.}} f(x)^k$$

$$\left\{ \sum_{n_1+n_2+n_3+\dots+n_k=n} c_{n_1} a_{n_2} \dots a_{n_k} \right\}$$

$$(c_0 + c_1 x + c_2 x^2 + \dots + c_n x^n + \dots) (c_0 + c_1 x + c_2 x^2 + \dots + c_n x^n + \dots)$$

consider $f(x) = \sum_{n \geq 0} c_n x^n$ &

$$g(x) \equiv f(x) - \sum_{n=0}^{h-1} a_n x^n = \sum_{n=h}^{\infty} c_n x^n$$

Then

$$g(x)^k \xrightarrow{\text{o.p.s.j.t.}} \left\{ \sum_{\substack{n_1+n_2+n_3+\dots+n_k=n \\ \text{+ all } n_i's \geq h}} a_{n_1} a_{n_2} a_{n_3} \dots a_{n_k} \right\} \quad \text{w/ } c_n \equiv 1$$

Thus what is the ~~coefficient~~ of n th coefficient of $g(x)^k$?

$$\left(\sum_{n=h}^{\infty} a_n x^n \right)^k = \left(\sum_{n=h}^{\infty} a_n x^n \right)^k$$

$$= \left(\frac{1}{1-x} - x^0 - x^1 - x^2 - x^3 - \dots - x^{h-1} \right)^k$$

$$= \left(\frac{1}{1-x} - x \sum_{p=0}^{h-1} x^p \right)^k = \left(\frac{1}{1-x} - \frac{x^h - 1}{x - 1} \right)^k$$

$$= \left(\frac{1 + (x^h - 1)}{1-x} \right)^k = \left(\frac{x^h}{1-x} \right)^k$$

$$= x^{k(h)} \frac{1}{(1-x)^k} = x^{k(h)} \sum_{n=0}^{\infty} \binom{n+k-1}{n} x^n$$

$$= \sum_{n=0}^{\infty} \binom{n+k-1}{n} x^{n+k(h)} = \sum_{n \geq 0} \binom{n+k-1}{k-1} x^{n+k(h)}$$

$$= \sum_{n \geq k(h)} \binom{n-k(h)+k-1}{k-1} x^n = \sum_{n \geq kh} \binom{n-kh+k-1}{k-1} x^n$$

rewriting expression

$$\therefore \text{~~the final result is~~} \quad f(n, k, h) = \binom{n-kh+k-1}{k-1}$$

(8)

$$(a) \overline{\lim}_{n \rightarrow \infty} x_n = 1$$

(i) $\forall \epsilon > 0$ all but finitely many members of x_n

$$\overline{x_n} < 1 + \epsilon \quad \text{True}$$

(ii) $\forall \epsilon > 0$ infinitely many members of the seq sat $x_n > 1 - \epsilon$ true ✓

$$(b) \overline{\lim}_{n \rightarrow \infty} x_n = +1$$

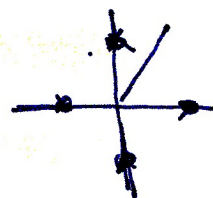
(i) $\forall \epsilon > 0$ all but finitely many members of x_n

$$x_n < 1 + \epsilon \quad \text{True}$$

(ii) $\forall \epsilon > 0$ ∞ many members of the seq satisfy $x_n > 1 - \epsilon$ true ✓

$$(c) \left\{ \cos\left(\frac{n\pi}{k}\right) \right\}_{n=0}^{\infty} \quad k \text{ fixed int.}$$

$$k=2: \left\{ \cos\left(\frac{n\pi}{2}\right) \right\}_0^{\infty} = 1, 0, -1, 0, 1, \dots$$



$$k=3: \left\{ \cos\left(\frac{n\pi}{3}\right) \right\}_0^{\infty} = 1, \frac{1}{2}, \cos\left(\frac{2\pi}{3}\right), \cos\left(\frac{3\pi}{3}\right), \cos\left(\frac{4\pi}{3}\right), \cos\left(\frac{5\pi}{3}\right), \dots$$

$\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$

$$\overline{\lim}_{n \rightarrow \infty} \cos\left(\frac{n\pi}{k}\right) = 1 \quad \text{Since } x_n = 1 \text{ for } n = p \cdot k, 2$$

$n = (2k)p$

(i) $\forall \epsilon > 0$ all but finitely many members of x_n

$$x_n < 1 + \epsilon \text{ true}$$

(ii) $\forall \epsilon > 0$ ∞ many members of x_n satisfy $x_n > 1 - \epsilon$ yes.

$$(d) \left\{ 1 + \frac{(-1)^n}{n} \right\}_{n \geq 1}$$

$$\lim_{n \rightarrow \infty} x_n = 1$$

(i) $\forall \epsilon > 0$ all but finitely many members of x_n

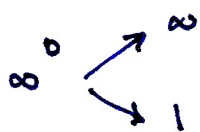
$$x_n < 1 + \epsilon$$

true



(ii) $\forall \epsilon > 0 \exists \infty$ # of elts of x_n $x_n > 1 - \epsilon$ true

$$(e) \{n^{x_n}\}_{n \geq 1}$$



$$\lim_{n \rightarrow \infty} n^{x_n} = \lim_{n \rightarrow \infty} \left(\frac{1}{n} \ln(n) \right)$$

$$\lim_{n \rightarrow \infty} \left(\frac{\ln(n)}{n} \right)$$

2/2

$$= e$$

$$= e$$

$$= \lim_{n \rightarrow \infty} \frac{y_n}{x_n} = 1$$

Claim $\overline{\lim}_{n \rightarrow \infty} x_n = 1$

(i) ~~$\forall \epsilon > 0$~~ $\forall \epsilon > 0$ all but finitely many x_n satisfy

$$x_n < 1 + \epsilon$$

$$n^{\frac{1}{n}} < 1 + \epsilon$$

$$n < (1 + \epsilon)^n \approx 1 + n\epsilon + O(n^2)$$

$$(1 - \epsilon)n < 1 + O(n^2)$$

$$n < \frac{1}{1 - \epsilon} \quad \text{looks like inequality goes the wrong way.}$$

(ii) $\forall \epsilon > 0 \exists$ an n # of x_n that satisfy

$$x_n > 1 - \epsilon$$

$$n^{\frac{1}{n}} > 1 - \epsilon$$

$$n > (1 - \epsilon)^n \approx 1 - \epsilon n + O(n^2)$$

$$(1 + \epsilon)n > 1 + O(n^2)$$

$$n > \frac{1}{1 + \epsilon} + O(n^2)$$

(9) let $\lim_{n \rightarrow \infty} x_n = L$

Then $\Leftrightarrow \forall \epsilon > 0 \exists N_0 \text{ s.t. } |x_n - L| < \epsilon \quad \forall n \geq N_0$

So check is for the $\overline{\lim} x_n$?

(i)

$\forall \epsilon > 0$ all but finite ~~the~~ many x_n satisfy

~~$x_n > L + \epsilon$~~ $x_n < L + \epsilon$

$$x_n - L < \epsilon$$

$$|x_n - L| < \epsilon \quad \text{pick } N_0(\epsilon) \text{ s.t. this statement is true}$$

Then the previous statement is true.

(ii) $\forall \epsilon > 0 \exists \text{ an } \infty \text{ of } x_n \text{ s.t.}$

$$x_n > L - \epsilon$$

$$x_n - L > -\epsilon$$

$$L - x_n < \epsilon$$

True.

... ...

... ...

(10) ~~Ex~~ Assume it holds: let $L_1 \neq L_2$

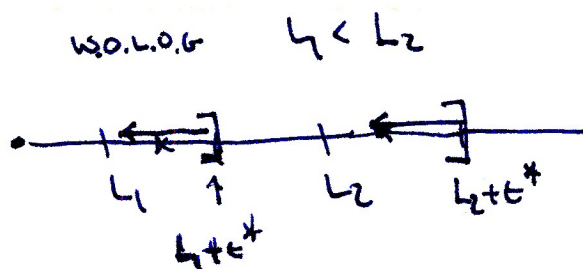
$$\overline{\lim}_{n \rightarrow \infty} x_n = L_1 \quad + \quad \overline{\lim}_{n \rightarrow \infty} x_n = L_2 \quad L_1 \neq L_2$$

(i) Then $\forall$ all but finite ~~many~~ x_n satisfy

$$\begin{aligned} x_n &< L_1 + \epsilon & + & x_n < L_2 + \epsilon \\ \forall n \geq n_1 & & \forall n \geq n_2 & \\ x_n - L_1 &< \epsilon & x_n - L_2 &< \epsilon \end{aligned} \quad \text{let } \bar{n} = \max(n_1, n_2)$$

$$\begin{cases} 2x_n - L_1 - L_2 < 2\epsilon \\ x_n - \frac{L_1 + L_2}{2} < \epsilon \end{cases}$$

let $\epsilon^* =$ ~~$\frac{|L_1 - L_2|}{2}$~~ $\frac{|L_1 - L_2|}{2}$



(ii) $\forall \epsilon > 0 \exists$ an ∞ # of x_n that satisfy

$$\del{x_n} > L_1 - \epsilon \quad x_n > L_2 - \epsilon$$

pick a $n_1^* + n_1^* \geq \bar{n}$

pick a

$$+ x_{n_1^*} > L_1 - \epsilon$$

(11) (a)

$$R = \frac{1}{\overline{\lim}_{n \rightarrow \infty} |a_n|^{1/n}}$$

$$R = \frac{1}{\overline{\lim}_{n \rightarrow \infty} \left| \frac{1}{n^2} \right|^{1/n}} = \frac{1}{\overline{\lim}_{n \rightarrow \infty} \frac{1}{n^{2/n}}} = \frac{1}{1} = 1$$

(b) $R = \frac{1}{\overline{\lim}_{n \rightarrow \infty} |a_n|^{1/n}}$

$$a_n = \begin{cases} 0 & n = 3k+1, 3k+2 \\ 1 & n = 3k \end{cases}$$

$$= 1$$

(c) $R = \frac{1}{\overline{\lim}_{n \rightarrow \infty} |a_n|^{1/n}}$

$$a_n = \begin{cases} 5^{n/2} & n = 0, 2, 4, 6, \dots \\ 0 & \text{otherwise} \end{cases}$$

$$= \frac{1}{\overline{\lim}_{n \rightarrow \infty} |5^{n/2}|^{1/n}} = \frac{1}{\overline{\lim}_{n \rightarrow \infty} 5^{1/2}} = \frac{1}{\sqrt{5}}$$

(d) $R = \frac{1}{\overline{\lim}_{n \rightarrow \infty} |a_n|^{1/n}}$

$$a_n = \begin{cases} n! & n = 0, 2, 4, \dots \\ 0 & \text{otherwise} \end{cases}$$

$$= \frac{1}{\overline{\lim}_{n \rightarrow \infty} (n!)^{1/n}} = \frac{1}{\infty} = 0.$$

$$(e) \quad \sum_{n=0}^{\infty} z^n = \frac{1}{\lim_{n \rightarrow \infty} |a_n|^{1/n}} = \frac{1}{\lim_{n \rightarrow \infty} |a_n|^{1/n}}$$

$$\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \rightarrow \infty} \frac{|x^{(n+1)!}|}{|x^{n!}|}$$

$$= \lim_{n \rightarrow \infty} \left| \frac{x^{n!+1}}{x^{n!}} \right| = \lim_{n \rightarrow \infty} \left| \frac{(x^{n!})^{n+1}}{x^{n!}} \right|$$

$$\lim_{n \rightarrow \infty} (x^{n!})^n < 1 \quad \text{if } |x| < 1 ? \quad \text{How check.}$$

(12) $R=0$; $R=\infty$

let $R=0$

$$\Rightarrow \overline{\lim}_{n \rightarrow \infty} |a_n|^{1/n} = \infty$$

Then $\forall M > 0 \exists n \text{ s.t. } |a_n|^{1/n} > M.$

pick $M > 1 \Rightarrow |a_n| > M^n$

+ n th term of series is $|a_n x^n| > (Mx)^n$ can be made to ~~not~~ $\rightarrow 0$

By picking $M > \max(1, 1/x)$ + $\therefore$ series diverges.

let $R = \infty$

$$\Rightarrow \overline{\lim}_{n \rightarrow \infty} |a_n|^{1/n} = 0$$

Then (i) $\forall \epsilon > 0$ all but finitely many members of the sequence
s.t. $x_n < \epsilon$

$$(13) \quad g(n \cdot m) = \sum_{d|n \cdot m} f(d)$$

w/ n & m relatively prime.

Let $n \cdot m = \prod_i p_i^{a_i}$ be decomposed into its prime decomposition.

$$\text{Then } \sum_{d|m \cdot n} f(d) = \sum_{d = \prod p_i^{a_i}} f(d) = \sum_{\vec{a}, \vec{p}} f(\prod p_i^{a_i}) = \sum_{\vec{a}, \vec{p}} \prod f(p_i^{a_i})$$

$d = a$ ~~product~~ sub product
of the full product
(sum over all such
sub products)

$$m \cdot n = p_1^{a_1} p_2^{a_2} \dots p_r^{a_r} = (p_1^{a_1} \dots) (p_r^{a_r})$$

Then all sub products are given by changing the vector $\vec{a}$
 $= (a_1, a_2, \dots, a_r)$ w/ each element going from 0 to its maximal allowed
 value.

(20)

$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$$

$$e^{t(x+y)} = \sum_{n \geq 0} \frac{t^n (x+y)^n}{n!} = e^{tx} \cdot e^{ty}$$

$$= \sum_{n \geq 0} \frac{t^n x^n}{n!} \cdot \sum_{n \geq 0} \frac{t^n y^n}{n!}$$

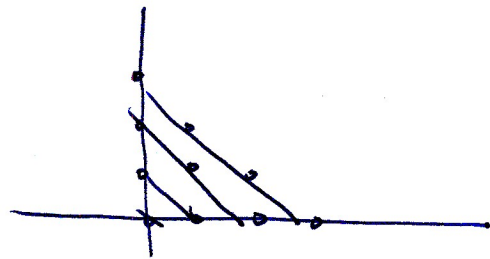
$$= \sum_{n_1, n_2 \geq 0} \frac{t^{n_1} \cdot t^{n_2} x^{n_1} y^{n_2}}{n_1! n_2!}$$

$$= \sum_{n_1, n_2 \geq 0} \frac{1}{n!} \left(\frac{n!}{n_1! n_2!} x^{n_1} y^{n_2} \right) t^n$$

$$n \equiv n_1 + n_2$$

$$n_1 = n - n_2$$

$$= \sum_{n \geq 0} \frac{1}{n!} \sum_{k=0}^n \frac{n!}{(n-k)! k!} x^{n-k} y^k$$



considering $e^{t(x_1 + x_2 + \dots + x_k)}$

$$= e^{tx_1} e^{tx_2} \dots e^{tx_k} \quad \text{using 2}$$

$$\text{f.e.} \rightarrow \left\{ \sum_{r_1 + r_2 + \dots + r_k = n} \frac{n!}{r_1! r_2! \dots r_k!} a_1^{r_1} a_2^{r_2} \dots a_k^{r_k} \right\}$$

(14) $\phi(n) = \#$ of integers m ($1 \leq m \leq n$) $\neq$ m is relatively prime to n .

Let $n = p_1^{a_1} \dots p_r^{a_r}$ be the prime factorization of n

Then all d that $\div n$ can be written as $d = p_1^{b_1} \dots p_r^{b_r}$

For a vector $\vec{b}$ of length r w/ elements b_i $0 \leq b_i \leq a_i$

~~$\phi(d) = \#$~~ $\phi(d) = \#$ of b_i 's that are non-zero.

Then

$$\sum_{d|n} \phi(d) =$$

?

(15) Let n & m be relatively prime

(a) $\phi(n \cdot m) =$

(b) $\phi(n) =$

?

(16) Thm 26.1

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_p \{1 + f(p)p^{-s} + f(p^2)p^{-2s} + f(p^3)p^{-3s} + \dots\}$$

~~3, 10~~ are relatively prime

$$\phi(p) = 2$$

$$\phi(p) = p$$

$$\phi(p^2) = 3$$

$$\phi(p^2) = p^2 - 1$$

$$\phi(p^3)$$

$$\phi(p^3) = p^3 - 2$$

$\vdots$

$$\sum_{n=1}^{\infty} \frac{\phi(n)}{n^s} = \prod_p \{ 1 + p \cdot p^{-s} + (p^2 - 1) p^{-2s} + (p^3 - 2) p^{-3s} + \dots \}$$

$$= \prod_p \{ 1 + p^{-s+1} + p^{-s} - \dots \}$$

(1) (a) $\sum_{n=1}^{\infty} \frac{n}{n^s} = \sum_{n=1}^{\infty} \frac{1}{n^{s-1}} = \zeta(s-1)$

(b) $\sum_{n=1}^{\infty} \frac{n^{\alpha}}{n^s} = \sum_{n=1}^{\infty} \frac{1}{n^{s-\alpha}} = \zeta(s-\alpha)$

(c) $\sum_{n=1}^{\infty} \frac{\log n}{n^s} = \sum_{n=1}^{\infty} \frac{\int_0^{\infty} \frac{e^{-xt}}{n} dt}{n^s} \quad ?$

$$(d) \sum_{n=1}^{\infty} \frac{\left(\sum_{d|n} 1 \right)^9}{n^s} = \left(\sum_{n=1}^{\infty} \frac{1}{n^s} \right) \left(\sum_{n=1}^{\infty} \frac{n^9}{n^s} \right)$$

take $b_n \equiv 1$ in 2.6.2.

+ ~~take~~ $a_n = n^9$ Then

$$= \zeta(s) \zeta(s-9)$$

(18) $n=1, 2, 3, 4, 5, 6, 7, 8$

(a) $\sum_{d|n} \phi(d) = n$

$n=1$ $1 \stackrel{?}{=} \sum_{d|1} \phi(d) = \phi(1) = 1 \quad \checkmark$

$2 \stackrel{?}{=} \sum_{d|2} \phi(d) = \phi(1) + \phi(2) = 1 + 1 \quad \checkmark$

$3 \stackrel{?}{=} \sum_{d|3} \phi(d) = \phi(1) + \phi(3) = 1 + 2 = 3 \quad \checkmark$

$4 \stackrel{?}{=} \sum_{d|4} \phi(d) = \phi(1) + \phi(2) + \phi(4) = 1 + 1 + 2 = 4 \quad \checkmark$

$5 \stackrel{?}{=} \sum_{d|5} \phi(d) = \phi(1) + \phi(5) = 1 + 4 = 5 \quad \checkmark$

Assume 1 is relatively prime to any integer m .

$$6 \stackrel{?}{=} \sum_{d|6} \phi(d) = \phi(1) + \phi(2) + \phi(3) + \phi(6) = 6 \quad \checkmark$$

$\{1\}$ $\{1\}$ $\{1, 2\}$ $\{1, 3, 5, 6\}$

$$7 \stackrel{?}{=} \sum_{d|7} \phi(d) = \phi(1) + \phi(7) = 7 \quad \checkmark$$

$\{1\}$ $\{1, 2, 3, 4, 5, 6\}$

$$8 \stackrel{?}{=} \sum_{d|8} \phi(d) = \phi(1) + \phi(2) + \phi(4) + \phi(8) = 8 \quad \checkmark$$

$\{1\}$ $\{1\}$ $\{1, 3\}$ $\{1, 3, 5, 7\}$

Deg of L.H.S is

$$\left\{ \sum_{d|n} a_d b_{\frac{n}{d}} \right\} \rightarrow f(s)g(s)$$

pick $a_n = \phi(n)$

$b_n \equiv 1$

Then $\left\{ \sum_{d|n} \phi(d) \right\}_1^\infty \rightarrow \left(\sum_{n=1}^\infty \frac{\phi(n)}{n^s} \right) \left(\sum_{n=1}^\infty \frac{1}{n^s} \right)$

$\cdot \zeta(s)$

+ Deg of R.H.S is $\chi(s-1)$

$$\sum_{d|7} \mu(d) = \mu(1) + \mu(7) = 0 \quad \checkmark$$

$$\sum_{d|8} \mu(d) = \mu(1) + \mu(2) + \mu(4) + \mu(8)$$

" "

$$1 + -1 + 0 + 0 = 0 \quad \checkmark$$

D.s.g.f of L.H.S is

$$\left\{ \sum_{n=1}^{\infty} a_n b_n \right\} \rightarrow f(s) g(s)$$

take again $a_n = \mu(n)$
 $b_n \equiv 1$

The D.s.g.f. of L.H.S $\Rightarrow \left(\sum_{n \geq 1} \frac{\mu(n)}{n^s} \right) \left(\sum_{n \geq 1} \frac{1}{n^s} \right)$

$$\frac{1}{\zeta(s)} \cdot \zeta(s) = 1 \neq 0 \quad ? \text{ what happens?}$$

(4) $\sum_{d|n}$

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(19) From 7 given

2.2.6

$$f(k) = \sum_{j=1}^k (k-j)f(j) + 1$$

$$\Rightarrow F(x) = \frac{1-2x}{1-3x+x^2} = \frac{1-2x}{(1-x)(1-x)}$$

$$\text{Fib}(x) = \frac{x}{1-x-x^2}$$

F_{2k-1} ~~with~~ $k \geq 1$ would be the selection of the odd coefficients from the p.s. generated by $\text{Fib}(x)$

Thus the p.s.g.f. for F_{2k-1} is given by

$$\begin{aligned} b(x) &= \frac{\text{Fib}(x) - \text{Fib}(-x)}{2} = \frac{1}{2} \sum_{n \geq 0} F_n x^n - \frac{1}{2} \sum_{n \geq 0} F_n (-1)^n x^n \\ &= \frac{1}{2} \sum_{n \geq 0} F_n (1 - (-1)^n) x^n = \sum_{k \geq 1} F_{2k-1} x^{2k-1} \\ &= \frac{1}{x} \sum_{k \geq 1} F_{2k-1} x^{2k} \end{aligned}$$

Then

$$b(\sqrt{x}) = \frac{\text{Fib}(\sqrt{x}) - \text{Fib}(-\sqrt{x})}{2} = \frac{1}{\sqrt{x}} \sum_{k \geq 1} F_{2k-1} (x^{1/2})^{2k} = \frac{1}{\sqrt{x}} \sum_{k \geq 1} F_{2k-1} x^k$$

Thus

$$\sum_{k \geq 1} F_{2k-1} x^k = \frac{\sqrt{x}}{2} (Fib(\sqrt{x}) - Fib(-\sqrt{x}))$$

$$= \frac{\sqrt{x}}{2} \left[Fib(\xi) - Fib(-\xi) \right]_{\xi = \sqrt{x}} = \frac{\sqrt{x}}{2} \left[\frac{\xi}{1-\xi-\xi^2} + \frac{\xi}{1+\xi-\xi^2} \right]_{\xi = \sqrt{x}}$$

$$= \frac{\sqrt{x}}{2} \cdot \xi \left[\frac{1+\xi-\xi^2 + 1-\xi-\xi^2}{(1-\xi-\xi^2)(1+\xi-\xi^2)} \right]_{\xi = \sqrt{x}}$$

$$= \frac{\sqrt{x}}{2} \cdot \xi \left[\frac{2(1-\xi^2)}{1+\xi-\xi^2-\xi-\xi^2+\xi^2-\xi^2-\xi^3+\xi^4} \right]_{\xi = \sqrt{x}}$$

$$= \frac{\sqrt{x}}{2} \cdot \xi \left[\frac{2(1-\xi^2)}{1-3\xi^2+\xi^4} \right]_{\xi = \sqrt{x}} = \frac{\cancel{\sqrt{x}}}{2} \cdot \frac{x}{2} \frac{2(1-x)}{(1-3x+x^2)} = \frac{x(1-x)}{1-3x+x^2}$$

Where is the error?

(21) $f(n, k, T) = \#$ of ordered rep of n as a sum of
?

(22) ?

(23)
$$\frac{x}{e^x - 1} = \sum_{n \geq 0} \frac{B_n x^n}{n!}$$

$$\frac{e^{mx} - 1}{e^x - 1} \sim \frac{r^{n+1} - 1}{r - 1} = \sum_{k=0}^n r^k$$

(24)

(25)

$$H_0(x, y) =$$

$$H_1(x, y) = \exp \left\{ \gamma \frac{d_1 x^1}{1!} \right\}$$

$$H_2(x, y) = \exp \left\{ \gamma \frac{d_2 x^2}{2!} \right\}$$

⋮

$$H(x, y) = \prod_{i=0}^{\infty} H_i(x, y) \quad \text{By Fundamental Lemma:}$$

$$= \prod_{i=0}^{\infty} \exp \left\{ \gamma \frac{d_i x^i}{i!} \right\}$$

$$= \exp \left\{ \gamma \sum_{i=0}^{\infty} \frac{d_i x^i}{i!} \right\} = \exp \left\{ \gamma D(x) \right\} \quad \text{eq 3.4.4}$$

$$nh_n = \sum_{k=0}^n \binom{n}{k} k d_k \cdot h_{n-k}$$

$$\therefore \binom{n}{2} = \sum_{k=0}^n \binom{n}{k} \cdot k \cdot d_k \cdot 2^{\binom{n-k}{2}}$$

let $n=1$:

$$1 \cdot 2^{\binom{1}{2}} = \sum_{k=0}^1 \binom{1}{k} \cdot k \cdot d_k \cdot 2^{\binom{1-k}{2}}$$

Think $\binom{1}{2} = 0 \Rightarrow$

$$1 = 0 + 1 \cdot 1 \cdot d_1 \cdot 2^{\binom{0}{2}}$$

Think $\binom{0}{2} = 0 \Rightarrow d_1 = 1$

let $n=2$:

$$2 \cdot 2^{\binom{2}{2}} = \sum_{k=0}^2 \binom{2}{k} \cdot k \cdot d_k \cdot 2^{\binom{2-k}{2}}$$

$$2^2 = 0 + \binom{2}{1} \cdot 1 \cdot d_1 \cdot 2^{\binom{1}{2}} + \binom{2}{2} \cdot 2 \cdot d_2 \cdot 2^{\binom{0}{2}}$$

$$4 = 0 + 2 \cdot 1 \cdot 1 + 1 \cdot 2 \cdot d_2 \cdot 1$$

$$4 = 2 + 2d_2 \Rightarrow d_2 = 1$$

let $n=3$:

$$3 \cdot 2^{\binom{3}{2}} = \sum_{k=0}^3 \binom{3}{k} k d_k \cdot 2^{\binom{3-k}{2}}$$

$$\binom{3}{2} = \frac{3!}{1!2!} = 3$$

$$3 \cdot 2^3 = 0 + \binom{3}{1} d_1 2^{\binom{2}{2}} + \binom{3}{2} \cdot 2 \cdot d_2 \cdot 2^{\binom{1}{2}} + \binom{3}{3} \cdot 3 \cdot d_3 \cdot 2^{\binom{0}{2}}$$

$$24 = \cancel{24} 3 \cdot 1 \cdot 2 + 3 \cdot 2 \cdot 1 + 1 \cdot 3 \cdot d_3 \cdot 1$$

$$24 = 6 + 6 + 3d_3$$

$$3d_3 = 12$$

$$d_3 = 4$$

①

$$n = n_1 + n_2 + \dots + n_k \quad n_1 > n_2 > n_3 > \dots > n_k \geq 1$$

a partition of n into distinct parts + a partition of n into odd parts.

$$n_1 - n_2 \geq (2p+1) \quad p=0,1,2,\dots$$

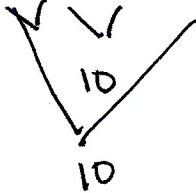
$$n = r_1 \cdot 1 + r_3 \cdot 3 + r_5 \cdot 5 + \dots$$

$$= (\overbrace{2+2+\dots}^{a_1} + \overbrace{2+2+\dots}^{b_1}) + 3$$

$$= (2^{a_1} + 2^{b_1} + \dots) + 3 \cdot (2^{a_3} + 2^{b_3} + \dots) + 5(2^{a_5} + 2^{b_5} + \dots)$$

$$= 2^{a_1} + 2^{b_1} + \dots + 3 \cdot 2^{a_3} + 3 \cdot 2^{b_3} + \dots + 5 \cdot 2^{a_5} + 5 \cdot 2^{b_5} + \dots$$

$$39 = 3 + 3 + 7 + 7 + 19 = \cancel{2 \cdot 3 + 2 \cdot 7 + 1 \cdot 19}$$



39 ✓

$$= \cancel{2 \cdot 3 + 2 \cdot 7} + 3 \cdot 2 + 7(2) + 19 \cdot 1$$

$$= 6 + 14 + 19$$

(2) ...

(3) $\{1, 2, \dots, n\} = \bigcup \underbrace{\{x, y, \dots, z\}}_{\# \text{ elts is prime.}}$

Hand enumerator $h(n, k) = ?$

Deck enumerator $d(n, k) = ?$

(4) $g^p = 1_n$ Γ group.

(a) $p = \lim_{n \rightarrow \infty} \frac{1}{n} (\text{all cycle lengths})$

(b) $g(n, k) \times \sum \tilde{g}(n, m)$

$$\tilde{g}(n, k) = \sum_{d|k} g(n, d)$$

$$\rightarrow g(n, k) = \sum_{d|k} \mu(k/d) \tilde{g}(n, d)$$

(5) $T_n = \# \text{ of involutions of } n \text{ letters.}$

$$(a) T(x) = \sum_{n \geq 0} \frac{t_n}{n!} x^n = e^{x + \frac{1}{2}x^2}$$

~~Alg~~ x Dlog method

$$x \frac{T'(x)}{T(x)} = x(1+x)$$

~~xxxxxx~~ $xT'(x) = x(1+x)T(x)$

$$\Rightarrow T'(x) = (1+x)T(x)$$

$$\sum_{n \geq 0} n \frac{t_n x^{n-1}}{n!} = \sum_{n \geq 0} \frac{t_n x^n}{n!} + \sum_{n \geq 0} \frac{t_n x^{n+1}}{n!}$$

$$\sum_{n \geq 1} \frac{(n+1)t_{n+1} x^n}{(n+1)!} = \sum_{n \geq 0} \frac{t_n x^n}{n!} + \sum_{n \geq 1} \frac{t_{n-1} x^n}{(n-1)!}$$

$$\sum_{n \geq 0} \left(\frac{(n+1)t_{n+1}}{(n+1)!} - \frac{t_n}{n!} - \frac{t_{n-1}}{(n-1)!} \right) x^n - t_0 = 0$$

$$t_0 = 1 \quad t_1 = 1! \frac{dT(0)}{dx} = e^{(1+2x)} \Big|_0 = 1$$

$$\therefore \frac{t_{n+1}}{n!} - \frac{t_n}{n!} - \frac{t_{n-1}}{(n-1)!} = 0 \quad n \geq 0$$

$$t_{n+1} - t_n - n t_{n-1} = 0$$

$$\Rightarrow t_n = t_{n-1} + (n-1)t_{n-2} \quad \checkmark$$

(b) ...

(c) Consider an ^{given} involution of n letters. Split ~~this~~ consider the n th letter. Separate all involutions into 2 piles, on the left

We have those involutions that are fixed points of n . Thus there are t_{n-1} of these.

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On the right we have involutions that ~~have~~ have cycles of degree 2 for element n . This means that the element n must switch places $\rightarrow$ some other element k from 1 to $n-1$. As each choice for k gives a different involution we have $(n-1)t_{n-2}$ involutions on the right side.

(6)?

$$D(x) = \sum_{n \geq 4} \frac{(n-1)! x^n}{n!} \dots$$

⑦ ?

⑧ (3,5,2) is ~~$\sum_k$~~ $\Leftrightarrow H(x,y) = \frac{1}{(1-x)y}$

Shifting #'s of 1st kind

$$S(y) = \sum_k \binom{n}{k} y^k$$

Ans: γ D log method $\Rightarrow \gamma$ D log $S = \gamma$ D log $\sum_k \binom{n}{k} y^k$

$$\gamma \frac{S'(y)}{S(y)} = \gamma \frac{\sum_k \binom{n}{k} k y^{k-1}}{\sum_k \binom{n}{k} y^k}$$

$$\sum_k \binom{n}{k} y^k = \left[\frac{x^n}{n!} \right] \dots$$

Not sure ??

⑨ know "hard" expression $h(n) = (n-1)!$
~~want~~ ~~but~~ want that expression "J(k)"
 By Exponential Formula

$$D(x) = \sum_{k \text{ even}, \dots}$$

(10) (3.6.2)

$$\{n\}_k = \left[\frac{x^n}{n!} \right] \left\{ \frac{(e^x - 1)^k}{k!} \right\}$$

$$= \left[\frac{x^n}{n!} \right] \left\{ \frac{1}{k!} \sum_{p=0}^k \binom{k}{p} e^{px} (-1)^{k-p} \right\}$$

$$= \frac{(-1)^k}{k!} \left[\frac{x^n}{n!} \right] \left\{ \sum_{p=0}^k \binom{k}{p} (-1)^p e^{px} \right\}$$

$$= \frac{(-1)^k}{k!} \left[\frac{x^n}{n!} \right] \left\{ \sum_{p=0}^k \binom{k}{p} (-1)^p \sum_{n=0}^{\infty} \frac{p^n x^n}{n!} \right\}$$

$$= \frac{(-1)^k}{k!} \left[\frac{x^n}{n!} \right] \left\{ \sum_{n=0}^{\infty} \left(\sum_{p=0}^k \binom{k}{p} (-1)^p p^n \right) \frac{x^n}{n!} \right\}$$

$$\Rightarrow \{n\}_k = \frac{(-1)^k}{k!} \sum_{p=0}^k \binom{k}{p} (-1)^p p^n$$

change sum for from $p \leftrightarrow r$ + $p=0$ has no cost

$$\{n\}_k = \frac{(-1)^k}{k!} \sum_{r=1}^k \binom{k}{r} (-1)^r r^n$$

$$\sum_k = \sum_{r=1}^k (-1)^{k-r} \frac{k!}{k!(k-r)!r!} r^n$$

$$= \sum_{r=1}^k \frac{(-1)^{k-r}}{(k-r)!(r-1)!} r^{n-1} \quad \checkmark$$

⑪ ?

⑫ $h(n) = (n-1)!$

$$D(x) = \sum$$

⑬ $T(x) = \sum_{n \geq 0} \frac{t_n x^n}{n!} = e^{(x + \frac{1}{2}x^2)}$

$$G(x) = \sum_{n \geq 0} \frac{g(n) x^n}{n!} = \frac{e^{-\frac{1}{2}x - \frac{1}{4}x^2}}{\sqrt{1-x}}$$

$$T(x)G(x)^2 = \frac{1}{1-x}$$

$$\left(\sum_{n \geq 0} \frac{t_n x^n}{n!} \right) \left(\sum_{n \geq 0} \frac{g(n) x^n}{n!} \right)^2 = \sum_{n \geq 0} x^n$$

$$\left(\sum_{n \geq 0} \frac{t_n x^n}{n!} \right) \left(\sum_{n, m} \frac{g(n)g(m) x^{n+m}}{n!m!} \right) = \sum_{n \geq 0} x^n$$

$$\sum_{l, m, n} \frac{t_l g(n) g(m)}{l! n! m!} x^{n+m+l} = \sum_{n \geq 0} x^n$$

$$\Rightarrow \sum_{n \geq 0} \sum_{l, m} \frac{t_l g(m) g(m+l-n)}{l! m! (m+l-n)!} x^n = \sum_{n \geq 0} x^n$$

$$\left\{ \begin{array}{l} \text{index} = n+m+l \\ \neg n = m+l - \text{index} \end{array} \right\}$$

$$\neg \sum_{l, m} \frac{t_l g(m) g(m+l-n)}{l! m! (m+l-n)!} = 1 \quad \forall n.$$

$$\Leftrightarrow \sum_{i+j+k=n} \frac{t_i g(j) g(k)}{i! j! k!} = 1 \quad \forall n \geq 0.$$

(14)

$$(a) \quad e^{\gamma D(x)} = \sum_n \frac{\phi_n(\gamma)}{n!} x^n$$

$$D_x \rightarrow D(x) e^{\gamma D(x)} = \sum_n \frac{\phi'_n(\gamma)}{n!} x^n$$

$$\rightarrow F(D_\gamma)$$

$$\rightarrow F(D_\gamma) e^{\gamma D(x)} = \sum_n$$

put f to be the inverse of $D^{-1}(D_\gamma)$

$$\rightarrow D^{-1}$$

(b)

$$e^{\gamma D(x)} = \frac{1}{(1-x)^\gamma} = \sum_n \binom{-\gamma}{n} (-1)^n x^n$$

$$\phi_n(\gamma) = (1-x)^{-\gamma} = \sum_n \binom{-\gamma}{n} (-1)^n x^n$$

$$= \frac{(-\gamma)(-\gamma-1)(-\gamma-2)\dots(-\gamma-n+1)}{n!} (-1)^n x^n$$

$$= \sum_n \frac{\gamma(\gamma+1)(\gamma+2)\dots(\gamma+n-1)}{n!} x^n$$

$$\therefore \phi_n(\gamma) = \gamma(\gamma+1)(\gamma+2)\dots(\gamma+n-1)$$

$$D(x) = -\log(1-x)$$

so

$$D^{(-)}(x) = 1 - e^{-x}$$

$$\begin{cases} \gamma = -\log(1-x) \\ e^{-\gamma} = 1-x \\ x = 1 - e^{-\gamma} \end{cases}$$

$$D^{(-)}(D_{\gamma}) \phi_n(\gamma) = \{1 - e^{-D_{\gamma}}\} \phi_n(\gamma)$$

$$\left\{ e^D \cdot f(\gamma) = \sum_n \frac{D^n}{n!} f(\gamma) = \sum_{n \geq 0} \frac{f^{(n)}(\gamma)}{n!} \right\}$$

$$\left\{ = \sum_{n \geq 0} \frac{f^{(n)}(\gamma)}{n!} (x-\gamma)^n \right\} = f(\gamma+1)$$

$$x = \gamma+1$$

$$\text{so } e^{-D} f(\gamma) = \sum_{n \geq 0} \frac{(-1)^n D^n}{n!} f(\gamma) = \sum_{n \geq 0} \frac{(-1)^n f^{(n)}(\gamma)}{n!}$$

$$= \sum_{n!} \frac{f^{(n)}(\gamma)}{n!} \underbrace{(-1)^n (x-\gamma)^n}_{x=\gamma+1} = f(-(\gamma+1))$$

* see next pg.

$$\sum_n \frac{f^{(n)}(\gamma)}{n!} (-x+\gamma)^n \Big|_{x=\gamma+1}$$

Why is this expression not correct?

Another way

$$\sum_{n \geq 0} \frac{f^{(n)}(\gamma)}{n!} (x-\gamma)^n \Big|_{x=\gamma-1} = f(\gamma-1)$$

So

$$\{1 - e^{-\partial_\gamma}\} \phi_n(\gamma) = \cancel{\{1 - \frac{1}{e}\} \phi_n(\gamma)}$$

$$= \cancel{\phi_n(\gamma)} - \phi_n(\gamma-1)$$

$$\gamma(\gamma+1)(\gamma+2) \dots (\gamma+n-1) - (\gamma-1)(\gamma)(\gamma+1) \dots (\gamma+n-2)$$

$$= \gamma(\gamma+1) \dots (\gamma+n-2) [\gamma+n-1 - (\gamma-1)]$$

$$= \gamma(\gamma+1) \dots (\gamma+n-2)(n) = n \phi_{n-1}(\gamma) \quad \checkmark$$

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(15) ?

(16) $x^x = ?$

(17) (3.5.2) ?

(18) ?

(19) —

(20) —

(21) $n = x \cdot 1 + y \cdot 2 + z \cdot 3$