Notes and Solutions for: The Mathematics of Financial Derivatives by Paul Wilmott, Sam Howison, and Jeff Dewynne

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Introduction

Here you'll find some notes that I wrote up as I worked through this excellent book. For some of the problems I used MATLAB to perform any needed calculations. The code snippets for various exercises can be found at the following location:

`http://waxworksmath.com/Authors/N_Z/Wilmott/MathOfFinancialDerivatives/wilmott.html`

I've worked hard to make these notes as good as I can, but I have no illusions that they are perfect. If you feel that that there is a better way to accomplish or explain an exercise or derivation presented in these notes; or that one or more of the explanations is unclear, incomplete, or misleading, please tell me. If you find an error of any kind – technical, grammatical, typographical, whatever – please tell me that, too. I'll gladly add to the acknowledgments in later printings the name of the first person to bring each problem to my attention.

Acknowledgements

Special thanks to (most recent comments are listed first): Felix Huber and Peter Hogeveen for helping improve these notes and solutions. All comments (no matter how small) are much appreciated.

Chapter 1 (Options and Markets)

Exercise 1 (stock splits)

If a company issues a stock split (doubling the number of stock shares) I would expect that the value of each share of stock to divide by two and the number of stock shares that a person owned to double keeping the net value of all the stock constant. I would also expect the options values to also divide their value by two.

As an example, of a stock split consider the value of a Rolls-Royce call (the option to buy on Rolls-Royce stock at a given price at a future date). According to Figure 1.1 from the book its value with a March expiration date is 11p. If Rolls-Royce issues a one-for-one stock split I would expect that the value of the March call option (for one share) to go to $11\text{p}/2 = 5.5\text{p}$.

For a two-for-one issue, I would expect the value of the underlying security to drop to one third its original price, since the owner of one share ends up owning three shares after the issue. In the same way I would expect that the value of options on the underlying would be reduced by a factor of three.

Exercise 2 (dividends)

I would assume that the price of the stock S *before* the dividend is issued must implicitly include the value of the dividend in the stocks evaluation. Thus after the dividend issue the stock price would decrease by an amount D to $S - D$.

Exercise 3 (the direction of uncertainty)

As the volatility of the underlying security increases it becomes *more* likely that the underlying will obtain its strike price. Since it is more likely to be exercised there is more value to the option.

Exercise 4 (zero-sum game)

What is meant by the statement "options transactions are a zero sum game" is that there is no loss or gain in the immediate purchase of an option. It can be sold immediately after its purchase for the same price. Thus options have an intrinsic value that does not change as they are purchased.

Chapter 2 (Asset Price Random Walks)

Exercise 1 (stochastic derivatives)

For this problem, we require Ito's lemma for a function $f(S)$, when S is by a stochastic process that satisfies $dS = \mu S dt + \sigma S dX$, with dX the random variable. Here we are using the notation that a capital letter represents a random variable and a lower case letter represents a deterministic variable. With that background Ito's lemma (for a function $f(S)$ with no explicit time dependence) is given by

$$\begin{aligned} df &= (\sigma S dX + \mu S dt) \frac{df}{dS} + \frac{1}{2}\sigma^2 S^2 \frac{d^2S}{dS^2} dt \\ &= \sigma S \frac{df}{dS} dX + \left(\mu S \frac{df}{dS} + \frac{1}{2}\sigma^2 S^2 \frac{d^2S}{dS^2} \right) dt \end{aligned} \quad (1)$$

Part (a): Here our function f is given by $f(S) = AS$, so $df/dS = A$, and $d^2f/dS^2 = 0$ so Ito's lemma then gives

$$\begin{aligned} df &= \sigma S A dX + \mu S A dt \\ &= \sigma f(S) dX + \mu f(S) dt \,. \end{aligned}$$

Part (b): Here our function f is given by $f(S) = S^n$, so $df/dS = nS^{n-1}$ and $d^2f/dS^2 = n(n-1)S^{n-2}$ and Ito's lemma becomes

$$\begin{aligned} df &= \sigma S(nS^{n-1}) dX + \left(\mu S n S^{n-1} + \frac{1}{2}\sigma^2 S^2 n(n-1) S^{n-2} \right) dt \\ &= n\sigma S^n dX + \left(\mu n S^n + \frac{n(n-1)}{2}\sigma^2 S^n \right) dt \\ &= n\sigma f(S) dX + \left(\mu n + \frac{n(n-1)}{2}\sigma^2 \right) f(S) dt \,. \end{aligned}$$

Exercise 2 (verifying a stochastic integral with Ito's lemma)

The Equation 2.12 is a stochastic integral given by

$$\int_{t_0}^{t} X(\tau) dX(\tau) = \frac{1}{2}(X(t)^2 - X(t_0)^2) - \frac{1}{2}(t - t_0) \,.$$

Note that this differs from what one would expect from non-stochastic calculus in the term linear in t. The differential of the left hand side of this expression is given by XdX. The differential of the right hand side is given symbolically by

$$d\left(\frac{1}{2}X^2\right) - \frac{1}{2}dt\,. \tag{2}$$

To evaluate the differential $d(\frac{1}{2}X^2)$ we must invoke Ito's lemma since the variable $X(t)$ is random. Now Ito's lemma for a function $f(X)$ is

$$df = \frac{df}{dx}dX + \frac{1}{2}\frac{d^2f}{dX^2}dX^2\,.$$

In our case $f(X) \equiv \frac{X^2}{2}$, so $\frac{df}{dX} = X$, $\frac{d^2f}{dX^2} = 1$. Finally using the heuristic that $dX^2 = dt$ we see that Eq. 2 becomes

$$df = XdX + \frac{1}{2}(1)dt - \frac{1}{2}dt = XdX\,,$$

proving the desired equivalence.

Exercise 3 (the density function for the log-normal random variable)

Considering S a stochastic variable with increments given by

$$dS = \sigma SdX + \mu Sdt\,,$$

And a function f defined by $f(S) = \log(S)$, Ito's lemma simplifies and we observe that f satisfies the following stochastic differential equation (see Page 28 of the text)

$$df = \sigma dX + (\mu - \frac{1}{2}\sigma^2)dt\,.$$

In the above expression, as dX is the only random variable it follows that the distribution of f can be obtained from the distribution of dX. To do this imagine integrating the above from some fixed point f_0 to f. This can be thought of as adding together a large number of independent identically distributed random variables dX. Because dX is assumed to be normally distributed with a mean of 0 and a variance t, we see that the variable f is normally distributed with mean $(\mu - \frac{1}{2}\sigma^2)t$ and variance equal to $\sigma^2 t$. The fact that the variance is given by $\sigma^2 t$ follows from the fact that a random variable defined as a sum of independent random variables has a variance given by the sum of the individual variances. Thus the probability density function (PDF) of f is given by

$$\frac{1}{\sqrt{2\pi}(\sigma\sqrt{t})}e^{-\frac{1}{2}\frac{(f-(f_0+(\mu-\frac{1}{2}\sigma^2)t)^2}{\sigma^2 t}}\,.$$

To compute the PDF of the random variable S given the PDF of the random variable f we use the following theorem involving transformations of random variables from probability theory

$$p_S(s) = p_F(f(s))\frac{df}{ds}\,,$$

Here $p_S(s)$ is the PDF of the random variable S and $p_F(f)$ is the PDF of the random variable F. Since when $f(S) = \log(S)$, the derivative is given by $\frac{df}{ds} = \frac{1}{s}$ using the above we get for $p_S(s)$ the following

$$\begin{aligned} p_S(s) &= \frac{1}{\sqrt{2\pi t}s\sigma}e^{-\frac{1}{2}\frac{(\log(s)-(\log(s_0)+(\mu-\frac{1}{2}\sigma^2)t)^2}{t\sigma^2}} \\ &= \frac{1}{\sigma s\sqrt{2\pi t}}e^{-\frac{1}{2}\frac{(\log(s/s_0)-(\mu-\frac{1}{2}\sigma^2)t)^2}{t\sigma^2}}\,, \end{aligned}$$

as claimed in the book.

Exercise 4 (a function of a random variable that has no drift)

We assume that our random variable G is defined in terms of X and t by a stochastic differential equation given by

$$dG = A(G,t)dX + B(G,t)dt\,.$$

Now lets consider a function of G say $f(G)$. We will apply Ito's lemma to f. Now Ito's lemma applied to the function f is given by

$$\begin{aligned} df &= \frac{df}{dG}dG + \frac{1}{2}\frac{d^2f}{dG^2}dG^2 \\ &= (A(G,t)dX + B(G,t)dt)\,\frac{df}{dG} + \frac{1}{2}\,(A(G,t)dX + B(G,t)dt)^2\,\frac{d^2f}{dG^2} \\ &= (A(G,t)dX + B(G,t)dt)\,\frac{df}{dG} + \frac{1}{2}A(G,t)^2dt\frac{d^2f}{dG^2} \\ &= A(G,t)\frac{df}{dG}dX + \left(B(G,t)\frac{df}{dG} + \frac{1}{2}A(G,t)^2\frac{d^2f}{dG^2}\right)dt \end{aligned}$$

Where in the above we have used the "rule" that $dX^2 = dt$. Now we can select f such that we eliminate the drift term by setting the coefficient of dt to zero. Specifically we require

$$B(G,t)\frac{df}{dG} + \frac{1}{2}A(G,t)^2\frac{d^2 f}{dG^2} = 0\,.$$

With this specification we find that the evolution of f follows a random with no drift given by

$$df = A(G,t)\frac{df}{dG}dX\,.$$

Thus given a random variable that has a drift term we can construct a function of G such that this function has no drift.

Exercise 5 (Ito's lemma for multidimensional functions)

We are told that the variables S_i satisfy the stochastic differential equations

$$dS_i = \sigma_i S_i dX_i + \mu_i S_i dt \quad \text{for} \quad i = 1, 2, \ldots, n\,,$$

where the random increments dX_i satisfy

$$\begin{aligned} E[dX_i] &= 0 \\ E[dX_i^2] &= dt \\ E[dX_i\, dX_j] &= \rho_{ij}\, dt\,, \end{aligned}$$

and we desire to compute the differential df of a multidimensional function $f = f(S_1, S_2, \ldots, S_n)$. For such a function using calculus we can write its differential as

$$df = \sum_{i=1}^{n}\frac{\partial f}{\partial S_i}dS_i + \frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2 f}{\partial S_i \partial S_j}dS_i dS_j \ldots$$

Splitting the second sum in the above (the one over i and j) into a diagonal term and the non-diagonal terms we find that

$$\begin{aligned} df &= \sum_{i=1}^{n}\frac{\partial f}{\partial S_i}dS_i \\ &+ \frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2 f}{\partial S_i^2}dS_i^2 \\ &+ \frac{1}{2}\sum_{i,j=1; i\neq j}^{n}\frac{\partial^2 f}{\partial S_i \partial S_j}dS_i dS_j \ldots \end{aligned}$$

To evaluate the diagonal sums in the above we use the two facts. The first is that $dS_i = \sigma_i S_i dX_i + \mu_i S_i dt$ and the second is that $dS_i^2 = \sigma_i^2 S_i^2 dX_i{}^2 = \sigma_i^2 S_i^2 dt$, to first order. To evaluate the non-diagonal terms we can expand the product above as

$$
\begin{aligned}
dS_i dS_j &= (\sigma_i S_i dX_i + \mu_i S_i dt)(\sigma_j S_j dX_j + \mu_j S_j dt) \\
&= \sigma_i \sigma_j S_i S_j dX_i dX_j + \sigma_i \mu_j S_i S_j dX_i dt \\
&+ \mu_i \sigma_j S_i S_j dt dX_j + \mu_i \mu_j S_i S_j dt^2 \,.
\end{aligned}
$$

Since dX_i is $O(\sqrt{dt})$ the terms in the above expression are ordered with increasing powers of dt, as dt, $dt^{3/2}$, $dt^{3/2}$, and dt^2. As always, we are concerned with the limit of this expression as $dt \to 0$ we see that to leading order only the term $\sigma_i \sigma_j S_i S_j dX_i dX_j$ remains. With these substitutions we obtain for df the following

$$
\begin{aligned}
df &= \sum_{i=1}^{n} \frac{\partial f}{\partial S_i} (\sigma_i S_i dX_i + \mu_i S_i dt) \\
&+ \frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2 f}{\partial S_i^2} \sigma_i^2 S_i^2 dt \\
&+ \frac{1}{2} \sum_{i,j=1; i \neq j}^{n} \frac{\partial^2 f}{\partial S_i \partial S_j} \sigma_i \sigma_j S_i S_j dX_i dX_j \ldots
\end{aligned}
$$

Using $dX_i dX_j \to \rho_{ij} dt$ and rearranging terms we have that Ito's lemma for a multivariate function f is given by

$$
\begin{aligned}
df &= \sum_{i=1}^{n} \frac{\partial f}{\partial S_i} \sigma_i \, S_i \, dX_i \\
&+ \sum_{i=1}^{n} \left(\frac{\partial f}{\partial S_i} \mu_i S_i + \frac{1}{2} \frac{\partial^2 f}{\partial S_i^2} \sigma_i^2 S_i^2 + \frac{1}{2} \sum_{i,j=1; i \neq j}^{n} \frac{\partial^2 f}{\partial S_i S_j} \sigma_i \sigma_j S_i S_j \rho_{ij} \right) dt \,.
\end{aligned}
$$

Chapter 3 (The Black-Scholes Model)

Additional Notes

The derivation of Δ for a European call

Here we perform the derivation of the expression for Δ for a European call option. As such, recall that a European call has a value given by

$$C(S,t) = SN(d_1) - Ee^{-r(T-t)}N(d_2)\,,$$

with d_1 and d_2 functions of S and t given by

$$\begin{aligned} d_1(S,t) &= \frac{\ln(S/E) + (r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = \frac{\log(S/E)}{\sigma\sqrt{T-t}} + \frac{(r+\frac{1}{2}\sigma^2)}{\sigma}\sqrt{T-t} \\ d_2(S,t) &= \frac{\ln(S/E) + (r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = \frac{\log(S/E)}{\sigma\sqrt{T-t}} + \frac{(r-\frac{1}{2}\sigma^2)}{\sigma}\sqrt{T-t}\,, \end{aligned}$$

with $N(x)$ the cumulative distribution function for the standard normal and is given by

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}y^2}dy\,. \tag{3}$$

From the definition of Δ we find that

$$\Delta \equiv \frac{\partial C}{\partial S} = N(d_1) + SN'(d_1)\frac{\partial d_1}{\partial S} - Ee^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial S}\,.$$

From the definitions of d_1 and d_2 we see that $d_1 - d_2 = \sigma\sqrt{T-t}$ and thus that

$$\frac{\partial d_1}{\partial S} = \frac{\partial d_2}{\partial S}\,.$$

Finally $N'(d) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}d^2}$. With these results the expression for $N'(d_2)$ is given by

$$\begin{aligned} N'(d_2) &= \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}d_2^2} = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(d_1-\sigma\sqrt{T-t})^2} \\ &= \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}d_1^2}\,e^{d_1\sigma\sqrt{T-t}}\,e^{-\frac{1}{2}\sigma^2(T-t)} \\ &= N'(d_1)e^{d_1\sigma\sqrt{T-t}}\,e^{-\frac{1}{2}\sigma^2(T-t)}\,. \end{aligned}$$

So the expression for the European calls delta becomes

$$\Delta = N(d_1) + \frac{\partial d_1}{\partial S} N'(d_1) \left[S - E e^{-r(T-t)} \, e^{d_1 \sigma \sqrt{T-t}} \, e^{-\frac{1}{2}\sigma^2 (T-t)} \right]$$

From the definition of d_1 given above, we see that the product $d_1 \sigma \sqrt{T-t}$ equals

$$d_1 \sigma \sqrt{T-t} = \log(S/E) + (r + \frac{1}{2}\sigma^2)(T-t) \, ,$$

so using this in the expression for our call's delta we see that the exponential product (the second term in the brackets above) becomes

$$\begin{aligned} e^{-r(T-t)} e^{d_1 \sigma \sqrt{T-t}} e^{-\frac{1}{2}\sigma^2 (T-t)} &= e^{-r(T-t) + \log(S/E) + (r + \frac{1}{2}\sigma^2)(T-t) - \frac{1}{2}\sigma^2 (T-t)} \\ &= e^{\log(S/E)} = S/E \, , \end{aligned}$$

and the expression in brackets vanishes finally giving

$$\Delta = N(d_1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_1} e^{-\frac{1}{2}y^2} dy \, ,$$

the expression claimed in the book.

Problem Solutions

Exercise 1 (speculation with calls and puts)

If the stock price of XYZ will *rise* 10% then its new share price will be $1.1 * 10 = 11$, while if it *falls* by 10% its new price will be $0.9 * 10 = 9$. As stated, if we anticipate that either of these two outcomes will happen after the election then we should buy one call (in anticipation that the price will rise) and one put (in anticipation that the price will fall). The entire portfolio at expiration will then have value of

$$\Pi = \max(S - E_1, 0) + \max(E_2 - S, 0) \, .$$

Where the first term is the payoff from the call with a strike price of E_1 and the second term is the payoff from the put with a strike price of E_2. From the above calculation to make money the call should have a strike price of $E_1 = 10.50 < 11.0$ and the put should have a strike price of $E_2 = 9.50 > 9.0$. With these values our portfolio above becomes

$$\Pi = \max(S - 10.5, 0) + \max(9.50 - S, 0) \, .$$

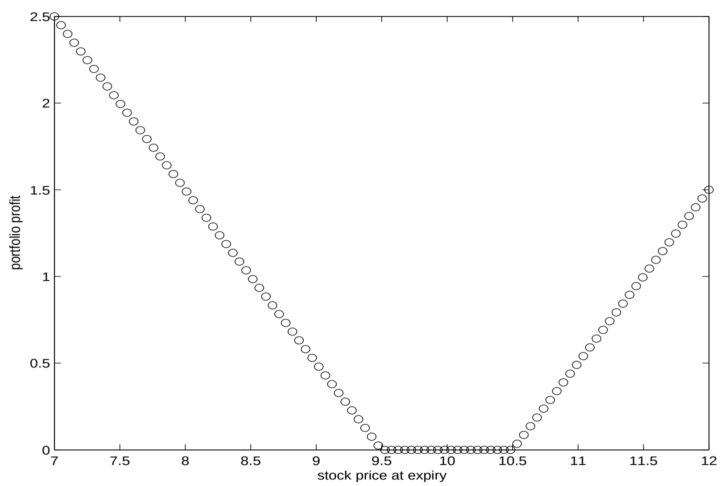

Figure 1: The payoff diagram for Problem 1.

Since the election is in December we should have the strike dates for the above options in December. The payoff diagram for such a portfolio is shown in Figure 1. Note that this level of profit is available at the cost of one call $C(10.5; \text{December})$, and one put $P(9.5; \text{December})$.

Exercise 2 (more payoff diagrams)

Part (a): When we short one share, long two calls with exercise price E our portfolio consists of

$$\Pi = -S + 2C\,.$$

At the expiry time T the value of this portfolio is then given by

$$\Pi = -S + 2\max(S - E, 0)\,.$$

From this we see that if at the expiry time T the underlying is of greater value than the exercise price (i.e. $S > E$) then

$$\Pi = -S + 2(S - E) = S - 2E\,.$$

If however, at the expiry time the underlying is of lesser value than the expiry price (i.e. $S < E$) then our portfolio is given by

$$\Pi = -S\,.$$

These two results combined gives the payoff diagram shown in Figure 2.
Part (b): When we long one call and one put, both with an exercise price E, our portfolio will consist of

$$\Pi = C + P\,.$$

At the expiry time T the value of this portfolio (since the call and the put both have the same exercise price E) is given by

$$\Pi = \max(S - E, 0) + \max(E - S, 0)\,.$$

From which we see that if at expiry time if $S > E$ our portfolio is worth $\Pi = S - E$. If at the expiry time $S < E$ then $\Pi = E - S$. Plotting this return gives the payoff diagram in Figure 2.
Part (c): Our portfolio will consist of

$$\Pi = C + 2P\,,$$

so at the expiry time T the value of this portfolio (if the call and the put both have the same exercise price E) is given by

$$\Pi = \max(S - E, 0) + 2\max(E - S, 0)\,.$$

Then at the expiry time if $S > E$ then our portfolio is worth $\Pi = S - E$. If at the expiry time $S < E$ then $\Pi = 2(E - S)$. Plotting this return gives the payoff diagram in Figure 2.
Part (d): Our portfolio will consist of

$$\Pi = P + 2C\,,$$

so at the expiry time T the value of this portfolio (if the call and the put both have the same exercise price E) is given by

$$\Pi = \max(S - E, 0) + 2\max(S - E, 0)\,.$$

Then at the expiry time if $S > E$ then our portfolio is worth $\Pi = 2(S - E)$. If at the expiry time $S < E$ then $\Pi = E - S$. Plotting this return gives the payoff diagram in Figure 2.
Part (e): Our portfolio will consist of

$$\Pi = C(S, t; E_1) + P(S, t; E_2)\,,$$

if $E_1 > E_2$ at the expiry time T the value of this portfolio is given by

$$\Pi = \max(S - E_1, 0) + \max(E_2 - S, 0)\,.$$

Depending on the value of the stock at the expiry time the portfolio can be shown equal to

$$\Pi = \begin{cases} S - E_1 & S > E_1 \\ 0 & E_2 < S < E_1 \\ E_2 - S & S < E_2 \end{cases}$$

Plotting this return gives the payoff diagram in Figure 3. If $E_1 = E_2 = E$ the portfolio can be shown equal to

$$\Pi = \begin{cases} S - E & S > E \\ E - S & S < E \end{cases}$$

Plotting this return gives the payoff diagram in Figure 4. If $E_1 < E_2$ the portfolio can be shown equal to

$$\Pi = \begin{cases} S - E_1 & S > E_2 \\ E_2 - E_1 & E_1 < S < E_2 \\ E_2 - S & S < E_1 \end{cases}$$

Plotting this return gives the payoff diagram in Figure 5.
Part (f): Our portfolio is given by the portfolio in part (e) with the addition of shorting one call and one put.

$$\Pi = C(S, t; E_1) + P(S, t; E_2) - C(S, t; E) - P(S, t; E)\,.$$

Assuming that $E_1 < E_2$ as in part (e) we have three cases for E, $E < E_1$, $E_1 < E < E_2$ or $E > E_2$.

Exercise 3 (some specific solutions to the Black-Scholes equation)

The Black-Scholes equation is given by

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0\,. \tag{4}$$

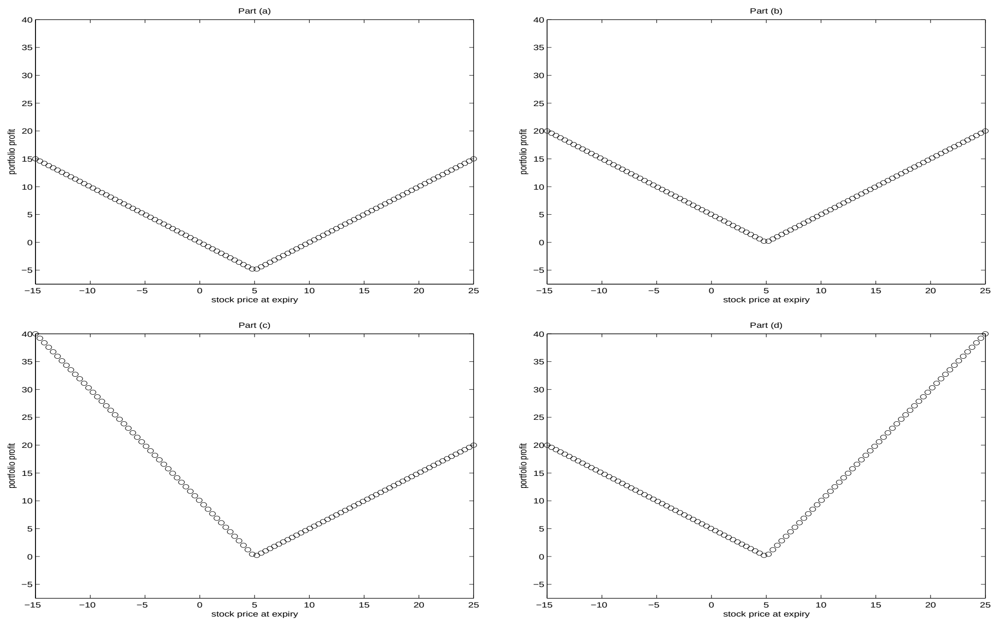

Figure 2: The expiry payoff diagrams for Problem 2 in Chapter 3. Clockwise from top to bottom we have Part (a), Part (b), Part (c), and Part (d).

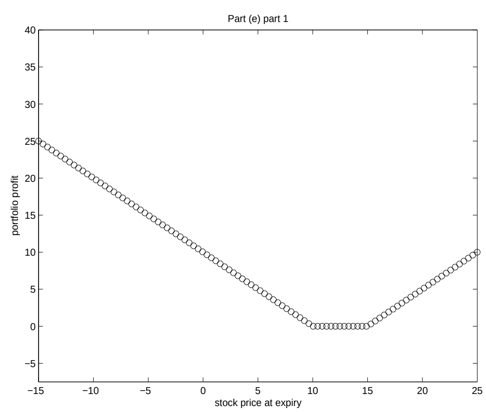

Figure 3: The payoff diagram for problem 2 part e part 1

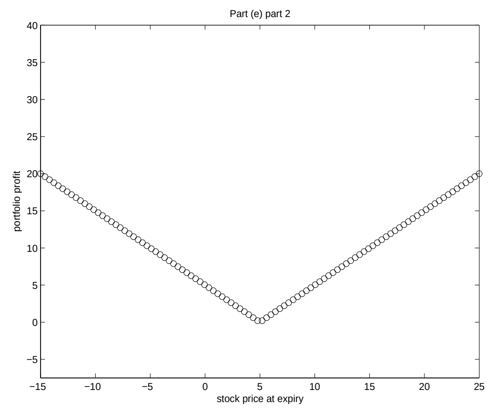

Figure 4: The payoff diagram for problem 2 part e part 2

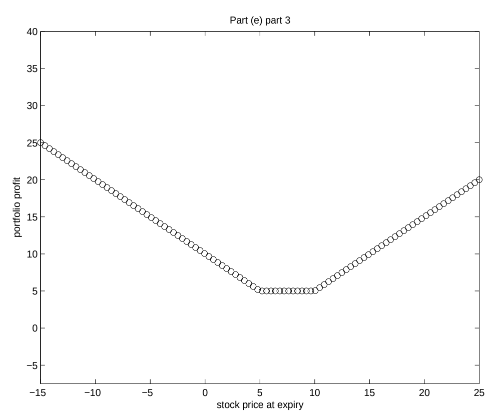

Figure 5: The payoff diagram for problem 2 part e part 3

Part (a): For $V(S,t) = AS$, we can calculated each of the required derivative on the left hand side of this expression as follows

$$\begin{aligned}\frac{\partial V}{\partial t} &= 0 \\ \frac{\partial V}{\partial S} &= A \\ \frac{\partial^2 V}{\partial S^2} &= 0\,.\end{aligned}$$

Thus substituting $V = AS$ into the left hand side of the Black-Scholes equation gives

$$0 + 0 + rSA - rAs = 0\,,$$

showing that $V = AS$ is a solution. We note that this solution represents a pure investment in the underlying. Note that also in this case

$$\Delta = \frac{\partial V}{\partial S} = A\,.$$

Part (b): For $V = Ae^{rt}$ we again evaluate each derivative in tern and find that

$$\begin{aligned}\frac{\partial V}{\partial t} &= rAe^{rt} \\ \frac{\partial V}{\partial S} &= 0 \\ \frac{\partial^2 V}{\partial S^2} &= 0\,,\end{aligned}$$

so placing $V = Ae^{rt}$ into the left hand side of the Black-Scholes equation we obtain

$$rAe^{rt} - rAe^{rt} = 0\,,$$

proving that $V = Ae^{rt}$ is a solution. This solution represents an investment in a fixed interest rate account like a bank account. Note that when $V = Ae^{rt}$ we have $\Delta = 0$.

Exercise 4 (satisfying the Black-Scholes equation)

The equation 3.17 for a call and 3.18 for a put are given by

$$\begin{aligned}C(S,t) &= SN(d_1) - Ee^{-r(T-t)}N(d_2) \qquad (5)\\ P(S,t) &= Ee^{-r(T-t)}N(-d_2) - SN(-d_1) \qquad (6)\end{aligned}$$

with d_1 and d_2 functions of S and t given by

$$\begin{aligned} d_1(S,t) &= \frac{\ln(S/E) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \\ &= \frac{\log(S/E)}{\sigma\sqrt{T-t}} + \frac{(r+\frac{1}{2}\sigma^2)}{\sigma}\sqrt{T-t} \qquad (7) \\ d_2(S,t) &= \frac{\ln(S/E) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \\ &= \frac{\log(S/E)}{\sigma\sqrt{T-t}} + \frac{(r-\frac{1}{2}\sigma^2)}{\sigma}\sqrt{T-t}\,, \qquad (8) \end{aligned}$$

and $N(x)$ given by

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2} dy\,. \qquad (9)$$

We will show that these expressions satisfy the Black-Scholes equation. To do this we will need some derivatives. We find that

$$\begin{aligned} \frac{\partial d_1}{\partial S} &= \frac{1}{S\sigma\sqrt{T-t}} = \frac{\partial d_2}{\partial S} \\ \frac{\partial d_1}{\partial t} &= +\frac{1}{2\sigma}\log(S/E)(T-t)^{-3/2} - \frac{(r+\frac{1}{2}\sigma^2)}{2\sigma}(T-t)^{-1/2} \\ \frac{\partial d_2}{\partial t} &= +\frac{1}{2\sigma}\log(S/E)(T-t)^{-3/2} - \frac{(r-\frac{1}{2}\sigma^2)}{2\sigma}(T-t)^{-1/2} \\ \frac{\partial^2 d_1}{\partial S^2} &= -\frac{1}{S^2\sigma\sqrt{T-t}} = \frac{\partial^2 d_2}{\partial S^2}\,. \end{aligned}$$

To show that the expression for C is a solution to the Black-Scholes equation means that it is in the nullspace of the following operator (the Black-Scholes equation)

$$\mathcal{BS}(V) = \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV\,.$$

To show this expression is zero for a European call we compute each required derivative in tern. We find that the time derivative is given by

$$\frac{\partial C}{\partial t} = SN'(d_1)\frac{\partial d_1}{\partial t} - Ere^{-r(T-t)}N(d_2) - Ee^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial t}\,.$$

The first derivative with respect to S is given by

$$\frac{\partial C}{\partial S} = N(d_1) + SN'(d_1)\frac{\partial d_1}{\partial S} - Ee^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial S}\,.$$

Using the product rule from calculus on the above we find the second derivative of C with respect to S given by

$$\begin{aligned}
\frac{\partial^2 C}{\partial S^2} &= N'(d_1)\frac{\partial d_1}{\partial S} \\
&+ N'(d_1)\frac{\partial d_1}{\partial S} + SN''(d_1)\left(\frac{\partial d_1}{\partial S}\right)^2 + SN'(d_1)\frac{\partial^2 d_1}{\partial S^2} \\
&- Ee^{-r(T-t)}N''(d_2)\left(\frac{\partial d_2}{\partial S}\right)^2 - Ee^{-r(T-t)}N'(d_2)\frac{\partial^2 d_2}{\partial S^2}\,.
\end{aligned}$$

Putting everything into the Black-Scholes equation we get

$$\begin{aligned}
\mathcal{BS}(V) &= SN'(d_1)\frac{\partial d_1}{\partial t} - Ere^{-r(T-t)}N(d_2) - Ee^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial t} \\
&+ \sigma^2 S^2 N'(d_1)\frac{\partial d_1}{\partial S} + \frac{1}{2}\sigma^2 S^3 N''(d_1)\left(\frac{\partial d_1}{\partial S}\right)^2 \\
&+ \frac{1}{2}\sigma^2 S^3 N'(d_1)\frac{\partial^2 d_1}{\partial S^2} - \frac{1}{2}\sigma^2 S^2 Ee^{-r(T-t)}N''(d_2)\left(\frac{\partial d_2}{\partial S}\right)^2 \\
&- \frac{1}{2}\sigma^2 S^2 Ee^{-r(T-t)}N'(d_2)\frac{\partial^2 d_2}{\partial S^2} \\
&+ rSN(d_1) + rS^2 N'(d_1)\frac{\partial d_1}{\partial S} - rSEe^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial S} \\
&- rSN(d_1) + rEe^{-r(T-t)}N(d_2)\,.
\end{aligned}$$

A few terms cancel immediately. To further simplify this, we will convert every expression involving d_2 into an equivalent expression involving d_1. From the definitions of d_1 and d_2 we know that $d_2 = d_1 - \sigma\sqrt{T-t}$, so that the derivatives with respect to S are equivalent i.e.

$$\frac{\partial d_2}{\partial S} = \frac{\partial d_1}{\partial S} \quad \text{and} \quad \frac{\partial^2 d_2}{\partial S^2} = \frac{\partial^2 d_1}{\partial S^2}\,,$$

while the derivative with respect to t are related as

$$\frac{\partial d_2}{\partial t} = \frac{\partial d_1}{\partial t} + \frac{\sigma}{2\sqrt{T-t}}\,. \tag{10}$$

We also need the following result which is derived earlier in this solution manual. In the section on the derivation of the analytic expression for a

European call we show that

$$N'(d_2) = N'(d_1)\left(\frac{S}{E}\right)e^{r(T-t)}\,. \tag{11}$$

From this, taking the S derivative gives

$$N''(d_2)\frac{\partial d_2}{\partial S} = \left(N''(d_1)\frac{\partial d_1}{\partial S}\left(\frac{S}{E}\right) + N'(d_1)\frac{1}{E}\right)e^{r(T-t)}\,.$$

Dividing by $\frac{\partial d_2}{\partial S}$ and remembering that $\frac{\partial d_2}{\partial S} = \frac{\partial d_1}{\partial S}$ we have that

$$N''(d_2) = \left(N''(d_1)\left(\frac{S}{E}\right) + \frac{1}{E}N'(d_1)\frac{1}{\frac{\partial d_1}{\partial S}}\right)e^{r(T-t)}\,.$$

With all of these substitutions the Black-Scholes equation becomes

$$\begin{aligned}
& SN'(d_1)\frac{\partial d_1}{\partial t} - SN'(d_1)\left(\frac{\partial d_1}{\partial t} + \frac{\sigma}{2\sqrt{T-t}}\right) \\
+ \;& \sigma^2 S^2 N'(d_1)\frac{\partial d_1}{\partial S} + \frac{1}{2}\sigma^2 S^3 N''(d_1)\left(\frac{\partial d_1}{\partial S}\right)^2 + \frac{1}{2}\sigma^2 S^3 N'(d_1)\left(\frac{\partial d_1^2}{\partial S^2}\right) \\
- \;& \frac{1}{2}\sigma^2 S^2\left[SN''(d_1) + \frac{N'(d_1)}{\left(\frac{\partial d_1}{\partial S}\right)}\right]\left(\frac{\partial d_1}{\partial S}\right)^2 \\
- \;& \frac{1}{2}\sigma^2 S^3 N'(d_1)\frac{\partial^2 d_1}{\partial S^2} \\
+ \;& rS^2 N'(d_1)\frac{\partial d_1}{\partial S} - rS^2 N'(d_1)\frac{\partial d_1}{\partial S} \\
= \;& -SN'(d_1)\frac{\sigma}{2\sqrt{T-t}} + \sigma^2 S^2 N'(d_1)\frac{\partial d_1}{\partial S} + \frac{1}{2}\sigma^2 S^3 N''(d_1)\left(\frac{\partial d_1}{\partial S}\right)^2 \\
+ \;& \frac{1}{2}\sigma^2 S^3 N'(d_1)\left(\frac{\partial d_1^2}{\partial S^2}\right) \\
- \;& \frac{1}{2}\sigma^2 S^3 N''(d_1)\left(\frac{\partial d_1}{\partial S}\right)^2 - \frac{1}{2}\sigma^2 S^2 N'(d_1)\frac{\partial d_1}{\partial S} \\
- \;& \frac{1}{2}\sigma^2 S^3 N'(d_1)\frac{\partial^2 d_1}{\partial S^2} \\
= \;& -SN'(d_1)\frac{\sigma}{2\sqrt{T-t}} + \frac{1}{2}\sigma^2 S^2 N'(d_1)\frac{\partial d_1}{\partial S} \\
= \;& \frac{N'(d_1)}{2}\left[-\frac{S\sigma}{\sqrt{T-t}} + \sigma^2 S^2\frac{\partial d_1}{\partial S}\right]\,.
\end{aligned}$$

Remembering that $\frac{\partial d_1}{\partial S} = \frac{1}{\sigma S \sqrt{T-t}}$, the above is seen equal to

$$\frac{N'(d_1)}{2}\left[-\frac{S\sigma}{\sqrt{T-t}} + \frac{S\sigma}{\sqrt{T-t}}\right] = 0\,,$$

as was to be shown.

To show that the analytic expression given in the book satisfy the required *boundary conditions* for a European call and put, recall that the boundary conditions for a European call are

$$C(0,t) = 0 \quad \text{and} \quad C(S,t) \sim S - Ee^{-r(T-t)} \quad \text{as} \quad S \to \infty\,.$$

and the boundary condition for a European put are given by

$$P(0,t) = Ee^{-r(T-t)} \quad \text{and} \quad P(S,t) \to 0 \quad \text{as} \quad S \to \infty\,.$$

To check that our analytic solutions for a European call satisfies the boundary conditions above, we begin by noting that for $S \sim 0$ both $d_1(S,t)$ and $d_2(S,t)$ satisfy $d_1(0,t) = d_2(0,t) = -\infty$ (since $\log(S) \to -\infty$ under that limit). This gives the behavior $C(0,t) = 0$ as required. At the other extreme where $S \to \infty$ both d_1 and d_2 as functions of S tend towards $+\infty$ so $N(d_1)$ and $N(d_2)$ both tends towards one. Thus for large S $C(S,t)$ goes like $S - Ee^{-r(T-t)}$ as required. Now lets consider the final condition on a European call. When $t = T$ we see that both d_1 and d_2 tend towards $+\infty$, if $S > E$ otherwise they both go to $-\infty$. Therefore we find that if $S > E$ then $C(S,T) \approx S\cdot 1 - E\cdot 1 = S - E$. While if $S < E$ then $C(S,T) \approx S\cdot 0 - E\cdot 0 = 0$. Combining these two results we can write $C(S,T) = \max(S-E,0)$.

For a European put we have an analytic solution given by

$$P(S,t) = Ee^{-r(T-t)}N(-d_2) - SN(-d_1)\,,$$

with the definition of d_1, d_2 and $N(\cdot)$ as for a European call. For $S \sim 0$ we again have that $d_1(0,t) = d_2(0,t) = -\infty$, so that $N_1(-d_1(0,t)) = N(-d_2(0,t)) = 1$. Therefore

$$P(S,t) \approx Ee^{-r(T-t)} \quad \text{as} \quad S \to 0\,,$$

as required. At the other extreme as $S \to +\infty$ both d_1 and d_2 both go to ∞ so $-d_1$ and $-d_2$ both go to negative ∞. One term in the expression for $P(S,t)$ is easy to evaluate for large S. We see that

$$Ee^{-r(T-t)}N(-d_2) \to 0 \quad \text{as} \quad S \to +\infty\,.$$

It remains to show that the second term in the analytic expression for $P(S,t)$ or the product $SN(-d_1)$ tends to zero as S tends to infinity. Since this is an indeterminate limit (one of the type $\infty \cdot 0$) we need to use L'Hôpital's rule to evaluate it. We find

$$\begin{aligned}
\lim_{S\to\infty} SN(-d_1) &= \lim_{S\to\infty} \frac{N(-d_1(S))}{(1/S)} \\
&= \lim_{S\to\infty} \frac{N'(-d_1(S))(-d_1'(S))}{(-1/S^2)} \\
&= \lim_{S\to\infty} S^2 N'(-d_1(S)) d_1'(S) \\
&= \lim_{S\to\infty} S^2 \left(\frac{1}{\sqrt{2\pi}}\right) e^{-\frac{d_1(S)^2}{2}} \frac{1}{S\sigma\sqrt{T-t}} \\
&= \lim_{S\to\infty} \frac{Se^{-\frac{d_1(S)^2}{2}}}{\sqrt{2\pi}\sigma\sqrt{T-t}} = 0\,.
\end{aligned}$$

Finally lets consider the final condition on a European put if $S > E$ we have $P(S,T) \sim 0 - S\cdot 0 = 0$, while if $S < E$ we have $P(S,T) \sim E\cdot 1\cdot 1 - S = E - S$. Thus we can conclude that for a European put $P(S,T) = \max(E-S, 0)$, as expected.

The Put-Call Parity Equation 3.2 in the book, requires that

$$S + P - C = Ee^{-r(T-t)}\,.$$

We can check if the explicit solutions for puts and calls, i.e. that $P(S,t)$ and $C(S,t)$ as given in the book indeed satisfy this formula. Inserting their analytic form into $P - C$ we have that

$$\begin{aligned}
P - C &= (Ee^{-r(T-t)}N(-d_2) - SN(-d_1)) - (SN(d_1) - Ee^{-r(T-t)}N(d_2)) \\
&= -S\left(N(-d_1) + N(d_1)\right) + Ee^{-r(T-t)}\left(N(-d_2) + N(d_2)\right)\,.
\end{aligned}$$

To simplify this we note that the cumulative normal $N(x)$ has the following symmetry property

$$\begin{aligned}
N(x) + N(-x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-x} e^{-\frac{y^2}{2}} dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy + \frac{1}{\sqrt{2\pi}} \int_{\infty}^{x} e^{-\frac{y^2}{2}} (-dy) \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy = 1\,.
\end{aligned}$$

Where in the second line above we have used the variable transformation $v = -y$ to transform the second integral to one with positive integration limits. The above Put-Call Parity formula $S + P - C$ simplifies to

$$S - S + Ee^{-r(T-t)} = Ee^{-r(T-t)}\,,$$

as required.

Exercise 6 (similarity solutions to the Black-Scholes equation)

The Black-Scholes equation is given by

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0\,.$$

Part (a): If our solution depends on S only i.e $V = V(S)$ when we put this solution into the Black-Scholes equation we find

$$\frac{1}{2}\sigma^2 S^2 \frac{d^2 V}{dS^2} + rS\frac{dV}{dS} - rV = 0\,.$$

This is a Euler differential equation and has solution given by $V(S) = S^p$ for some p. Taking the S derivative of this ansatz gives

$$\begin{aligned} \frac{dV}{dS} &= pS^{p-1} \\ \frac{d^2V}{dS^2} &= p(p-1)S^{p-2}\,, \end{aligned}$$

which we can put into the equation above to obtain

$$\frac{1}{2}\sigma^2 S^2 p(p-1)S^{p-2} + rSpS^{p-1} - rS^p = 0\,.$$

Factoring S^p from the above equation we see that p must satisfy

$$\frac{1}{2}\sigma^2 p(p-1) + rp - r = 0\,, \tag{12}$$

or grouping powers of p we find that p solves

$$\frac{\sigma^2}{2}p^2 + \left(r - \frac{\sigma^2}{2}\right)p - r = 0\,.$$

Solving for p using the quadratic equation we find that two possible values for p are given by

$$p = \frac{-\left(r - \frac{\sigma^2}{2}\right) \pm \sqrt{(r - \frac{\sigma^2}{2})^2 + 4\left(\frac{\sigma^2}{2}\right) r}}{2\left(\frac{\sigma^2}{2}\right)} = \frac{\left(\frac{\sigma^2}{2} - r\right) \pm \left(r + \frac{\sigma^2}{2}\right)}{\sigma^2}.$$

Taking the plus sign in the above we find that one value for p is

$$p_+ = 1,$$

while if we take the minus sign we find that another value for p is

$$p_- = -\frac{2r}{\sigma^2}.$$

Thus the complete solution in this case is given by

$$V(S) = C_1 S + C_2 S^{-\frac{2r}{\sigma^2}},$$

for two arbitrary constants C_1 and C_2, which would need to be determined from initial conditions on $V(S)$.

Part (b): If we assume that our solution can be written as $V = A(t)B(S)$ the required t and S derivatives will give us

$$\begin{aligned}
\frac{\partial V}{\partial t} &= A'(t)B(S) \\
\frac{\partial V}{\partial S} &= A(t)B'(S) \\
\frac{\partial^2 V}{\partial S^2} &= A(t)B''(S).
\end{aligned}$$

When we put these expressions into the Black-Scholes equation we find

$$A'(t)B(S) + \frac{1}{2}\sigma^2 S^2 A(t)B''(S) + rSA(t)B'(S) - rA(t)B(S) = 0.$$

Now dividing the above by $A(t)B(S)$ we find

$$\frac{A'(t)}{A(t)} + \frac{1}{2}\sigma^2 S^2 \frac{B''(S)}{B(S)} + rS\frac{B'(S)}{B(S)} - r = 0.$$

This is in a common form found in many equations. Specifically it is one where we can separate the two dependent variables on either side of the equation. Separating terms that depend on t and S we have that the above equals the following

$$-\frac{A'(t)}{A(t)} = \frac{1}{2}\sigma^2 S^2 \frac{B''(S)}{B(S)} + rS\frac{B'(S)}{B(S)} - r\,.$$

The only way this can be true is if each side of this expression is equal to a constant. Setting each side equal to the constant λ we have then two equations. The first is

$$-\frac{A'(t)}{A(t)} = \lambda$$

so that the differential equation for A is given by

$$A'(t) + \lambda A(t) = 0\,,$$

which has a solution given by $A(t) = C_1 e^{-\lambda t}$, where C_1 is an arbitrary constant. The second equation (the one involving S) is then given by

$$\frac{1}{2}\sigma^2 S^2 B''(S) + rSB'(S) - (r+\lambda)B(S) = 0\,,$$

which is the same type of equation as in Part (a) of this problem. To solve this equation we let $B(S) = S^p$, which when we substitute in and divide the resulting equation by $B(S)$ we find that p that must satisfy

$$\frac{1}{2}\sigma^2 p(p-1) + rp - (r+\lambda) = 0\,,$$

or converting this to a direct quadratic equation for p we find

$$\frac{1}{2}\sigma^2 p^2 + (r - \frac{1}{2}\sigma^2)p - (r+\lambda) = 0\,.$$

When we use the quadratic equation to solve for p we find that p must satisfy (only a few steps of this algebra are shown)

$$\begin{aligned} p &= \frac{-(r-\frac{1}{2}\sigma^2) \pm \sqrt{(r-\frac{1}{2}\sigma^2)^2 + 4\left(\frac{1}{2}\right)\sigma^2(r+\lambda)}}{2\left(\frac{1}{2}\sigma^2\right)} \\ &= \frac{-(r-\frac{1}{2}\sigma^2) \pm \sqrt{(r+\frac{\sigma^2}{2})^2 + 2\sigma^2\lambda}}{\sigma^2}\,. \end{aligned}$$

Then with these two roots p_- and p_+ the general solution is given by

$$V(S,t) = C_1 S^{p_+} e^{-\lambda t} + C_2 S^{p_-} e^{-\lambda t}\,.$$

Note that here p_+ and p_- are functions of r, σ, and λ. The total solution to V can be obtained by superimposing the solutions for various λ's.

Verification: put-call parity greeks (page 48)

From the Put-Call Parity formula $S + P - C = Ee^{-r(T-t)}$, taking the partial derivative of this expression with respect to S on both sides we obtain

$$1 + \frac{\partial P}{\partial S} - \frac{\partial C}{\partial S} = 0\,.$$

Thus if $\frac{\partial C}{\partial S} = N(d_1)$ we have that the derivative of P with respect to S can be expressed as

$$\frac{\partial P}{\partial S} = \frac{\partial C}{\partial S} - 1 = N(d_1) - 1\,,$$

as claimed.

Chapter 4 (Partial Differential Equations)

Exercise 1 (uniqueness of the initial value problem for the heat equation)

Let u_1 and u_2 satisfy Equation 4.3-4.7. That is

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2},$$

$$u(x,0) = u_0(x).$$

With $u_0(x)$ sufficiently well behaved such that $\lim_{|x|\to\infty} u_0(x)e^{-ax^2} = 0$ and finally,

$$\lim_{|x|\to\infty} u(x,\tau)e^{-ax^2} = 0,$$

for any $a > 0$ and $\tau > 0$. Now define the variable v in terms of u_i as $v \equiv u_1 - u_2$. Then computing the derivatives of v we see that

$$\begin{aligned}
\frac{\partial v}{\partial \tau} &= \frac{\partial u_1}{\partial \tau} - \frac{\partial u_2}{\partial \tau} \\
\frac{\partial^2 v}{\partial x^2} &= \frac{\partial^2 u_1}{\partial x^2} - \frac{\partial^2 u_2}{\partial x^2}.
\end{aligned}$$

So that we compute $\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}$ is equivalent to

$$\frac{\partial u_1}{\partial \tau} - \frac{\partial u_2}{\partial \tau} = \frac{\partial^2 u_1}{\partial x^2} - \frac{\partial^2 u_2}{\partial x^2},$$

or

$$\frac{\partial u_1}{\partial \tau} - \frac{\partial^2 u_1}{\partial x^2} = \frac{\partial u_2}{\partial \tau} - \frac{\partial^2 u_2}{\partial x^2}.$$

Now since both sides are zero since each of the individual functions u_i satisfy the diffusion equation.

Since $v \equiv u_1 - u_2$, v has an initial condition given by

$$v(x,0) = u_1(x,0) - u_2(x,0) = u_0(x) - u_0(x) = 0.$$

Now following the suggestions in the book define a function $E(\tau)$ to be

$$E(\tau) = \int_{-\infty}^{\infty} v^2(x,\tau)dx.$$

Now since anything squared is positive (or zero) we see that $E(\tau) \geq 0$ for all τ. Also evaluating E at $\tau = 0$ gives

$$E(0) = \int_{-\infty}^{\infty} v^2(x,0)dx = \int_{-\infty}^{\infty} 0dx = 0$$

since the initial condition for v is zero. Again as suggested in the text, consider $\frac{dE}{d\tau}$. We find (taking the derivative inside the integration) that

$$\frac{dE}{d\tau} = \int_{-\infty}^{\infty} 2v(x,\tau)v_\tau(x,\tau)dx\,.$$

Since $v(x,\tau)$ satisfies the heat equation we know its τ derivative must satisfy $v_\tau = v_{xx}$ and the above is equal to (and then integrating by parts)

$$\begin{aligned}
\frac{dE}{d\tau} &= 2\int_{-\infty}^{\infty} v(x,\tau)v_{xx}(x,\tau)dx \\
&= 2v(x,\tau)v_x(x,\tau)|_{-\infty}^{\infty} - 2\int_{-\infty}^{\infty} v_x(x,\tau)^2 dx\,.
\end{aligned}$$

Assuming that $v(\cdot)$ *has* an x derivative and decays to zero as $|x| \to \infty$ the first term in the above goes to zero and we are left with

$$\frac{dE(\tau)}{d\tau} = -2\int_{-\infty}^{\infty} v_x(x,\tau)^2 dx \leq 0\,.$$

From all these facts we have that the function $E(\tau)$ must satisfy the following $E(0) = 0$, $E(\tau) \geq 0$, and $E'(\tau) \leq 0$. The only function that can satisfy all three of these is $E(\tau) = 0$.

Exercise 2 (stable solutions to the forward/backwards diffusion equation)

Let $v(x,\tau) \equiv \sin(nx)e^{-n^2\tau}$. Then the derivatives of v are given by

$$\begin{aligned}
v_\tau(x,\tau) &= -n^2\sin(nx)e^{-n^2\tau} = -n^2 v(x,\tau) \\
v_x(x,\tau) &= n\cos(nx)e^{-n^2\tau} \\
v_{xx}(x,\tau) &= -n^2\sin(nx)e^{-n^2\tau} = -n^2 v(x,\tau)\,.
\end{aligned}$$

From which we see that $v_\tau = v_{xx}$ and $v(x,\tau)$ satisfies the forward diffusion equation.

Now let $u(x,\tau) \equiv \sin(nx)e^{n^2\tau}$. Then the derivatives of u are given by

$$\begin{aligned} u_\tau(x,\tau) &= n^2\sin(nx)e^{n^2\tau} = n^2 u(x,\tau) \\ u_x(x,\tau) &= n\cos(nx)e^{n^2\tau} \\ u_{xx}(x,\tau) &= -n^2\sin(nx)e^{n^2\tau} = -n^2 u(x,\tau)\,. \end{aligned}$$

So we see that $u_\tau = -u_{xx}$ so $u(x,\tau)$ satisfies the backwards diffusion equation.

As suggested in the book, lets consider the forward diffusion equation $v_t = v_{xx}$ on the interval $-\pi < x < \pi$ with $u = 0$ on the boundaries $x = \pm\pi$ and $u(x,0)$ given. Now $\sin(nx)e^{-n^2\tau}$ is one solution to the linear forward diffusion equation and it turns out that we can create *all* additional solutions superimposing these solutions. We have that any solution to the quoted problem can be written in the form

$$v(x,\tau) = \sum_{n=1}^{\infty} b_n \sin(nx)e^{-n^2\tau}\,.$$

For some set of b_n's. Now given an arbitrary initial condition $v(x,0)$ our task is to select the coefficients b_n so that they satisfy a Fourier sin series i.e.

$$v(x,0) = \sum_{n=1}^{\infty} b_n \sin(nx)\,.$$

Exercise 3 (the fundamental solution satisfied the heat equation)

With $u_\delta(x,\tau)$ defined as

$$u_\delta(x,\tau) \equiv \frac{1}{2\sqrt{\pi\tau}}e^{-\frac{x^2}{4\tau}} \quad -\infty < x < \infty, \tau > 0\,.$$

We have that our τ derivative of this function given by

$$\begin{aligned} \frac{\partial u_\delta}{\partial \tau} &= -\frac{1}{2}\frac{1}{2\sqrt{\pi}}\tau^{-3/2}e^{-\frac{x^2}{4\tau}} + \frac{1}{2\sqrt{\pi\tau}}e^{-\frac{x^2}{4\tau}}\left(\frac{x^2}{4\tau^2}\right) \\ &= \left(-\frac{1}{2\tau} + \frac{x^2}{4\tau^2}\right)\frac{1}{2\sqrt{\pi\tau}}e^{-\frac{x^2}{4\tau}}\,. \end{aligned}$$

Next, we have our first derivative with respect to x is given by

$$\frac{\partial u_\delta}{\partial x} = \frac{1}{2\sqrt{\pi\tau}}\left(\frac{-2x}{4\tau}\right)e^{-\frac{x^2}{4\tau}} = \left(\frac{-x}{2\tau}\right)\frac{1}{2\sqrt{\pi\tau}}e^{-\frac{x^2}{4\tau}}\,.$$

So that our second derivative with respect to x is given by

$$\begin{aligned}\frac{\partial^2 u_\delta}{\partial x^2} &= -\frac{1}{2\tau}\frac{1}{2\sqrt{\pi\tau}}e^{-\frac{x^2}{4\tau}} - \left(\frac{x}{2\tau}\right)\left(\frac{-x}{2\tau}\right)\frac{1}{2\sqrt{\pi\tau}}e^{-\frac{x^2}{4\tau}} \\ &= \left(-\frac{1}{2\tau} + \frac{x^2}{4\tau^2}\right)\frac{1}{2\sqrt{\pi\tau}}e^{-\frac{x^2}{4\tau}}\,.\end{aligned}$$

Using all of the above we see that

$$\frac{\partial u_\delta}{\partial \tau} = \frac{\partial^2 u_\delta}{\partial x^2}\,,$$

thus $u_\delta(x,\tau)$ satisfies the forward diffusion equation as we were asked to show.

Chapter 5 (The Black-Scholes Formula)

Additional Notes and Derivations

The non-dimensional Black-Scholes equation

As suggested in the book lets define unitless parameters x, τ, and v in terms of the given financial parameters E, T, t, S etc. as

$$\begin{aligned} S &= Ee^x \Rightarrow x = \log(S/E) \\ t &= T - \frac{\tau}{\frac{1}{2}\sigma^2} \Rightarrow \tau = \frac{1}{2}\sigma^2(T-t) \\ C(S,t) &= Ev(x,t)\,. \end{aligned}$$

Then with these definitions the time derivatives in Black-Scholes are easily transformed since

$$dt = -\frac{d\tau}{\frac{1}{2}\sigma^2} \Rightarrow d\tau = -\frac{1}{2}\sigma^2 dt\,,$$

while the S derivatives transform using the usual chain rule of calculus as

$$\begin{aligned} \frac{\partial}{\partial S} &= \frac{\partial x}{\partial S}\frac{\partial}{\partial x} = \frac{1}{S}\frac{\partial}{\partial x} \\ \frac{\partial^2}{\partial S^2} &= \frac{\partial}{\partial S}\left(\frac{\partial}{\partial S}\right) = \frac{\partial}{\partial S}\left(\frac{1}{S}\frac{\partial}{\partial x}\right) = -\frac{1}{S^2}\frac{\partial}{\partial x} + \frac{1}{S}\frac{\partial}{\partial S}\left(\frac{\partial}{\partial x}\right) \\ &= -\frac{1}{S^2}\frac{\partial}{\partial x} + \frac{1}{S^2}\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\right)\,. \end{aligned}$$

With these substitutions the Black-Scholes equation becomes

$$-\frac{1}{2}\sigma^2\frac{\partial v}{\partial \tau} + \frac{1}{2}\sigma^2 S^2\left[-\frac{1}{S^2}\frac{\partial v}{\partial x} + \frac{1}{S^2}\frac{\partial^2 v}{\partial x^2}\,.\right] + rS\left[\frac{1}{S}\frac{\partial v}{\partial x}\right] - rv = 0\,,$$

or

$$\frac{\partial v}{\partial \tau} + \frac{\partial v}{\partial x} - \frac{\partial^2 v}{\partial x^2} - \frac{r}{\frac{1}{2}\sigma^2}\frac{\partial v}{\partial x} + \frac{r}{\frac{1}{2}\sigma^2}v = 0\,.$$

Defining $k \equiv \frac{r}{\frac{1}{2}\sigma^2}$ the above becomes

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k-1)\frac{\partial v}{\partial x} - kv = 0\,, \qquad (13)$$

or Equation 5.10 in the book. The initial at $t = T$ conditions at on our European call of $C(T, S) = \max(S - E, 0)$ becomes (remembering that when $t = T$ we have $\tau = 0$)

$$Ev(x, 0) = \max(Ee^x - E, 0) \Rightarrow v(x, 0) = \max(e^x - 1, 0) .$$

Next we will use the substitution $v(x, \tau) = e^{\alpha x + \beta \tau} u(x, \tau)$ to simplify further our differential equation by removing the lower order terms. We find the derivatives of $v(x, \tau)$, defined this way given by

$$\begin{aligned}
v_\tau &= \beta e^{\alpha x + \beta \tau} u(x, \tau) + e^{\alpha x + \beta \tau} u_\tau(x, \tau) \\
v_x &= \alpha e^{\alpha x + \beta \tau} u(x, \tau) + e^{\alpha x + \beta \tau} u_x(x, \tau) \\
v_{xx} &= \alpha^2 e^{\alpha x + \beta \tau} u(x, \tau) + 2\alpha e^{\alpha x + \beta \tau} u_x(x, \tau) + e^{\alpha x + \beta \tau} u_{xx}(x, \tau) .
\end{aligned}$$

Putting these expressions into the above differential equation we get (dividing by $e^{\alpha x + \beta \tau}$) we get

$$\beta u + u_\tau = \alpha^2 u + 2\alpha u_x + u_{xx} + (k - 1)(\alpha u + u_x) - ku .$$

or

$$u_\tau = (-\beta + \alpha^2 + \alpha(k - 1) - k)u + (2\alpha + k - 1)u_x + u_{xx} .$$

From which we see that we can eliminate the lower order terms and produce just the diffusion equation by setting $\beta = \alpha^2 + \alpha(k-1) - k$, and $2\alpha + k - 1 = 0$. Solving the last equation for α gives $\alpha = \frac{1-k}{2}$ which when put into the first equation gives

$$\beta = \left(\frac{1 - k}{2}\right)^2 + \left(\frac{1 - k}{2}\right)(k - 1) - k = -\frac{1}{4}(1 + k)^2 ,$$

as claimed in the book. The initial condition on v transform into an initial condition on u as follows

$$\begin{aligned}
v(x, 0) &= \max(e^x - 1, 0) \\
e^{\alpha x} u(x, 0) &= \max(e^x - 1, 0) \\
e^{-\frac{1}{2}(k-1)x} u(x, 0) &= \max(e^x - 1, 0) \\
u(x, 0) &= \max(e^{\frac{1}{2}(k+1)x} - e^{-\frac{1}{2}(k-1)x}, 0) ,
\end{aligned}$$

Which is Equation 5.11 in the book. Given this initial condition we can use the Greens function for the diffusion equation to explicitly solve the Black-Scholes equation. Following the book we see that the solution $u(x,\tau)$ is given by

$$u(x,\tau) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u_0(x+\sqrt{2\tau}x')e^{-\frac{x'^2}{2}}dx' .$$

To evaluate this integral consider the fact that the maximum in the definition of the integrand $u_0(x)$ above will be the exponential part and not zero (by definition) when

$$e^{\frac{1}{2}(k+1)x} - e^{\frac{1}{2}(k-1)x} > 0 ,$$

or taking logarithms of this and simplifying some

$$(k+1)x > (k-1)x ,$$

by subtracting kx from both sides this becomes $x > -x$ which is true when $x > 0$. Thus the integrand is $u_0(\cdot)$ evaluated at $x + \sqrt{2\tau}x'$ has as its max the exponential terms (and not zero) when

$$x + \sqrt{2\tau}x' > 0 \Rightarrow x' > -\frac{x}{\sqrt{2\tau}} .$$

Thus the above integral becomes

$$\begin{aligned} u(x,\tau) &= \frac{1}{\sqrt{2\pi}}\int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} \left(e^{\frac{1}{2}(k+1)(x'\sqrt{2\tau}+x)} - e^{\frac{1}{2}(k-1)(x'\sqrt{2\tau}+x)}\right) e^{-\frac{1}{2}x'^2}dx' \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{\frac{1}{2}(k+1)(x'\sqrt{2\tau}+x)}e^{-\frac{1}{2}x'^2}dx' \\ &- \frac{1}{\sqrt{2\pi}}\int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{\frac{1}{2}(k-1)(x'\sqrt{2\tau}+x)}e^{-\frac{1}{2}x'^2}dx' , \end{aligned}$$

which is the expression given in the text. Defining exponential of the first integral as $\eta \equiv -\frac{1}{2}x'^2 + \frac{1}{2}(k+1)\sqrt{2\tau}x' + \frac{1}{2}(k+1)x$ we see that it simplifies as follows

$$\begin{aligned}
\eta &= -\frac{1}{2}\left(x'^2-(k+1)\sqrt{2\tau}x'\right)+\frac{1}{2}(k+1)x \\
&= -\frac{1}{2}\left(x'^2-(k+1)\sqrt{2\tau}x'+\left(\frac{(k+1)\sqrt{2\tau}}{2}\right)^2\right) \\
&+ \frac{1}{2}\left(\frac{(k+1)\sqrt{2\tau}}{2}\right)^2+\frac{1}{2}(k+1)x \\
&= -\frac{1}{2}\left(x'-\frac{(k+1)\sqrt{2\tau}}{2}\right)^2+\frac{1}{2}(k+1)x+\frac{1}{4}(k+1)^2\tau\,.
\end{aligned}$$

Using this, the first integral above $I_1 \equiv \frac{1}{\sqrt{2\pi}}\int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{\frac{1}{2}(k+1)(x'\sqrt{2\tau}+x)}e^{-\frac{1}{2}x'^2}dx'$ becomes

$$\begin{aligned}
I_1 &= \frac{1}{\sqrt{2\pi}}\int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{-\frac{1}{2}\left(x'-\frac{(k+1)\sqrt{2\tau}}{2}\right)^2}e^{\frac{1}{2}(k+1)x+\frac{1}{4}(k+1)^2\tau}dx' \\
&= \frac{e^{\frac{1}{2}(k+1)x+\frac{1}{4}(k+1)^2\tau}}{2\sqrt{\pi}}\int_{-\frac{x}{\sqrt{2\tau}}-\frac{1}{2}(k+1)\sqrt{2\tau}}^{\infty} e^{-\frac{1}{2}\rho^2}d\rho \\
&= e^{\frac{1}{2}(k+1)x+\frac{1}{4}(k+1)^2\tau}N(d_1)\,,
\end{aligned}$$

with d_1 defined as

$$d_1(x,\tau)=\frac{x}{\sqrt{2\tau}}+\frac{1}{2}(k+1)\sqrt{2\tau}\,,$$

and $N(\cdot)$ the cumulative distribution function for the standard normal, i.e.

$$N(d)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d} e^{-\frac{1}{2}s^2}ds\,.$$

The second integral is integrated in exactly the same way as the first but with $k+1$ replaced with $k-1$. With the explicit solution to the pure diffusion equation given above we can now begin to extract the solution to the Black-Scholes equation in terms of the financial variables. To do this we compute the function $v(x,\tau)$ from the function $u(x,\tau)$. We find

$$v(x,\tau)=e^{-\frac{1}{2}(k-1)x-\frac{1}{4}(k+1)^2\tau}\left(e^{\frac{1}{2}(k+1)x+\frac{1}{4}(k+1)^2\tau}N(d_1)-e^{\frac{1}{2}(k-1)x+\frac{1}{4}(k-1)^2\tau}N(d_2)\right)\,,$$

with

$$d_2(x,\tau) = \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k-1)\sqrt{2\tau}\,.$$

Performing algebraic simplifications on the above we find that

$$\begin{aligned} v(x,\tau) &= e^x N(d_1) - e^{\frac{1}{4}((k-1)^2-(k+1)^2)\tau} N(d_2) \\ &= e^x N(d_1) - e^{\frac{1}{4}(-4k)\tau} N(d_2) \\ &= e^x N(d_1) - e^{-k\tau} N(d_2)\,. \end{aligned}$$

In terms of the original variables $x = \log(S/E)$, $\tau = \frac{1}{2}\sigma^2(T-t)$, $C = Ev(x,\tau)$ and $k = \frac{r}{\frac{1}{2}\sigma^2}$, and we find

$$\begin{aligned} C(S,t)/E &= \frac{S}{E} N(d_1(S,t)) - e^{-(T-t)r} N(d_2(S,t)) \quad \text{or} \\ C(S,t) &= SN(d_1(S,t)) - Ee^{-r(T-t)} N(d_2(S,t))\,. \end{aligned}$$

Here d_1 is considered as a function of S and t given by

$$\begin{aligned} d_1(S,t) &= \frac{\log(S/E)}{\sigma\sqrt{T-t}} + \frac{1}{2}\left(\frac{r}{\frac{1}{2}\sigma^2} + 1\right)\sigma\sqrt{T-t} \quad \text{or} \\ &= \frac{\log(S/E) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\,. \end{aligned}$$

The expression for d_2 is computed in the same way.

For a European put, the payoff in terms of the original financial variables is $P(S,T) = \max(0, E-S)$. Using the same non-dimensionalization substitution as before we let $v(x,\tau) = \frac{P(S,t)}{E}$ so that when $t = T$ we have $\tau = 0$ and the final condition or payoff on $P(S,t)$ becomes an initial condition on $v(x,\tau)$ of

$$v(x,0) = \frac{P(S,T)}{E} = \max(0, 1 - \frac{S}{E})\,.$$

with $S = Ee^x$, this simplifies further to

$$v(x,0) = \max(0, 1 - e^x)\,.$$

Since the functions $v(x,\tau)$ and $u(x,\tau)$ are related by $v(x,\tau) = e^{\alpha x + \beta\tau} u(x,\tau)$, this initial condition on v becomes an initial condition on u. Namely,

$$v(x,0) = \max(0, 1-e^x) = e^{\alpha x} u(x,0) = e^{-\frac{1}{2}(k-1)x} u(x,0)\,,$$

which gives an initial condition on u of

$$u(x,0) = \max(e^{\frac{1}{2}(k-1)x} - e^{\frac{1}{2}(k+1)x}, 0)\,,$$

as claimed in the text.

Since we know the analytical expression for a European call we can use Put-Call parity to derive the analytic expression for a European put. From Put-Call parity we have that $P = C - S + Ee^{-r(T-t)}$, when we put in the expression for $C(S,t)$ we find

$$\begin{aligned} P(S,t) &= S(N(d_1) - 1) - Ee^{-r(T-t)}(N(d_2) - 1) \\ &= -SN(-d_1) + Ee^{-r(T-t)}N(-d_2)\,, \end{aligned}$$

where we have used the fact that $N(d) + N(-d) = 1$.

If we assume a general option payoff given by $\Lambda(S)$ then using the Black-Scholes framework above we can price an option with this payoff. By definition this payoff function is the value of this option when $t = T$, that is $V(S,T) = \Lambda(S)$. For *arbitrary* times the function $V(S,t)$ is related to our unitless function and solution to the pure diffusion equation $u(x,\tau)$ by

$$V(S,t) = Ee^{\alpha x + \beta\tau}u(x,\tau) \Rightarrow u(x,\tau) = e^{-\alpha x - \beta\tau}\left(\frac{V(S,t)}{E}\right)\,.$$

When $\tau = 0$ (equivalently $t = T$) the initial condition on $u(x,\tau)$ is given by

$$\begin{aligned} u(x,0) &= e^{-\alpha x}\left(\frac{V(S,T)}{E}\right) = e^{\frac{1}{2}(k-1)x}\left(\frac{\Lambda(S)}{E}\right) \\ &= e^{\frac{1}{2}(k-1)x}\left(\frac{\Lambda(Ee^x)}{E}\right)\,. \end{aligned}$$

Where in the last step we remember that S is a function of x related by $S = Ee^x$. Using the Green's function representation of the solution to the pure diffusion equation with the above initial condition to determine $u(x,\tau)$ we find that

$$u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}}\int_{-\infty}^{\infty} e^{\frac{1}{2}(k-1)\xi}\left(\frac{\Lambda(Ee^\xi)}{E}\right)e^{-\frac{(x-\xi)^2}{4\tau}}d\xi\,.$$

For $V(S,t)$ this becomes the following

$$\begin{aligned} V(S,t) &= Ee^{\alpha x + \beta t}u(x,\tau) \\ &= \frac{Ee^{\alpha x}e^{\beta\tau}}{2\sqrt{\pi\tau}}\int_{-\infty}^{\infty}\left(\frac{\Lambda(Ee^\xi)}{E}\right)e^{\frac{1}{2}(k-1)\xi}e^{-\frac{(x-\xi)^2}{4\tau}}d\xi \end{aligned}$$

As a guide to the following transformations we will perform to simplify this expression we will keep the $e^{\beta\tau}$ term outside the integral while we will bring the $e^{\alpha x}$ term inside and complete the square on the expression in the exponent. That is, since $\alpha = -\frac{1}{2}(k-1)$ we see that we are considering

$$V(S,t) = \frac{e^{\beta\tau}}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} \Lambda(Ee^{\xi}) e^{\frac{1}{2}(k-1)(\xi-x)} e^{-\frac{(x-\xi)^2}{4\tau}} d\xi \,.$$

If we define the exponent of the exponential integrand above as $\mathcal{P}$ (for power) it can be simplified by completing the square in the expression $\xi - x$ as

$$\begin{aligned}
\mathcal{P} &\equiv -\frac{(x-\xi)^2}{4\tau} + \frac{1}{2}(k-1)(\xi-x) \\
&= -\frac{1}{4\tau}\left[(\xi-x)^2 - 2\tau(k-1)(\xi-x)\right] \\
&= -\frac{1}{4\tau}\left[(\xi-x)^2 - 2\tau(k-1)(\xi-x) + \tau^2(k-1)^2\right] + \frac{\tau(k-1)^2}{4} \\
&= -\frac{1}{4\tau}\left[(\xi-x) - \tau(k-1)\right]^2 + \frac{\tau(k-1)^2}{4} \,.
\end{aligned}$$

Thus our expression for $V(S,t)$ becomes

$$V(S,t) = \frac{e^{\beta\tau} e^{\frac{1}{4}(k-1)^2\tau}}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} \Lambda(Ee^{\xi}) e^{-\frac{1}{4\tau}[(\xi-x)-\tau(k-1)]^2} d\xi$$

We now recall the definitions of all constants involved. We have $\beta = -\frac{1}{4}(k+1)^2$ so that the exponential function in the front of our integral becomes

$$e^{(\beta+\frac{1}{4}(k-1)^2)\tau} = e^{-k\tau} \,.$$

Since $k = \frac{r}{\frac{1}{2}\sigma^2}$ and $\tau = \frac{\sigma^2}{2}(T-t)$ the product of k and τ is $r(T-t)$. We also see that the expression $\tau(k-1)$ becomes,

$$\tau(k-1) = \frac{\sigma^2}{2}(T-t)\left(\frac{r}{\frac{1}{2}\sigma^2} - 1\right) = (T-t)\left(r - \frac{1}{2}\sigma^2\right) \,,$$

and the expression 4τ is $2\sigma^2(T-t)$. Finally letting $S' = Ee^{\xi}$ and $S = Ee^{x}$ so that $\xi - x = \log(S'/S)$, and $dS' = Ee^{\xi}d\xi = S'd\xi$ our integral above becomes

$$V(S,t) = \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_0^{\infty} \Lambda(S') e^{-[\log(S'/S)-(r-\frac{1}{2}\sigma^2)(T-t)]^2/2\sigma^2(T-t)} \frac{dS'}{S'} \,,$$

the same expression as in the book in the section on binary options.

To compute the Δ of this general expression for the option price we take the partial derivative of this expression with respect to S. We find that

$$\begin{aligned}
\Delta &= \frac{\partial V}{\partial S} = \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_0^\infty \Lambda(S') e^{-[\log(S'/S)-(r-\frac{1}{2}\sigma^2)(T-t)]^2/2\sigma^2(T-t)} \\
&\times \left(\frac{\log(S'/S)-(r-\frac{1}{2}\sigma^2)(T-t)}{\sigma^2(T-t)S}\right)\frac{dS'}{S'} \\
&= \frac{e^{-r(T-t)}}{\sigma^3\sqrt{2\pi}(T-t)^{3/2}S} \int_0^\infty \Lambda(S')\log(S'/S) e^{-[\log(S'/S)-(r-\frac{1}{2}\sigma^2)(T-t)]^2/2\sigma^2(T-t)} \frac{dS'}{S'} \\
&- \frac{e^{-r(T-t)}(r-\frac{1}{2}\sigma^2)}{\sigma^3\sqrt{2\pi(T-t)}S} \int_0^\infty \Lambda(S') e^{-[\log(S'/S)-(r-\frac{1}{2}\sigma^2)(T-t)]^2/2\sigma^2(T-t)} \frac{dS'}{S'} \\
&= \frac{e^{-r(T-t)}}{\sigma^3\sqrt{2\pi}(T-t)^{3/2}S} \int_0^\infty \Lambda(S')\log(S'/S) e^{-[\log(S'/S)-(r-\frac{1}{2}\sigma^2)(T-t)]^2/2\sigma^2(T-t)} \frac{dS'}{S'} \\
&- \frac{1}{\sigma^2 S}\left(r-\frac{1}{2}\sigma^2\right)V(S,t)\,.
\end{aligned}$$

Given a specific functional form for $\Lambda(S)$ the former integral could be evaluated with the substitution $v = \log(S'/S)$, by writing S' in terms of $\log(S'/S)$ as

$$\Lambda(S') = \Lambda\left(S\left(\frac{S'}{S}\right)\right) = \Lambda\left(Se^{\log(S'/S')}\right)\,.$$

Any $\Lambda(\cdot)$ that is a polynomial in its argument could be evaluated using similar methods from this section.

If our payoff $\Lambda(S)$ is a step function at the strike E i.e. $\Lambda(S) = B\mathcal{H}(S-E)$ (as in the definition of a binary option) from the general expression for the evaluation of the option price above we see that

$$\begin{aligned}
V(S,t) &= \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_0^\infty B\mathcal{H}(S'-E) e^{-[\log(S'/S)-(r-\frac{1}{2}\sigma^2)(T-t)]^2/2\sigma^2(T-t)} \frac{dS'}{S'} \\
&= \frac{Be^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_E^\infty e^{-[\log(S'/S)-(r-\frac{1}{2}\sigma^2)(T-t)]^2/2\sigma^2(T-t)} \frac{dS'}{S'}\,.
\end{aligned}$$

To evaluate this integral introduce an integration variable v (unrelated to the variable V for option price) such that

$$\begin{aligned} v &= \frac{-\log(S'/S) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \quad \text{and} \\ dv &= \frac{dv}{dS'}dS' = -\frac{1}{\sigma\sqrt{T-t}}\frac{dS'}{S'}\,. \end{aligned}$$

With this our logarithmic differential becomes $\frac{dS'}{S'} = -\sigma\sqrt{T-t}\,dv$ and our integral above transforms to

$$V(S,t) = -\frac{Be^{-r(T-t)}}{\sqrt{2\pi}}\int_{d_2(S,t)}^{-\infty} e^{-v^2/2}dv = Be^{-r(T-t)}N(d_2(S,t))\,.$$

when we recall the definition of $d_2(S,t)$. This is the same expression quoted in the book.

Problem Solutions

Exercise 1 (similarity solutions)

We desire to find a similarity solution to

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \quad -\infty < x < \infty, \tau > 0\,,$$

with initial conditions given by the Heaviside step function $H(x)$ defined as

$$H(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}\,,$$

Following the discussion in the book we seek a solution that has the following functional form $u(x,t) = \tau^\alpha U(x/\tau^\beta)$ for some to be determined function $U(\cdot)$. A solution of this form must obviously satisfy the differential equation, the boundary conditions, and the initial conditions. We begin by discussing requirements on U needed to have the *initial condition* depend only on U's argument, $\xi \equiv x/\tau^\beta$. When $\tau = 0$ the expression $x/\tau^\beta = \pm\infty$, with a sign that depends on the sign of the variable x. If $x > 0$ we have

$$\lim_{\tau\to 0^+} x/\tau^\beta = +\infty\,,$$

while when $x < 0$ we have

$$\lim_{\tau \to 0^+} x/\tau^\beta = -\infty\,.$$

So to have the entire expression for u i.e. $\tau^\alpha U(x/\tau^\beta)$ depend only on ξ when $\tau = 0$ we have two conditions. The the requirement when $x < 0$ is that $U(-\infty)$ must be *bounded* so that the entire expression equals zero as required by the initial condition when x is negative. For positive x we require that

$$\lim_{\tau \to 0} \tau^\alpha U(x/\tau^\beta) = \lim_{\tau \to 0} \tau^\alpha U(+\infty) = +1\,.$$

This condition forces us to take $\alpha = 0$. Using this value for α when $x < 0$ we conclude that U itself must vanish and we have that $U(-\infty) = 0$. Thus $U(\cdot)$ must satisfy the boundary conditions

$$U(-\infty) = 0 \quad \text{and} \quad U(+\infty) = 1\,.$$

Since $\alpha = 0$ the functional form for $u(x,t)$ is $U(x/\tau^\beta)$. The derivatives of this expression are given by

$$\begin{aligned} u_t &= U'(x/\tau^\beta)(-\beta)x\tau^{-\beta-1} \\ u_x &= U'(x/\tau^\beta)1/\tau^\beta \\ u_{xx} &= U''(x/\tau^\beta)1/\tau^{2\beta}\,, \end{aligned}$$

which when we put into the differential equation gives

$$\frac{1}{\tau^{2\beta}} U''(\xi) = U'(\xi)(-\beta)x\tau^{-\beta-1}\,.$$

Solving for $U''(\xi)$ and introducing the variable ξ we obtain

$$\begin{aligned} U''(\xi) &= -\beta U'(\xi)\left(\frac{x}{\tau^\beta}\right)\frac{\tau^{2\beta}}{\tau} \\ &= -\beta U'(\xi)\xi\tau^{2\beta-1}\,. \end{aligned}$$

For this equation to be independent of τ we must take $2\beta - 1 = 0$, or $\beta = 1/2$. Our differential equation for $U(\cdot)$ now becomes

$$U''(\xi) = -\frac{1}{2}\xi U'(\xi)\,.$$

This is the *same* equation as 5.5 from the book but with the boundary conditions given by $U(-\infty) = 0$ and $U(\infty) = +1$. Using the solution of the above differential equation presented in the book we have that $U(\cdot)$ is given by

$$U(\xi) = C \int_0^{\xi} e^{-s^2/4} ds + D \, .$$

Evaluating U at the boundary condition $\xi = -\infty$ gives

$$U(-\infty) = C \int_0^{-\infty} e^{-s^2/4} ds + D = -C\sqrt{\pi} + D = 0 \, .$$

While the boundary condition of $\xi = +\infty$ gives

$$U(\infty) = C \int_0^{\infty} e^{-s^2/4} ds + D = C\sqrt{\pi} + D = 1 \, .$$

Solving these two equations for C and D using Crammer's rule we have

$$\begin{aligned} C &= \frac{\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} -\sqrt{\pi} & 1 \\ \sqrt{\pi} & 1 \end{vmatrix}} = \frac{-1}{-\sqrt{\pi} - \sqrt{-\pi}} = \frac{1}{2\sqrt{\pi}} \\ D &= \frac{1}{-2\sqrt{\pi}} \begin{vmatrix} -\sqrt{\pi} & 0 \\ \sqrt{\pi} & 1 \end{vmatrix} = \frac{-\sqrt{\pi}}{-2\sqrt{\pi}} = \frac{1}{2} \, . \end{aligned}$$

So we finally arrive at $U(\cdot)$ the following expression

$$U(\xi) = \frac{1}{2\sqrt{\pi}} \int_0^{\xi} e^{-s^2/4} ds + \frac{1}{2} \, .$$

Converting back to x and t this is given by

$$u(x,t) = \frac{1}{2\sqrt{\pi}} \int_0^{x/\sqrt{\tau}} e^{-s^2/4} ds + \frac{1}{2} \, .$$

Lets now compute $\frac{\partial u}{\partial x}$ by direct differentiation. We see that it is given by

$$\frac{\partial u}{\partial x} = \frac{1}{2\sqrt{\pi}} e^{-x^2/4\tau} \frac{1}{\sqrt{\tau}} = \frac{1}{2\sqrt{\pi\tau}} e^{-x^2/4\tau} \, .$$

which is *equal* to $u_\delta(x,t)$ the fundamental solution to the heat equation.

Following the other path suggested by the book we can also derive the differential equation that $\frac{\partial u}{\partial x}$ satisfies from the given differential equation and boundary conditions. To construct an initial value problem for $\frac{\partial u}{\partial x}$ take the x derivative of the given heat equation $u_\tau = u_{xx}$ to obtain

$$\frac{\partial}{\partial \tau}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial x}\right) .$$

Taking the x derivative of the initial conditions and remembering that the Heaviside function $H(x)$ can be written as

$$H(x) \equiv \int_{-\infty}^{x} \delta(\xi) d\xi \, ,$$

where $\delta(\xi)$ is the delta function centered at zero. Thus we see that

$$\frac{\partial H(x)}{\partial x} = \delta(x) .$$

Thus $\frac{\partial u}{\partial x}$ has an initial condition given by $\frac{\partial u}{\partial x}(x,0) = \delta(x)$. Combining all of this we see that $\frac{\partial u}{\partial x}$ satisfies the same equation and initial conditions as $u_\delta(x,\tau)$ so by uniqueness of solutions to linear differential equations it must equal it.

Exercise 2 (another example Green's function)

We are told that $u(x,\tau)$ satisfies the following diffusion differential equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \quad x > 0, \tau > 0 \, ,$$

with a semi-infinite initial value given by $u(x,0) = u_0(x)$ for $x > 0$ and $u(0,\tau) = 0$, for $\tau > 0$. As suggested in the book define $v(x,\tau)$ to be a reflection across the line $x = 0$ so that

$$v(x,\tau) = \begin{cases} u(x,\tau) & x > 0 \\ -u(-x,\tau) & x < 0 \end{cases}$$

Evaluating v at $x = 0$ (with limits from the left and right if needed), we have that $v(0,\tau) = u(0,\tau)$ or $-u(0,\tau)$, depending on the limiting direction taken. In either case both of these expressions are zero so that $v(0,\tau) = 0$. To use

Equation 5.7 in this context we note that v is defined *for all* x, satisfies the diffusion equation (this can be seen by taking the required derivatives) and has an initial value function $v_0(x)$ given by

$$v_0(x) \equiv v(x,0) = \begin{cases} u(x,0) = u_0(x) & x > 0 \\ -u(-x,0) = -u_0(-x) & x < 0 \end{cases} .$$

Thus $v(x,\tau)$ can be written using Equation 5.7 (the Green's function for the infinite interval problem) as

$$v(x,\tau) = \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} v_0(s) e^{-\frac{(x-s)^2}{4\tau}} ds .$$

Breaking this integral up into two parts across zeros based on the definition of $v_0(\cdot)$ given above we have

$$\begin{aligned} v(x,\tau) &= \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{0} v_0(s) e^{-\frac{(x-s)^2}{4\tau}} ds + \frac{1}{2\sqrt{\pi\tau}} \int_{0}^{\infty} v_0(s) e^{-\frac{(x-s)^2}{4\tau}} ds \\ &= -\frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{0} u_0(-s) e^{-\frac{(x-s)^2}{4\tau}} ds + \frac{1}{2\sqrt{\pi\tau}} \int_{0}^{\infty} v_0(s) e^{-\frac{(x-s)^2}{4\tau}} ds . \end{aligned}$$

In the first integral make the change of variables $s' = -s$, so that $ds' = -ds$ and find that it equals

$$-\frac{1}{2\sqrt{\pi\tau}} \int_{\infty}^{0} u_0(s') e^{-\frac{(x+s')^2}{4\tau}} (-ds') = -\frac{1}{2\sqrt{\pi\tau}} \int_{0}^{\infty} u_0(s') e^{-\frac{(x+s')^2}{4\tau}} ds' .$$

When we put this back with the other part of $v(x,\tau)$ we get a total for $v(x,\tau)$ the following

$$v(x,\tau) = \frac{1}{2\sqrt{\pi\tau}} \int_{0}^{\infty} u_0(s) \left(e^{-\frac{(x-s)^2}{4\tau}} - e^{-\frac{(x+s)^2}{4\tau}} \right) ds ,$$

as we were to show.

Exercise 3 (similarity solutions of the forced diffusion equation)

Given the partial differential equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + F(x) \quad x > 0, \tau > 0 ,$$

with $u(x,0)=0$ for $x>0$ and $u(0,\tau)=0$ for $\tau>0$ we will attempt to use the method of similarity solutions to solve this equation analytically. Following the technical point in this chapter we seek a solution to this equation of the form $u(x,\tau)=\tau^\alpha f(x/\tau^\beta)$. The condition $u(0,\tau)=0$ requires that $f(0)=0$. The condition $u(x,0)=0$ requires that $\lim_{\tau\to 0}\tau^\alpha f(x/\tau^\beta)=0$. Taking the derivatives of the hypothesized expression for $u(x,\tau)$ we find

$$\begin{aligned}
u_\tau &= \alpha\tau^{\alpha-1}f(x/\tau^\beta)+\tau^\alpha f'(x/\tau^\beta)(-\beta x\tau^{-\beta-1}) \\
u_x &= \tau^\alpha f'(x/\tau^\beta)1/\tau^\beta \\
u_{xx} &= f''(x/\tau^\beta)\tau^\alpha/\tau^{2\beta}=\tau^{\alpha-2\beta}f''(x/\tau^\beta)\,,
\end{aligned}$$

Defining $\xi\equiv x/\tau^\beta$ and putting these expressions into the differential equation gives

$$\alpha\tau^{\alpha-1}f(\xi)-\beta x\tau^{\alpha-\beta-1}f'(\xi)=\tau^{\alpha-2\beta}f''(\xi)+F(x)\,.$$

Since $x=\xi\tau^\beta$ the above becomes

$$\alpha\tau^{\alpha-1}f(\xi)-\beta\xi\tau^{\alpha-1}f'(\xi)=\tau^{\alpha-2\beta}f''(\xi)+F(\xi\tau^\beta)\,.$$

Part (a): If $F(x)=x$, and after dividing this equation by $\tau^{\alpha-1}$ the above becomes

$$\alpha f(\xi)-\beta\xi f'(\xi)=\tau^{-2\beta+1}f''(\xi)+\xi\tau^{\beta-\alpha+1}\,.$$

To have this expression independent of τ and x requires that

$$\begin{aligned}
-2\beta+1 &= 0 \\
\beta-\alpha+1 &= 0\,.
\end{aligned}$$

Thus $\beta=\frac{1}{2}$ and $\alpha=\beta+1=\frac{3}{2}$ and our final functional form for $u(x,\tau)$ looks like

$$u(x,\tau)=\tau^{3/2}u(x/\sqrt{\tau})\,,$$

with the final differential equation for $f(\cdot)$ given by

$$\frac{3}{2}f(\xi)-\frac{\xi}{2}f'(\xi)=f''(\xi)+\xi\,.$$

Putting all of the derivative terms to one side gives

$$f''(\xi)+\frac{\xi}{2}f'(\xi)-\frac{3}{2}f(\xi)=-\xi\,. \tag{14}$$

To solve this we first consider the homogeneous equation obtained by setting the right hand side of the above equal to zero or

$$f''(\xi) + \frac{\xi}{2} f'(\xi) - \frac{3}{2} f(\xi) = 0\,.$$

Next we solve the inhomogeneous problem by using a trial solution of $f_{\text{ih}}(\xi) = A\xi + B$. Putting such a solution into the above equation gives

$$\frac{\xi}{2} A - \frac{3}{2}(A\xi + B) = -\xi\,,$$

which requires that $A = 1$ and $B = 0$, so $f_{\text{ih}}(\xi) = \xi$.
Part (b): If $F(x) = 1$ the above becomes

$$\alpha\tau^{\alpha-1} f(\xi) - \beta\xi\tau^{\alpha-1} f'(\xi) = \tau^{\alpha-2\beta} f''(\xi) + 1\,.$$

To have this equation independent of x and τ requires $\alpha = 1$ and $\beta = \frac{1}{2}$ giving

$$f(\xi) - \frac{1}{2}\xi f'(\xi) = f''(\xi) + 1\,.$$

Putting all of the derivative terms to one side gives

$$f''(\xi) + \frac{\xi}{2} f'(\xi) - f(\xi) = -1\,. \tag{15}$$

To solve this we first consider the homogeneous equation obtained by setting the right hand side of the above equal to zero or

$$f''(\xi) + \frac{\xi}{2} f'(\xi) - f(\xi) = 0\,.$$

Next we solve the inhomogeneous problem by using a trial solution of $f_{\text{ih}}(\xi) = A$. Putting such a solution into the above equation gives

$$-A = -1 \Rightarrow A = 1\,,$$

so $f_{\text{ih}}(\xi) = 1$.

Exercise 4 (reducing to the pure diffusion equation)

Consider the general parabolic equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + a\frac{\partial u}{\partial x} + bu\,.$$

with a and b constants. In this problem we will show how to reduce this equation to the pure diffusion equation

$$\frac{\partial \hat{u}}{\partial \hat{\tau}} = \frac{\partial^2 \hat{u}}{\partial \hat{x}^2}\,,$$

through a suitable change of coordinates. We begin by moving the term bu to the left hand side of the equation as

$$\frac{\partial u}{\partial \tau} - bu = \frac{\partial^2 u}{\partial x^2} + a\frac{\partial u}{\partial x}\,.$$

This left hand side has an integrating factor given by $e^{-b\tau}$. Thus multiply both sides by this factor to obtain

$$e^{-b\tau}\frac{\partial u}{\partial \tau} - bue^{-b\tau} = e^{-b\tau}\frac{\partial^2 u}{\partial x^2} + e^{-b\tau}a\frac{\partial u}{\partial x}\,,$$

or

$$\frac{\partial(e^{-b\tau}u)}{\partial \tau} = \frac{\partial^2(e^{-b\tau}u)}{\partial x^2} + a\frac{\partial(e^{-b\tau}u)}{\partial x}\,.$$

From this we see that we should define our first transformation to be on u itself as $\hat{u} = e^{-b\tau}u$, and we obtain our first modified equation of

$$\frac{\partial \hat{u}}{\partial \tau} = \frac{\partial^2 \hat{u}}{\partial x^2} + a\frac{\partial \hat{u}}{\partial x}\,.$$

To motivate the next transformation move the term $a\frac{\partial \hat{u}}{\partial x}$ to the left hand side in the above equation obtaining

$$\frac{\partial \hat{u}}{\partial \tau} - a\frac{\partial \hat{u}}{\partial x} = \frac{\partial^2 \hat{u}}{\partial x^2}\,,$$

which we recognize (due to the wave like operator $\frac{\partial}{\partial \tau} - a\frac{\partial}{\partial x}$ on the left hand side) as a diffusion equation but in a translated coordinate system. To un-translate this coordinate system we transform the variables (x, τ) to new variables $(\hat{x}, \hat{\tau})$ as follows

$$\begin{aligned}\hat{x} &= x + a\tau \\ \hat{\tau} &= \tau\,.\end{aligned}$$

This transformation has an inverse given by

$$\begin{aligned} x &= \hat{x} - a\hat{\tau} \\ \tau &= \hat{\tau}\,. \end{aligned}$$

With these definitions, the transformation of the derivatives in our differential equation are given by

$$\begin{aligned} \frac{\partial}{\partial \tau} &= \frac{\partial \hat{\tau}}{\partial \tau}\frac{\partial}{\partial \hat{\tau}} + \frac{\partial \hat{x}}{\partial \tau}\frac{\partial}{\partial \hat{x}} = 1\frac{\partial}{\partial \hat{\tau}} + a\frac{\partial}{\partial \hat{x}} \\ \frac{\partial}{\partial x} &= \frac{\partial \hat{\tau}}{\partial x}\frac{\partial}{\partial \hat{\tau}} + \frac{\partial \hat{x}}{\partial x}\frac{\partial}{\partial \hat{x}} = 0 + 1\frac{\partial}{\partial \hat{x}}\,. \end{aligned}$$

So our partial differential equation becomes

$$\frac{\partial \hat{u}}{\partial \hat{\tau}} + a\frac{\partial \hat{u}}{\partial \hat{x}} - a\frac{\partial \hat{u}}{\partial \hat{x}} = \frac{\partial^2 \hat{u}}{\partial \hat{x}^2}\,,$$

or

$$\frac{\partial \hat{u}}{\partial \hat{\tau}} = \frac{\partial^2 \hat{u}}{\partial \hat{x}^2}\,,$$

the pure diffusion equation. Thus in summary if $u(x,\tau)$ satisfies the *original* convection diffusion partial differential equation, then the function $\hat{u}(\hat{x},\hat{\tau})$ satisfies the pure diffusion equation. Solving this equation for $\hat{u}$ using the analytic solution to the diffusion equation we find that

$$\hat{u}(\hat{x},\hat{\tau}) = \frac{1}{2\sqrt{\pi\hat{\tau}}}\int_{-\infty}^{\infty} \hat{u}_0(s) e^{-\frac{(\hat{x}-s)^2}{2\hat{\tau}}} ds\,.$$

For the initial condition $\hat{u}_0(s)$ we see that

$$\hat{u}_0(s) \equiv \hat{u}(s,0) = u(s,0) = u_0(s)\,,$$

or the initial conditions on the original equation. Replacing $\hat{x}$, $\hat{\tau}$ and $\hat{u}$ with their definitions we see that

$$u(x,\tau) = \frac{e^{b\tau}}{2\sqrt{\pi\tau}}\int_{-\infty}^{\infty} u_0(s) e^{-\frac{(x+a\tau-s)^2}{2\tau}} ds\,,$$

as the full solution to the original problem.

Now consider the second differential equation given by

$$c(\tau)\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}\,.$$

To motivate the transformation we will perform on this equation we write this equation as

$$\frac{\partial u}{\left(\frac{1}{c(\tau)}\right)\partial\tau} = \frac{\partial^2 u}{\partial x^2}\,.$$

Then we want to introduce change of time variable $\hat{\tau} = \hat{\tau}(\tau)$ such that the denominator of the first fraction is a differential. That is we require $\hat{\tau}$ to satisfy

$$d\hat{\tau} = \frac{1}{c(\tau)}d\tau\,.$$

This means that we should define our mapping of τ to $\hat{\tau}$ as

$$\hat{\tau}(\tau) = \int_0^\tau \frac{d\tau'}{c(\tau')}d\tau'\,,$$

as the appropriate transformation, to reduce the original equation to $\frac{\partial u}{\partial\hat{\tau}} = \frac{\partial^2 u}{\partial x^2}$, in this case.

Exercise 5 (time dependent interest rates and volatility)

Recalling the Black-Scholes equation for a European call and assuming time dependent interest rates and stock volatility we have

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma(t)^2 S^2 \frac{\partial^2 C}{\partial S^2} + r(t)S\frac{\partial C}{\partial S} - r(t)C = 0\,.$$

Part (a): As suggested let $S = Ee^x$, $C = Ev$, and $t = T - t'$ be a transformation of the variables (t, S, C). Note that these imply that $x = \ln(S/E)$. To perform these substitution we begin by transforming derivative of the independent variables as follows. First the time derivative becomes

$$\frac{\partial}{\partial t} = \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'} = -\frac{\partial}{\partial t'}\,.$$

Next, the first derivative with respect to S becomes

$$\frac{\partial}{\partial S} = \frac{\partial x}{\partial S}\frac{\partial}{\partial x} = \frac{\partial \ln(S/E)}{\partial S}\frac{\partial}{\partial x} = \frac{1}{S}\frac{\partial}{\partial x} = \frac{1}{Ee^x}\frac{\partial}{\partial x}\,.$$

From which, we can next calculate the second derivative with respect to S in terms of the new variable x. We find that

$$\begin{aligned}
\frac{\partial^2}{\partial S^2} &= \frac{\partial}{\partial S}\left(\frac{\partial}{\partial S}\right) = \frac{1}{Ee^x}\frac{\partial}{\partial x}\left(\frac{1}{Ee^x}\frac{\partial}{\partial x}\right) = \frac{e^{-x}}{E^2}\frac{\partial}{\partial x}\left(e^{-x}\frac{\partial}{\partial x}\right) \\
&= \frac{e^{-x}}{E^2}\left[-e^{-x}\frac{\partial}{\partial x} + e^{-x}\frac{\partial^2}{\partial x^2}\right] \\
&= \frac{e^{-2x}}{E^2}\frac{\partial^2}{\partial x^2} - \frac{e^{-2x}}{E^2}\frac{\partial}{\partial x}\,.
\end{aligned}$$

When we put these expressions into the Black-Scholes equation above we find

$$\begin{aligned}
&- \frac{\partial(Ev)}{\partial t'} + \frac{1}{2}\sigma^2(t')(E^2e^{2x})\left[\frac{e^{-2x}}{E^2}\frac{\partial^2(Ev)}{\partial x^2} - \frac{e^{-2x}}{E^2}\frac{\partial(Ev)}{\partial x}\right] \\
&+ r(t')(Ee^x)\left[\frac{e^{-x}}{E}\frac{\partial(Ev)}{\partial x}\right] - r(t')Ev = 0\,.
\end{aligned}$$

where we have assumed that $\sigma(\cdot)$ and $r(\cdot)$ are now considered functions of t' rather than t. When we simplify this we find that

$$\frac{\partial v}{\partial t'} = \frac{1}{2}\sigma^2(t')\frac{\partial^2 v}{\partial x^2} + (r(t') - \frac{1}{2}\sigma^2(t'))\frac{\partial v}{\partial x} - r(t')v\,.$$

which is the same expression as in the book.

Part (b): Motivated by the differential equation above introduce a new time variable $\hat{\tau}$ such that its differential is given by $\frac{1}{2}\sigma^2(t')dt' = d\hat{\tau}$. By integrating both sides of this we see that

$$\hat{\tau}(t') = \int_0^{t'} \frac{1}{2}\sigma^2(s)ds\,, \tag{16}$$

is the functional form for $\hat{\tau}(\cdot)$. Then the derivative with respect to t' is equal to

$$\frac{\partial}{\partial t'} = \frac{\partial\hat{\tau}}{\partial t'}\frac{\partial}{\partial\hat{\tau}} = \frac{1}{2}\sigma^2(t')\frac{\partial}{\partial\hat{\tau}}\,,$$

so the equation at this point becomes

$$\frac{1}{2}\sigma^2(t')\frac{\partial v}{\partial\hat{\tau}} = \frac{1}{2}\sigma^2(t')\frac{\partial^2 v}{\partial x^2} + (r(t') - \frac{1}{2}\sigma^2(t'))\frac{\partial v}{\partial x} - r(t')v = 0\,,$$

or on dividing by $\frac{1}{2}\sigma^2(t')$ we find

$$\frac{\partial v}{\partial \hat{\tau}} = \frac{\partial^2 v}{\partial x^2} + \left(\frac{2r(t')}{\sigma(t')^2} - 1\right)\frac{\partial v}{\partial x} - \frac{2r(t')}{\sigma(t')^2}v = 0\,.$$

Now defining $a(\hat{\tau})$ and $b(\hat{\tau})$ as[1]

$$\begin{aligned} a(\hat{\tau}) &= \frac{2r(t')}{\sigma(t')^2} - 1 \\ b(\hat{\tau}) &= \frac{2r(t')}{\sigma(t')^2}\,, \end{aligned}$$

With these definitions of a and b our partial differential equation for v becomes

$$\frac{\partial v}{\partial \hat{\tau}} = \frac{\partial^2 v}{\partial x^2} + a(\hat{\tau})\frac{\partial v}{\partial x} - b(\hat{\tau})v\,. \tag{17}$$

Part (c): If we drop the second derivative term in the above we are left with the equation

$$\frac{\partial v}{\partial \hat{\tau}} = a(\hat{\tau})\frac{\partial v}{\partial x} - b(\hat{\tau})v\,.$$

This equation can be solved using the method of characteristics, but its easier to just verify that the proposed solution does indeed satisfy it. Consider the required derivatives of the proposed functional form for v

$$\begin{aligned} v_{\hat{\tau}} &= F'(x + A(\hat{\tau}))A'(\hat{\tau})e^{-B(\hat{\tau})} - F(x + A(\hat{\tau}))B'(\hat{\tau})e^{-B(\hat{\tau})} \\ &= F'(x + A(\hat{\tau}))a(\hat{\tau})e^{-B(\hat{\tau})} - F(x + A(\hat{\tau}))b(\hat{\tau})e^{-B(\hat{\tau})}\,. \end{aligned}$$

and

$$v_x = F'(x + A(\hat{\tau}))e^{-B(\hat{\tau})}\,.$$

[1]In these expressions for the variable t' we envision invoking the *inverse* of the transformation $t' \to \hat{\tau}(t')$. That is, if we define a function $F(\cdot)$ as

$$F(t') = \int_0^{t'} \frac{1}{2}\sigma^2(s)ds\,,$$

such that the variable $\hat{\tau}$ in terms of t' is given by $\hat{\tau} = F(t')$ we would need to compute F^{-1} such that $t' = F^{-1}(\hat{\tau})$. With this function F^{-1} the expression $r(t')$ in the above should be considered as

$$r(t') = r(F^{-1}(\hat{\tau})) = r(\hat{\tau})\,.$$

With these two results computing the expression $v_{\hat{\tau}} - a(\hat{\tau}) + b(\hat{\tau})$ we obtain

$$\begin{array}{rl} & F'(x+A(\hat{\tau}))a(\hat{\tau})e^{-B(\hat{\tau})} - F(x+A(\hat{\tau}))b(\hat{\tau})e^{-B(\hat{\tau})} \\ - & F'(x+A(\hat{\tau}))a(\hat{\tau})e^{-B(\hat{\tau})} + b(\hat{\tau})F(x+A(\hat{\tau}))e^{-B(\hat{\tau})} = 0\,, \end{array}$$

showing that the proposed v is indeed a solution.

Part (d): We seek a solution to the full advection diffusion Equation 17 above. We will use the result from Part (c) above to eliminate the lower order terms in this equation and reduce it to the pure diffusion equation. To do this, introduce the variable V defined in terms of v as

$$v(x,\hat{\tau}) = e^{-B(\hat{\tau})}V(\hat{x},\hat{\tau}) = e^{-B(\hat{\tau})}V(x+A(\hat{\tau}),\hat{\tau})\,.$$

Then the derivatives of $v(x,\hat{\tau})$ are given as

$$\begin{array}{rcl} v_x & = & e^{-B(\hat{\tau})}V_{\hat{x}} \quad \text{and} \quad v_{xx} = e^{-B(\hat{\tau})}V_{\hat{x}\hat{x}} \quad \text{while} \\ v_{\hat{\tau}} & = & -b(\hat{\tau})e^{-B(\hat{\tau})}V + e^{-B(\hat{\tau})}V_{\hat{x}}\,a(\hat{\tau}) + e^{-B(\hat{\tau})}V_{\hat{\tau}}\,. \end{array}$$

When these expressions are put into Equation 17 we obtain $V_{\hat{\tau}} = V_{\hat{x}\hat{x}}$ the pure diffusion equation results. Since we have an explicit analytic formula for the pure diffusion equation we can compute V as

$$V(\hat{x},\hat{\tau}) = \frac{1}{2\sqrt{\pi\hat{\tau}}}\int_{-\infty}^{\infty} V_0(s)e^{-(\hat{x}-s)^2/4\hat{\tau}}ds\,.$$

We take note of a few things before continuing. First in terms of v the function V is given by $V(\hat{x},\hat{\tau}) = e^{B(\hat{\tau})}v(x-A(\hat{\tau}),\hat{\tau})$ and the functions A and B are defined by differential equations such that when $\hat{\tau} = 0$ (equivalently $t' = 0$) we may take $A(0) = 0$ and $B(0) = 0$, so that our initial condition V_0 on V in terms of the initial condition on v becomes

$$V_0(s) \equiv V(s,0) = e^{B(0)}v(s-A(0),0) = v(s,0) = v_0(s)\,,$$

So V and v have the *same* initial data and we have

$$v(x,\hat{\tau}) = \frac{e^{-B(\hat{\tau})}}{2\sqrt{\pi\hat{\tau}}}\int_{-\infty}^{\infty} v_0(s)e^{-(x+A(\hat{\tau})-s)^2/4\hat{\tau}}ds\,.$$

The expression for the *call* in terms of the original variables (x,t) is

$$\begin{array}{rcl} C(x,t) & = & \dfrac{e^{-B(\hat{\tau})}}{2\sqrt{\pi\hat{\tau}}}\displaystyle\int_{-\infty}^{\infty} C(Ee^s,T)e^{-(x+A(\hat{\tau})-s)^2/4\hat{\tau}}ds \\ & = & \dfrac{e^{-B(\hat{\tau})}}{2\sqrt{\pi\hat{\tau}}}\displaystyle\int_{-\infty}^{\infty} \max(0,Ee^s-E)e^{-(x+A(\hat{\tau})-s)^2/4\hat{\tau}}ds\,. \end{array}$$

Which may seem complicated but its evaluation is relatively simple and will be explained in more detail below. Before presenting a procedural method for evaluating this expression at a given (S, t) we first present a simplified way to evaluate the functions A and B. Recalling that they are defined in terms of two differential equations as

$$\begin{aligned} \frac{dA}{d\hat{\tau}} &= \frac{2r(\hat{\tau})}{\sigma^2(\hat{\tau})} - 1 \quad \text{with} \quad A(0) = 0 \\ \frac{dB}{d\hat{\tau}} &= \frac{2r(\hat{\tau})}{\sigma^2(\hat{\tau})} \quad \text{with} \quad B(0) = 0\,, \end{aligned}$$

we can use Equation 16 above to transfer these into differential equations with respect to t' (rather than $\hat{\tau}$). Since

$$\frac{d}{d\hat{\tau}} = \frac{dt'}{d\hat{\tau}}\frac{d}{dt'} = \frac{2}{\sigma^2(t')}\frac{d}{dt'}\,,$$

we find the differential equations for A and B in terms of t' become

$$\begin{aligned} \frac{dA}{dt'} &= r(t') - \frac{1}{2}\sigma^2(t') \quad \text{with} \quad A(0) = 0 \\ \frac{dB}{dt'} &= r(t') \quad \text{with} \quad B(0) = 0\,, \end{aligned}$$

This shows that A and B can be expressed as integrals and are given by

$$A(t') = \int_0^{t'} \left(r(s) - \frac{1}{2}\sigma^2(s) \right) ds \quad B(t') = \int_0^{t'} r(s) ds\,. \tag{18}$$

Using these simplification our call is evaluated at the point (S, t) where $0 < S < \infty$ and $0 < t < T$, using the following algorithm:

1. Compute $t' = T - t$ and $x = \log(S/E)$
2. Compute $A(t')$ and $B(t')$ using Equations 18
3. Compute the desired call price $C(S, t) \equiv C(x, t)$ using Equation 18

Exercise 6

Equation 5.10 in the book is given by

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k-1)\frac{\partial v}{\partial x} - kv = 0\,.$$

If we define v in terms of another function V as $v(x,\tau) = e^{-k\tau}V(\xi,\tau)$ with $\xi = x + (k-1)\tau$ then we see that

$$\begin{aligned} v_x &= e^{-k\tau}V_\xi \quad v_{xx} = e^{-k\tau}V_{\xi\xi} \\ u_\tau &= -ke^{-k\tau}V + e^{-k\tau}V_\tau + e^{-k\tau}V_\xi\,(k-1)\,. \end{aligned}$$

When these expressions are put into Equation 5.10 above many things cancel and we end with $V_\tau = V_{\xi\xi}$, the pure diffusion equation. The latter equation for V could be solved using the known Green's function for the diffusion equation.

Exercise 7

In Problem 4 from Chapter 3 we have shown that the analytic expressions for C and P individually satisfy the Black-Scholes equation. Since the Black-Scholes equation is linear we know that the expression $C-P$ will also satisfy the Black-Scholes equation. The final condition for this expression (when $t = T$) is given by

$$C(S,T) - P(S,T) = \max(0, S-E) - \max(0, E-S)\,.$$

To evaluate this expression we plot the two option payoffs $\max(0, S-E)$, $\max(0, E-S)$, and the negative of the put payoff $-\max(0, E-S)$ in Figure 6. When we flip the sign of the payoff for a put we see that the *total* payoff is a straight line through E. Thus the expression $C(S,T) - P(S,T)$ simplifies to $S-E$. This can also be seen by considering what the expression $\max(0, S-E) - \max(0, E-S)$ evaluates to in the two cases $S < E$ and $S > E$. In both cases the expression $C(S,T) - P(S,T)$ simplifies to $S-E$. From put-call parity we know that $P - C = Ee^{-r(T-t)} - S$, thus the the expression $Ee^{-r(T-t)} - S$ is also a solution to the Black-Scholes equation. This expression is the value of a combined portfolio consisting of a bond that pay's E at time $t = T$ offering interest rate r and short sell one share of the stock.

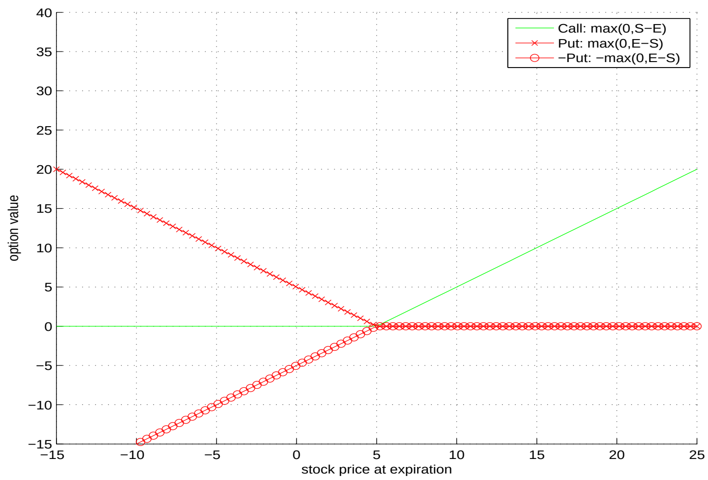

Figure 6: Plots of the payoff expressions for a European call $\max(0, S - E)$, a European put $\max(0, E - S)$, and the negative of the European put $-\max(0, E - S)$. Here for illustration we take the strike $E = 5$.

Exercise 8 (the European Put)

Most of the manipulations performed in the section entitled "The Black-Scholes Formula Derived" still hold true when we consider a European put. The one exception is that now the transformed payoff for the function u is given by

$$u_0(x) \equiv u(x, 0) = \max(e^{\frac{1}{2}(k-1)x} - e^{\frac{1}{2}(k+1)x}, 0)\,,$$

which is derived in these notes in that section, see page 35. We can repeat the same arguments used to derive the value of a European call to derive the expression for a European put. Namely, a European put has a transformed solution $u(x, \tau)$ given by

$$u(x, \tau) \;=\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_0(x + \sqrt{2\tau}x') e^{-\frac{x'^2}{2}} dx'\,.$$

To evaluate this integral consider the fact that the maximum in the definition of the integrand, $u_0(x)$, above will be the exponential part and not zero by definition when

$$e^{\frac{1}{2}(k-1)x} - e^{\frac{1}{2}(k+1)x} > 0\,,$$

or on taking logarithms of this and simplifying some gives

$$(k-1)x > (k+1)x \Rightarrow -x > x \Rightarrow x < 0\,.$$

Thus the integrand $u_0(\cdot)$ (when evaluated at $x + \sqrt{2\tau}x'$) has as its maximum the exponential terms (and not zero) when

$$x + \sqrt{2\tau}x' < 0 \Rightarrow x' < -\frac{x}{\sqrt{2\tau}}\,.$$

Thus the above integral expression for $u(x,\tau)$ becomes

$$\begin{aligned}
u(x,\tau) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-\frac{x}{\sqrt{2\tau}}}\left(e^{\frac{1}{2}(k-1)(x'\sqrt{2\tau}+x)} - e^{\frac{1}{2}(k+1)(x'\sqrt{2\tau}+x)}\right)e^{-\frac{1}{2}x'^2}dx' \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-\frac{x}{\sqrt{2\tau}}} e^{\frac{1}{2}(k-1)(x'\sqrt{2\tau}+x)}e^{-\frac{1}{2}x'^2}dx' \\
&- \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-\frac{x}{\sqrt{2\tau}}} e^{\frac{1}{2}(k+1)(x'\sqrt{2\tau}+x)}e^{-\frac{1}{2}x'^2}dx'\,.
\end{aligned}$$

Defining the exponent of the first integral above as $\eta \equiv -\frac{1}{2}x'^2 + \frac{1}{2}(k-1)\sqrt{2\tau}x' + \frac{1}{2}(k-1)x$ we complete the square in the variable x' as follows

$$\begin{aligned}
\eta &= -\frac{1}{2}\left(x'^2 - (k-1)\sqrt{2\tau}x'\right) + \frac{1}{2}(k-1)x \\
&= -\frac{1}{2}\left(x'^2 - (k-1)\sqrt{2\tau}x' + \left(\frac{(k-1)\sqrt{2\tau}}{2}\right)^2\right) \\
&+ \frac{1}{2}\left(\frac{(k-1)\sqrt{2\tau}}{2}\right)^2 + \frac{1}{2}(k-1)x \\
&= -\frac{1}{2}\left(x' - \frac{(k-1)\sqrt{2\tau}}{2}\right)^2 + \frac{1}{2}(k-1)x + \frac{1}{4}(k-1)^2\tau\,.
\end{aligned}$$

Using this, the first integral above $I_1 \equiv \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-\frac{x}{\sqrt{2\tau}}} e^{\frac{1}{2}(k-1)(x'\sqrt{2\tau}+x)}e^{-\frac{1}{2}x'^2}dx'$ becomes

$$\begin{aligned}
I_1 &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-\frac{x}{\sqrt{2\tau}}} e^{-\frac{1}{2}\left(x' - \frac{(k-1)\sqrt{2\tau}}{2}\right)^2} e^{\frac{1}{2}(k-1)x + \frac{1}{4}(k-1)^2\tau}dx' \\
&= \frac{e^{\frac{1}{2}(k-1)x + \frac{1}{4}(k-1)^2\tau}}{\sqrt{2\pi}}\int_{-\infty}^{-\frac{x}{\sqrt{2\tau}} - \frac{1}{2}(k-1)\sqrt{2\tau}} e^{-\frac{1}{2}s^2}ds \\
&= e^{\frac{1}{2}(k-1)x + \frac{1}{4}(k-1)^2\tau}N(-d_2)\,,
\end{aligned}$$

with d_2 defined as

$$d_2(x,\tau) = \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k-1)\sqrt{2\tau}\,,$$

and $N(\cdot)$ the cumulative distribution function for the standard normal, i.e.

$$N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-\frac{1}{2}s^2} ds\,.$$

The second integral is integrated in exactly the same way as the first but with $k-1$ replaced with $k+1$.

With the explicit solution to the pure diffusion equation above we can now extract the solution to the Black-Scholes equation in terms of the financial variables of interest. To do this we begin by computing the function $v(x,\tau)$ from the function $u(x,\tau)$. Recalling that $v = e^{-\frac{1}{2}(k-1)x - \frac{1}{4}(k+1)^2\tau} u(x,\tau)$. We find

$$v(x,\tau) = e^{-\frac{1}{2}(k-1)x-\frac{1}{4}(k+1)^2\tau}\left(e^{\frac{1}{2}(k-1)x+\frac{1}{4}(k-1)^2\tau}N(-d_2) - e^{\frac{1}{2}(k+1)x+\frac{1}{4}(k+1)^2\tau}N(-d_1)\right)\,,$$

with

$$d_1(x,\tau) = \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k+1)\sqrt{2\tau}\,.$$

Performing algebraic simplifications on the above we find that

$$v(x,\tau) = e^{-k\tau}N(-d_2) - e^{x}N(-d_1)\,.$$

In terms of the original variables $x = \log(S/E)$, $\tau = \frac{1}{2}\sigma^2(T-t)$, $P = Ev(x,\tau)$ and $k = \frac{r}{\frac{1}{2}\sigma^2}$, and we find

$$\begin{aligned} P(S,t)/E &= e^{-(T-t)r}N(-d_2(S,t)) - \frac{S}{E}N(-d_1(S,t)) \quad \text{or} \\ P(S,t) &= Ee^{-r(T-t)}N(-d_2(S,t)) - SN(-d_1(S,t))\,. \end{aligned}$$

Here the expressions d_1 and d_2 are considered as functions of S and t in the normal way.

Exercise 9 (the European Greeks)

For this problem we desire to compute the gamma, theta, vega, and rho for a European call and put, given by Equations 5 and 6 above respectively.

To do this we begin by remembering that the definition of the **gamma**, denoted by Γ, for a European call is given by

$$\begin{aligned}
\Gamma_C &= \frac{\partial^2 C}{\partial S^2} = \frac{\partial \Delta_C}{\partial S} = \frac{\partial N(d_1)}{\partial S} = N'(d_1)\frac{\partial d_1}{\partial S} \\
&= N'(d_1)\left(\frac{1}{\sigma S\sqrt{T-t}}\right) = \left(\frac{e^{-d_1^2/2}}{\sqrt{2\pi}}\right)\left(\frac{1}{\sigma S\sqrt{T-t}}\right), \qquad (19)
\end{aligned}$$

since $\Delta_C = N(d_1)$. For a European put we have

$$\begin{aligned}
\Gamma_P &= \frac{\partial^2 P}{\partial S^2} = \frac{\partial \Delta_P}{\partial S} = \frac{\partial}{\partial S}(N(d_1)-1) = \frac{\partial^2 C}{\partial S^2} \\
&= \left(\frac{e^{-d_1^2/2}}{\sqrt{2\pi}}\right)\left(\frac{1}{\sigma S\sqrt{T-t}}\right), \qquad (20)
\end{aligned}$$

the *same* as for a European call.

The greek **theta** denoted by θ, for an option V is defined as $\theta \equiv -\frac{\partial V}{\partial t}$, which using the Black-Scholes equation is given in terms of derivatives of S as

$$\theta = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = \frac{1}{2}\sigma^2 S^2 \Gamma + rS\Delta - rV\,.$$

For a European call we have $V = C = SN(d_1) - Ee^{-r(T-t)}N(d_2)$, $\Delta_C = N(d_1)$, and $\Gamma_C = \frac{N'(d_1)}{\sigma S\sqrt{T-t}}$, so the above becomes

$$\begin{aligned}
\theta_C &= \frac{1}{2}\sigma^2 S^2\left(\frac{N'(d_1)}{\sigma S\sqrt{T-t}}\right) + rSN(d_1) - r(SN(d_1) - Ee^{-r(T-t)}N(d_2)) \\
&= \frac{\sigma S N'(d_1)}{2\sqrt{T-t}} + rEe^{-r(T-t)}N(d_2)\,. \qquad (21)
\end{aligned}$$

For a European put $V = P = Ee^{-r(T-t)}N(-d_2) - SN(-d_1)$, $\Delta_P = N(d_1) - 1$, and $\Gamma_P = \frac{N'(d_1)}{\sigma S\sqrt{T-t}}$, so the above becomes

$$\begin{aligned}
\theta_P &= \frac{1}{2}\sigma^2 S^2\left(\frac{N'(d_1)}{\sigma S\sqrt{T-t}}\right) + rS(N(d_1)-1) - r(Ee^{-r(T-t)}N(-d_2) - SN(-d_1)) \\
&= \frac{\sigma S N'(d_1)}{2\sqrt{T-t}} + rSN(d_1) + rSN(-d_1) - rS - rEe^{-r(T-t)}N(-d_2)\,.
\end{aligned}$$

Since $N(-x)+N(x) = 1$ the three term expression $rSN(d_1)+rSN(-d_1)-rS$ above is zero, and the above simplifies to

$$\begin{aligned}
\theta_P &= \frac{\sigma S N'(d_1)}{2\sqrt{T-t}} - rEe^{-r(T-t)}N(-d_2) \qquad (22)\\
&= \frac{\sigma S N'(d_1)}{2\sqrt{T-t}} - rEe^{-r(T-t)}(1-N(d_2))\\
&= \theta_C - rEe^{-r(T-t)}\,. \qquad (23)
\end{aligned}$$

This last equation could also be obtained from put-call parity by taking the (negative) time derivative of the put-call parity expression $C - P = S - Ee^{-r(T-t)}$. Taking this derivative we see that $\theta_C = \theta_P + rEe^{-r(T-t)}$, an expression equivalent to the above.

The definition of **vega** is the derivative of the option price with respect to the volatility σ. For a European call we then have

$$\frac{\partial C}{\partial \sigma} = SN'(d_1)\frac{\partial d_1}{\partial \sigma} - Ee^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial \sigma}\,.$$

Using Equation 11 we change the factor $N'(d_2)$ into a factor in terms of $N'(d_1)$ to get

$$\begin{aligned}
\frac{\partial C}{\partial \sigma} &= SN'(d_1)\frac{\partial d_1}{\partial \sigma} - Ee^{-r(T-t)}\left[\frac{S}{E}e^{r(T-t)}N'(d_1)\right]\frac{\partial d_2}{\partial \sigma}\\
&= SN'(d_1)\left[\frac{\partial d_1}{\partial \sigma} - \frac{\partial d_2}{\partial \sigma}\right]\,.
\end{aligned}$$

From the definition of d_1 and d_2 given by Equations 7 and 8 above we find that

$$\frac{\partial d_1}{\partial \sigma} = -\frac{d_1}{\sigma} + \sqrt{T-t} \quad \text{and} \quad \frac{\partial d_2}{\partial \sigma} = -\frac{d_2}{\sigma} - \sqrt{T-t}\,,$$

so their difference is given by

$$\frac{\partial d_1}{\partial \sigma} - \frac{\partial d_2}{\partial \sigma} = \frac{d_2 - d_1}{\sigma} + 2\sqrt{T-t} = -\sqrt{T-t} + 2\sqrt{T-t} = \sqrt{T-t}\,,$$

when we recall that $d_2 = d_1 - \sigma\sqrt{T-t}$. Thus we have shown that

$$\frac{\partial C}{\partial \sigma} = SN'(d_1)\sqrt{T-t}\,, \qquad (24)$$

as the expression for the vega of a European call. For a European put taking the σ derivative of the put-call parody relationship gives

$$\frac{\partial P}{\partial \sigma} = \frac{\partial C}{\partial \sigma}\,.$$

The **rho** for a European call is defined as $\frac{\partial C}{\partial r}$ and is given by

$$\frac{\partial C}{\partial r} = SN'(d_1)\frac{\partial d_1}{\partial r} + E(T-t)e^{-r(T-t)}N(d_2) - Ee^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial r}\,.$$

Since $\frac{\partial d_1}{\partial r} = \frac{\sqrt{T-t}}{\sigma} = \frac{\partial d_2}{\partial r}$ so that $\frac{\partial C}{\partial r}$ simplifies to

$$\begin{aligned} \frac{\partial C}{\partial r} &= (SN'(d_1) - Ee^{-r(T-t)}N'(d_2))\frac{\partial d_1}{\partial r} + E(T-t)e^{-r(T-t)}N(d_2) \\ &= E(T-t)e^{-r(T-t)}N(d_2)\,. \end{aligned} \tag{25}$$

Where we have again used Equation 11 to eliminate the first term. Taking the r derivative of the put-call parody relationship shows that the rho for a European put is given by

$$\begin{aligned} \frac{\partial P}{\partial r} &= \frac{\partial C}{\partial r} - E(T-t)e^{-r(T-t)} = E(T-t)e^{-r(T-t)}(N(d_2)-1) \\ &= -E(T-t)e^{-r(T-t)}N(-d_2)\,. \end{aligned} \tag{26}$$

Exercise 10 (visualization of the European Greeks)

These are plotted with the Matlab function `prob_5_10.m`.

Exercise 11 (the random walk followed by a European call)

Using Ito's lemma the differential of a function $C(S,t)$ where S is a stochastic random variable is given by

$$dC = C_t dt + C_s dS + \frac{1}{2}C_{SS}dS^2\,.$$

Since the variation in S is assumed to follow geometric Brownian motion i.e. $\frac{dS}{S} = \mu dt + \sigma dX$, we see that to first order in dt that

$$dS^2 = S^2\sigma^2 dX^2 = S^2\sigma^2 dt\,,$$

where we have used the rule of thumb that $dX^2 = dt$. Using this expression for dS^2 in Ito's lemma we have that dC is given by

$$\begin{aligned} dC &= C_t dt + C_S dS + \frac{1}{2} S^2 \sigma^2 C_{SS} dt \\ &= \left(C_t + \mu S C_S + \frac{1}{2} S^2 \sigma^2 C_{SS} \right) dt + S \sigma C_S dX \,. \end{aligned}$$

Since C satisfies the Black-Scholes equation we know that

$$C_t + \frac{1}{2} S^2 \sigma^2 C_{SS} = rC - rSC_S \,,$$

and the differential, dC becomes

$$dC = (rC - (r - \mu) S C_S) dt + S \sigma C_S dX \,.$$

Here C_S is the value of the delta for a European call option and is given by the standard formulas i.e. $N(d_1)$.

Exercise 12 (positivity of the Black-Scholes solution)

For the diffusion or heat equation an explicit solution is given in terms of the Greens function. The expression for the solution on an infinite interval is

$$u(x, \tau) = \frac{1}{2\sqrt{\pi \tau}} \int_{-\infty}^{\infty} u(s, 0) e^{-(x-s)^2/4\tau} ds \,.$$

From which we see that if our initial condition is always positive i.e. $u(s, 0) \geq 0$, then $u(x, \tau) > 0$ for all $\tau > 0$. Because we can reduce the Black-Scholes equation to the diffusion equation if our initial payoff $u(s, 0)$ is positive, the option price will be positive also.

Exercise 14 (nondimensionalization)

We can nondimensionalize X with the length of the bar that is $x = \frac{X}{L}$, nondimensionalize U with the given typical value of the temperature variations i.e. $u = \frac{U}{U_0}$. To nondimensionalize the time variable T we need a typical value for time. We could construct a nondimenaionsional time directly from the given physical parameters by relating each of them in their base units say in the MKS system. It is easier however to propose a nondimensional time T_0

such that $t = \frac{T}{T_0}$ and to the derive what this should be. Putting these three new variables t, x, and u into our differential equation we have

$$\rho c T_0 \frac{\partial u}{\partial t} = k L^2 \frac{\partial^2 u}{\partial x^2} \, ,$$

or

$$\frac{\partial u}{\partial t} = \frac{k L^2}{\rho c T_0} \frac{\partial^2 u}{\partial x^2} \, .$$

This leads us pick the constant T_0 such that the coefficient of $\frac{\partial^2 u}{\partial x^2}$ is unity. That is take T_0 such that

$$\frac{k L^2}{\rho c T_0} = 1 \Rightarrow T_0 = \frac{\rho c}{k L^2} \, .$$

Exercise 15 (supershares)

If our payoff is $B\mathcal{H}(E-S)$ we have (using the general option pricing formula) that

$$\begin{aligned} V(S,t) &= \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_0^\infty B\mathcal{H}(E-S') e^{-[\log(S'/S)-(r-\frac{1}{2}\sigma^2)(T-t)]^2/2\sigma^2(T-t)} \frac{dS'}{S'} \\ &= \frac{B e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_0^E e^{-[\log(S'/S)-(r-\frac{1}{2}\sigma^2)(T-t)]^2/2\sigma^2(T-t)} \frac{dS'}{S'} \, . \end{aligned}$$

To evaluate this integral introduce an integration variable v such that

$$\begin{aligned} v &= \frac{\log(S'/S) - (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \quad \text{and} \\ dv &= \frac{dv}{dS'} dS' = -\frac{1}{\sigma\sqrt{T-t}} \frac{dS'}{S'} \Rightarrow \frac{dS'}{S'} = \sigma\sqrt{T-t} \, dv \, , \end{aligned}$$

and our integral above transforms to

$$-\frac{B e^{-r(T-t)}}{\sqrt{2\pi}} \int_{-\infty}^{-d_2(S,t)} e^{-v^2/2} dv = B e^{-r(T-t)} N(-d_2(S,t)) \, ,$$

when we recall the definition of $d_2(S,t)$. As suggested in the book another way to do this problem is to recognize that $\mathcal{H}(S-E) + \mathcal{H}(E-S) = 1$ and thus our payoff function $\Lambda(S)$ in this case can be written as

$$\Lambda(S) = B\mathcal{H}(E-S) = B - B\mathcal{H}(S-E) \, ,$$

so the option value is the superposition of the option value for a payoff of the constant B and a payoff of $B\mathcal{H}(S-E)$. The latter is calculated in this chapter and is given by $Be^{-r(T-t)}N(d_2(S,t))$. The option value for the payoff B from an investment now that pays B one with certainty at time $T-t$ later is given by $Be^{-r(T-t)}$. Thus we have that our option value $V(S,t)$ is

$$\begin{aligned} V(S,t) &= Be^{-r(T-t)} - Be^{-r(T-t)}N(d_2(S,t)) \\ &= Be^{-r(T-t)}(1 - N(d_2(S,t))) = Be^{-r(T-t)}N(-d_2(S,t)) \,, \end{aligned}$$

the same result as before.

A supershare has a payoff given by $1/d$ if $E < S < E + d$ at expiry and zero otherwise. In terms of the Heaviside function $\mathcal{H}$ this is then

$$\Lambda(S) = \frac{1}{d}\left(\mathcal{H}(S-E) - \mathcal{H}(S-E-d)\right) .$$

Since this is a linear superposition of the payoffs from two European binary options with individual payoffs functions of $\frac{1}{d}\mathcal{H}(S-E)$ and $-\frac{1}{d}(\mathcal{H}(S-E-d)$ the options value of the supershare is the sum of the option values with these two individual payoffs. Since we know the analytic expression for the option value of a European binary option with payoff $B\mathcal{H}(S-E)$ (see the notes for the section of the book on binary options) the value of a supershare is

$$\begin{aligned} V(S,t) &= \frac{1}{d}e^{-r(T-t)}N\left(\frac{\log(\frac{S}{E}) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right) \\ &- \frac{1}{d}e^{-r(T-t)}N\left(\frac{\log(\frac{S}{E+d}) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right) . \end{aligned}$$

Exercise 16 (European asset-or-nothing)

We are told that the European asset-or-nothing call pays S if $S > E$ at expiry and nothing if $S \leq E$. So its payoff function can be written in terms of the the Heaviside function as $\Lambda(S) = S\mathcal{H}(S-E)$. From the general option valuation formula presented earlier we have that

$$
\begin{aligned}
V(S,t) &= \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}}\int_0^\infty S'\mathcal{H}(S'-E)e^{-[\log(S'/S)-(r-\frac{1}{2}\sigma^2)(T-t)]^2/2\sigma^2(T-t)}\frac{dS'}{S'} \\
&= \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}}\int_E^\infty S'e^{-[\log(S'/S)-(r-\frac{1}{2}\sigma^2)(T-t)]^2/2\sigma^2(T-t)}\frac{dS'}{S'} \\
&= \frac{Se^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}}\int_E^\infty \left(\frac{S'}{S}\right)e^{-[\log(S'/S)-(r-\frac{1}{2}\sigma^2)(T-t)]^2/2\sigma^2(T-t)}\frac{dS'}{S'}\,.
\end{aligned}
$$

To further evaluate this integral we will write $\frac{S'}{S} = e^{\log(S'/S)}$ and complete the square of the expression in the exponent with respect to the variable $\log(S'/S)$. Before continuing we note that this technique will also work in general for any polynomial expression of S' that we desire to integrate this quadratic exponential against. Defining the total exponent of the above (but ignoring for now the fraction $-\frac{1}{2\sigma^2(T-t)}$) to be $\mathcal{P}$ (for power) we find that the exponent of the above becomes

$$
\begin{aligned}
\mathcal{P} &= \log(\frac{S'}{S})^2 - 2\left(r-\frac{1}{2}\sigma^2\right)(T-t)\log(\frac{S'}{S}) \\
&- 2\sigma^2(T-t)\log(\frac{S'}{S}) + \left(r-\frac{1}{2}\sigma^2\right)^2(T-t)^2 \\
&= \log(\frac{S'}{S})^2 - 2\left(r+\frac{1}{2}\sigma^2\right)(T-t)\log(\frac{S'}{S}) + \left(r-\frac{1}{2}\sigma^2\right)^2(T-t)^2 \\
&= \log(\frac{S'}{S})^2 - 2\left(r+\frac{1}{2}\sigma^2\right)(T-t)\log(\frac{S'}{S}) + (T-t)^2\left(r+\frac{1}{2}\sigma^2\right)^2 \\
&- (T-t)^2\left(r+\frac{1}{2}\sigma^2\right)^2 + \left(r-\frac{1}{2}\sigma^2\right)^2(T-t)^2 \\
&= \left[\log(\frac{S'}{S})^2 - \left(r+\frac{1}{2}\sigma^2\right)(T-t)\right]^2 - 2r(T-t)^2\sigma^2\,.
\end{aligned}
$$

In the second to last step we have added and subtracted the expression $(T-t)^2\left(r+\frac{1}{2}\sigma^2\right)^2$ to complete the square as shown in the last equation above. Multiplying this expression by $-\frac{1}{2\sigma^2(T-t)}$ we have

$$
-\frac{\mathcal{P}}{2\sigma^2(T-t)} = \frac{-1}{2\sigma^2(T-t)}\left[\log(S'/S)^2 - \left(r+\frac{1}{2}\sigma^2\right)(T-t)\right]^2 + r(T-t)\,.
$$

With this our expression for $V(S,t)$ becomes

$$V(S,t) = \frac{S}{\sigma\sqrt{2\pi(T-t)}} \int_E^\infty e^{-[\log(S'/S)-(r+\frac{1}{2}\sigma^2)(T-t)]^2/2\sigma^2(T-t)} \frac{dS'}{S'} \,.$$

To evaluate this integral introduce an integration variable v such that

$$\begin{aligned} v &= \frac{-\log(S'/S) + (r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \quad \text{so} \\ dv &= \frac{dv}{dS'}dS' = -\frac{1}{\sigma\sqrt{T-t}}\frac{dS'}{S'} \,. \end{aligned}$$

With this our logarithmic differential becomes $\frac{dS'}{S'} = -\sigma\sqrt{T-t}\,dv$ and our integral above transforms to

$$V(S,t) = -\frac{S}{\sqrt{2\pi}} \int_{d_1(S,t)}^{-\infty} e^{-v^2/2} dv = SN(d_1(S,t)) \,,$$

when we recall the definition of $d_1(S,t)$.

Exercise 17 (the probability of exercise)

As discussed in the text the probability of exercise can be found by computing the expectation of $\mathcal{H}(S-E)$ or

$$\int_0^\infty \mathcal{H}(s'-E) p_S(s') ds' = \int_E^\infty p_S(s') ds' \,,$$

where $p_S(s')$ is the log-normal probability density followed by our asset. That is, if our stock starts with a price of S the probability density that it takes a value of s' at time t later is given by

$$p_S(s') = \frac{1}{\sigma s'\sqrt{2\pi t}} e^{-(\log(s'/S)-(\mu-\frac{1}{2}\sigma^2)t)^2/2\sigma^2 t} \,.$$

To evaluate the probability that we expire in the money at time T or $T-t$ from now we must compute

$$\int_E^\infty \frac{1}{\sigma s'\sqrt{2\pi(T-t)}} e^{-(\log(s'/S)-(\mu-\frac{1}{2}\sigma^2)(T-t))^2/2\sigma^2(T-t)} \,.$$

We can do this by introducing the integration variable $v = \frac{\log(s'/S)-(\mu-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$, so that $dv = \frac{ds'}{s'}\frac{1}{\sigma\sqrt{T-t}}$ and our integral (and probability p) above becomes

$$
\begin{aligned}
p &= \frac{1}{\sqrt{2\pi}} \int_{\frac{\log(E/S)-(\mu-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}}^{\infty} e^{-v^2/2} dv = \frac{1}{\sqrt{2\pi}} \int_{-\tilde{d}_2(S,t)}^{\infty} e^{-v^2/2} dv \\
&= 1 - \int_{-\infty}^{\tilde{d}_2(S,t)} e^{-v^2/2} dv = 1 - N(\tilde{d}_2(S,t)) \, .
\end{aligned}
$$

Here we have used the fact that $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-v^2/2} dv = 1$ and we have defined $\tilde{d}_2(S,t)$ identical to d_2 but with μ replacing r. That is

$$
\tilde{d}_2(S,t) = \frac{\log(S/E) + (\mu - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \, .
$$

Exercise 18 (synthesis of options from standard calls)

Assume that we *can* synthesize the general option value $V(S,t)$ with a strike density function $f(E)$ and European calls $C(S,t;E)$ i.e. that

$$
V(S,t) = \int_0^{\infty} f(E) C(S,t;E) dE
$$

To find the function $f(\cdot)$ we note that at expiration our option has a payoff $\Lambda(S)$. That is when $t = T$ we have $V(S,T) = \Lambda(S)$ and $C(S,T;E) = \max(S-E,0)$ and the above becomes

$$
\Lambda(S) = \int_0^{\infty} f(E) \max(S-E,0) dE = \int_0^{S} f(E)(S-E) dE \, .
$$

To find the $f(\cdot)$ that will satisfy this take the derivative of the above with respect to S and use the identity [4]

$$
\frac{d}{dt} \int_{\alpha(t)}^{\beta(t)} f(x,t) dx = \frac{d\beta(t)}{dt} f(\beta,t) - \frac{d\alpha(t)}{dt} f(\alpha,t) + \int_{\alpha(t)}^{\beta(t)} \frac{\partial f}{\partial t}(x,t) dx
$$

we see that (in baby steps)

$$
\Lambda'(S) = f(S)(S-S) + \int_0^{S} f(E) dE = \int_0^{S} f(E) dE \, .
$$

By taking another derivative and using the fundamental theorem of calculus we find that the strike density $f(\cdot)$ in terms of the payoff function $\Lambda(S)$ should be

$$f(S) = \Lambda''(S)\,.$$

Here the derivatives of Λ are taken with respect to S. Any other variable in Λ is held constant when we take the derivative.

Part (a): To verify this is correct consider the payoff $\Lambda(S) = \max(S - \hat{E}, 0)$ or that from a vanilla European call with a strike of $\hat{E}$. Then the first derivative of $\Lambda(S)$ is the Heaviside function. That is

$$\Lambda'(S) = \left\{ \begin{array}{cc} 0 & S < \hat{E} \\ 1 & S > \hat{E} \end{array} \right. = \mathcal{H}(S - \hat{E})\,,$$

while the second derivative is the delta function by $\Lambda''(S) = \delta(S - \hat{E})$. We can verify that this density f is correct by verifying the integration above. We find our option $V(S,t)$ given by

$$V(S,t) = \int_0^\infty \delta(E - \hat{E}) C(S,t;E) dE = C(S,t;\hat{E})\,.$$

Which states the our option V is in fact a European call with strike at $\hat{E}$.

Part (b): If $\Lambda(S) = S$, following the book we will write this as $\Lambda(S) = \max(S, 0)$, so that

$$\Lambda'(S) = \left\{ \begin{array}{cc} 0 & S < 0 \\ 1 & S > 0 \end{array} \right. = \mathcal{H}(S)\,,$$

so that $\Lambda''(S) = \delta(S) \equiv f(S)$. So to verify our decomposition we find that

$$V(S,t) = \int_0^\infty \delta(E) C(S,t;E) dE = C(S,t;0)\,,$$

a call with exercise price zero or a portfolio of just a single share of our stock.

The cash-or-nothing call, (denoted here as $\tilde{C}$) has a payoff given by $\Lambda(S) = \mathcal{H}(S - E)$, and an option value of $\tilde{C}(S,t) = e^{-r(T-t)}N(d_2)$. This is discussed and derived on page 38. The cash or nothing density f must satisfy

$$V(S,t) = \int_0^\infty f(E) \tilde{C}(S,t;E) dE\,,$$

so that at expiration $t = T$ $f(\cdot)$ must satisfy

$$\Lambda(S) = \int_0^\infty f(E)\mathcal{H}(S-E)dE = \int_0^S f(E)dE\,,$$

so taking the first derivative of this with respect to S shows that f must satisfy

$$f(E) = \Lambda'(S)\,.$$

Repeating parts (a) and (b) with cash or nothing calls we have that if our payoff is a European call with strike $\hat{E}$ so that $\Lambda(S) = \max(S-\hat{E},0)$ then $f(S) = \Lambda'(S) = \mathcal{H}(S-\hat{E})$. To check that this is a valid result we consider

$$\begin{aligned}\int_0^\infty \mathcal{H}(E-\hat{E})\tilde{C}(S,t;E)dE &= \int_{\hat{E}}^\infty \tilde{C}(S,t,E)dE = \int_0^{\hat{E}} e^{-r(T-t)}N(d_2)dE \\ &= e^{-r(T-t)}\int_0^{\hat{E}} N(d_2)dE = C(S,t;\hat{E})\end{aligned}$$

For part (b) if our payoff $\Lambda(S) = S$ then $f(S) = \Lambda'(S) = 1$ so that we need to verify that

$$\begin{aligned}V(S,t) &= \int_0^\infty 1\tilde{C}(S,t;E)dE \\ &= \int_0^\infty e^{-r(T-t)}N(d_2)dE = S \\ &= e^{-r(T-t)}\int_0^\infty N(d_2)dE\,,\end{aligned}$$

Exercise 19 (European calls as a function of strike)

We know that a European call has a value given by Equation 5, where d_1 and d_2 depend on the strike E. Assuming that we can take derivatives with respect to E we have that

$$\frac{\partial C}{\partial E} = SN'(d_1)\frac{\partial d_1}{\partial E} - e^{-r(T-t)}N(d_2) - Ee^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial E}\,.$$

Since

$$\frac{\partial d_1}{\partial E} = -\frac{1}{E\sigma\sqrt{T-t}} = \frac{\partial d_2}{\partial E}\,,$$

we see that

$$\frac{\partial C}{\partial E} = (SN'(d_1) - Ee^{-r(T-t)}N'(d_2))\frac{\partial d_1}{\partial E} - e^{-r(T-t)}N(d_2)\,.$$

From Equation 11 the first expression vanishes and we are left with

$$\frac{\partial C}{\partial E} = -e^{-r(T-t)}N(d_2)\,. \tag{27}$$

Which gives a second derivative of

$$\frac{\partial^2 C}{\partial E^2} = -e^{-r(T-t)}N'(d_2)\frac{\partial d_1}{\partial E} = \frac{e^{-r(T-t)}N'(d_2)}{E\sigma\sqrt{T-t}}\,. \tag{28}$$

Taking the time derivative of C we have

$$\frac{\partial C}{\partial t} = SN'(d_1)\frac{\partial d_1}{\partial t} - Ee^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial t} - Ere^{-r(T-t)}N(d_2)\,.$$

We use Equation 10 for the first simplification to find

$$\frac{\partial C}{\partial t} = \left(SN'(d_1) - Ee^{-r(T-t)}N'(d_2)\right)\frac{\partial d_2}{\partial t} - \frac{SN'(d_1)\sigma}{2\sqrt{T-t}} - Ere^{-r(T-t)}N(d_2)\,.$$

We now use Equation 11 in two ways. The first is to recognize that the first term in parenthesis above vanishes and the second is to convert the $N'(d_1)$ into an equivalent expression in terms of $N'(d_2)$. These transformations give

$$\frac{\partial C}{\partial t} = -\frac{E\sigma e^{-r(T-t)}N'(d_2)}{2\sqrt{T-t}} - Ere^{-r(T-t)}N(d_2)\,.$$

Recalling Equation 27 and 28 we finally end up with

$$\frac{\partial C}{\partial t} = -\frac{1}{2}\sigma^2 E^2\frac{\partial^2 C}{\partial E^2} + Er\frac{\partial C}{\partial E}\,,$$

the desired expression.

Chapter 6 (Variations on the Black-Scholes Model)

Additional Notes and Derivations

The solution of the dividend modified Black-Scholes equation

Notice that Equation 6.3 is *not* simply the standard Black-Scholes equation with r replaced with $r - D_0$ because the coefficient of the V term is r and not $r - D_0$. Thus to use the known solution to the Black-Scholes equation we must have the coefficients of $S\frac{\partial V}{\partial S}$ and V to have the same value. Since this is not yet true we need another way to proceed. As suggested in the book for the case of European calls with $V = C$ we can let $C = e^{-D_0(T-t)}C_1$, so that the t derivative of C in terms of C_1 becomes

$$C_t = D_0 e^{-D_0(T-t)} C_1 + e^{-D_0(T-t)} C_{1t} \, .$$

Putting this value of C_t into the modified Black-Scholes equation our equation for C_1 becomes

$$\begin{aligned} 0 &= D_0 e^{-D_0(T-t)} C_1 + e^{-D_0(T-t)} C_{1t} + \frac{1}{2}\sigma^2 S^2 e^{-D_0(T-t)} \frac{\partial^2 C_1}{\partial S^2} \\ &+ (r - D_0) e^{-D_0(T-t)} S \frac{\partial C_1}{\partial S} - r e^{-D_0(T-t)} C_1 \, , \end{aligned}$$

or by multiplying by $e^{D_0(T-t)}$ we have that C_1 satisfies

$$C_{1t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C_1}{\partial S^2} + (r - D_0) S \frac{\partial C_1}{\partial S} - (r - D_0) C_1 = 0 \, ,$$

which we see is the *exact* Black-Scholes equation for a European call with an "interest rate" parameter of $r - D_0$ rather than r. From the known solution to the Black-Scholes equation the solution for C_1 is

$$C_1(S,t) = SN(d_{10}) - Ee^{-(r-D_0)(T-t)} N(d_{20}) \, ,$$

with d_{10} and d_{20} defined in the same way as the previous definitions of d_1 and d_2 but with the interest rate parameter r there replaced with $D_0 - r$. So that the full expression for $C(S,t)$ is given by

$$\begin{aligned} C(S,t) &= e^{-D_0(T-t)} C_1(S,t) \\ &= e^{-D_0(T-t)} SN(d_{10}) - Ee^{-r(T-t)} N(d_{20}) \, . \end{aligned}$$

Another solution method

We begin with the dividend modified Black-Scholes equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0) S \frac{\partial V}{\partial S} - rV = 0\,.$$

We will simplify this by changing the variables from (S, t) to $(\hat{S}, \hat{t})$ defined as $\hat{S} = Se^{-D_0(T-t)}$ and $\hat{t} = t$. The derivatives of the original variables transform using the standard change of variables formula

$$\begin{aligned}
\frac{\partial}{\partial t} &= \frac{\partial \hat{S}}{\partial t}\frac{\partial}{\partial \hat{S}} + \frac{\partial \hat{t}}{\partial t}\frac{\partial}{\partial \hat{t}} = D_0 S e^{-D_0(T-t)}\frac{\partial}{\partial \hat{S}} + \frac{\partial}{\partial \hat{t}} = D_0 \hat{S}\frac{\partial}{\partial \hat{S}} + \frac{\partial}{\partial \hat{t}} \\
\frac{\partial}{\partial S} &= \frac{\partial \hat{S}}{\partial S}\frac{\partial}{\partial \hat{S}} + \frac{\partial \hat{t}}{\partial S}\frac{\partial}{\partial \hat{t}} = e^{-D_0(T-t)}\frac{\partial}{\partial \hat{S}}\,.
\end{aligned}$$

From which we see that the second derivative with respect to S becomes

$$\frac{\partial^2}{\partial S^2} = e^{-D_0(T-t)}\frac{\partial}{\partial \hat{S}}\left(e^{-D_0(T-t)}\frac{\partial}{\partial \hat{S}}\right) = e^{-2D_0(T-t)}\frac{\partial^2}{\partial \hat{S}^2}\,.$$

In terms of these new variables the dividend modified Black-Scholes equation above becomes

$$\frac{\partial V}{\partial \hat{t}} + \frac{1}{2}\sigma^2 \hat{S}^2 \frac{\partial^2 V}{\partial \hat{S}^2} + r\hat{S}\frac{\partial V}{\partial \hat{S}} - rV = 0\,.$$

We recognized this as the regular Black-Scholes equation for which we know the solution (considering the case where V represents a European call option) i.e.

$$\begin{aligned}
V &= \hat{S}N(\hat{d}_1) - Ee^{-r(T-\hat{t})}N(\hat{d}_2) \quad \text{with} \\
\hat{d}_1 &= \frac{\log(\hat{S}/E) + (r + \frac{1}{2}\sigma^2)(T - \hat{t})}{\sigma\sqrt{T - \hat{t}}}\,,
\end{aligned}$$

and a similar expression for $\hat{d}_2$. When we convert back to the financial variables S and t we have that

$$\begin{aligned}
\hat{d}_1 &= \frac{\log(Se^{-D_0(T-t)}/E) + (r + \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}} \\
&= \frac{\log(S/E) + (r - D_0 + \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}} = d_{10}\,,
\end{aligned}$$

with a similar transformation for $\hat{d}_2$. Then V becomes

$$V = e^{-D_0(T-t)} S N(d_{10}) - E e^{-r(T-t)} N(d_{20}) \,,$$

the same expressions as before.

Time dependent interest rates and volatility

Given the desire to introduce the variables $\bar{S}$, $\bar{V}$ and $\bar{t}$ defined as $\bar{S} = S e^{\alpha(t)}$, $\bar{V} = V e^{\beta(t)}$, and $\bar{t} = \gamma(t)$. Do do this we begin by computing how the derivatives will transformation under this change of variables. For the t derivative of V we find

$$\frac{\partial V}{\partial t} = \frac{\partial}{\partial t}(e^{-\beta(t)}\bar{V}) = e^{-\beta(t)}\frac{\partial \bar{V}}{\partial t} - \dot{\beta}(t) e^{-\beta(t)}\bar{V} \,.$$

Next we transform directly the various partial derivatives using the normal change of variable formulas

$$\begin{aligned}
\frac{\partial}{\partial t} &= \frac{\partial \bar{t}}{\partial t}\frac{\partial}{\partial \bar{t}} + \frac{\partial \bar{S}}{\partial t}\frac{\partial}{\partial \bar{S}} = \dot{\gamma}(t)\frac{\partial}{\partial \bar{t}} + S e^{\alpha(t)}\dot{\alpha}(t)\frac{\partial}{\partial \bar{S}} \\
&= \dot{\gamma}(t)\frac{\partial}{\partial \bar{t}} + \bar{S}\dot{\alpha}(t)\frac{\partial}{\partial \bar{S}} \\
\frac{\partial}{\partial S} &= \frac{\partial \bar{t}}{\partial S}\frac{\partial}{\partial \bar{t}} + \frac{\partial \bar{S}}{\partial S}\frac{\partial}{\partial \bar{S}} = e^{\alpha(t)}\frac{\partial}{\partial \bar{S}} \,.
\end{aligned}$$

Now since t depends only on $\bar{t}$ and not $\bar{S}$ we have that the second derivative of S in terms of $\bar{S}$ becomes

$$\frac{\partial^2}{\partial S^2} = e^{2\alpha(t)}\frac{\partial^2}{\partial \bar{S}^2} \,.$$

With all of these modifications the Black-Scholes equation becomes

$$\begin{aligned}
0 &= e^{-\beta(t)}\left[\dot{\gamma}(t)\frac{\partial \bar{V}}{\partial \bar{t}} + \bar{S}\dot{\alpha}(t)\frac{\partial \bar{V}}{\partial \bar{S}}\right] - \dot{\beta}(t)e^{-\beta(t)}\bar{V} \\
&+ \frac{1}{2}\sigma(t)^2\bar{S}^2 e^{-2\alpha(t)}\left[e^{2\alpha(t)}\frac{\partial^2(e^{-\beta(t)}\bar{V})}{\partial \bar{S}^2}\right] + r(t)(e^{-\alpha(t)}\bar{S})e^{\alpha(t)}\frac{\partial(e^{-\beta(t)}\bar{V})}{\partial \bar{S}} \\
&- r(t)e^{-\beta(t)}\bar{V} \,.
\end{aligned}$$

or multiplying by $e^{\beta(t)}$ and simplifying some we have

$$\dot{\gamma}(t)\frac{\partial \bar{V}}{\partial \bar{t}} + \bar{S}\dot{\alpha}(t)\frac{\partial \bar{V}}{\partial \bar{S}} - \dot{\beta}(t)\bar{V} + \frac{1}{2}\sigma^2(t)\bar{S}^2\frac{\partial^2 \bar{V}}{\partial \bar{S}^2} + r(t)\bar{S}\frac{\partial \bar{V}}{\partial \bar{S}} - r(t)\bar{V} = 0\,.$$

Now grouping all of the terms with the same derivative expression together we find that the above becomes

$$\dot{\gamma}(t)\frac{\partial \bar{V}}{\partial \bar{t}} + \frac{1}{2}\sigma^2(t)\bar{S}^2\frac{\partial^2 \bar{V}}{\partial \bar{S}^2} + (r(t) + \dot{\alpha}(t))\bar{S}\frac{\partial \bar{V}}{\partial \bar{S}} - (r(t) + \dot{\beta}(t))\bar{V} = 0\,,$$

which is equation 6.16 from the book.

Problem Solutions

Problem 1 (put-call parity on options with a continuous dividend yield)

From the discussions on Page 69 in these notes recall that when continuous dividends are assumed on the underling both the call and put option prices (C and P) satisfy the differential equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + (r - D_0)S\frac{\partial V}{\partial S} - rV = 0\,. \qquad (29)$$

Since this is a linear equation the difference of two solutions or the expression $V \equiv P - C$, also satisfies this equation and has a terminal condition when $t = T$ given by

$$V(S, T) = \max(0, E - S) - \max(0, S - E) = E - S\,.$$

Thus for this problem we want to solve the dividend modified Black-Scholes Equation 29 for V with the terminal condition $V(S, T) = E - S$. To do this we will first transform the equation for V into a equation for V_1 by defining V_1 in terms of V as

$$V(S, t) = e^{-D_0(T-t)}V_1(S, t)\,.$$

Then as shown, in the sections above, V_1 satisfies the Black-Scholes equation with r replaced by $r - D_0$, that is the equation

$$\frac{\partial V_1}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V_1}{\partial S^2} + (r - D_0)S\frac{\partial V_1}{\partial S} - (r - D_0)V_1 = 0\,, \qquad (30)$$

with a terminal condition on V_1 given by $V_1(S,T) = E - S$. The solution to this equation is similar to the derivation of the put-call parity relationship for options on *non-dividend* paying stock. There we found that $P - C = Ee^{-\hat{r}(T-t)} - S$, if the interest rate was $\hat{r}$. This idea applied to solve for V_1 means that its solution is

$$V_1(S,t) = Ee^{-(r-D_0)(T-t)} - S\,,$$

since in Equation 30 the "interest rate" is $r - D_0$. We see that this means

$$Ve^{D_0(T-t)} = Ee^{-(r-D_0)(T-t)} - S$$

or solving for V and recalling that $V = P - C$ the put-call relationship for options on an asset that pay a continuous dividend yield is

$$P - C = Ee^{-r(T-t)} - Se^{-D_0(T-t)}\,. \tag{31}$$

Problem 2 (the delta on an option with an underlying having a continuous dividend yield)

Recall that the delta of a European call option is defined as $\Delta_C = \frac{\partial C}{\partial S}$, but in this situation $C(S,t) = e^{-D_0(T-t)}C_1(S,t)$ where $C_1(S,t)$ satisfies the Black-Scholes equation with r replaced with $r - D_0$. Thus $\frac{\partial C_1}{\partial S} = N(d_{10})$, with d_{10} defined earlier in this chapter. Combining these results we have

$$\Delta_C = e^{-D_0(T-t)}\frac{\partial C_1}{\partial S} = e^{-D_0(T-t)}N(d_{10})\,.$$

Problem 3 (transforming the continuous dividend Black-Scholes equation)

We begin by recalling the Black-Scholes equation with continuous dividend yield given in Equation 29. To reduce this to the diffusion equation proceed as in the previous chapter by defining a new set of variables x, τ and v defined as $S = Ee^x$, $t = T - \tau/\left(\frac{1}{2}\sigma^2\right)$, and $V = Ev(x,t)$. When we do this we get (in the same way as derived in these notes) the following equation for v

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + \left(\frac{r - D_0}{\frac{1}{2}\sigma^2} - 1\right)\frac{\partial v}{\partial x} - \frac{r}{\frac{1}{2}\sigma^2}v\,.$$

Defining $k = \frac{r}{\frac{1}{2}\sigma^2}$ and $k' = \frac{r-D_0}{\frac{1}{2}\sigma^2}$ the above becomes

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k'-1)\frac{\partial v}{\partial x} - kv\,.$$

As before let $v = e^{\alpha x + \beta \tau} u(x,\tau)$ to get

$$\beta u + \frac{\partial u}{\partial \tau} = \alpha^2 u + 2\alpha \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + (k'-1)\left(\alpha u + \frac{\partial u}{\partial x}\right) - ku\,,$$

from which we can obtain an equation with no u term by taking

$$\beta = \alpha^2 + (k'-1)\alpha - k\,,$$

while we can also eliminate the $\frac{\partial u}{\partial x}$ term by taking α such that it satisfies

$$0 = 2\alpha + k' - 1\,.$$

Solving these two expressions for α and β give $\alpha = -\frac{k'-1}{2}$ and

$$\beta \;=\; \left(\frac{k'-1}{2}\right)^2 - \frac{(k'-1)^2}{2} - k = -\frac{(k'-1)^2}{4} - k\,,$$

to give the pure diffusion equation for the unknown u. In summary we have our option price given by

$$V(S,t) = Ee^{-\left[\left(\frac{k'-1}{2}\right)x + \left(\frac{(k'-1)^2}{4} + k\right)\tau\right]} u(x,\tau)\,.$$

From which we see that we now have three dimensionless parameters k, k', and $\frac{1}{2}\sigma^2 T$. We now compute how these transformation affect the payoff. We see that the transformed final payoff for a call is given by

$$\begin{aligned} u(x,0) &= e^{-\alpha x} v(x,0) = e^{-\alpha x}\frac{V(S,T)}{E} \\ &= \frac{e^{-\alpha x}}{E}\max(0, S-E) = e^{-\alpha x}\max(0, \frac{S}{E} - 1) \\ &= e^{-\alpha x}\max(0, e^x - 1) = \max(0, e^{(1-\alpha)x} - e^{-\alpha x})\,, \end{aligned}$$

with $\alpha = -\frac{k'-1}{2}$, we have $1-\alpha = \frac{k'+1}{2}$, and the payoff function above becomes

$$u(x,0) = \max(0, e^{\left(\frac{k'+1}{2}\right)x} - e^{\left(\frac{k'-1}{2}\right)x})\,.$$

Problem 4 (bounds on calls on assets with continuous dividends)

Part (a): For this problem we want to show that when $C(S,t)$ is the value of a European call option on an underlying paying a continuous dividend that

$$\lim_{S\to\infty} \left(C(S,t) \leq \max(0, S-E)\right) \,,$$

which when $S \to \infty$ is equivalent to $C(S,t) \leq S - E$ for S large. The expression for $C(S,t)$ in this case is derived in the text. Using the first step in that derivation the above desired inequality becomes

$$e^{-D_0(T-t)} C_1(S,t) \leq S - E \,,$$

with $C_1(S,t)$ the solution to the Black-Scholes equation with r replaced with $r - D_0$. As discussed in the subsection entitled "Technical Point: Boundary Conditions at Infinity" in Chapter 3 because of this we have that

$$C_1(S,t) \propto S - Ee^{-(r-D_0)(T-t)} \quad \text{for} \quad S \to \infty \,,$$

so that we see that

$$\begin{aligned} \lim_{S\to\infty} \left(e^{-D_0(T-t)} C_1(S,t)\right) &= \lim_{S\to\infty} \left(e^{-D_0(T-t)}(S - Ee^{-(r-D_0)(T-t)})\right) \\ &= \lim_{S\to\infty} \left(Se^{-D_0(T-t)} - Ee^{-r(T-t)}\right) . \end{aligned}$$

Thus our problem simplifies to asking if the following inequality is true for large S

$$Se^{-D_0(T-t)} - Ee^{-r(T-t)} \leq S - E \,.$$

To do this we will use reversible transformations to convert the above inequality into an inequality we trivially see is true. To do this we begin by dividing by E and rearranging some to get

$$-\left(\frac{S}{E}\right)(1 - e^{-D_0(T-t)}) + (1 - e^{-r(T-t)})) \leq 0 \,.$$

Now since $e^{-D_0(T-t)} \leq 1$ and $e^{-r(T-t)} \leq 1$ the above can be converted into

$$\frac{S}{E} \geq \frac{1 - e^{-r(T-t)}}{1 - e^{-D_0(T-t)}} \,,$$

which we see will be trivially true if S is taken large enough.

Part (b): We next desire to show that a call on an asset *with* dividends is less valuable than the call on an asset without dividends. To show this we note that the value of a call option on an asset with dividends is given by the solution to

$$\left[\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV\right] - D_0 S\frac{\partial V}{\partial S} = 0\,,$$

where the term in braces is the standard Black-Scholes operator (defined as $\mathcal{BS}(\cdot)$) and is exactly zero when apply it to a European call on an underlying that does not pay dividends. Next note that $D_0 S\frac{\partial V}{\partial S}$ is positive for all S. To sum to zero the the Black-Scholes operator in brackets must be non-negative when operating on a European call option (defined as V_1) on an underlying that pays dividends that is we know that

$$\mathcal{BS}(V_1) \geq 0\,.$$

while for the the call-option on the underlying that pays no dividends (defined as V_2) requires

$$\mathcal{BS}(V_2) = 0\,.$$

Due to the $-rV$ term in the Black-Scholes equation this implies that $V_1 \leq V_2$ and we have shown the required inequality.

Problem 5 (put options on an underlying that pays dividends)

For a *put* option on an underlying that pays a continuous dividend yield of D_0, we will have the same governing differential equation that we have been considering (the dividend modified Black-Scholes equation)

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial S^2} + (r - D_0)S\frac{\partial V}{\partial S} - rV = 0\,,$$

but with a terminal condition appropriate for a put option. Following the steps outlined above the substitution $P(S,t) = e^{-D_0(T-t)}P_1(S,t)$ reduces this equation to the basic Black-Scholes equation but with r replaced by $r - D_0$. Thus the value of a European put option on an asset that pays a continuous dividend yield D_0 is found to be

$$\begin{aligned} P(S,t) &= e^{-D_0(T-t)}\left(Ee^{-(r-D_0)(T-t)}N(-d_{20}) - SN(-d_{10})\right) \\ &= Ee^{-r(T-t)}N(-d_{20}) - Se^{-D_0(T-t)}N(-d_{10})\,. \end{aligned}$$

Here d_{10} and d_{20} are defined in the text in the same way they are defined earlier. We next consider the valuation of a put option that pays a discrete dividend at t_d.

To value a put option on an underlying that pays a dividend at time t_d, we will follow the discussion in the book which focused on valuing a call option under a similar situation. Specifically we perform the following algorithmic steps:

- Solve the Black-Scholes equation backwards from expiry $t = T$ until just *after* the dividend date $t = t_d^+$.
- Implement the jump condition across the dividend payment time $t = t_d$ to find the option value at the time just *before* the dividend date $t = t_d^-$.
- Solve the Black-Scholes equation backwards from this time $t = t_d^-$ with terminal conditions that are computed in the above step.

For the first step (and using the notation $P(S, t; E)$ to represent the pure Black-Scholes put solution and $P_d(S, t; E)$ the same but on an dividend paying underlying) when solving from $t = T$ to $t = t_d^+$ we have

$$P_d(S, t) = P(S, t; E) \quad \text{for} \quad t_d^+ \leq t \leq T \,,$$

simply the put solution to the normal Black-Scholes equation for times between t_d^+ and T. The jump condition requires that across t_d we have

$$\begin{aligned} P_d(S, t_d^-) &= P_d(S(1 - d_y), t_d^+) \\ &= P(S(1 - d_y), t_d^+) \,. \end{aligned}$$

To complete this procedure we need to solve the Black-Scholes equation for a put which has a terminal condition at t_d^- of $P(S(1-d_y), t_d^+; E)$. As discussed in the book (in the case of a European call) this multiplication of $1 - d_y$ is a uniform scaling of S which leaves the Black-Scholes equation invariant so the solution for $0 \leq t \leq t_d^-$ is $P(S(1-d_y), t; E)$ simply the normal Black-Scholes European put solution evaluated at $(1 - d_y)S$. Evaluating this expression at $t = T$ gives the following

$$\begin{aligned} P(S(1 - d_y), T; E) &= \max(E - S(1 - d_y), 0) \\ &= (1 - d_y) \max(E(1 - d_y)^{-1} - S, 0) \,, \end{aligned}$$

which conveys the fact that $P(S(1-d_y), t; E)$ can also be represented as the value of $(1-d_y)$ standard puts, with an exercise price of $E(1-d_y)^{-1}$. That is

$$P_d(S,t) = (1-d_y)P(S,t;E(1-d_y)^{-1}) \quad \text{for} \quad 0 \le t \le t_d^- .$$

In summary we have that

$$P_d(S,t) = \begin{cases} (1-d_y)P(S,t;E(1-d_y)^{-1}) & 0 \le t \le t_d^- \\ P(S,t;E) & t_d^+ \le t \le T \end{cases} .$$

Dividends *increase* the value of a put option since they decrease the value of the underlying by an amount equal to the dividend and make it more likely that the option will expire in the money.

Problem 6 (a call option on an option that pays two dividends)

We follow the general procedure for dealing with options on underlying that pay discrete dividends i.e. we work backwards from expiration at $t = T$ and apply jump conditions as we cross each interface. If our two dividends are paid at the times $t = t_1$ and $t = t_2$ ($t_1 < t_2$) with dividend values d_1 and d_2 then using the procedure of working backwards we then have

$$C_d(S,t) = C(S,t;E) \quad t_2 \le t \le T .$$

For the first jump condition at $t = t_2$ as in the book we have

$$C_d(S,t) = (1-d_2)C(S,t;E(1-d_2)^{-1}) \quad \text{for} \quad t_1^+ \le t \le t_2^- .$$

To work from t_1^+ back down to 0 we need to apply the jump condition $V(S,t_d^-) = V(S(1-d_y), t_d^+)$ which gives in this case (since we are working across t_1) the following

$$\begin{aligned} C_d(S,t_1^-) &= C_d(S(1-d_1), t_1^+) \\ &= (1-d_2)C(S(1-d_1), t_1^+; E(1-d_2)^{-1}) . \end{aligned}$$

From the multiplicative invariants of S in the Black-Scholes equation we can show that $V(\alpha S, t; E) = \alpha V(S, t; E\alpha^{-1})$, so that the above becomes

$$C_d(S;t_1^-) = (1-d_2)^{-1}(1-d_1)^{-1}C(S,t_1^+;E(1-d_2)^{-1}(1-d_1)^{-1}) .$$

Thus in summary our call option on an asset that pays two discrete dividends is given by

$$C_d(S,t) = \begin{cases} (1-d_2)^{-1}(1-d_1)^{-1}C(S,t;E(1-d_2)^{-1}(1-d_1)^{-1}) & 0 \le t \le t_1^- \\ (1-d_2)^{-1}C(S,t;E(1-d_2)^{-1}) & t_1^+ \le t \le t_2^- \\ C(S,t;E) & t_2^+ \le t \le T \end{cases} .$$

Problem 7 (a constant dividend model)

A constant value of D for a dividend at time t_d would cause the stock price to decrease by exactly D at this time. That is $S(t_d^+) = S(t_d^-) - D$. From the option continuity conditions we will still require that $V(S, t_d^-) = V(S-D, t_d^+)$, as a jump condition required on our derivative product. To evaluate the value of a call option we will work backwards from expiry at $t = T$ to $t = t_d$ and then from t_d to 0 using the above jump condition. We begin by noting that

$$C_d(S,t) = C(S,t;E) \quad \text{for} \quad t_d^+ \leq t \leq T\,.$$

Our function C_d must have a boundary condition at $t = t_d$ consistent with the above or

$$\begin{aligned} C_d(S, t_d^-) &= C_d(S - D, t_d^+) \\ &= C(S - D, t_d^+; E)\,. \end{aligned}$$

Now this translation (by D) of the underlying price S is an invariant transformation in the Black-Scholes equation so we can evaluate the *function* $C(S - D, t; E)$ by observing that at expiry it has a value of

$$\begin{aligned} C(S - D, T; E) &= \max(S - D - E, 0) \\ &= \max(S - (D + E), 0)\,, \end{aligned}$$

which we recognize as the payoff for a standard call option with an expiration $D + E$. Thus before t_d we have our option price given by $C(S, t; D + E)$. To summarize then the value of a European call on an underlying paying a fixed dividend with value D at time t_d is given by

$$C_d(S,t) = \begin{cases} C(S,t;D+E) & 0 \leq t \leq t_d^- \\ C(S,t;E) & t_d^+ \leq t \leq T\,. \end{cases}$$

Having completed our analysis on a European call, we now perform the same but for a European put.

For a European put the only step in the above derivation that might change is the evaluation of the function $P(S - D, t; E)$. In this case we by observing that at expiration we have

$$\begin{aligned} P(S - D, T; E) &= \max(E - (S - D), 0) \\ &= \max(E + D - S, 0)\,, \end{aligned}$$

which we recognize as the standard payoff of a European put with expiration $E+D$. Thus our put option evaluates to

$$P_d(S,t) = \begin{cases} P(S,t;D+E) & 0 \le t \le t_d^- \\ P(S,t;E) & t_d^+ \le t \le T \end{cases} .$$

This expression is analogous to the earlier expression for a call. A problem with this dividend model is that it can result in negative stock values if $S < D$.

Problem 8 (paying a premium to enter a forward contract)

One way to solve this problem is to use an arbitrage argument as follows. Since the holder of the short forward contract (the person who guarantees the delivery of the stock at time T) receives an amount Z at the beginning of the contract they must only now borrow an amount $S(t)-Z$ to buy the asset to hedge all risk away. Now at any time $T-t$ later the holder of the short position must pay the "bank" back an amount $(S(t)-Z)e^{r(T-t)}$. For there to be no arbitrage this value must also *be* the price of the forward contract so that

$$F = (S(t)-Z)e^{r(T-t)} .$$

Problem 9 (the random walk followed by the futures price F)

The futures price in terms of the underlying price S is given by

$$F = Se^{r(T-t)} ,$$

so that the stochastic differential of F is given by (and the random walk that it satisfies)

$$\begin{aligned} dF &= \frac{\partial F}{\partial S} dS + \frac{\partial F}{\partial t} dt \\ &= e^{r(T-t)}(\mu S dt + S\sigma dX) - rSe^{r(T-t)} dt \\ &= (\mu - r)Se^{r(T-t)} dt + \sigma S e^{r(T-t)} dX \\ &= (\mu - r)F dt + \sigma F dX \end{aligned}$$

or

$$\frac{dF}{F} = (\mu - r)dt + \sigma dX .$$

Problem 10 (the forward/futures put-call parity relationship)

We will use the Black-Scholes portfolio evaluation approach to derive the put-call parity result here. Consider a portfolio long one call and short one put given by $\Pi = C - P$, then at expiration this portfolio has a value given by $\Pi(F, T)$ or

$$\max(F-E,0)-\max(E-F,0) = \begin{cases} 0-(E-F) = F-E & \text{when} \quad E > F \\ (F-E)-0 = F-E & \text{when} \quad E < F \end{cases} .$$

Thus independent of the relationship between E and F the portfolio at expiration is valued at $F-E$ at a time $T-t$ from the current time t. To avoid arbitrage, this portfolio must be worth the present value of an amount $F-E$ at a time $T-t$ from now. Thus we have

$$C - P = (F - E)e^{-r(T-t)} ,$$

as the put-call parody relationship. By analogy with the results in this chapter if the asset pays a continuous dividend yield of D_0 the put-call parody relationship above is given by

$$C - P = (F - E)e^{-(r-D_0)(T-t)} .$$

Problem 11 (the forward price on an asset that pays a dividend)

Consider an option V on this underlying. Now the holder of the long side of this futures contract (at time T) receives an asset worth S and pays an amount F. Thus at expiration the value of this option is $V(S, T) = S - F$. Working backwards from $t = T$ to the dividend date $t = t_d^+$ we have the value of this option given by

$$S - Fe^{-r(T-t)} \quad \text{for} \quad t_d^+ \leq t \leq T ,$$

Across the dividend date from the continuity of option prices we know that $V(S, t_d^-) = V(S(1 - d_y), t_d^+)$ and applying the dividend continuity equation above we see that

$$\begin{aligned} V(S, t_d^-) &= S(1 - d_y) - Fe^{-r(T-t_d^-)} \\ &= (1 - d_y)\left[S - \frac{F}{1 - d_y}e^{-r(T-t_d^-)}\right] . \end{aligned}$$

Now this expression can be seen to be $1 - d_y$ forward options, each with a forward price of $\frac{F}{1-d_y}$. This expression must hold down to $t = 0$ where the option is valued at zero (as there is no cost to entering the forward option contract). Equating this option value to zero gives

$$(1 - d_y)\left[S - \frac{F}{1 - d_y} e^{-r(T - t_d^-)}\right] = 0\,,$$

which when we solve for F gives

$$F = (1 - d_y) S e^{r(T-t)}\,,$$

as the price of a forward contract that pays a dividend $S(t_d)d_y$ at some point before expiration $t = T$.

Problem 12 (the range forward contract)

Let V be the value of a range forward contract. Then at expiry we see that V has a payoff function of

$$V(S,T) = \begin{cases} S - E_1 = -(E_1 - S) & S < E_1 \\ 0 & E_1 \leq S \leq E_2 \\ S - E_2 & E_2 < S \end{cases}$$

Which if we plot this payoff V as a function of S we obtain a plot like that in Figure 7. From this plot we see that the range forward contract looks like a combination of a long call with strike price E_2 and short a put with strike price E_1. Using the uniqueness of the Black-Scholes equation this means that the value of this option is given (using the notation of $P(S,t;E)$ and $C(S,t;E)$ for standard Black-Scholes values of a put and a call respectively) by

$$V(S,t) = -P(S,t;E_1) + C(S,t;E_2)\,.$$

Problem 13 (a time-varying dividend yield)

Our random walk (followed by S) will now become

$$dS = \sigma(t) S dX + (\mu - D(t)) S dt\,,$$

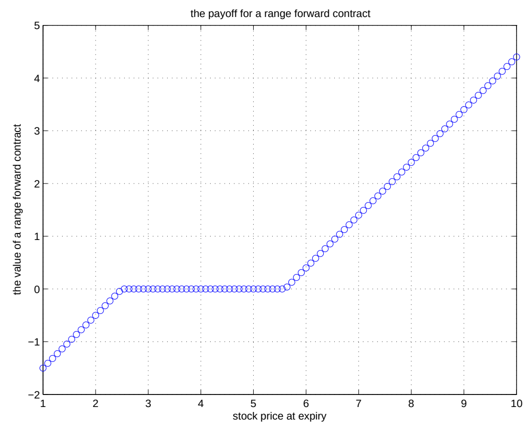

Figure 7: The payoff from a range forward contract with $E_1 = 2.5$ and $E_2 = 5.6$. Note that this is very much like a portfolio that is long one call with a strike price E_2 and short a put with strike price E_1.

while our partial differential equation satisfied by our option price is now modified in the expected way

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma(t)^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r(t) - D(t))S\frac{\partial V}{\partial S} - r(t)V = 0\,.$$

Attempting the substitution defined as

$$V(S,t) = e^{-\int_t^T D(\tau)d\tau} V_1(S,t)\,.$$

We see that the derivatives in the Black-Scholes transform as

$$\frac{\partial V}{\partial S} = e^{-\int_t^T D(\tau)d\tau}\frac{\partial V_1}{\partial S} \quad \text{with} \quad \frac{\partial^2 V}{\partial S^2} = e^{-\int_t^T D(\tau)d\tau}\frac{\partial^2 V_1}{\partial S^2}\,,$$

and

$$\frac{\partial V}{\partial t} = e^{-\int_t^T D(\tau)d\tau}\frac{\partial V_1}{\partial t} + D(t)e^{-\int_t^T D(\tau)d\tau}V_1\,.$$

When these expressions are put into the equation above, followed by multiplying by $e^{\int_t^T D(\tau)d\tau}$ on both sides we obtain the equation

$$\frac{\partial V_1}{\partial S} + \frac{1}{2}\sigma(t)^2 S^2 \frac{\partial^2 V_1}{\partial S^2} + (r(t) - D(t))S\frac{\partial V_1}{\partial S} - (r(t) - D(t))V_1 = 0\,.$$

Thus in terms of the time varying Black-Scholes equation discussed in Section 6.5 we have coefficients $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ given by

$$\begin{aligned}
\alpha(t) &= \int_t^T (r(\tau) - D(\tau))d\tau \\
\beta(t) &= \int_t^T (r(\tau) - D(\tau))d\tau \\
\gamma(t) &= \int_t^T \sigma(\tau)^2 d\tau \,.
\end{aligned}$$

Chapter 7 (American Options)

Additional Notes and Derivations

The function $\max(S - E, 0)$

In this subsection of these notes we show that $\max(S-E, 0)$ is not a solution to the Black-Scholes dividend modified equation 29. To do this let $V = \max(S - E, 0)$, then $\frac{\partial V}{\partial t} = 0$ and

$$\frac{\partial V}{\partial S} = \begin{cases} 0 & S < E \\ 1 & S > E \end{cases} = \mathcal{H}(S - E)\,,$$

where $\mathcal{H}(\cdot)$ is the Heaviside step function. From this it follows that $\frac{\partial^2 V}{\partial S^2} = \delta(S-E)$, with $\delta(\cdot)$ the Dirac delta function. When these expressions are put into the Black-Scholes dividend modified equation 29 we get

$$\frac{1}{2}\sigma^2 S^2 \delta(S - E) + (r - D_0)S\mathcal{H}(S - E) - r\max(S - E, 0)\,.$$

When $S < E$ this expression becomes

$$0 + (r - D_0)S\,(0) - r\,(0) = 0\,.$$

While if $S > E$ this becomes

$$0 + (r - D_0)S\,(1) - r\,(S - E) = -D_0 S + rE \neq 0\,.$$

This shows that $\max(S-E, 0)$ is not a solution to the Black-Scholes dividend modified equation (when $S > E$ at least).

The non-dimensional BS equation for American options

In this section of notes we derive the non-dimensional equation for an American call. Defining $S = Ee^x$, $t = T - \frac{\tau}{\frac{1}{2}\sigma^2}$, and $C(S,t) = S - E + Ec(x,\tau)$, as a first step in transforming Equation 29 into non-dimensional form the t and S derivatives of C in terms of c becomes

$$\frac{\partial C}{\partial t} = E\frac{\partial c}{\partial t} \quad \text{and} \quad \frac{\partial C}{\partial S} = 1 + E\frac{\partial c}{\partial S} \quad \text{so} \quad \frac{\partial^2 C}{\partial S^2} = E\frac{\partial^2 c}{\partial S^2}\,,$$

so that the dividend modified Black-Scholes equation in terms of the variable c becomes (after dividing by E) the following

$$\frac{\partial c}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 c}{\partial S^2} + (r - D_0)\frac{S}{E}\left(1 + E\frac{\partial c}{\partial t}\right) - r\left(\frac{S}{E} - 1 + c\right) = 0\,.$$

We now need to convert the derivatives with respect to t and S into derivatives with respect to τ and x. To do this we will use the standard transformation of coordinates. This is done exactly as in subsection and results in the following equation

$$\frac{\partial c}{\partial \tau} = \frac{\partial^2 c}{\partial x^2} + (k' - 1)\frac{\partial c}{\partial x} - kc + f(x)\,. \tag{32}$$

The non-dimensional continuity conditions for American options

We will now transform the two continuity conditions for American call options into a non-dimensional form. The continuity of the option price requires

$$C(S_f(t), t) = S_f(t) - E\,, \tag{33}$$

and the continuity of the first derivative requires

$$\frac{\partial C}{\partial S}(S_f(t), t) = 1\,. \tag{34}$$

The free boundary $S_f(t)$ under the non-dimensionalization transformations is mapped to a function $x_f(\tau)$ so that the first continuity condition Equation 33 transforms as

$$C(S_f(t), t) = S_f(t) - E \Rightarrow c(x_f(\tau), \tau) = 0\,.$$

The second continuity condition or Equation 34 becomes

$$\frac{\partial C}{\partial S}(S_f(t), t) = 1 \Rightarrow 1 + E\frac{\partial c}{\partial S} = 1\,,$$

or

$$\frac{\partial c}{\partial S} = 0 \Rightarrow \frac{\partial c}{\partial x}(x_f(\tau), \tau) = 0\,,$$

when we use the definition that $S = Ee^x$ to transform the first derivative. Finally, we transform the inequality constraint on an American call $C \geq \max(S-E,0)$ into non-dimensional variables. This constraint becomes

$$
\begin{aligned}
S - E + Ec(x,\tau) &\geq \max(S-E,0) \quad \text{or} \\
c(x,\tau) &\geq \frac{1}{E}(\max(S-E,0) - S + E) \\
&= \begin{cases} 0 & S > E \\ \frac{1}{E}(-S+E) & S < E \end{cases} \\
&= \begin{cases} 0 & x > 0 \\ 1 - e^x & x < 0 \end{cases} = \max(0, 1-e^x)\,.
\end{aligned}
$$

Removal of the lower order terms

In this section of these notes we will apply transformations that simplify Equation 32 by removing lower order terms. We begin with the transformation that removes the $\frac{\partial u}{\partial \xi}$ term from this equation. Given Equation 32 or Equation 7.19 from the book

$$
\frac{\partial c}{\partial \tau} = \frac{\partial^2 c}{\partial x^2} + (k'-1)\frac{\partial c}{\partial x} - kc + f(x)\,,
$$

we transform to a new coordinate system $(\xi, \hat{\tau})$ defined in terms of (x, τ) as

$$
\xi = x + (k'-1)\tau \quad \text{and} \quad \hat{\tau} = \tau\,.
$$

To do this we note that the derivatives transformation as follows

$$
\begin{aligned}
\frac{\partial}{\partial x} &= \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \hat{\tau}}{\partial x}\frac{\partial}{\partial \hat{\tau}} = \frac{\partial}{\partial \xi} \\
\frac{\partial}{\partial \tau} &= \frac{\partial \xi}{\partial \tau}\frac{\partial}{\partial \xi} + \frac{\partial \tau}{\partial x}\frac{\partial}{\partial \hat{\tau}} = (k'-1)\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \hat{\tau}}\,.
\end{aligned}
$$

Which when put into Equation 7.19 gives

$$
(k'-1)\frac{\partial c}{\partial \xi} + \frac{\partial c}{\partial \hat{\tau}} = \frac{\partial^2 c}{\partial \xi^2} + (k'-1)\frac{\partial c}{\partial \xi} - kc + f(x)\,,
$$

and dropping the hat on τ for notational simplicity we find that

$$
\frac{\partial c}{\partial \tau} = \frac{\partial^2 c}{\partial \xi^2} - kc + f(\xi - (k'-1)\tau)\,, \tag{35}
$$

or same the expression in the book. Next we will follow this transformation with another aimed at removing the term $-kc$ from the above equation. To do this let $c(\xi, \tau) = e^{-k\tau} w(\xi, \tau)$ so that the derivatives of c become (in terms of w) the following

$$\begin{aligned} \frac{\partial c}{\partial \xi} &= e^{-k\tau} \frac{\partial w}{\partial \xi} \\ \frac{\partial c}{\partial \tau} &= -ke^{-k\tau} w + e^{-k\tau} \frac{\partial w}{\partial \tau} . \end{aligned}$$

When we put these into Equation 35 we obtain

$$-ke^{-k\tau} w + e^{-k\tau} \frac{\partial w}{\partial \tau} = e^{-k\tau} \frac{\partial^2 w}{\partial \xi^2} - ke^{-k\tau} w - f(\xi - (k' - 1)\tau) ,$$

or simplifying some we obtain

$$\frac{\partial w}{\partial \tau} = \frac{\partial^2 w}{\partial \xi^2} - e^{k\tau} f(\xi - (k' - 1)\tau) , \quad (36)$$

the same equation as in the book.

The Algebra in the Local Analysis of the Free Boundary

In this section of these notes we will derive and extend the section of the book that deals with the local analysis of the free boundary $S_f(t)$ for American options. We begin by considering the free boundary to initially be at x_0 where x_0 satisfies $f(x_0) = 0$ and derive a partial differential equation satisfied by c around this point. We do this by considering the magnitude of every term $\frac{\partial c}{\partial \tau}$, $\frac{\partial^2 c}{\partial x^2}$, $(k' - 1)\frac{\partial c}{\partial x}$, $-kc$, and $f(x)$ in Equation 32. Hypothesizing that $\frac{\partial c}{\partial x}$ and c will be of a much smaller magnitude than the other terms we will drop these terms from this equation. For the term $f(x)$ we will replace it with an Taylor expansion about the point x_0. Since $f(x_0) = 0$ by the definition of x_0 a Taylor expansion of the function f about the point x_0 gives

$$\begin{aligned} f(x) &= f(x_0) + f'(x_0)(x - x_0) + O((x - x_0)^2) \\ &= (k' - k)e^{x_0}(x - x_0) + O((x - x_0)^2) . \end{aligned}$$

Since x_0 is a zero of $f(\cdot)$ we have that $(k' - k)e^{x_0} + k = 0$, showing that the expression $(k' - k)e^{x_0} = -k$, so an approximation to f that is valid near x_0 is given by $f(x) \approx -k(x - x_0)$.

As we are dropping the terms $\frac{\partial c}{\partial x}$ and c relative to $\frac{\partial^2 c}{\partial x^2}$ and f we have an approximate local solution $\hat{c}$ given by the solution to

$$\frac{\partial \hat{c}}{\partial \tau} = \frac{\partial^2 \hat{c}}{\partial x^2} - k(x - x_0)\,. \tag{37}$$

To solve this equation, as suggested in the book, we try a similarity solution in the variable $\xi = \frac{x-x_0}{\sqrt{\tau}}$ with a functional form for $\hat{c}(\xi)$ given by $\hat{c}(\xi) = \tau^{3/2} c^*(\xi)$. Computing first the τ derivative of ξ (which will be needed later) we find

$$\frac{\partial \xi}{\partial \tau} = -\frac{1}{2}(x - x_0)\tau^{-3/2} = -\frac{1}{2}\left(\frac{(x - x_0)}{\sqrt{\tau}}\right)\tau^{-1} = -\frac{\xi}{2\tau}\,.$$

We then have that the τ derivative of $\hat{c}$ given by

$$\frac{\partial \hat{c}}{\partial \tau} = \frac{3}{2}\tau^{1/2} c^*(\xi) + \tau^{3/2}\frac{\partial c^*}{\partial \xi}\frac{\partial \xi}{\partial \tau} = \frac{3}{2}\tau^{1/2} c^*(\xi) + \tau^{3/2}\frac{\partial c^*}{\partial \xi}\left(-\frac{\xi}{2\tau}\right)\,.$$

The two x derivatives of $\hat{c}$ are given by

$$\begin{aligned}
\frac{\partial \hat{c}}{\partial x} &= \tau^{3/2}\frac{\partial c^*}{\partial \xi}\frac{\partial \xi}{\partial x} = \tau^{3/2}\frac{\partial c^*}{\partial \xi}\left(\frac{1}{\sqrt{\tau}}\right) = \tau\frac{\partial c^*}{\partial \xi} \quad \text{so} \\
\frac{\partial^2 \hat{c}}{\partial x^2} &= \tau\frac{\partial^2 c^*}{\partial \xi^2}\frac{\partial \xi}{\partial x} = \tau\frac{\partial^2 c^*}{\partial \xi^2}\left(\frac{1}{\sqrt{\tau}}\right) = \tau^{1/2}\frac{\partial^2 c^*}{\partial \xi^2}\,.
\end{aligned}$$

When these expressions are put into Equation 37 we get

$$\frac{3}{2}\tau^{1/2} c^* - \frac{\xi}{2}\tau^{1/2}\frac{\partial c^*}{\partial \xi} = \tau^{1/2}\frac{\partial^2 c^*}{\partial \xi^2} - k\tau^{1/2}\xi\,,$$

or

$$\frac{3}{2}c^* - \frac{1}{2}\xi\frac{\partial c^*}{\partial \xi} = \frac{\partial^2 c^*}{\partial \xi^2} - k\xi\,, \tag{38}$$

which is Equation 7.23 in the book.

We now need to determine the limiting conditions on c^* under various limiting conditions on ξ. when $\xi \to -\infty$ we have that $x \to -\infty$ and we expect $\frac{\partial^2 \hat{c}}{\partial x^2} \to 0$. With these substitutions Equation 38 for $\hat{c}$ becomes

$$\frac{\partial \hat{c}}{\partial \tau} = -kx \quad \text{when} \quad x \to -\infty\,,$$

so integrating this expression once with respect to τ gives $\hat{c} \sim -k\tau x$. In terms of the function $c^* = \tau^{-3/2}\hat{c}$ this means that $\tau^{3/2}c^* \sim -k\tau x$ or

$$c^* \sim -k\tau^{-1/2}x \sim -k\xi$$

as the asymptotic condition on c^* when $\xi \to -\infty$.

We now return to solving the Equation 38,

$$\frac{\partial^2 c^*}{\partial \xi^2} + \frac{1}{2}\xi\frac{\partial c^*}{\partial \xi} - \frac{3}{2}c^* = k\xi\,.$$

We do this by first solving the homogeneous equation and then next the inhomogeneous equation. To solve the homogeneous equation we trying a solution given by a polynomial of the form $c^*(\xi) = A + B\xi + C\xi^2 + D\xi^3$. From which after taking derivatives of this expression and putting them into the above equation results in the following constraints on A, B, C, and D

$$(2C + 6D\xi) + \frac{1}{2}\xi(B + 2C\xi + 3D\xi^2) - \frac{3}{2}(A + B\xi + C\xi^2 + C\xi^3) = 0\,.$$

Equating powers of ξ on both sides of the above to have a solution requires that

$$\begin{aligned}
2C - \frac{3}{2}A &= 0 \\
6D + \frac{1}{2}B - \frac{3}{2}B &= 0 \\
C - \frac{3}{2}C &= 0 \\
\frac{3}{2}D - \frac{3}{2}D &= 0\,.
\end{aligned}$$

These equations require $A = 0$, $B = 6D$, $C = 0$, while D is arbitrary. Since D is arbitrary and we can take it to be 1. So *one* homogeneous solution is then

$$c^*(\xi) = 6\xi + \xi^3\,.$$

To find a second homogeneous solution we let $c_2^*(\xi) = c_1^*(\xi)a(\xi)$, put this expression into Equation 38 and derive an equation for $a(\xi)$ which we solve. When we put this expression into the *homogeneous* version of Equation 38 we find that $a(\xi)$ must satisfy the following ordinary differential equation

$$\xi(6 + \xi^2)\frac{d^2a}{d\xi^2} + \frac{1}{2}(24 + 18\xi^2 + \xi^4)\frac{da}{d\xi} = 0\,.$$

We recognize this equation as a linear first order equation in $a'(\xi)$. To solve it let $v(\xi)$ be defined as $v(\xi) = \frac{da(\xi)}{d\xi}$ and we see that $v(\xi)$ satisfies

$$\frac{dv(\xi)}{v(\xi)} = -\frac{1}{2}\left(\frac{24 + 18\xi^2 + \xi^4}{\xi(6+\xi^2)}\right) dx\,.$$

We next integrate both sides of this expression to obtain

$$\log(v(\xi)) = -\frac{1}{2}\left(\frac{\xi^2}{2} + 4\log(\xi) + 4\log(6+\xi^2)\right) + C\,,$$

with C a integration constant. This then gives that $v(\xi)$ as

$$v(\xi) = \frac{da(\xi)}{d\xi} = \frac{1}{\xi^2}\left(\frac{1}{(6+\xi^2)^2}\right) e^{-\frac{\xi^2}{4}}\,.$$

Integrating this expression we find that $a(\xi)$ is given by

$$a(\xi) = e^{-\frac{\xi^2}{4}}\left(-\frac{1}{36\xi} - \frac{\xi}{72(6+\xi^2)}\right) - \frac{1}{48}\sqrt{\pi}\mathrm{Erf}(\frac{\xi}{2})\,.$$

here $\mathrm{Erf}(x)$ is defined as $\mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$. Now extending the lower limit of integration in this definition to $-\infty$ will only add a constant to the expression for $\mathrm{Erf}(x)$ and correspondingly only add a constant to the expression for $a(\xi)$. The inclusion of this constant when we compute $c_2^*(\xi)$ (defined as $c_1^*(\xi)a(\xi)$) will only add a multiple of $c_1^*(\xi)$ which we already know if a solution. Thus without loss we can extend the lower limit in the definition of the error function to $-\infty$. Doing this we have that $c_2^*(\xi)$ is given by The algebra for this can be found in the Mathematica file `loc_amer_red_of_order.nb`.

$$\begin{aligned}
c_2^*(\xi) &= c_1^*(\xi)a(\xi) \\
&= -(\xi^3+6\xi)e^{-\frac{\xi^2}{4}}\left(\frac{1}{36\xi} + \frac{\xi}{72(6+\xi^2)}\right) - \frac{1}{24}(\xi^3+6\xi)\int_0^{\xi/2} e^{-t^2}dt \\
&= -e^{-\xi^2/4}\left(\frac{\xi^2+6}{36} + \frac{\xi^2}{72}\right) - \frac{1}{24}(\xi^3+6\xi)\int_{-\infty}^{\xi} e^{-v^2/2}\frac{dv}{2} \\
&= -\frac{1}{24}\left((\xi^2+4)e^{-\frac{\xi^2}{4}} + \frac{1}{2}(\xi^3+6\xi)\int_{-\infty}^{\xi} e^{-v^2/2}dv\right)\,.
\end{aligned}$$

Taking the derivative required for the expression $c_2^*(\xi_0) = \xi_0 \frac{dc_2^*(\xi_0)}{d\xi}$ we get

$$\begin{aligned}
\frac{dc_2^*}{d\xi} &= 2\xi e^{-\frac{\xi^2}{4}} + (\xi^2+4)e^{-\frac{\xi^2}{4}}\left(-\frac{1}{2}\xi\right) \\
&+ \frac{1}{2}(3\xi^2+6)\int_{-\infty}^{\xi} e^{-\frac{1}{4}s^2}ds + \frac{1}{2}(\xi^3+6\xi)e^{-\frac{1}{4}\xi^2} \\
&= 3\xi e^{-\frac{1}{4}\xi^2} + \left(\frac{3}{2}\xi^2+3\right)\int_{-\infty}^{\xi} e^{-\frac{1}{4}s^2}ds\,.
\end{aligned}$$

Thus the equation $c_2^*(\xi_0) = \xi_0 \frac{dc_2^*(\xi_0)}{d\xi}$ becomes

$$(\xi^2+4-3\xi^2)e^{-\frac{1}{4}\xi^2} + (\frac{1}{2}\xi^3+3\xi-\frac{3}{2}\xi^3-3\xi)\int_{-\infty}^{\xi} e^{-\frac{1}{4}s^2}ds = 0\,.$$

or

$$\xi^3 e^{\frac{1}{4}\xi^2}\int_{-\infty}^{\xi} e^{-\frac{1}{4}s^2}ds = 2(2-\xi^2)\,, \tag{39}$$

which is the transcendental Equation 7.24 in the book. The constant B is then given in terms of the root of Equation 39 (denoted ξ_0) by $B = \frac{k\xi_0}{c_2^*(\xi_0)}$.

Solving the transcendental equation numerically

In this section of these notes we will solve the transcendental Equation 39 numerically using the MATLAB function `fsolve`. To do this we recognize that the MATLAB function `normcdf` is defined as

$$\text{normcdf}(x,\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{(t-\mu)^2}{2\sigma^2}}dt\,,$$

So that in terms of this function Equation 39 becomes

$$2\sqrt{\pi}\xi_0^3 e^{\frac{1}{4}\xi_0^2}\text{normcdf}(\xi_0,0,\sqrt{2}) = 2(2-\xi_0^2)\,,$$

In the MATLAB file `solve_loc_am_bl.m` we begin by graphically plotting the equation above observing that the root is indeed near 0.9. We then use this as an initial guess in the `fsolve` function. Doing this we find a value for the root of $\xi_0 = 0.903447$ in agreement with the book.

Problem Solutions

Problem 1 (installment options)

Let V be the value of the installment option and consider a portfolio constructed with Δ shares of the underlying (each share with the value S), used to hedge away the risk of owning V i.e. our total portfolio Π is given by $\Pi = V - \Delta S$. Then over one timestep, dt, our portfolio would normally change as $d\Pi = dV - \Delta dS$ but since to own this option over this amount of time we also have to pay the installment of $L(t)dt$. With this included our portfolio change becomes

$$d\Pi = dV - \Delta dS - Ldt\,.$$

Since this portfolio cannot generate a return *greater* than the risk free rate r we must have $d\Pi = r\Pi dt$ which gives

$$dV - \Delta dS - Ldt = r\Pi dt\,.$$

This becomes a differential equation given by $\mathcal{BS}(V) = L$, where $\mathcal{BS}(\cdot)$ is the Black-Scholes operator, for the value of an installment option V. Obviously, $V \geq 0$ since if the option value ever went negative, the option holder would simply stop the installment payments and the option would lapse. Thus when $V \geq 0$ we have $\mathcal{BS}(V) = L$, and otherwise $V = 0$.

Problem 2 (some inequalities for American options)

That the put price must be larger than its payoff i.e. $P \geq \max(E - S, 0)$ has been shown with an arbitrage argument in the beginning of this chapter.

To show the inequality $C \geq S - Ee^{-r(T-t)}$ consider a portfolio Π given by $\Pi = C - S + Ee^{-r(T-t)}$. That is our portfolio is long one call, short one unit of stock, and long an amount of cash that will pay E at time T. If we exercise the American call option, C, *before* expiry, we payout an amount of E and receive an asset valued at S, then our portfolio becomes

$$\Pi' = -E + S - S + Ee^{-r(T-t)} = -E + Ee^{-r(T-t)} < 0\,,$$

showing that by exercising our American call early we have a negative net value. If we wait until the expiration time $t = T$ then $\Pi(T) = C - S + E$, which if $S \geq E$, we will exercise our call, and obtain $\Pi(T) = -E+S-S+E =$

0, while if $S < E$ we will not, and obtain $\Pi(T) = -S + E$, which is positive. Thus if we exercise before $t = T$ we loose money and it would be preferable to wait until $t = T$. The upside of this discussion is that it is *not* optimal to exercise early for an American call. Since a rational investor (who had this portfolio) would seek to maximize his/her profits we must have $\Pi \geq 0$ or

$$C \geq S - Ee^{-r(T-t)}\,.$$

From this discussion we see that an American put is *more* valuable than an European put, while an American call has the *same* value as its European counterpart. If we denote American option prices with no subscript and European option prices with the subscript E from this statement we have that

$$C = C_E \quad \text{and} \quad P \geq P_E\,.$$

So that by adding the negative of the second inequality to the first

$$C - P \leq C_E - P_E\,.$$

The right hand side of this (from the normal European put-call parity relationship) is given by $S - Ee^{-r(T-t)}$ giving the inequality of

$$C - P \leq S - Ee^{-r(T-t)}\,,$$

or the second half of the third desired inequality.

Finally, to show the inequality $S - E \leq C - P$, consider the portfolio Π defined by $\Pi = C - P - S + E$. Since it is not optimal to exercise the America call early, we can assume that this will not happen, while if the put is exercised early our portfolio goes to

$$\Pi' = C - (E - S) - S + E = C \geq 0\,.$$

Since when exercising an American put early we sell an asset worth S at the exercise price E (the net effect on our portfolio from this transaction is $E - S$). At expiration this portfolio is valued

$$\Pi(T) = C(T) - P(T) - S + E = \begin{cases} 0 - (E - S) - S + E = 0 & S < E \\ S - E - 0 - S + E = 0 & S > E \end{cases}$$

showing that $\Pi(T) = 0$. In all cases $\Pi(t) \geq 0$, showing that

$$C - P - S + E \geq 0\,,$$

the desired inequality.

Problem 3 (solving obstacle problems)

For this problem one could use symmetry to recognize that $x_P = -x_Q$ and avoid having to do twice as much work. For exposition in the solution below I set up the required subproblems for both x_P and x_Q.

A differential equation formulation for the obstacle problem is given in the book by Equation 7.4 given by

$$\begin{array}{ccc} & u(-1) = 0 & \\ u''(x) = 0 & \text{for} & -1 < x < x_P \\ u(x_P) = f(x_P) & \text{and} & u'(x_P) = f'(x_P) \\ u(x) = f(x) & \text{for} & x_P < x < x_Q \\ u(x_Q) = f(x_Q) & \text{and} & u'(x_Q) = f'(x_Q) \\ u''(x) = 0 & \text{for} & x_Q < x < 1 \\ & u(1) = 0\,. & \end{array}$$

As discussed in the book, the location of the points x_P and x_Q are unknown and need to be defined by the solutions to the above equations. In the regions $-1 < x < x_P$ and $x_Q < x < 1$ we also require our unknown function to be *linear*.

Part(a): With $f(x) = \frac{1}{2} - x^2$, we will take $u(x) = A + Bx$ for $-1 < x < x_P$ and $u(x) = C + Dx$ for $x_Q < x < 1$. With these the above equations specialize to

$$\begin{array}{ccc} & A - B = 0 & \\ u'' = 0 & & -1 < x < x_P \\ A + Bx_P = \frac{1}{2} - x_P^2 & & B = -2x_P \\ u(x) = \frac{1}{2} - x^2 & & x_P < x < x_Q \\ C + Dx_Q = \frac{1}{2} - x_Q^2 & & D = -2x_Q \\ u'' = 0 & & x_Q < x < 1 \\ & C + D = 0\,. & \end{array}$$

We thus conclude that $A = B = -2x_P$ and $C = -D = 2x_Q$. Using the required continuity conditions at $x = x_P$ and $x = x_Q$ we have that x_P and x_Q must satisfy

$$-2x_P - 2x_P^2 = \frac{1}{2} - x_P^2 \quad \text{and} \quad 2x_Q - 2x_Q^2 = \frac{1}{2} - x_Q^2\,.$$

From which we find that x_P satisfies the following quadratic equation, and has a solution given by

$$x_P^2+2x_P+\frac{1}{2}=0 \Rightarrow x_P=\frac{-2\pm\sqrt{4-4(1)(1/2)}}{2}=-1\pm\frac{1}{\sqrt{2}}=\begin{cases}-1.7071\\-0.2929\end{cases},$$

While for x_Q we find

$$x_Q^2-2x_Q+\frac{1}{2}=0 \Rightarrow x_Q=\frac{2\pm\sqrt{4-4(1)(1/2)}}{2}=1\pm\frac{1}{\sqrt{2}}=\begin{cases}0.2929\\1.7071\end{cases}.$$

Since we must have $x_P > -1$ and $x_Q < 1$ we conclude that $x_P = -0.2929$ and $x_Q = +0.2929$ (demonstrating the symmetry mentioned above). From these results we plot the solution to this obstacle problem in Figure 8 (left).

Part (b): When $f(x) = \frac{1}{2} - \sin^2\left(\frac{\pi x}{2}\right)$ following the same procedure as above we have that the differential formulation again requires $A = B$ and $C = -D$. Before determining the equations for x_P and x_Q we notice the following identity holds on the derivative of $f(x)$

$$\begin{aligned} f'(x) &= -2\sin(\frac{\pi x}{2})\cos(\frac{\pi x}{2})\frac{\pi}{2} = -\pi\sin(\frac{\pi x}{2})\cos(\frac{\pi x}{2}) = -\frac{\pi}{2}\sin(2\left(\frac{\pi x}{2}\right)) \\ &= -\frac{\pi}{2}\sin(\pi x)\,. \end{aligned}$$

Using this result, the continuity condition at $x = x_P$ becomes

$$A + Bx_P = \frac{1}{2} - \sin^2\left(\frac{\pi x_P}{2}\right) \quad \text{and} \quad B = -\frac{\pi}{2}\sin(\pi x_P)\,.$$

Putting the second equation into the first equation and remembering that $A = B$ we find that the value of x_P that satisfies

$$-\frac{\pi}{2}(1+x_P)\sin(\pi x_P) = \frac{1}{2} - \sin^2(\frac{\pi x_P}{2})\,.$$

This equation would have to be solved numerically. For the point x_Q the continuity condition becomes

$$C + Dx_Q = \frac{1}{2} - \sin^2(\frac{\pi x_Q}{2}) \quad \text{and} \quad D = -\frac{\pi}{2}\sin(\pi x_Q)\,.$$

Using the above relations for C and D this gives the equation for x_Q of

$$-\frac{\pi}{2}(-1+x_Q)\sin(\pi x_Q) = \frac{1}{2} - \sin^2(\frac{\pi x_Q}{2})\,.$$

The solutions to this obstacle problem is plotted in Figure 8 (right). The calculations and plots for this problem are performed in the Matlab functions `prob_7_3_pt_a.m` and `prob_7_3_pt_b.m` respectively.

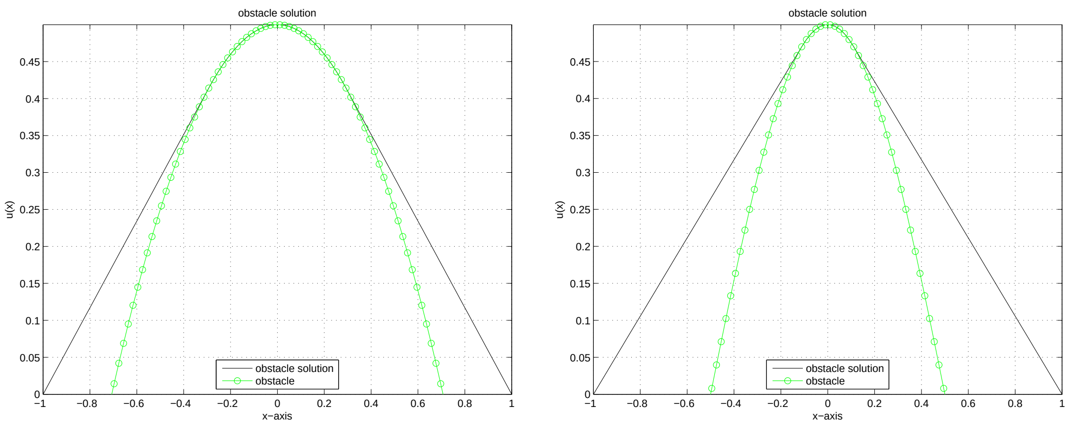

Figure 8: The solution to the obstacle problem. **Left:** For an obstacle given by $f(x) = \frac{1}{2} - x^2$. **Right:** For an obstacle with a profile given by $f(x) = \frac{1}{2} - \sin(\pi x/2)^2$.

Problem 4 (the variational inequality)

If u and v are both elements of $\mathcal{K}$ and defining $g(x)$ to be $(1-\lambda)u + \lambda v$ with $0 \le \lambda \le 1$ we see that g is piecewise continually differentiable since both u and v are and g is a linear combination of these two functions. At the two end points ± 1 we have

$$g(\pm 1) = (1-\lambda)u(\pm 1) + \lambda v(\pm 1) = (1-\lambda)0 + \lambda 0 = 0\,.$$

Also since $u \ge f$ and $v \ge f$, by multiplying the first inequality by $1-\lambda$, the second by λ, and adding we get

$$(1-\lambda)u + \lambda v \ge (1-\lambda)f + \lambda f = f\,.$$

Thus the space $\mathcal{K}$ is compact.

Since we assume that u is the minimizer of $E[\cdot]$ letting v be any other test function then $E[\cdot]$ evaluated at $u + \lambda(v-u)$, (since it is a different function than the minimizer u) must have a larger value than $E[u]$. That is

$$\begin{aligned} E[u + \lambda(v-u)] &\ge E[u] \quad \text{or} \\ E[(1-\lambda)u + \lambda u] &\ge E[u] \quad \text{or} \\ E[(1-\lambda)u + \lambda u] - E[u] &\ge 0\,. \end{aligned}$$

Writing this expression in terms of its integral equivalent we have

$$\frac{1}{2}\int_{-1}^{+1}((1-\lambda)u' + \lambda v')^2 - u'^2)dx \geq 0\,.$$

Rewriting the argument of the integrand in terms of $u' + \lambda(v' - u')$ as

$$\frac{1}{2}\int_{-1}^{+1}(u'\lambda(v' - u'))^2 - u'^2)dx \geq 0\,.$$

Expanding the quadratic, canceling, and dividing by λ we have

$$2\int_{-1}^{+1} u'(v' - u')dx \geq -\lambda \int_{-1}^{+1}(v' - u')^2 dx\,.$$

By taking λ smaller and smaller (but still positive and nonzero) the right hand side of the above approaches zero from below, while the left hand side is independent of λ. The limiting case gives the desired inequality

$$\int_{-1}^{+1} u'(v' - u')dx \geq 0\,.$$

Problem 7 (American call and put problems as (S, t) linear complementary problems)

An American put requires that

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP \leq 0\,,$$

and $P \geq \max(E - S, 0)$, so the linear complementary problem for an American put option in terms of the *financial* variables (S, t) is given by

$$\begin{aligned}
-\left(\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP\right)(P - \max(E - S, 0)) &= 0 \\
-\left(\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP\right) &\geq 0 \\
P - \max(E - S, 0) &\geq 0\,.
\end{aligned}$$

For an American call, we have $C \geq \max(S - E, 0)$ and the standard Black-Scholes inequality

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC \leq 0\,,$$

which translates into a linear complementary problem exactly as an American put (done above).

Problem 8 (solving the Stefan problem)

Given the proposed similarity solution $u(x,\tau) = u^*(x/\sqrt{\tau})$ and $x_f(\tau) = \xi_0\sqrt{\tau}$, we see that

$$
\begin{aligned}
u_x &= u^*_\xi(\xi)\left(\frac{1}{\sqrt{\tau}}\right) \quad \text{so} \quad u_{xx} = u^*_{\xi\xi}(\xi)\left(\frac{1}{\tau}\right) \\
u_\tau &= u^*_\xi(\xi)\left(-\frac{1}{2}x\tau^{-3/2}\right) = u^*_\xi(\xi)\left(-\frac{1}{2}\frac{\xi}{\tau}\right).
\end{aligned}
$$

When these expressions are put into the given diffusion equation we obtain

$$
-\frac{1}{2}\left(\frac{\xi}{\tau}\right)u^*_\xi = \frac{1}{\tau}u^*_{\xi\xi}\,,
$$

or multiplying by τ and rearranging we obtain

$$
u^*_{\xi\xi} + \frac{1}{2}\xi u^*_\xi = 0\,, \tag{40}
$$

as our ordinary differential equation for u^*. The free boundary conditions for u reduce to boundary conditions on u^*. For example, the condition $u(x_f(\tau),\tau) = 0$ reduces to

$$
u(x_f(\tau),\tau) = u^*\left(\frac{x_f(\tau)}{\sqrt{\tau}}\right) = u^*\left(\frac{\xi_0\sqrt{\tau}}{\sqrt{\tau}}\right) = u^*(\xi_0) = 0\,. \tag{41}
$$

While the condition $\frac{\partial u}{\partial x}(x_f(\tau),\tau) = -\frac{dx_f(\tau)}{d\tau}$ reduces to (using the derivative u_ξ above)

$$
\frac{1}{\sqrt{\tau}}u^*_\xi(\xi_0) = -\frac{d}{d\tau}(\xi_0\sqrt{\tau}) = -\frac{\xi_0}{2}\frac{1}{\sqrt{\tau}}\,,
$$

or

$$
u^*_\xi(\xi_0) = -\frac{1}{2}\,. \tag{42}
$$

Now we recognized that Equation 40 is a linear equation in u^*_ξ and has an integrating factor [1] given by

$$
\exp\left\{\int_0^\xi \frac{1}{2}s\,ds\right\} = \exp\left\{\frac{1}{4}\xi^2\right\}\,.
$$

Multiplying both sides of Equation 40 by this integrating factor we recognize that it can be written as

$$\frac{d}{d\xi}\left(e^{\xi^2/4}u_\xi^*\right) = 0\,.$$

When integrated once this gives the following for u_ξ^*

$$u_\xi^* = Ce^{-\xi^2/4}\,,$$

with C an integration constant. Integrating a second time we find that the function $u^*(\xi)$ is given by

$$u^*(\xi) = C\int_0^\xi e^{-s^2/4}ds + D\,,$$

with D another integrating constant. To evaluate these integration constants we recognized that the boundary condition $u(0,\tau) = 1$ in terms of u^* is $u^*(0) = 1$, which in tern requires that $D = 1$. Thus $u^*(\xi)$ now looks like

$$u^*(\xi) = C\int_0^\xi e^{-s^2/4}ds + 1\,.$$

The two free surface conditions $u(x_f(\tau),\tau) = 0$ and $\frac{\partial u}{\partial x}(x_f(\tau),\tau) = -\frac{dx_f(\tau)}{d\tau}$ in terms of $u^*(\xi)$ using Equations 41 and 42 now become

$$\begin{aligned} C\int_0^{\xi_0} e^{-s^2/4}ds + 1 &= 0 \\ Ce^{-\xi_0^2/4} &= -\frac{1}{2}\,. \end{aligned}$$

Solving the second equation for C and putting this into the first equation gives a single equation for ξ_0 of

$$-\frac{1}{2}e^{\xi_0^2/4}\int_0^{\xi_0} e^{-s^2/4}ds + 1 = 0\,, \tag{43}$$

the same equation as claimed in the book. To solve this equation numerically in MATLAB we recall the definition of the error function `erf`

$$\text{erf}(x) \equiv \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt = \frac{1}{\sqrt{\pi}}\int_0^{2x} e^{-s^2/4}ds\,.$$

Thus the integral we are evaluating in Equation 43 can be written in terms of the error function `erf` as

$$\int_0^{\xi_0} e^{-s^2/4} ds = \sqrt{\pi}\, \mathrm{erf}(\xi_0/2)\,.$$

Using this equivalence we can use the MATLAB command `fsolve` to solve the following nonlinear equation

$$-\frac{\sqrt{\pi}}{2} e^{\xi_0^2/4} \mathrm{erf}(\xi_0/2) + 1 = 0\,,$$

for ξ_0. This is done in the MATLAB script `prob_7_8.m` where we find $\xi_0 = 1.416281$.

Problem 9 (another Stefan like problem)

Given the proposed similarity solution $c(x,\tau) = \tau c^*(x/\sqrt{\tau})$, we see that its derivatives are given by

$$\begin{aligned}
c_x &= \tau c^*_\xi(\xi) \left(\frac{1}{\sqrt{\tau}}\right) = \sqrt{\tau} c^*_\xi(\xi) \quad \text{so} \quad c_{xx} = c^*_{\xi\xi}(\xi) \\
c_\tau &= c^*(\xi) + \tau c^*_\xi(\xi) \left(-\frac{1}{2} x \tau^{-3/2}\right) = c^*(\xi) - \frac{\xi}{2} c^*_\xi(\xi)\,.
\end{aligned}$$

When these expressions are put into the given partial differential equation for c we obtain

$$c^* - \frac{\xi}{2} c^*_\xi = c^*_{\xi\xi} - 1\,,$$

or

$$c^*_{\xi\xi} + \frac{\xi}{2} c^*_\xi - c^* = 1\,, \tag{44}$$

as our ordinary differential equation for $c^*(\xi)$. As before, the boundary conditions for $c(x,\tau)$ reduced to corresponding conditions on $c^*(\xi)$. The condition $c(x_f(\tau),\tau) = 0$ reduces to

$$\begin{aligned}
c(x_f(\tau),\tau) &= \tau c^*\left(\frac{x_f(\tau)}{\sqrt{\tau}}\right) = \tau c^*\left(\frac{\xi_0\sqrt{\tau}}{\sqrt{\tau}}\right) = \tau c^*(\xi_0) = 0 \\
&\Rightarrow \; c^*(\xi_0) = 0\,.
\end{aligned} \tag{45}$$

While the condition $\frac{\partial c}{\partial x}(x_f(\tau), \tau) = 0$ reduces to

$$c^*_\xi(\xi_0) = 0\,. \tag{46}$$

Finally, the condition $c(0, \tau) = \tau$ becomes $\tau c^*(0) = \tau$ or

$$c^*(0) = 1\,. \tag{47}$$

To continue in solving for $c(\xi)$, we recognized that Equation 44 is a second order linear equation in c^*. To solve the homogeneous equation we begin by trying a low order polynomial; say $c^*(\xi) = A + B\xi + C\xi^2 + D\xi^3$. When we take the needed derivative of this expression and put them into Equation 44 we get

$$2C + 6D\xi + \frac{\xi}{2}(B + 2C\xi + 3D\xi^2) - (A + B\xi + C\xi^2 + D\xi^3) = 0\,.$$

Equating powers of ξ we obtain

$$\begin{aligned} 2C - A &= 0 \quad \text{for} \quad \xi^0 \\ 6D + \frac{B}{2} - B &= 0 \quad \text{for} \quad \xi^1 \\ C - C &= 0 \quad \text{for} \quad \xi^2 \\ \frac{3}{2}D - D &= 0 \quad \text{for} \quad \xi^3\,. \end{aligned}$$

From these equations we see that $D = 0$, $B = 0$, C is arbitrary, and $A = 2C$. Thus one homogeneous solution is

$$c^*_1(\xi) = 2 + \xi^2\,.$$

To find a second solution we will use the method of reduction of order. Let $c^*_2(\xi) = c^*_1(\xi)a(\xi)$ and use Equation 44 to derive an equation for $a(\xi)$. When we put the expression for $c^*_2(\xi)$ into our differential equation we get that $a(\xi)$ must satisfy

$$(2 + \xi^2)a''(\xi) + \frac{1}{2}\xi(10 + \xi^2)a'(\xi) = 0\,,$$

or

$$\frac{d(a')}{a'} = -\frac{1}{2}\left(\frac{\xi(10 + \xi^2)}{2 + \xi^2}\right) d\xi\,.$$

Integrating both sides of this we obtain

$$\ln(a') = -\frac{1}{2}\left(\frac{\xi^2}{2} + 4\ln(2+\xi^2)\right) + C_1 \,,$$

with C_1 our first integration constant. Solving for $a'(\xi)$ and integrating a second time we obtain

$$a(\xi) = C_2 e^{-\frac{\xi^2}{4}}\left(\frac{\xi}{4(2+\xi^2)} + \frac{\sqrt{\pi}}{8} e^{\frac{\xi^2}{4}} \mathrm{Erf}(\frac{\xi}{2})\right) .$$

Here $\mathrm{Erf}(x)$ is the error function defined as $\mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$. Doing this, we conclude (after multiplying by $c_1^*(\xi)$) that $c_2^*(\xi)$ is given by

$$c_2^*(\xi) = \frac{\xi}{4} e^{-\frac{\xi^2}{4}} + \frac{\sqrt{\pi}}{8}(2+\xi^2)\mathrm{Erf}(\frac{\xi}{2}) .$$

We will keep this form of the second solution (rather than extending the integral in the definition of the error function to infinity) because one of our boundary conditions, Equation 47, is at $\xi = 0$, and from the above expression we see that $c_2^*(0) = 0$. The general solution to the homogeneous Equation 44 is therefore given by

$$c^*(\xi) = Ac_1^*(\xi) + Bc_2^*(\xi) .$$

To solve the inhomogeneous equation, by observation, we see that a particular solution to the inhomogeneous equation is $c^*(\xi) = -1$. Thus the total solution to Equation 44 is given by

$$c^*(\xi) = -1 + Ac_1^*(\xi) + Bc_2^*(\xi) .$$

We now seek to determine A, B, and ξ_0 using the three boundary conditions in Equations 47 46, and 45 one at a time. Equation 47 requires that the constant A satisfy (since $c_2^*(0) = 0$) that

$$-1 + 2A = 1 \quad \text{or} \quad A = 1 .$$

With this information $c^*(\xi)$ now becomes

$$c^*(\xi) = 1 + \xi^2 + B\left(\frac{\xi}{4}e^{-\frac{\xi^2}{4}} + \frac{\sqrt{\pi}}{8}(2+\xi^2)\mathrm{Erf}(\frac{\xi}{2})\right) .$$

Equation 45 requires

$$1+\xi_0^2+B\left(\frac{\xi_0}{4}e^{-\frac{\xi_0^2}{4}}+\frac{\sqrt{\pi}}{8}(2+\xi_0^2)\text{Erf}(\frac{\xi_0}{2})\right)=0\,.$$

Solving this for B we find the obvious expression

$$B=-\frac{1+\xi_0^2}{\left(\frac{\xi_0}{4}e^{-\frac{\xi_0^2}{4}}+\frac{\sqrt{\pi}}{8}(2+\xi_0^2)\text{Erf}(\frac{\xi_0}{2})\right)}\,.$$

Finally, Equation 46 gives a messy expression for B and ξ_0 which we won't include here. However, when put the above expression for B above into what results and simplify we obtain the following

$$\frac{-4+2e^{\xi_0^2/4}\sqrt{\pi}\xi_0\text{Erf}(\frac{\xi_0}{2})}{2\xi_0+e^{\xi_0^2/4}\sqrt{\pi}(2+\xi_0^2)\text{Erf}(\frac{\xi_0}{2})}=0\,,$$

or the result that ξ_0 must satisfy (by canceling the denominator in the above)

$$\begin{aligned}1 &= \frac{\sqrt{\pi}}{2}e^{\xi_0^2/4}\text{Erf}(\xi_0/2)\\ &= e^{\xi_0^2/4}\int_0^{\xi_0/2}e^{-t^2}dt\\ &= \frac{1}{2}e^{\xi_0^2/4}\int_0^{\xi_0}e^{-v^2/4}dv\,.\end{aligned}$$

The last integral follows from the previous one by using the substitution $v=2t$. This is the same integral equation for ξ_0 as was found in the previous problem. The algebra for this problem can be found in the Mathematica file `prob_7_9_algebra.nb`.

Problem 10 (the equivalence of the two Stefan problems)

We begin by recalling the system of equations from Problem 9. The partial differential equation is given by

$$\frac{\partial c}{\partial\tau}=\frac{\partial^2 c}{\partial x^2}-1 \quad \text{where} \quad 0<x<x_f(\tau)\,,$$

with three boundary conditions given by

$$c(x_f(\tau), \tau) = 0 \quad \frac{\partial c}{\partial x}(x_f(\tau), \tau) = 0 \quad c(0, \tau) = \tau \,.$$

If we take the τ derivative of the partial differential equation above we get

$$\frac{\partial}{\partial \tau}\left(\frac{\partial c}{\partial \tau}\right) = \frac{\partial^2}{\partial x^2}\left(\frac{\partial c}{\partial \tau}\right) \,.$$

Taking the τ derivative of the first boundary condition $c(x_f(\tau), \tau) = 0$ gives

$$\frac{\partial c}{\partial x}(x_f(\tau), \tau)\frac{dx_f}{d\tau} + \frac{\partial c}{\partial \tau} = 0 \cdot \frac{dx_f}{d\tau} + \frac{\partial c}{\partial \tau} = 0 \,,$$

Where we have used the second boundary condition $\frac{\partial c}{\partial x}(x_f(\tau), \tau)) = 0$ to simplify the above resulting in

$$\frac{\partial c}{\partial \tau}(x_f(\tau), \tau)) = 0 \,. \tag{48}$$

Taking the τ derivative of the condition $\frac{\partial c}{\partial x}(x_f(\tau), \tau) = 0$, gives

$$\frac{\partial^2 c}{\partial x^2}(x_f(\tau), \tau)\frac{dx_f}{d\tau} + \frac{\partial}{\partial x}\left(\frac{\partial c}{\partial \tau}\right) = 0 \,.$$

Now from the differential equation for c we have that $\frac{\partial^2 c}{\partial x^2} = \frac{\partial c}{\partial \tau} + 1$ which when put into the above gives

$$\left(\frac{\partial c}{\partial \tau}(x_f(\tau), \tau) + 1\right)\frac{dx_f}{d\tau} + \frac{\partial}{\partial x}\left(\frac{\partial c}{\partial \tau}\right) = 0 \,.$$

Using Equation 48 the above becomes

$$\frac{\partial}{\partial x}\left(\frac{\partial c}{\partial \tau}\right) = -\frac{dx_f}{d\tau} \,.$$

Finally, taking the τ derivative of the third boundary condition $c(0, \tau) = \tau$, we have

$$\frac{\partial c}{\partial \tau}(0, \tau) = 1 \,.$$

Thus the unknown $\frac{\partial c}{\partial \tau}$ must solve the following system

$$\frac{\partial}{\partial \tau}\left(\frac{\partial c}{\partial \tau}\right) = \frac{\partial^2}{\partial x^2}\left(\frac{\partial c}{\partial \tau}\right) \quad \text{where} \quad 0 < x < x_f(\tau)\,,$$

with boundary conditions given by

$$\frac{\partial c}{\partial \tau}(x_f(\tau), \tau) = 0\,, \quad \frac{\partial c}{\partial x}\left(\frac{\partial c}{\partial \tau}(x_f(\tau), \tau)\right) = -\frac{dx_f(\tau)}{d\tau}\,, \quad \frac{\partial c}{\partial \tau}(0, \tau) = 1\,,$$

which we recognize as exactly the problem 8 if we make the substitution $u = \frac{\partial c}{\partial \tau}$ as suggested.

Chapter 8 (Finite-difference Methods)

Additional Notes and Derivations

The Black-Scholes solution from the pure diffusion equation

From Section 5.4 in the book the change of variables

$$S = Ee^x \,, \quad t = T - \frac{\tau}{\frac{1}{2}\sigma^2} \,, \quad V(S,t) = Ev(x,\tau) = Ee^{\alpha x + \beta\tau} u(x,\tau) \,,$$

with appropriate values for α and β reduce the Black-Scholes Equation 4 in the financial variables (S,t) to the pure diffusion equation $\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$, in dimensionless variables (x,τ). A consequence of this fact is that given a solution to the diffusion equation $u(x,\tau)$ a solution to the more complicated Black-Scholes equation $V(S,t)$ is given by

$$\begin{aligned} V &= Ee^{-\frac{1}{2}(k-1)x} e^{-\frac{1}{4}(k+1)^2\tau} u\,(x,\tau) & (49) \\ &= Ee^{-\frac{1}{2}(k-1)x} e^{-\frac{1}{4}(k+1)^2\tau} u \left(\log(\frac{S}{E}), \frac{1}{2}\sigma^2 (T-t) \right) \\ &= E^{\frac{1}{2}(k+1)} S^{-\frac{1}{2}(k-1)} e^{-\frac{1}{8}(k+1)^2\sigma^2(T-t)} u \left(\log(\frac{S}{E}), \frac{1}{2}\sigma^2 (T-t) \right) \,, & (50) \end{aligned}$$

where we have used $\alpha = -\frac{1}{2}(k-1)$, $\beta = -\frac{1}{4}(k+1)^2$, and $k = \frac{r}{\frac{1}{2}\sigma^2}$. This is the expression given in the book. Equation 49 is a representation of V in terms of the variables (x,τ), and Equation 50 is the representation of V in terms of the financial variables (S,t) but still referencing the diffusion solution u. A few things should be noted about using the prescription above to derive *numerical* solutions to the Black-Scholes equation. The first is that the options expiration corresponds to the financial time $t = T$ which in scaled time is given by $\tau = 0$. To solve for $u(x,\tau)$ for $\tau > 0$ we integrate the pure diffusion equation from $\tau = 0$ to the time τ_{max} that corresponds to the financial time $t = 0$. Using the above transformation between time variables this corresponds to integrating u until a time given by

$$\tau_{\text{max}} = \frac{1}{2}\sigma^2 T \,.$$

Another important numerical consideration is the explicit specification of the transformation of the boundary conditions from financial variables to their representation in terms of the transformed variables. For convenience we list these functions here.

A put option

For a put option, the standard boundary conditions in terms of the financial variables and a given finite spatial computational grid $[S_{\min}, S_{\max}]$ are given by

$$\begin{aligned} P(S_{\min}, t) &\approx Ee^{-r(T-t)} \quad \text{as} \quad S_{\min} \to 0 \\ P(S_{\max}, t) &\approx 0 \quad \text{as} \quad S_{\max} \to \infty\,, \end{aligned}$$

from which we can derive boundary conditions on u relative to a transformed grid $[x_{\min}, x_{\max}]$ as

$$\begin{aligned} u(x_{\min}, \tau) &= e^{\frac{1}{4}(k+1)^2\tau} e^{\frac{1}{2}(k-1)x_{\min}} e^{-k\tau} \\ u(x_{\max}, \tau) &= 0\,. \end{aligned}$$

These boundary conditions are implemented in the MATLAB functions `u_m_inf_put.m` and `u_p_inf_put.m`. The general financial payoff function for a put is given by the well known expression

$$P(S, T) = \max(E - S, 0)\,,$$

which when we use Equation 49 to derive the corresponding payoff in terms of (x, τ) we find

$$g(x, \tau) = e^{\frac{1}{4}(k+1)^2\tau} \max(e^{\frac{1}{2}(k-1)x} - e^{\frac{1}{2}(k+1)x}, 0)\,. \tag{51}$$

This expression is coded in the MATLAB function `tran_payoff_put.m`.

A call option

For a call option, the standard boundary conditions in terms of the financial variables and a fixed finite computational grid $[S_{\min}, S_{\max}]$ are given by

$$\begin{aligned} C(S_{\min}, t) &\approx 0 \quad \text{as} \quad S_{\min} \to 0 \\ C(S_{\max}, t) &\approx S_{\max} - E \quad \text{as} \quad S_{\max} \to \infty\,, \end{aligned} \tag{52}$$

from which one could derive boundary conditions for u naively on the transformed computational grid $[x_{\min}, x_{\max}]$ as

$$\begin{aligned} u(x_{\min}, \tau) &= 0 \\ u(x_{\max}, \tau) &= e^{\frac{1}{4}(k+1)^2\tau}\big(e^{\frac{1}{2}(k+1)x_{\max}} - e^{\frac{1}{2}(k-1)x_{\max}}\big)\,. \end{aligned}$$

The problem with these boundary conditions is that they require one to take $S_{\max}$ quite large to obtain accurate convergence of the numerical methods to the exact solution. Another issue with using Equation 52 is that the boundary conditions have t or τ dependence. A much better approximation can be obtained if we take for our boundary conditions an expression which is more exact for $S_{\max}$ large but still finite. One such expression can be obtained from the put-call parody result

$$C - P = S - Ee^{-r(T-t)}\,. \tag{53}$$

We see that for S large and using the assumption that $P(S_{\max}, t) \approx 0$ we have

$$C(S_{\max}, t) \approx S_{\max} - Ee^{-r(T-t)} \quad \text{as} \quad S_{\max} \to \infty \tag{54}$$

This later condition transforms in the (x, τ) domain into

$$u(x_{\max}, \tau) \approx e^{\frac{1}{4}(k+1)^2\tau}\left(e^{\frac{1}{2}(k+1)x_{\max}} - e^{\frac{1}{2}(k-1)x_{\max}}e^{-k\tau}\right)\,. \tag{55}$$

These boundary conditions are implemented in the MATLAB functions `u_m_inf_call.m` and `u_p_inf_call.m`. The general financial payoff function for a call is given by the well known expression

$$C(S, T) = \max(S - E, 0)\,,$$

which when we use Equation 49 to derive the corresponding payoff in terms of (x, τ) we find

$$g(x, \tau) = e^{\frac{1}{4}(k+1)^2\tau}\max(e^{\frac{1}{2}(k+1)x} - e^{\frac{1}{2}(k-1)x}, 0)\,. \tag{56}$$

The general payoff function for a call is coded in the MATLAB function `tran_payoff_call.m`.

A cash-or-nothing put option

For a cash-or-nothing put the financial boundary conditions

$$\begin{aligned} V(S_{\min}, t) &\approx B \quad \text{as} \quad S_{\min} \to 0 \\ V(S_{\max}, t) &\approx 0 \quad \text{as} \quad S_{\max} \to \infty\,. \end{aligned}$$

These give rise to the transformed boundary conditions of

$$\begin{aligned} u(x_{\min}, \tau) &= be^{\frac{1}{4}(k+1)^2\tau}e^{\frac{1}{2}(k-1)x_{\min}} \\ u(x_{\max}, \tau) &= 0\,. \end{aligned}$$

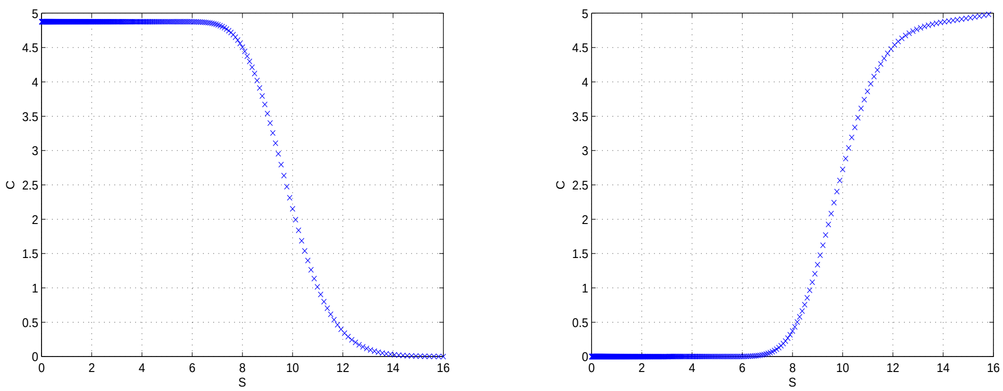

Figure 9: Numerical solutions for European cash-or-nothing options. The financial variables were taken to be $E = 10$, $r = 0.05$, $\sigma = 0.2$, and $T = 0.5$ (six months to expiration) and $B = E/2 = 5$. **Left:** A cash-or-nothing put. **Right:** A cash-or-nothing call.

We found through experimentation that taking the boundary condition above for x_{max} performed poorly. A much better boundary condition can be derived by considering the put-call parody results for cash-or-nothing put/call options. This is given by

$$C + P = Be^{-r(T-t)} \,.$$

If we solve for P and assume that $C \approx 0$ when $S_{\text{max}} \to 0$ we obtain the following

$$u(x_{\text{min}}, \tau) = be^{\frac{1}{4}(k+1)^2\tau} e^{\frac{1}{2}(k-1)x_{\text{min}}} e^{-k\tau} \,.$$

These boundary conditions are implemented in the MATLAB functions `u_m_inf_CON_put.m` and `u_p_inf_CON_put.m`. The financial terminal conditions are given by

$$\Lambda(S,T) = \begin{cases} B & S < E \\ 0 & S \geq E \end{cases} \,, \qquad (57)$$

which when we use Equation 49 to derive the corresponding payoff in terms of (x, τ) we find

$$g(x,\tau) = e^{\frac{1}{4}(k+1)^2\tau} \begin{cases} be^{\frac{1}{2}(k-1)x} & x < 0 \\ 0 & x \geq 0 \end{cases} \,. \qquad (58)$$

In these expressions $b = \frac{B}{E}$. The general payoff function for a cash-or-nothing put is coded in the MATLAB function `tran_payoff_CON_put.m`. For an example of the output when running these numerical codes see Figure 9 (left).

A cash-or-nothing call option

For a cash-or-nothing call the financial boundary conditions

$$\begin{aligned} V(S_{\min}, t) &\approx 0 \quad \text{as} \quad S_{\min} \to 0 \\ V(S_{\max}, t) &\approx B \quad \text{as} \quad S_{\max} \to \infty \,. \end{aligned}$$

These give rise to the transformed boundary conditions of

$$\begin{aligned} u(x_{\min}, \tau) &= 0 \\ u(x_{\max}, \tau) &= b e^{\frac{1}{4}(k+1)^2\tau} e^{\frac{1}{2}(k-1)x_{\max}} \,. \end{aligned}$$

The financial terminal conditions are given by

$$\Lambda(S,T) = \begin{cases} 0 & S < E \\ B & S \geq E \end{cases} \tag{59}$$

which when we use Equation 49 to derive the corresponding payoff in terms of (x, τ) we find

$$g(x,\tau) = e^{\frac{1}{4}(k+1)^2\tau} \begin{cases} 0 & x < 0 \\ b e^{\frac{1}{2}(k-1)x} & x \geq 0 \end{cases} \,, \tag{60}$$

here again $b = \frac{B}{E}$. The general payoff function for a cash-or-nothing call is coded in the MATLAB function `tran_payoff_CON_call.m`. For an example of the output when running these numerical codes see Figure 9 (right).

Exercise Solutions

Exercise 1 (expansions of finite-differences)

Taylor's theorem applied to $u(x, \tau + \delta\tau)$ gives

$$u(x, \tau + \delta\tau) = u(x,\tau) + u_\tau(x,\tau)\delta\tau + \frac{1}{2} u_{\tau\tau}(x, \tau + \lambda\delta\tau)\delta\tau^2 \,,$$

where λ is a real number such that $0 \leq \lambda \leq 1$. Computing the required forward difference from this expression then gives

$$\frac{u(x,\tau+\delta\tau)-u(x,\tau)}{\delta\tau} = u_\tau(x,\tau) + \frac{1}{2}u_{\tau\tau}(x,\tau+\lambda\delta\tau)\delta\tau\,,$$

which is the desired expression. Doing similar manipulations for the expression $u(x,\tau-\delta\tau)$ we find its Taylor expansion given by

$$u(x,\tau-\delta\tau) = u(x,\tau) - u_\tau(x,\tau)\delta\tau + \frac{1}{2}u_{\tau\tau}(x,\tau-\lambda\delta\tau)\delta\tau^2\,,$$

again with $0 \leq \lambda \leq 1$. Computing the backwards difference required in Equation 8.2 we find

$$-\left(\frac{u(x,\tau-\delta\tau)-u(x,\tau)}{\delta\tau}\right) = u_\tau(x,\tau) - \frac{1}{2}u_{\tau\tau}(x,\tau-\lambda\delta\tau)\delta\tau\,,$$

which is the desired expression.

Exercise 2 (centered finite-difference approximations)

We begin this problem by expanding $u(x,\tau+\delta\tau)$ and $u(x,\tau-\delta\tau)$ as in Exercise 1 but now including terms up to third order in $\delta\tau$. We find

$$\begin{aligned}
u(x,\tau+\delta\tau) &= u(x,\tau) + u_\tau(x,\tau)\delta\tau + \frac{1}{2}u_{\tau\tau}(x,\tau)\delta\tau^2 + \frac{1}{6}u_{\tau\tau\tau}(x,\tau+\lambda\delta\tau)\delta\tau^3 \\
u(x,\tau-\delta\tau) &= u(x,\tau) - u_\tau(x,\tau)\delta\tau + \frac{1}{2}u_{\tau\tau}(x,\tau)\delta\tau^2 - \frac{1}{6}u_{\tau\tau\tau}(x,\tau-\lambda\delta\tau)\delta\tau^3\,.
\end{aligned}$$

Where λ is a real number between 0 and 1. Computing the required central finite-difference we see that

$$\begin{aligned}
\frac{u(x,\tau+\delta\tau)-u(x,\tau-\delta\tau)}{2\delta\tau} &= u_\tau(x,\tau) \\
&+ \frac{1}{12}\left(u_{\tau\tau\tau}(x,\tau+\lambda\delta\tau) + u_{\tau\tau\tau}(x,\tau-\lambda\delta\tau)\right)\delta\tau^2\,.
\end{aligned}$$

This later expression is $u_\tau(x,\tau) + O(\delta\tau^2)$ as claimed. To show Equation 8.4 we can simply use the above expansion but evaluated under the substitution $\delta\tau \to \frac{1}{2}\delta\tau$. Doing so gives

$$\begin{aligned}
\frac{u(x,\tau+\delta\tau/2)-u(x,\tau-\delta\tau/2)}{\delta\tau} &= u_\tau(x,\tau) \\
&+ \frac{1}{48}\left(u_{\tau\tau\tau}(x,\tau+\lambda\frac{\delta\tau}{2}) + u_{\tau\tau\tau}(x,\tau-\lambda\frac{\delta\tau}{2})\right)\delta\tau^2\,.
\end{aligned}$$

which again is $u_\tau(x,\tau) + O(\delta\tau^2)$ as claimed.

Exercise 3 (the central-symmetric-difference equation)

We can show that the required expression is second order by expanding the finite difference expression given. Using the Taylor expansions from Exercise 2 (but in terms of x rather than τ) we can evaluate the central-symmetric finite-difference expression D, defined as $D \equiv u_{i+1} - 2u_i + u_{i-1}$ as

$$\begin{aligned} D &= u_i + (\delta x)(u_x)_i + \frac{(\delta x)^2}{2}(u_{xx})_i + \frac{(\delta x)^3}{3!}(u_{xxx})_i + O((\delta x)^4) \\ &- 2u_i \\ &+ u_i - (\delta x)(u_x)_i + \frac{(\delta x)^2}{2}(u_{xx})_i - \frac{(\delta x)^3}{3!}(u_{xxx})_i + O((\delta x)^4) \\ &= (\delta x)^2(u_{xx})_i + \frac{2}{4!}(\delta x)^4(u_{xxxx})_i + O((\delta x)^6)\,. \end{aligned}$$

When we divide this expression by $(\delta x)^2$ we see that

$$\frac{1}{(\delta x)^2}(u_{i+1} - 2u_i + u_{i-1}) = (u_{xx})_i + \frac{1}{12}(\delta x)^2(u_{xxxx})_i + \text{H.O.T.}\,,$$

where H.O.T. stands for higher order terms. This shows the given finite-difference approximation is accurate to second order.

Exercise 4 (operations in the explicit finite-difference algorithm)

The grid update equation for our explicit finite-difference algorithm in this case is given by

$$u_n^{m+1} = \alpha u_{n+1}^m + (1 - 2\alpha)u_n^m + \alpha u_{n-1}^m\,. \tag{61}$$

As initially written, we would conclude that (assuming the number $1 - 2\alpha$ is pre-computed) that we need three multiplications and two additions to compute u_n^{m+1}. Since this must be done at each of our n internal nodes where $N^- < n < N^+$ of which there are $N = N^+ - N^- - 1$ of them this results in $3N$ multiplications and $2N$ additions. The number of multiplications can be reduced if we rewrite Equation 61 as

$$u_n^{n+1} = (1 - 2\alpha)u_n^m + \alpha(u_{n+1}^m + u_{n-1}^m)\,, \tag{62}$$

which requires only *two* multiplications and *two* additions per timestep.

Exercise 5 (stability of the explicit finite-difference scheme)

Taking $e_n^m = \lambda^m \sin(n\omega)$ and putting this into Equation 61 above we find the increment of m in e_n^m given by

$$e_n^{m+1} = \lambda^{m+1} \sin(n\omega) = \lambda e_n^m ,$$

and the two increments n in e_n^m given by

$$\begin{aligned} e_{n\pm 1}^m &= \lambda^m \sin((n \pm 1)\omega) \\ &= \lambda^m (\sin(n\omega)\cos(\omega) \pm \cos(n\omega)\sin(\omega)) \\ &= \cos(\omega) e_n^m \pm \lambda^m \sin(\omega)\cos(n\omega) . \end{aligned} \tag{63}$$

When we put these into the Equation 61 we get the following

$$\begin{aligned} \lambda e_n^m &= \alpha \cos(\omega) e_n^m + \alpha \lambda^m \sin(\omega)\cos(n\omega) \\ &+ (1 + 2\alpha) e_n^m \\ &+ \alpha \cos(\omega) e_n^m - \alpha \lambda^m \sin(\omega)\cos(n\omega) \\ &= 2\alpha \cos(\omega) e_n^m + (1 - 2\alpha) e_n^m . \end{aligned}$$

Solving for λ gives (after we divide by e_n^m) the following

$$\lambda = 2\alpha \cos(\omega) + 1 - 2\alpha .$$

When we use the trigonometric identity $\cos(\omega) = 1 - 2\sin^2(\omega/2)$, the above becomes

$$\lambda = 1 - 4\alpha \sin^2(\omega/2) . \tag{64}$$

For stability of our numerical scheme we must have $|\lambda| < 1$ or

$$|1 - 4\alpha \sin^2(\omega/2)| < 1 ,$$

which will be true if

$$-1 < 1 - 4\alpha \sin^2(\omega/2) < 1 ,$$

is true. This last inequality will be true if the following inequality on α is true

$$0 < \alpha < \frac{1}{2\sin^2(\omega/2)} .$$

Since $0 < \sin^2(\omega/2) < 1$ we see that the fraction on the right-hand side of the above is bounded as

$$\frac{1}{2} < \frac{1}{2\sin^2(\omega/2)} < \infty \,.$$

Thus if we *require* that $\alpha < \frac{1}{2}$ we can conclude that

$$0 < \alpha < \frac{1}{2} < \frac{1}{2\sin^2(\omega/2)} \,,$$

and all of the manipulations of our inequalities can be performed backwards forcing stability of our numerical scheme.

Exercise 6 (grid refinement in an explicit method)

If we increase the number of x grid points by a factor of K, the grid spacing δx must decrease by a factor of K. From exercise 4 above, the number of operations per timestep (multiplications or additions) will then increase by a factor of K. If we keep $\alpha \equiv \frac{\delta\tau}{(\delta x)^2}$ fixed, since δx decrease by a factor of K i.e. $\delta x' = \frac{\delta x}{K}$ we see that $\delta\tau$ must decrease by a factor of K^2 that is $\delta\tau' = \frac{\delta\tau}{K^2}$. To integrate to the same expiration time T, as before, requires that we perform M time update steps where, $M = \frac{T}{\delta\tau}$. Because of this when $\delta\tau$ decreases by a factor of K^2 we see that M must increase by the same factor. In summary then, by increasing the number of nodes by a factor or K the work per timestep increased as a factor of K, and the total number of integration steps increased by a factor of K^2 so the *total* work has increased by $K(K^2) = K^3$ as we were to show.

Exercise 7 (evaluating a bullish vertical spread)

From the discussion in Chapter 3 the payoff diagram for a bullish vertical spread is similar to that of a cash-or-nothing call except that there are now two strikes E_1 and E_2 and the payoff function is linear between them. That is its payoff function is given by

$$V(S,T) = \max(S - E_1, 0) - \max(S - E_2, 0) \,.$$

The bullish vertical spread has financial boundary conditions (in the variables S and t) that are the similar to the cash-or-nothing call i.e.

$$V(0,t) = 0 \quad \text{and} \quad V(+\infty, t) = E_2 - E_1 \,.$$

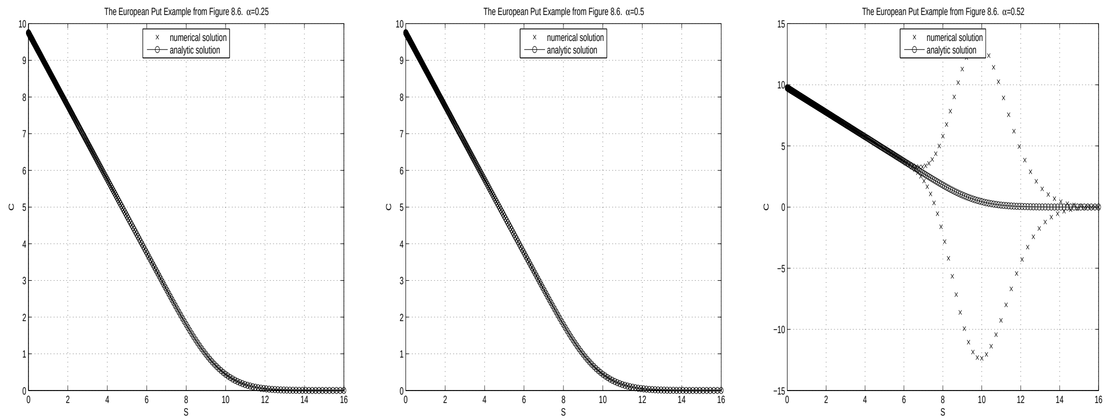

Figure 10: Numerical solutions via a explicit finite-difference method compared to analytic solutions for a European put option. The financial variables for this example were taken to be $E = 10$, $r = 0.05$, $\sigma = 0.2$, and $T = 0.5$ (six months to expiration). These numbers were chosen to duplicate the results presented in Figure 8.6 from the book. In many cases the results are indistinguishable from each other. The three different results correspond to different values of $\alpha \equiv \frac{\delta\tau}{(\delta x)^2}$ **Left:** With $\alpha = 0.25$. **Center:** With $\alpha = 0.5$. **Right:** With $\alpha = 0.52$, notice the instability that develops.

Thus the only change needed to evaluate this option in the finite-difference framework is to change the transformed payoff function.

Exercise 8 (explicit schemes for evaluating European options)

For this chapter we developed a suite of MATLAB codes to price European options using finite-differences. This problem asked about explicit methods. See Exercise 12 for similar results using implicit methods.

In the MATLAB function `explicit_fd.m` we implemented the explicit finite-difference scheme corresponding to the pseudo-code in Figure 8.5 from the book. This code can be executed by calling the MATLAB script `explicit_fd_driver.m` from the command prompt and it duplicates the results found in Figure 8.6 from the book. Plots of the comparison between the numerical solution and the analytical solution for three values of $\alpha = \frac{\delta t}{(\delta x)^2}$ and demonstrating the potential instability in explicit methods are shown in Figure 10. Using this code one can price calls, puts, cash-or-nothing calls, and cash-or-nothing puts simply by changing the functions that implement the initial and boundary conditions. See the MATLAB code `explicit_fd_driver.m` for examples of each of these options.

Exercise 9 (an explicit scheme for the Black-Scholes equation)

We desire to derive an explicit finite-difference scheme directly from the dividend modified Black-Scholes equation. To discretize this equation directly requires that we integrate from the financial time $t = T$ backwards to the current time $t = 0$. We begin with a finite-difference approximation of the τ derivative, $\frac{\partial V}{\partial \tau}$, given by

$$\frac{\partial V}{\partial \tau} \approx \frac{V_n^{m+1} - V_n^m}{\delta t},$$

and a spatial approximation of $\frac{\partial V}{\partial S}$ to be evaluated at $t^{m+1} = (m+1)\delta t$ using central differences as

$$\frac{\partial V}{\partial S} \approx \frac{V_{n+1}^{m+1} - V_{n-1}^{m+1}}{2\delta S}.$$

Finally, we take our approximation of the second derivative $\frac{\partial^2 V}{\partial S^2}$ using central differences and again evaluated at t^{m+1} as

$$\frac{\partial^2 V}{\partial S^2} \approx \frac{V_{n+1}^{m+1} - 2V_n^{m+1} + V_{n-1}^{m+1}}{(\delta S)^2}.$$

When these are put into the Black-Scholes Equation 29 with a discrete spatial discretization of $S_n = n\delta S$ we obtain the following difference equation

$$\begin{aligned} \frac{V_n^{m+1} - V_n^m}{\delta t} &+ \frac{1}{2}\sigma^2 (n\delta S)^2 \left(\frac{V_{n+1}^{m+1} - 2V_n^{m+1} + V_{n-1}^{m+1}}{(\delta S)^2} \right) \\ &+ (r - D_0)(n\delta S) \left(\frac{V_{n+1}^{m+1} - V_{n-1}^{m+1}}{2\delta S} \right) - rV_n^{m+1} = 0. \end{aligned}$$

Solving this expression for V_n^m in terms of V_n^{m+1} we find

$$\begin{aligned} V_n^m &= \delta t \left(\frac{1}{2}\sigma^2 n^2 - \frac{1}{2}(r - D_0)n \right) V_{n-1}^{m+1} \\ &+ V_n^{m+1} + \delta t \left(-\sigma^2 n^2 - r \right) V_n^{m+1} \\ &+ \delta t \left(\frac{1}{2}\sigma^2 n^2 + \frac{1}{2}(r - D_0)n \right) V_{n+1}^{m+1}. \end{aligned}$$

Defining a_n, b_n, and c_n as suggested in the text we arrive at the difference equation proposed. The limits on the index n go from $n = 0$ to $n = +\infty$ and the limits on the index m go from $m = M$ (which corresponds to $t = T$ or expiration) down to $m = 0$ (which corresponds to the time $t = 0$). Since our difference equation marches backwards in time at each step timestep we know the value of V_n^{m+1} and from it compute the value of V_n^m at the previous time from it directly. This is the definition of an explicit method. This method of solution will suffer from all the stability problems that explicit methods have i.e. we may have to enforce a timestep restriction on δt to ensure convergence.

Exercise 10 (minimum algorithm count for the implicit algorithm)

For this problem we desire to determine minimum the number of multiplications/divisions required per time-step when we solve for u_n^{m+1} given u_n^m using the LU algorithm using Equation 65. The LU algorithm is presented at the end of the section in the text that discusses it and consists of three main steps per time-step. In an implementation we first compute y_n from

$$\begin{aligned} y_{N^-+1} &= 1 + 2\alpha \\ y_n &= 1 + 2\alpha - \frac{\alpha^2}{y_{n-1}} \quad \text{for} \quad n = N^- + 2, \cdots, N^+ - 1\,. \end{aligned}$$

These calculations require approximately $N \equiv N^+ - N^- + 1$ multiplications/divisions. Then given the known b^m we compute q^m using

$$\begin{aligned} q_{N^-+1}^m &= b_{N^-+1}^m \\ q_n^m &= b_n^m + \frac{\alpha q_{n-1}^m}{y_{n-1}} \quad \text{for} \quad n = N^- + 2, \cdots, N^+ - 1\,, \end{aligned}$$

which requires approximately $2N$ multiplications/divisions. In the third and final step we compute u^{m+1} using

$$\begin{aligned} u_{N^+-1}^{m+1} &= \frac{q_{N^+-1}^m}{y_{N^+-1}} \\ u_n^{m+1} &= \frac{q_n^m + \alpha u_{n+1}^{m+1}}{y_n} \quad \text{for} \quad n = N^+ - 2, \cdots, N^- + 1\,, \end{aligned}$$

which requires another $2N$ multiplications/divisions. The three steps taken together then require $O(5N)$ multiplications/divisions in total.

Exercise 11 (stability of the implicit finite-difference scheme)

The difference equation derived when we discretize the diffusion equation $u_\tau = u_{xx}$ using an implicit finite-difference scheme is given by

$$-\alpha u_{n-1}^{m+1} + (1+2\alpha)u_n^{m+1} - \alpha u_{n+1}^{m+1} = u_n^m \,. \tag{65}$$

Since this is linear the error at node n and timestep m denoted e_n^m, must also satisfy this equation. From Fourier theory we can take as the function form for e_n^m the following expression

$$e_n^m = \lambda^m \sin(n\omega)\,. \tag{66}$$

The spatial increments $e_{n\pm1}^m$ are computed as in Equation 63. With these expressions then Equation 65 becomes

$$\begin{aligned} e_n^m &= -\alpha\lambda(e_n^m \cos(\omega) - \lambda^m \cos(n\omega)\sin(\omega)) \\ &+ (1+2\alpha)\lambda e_n^m \\ &- \alpha\lambda(e_n^m \cos(\omega) + \lambda^m \cos(n\omega)\sin(\omega))\,, \end{aligned}$$

When we cancel the expression involving $\cos(n\omega)$, and divide by e_n^m we obtain

$$-2\alpha\lambda\cos(\omega) + (1+2\alpha)\lambda = 1\,.$$

Solving for λ gives

$$\lambda = \frac{1}{1+2\alpha-2\alpha\cos(\omega)}\,.$$

If we use the trigonometric identity $\cos(\omega) = 1 - 2\sin^2(\omega/2)$, the above expression for λ then becomes

$$\lambda = \frac{1}{1+4\alpha\sin^2(\omega/2)}\,, \tag{67}$$

the expression given in the book. Since from this we can see that $\lambda < 1$ for all values of ω this scheme is stable.

Exercise 12 (implicit schemes for evaluating European options)

For this chapter we developed a suite of MATLAB codes to price European options using finite-differences. This problem asked about implicit methods. See Exercise 8 for examples of option pricing using explicit methods.

In the MATLAB functions developed here we implement the fully implicit finite-difference method with two matrix solver. The first matrix solver uses the LU decomposition and is implemented in the MATLAB function `implicit_fd_LU.m`. This corresponds to the pseudo-code found in Figure 8.10 from the book. The second matrix solver method implements the iterative method of successive over relaxation or SOR and is found in the MATLAB function `implicit_fd_SOR.m`. This algorithm corresponds to the pseudo-code found in Figure 8.11 from the book. These function can be executed from the command prompt by calling `implicit_fd_LU_driver.m` or `implicit_fd_SOR_driver.m` respectively. They both duplicate the numerical results given in Figure 8.12 from the book.

Finally, in the MATLAB functions `crank_fd_LU.m` and `crank_fd_SOR.m` we implement the Crank-Nicolson method with solution via the LU decomposition or successive over-relaxation SOR. They correspond to the pseudo-code in Figures 8.13 and 8.14 respectively from the book. They can also be run via the same driver functions as earlier and in this case duplicate the numerical results found in Figure 8.15.

These codes can price calls, puts, cash-or-nothing calls, and cash-or-nothing puts simply by changing the functions that implement the initial and boundary conditions. See the MATLAB codes for examples of each of these options.

Exercise 13 (an implicit scheme for the Black-Scholes equation)

In this problem we derive an implicit discretization of the dividend modified Black-Scholes Equation given by Equation 29. To do this we will evaluate our spatial (i.e. S) derivatives at the time $t^m = m\delta t$ rather than at the current timestep time $t^{m+1} = (m+1)\delta t$. The latter being the time where we evaluated the spatial derivatives in the explicit scheme in Exercise 12. With central differences for the spatial derivatives (evaluated at t^m) we have

$$
\begin{aligned}
\frac{\partial V}{\partial S} &\approx \frac{V_{n+1}^m - V_{n-1}^m}{2\delta S} \\
\frac{\partial^2 V}{\partial S^2} &\approx \frac{V_{n+1}^m - 2V_n^m + V_{n-1}^m}{(\delta S)^2} .
\end{aligned}
$$

When we put these expressions into the Black-Scholes equation we obtain

$$\begin{aligned}\frac{V_n^{m+1}-V_n^m}{\delta t} &+ \frac{1}{2}\sigma^2(n\delta S)^2\left(\frac{V_{n+1}^m-2V_n^m+V_{n-1}^m}{(\delta S)^2}\right)\\ &+ (r-D_0)(n\delta S)\left(\frac{V_{n+1}^m-V_{n-1}^m}{2\delta S}\right)-rV_n^m=0\,.\end{aligned}$$

Putting the V_n^m terms on the right-hand side and the V_n^{m+1} terms on the left hand side we obtain the difference equation

$$\begin{aligned}-V_n^{m+1} &= \delta t\left(\frac{1}{2}\sigma^2n^2-\frac{1}{2}(r-D_0)n\right)V_{n-1}^m\\ &- V_n^m-\delta t\left(\sigma^2n^2+r\right)V_n^m\\ &+ \delta t\left(\frac{1}{2}\sigma^2n^2+\frac{1}{2}(r-D_0)n\right)V_{n+1}^m\,.\end{aligned}\tag{68}$$

On defining A_n, B_n, and C_n as in the text we obtain the required discretization for the "interior" nodes $n\geq 1$. On the boundary nodes when $n=0$ we use a slightly modified finite differences. We begin with the same expressions as earlier but evaluated at $n=0$

$$\begin{aligned}\frac{\partial V}{\partial S} &\approx \frac{V_1^m-V_{-1}^m}{2\delta S}\\ \frac{\partial^2V}{\partial S^2} &\approx \frac{V_1^m-2V_0^m+V_{-1}^m}{(\delta S)^2}\,.\end{aligned}$$

Assuming that we can take $V_0^m=V_{-1}^m$ which would be the case if we had Dirichlet boundary conditions on the left end of the computational domain (corresponding to small stock prices) we obtain for the $i=0$ node then the following finite-difference approximations

$$\begin{aligned}\frac{\partial V}{\partial S} &\approx \frac{V_1^m-V_0^m}{2\delta S}\\ \frac{\partial^2V}{\partial S^2} &\approx \frac{V_1^m-V_0^m}{(\delta S)^2}\,.\end{aligned}$$

When we put this into the Black-Scholes equation the result is Equation 68 evaluated at $n=0$ and $V_{-1}^m=V_0^m$. This is (since A_n evaluated at $n=0$ is zero) the equation

$$B_0V_0^m+C_0V_1^m=V_0^{m+1}\,,$$

as claimed in the book.

Exercise 14 (the LU factorization of the implicit matrix)

The system given in this problem is a direct matrix representation of the equations found in problem 13. We desire to solve it by factoring the tridiagonal coefficient matrix (called here M) into factors L and U such that $M = LU$. Writing the matrix system for Exercise 13 as $Mv^m = b^m$ where $b^m = v^{m+1}$, if we can factor M as LU then we can solve the original system $Mv^m = b^m$ as two triangular systems

$$Lq^m = b^m \quad \text{and} \quad Uv^m = q^m \, .$$

To derive this factorization we will write the factorization of M into LU in terms of its components A_n, B_n, and C_n, a lower triangular matrix L and an upper triangular matrix U as

$$\begin{bmatrix} B_0 & C_0 & 0 & \cdots & & \cdots & 0 \\ A_1 & B_1 & C_1 & & & & \vdots \\ 0 & A_2 & B_2 & \ddots & & & \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ & & & \ddots & B_{N-2} & C_{N-2} & 0 \\ \vdots & & & & A_{N-1} & B_{N-1} & C_{N-1} \\ 0 & \cdots & & \cdots & 0 & A_N & B_N \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 & \cdots & & & & \cdots & 0 \\ l_1 & 1 & 0 & & & & & \vdots \\ 0 & l_2 & 1 & 0 & & & & \\ \vdots & & l_3 & 1 & 0 & & & \\ & & & \ddots & \ddots & \ddots & & \vdots \\ & & & & \ddots & 1 & 0 & 0 \\ \vdots & & & & & l_{N-1} & 1 & 0 \\ 0 & \cdots & & & \cdots & 0 & l_N & 1 \end{bmatrix} \begin{bmatrix} F_0 & z_0 & 0 & \cdots & & & \cdots & 0 \\ 0 & F_1 & z_1 & 0 & & & & \vdots \\ \vdots & 0 & F_2 & z_2 & 0 & & & \\ & & 0 & F_3 & \ddots & & & \\ & & & & \ddots & \ddots & 0 & \vdots \\ & & & & & F_{N-2} & z_{N-2} & 0 \\ \vdots & & & & & 0 & F_{N-1} & z_{N-1} \\ 0 & \cdots & & & & \cdots & 0 & F_N \end{bmatrix} .$$

Multiplying the two matrices on the right-hand side we obtain

$$
\begin{bmatrix}
F_0 & z_0 & 0 & \cdots & & & \cdots & 0 \\
l_1F_0 & l_1z_0+F_1 & z_1 & 0 & & & & \vdots \\
0 & l_2F_1 & l_2z_1+F_2 & z_2 & 0 & & & \\
\vdots & 0 & l_3F_2 & \ddots & \ddots & & & \\
 & & & \ddots & \ddots & z_{N-3} & 0 & \vdots \\
 & & & & \ddots & l_{N-2}z_{N-3}+F_{N-2} & z_{N-2} & 0 \\
\vdots & & & & 0 & l_{N-1}F_{N-2} & l_{N-1}z_{N-2}+F_{N-1} & z_{N-1} \\
0 & \cdots & & & \cdots & 0 & l_NF_{N-1} & l_Nz_{N-1}+F_N
\end{bmatrix}.
$$

Equating the elements of the upper subdiagonal of this matrix and the original matrix M gives

$$C_n = z_n \quad \text{for} \quad n = 0, 1, 2, \cdots, N-1\,, \tag{69}$$

and is the expression used to evaluate z_n in terms of known coefficients of M. Next, equating the elements of the lower subdiagonal gives

$$A_n = l_n F_{n-1} \quad \text{for} \quad n = 1, 2, \cdots, N\,. \tag{70}$$

Finally, equating the elements on the diagonal between the two matrices gives

$$
\begin{aligned}
B_0 &= F_0 \\
B_n &= l_n z_{n-1} + F_n \quad \text{for} \quad n = 1, 2, \cdots, N\,.
\end{aligned}
$$

Since we know z_n from Equation 69 this last equation becomes

$$B_n = l_n C_{n-1} + F_n \quad \text{for} \quad n = 1, 2, \cdots, N\,. \tag{71}$$

From Equation 70 we have that $l_n = \frac{A_n}{F_{n-1}}$ which when we put this into Equation 71 gives

$$F_n = B_n - \left(\frac{A_n}{F_{n-1}}\right) C_{n-1} \quad \text{for} \quad n = 1, 2, \cdots, N\,.$$

We now have all of the pieces to complete our LU factorization. Explicitly we need to determine expressions z_n, l_n, and F_n in terms of A_n, B_n, and C_n.

Since z_n is determined by Equation 69, to determine the other two elements of our factorization we see that we can first compute F_n as

$$\begin{aligned} F_0 &= B_0 \\ F_n &= B_n - \left(\frac{A_n}{F_{n-1}}\right) C_{n-1} \quad \text{for} \quad n = 1, 2, \cdots, N\,, \end{aligned}$$

and then compute l_n from F_n using Equation 70 or

$$l_n = \frac{A_n}{F_{n-1}} \quad \text{for} \quad n = 1, 2, \cdots, N\,.$$

Now that we have expressions for l_n, F_n, and z_n we can solve $Lq^m = b^m$ which is the system

$$\begin{bmatrix} 1 & 0 & \cdots & & & & \cdots & 0 \\ l_1 & 1 & 0 & & & & & \vdots \\ 0 & l_2 & 1 & 0 & & & & \\ \vdots & & l_3 & 1 & 0 & & & \\ & & & \ddots & \ddots & \ddots & & \vdots \\ & & & & \ddots & 1 & 0 & 0 \\ \vdots & & & & & l_{N-1} & 1 & 0 \\ 0 & \cdots & & & \cdots & 0 & l_N & 1 \end{bmatrix} \begin{bmatrix} q_0^m \\ q_1^m \\ \\ \vdots \\ \\ q_{N-1}^m \\ q_N^m \end{bmatrix} = \begin{bmatrix} V_0^{m+1} \\ V_1^{m+1} \\ \\ \vdots \\ \\ V_{N-1}^{m+1} \\ V_N^{m+1} \end{bmatrix},$$

as

$$\begin{aligned} q_0^m &= V_0^{m+1} \\ q_n^m &= V_n^{m+1} - l_n q_{n-1}^m = V_n^{m+1} - \left(\frac{A_n}{F_{n-1}}\right) q_{n-1}^m \quad \text{for} \quad n = 1, 2, \cdots, N\,. \end{aligned}$$

Next we solve $Uv^m = q^m$ which is the system

$$\begin{bmatrix} F_0 & z_0 & 0 & \cdots & & & \cdots & 0 \\ 0 & F_1 & z_1 & 0 & & & & \vdots \\ \vdots & 0 & F_2 & z_2 & 0 & & & \\ & & 0 & F_3 & \ddots & & & \\ & & & & \ddots & \ddots & 0 & \vdots \\ & & & & & F_{N-2} & z_{N-2} & 0 \\ \vdots & & & & & 0 & F_{N-1} & z_{N-1} \\ 0 & \cdots & & & & \cdots & 0 & F_N \end{bmatrix} \begin{bmatrix} V_0^m \\ V_1^m \\ \\ \vdots \\ \\ V_{N-1}^m \\ V_N^m \end{bmatrix} = \begin{bmatrix} q_0^m \\ q_1^m \\ \\ \vdots \\ \\ q_{N-1}^m \\ q_N^m \end{bmatrix}$$

Solving the Nth equation first, the $N-1$th equation next, up to the first equation gives the following recurrence relations for V_n^m

$$\begin{aligned}
V_N^m &= \frac{q_N^m}{F_N} \\
V_n^m &= \left(\frac{1}{F_n}\right)(q_n^m - z_n V_{n+1}^m) \\
&= \left(\frac{1}{F_n}\right)(q_n^m - C_n V_{n+1}^m) \quad \text{for} \quad n = N-1, N-2, \cdots, 1, 0\,.
\end{aligned}$$

Where in the above we have recalled that $z_n = C_n$. These give the desired expressions.

Exercise 15 (iterative methods for the B-S equation)

The Jacobi method for the system in Exercise 13 and 14 is given by beginning with an initial guess for V_n^m here denoted $V_n^{m,0}$. We then solve for the diagonal terms in relation to the non-diagonal terms as

$$V_n^m = \frac{1}{B_n}\left(V_n^{m+1} - A_n V_{n-1}^m - C_n V_{n+1}^m\right) \quad \text{for} \quad 1 < n < N\,. \tag{72}$$

Given this expression we then iterate it by putting in $V_n^{m,k}$ on the right-hand side and obtaining from it $V_n^{m,k+1}$ on the left-hand-side. As an equation this is given by

$$V_n^{m,k+1} = \frac{1}{B_n}\left(V_n^{m+1} - A_n V_{n-1}^{m,k} - C_n V_{n+1}^{m,k}\right)\,. \tag{73}$$

We begin these iterations using $V_n^{m,0}$ as the initial guess. When the iterations fail to change the value of $V_n^{m,k}$ significantly i.e. if for a user specified tolerance ϵ we have

$$||V_n^{m,k+1} - V_n^{m,k}|| < \epsilon\,,$$

for all m and n then we will stop performing iterations and take the last value of $V_n^{m,k}$ as the value of V_n^m to use at the next time-step.

The Gauss-Seidel algorithm is a simple modification of the above where we recognize that in calculating $V_n^{m,k+1}$ on the right-hand side of Equation 73 we have already calculated an updated value for $V_{n-1}^{m,k+1}$. Thus we can use this value immediately in our iteration scheme as follows

$$V_n^{m,k+1} = \frac{1}{B_n}\left(V_n^{m+1} - A_n V_{n-1}^{m,k+1} - C_n V_{n+1}^{m,k}\right)\,. \tag{74}$$

This simple modification greatly improves the convergence of this algorithm.

Finally, for the SOR algorithm, we can derive this iterative method using one Gauss-Seidel step to compute a "trial" value $Y_n^{m,k+1}$ for the unknown $V_n^{m,k+1}$ and then modify this value by "over-correcting" the previous estimate $V_n^{m,k}$ by this value. Mathematically, this consists of iterating the following two steps

$$\begin{aligned} Y_n^{m,k+1} &= \frac{1}{B_n}\left(V_n^{m+1} - A_n V_{n-1}^{m,k+1} - C_n V_{n+1}^{m,k}\right) \\ V_n^{m,k+1} &= V_n^{m,k} + \omega(Y_n^{m,k+1} - V_n^{m,k}) \,. \end{aligned}$$

Here ω is the over-relaxation parameter and is chosen to be $1 < \omega < 2$. Again these equations are iterated until convergence.

Exercise 16 (accuracy at the locations between grid points)

The right-hand side of Equation 8.28 is

$$\frac{1}{2}\left(\frac{u_{n+1}^m - 2u_n^m + u_{n-1}^m}{(\delta x)^2} + \frac{u_{n+1}^{m+1} - 2u_n^{m+1} + u_{n-1}^{m+1}}{(\delta x)^2}\right) + O((\delta x)^2)\,. \tag{75}$$

We have shown in Exercise 3 that the first term in parenthesis approximated the second derivative at (x, τ) as

$$\frac{u_{n+1}^m - 2u_n^m + u_{n-1}^m}{(\delta x)^2} = \frac{\partial^2 u(x,\tau)}{\partial x^2} + O((\delta x)^2)\,. \tag{76}$$

In the same way the second term in the above approximates the second derivative at $(x, \tau + \delta\tau)$ as

$$\frac{u_{n+1}^{m+1} - 2u_n^{m+1} + u_{n-1}^{m+1}}{(\delta x)^2} = \frac{\partial^2 u(x,\tau+\delta\tau)}{\partial x^2} + O((\delta x)^2)\,, \tag{77}$$

simply because we are evaluating u now at the time $\tau + \delta\tau$. These together show that Equation 75 is a finite difference approximation to the requested differential operator.

To show the next result we will Taylor expand each derivative about the point $(x, \tau + \frac{\delta\tau}{2})$. To provide more detail, we begin by Taylor expanding the expression $\frac{\partial^2 u}{\partial x^2}(x, \tau)$ as follows

$$\begin{aligned} \frac{\partial^2 u(x,\tau)}{\partial x^2} &= \frac{\partial^2 u(x, \tau + \frac{\delta\tau}{2} - \frac{\delta\tau}{2})}{\partial x^2} \\ &= \frac{\partial^2 u(x, \tau + \frac{\delta\tau}{2})}{\partial x^2} - \frac{\partial^3 u(x, \tau + \frac{\delta\tau}{2})}{\partial\tau\partial x^2}\left(\frac{\delta\tau}{2}\right) + O((\delta\tau)^2)\,. \end{aligned}$$

Here we have Taylor expanded about the point $(x, \tau + \frac{\delta\tau}{2})$ by the small increment $-\frac{\delta\tau}{2}$. Next, using the same trick we will simplify the expression $\frac{\partial^2 u}{\partial x^2}(x, \tau + \delta\tau)$ as follows

$$\begin{aligned}
\frac{\partial^2 u(x, \tau + \delta\tau)}{\partial x^2} &= \frac{\partial^2 u(x, \tau + \frac{\delta\tau}{2} + \frac{\delta\tau}{2})}{\partial x^2} \\
&= \frac{\partial^2 u(x, \tau + \frac{\delta\tau}{2})}{\partial x^2} + \frac{\partial^3 u(x, \tau + \frac{\delta\tau}{2})}{\partial\tau \partial x^2} \left(\frac{\delta\tau}{2}\right) + O((\delta\tau)^2) \,.
\end{aligned}$$

In this case we have Taylor expanded about the point $(x, \tau + \frac{\delta\tau}{2})$ by the small increment $\frac{\delta\tau}{2}$. If we then average the previous two expressions the term linear in $\delta\tau$ cancels and we derive the following

$$\frac{1}{2}\left(\frac{\partial^2 u(x,\tau)}{\partial x^2} + \frac{\partial^2 u(x, \tau + \delta\tau)}{\partial x^2}\right) = \frac{\partial^2 u(x, \tau + \frac{\delta\tau}{2})}{\partial x^2} + O(\delta\tau^2) \,, \tag{78}$$

as we were to show.

We can use the same trick of evaluating each derivatives at the midpoint between them to show the next result. Specifically, we have a Taylor series for $u(x, \tau + \delta\tau)$ given by

$$\begin{aligned}
u(x, \tau + \delta\tau) &= u(x, \tau + \frac{\delta\tau}{2} + \frac{\delta\tau}{2}) \\
&= u(x, \tau + \frac{\delta\tau}{2}) + u_\tau(x, \tau + \frac{\delta\tau}{2}) \left(\frac{\delta\tau}{2}\right) \\
&+ \frac{1}{2} u_{\tau\tau}(x, \tau + \frac{\delta\tau}{2}) \left(\frac{\delta\tau}{2}\right)^2 + O(\delta\tau^3) \,.
\end{aligned}$$

While for $u(x, \tau)$ we find

$$\begin{aligned}
u(x, \tau) &= u(x, \tau + \frac{\delta\tau}{2} - \frac{\delta\tau}{2}) \\
&= u(x, \tau + \frac{\delta\tau}{2}) - u_\tau(x, \tau + \frac{\delta\tau}{2}) \left(\frac{\delta\tau}{2}\right) \\
&+ \frac{1}{2} u_{\tau\tau}(x, \tau + \frac{\delta\tau}{2}) \left(\frac{\delta\tau}{2}\right)^2 + O(\delta\tau^3) \,.
\end{aligned}$$

Computing the required forward difference we find

$$\frac{u(x, \tau + \delta\tau) - u(x, \tau)}{\delta\tau} = u_\tau(x, \tau + \frac{\delta\tau}{2}) + O(\delta\tau^2) \,, \tag{79}$$

or the requested expression. Then using the above identities in reverse we see that the differential equation evaluated at a point midway between two grid points or

$$\frac{\partial u(x, \tau + \frac{\delta\tau}{2})}{\partial \tau} = \frac{\partial^2 u(x, \tau + \frac{\delta\tau}{2})}{\partial x^2},$$

is equivalent to

$$\frac{u(x, \tau + \delta\tau) - u(x, \tau)}{\delta\tau} + O((\delta\tau)^2) = \frac{1}{2}\left(\frac{\partial^2 u(x, \tau)}{\partial x^2} + \frac{\partial^2 u(x, \tau + \delta\tau)}{\partial x^2}\right) + O((\delta\tau)^2).$$

By Equation 79 on the left and Equation 78 on the right. Then using Equation 76 and 77 we have the above is equivalent to

$$\frac{u_n^{m+1} - u_n^m}{\delta\tau} = \frac{1}{2}\left(\frac{u_{n+1}^m - 2u_n^m + u_{n-1}^m}{\delta x^2} + \frac{u_{n+1}^{m+1} - 2u_n^{m+1} + u_{n-1}^{m+1}}{\delta x^2}\right) + O((\delta x)^2) + O((\delta t)^2),$$

showing that the Crank-Nicolson scheme is approximating

$$\frac{\partial u(x, \tau + \frac{\delta\tau}{2})}{\partial \tau} = \frac{\partial^2 u(x, \tau + \frac{\delta\tau}{2})}{\partial x^2},$$

to an accuracy of $O((\delta x)^2) + O((\delta t)^2)$.

Exercise 17 (stability of the Crank-Nicholson scheme)

The Crank-Nicholson equation for the pure diffusion equation $u_\tau = u_{xx}$ is given by

$$(1+\alpha)u_n^{m+1} - \frac{1}{2}\alpha(u_{n-1}^{m+1} + u_{n+1}^{m+1}) = (1-\alpha)u_n^m + \frac{1}{2}\alpha(u_{n-1}^m + u_{n+1}^m). \quad (80)$$

Since this is a linear equation it is also the equation for the error e_n^m. Using Fourier analysis we can consider this equation when our error term has the specific functional form given by

$$e_n^m = \lambda^m \sin(n\omega).$$

When we put this expression into the Equation 80 and solve for λ we find

$$\lambda = \frac{1 - \alpha + \alpha\cos(\omega)}{1 + \alpha - \alpha\cos(\omega)}.$$

If we then use the trigonometric identity of $\cos(\omega) = 1 - 2\sin^2(\omega)$, the expression for λ above becomes

$$\lambda = \frac{1 - 2\alpha \sin^2(\omega)}{1 + 2\alpha \sin^2(\omega)},$$

or the required expression. Now taking the absolute value of λ we find

$$|\lambda| = \frac{|1 - 2\alpha \sin^2(\omega)|}{|1 + 2\alpha \sin^2(\omega)|} \leq \frac{1 + 2\alpha \sin^2(\omega)}{1 + 2\alpha \sin^2(\omega)} = 1,$$

showing that $|\lambda| \leq 1$ and that the Crank-Nicholson method is unconditionally stable.

Exercise 18 (the Crank-Nicholson scheme for the B-S equation)

The Crank-Nicholson scheme is a hybrid of the explicit finite-difference scheme and the implicit finite-difference scheme. From Exercise 9 above we have shown that the explicit finite-difference scheme is given by

$$V_n^m = a_n V_{n-1}^{m+1} + b_n V_n^{m+1} + c_n V_{n+1}^{m+1} \quad \text{for} \quad n = 1, 2, \cdots \tag{81}$$

with

$$\begin{aligned} a_n &= \frac{1}{2}(\sigma^2 n^2 - (r - D_0)n)\delta t \\ b_n &= 1 - (\sigma^2 n^2 + r)\delta t \\ c_n &= \frac{1}{2}(\sigma^2 n^2 + (r - D_0)n)\delta t . \end{aligned}$$

While the implicit discretization of the B-S equation is developed in Exercise 13 (above) and is given by

$$A_n V_{n-1}^m + B_n V_n^m + C_n V_{n+1}^m = V_n^{m+1} \quad \text{for} \quad n = 1, 2, \cdots \tag{82}$$

with

$$\begin{aligned} A_n &= -a_n \\ B_n &= 1 + (\sigma^2 n^2 + r)\delta t \\ C_n &= -c_n . \end{aligned}$$

Now the Crank-Nicholson finite-difference scheme can be obtained by averaging the explicit and implicit finite difference approximations above as follows. We first write the explicit and implicit finite-difference equations in terms of their finite-difference approximations for $\frac{\partial V}{\partial t}$ as

$$
\begin{aligned}
\frac{V_n^{m+1} - V_n^m}{\delta t} &= -\frac{a_n}{\delta t} V_{n-1}^{m+1} + (\sigma^2 n^2 + r) V_n^{m+1} - \frac{c_n}{\delta t} V_{n+1}^{m+1} \\
\frac{V_n^{m+1} - V_n^m}{\delta t} &= \frac{A_n}{\delta t} V_{n-1}^{m} + (\sigma^2 n^2 + r) V_n^{m} + \frac{C_n}{\delta t} V_{n+1}^{m} \,.
\end{aligned}
$$

Taking the average of the expressions on the right-hand side of these two finite-difference approximations of $\frac{\partial V}{\partial t}$ gives the Crank-Nicholson scheme of

$$
\begin{aligned}
\frac{V_n^{m+1} - V_n^m}{\delta t} &= -\frac{1}{2}\frac{a_n}{\delta t} V_{n-1}^{m+1} + \frac{1}{2}\frac{A_n}{\delta t} V_{n-1}^{m} \\
&+ \frac{1}{2}(\sigma^2 n^2 + r)(V_n^{m+1} + V_n^m) \\
&- \frac{1}{2}\frac{c_n}{\delta t} V_{n+1}^{m+1} + \frac{1}{2}\frac{C_n}{\delta t} V_{n+1}^{m} \,.
\end{aligned}
$$

Multiplying by δt and moving all terms evaluated at $m\delta t$ to the left-hand-side we obtain

$$
\begin{aligned}
& -V_n^m - \frac{1}{2} A_n V_{n-1}^m - \frac{\delta t}{2}(\sigma^2 n^2 + r) V_n^m - \frac{1}{2} C_n V_{n+1}^m \\
= \; & -V_n^{m+1} - \frac{1}{2} a_n V_{n-1}^{m+1} + \frac{\delta t}{2}(\sigma^2 n^2 + r) V_n^{m+1} - \frac{1}{2} c_n V_{n+1}^{m+1} \,.
\end{aligned}
$$

Multiplying by -2 and grouping terms we obtain

$$
\begin{aligned}
& A_n V_{n-1}^m + \left(2 + (\sigma^2 n^2 + r)\delta t\right) V_n^m + C_n V_{n+1}^m \\
= \; & a_n V_{n-1}^{m+1} + \left(2 - (\sigma^2 n^2 + r)\delta t\right) V_n^{m+1} + c_n V_{n+1}^{m+1} \,. \qquad (83)
\end{aligned}
$$

Now this is exactly the same equation as if we had simply added together Equations 81 and 82 the two finite difference equations and divided the resulting sum by 2. This last observation may be an easier way to generate the Crank-Nicholson scheme given the explicit and implicit schemes.

To derive the LU decomposition algorithm for this system we recognize that that it is exactly like in Exercise 14 except that now the elements on the diagonal of the system matrix and the elements of the vector on the right-hand side are slightly different (rather than simply V_n^{m+1}). If we denote these new right-hand side elements by Z_n^{m+1} our system becomes the following

$$Z_n^{m+1} \equiv a_n V_{n-1}^{m+1} + \left(2 - (\sigma^2 n^2 + r)\delta t\right) V_n^{m+1} + c_n V_{n+1}^{m+1},$$

and our new diagonal elements become $\tilde{B}_n = 2 + (\sigma^2 n^2 + r)\delta t$. With this notation the system we are solving is given by

$$\begin{bmatrix} \tilde{B}_0 & C_0 & 0 & \cdots & & \cdots & 0 \\ A_1 & \tilde{B}_1 & C_1 & & & & \vdots \\ 0 & A_2 & \tilde{B}_2 & \ddots & & & \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ & & & \ddots & \tilde{B}_{N-2} & C_{N-2} & 0 \\ \vdots & & & & A_{N-1} & \tilde{B}_{N-1} & C_{N-1} \\ 0 & \cdots & & \cdots & 0 & A_N & \tilde{B}_N \end{bmatrix} \begin{bmatrix} V_0^m \\ V_1^m \\ V_2^m \\ \vdots \\ \vdots \\ V_{N-2}^m \\ V_{N-1}^m \\ V_N^m \end{bmatrix} = \begin{bmatrix} Z_0^{m+1} \\ Z_1^{m+1} \\ Z_2^{m+1} \\ \vdots \\ \vdots \\ Z_{N-2}^{m+1} \\ Z_{N-1}^{m+1} \\ Z_N^{m+1} \end{bmatrix}.$$

Since this is effectively the same system as in Exercise 14 the LU algorithm developed there will also solve this system with the appropriate modifications. The SOR algorithm that was developed in Exercise 15 will also solve for V_n^m once we have made the required changes.

Exercise 19 (the 2D diffusion equation)

For this exercise we will derive several finite-difference equations for the 2D diffusion equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

An explicit scheme

To begin using forward differences for the time derivative and centered differences for the spatial derivatives by defining $u_{i,j}^m \equiv u(i\delta x, j\delta y, m\delta t)$ we obtain

$$\frac{u_{ij}^{m+1} - u_{ij}^m}{\delta\tau} = \frac{u_{i+1,j}^m - 2u_{ij}^m + u_{i-1,j}^m}{\delta x^2} + \frac{u_{i,j+1}^m - 2u_{ij}^m + u_{i,j-1}^m}{\delta y^2}. \qquad (84)$$

If we define $\alpha_x = \frac{\delta\tau}{\delta x^2}$ and $\alpha_y = \frac{\delta\tau}{\delta y^2}$ the above becomes

$$\begin{aligned} u_{ij}^{m+1} &= u_{ij}^m + \alpha_x(u_{i+1,j}^m - 2u_{ij}^m + u_{i-1,j}^m) + \alpha_y(u_{i,j+1}^m - 2u_{ij}^m + u_{i,j-1}^m) \\ &= -2(\alpha_x + \alpha_y)u_{ij}^m + \alpha_x u_{i-1,j}^m + \alpha_x u_{i+1,j}^m + \alpha_y u_{i,j-1}^m + \alpha_y u_{i,j+1}^m \,, \end{aligned}$$

a fully explicit scheme.

An implicit scheme

To implement a fully implicit scheme requires simply to evaluate the centered spatial derivatives at the time $\tau + \delta\tau$. When we do this we get

$$\frac{u_{ij}^{m+1} - u_{ij}^m}{\delta\tau} = \frac{u_{i+1,j}^{m+1} - 2u_{ij}^{m+1} + u_{i-1,j}^{m+1}}{\delta x^2} + \frac{u_{i,j+1}^{m+1} - 2u_{ij}^{m+1} + u_{i,j-1}^{m+1}}{\delta y^2} \,. \tag{85}$$

The Crank-Nicholson scheme

The Crank-Nicholson scheme is obtained by taking an average of the right-hand side of Equations 84 and 85 to get

$$\begin{aligned} \frac{u_{ij}^{m+1} - u_{ij}^m}{\delta\tau} &= \frac{1}{2}\left(\frac{u_{i+1,j}^m - 2u_{ij}^m + u_{i-1,j}^m}{\delta x^2} + \frac{u_{i,j+1}^m - 2u_{ij}^m + u_{i,j-1}^m}{\delta y^2}\right) \\ &+ \frac{1}{2}\left(\frac{u_{i+1,j}^{m+1} - 2u_{ij}^{m+1} + u_{i-1,j}^{m+1}}{\delta x^2} + \frac{u_{i,j+1}^{m+1} - 2u_{ij}^{m+1} + u_{i,j-1}^{m+1}}{\delta y^2}\right) \,. \end{aligned}$$

Chapter 9 (Methods for American Options)

Exercise Solutions

Exercise 1 (the projected SOR solver)

See the MATLAB function `PSOR_solver.m` where we implement the projected successive overrelaxation (PSOR) solver for American options discretized using a Crank-Nicholson finite-difference formulation.

Exercise 2 (an explicit discretization of the LCP)

The continuous linear complementary problem (LCP) is given by the following set of equations

$$\left(\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2}\right) \geq 0\,, \quad u(x,\tau) - g(x,\tau) \geq 0\,, \quad \text{and} \tag{86}$$

$$\left(\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2}\right)(u(x,\tau) - g(x,\tau)) = 0\,. \tag{87}$$

To discretize these inequalities explicitly, requires we evaluate all spatial derivatives at the discrete time $t^m = m\delta\tau$. Thus we consider the finite-difference approximations given by

$$\frac{\partial u}{\partial \tau} \approx \frac{u_n^{m+1} - u_n^m}{\delta\tau} + O(\delta\tau)\,,$$

and

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{n+1}^m - 2u_n^m + u_{n-1}^m}{\delta x^2} + O((\delta x)^2)\,.$$

Then considering the equation $\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2} = 0$ with the finite difference approximations above we find

$$u_n^{m+1} = u_n^m + \left(\frac{\delta\tau}{(\delta x)^2}\right)(u_{n+1}^m - 2u_n^m + u_{n-1}^m)\,.$$

The right-hand-side of this equation (using the definition $\alpha = \frac{\delta\tau}{\delta x^2}$) becomes

$$u_n^m + \alpha(u_{n+1}^m - 2u_n^m + u_{n-1}^m) = (1-2\alpha)u_n^m + \alpha(u_{n+1}^m + u_{n-1}^m)\,.$$

We will define this later expression as y_n^{m+1} (a proposed value for u_n^{m+1}). We can be guaranteed to satisfy the constraint $u(x,\tau) \geq g(x,\tau)$ by taking

$$u_n^{m+1} = \max(g_n^{m+1}, y_n^{m+1}) \, .$$

To further explain this technique note that if in this maximum we select g_n^{m+1} for the value of u_n^{m+1} (that is the condition $g_n^{m+1} > y_n^{m+1}$ holds) then $u_n^{m+1} = g_n^{m+1}$ and the discrete form of $u(x,\tau) = g(x,\tau)$ holds. Thus the right-most factor $u(x,\tau) - g(x,\tau)$ in Equation 87 vanishes. If in the maximum we select y_n^{m+1} (that is the condition $g_n^{m+1} < y_n^{m+1}$ holds) the discrete form of $\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2}$ vanishes and the left-most factor in Equation 87 is zero. In either case we have that the discrete form of

$$\left(\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial \tau^2} \right) (u(x,\tau) - g(x,\tau)) = 0 \, ,$$

holds.

Exercise 3 (implicit discretization of the LCP)

The continuous linear complementary problem (LCP) formulation is given in the Equations 86 and 87 above. Using an implicit discretization for the equalities above gives

$$\frac{u_n^m - u_n^{m-1}}{\delta t} - \frac{u_{n+1}^m - 2u_n^m + u_{n-1}^m}{(\delta x)^2} = 0 \, .$$

Simplification of this (using the definition of α defined as $\alpha = \frac{\delta t}{(\delta x)^2}$) we have

$$-\alpha u_{n+1}^m + (1 + 2\alpha) u_n^m - \alpha u_{n-1}^m = u_n^{m-1} \, , \tag{88}$$

for $n = N^- + 1, N^- + 2, \cdots, N^+ - 2, N^+ - 1$. When $n = N^-$ or $n = N^+$ we must enforce the continuous boundary conditions of

$$\lim_{x \to \pm\infty} u(x,\tau) = \lim_{x \to \pm\infty} g(x,\tau) \, ,$$

which in the discrete case become

$$u_{N^-}^m = g_{N^-}^m \quad \text{and} \quad u_{N^+}^m = g_{N^+}^m \tag{89}$$

With this, on defining the vectors $\mathbf{u}^m$ and $\mathbf{g}^m$ as in equation 9.10 in the text and $\mathbf{C}$ and $\mathbf{b}^m$ as given in the problem, Equation 88 with the boundary conditions in Equation 89 in matrix form becomes

$$\begin{aligned} &\mathbf{C}\mathbf{u}^m \geq \mathbf{b}^m \,, \mathbf{u}^m \geq \mathbf{g}^m \\ &(\mathbf{u}^m - \mathbf{g}^m)(\mathbf{C}\mathbf{u}^m - \mathbf{b}^m) = 0 \,, \end{aligned}$$

as claimed in the book.

Exercise 4 (the PSOR method for the implicit discretization)

The projected SOR method for the equations in Exercise 3 is given by iterating until convergence two steps. The first, is a normal Gauss-Seidel step followed by a modified overrelaxation step. Exercise 3 specifies that we solve the implicit equation

$$-\alpha u_{n-1}^m + (1 + 2\alpha) u_n^m - \alpha u_{n+1}^m = b_n^m \,,$$

for u_n^m. The normal Gauss-Seidel step for this equation is given by computing

$$y_n^{m,k+1} = \frac{1}{1+2\alpha}(b_n^m + \alpha u_{n-1}^{m,k+1} + \alpha u_{n+1}^{m,k}) \,.$$

Then the modified overrelaxation step is given by correcting this value y_n^{m+1} as

$$u_n^{m,k+1} = \max(g_n^m, u_n^{m,k} + \omega(y_n^{m,k+1} - u_n^{m,k})) \,.$$

Here ω is the overrelaxation parameter $1 < \omega < 2$. The two equations for $y_n^{m,k}$ and $u_n^{m,k}$ are repeatedly iterated until convergence of $u_n^{m,k}$.

Exercise 5 (the unmodified explicit LCP)

The unmodified linear complementary problem LCP for an American put $V(S, t)$ in terms of financial variables is given by two inequalities

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0) S \frac{\partial V}{\partial S} - rV \leq 0 \qquad (90)$$

$$V \geq \max(E - S, 0) \,, \qquad (91)$$

with the equality constraint that

$$\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0) S \frac{\partial V}{\partial S} - rV\right)(V - \max(E - S, 0)) = 0 \,.$$

To discretize Equation 90 we follow the steps from Exercise 9 in the previous chapter. Defining W_n^m be the explicit finite-difference solution to Equation 90 (taken as an equality) we enforce the second inequality constraint $V \geq \max(E - S, 0)$ by taking for V_n^m the following

$$\begin{aligned}
V_n^m &= \max(W_n^m, \max(E - n\delta S, 0)) \\
&= \max(W_n^m, E - n\delta S, 0) \\
&= \max(W_n^m, E - n\delta S)\,.
\end{aligned}$$

The changes to the above method needed to price an American call would be that the inequality in Equation 91 would now become

$$V(S,t) \geq \max(S - E, 0)\,, \tag{92}$$

which translates into the discrete representation as

$$\begin{aligned}
V_n^m &= \max(W_n^m, \max(n\delta S - E, 0)) \\
&= \max(W_n^m, n\delta S - E)\,.
\end{aligned}$$

For an American cash-or-nothing option the modification needed for pricing is to change the constraint imposed by the payoff function. As an example for a cash-or-nothing call we have a continuous constraint on the value of our American option V given by

$$V(S,t) \geq B\mathcal{H}(S - E)\,.$$

This requires that when we compute our numerical solution that we take

$$V_n^m = \max(W_n^m, B\mathcal{H}(n\delta S - E))\,.$$

Exercise 6 (the unmodified implicit LCP)

The linearly complementary problem (LCP) for an American put was discussed in Exercise 5 of this chapter and the fully implicit finite-difference discretization was derived in Exercise 13 in the previous chapter. Given the notation in this problem and the results from the previous exercises the Equations 86 would become

$$\mathbf{M}\mathbf{V}^m \geq \mathbf{V}^{m+1} \quad \text{and} \quad \mathbf{V}^m \geq \boldsymbol{\Lambda}\,,$$

with $\mathbf{V}^m$ the vector with components of V_n^m for $n = 0, 1, \cdots, N$, and $\mathbf{\Lambda}$ a vector with elements containing the payoff for an American put option evaluated at $S_n = n\delta S$ for $n = 0, 1, 2, \cdots, N$. That is

$$\Lambda = (0, \max(\delta S - E, 0), \max(2\delta S - E, 0), \cdots, \max(N\delta S - E, 0))^T .$$

The remaining expression

$$(\mathbf{V}^m - \mathbf{\Lambda})(\mathbf{M}\mathbf{V}^m - \mathbf{V}^{m+1}) = 0 ,$$

is just the transformed linear complementary problem constraint Equation 87 written in terms of the discrete vector variable $\mathbf{V}^m$.

Exercise 7 (the PSOR method for the implicit LCP)

The projected SOR algorithm from Exercise 6 consists of iterating the steps of solving the matrix system $\mathbf{M}\mathbf{V}^m = \mathbf{V}^{m+1}$ for $\mathbf{V}^m$ and then "correcting" its solution to enforce that $\mathbf{V}^m \geq \mathbf{\Lambda}$ and repeating these two steps until convergence. As an iterative scheme this means that we repeat the following two steps

$$\begin{aligned} Y_n^{m,k+1} &= \frac{1}{B_n}(V_n^{m+1} - A_n V_{n-1}^{m,k+1} - C_n V_{n+1}^{m,k}) \\ V_n^{m,k+1} &= \max(V_n^{m,k} - \omega(Y_n^{m,k+1} - V_n^{m,k}), \Lambda(n\delta S)) \\ &= \max(V_n^{m,k} - \omega(Y_n^{m,k+1} - V_n^{m,k}), \max(0, E - n\delta S)) . \end{aligned}$$

The first equation for $Y_n^{m,k+1}$ is a Gauss-Seidel step and is where we "solve" $\mathbf{M}\mathbf{V}^m = \mathbf{V}^{m+1}$ for $\mathbf{V}^m$. One step of the Gauss-Seidel method does not result in a very accurate solution requiring an overrelaxation step. The second equation above (the one for $V_n^{m,k+1}$) we "correct" via. the outer maximization function any proposed value for $V_n^{m,k+1}$ that would not satisfy the early option constraint $V_n^m \geq \Lambda(n\delta S)$.

Exercise 8 (numerical solutions for some American options)

Methods aimed at the solution to the Black-Scholes equation were discussed in previous chapters. The basic formulation presented was

1. Transform the Black-Scholes equation into the pure diffusion equation $u_\tau = u_{xx}$ with the final option payoff becoming the initial condition of the diffusion equation.

2. Solve the diffusion equation using any number of techniques.

3. Transform the solution of the diffusion equation back into the financial variables of interest using

$$V(S,t) = E^{\frac{1}{2}(k+1)} S^{\frac{1}{2}(1-k)} e^{-\frac{1}{8}(k+1)^2\sigma^2(T-t)} u\left(\log(S/E), \frac{1}{2}\sigma^2(T-t)\right).$$

   where $0 \le t \le T$ and $0 \le S \le \infty$.

These expressions and their derivations in addition to the transformed payoff functions are discussed in much more detail in the book and also in the discussions on Chapter 8 found in these notes.

To price options on a stock with a constant dividend yield D_0 we recall from Chapter 6 that its price is given by $V(S,t) = e^{-D_0(T-t)} V_1(S,t)$, where V_1 is the "normal" Black-Scholes price obtained numerically or analytically using an interest rate $r - D_0$ rather than r. Procedurally, this means that we solve the Black-Scholes equation with an interest rate of $r - D_0$ to get $V_1(S,t)$ and then rescale this this solution by the factor $e^{-D_0(T-t)}$ to get the complete solution. For this problem we implemented the Crank-Nicholson numerical scheme for pricing American options using the linear complementary framework. The code for the implementation of the projected SOR numerical method can be found in the MATLAB functions `PSOR_solver.m`. The code for the implementation of the timestepping can be found in the function `crank_fd_PSOR.m`. Examples of how to call these functions can be seen by calling the routines `dup_fig_9_4.m` and `dup_fig_9_5.m`. The first script `dup_fig_9_4.m` duplicates the table of option prices presented in Figures 9.4 for an American put option. While the second script `dup_fig_9_5.m` duplicates the plot of an American call option shown in Figure 9.5. Plots of the various profiles produced are shown in Figure 11.

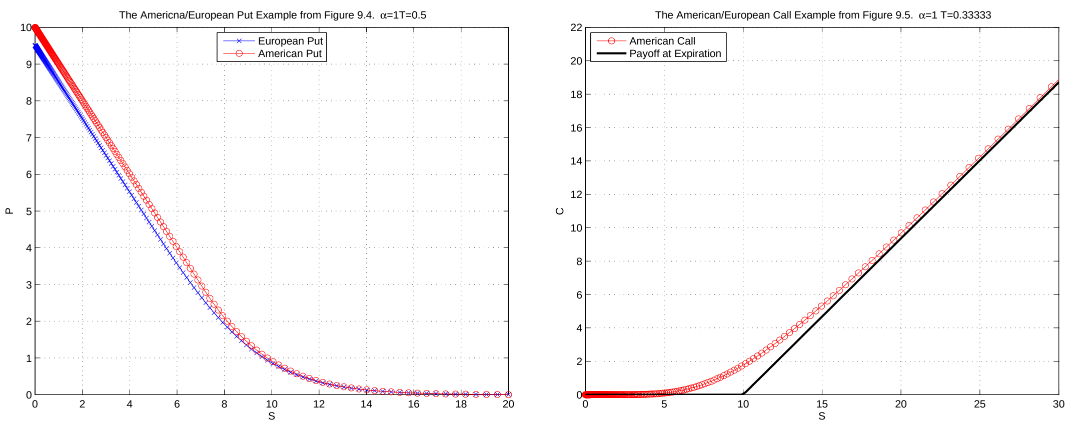

Figure 11: Example numerical solutions for American put and call options. **Left:** The numerical solution of an American put option with parameters taken to duplicate the results found in figure 9.4 from the book. **Right:** The numerical solution of an American call option with parameters taken to duplicate the results found in figure 9.5 from the book.

Chapter 10 (Binomial Methods)

Additional Notes on the Text

The discrete random walk

After having computed expected value of S^{m+1} and of $(S^{m+1})^2$ in Exercise 1 below the variance of the continuous random walk is

$$\begin{aligned} \text{var}_c[S^{m+1}|S^m] &= \text{E}[(S^{m+1})^2|S^m] - \text{E}[S^{m+1}|S^m]^2 \\ &= e^{(2r+\sigma^2)\delta t}(S^m)^2 - e^{2r\delta t}(S^m)^2 \\ &= (S^m)^2 e^{2r\delta t}(e^{\sigma^2\delta t} - 1)\,. \end{aligned} \tag{93}$$

To compute the variance of S^{m+1} under the discrete binomial process requires the calculation of $\mathcal{E}_b[(S^{m+1})^2|S^m]$. Using the definition of our binomial tree we find this expectation given by

$$\begin{aligned} \mathcal{E}_b[(S^{m+1})^2|S^m] &= p(uS^m)^2 + (1-p)(dS^m)^2 \\ &= (pu^2 + (1-p)d^2)(S^m)^2\,. \end{aligned} \tag{94}$$

Using this, the variance is then given by

$$\begin{aligned} \text{var}_b[S^{m+1}|S^m] &= (pu^2 + (1-p)d^2)(S^m)^2 - (S^m)^2 e^{2r\delta t} \\ &= (S^m)^2(pu^2 + (1-p)d^2 - e^{2r\delta t})\,. \end{aligned} \tag{95}$$

Setting this expression equal to the continuous variance expression in Equation 93 requires

$$pu^2 + (1-p)d^2 - e^{2r\delta t} = e^{\sigma^2\delta t + 2r\delta t} - e^{2r\delta t}\,.$$

This then becomes

$$pu^2 + (1-p)d^2 = e^{(2r+\sigma^2)\delta t}\,, \tag{96}$$

which is equation 10.5 in the book.

the case $u = 1/d$

If we solve for p in the expression obtained when we set the expectations equal (i.e. equation 10.4 in the book) we find that

$$pu + d - pd = e^{r\delta t}\,.$$

or solving for p that p can be expressed as

$$p = \frac{e^{r\delta t} - d}{u - d}\,. \tag{97}$$

Doing the same thing when we equate the variances Equation 96 (equation 10.5 in the book) gives the following for p in terms of u and d

$$p = \frac{e^{(2r+\sigma^2)\delta t} - d^2}{u^2 - d^2}\,. \tag{98}$$

This is equation 10.8 in the book. Dividing Equation 97 by Equation 98 we obtain

$$\left(\frac{u^2 - d^2}{u - d}\right)\left(\frac{e^{r\delta t} - d}{e^{(2r+\sigma^2)\delta t} - d^2}\right) = 1\,.$$

This allows us to solve for $u + d$ and we find

$$u + d = \frac{e^{(2r+\sigma^2)\delta t} - d^2}{e^{r\delta t} - d}\,,$$

which is the equation in the book presented below 10.8. When we multiply by $e^{r\delta t} - d$ on both side this expression becomes

$$(u + d)e^{r\delta t} - ud - d^2 = e^{(2r+\sigma^2)\delta t} - d^2\,,$$

or

$$(u + d)e^{r\delta t} - ud = e^{(2r+\sigma^2)\delta t}\,.$$

If we now enforce the constraint the the up returns u and the down returns d are equal in strength we take $u = \frac{1}{d}$ and obtain (after multiplying by $de^{-r\delta t}$) the equation

$$d^2 - (e^{-r\delta t} + e^{(r+\sigma^2)\delta t})d + 1 = 0\,.$$

If we define a variable A as

$$A = \frac{1}{2}\left(e^{-r\delta t} + e^{(r+\sigma^2)\delta t}\right)\,, \tag{99}$$

we obtain a quadratic equation for d given by

$$d^2 - 2Ad + 1 = 0\,. \tag{100}$$

This has solutions given by the quadratic formula. We find

$$d = A \pm \sqrt{A^2 - 1}\,. \tag{101}$$

Note that u since it equals $1/d$ is then given by

$$\begin{aligned} u &= \frac{1}{d} = \frac{1}{A \pm \sqrt{A^2 - 1}} \\ &= \left(\frac{1}{A \pm \sqrt{A^2 - 1}} \right) \left(\frac{A \mp \sqrt{A^2 - 1}}{A \mp \sqrt{A^2 - 1}} \right) \\ &= A \mp \sqrt{A^2 - 1}\,. \end{aligned} \tag{102}$$

Since $d < u$ we must take the negative sign in the expression 101 for d and the positive sign for u in the expression 102.

Now since $u = \frac{1}{d}$ the quadratic equation for u is given by setting $d = \frac{1}{u}$ in Equation 100 giving

$$\frac{1}{u^2} - 2\left(\frac{1}{u}\right) + 1 = 0\,.$$

Multiplying this equation by u and rearranging gives

$$1 - 2Au + u^2 = 0\,,$$

the same quadratic equation we had for d in 100.

the case $p = \frac{1}{2}$

When $p = \frac{1}{2}$ from the equation obtained by equating the mean of the continuous process and the binomial tree approximation (equation 10.4) we have

$$u + d = 2e^{r\delta t}\,. \tag{103}$$

and by equating the variance of the continuous random process and our discrete binomial tree given by Equation 96 we find

$$u^2 + d^2 = 2e^{(2r+\sigma^2)\delta t}\,, \tag{104}$$

which is equation 10.11 in the book. To solve these two equations for u and d we let $u = B + C$ and $d = B - C$ and then solve for B and C. With these substitutions Equation 103 becomes

$$B + C + B - C = 2B = 2e^{r\delta t} \quad \text{or} \quad B = e^{r\delta t}\,.$$

While Equation 104 becomes

$$(B+C)^2 + (B-C)^2 = 2e^{(2r+\sigma^2)\delta t},$$

or

$$C^2 = e^{(2r+\sigma^2)\delta t} - B^2 = e^{2r\delta t}(e^{\sigma^2\delta t} - 1),$$

so that C then becomes

$$C = e^{r\delta t}\sqrt{e^{\sigma^2\delta t} - 1}\,. \tag{105}$$

With this for C, u and d then become

$$\begin{aligned} u &= B + C = e^{r\delta t}\left(1 + \sqrt{e^{\sigma^2\delta t} - 1}\right) \\ d &= B - C = e^{r\delta t}\left(1 - \sqrt{e^{\sigma^2\delta t} - 1}\right), \end{aligned}$$

which are the equations 10.12 in the book.

Implementing the binomial tree algorithm

In this subsection we comment on the pseudo-code implementation of the discrete binomial model for a European option and found in figure 10.4 of the book. Most of the code presented there is straightforward with the exception of the initial `for` loop that fills the `array` data structure. The loop we wish to discuss here is inside a loop over timestep (the outer `m` loop) and is specifically given by

```
for( n=m; n>0; --n )
  array[n] = u*array[n-1];
array[0] = d*array[0];
```

This loop uses the fact that at the iteration `m-1` in the outer timestep loop we have the array `array[0:m-1]` filled the stock prices $S_n^{m-1} = d^{m-1-n}u^n S_0^0$ for $n = 0, 1, 2, \cdots, m-1$, and the observation that we can get most of the stock prices required for the *next* timestep `m` by multiplying every value in this array by the value of u which then gives for values of `array[1:m]` the following numbers

$$uS_{n-1}^{m-1} = d^{m-1-(n-1)}u^n S_0^0 = d^{m-n}u^n S_0^0 \quad \text{for} \quad n = 1, \cdots, m\,.$$

This calculation is exactly what the `for` loop over `n` in the above code does. With these values filling the array in the locations `array[1:m]` is almost what we need for the timestep m, where we are only missing the $n = 0$ term in the expression $d^{m-n}u^n S_0^0$. Now when $n = 0$ the original value of `array[0]` from timestep $m-1$ had not been modified and had the value $d^{m-1}S_0^0$ which is missing one d from the required expression for our discrete stock prices at the timestep m, i.e. the expression $d^{m-n}u^n S_0^0$ evaluated at $n = 0$. So to compensate for this we multiply the value of `array[0]` by d to add the stock price

$$d(d^{m-1}S_0^0) = d^m S_0^0\,,$$

which is the required $n = 0$ element. Thus we have filled the data structure `array[]` with the required discrete stock values at timestep m.

Exercise Solutions

Exercise 1 (the expectation of S^{m+1} and $(S^{m+1})^2$)

Part (a): The probability density of an assets price S' at time t' given its initial price S at the time t is given by $p(S, t; S', t')$ in equation 10.3 in the book. Repeated here for convenience we recall

$$p(S,t;S',t') = \frac{1}{\sigma S'\sqrt{2\pi(t'-t)}}\exp\{-\frac{\left(\log(S'/S) - (r-\frac{1}{2}\sigma^2)(t'-t)\right)^2}{2\sigma^2(t'-t)}\}\,,$$

Now (S, t) are the current state of our system and we move to the new state (S', t'). From this we define the expectation of new asset price S' as

$$\mathcal{E}[S^{m+1}|S^m] = \int_0^\infty S' p(S^m, m\delta t; S', (m+1)\delta t) dS'\,.$$

Thus using the above expression for p we have

$$\mathcal{E}[S^{m+1}|S^m] = \frac{1}{\sigma\sqrt{2\pi(\delta t)}}\int_0^\infty \exp\{-\frac{\left(\log(S'/S^m) - (r-\frac{1}{2}\sigma^2)(\delta t)\right)^2}{2\sigma^2(\delta t)}\}dS'\,.$$

To evaluate this integral let

$$\begin{aligned} v &= \frac{\log(S'/S^m) - (r-\frac{1}{2}\sigma^2)\delta t}{\sigma\sqrt{\delta t}} \quad \text{so that} \\ dv &= \frac{1}{\sigma\sqrt{\delta t}}\frac{dS'}{S'}\,, \end{aligned}$$

and S' in terms of v is

$$S' = S^m e^{\sigma\sqrt{\delta t}v + (r - \frac{1}{2}\sigma^2)\delta t}.$$

With all of these substitutions our expectation becomes

$$\begin{aligned}
\mathcal{E}[S^{m+1}|S^m] &= \frac{1}{\sigma\sqrt{2\pi(\delta t)}}\int_{-\infty}^{+\infty} e^{-v^2/2}\sqrt{\sigma^2\delta t}S^m e^{\sigma\sqrt{\delta t}v+\left(r-\frac{1}{2}\sigma^2\right)\delta t}dv \\
&= \frac{S^m e^{(r-\frac{1}{2}\sigma^2)\delta t}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}v^2+\sigma\sqrt{\delta t}v}dv.
\end{aligned}$$

Next lets consider the exponent of the exponential function in our integrand. By completing the square in terms of v we find

$$\begin{aligned}
-\frac{1}{2}(v^2 - 2\sigma\sqrt{\delta t}v) &= -\frac{1}{2}(v^2 - 2\sigma\sqrt{\delta t}v + \sigma^2\delta t - \sigma^2\delta t) \\
&= -\frac{1}{2}((v - \sigma\sqrt{\delta t})^2 - \sigma^2\delta t).
\end{aligned}$$

With this substitution the above expectation becomes

$$\frac{S^m e^{(r-\frac{1}{2}\sigma^2)\delta t}}{\sqrt{2\pi}} e^{\frac{1}{2}\sigma^2\delta t}\int_{-\infty}^{+\infty} e^{-\frac{1}{2}(v-\sigma\sqrt{\delta t})^2}dv = S^m e^{r\delta t},$$

the expression we were to show.

Part (b): Next we consider the expectation of $(S^{m+1})^2$ given by

$$\begin{aligned}
\mathcal{E}[(S^{m+1})^2|S^m] &= \int_0^{\infty} S'^2 p(S^m, m\delta t; S', (m+1)\delta t)dS' \\
&= \frac{1}{\sigma\sqrt{2\pi\delta t}}\int_0^{\infty} S' \exp\{-\frac{\left(\log(S'/S^m) - \left(r - \frac{1}{2}\sigma^2\right)\delta t\right)^2}{2\sigma^2\delta t}\}dS'
\end{aligned}$$

Again making the substitution for v as before this integral becomes

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}v^2}(S^m)^2 e^{2\sigma\sqrt{\delta t}v+2(r-\frac{1}{2}\sigma^2)\delta t}dv = \frac{(S^m)^2 e^{2(r-\frac{1}{2}\sigma^2)\delta t}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}v^2}e^{2\sigma\sqrt{\delta t}v}dv.$$

The exponent in this integral becomes

$$\begin{aligned}
-\frac{1}{2}(v^2 - 4\sigma\sqrt{\delta t}v) &= -\frac{1}{2}\left(v^2 - 4\sigma\sqrt{\delta t}v + 4\sigma^2\delta t - 4\sigma^2\delta t\right) \\
&= -\frac{1}{2}((v - 2\sigma\sqrt{\delta t})^2 - 4\sigma^2\delta t),
\end{aligned}$$

so our integral becomes

$$\frac{(S^m)^2 e^{2(r-\frac{1}{2}\sigma^2)\delta t} e^{2\sigma^2\delta t}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}(v-2\sigma\sqrt{\delta t})^2} dv = (S^m)^2 e^{2r\delta t} e^{\sigma^2 \delta t},$$

the expression we were to prove.

Exercise 2 (notes on European options with binomial trees)

That the memory required to evaluate a European option varies linearly with the number of time steps can be seen by looking at the pseudo-code for the pricing of a European option presented in this chapter (also see the discussion on Page 143). The only storage required is the array `array`, which is used to store the distribution of final asset prices

$$S_n^M = d^{M-n} u^n S_0^0 \quad n = 0, 1, 2, \cdots, M,$$

and then from these prices the option values V_n^M at this timestep

$$V_n^M = \max(S_n^M - E, 0) \quad \text{for} \quad n = 0, 1, \cdots, M.$$

As there is no other storage the memory is linear in the number of timesteps M. That the execution time is quadratic can be seen by again considering the pseudo-code. To find the value of V at given time ($m = 0$) we need to average the two values of V further up the binary tree. That is we need to compute

$$e^{r\delta t} V_n^m = p V_{n+1}^{m+1} + (1-p) V_n^{m+1},$$

for $m = M$ down to $m = 0$. This is accomplished with the following for loop

```
for(m=M; m>0; --m)
{
  for( n=0; n<m; ++n )
  {
    tmp = p*array[n+1] + (1-p)*array[n];
    array[n] = discount*tmp;
  }
}
```

which is clearly quadratic (two for loops) in time.

Exercise 6 (options on shares that pay discrete dividends)

To modify binomial tree methods to incorporate discrete dividends is quite simple. In the tree construction phase of binomial tree method one constructs values of S_n^m as done earlier using $S_n^m = d^{m-n}u^n S_0^0$ for $n = 0, 1, 2, \cdots, m$ and each time slice index m until reaching the dividend date/time given by $t_d = l\delta t$. We assume that our timestep δt is chosen such that binomial tree has $t_d = l\delta t$ for some index l. To continue the binomial tree for times after t_d, we simply decrement all stock values by the dividend amount $d_y S$. That is coming from the left into any node situated at a time of t_d we have an asset value given by S_n^m while exiting the same node we have an asset value given by $(1 - d_y)S_n^m$. To implement this in pseudo-code for the asset price code generation given in figure 10.7 in the book (with added comments) would change as follows

```
s[0][0] = S0;
for( m=1; m<=M; ++m) /* timestep loop */
{
  for(n=m+1; n>0; --n) /* stock price loop */
    s[m][n] = u*s[m-1][n-1];
 s[m][0] = d*s[m-1][0];
 /* If we have reached the dividend pay-out date before
    going further adjust all stock prices ex-dividend */
 if( m==l ){
   s[m][n] = s[m][n] * ( 1 - dy );
 }
}
```

Chapter 11 (Exotic and Path-dependent Options)

Exercise Solutions

Exercise 1 (put-call-parody for compound options)

To derive the put-call parity result for compound options (an option on an option) specifically for a call-on-a-call and a put-on-a-call we consider a portfolio Π, in which we are long one call (with strike E_2 and expiration time T_2), long one call-on-a-call and short one put-on-a-call both with the same expiration T_1 and strike E_1. To begin with we need to be able to evaluate the payoff on the put-on-a-call. Then following the same arguments as in the chapter we see that at the time T_1 (when we must decide to exercise) the value of the underlying call at that time is given by the standard Black-Scholes formula denoted $C(S, T_1)$. If at T_1 the stock price S was such that we had $C(S, T_1) < E_1$ we would choose to exercise this put-on-a-call since it worth an amount $E_1 - C(S, T_1)$. If on the other hand $C(S, T_1) > E_1$ we would not exercise our put-on-a-call since it is worthless. In either case, we see that the value of a put-on-a-call at $t = T_1$ is given by

$$\max(E_1 - C(S, T_1), 0)\,.$$

Revisiting our original problem with a portfolio Π described above, at the time $t = T_1$ since our initial portfolio was long one call, long one call-on-a-call and short one put-on-a-call it is worth

$$\begin{aligned}
\Pi(S, T_1) &= C(S, T_1) + \max(E_1 - C(S, T_1), 0) - \max(C(S, T_1) - E_1, 0) \\
&= C(S, T_1) - (C(S, T_1) - E_1) \\
&= E_1\,,
\end{aligned}$$

The asset that pays the fixed amount E_1 at the time T_1 is

$$E_1 e^{-r(T_1 - t)}\,.$$

Thus the put-call-parody relationship between call-on-a-call and a put-on-a-call is given by

$$C(S, t) + \text{PoC} - \text{CoC} = E_1 e^{-r(T_1 - t)}\,. \tag{106}$$

Where we have used the notation CoC and PoC to denote a call-on-a-call and a put-on-a-call respectively.

Exercise 2 (compound options priced based on their underlying)

The stochastic differential equation satisfied by an option's "random walk" is more complicated than that satisfied by an equity.

Exercise 3 (the derivation of the price for compound options)

Following the discussion (and using the notation) in the section on European compound options in this chapter, the call-on-a-call at the time T_1 has a payoff given by

$$\Lambda(S) = \max(C_{\text{BS}}(S, T_1) - E_1, 0)\,.$$

Here $C_{\text{BS}}(S, T_1)$ is the value of a call option that has an expiration at the time T_2 with an expiration price of E_2. To value our compound option we will use Formula 5.16 in the book which gives the value of an option at times $t < T_1$, when we know the payoff (or terminal condition) $\Lambda(S)$ at the terminal time T_1. This formula is

$$V(S,t) = \frac{e^{-r(T_1-t)}}{\sigma\sqrt{2\pi(T_1-t)}} \int_0^\infty \Lambda(S') e^{-\left(\log(S'/S)-(r-\frac{1}{2}\sigma^2)(T_1-t)\right)^2/2\sigma^2(T_1-t)} \frac{dS'}{S'}\,. \tag{107}$$

To evaluate this integral it will be helpful to define S^* so that if our stocks price is greater than S^* one would exercise our call-on-a-call option while if it is less than S^* one would not. That is the value of S^* should satisfy

$$C_{\text{BS}}(S^*, T_1) = E_1\,.$$

With this definition of S^* the integral for $V(S,t)$ above expression becomes

$$V(S,t) = \frac{e^{-r(T_1-t)}}{\sigma\sqrt{2\pi(T_1-t)}} \int_{S^*}^\infty (C_{\text{BS}}(S', T_1) - E_1) e^{-\left(\log(S'/S)-(r-\frac{1}{2}\sigma^2)(T_1-t)\right)^2/2\sigma^2(T_1-t)} \frac{dS'}{S'}\,. \tag{108}$$

It is easiest to evaluate this integral by breaking it into two parts. The second term, which we will denote by I_2, will be defined as

$$I_2 \equiv -\frac{E_1 e^{-r(T_1-t)}}{\sigma\sqrt{2\pi(T_1-t)}} \int_{S^*}^\infty e^{-\left(\log(S'/S)-(r-\frac{1}{2}\sigma^2)(T_1-t)\right)^2/2\sigma^2(T_1-t)} \frac{dS'}{S'}\,.$$

This integral can be evaluated by using the substitution

$$\begin{aligned} v &= \frac{\log(\frac{S'}{S}) - \left(r - \frac{1}{2}\sigma^2\right)(T_1 - t)}{\sigma\sqrt{T_1 - t}} \quad \text{so that} \qquad (109) \\ dv &= \left(\frac{1}{\sigma\sqrt{T_1 - t}}\right)\frac{dS'}{S'}\,, \end{aligned}$$

and the integral expression for I_2 becomes

$$\begin{aligned} I_2 &= -\frac{E_1 e^{-r(T_1-t)}}{\sqrt{2\pi}} \int_{v^*}^{\infty} e^{-\frac{v^2}{2}} dv = -\frac{E_1 e^{-r(T_1-t)}}{\sqrt{2\pi}} \int_{-\infty}^{-v^*} e^{-\frac{v^2}{2}} dv \\ &= -E_1 e^{-r(T_1-t)} N(-v^*)\,, \end{aligned}$$

where $N(x)$ is the cumulative distribution function for the standard normal given by Equation 3, and v^* is given by Equation 109 evaluated at $S' = S^*$. In summary then, the term I_2 is given by

$$I_2 = -E_1 e^{-r(T_1-t)} N\left(\frac{\log(\frac{S}{S^*}) + \left(r - \frac{1}{2}\sigma^2\right)(T_1 - t)}{\sigma\sqrt{T_1 - t}}\right)\,. \qquad (110)$$

Now the first term in Equation 108, which we will denote by I_1, is given by

$$I_1 \equiv \frac{e^{-r(T_1-t)}}{\sigma\sqrt{2\pi(T_1 - t)}} \int_{S^*}^{\infty} C_{\text{BS}}(S', T_1) e^{-\left(\log(S'/S) - (r - \frac{1}{2}\sigma^2)(T_1 - t)\right)^2 / 2\sigma^2(T_1 - t)} \frac{dS'}{S'}\,.$$

Recalling the formula for $C_{\text{BS}}(S', T_1)$ the value of the Black-Scholes call with strike E_2 and expiration T_2 evaluated at the time T_1 we have

$$C_{\text{BS}}(S', T_1) = S' N(d_1(S', T_1; E_2, T_2)) - E_2 e^{-r(T_2-T_1)} N(d_2(S', T_1; E_2, T_2))\,,$$

where again $N(\cdot)$ is the cumulative distribution function for the standard normal and d_1 and d_2 are given by their standard formulas expressed here for convenience

$$\begin{aligned} d_1(S', T_1; E_2, T_2) &= \frac{\log(S'/E_2) + \left(r + \frac{1}{2}\sigma^2\right)(T_2 - T_1)}{\sigma\sqrt{T_2 - T_1}} \\ d_2(S', T_1; E_2, T_2) &= \frac{\log(S'/E_2) + \left(r - \frac{1}{2}\sigma^2\right)(T_2 - T_1)}{\sigma\sqrt{T_2 - T_1}}\,. \end{aligned}$$

When this expression is put into the expression for I_2 we can decompose the result into two additional integral expressions, I_{11} and I_{12} defined by

$$
\begin{aligned}
I_{11} &= \frac{e^{-r(T_1-t)}}{\sigma\sqrt{2\pi(T_1-t)}}\int_{S^*}^{\infty} N(d_1(S',T_1;E_2,T_2))e^{-\left(\log(S'/S)-(r-\frac{1}{2}\sigma^2)(T_1-t)\right)^2/2\sigma^2(T_1-t)}dS' \\
I_{12} &= -\frac{E_2 e^{-r(T_2-t)}}{\sigma\sqrt{2\pi(T_1-t)}}\int_{S^*}^{\infty} N(d_2(S',T_1;E_2,T_2))e^{-\left(\log(S'/S)-(r-\frac{1}{2}\sigma^2)(T_1-t)\right)^2/2\sigma^2(T_1-t)}\frac{dS'}{S'}\,.
\end{aligned}
$$

To evaluate these integrals we will first derive an identity involving integrals of the Gaussian function, $\hat{g}(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$, against the cumulative normal distribution $N(x)$ in relation to the *bivariate* standard normal cumulative distribution function[2]. The identity we derive and use is similar to a convolution integral and is given by

$$
M(h,k;\rho) = \int_{-\infty}^{h} \hat{g}(x) N\left(\frac{k-\rho x}{\sqrt{1-\rho^2}}\right) dx\,. \tag{111}
$$

Where $M(h,k;\rho)$ is the cumulative distribution function for the *bivariate* normal random variables (x,y) having a correlation coefficient of ρ. Specifically, this function, $M(h,k;\rho)$, is defined as

$$
M(h,k;\rho) = \frac{1}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{h}\int_{-\infty}^{k} e^{-\frac{1}{2}\left(\frac{x^2-2\rho xy+y^2}{1-\rho^2}\right)} dy dx\,. \tag{112}
$$

This integral identity is stated in the appendix in the classic reference on compound options [3]. To show this we will begin by performing a change of variables on the two dimensional integral on the right-hand-side of Equation 112. To do this we first recall the change of variable formula for two dimensional integrals from [6]

$$
\iint_R f(x,y) dx dy = \iint_{R'} f(x(u,v),y(u,v))|D| du dv\,. \tag{113}
$$

In this identity $f(x,y)$ is an arbitrary two dimensional function, and D is the Jacobian of the mapping from (u,v) to (x,y) given by

$$
D = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \left(\frac{\partial x}{\partial u}\right)\left(\frac{\partial y}{\partial v}\right) - \left(\frac{\partial x}{\partial v}\right)\left(\frac{\partial y}{\partial u}\right)\,.
$$

[2]these types of integrals also correspond to the type of integrals we are trying to evaluate.

To use this expression we will change variables from the (x, y) pair to the (u, v) pair with the transformation

$$u = x \quad \text{and} \quad v = \frac{y - \rho x}{\sqrt{1-\rho^2}}\,.$$

This mapping has an inverse transformation given by

$$x = u \quad \text{and} \quad y = v\sqrt{1-\rho^2} + \rho u\,.$$

For this mapping we find that D is given by

$$D = \left| \begin{array}{cc} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{array} \right| = \sqrt{1-\rho^2}\,.$$

Using this we see that the exponential argument in the definition of $M(h, k; \rho)$ in Equation 112 will become in terms of u and v

$$\begin{aligned} x^2 - 2\rho xy + y^2 &= u^2 - 2\rho u(v\sqrt{1-\rho^2} + \rho u) + v^2(1-\rho^2) + 2\rho\sqrt{1-\rho^2}uv + \rho^2 u^2 \\ &= (1-\rho^2)u^2 + v^2(1-\rho^2)\,. \end{aligned}$$

Using this expression we can see that our transformed integrand of $M(h, k; \rho)$ is $e^{-\frac{1}{2}(u^2+v^2)}$, and our integral then becomes

$$\begin{aligned} M(h, k; \rho) &= \frac{1}{2\pi}\int_{-\infty}^{h}\int_{-\infty}^{\frac{k-\rho u}{\sqrt{1-\rho^2}}} e^{-\frac{1}{2}v^2}e^{-\frac{1}{2}u^2}dvdu \\ &= \int_{-\infty}^{h}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}u^2}\int_{-\infty}^{\frac{k-\rho u}{\sqrt{1-\rho^2}}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}v^2}dvdu \\ &= \int_{-\infty}^{h}\hat{g}(u)N\left(\frac{k-\rho u}{\sqrt{1-\rho^2}}\right)du\,, \end{aligned}$$

the desired identity.

Back to the evaluation of the integrals I_{11} and I_{12} where we will begin by first evaluating I_{12}. Recall I_{12} is given by (ignoring the constants for a moment)

$$I_{12} \propto \int_{S^*}^{\infty} N\left(\frac{\log(S'/E_2) + (r - \frac{1}{2}\sigma^2)(T_2 - T_1)}{\sigma\sqrt{T_2 - T_1}}\right) e^{-\frac{(\log(S'/S)-(r-\frac{1}{2}\sigma^2)(T_1-t))^2}{2\sigma^2(T_1-t)}}\frac{dS'}{S'}\,.$$

We will make a series of transformation that will convert this integral into the desired form so that we can use the integral identity above. We begin with the transformation

$$v = \log(S') \quad \text{so} \quad dv = \frac{dS'}{S'} \,,$$

where under this transformation I_{12} now becomes proportional to

$$\int_{\log(S^*)}^{\infty} N\left(\frac{v - \log(E_2) + (r - \frac{1}{2}\sigma^2)(T_2 - T_1)}{\sigma\sqrt{T_2 - T_1}}\right) e^{-\frac{(v - \log(S) - (r - \frac{1}{2}\sigma^2)(T_1 - t))^2}{2\sigma^2(T_1 - t)}} dv \,.$$

As the next (somewhat trivial) transformation we take $u = -v$ so that $du = -dv$ and the above becomes

$$\int_{-\infty}^{-\log(S^*)} N\left(\frac{-u - \log(E_2) + (r - \frac{1}{2}\sigma^2)(T_2 - T_1)}{\sigma\sqrt{T_2 - T_1}}\right) e^{-\frac{(u + \log(S) + (r - \frac{1}{2}\sigma^2)(T_1 - t))^2}{2\sigma^2(T_1 - t)}} du \,.$$

Next we perform the following transformation aimed at introducing the Gaussian function $e^{-\frac{1}{2}x^2}$ into the integrand. We let

$$\begin{aligned}
w &= \frac{u + \log(S) + (r - \frac{1}{2}\sigma^2)(T_1 - t)}{\sigma\sqrt{T_1 - t}} \quad \text{so} \\
dw &= \frac{du}{\sigma\sqrt{T_1 - t}} \quad \text{and} \\
u &= -\log(S) - (r - \frac{1}{2}\sigma^2)(T_1 - t) + \sigma w\sqrt{T_1 - t} \,.
\end{aligned}$$

With these expressions we obtain limits of the integral I_{12} going from $-\infty$ to an upper limit of h_- defined as

$$h_- \equiv \frac{-\log(S^*) + \log(S) + (r - \frac{1}{2}\sigma^2)(T_1 - t)}{\sigma\sqrt{T_1 - t}} \,,$$

and the *argument* of the cumulative normal N becomes

$$\begin{aligned}
& \frac{\log(S) + (r - \frac{1}{2}\sigma^2)(T_1 - t) - \sigma w\sqrt{T_1 - t} - \log(E_2) + (r - \frac{1}{2}\sigma^2)(T_2 - T_1)}{\sigma\sqrt{T_2 - T_1}} \\
= \; & \frac{\log(S/E_2) + (r - \frac{1}{2}\sigma^2)(T_2 - t) - \sigma w\sqrt{T_1 - t}}{\sigma\sqrt{T_2 - T_1}} \,.
\end{aligned}$$

For ease of notation we will define the terms without the variable w in the numerator of this fraction to be l_-. That is we take l_- to be given by

$$l_- \equiv \log(S/E_2) + (r - \frac{1}{2}\sigma^2)(T_2 - t) .$$

Thus we now have

$$\begin{aligned} I_{12} &\propto \int_{-\infty}^{h_-} N\left(\frac{l_- - \sigma w\sqrt{T_1 - t}}{\sigma\sqrt{T_2 - T_1}}\right) e^{-\frac{w^2}{2}}\, dw\, \sigma\sqrt{T_1 - t} \\ &= \sigma\sqrt{2\pi(T_1 - t)} \int_{-\infty}^{h_-} \left(\frac{1}{\sqrt{2\pi}}\right) e^{-\frac{w^2}{2}} N\left(\frac{l_- - \sigma w\sqrt{T_1 - t}}{\sigma\sqrt{T_2 - T_1}}\right) dw . \end{aligned}$$

To make this integral of the cumulative normal match the one derived above requires that we find values for ρ and k that make the arguments of the cumulative normal satisfies

$$\frac{l_- - \sigma\sqrt{T_1 - t}\, w}{\sigma\sqrt{T_2 - T_1}} = \frac{k_- - \rho w}{\sqrt{1 - \rho^2}} . \tag{114}$$

Setting the coefficients on both sides of w equal in this expression requires that ρ satisfy

$$\frac{\rho}{\sqrt{1 - \rho^2}} = \frac{\sqrt{T_1 - t}}{\sqrt{T_2 - T_1}} .$$

Solving this equation for ρ we find

$$\rho = \sqrt{\frac{T_1 - t}{T_2 - t}} , \tag{115}$$

and from this we can compute that the expression $1 - \rho^2$ is

$$1 - \rho^2 = \frac{T_2 - T_1}{T_2 - t} .$$

Finally, the value of k_- needed to match to the identity in Equation 114 requires

$$\frac{k_-}{\sqrt{1 - \rho^2}} = \frac{l_-}{\sigma\sqrt{T_2 - T_1}} \quad \text{so} \quad k_- = \frac{l_-}{\sigma\sqrt{T_2 - t}} .$$

Then with ρ and k_- defined as above we see that our integral now becomes

$$I_{12} \propto \sigma\sqrt{2\pi(T_1 - t)} M(h_-, k_-; \rho) .$$

When we recall the proportionality constant for I_{12} we finally arrive at a complete specification of its value

$$I_{12} = -E_2 e^{-r(T_2-t)} M(h_-, k_-; \rho) \tag{116}$$

$$h_- = \frac{\log(S/S^*) + (r - \frac{1}{2}\sigma^2)(T_1 - t)}{\sigma\sqrt{T_1 - t}} \quad \text{and} \tag{117}$$

$$k_- = \frac{\log(S/E_2) + (r - \frac{1}{2}\sigma^2)(T_2 - t)}{\sigma\sqrt{T_2 - t}}\,. \tag{118}$$

For the evaluation of I_{11} recall that its integral is proportional to

$$I_{11} \propto \int_{S^*}^{\infty} N\left(\frac{\log(S'/E_2) + (r + \frac{1}{2}\sigma^2)(T_2 - T_1)}{\sigma\sqrt{T_2 - T_1}}\right) e^{-\frac{(\log(S'/S) - (r - \frac{1}{2}\sigma^2)(T_1 - t))^2}{2\sigma^2(T_1 - t)}} dS'\,.$$

To evaluate this integral we again set $v = \log(S')$ so $S' = e^v$, $dv = \frac{dS'}{S'}$, and $dS' = e^v dv$ and we have

$$I_{11} \propto \int_{\log(S^*)}^{\infty} N\left(\frac{v - \log(E_2) + (r + \frac{1}{2}\sigma^2)(T_2 - T_1)}{\sigma\sqrt{T_2 - T_1}}\right) e^{-\frac{(v - \log(S) - (r - \frac{1}{2}\sigma^2)(T_1 - t))^2}{2\sigma^2(T_1 - t)}} e^v dv\,.$$

Combining the exponents of the two exponentials above we obtain the *negative* of

$$\frac{(v - \log(S) - (r - \frac{1}{2}\sigma^2)(T_1 - t))^2 - 2\sigma^2(T_1 - t)v}{2\sigma^2(T_1 - t)}\,.$$

Expanding the square of the expression in the numerator and simplifying we see that the numerator of the above expression becomes

$$\begin{aligned}
\text{numerator} &= v^2 - 2v\left[\log(S) + (r - \frac{1}{2}\sigma^2)(T_1 - t)\right] + \left[\log(S) + (r - \frac{1}{2}\sigma^2)(T_1 - t)\right]^2 \\
&- 2\sigma^2(T_1 - t)v \\
&= v^2 - 2v\left[\log(S) + (r - \frac{1}{2}\sigma^2)(T_1 - t) + \sigma^2(T_1 - t)\right] \\
&+ \left[\log(S) + (r - \frac{1}{2}\sigma^2)(T_1 - t)\right]^2 \\
&= v^2 - 2v\left[\log(S) + (r + \frac{1}{2}\sigma^2)(T_1 - t)\right] + \left[\log(S) + (r - \frac{1}{2}\sigma^2)(T_1 - t)\right]^2 \\
&= v^2 - 2v\left[\log(S) + (r + \frac{1}{2}\sigma^2)(T_1 - t)\right] + \left[\log(S) + (r + \frac{1}{2}\sigma^2)(T_1 - t)\right]^2 \\
&- \left[\log(S) + (r + \frac{1}{2}\sigma^2)(T_1 - t)\right]^2 + \left[\log(S) + (r - \frac{1}{2}\sigma^2)(T_1 - t)\right]^2 \\
&= \left(v - \log(S) - (r + \frac{1}{2}\sigma^2)(T_1 - t)\right)^2 - 2\sigma^2(T_1 - t)(\log(S) + r(T_1 - t))\,.
\end{aligned}$$

Where in the last step we have used the factorization $a^2 - b^2 = (a+b)(a-b)$. When we divide this expression by the $2\sigma^2(T_1 - t)$ we find that the expression for I_{11} now becomes

$$I_{11} \propto Se^{r(T_1-t)} \int_{\log(S^*)}^{\infty} N\left(\frac{v - \log(E_2) + (r + \frac{1}{2}\sigma^2)(T_2 - T_1)}{\sigma\sqrt{T_2 - T_1}}\right) e^{-\frac{(v - \log(S) - (r + \frac{1}{2}\sigma^2)(T_1 - t))^2}{2\sigma^2(T_1 - t)}} dv\,.$$

Letting $u = -v$ we find $du = -dv$ we obtain that I_{11} is proportional to

$$Se^{r(T_1-t)} \int_{-\infty}^{-\log(S^*)} N\left(\frac{-u - \log(E_2) + (r + \frac{1}{2}\sigma^2)(T_2 - T_1)}{\sigma\sqrt{T_2 - T_1}}\right) e^{-\frac{(u + \log(S) + (r + \frac{1}{2}\sigma^2)(T_1 - t))^2}{2\sigma^2(T_1 - t)}} du\,.$$

Next we perform the following transformation aimed again at introducing the Gaussian function $e^{-\frac{1}{2}x^2}$ in the integrand. We let

$$\begin{aligned}
w &= \frac{u + \log(S) + (r + \frac{1}{2}\sigma^2)(T_1 - t)}{\sigma\sqrt{T_1 - t}} \quad \text{so} \\
dw &= \frac{du}{\sigma\sqrt{T_1 - t}} \quad \text{and} \\
u &= -\log(S) - (r + \frac{1}{2}\sigma^2)(T_1 - t) + \sigma w\sqrt{T_1 - t}\,.
\end{aligned}$$

With these expressions we obtain limits of the integral I_{11} going from $-\infty$ to an upper limit of h_+ defined as

$$h_+ \equiv \frac{-\log(S^*) + \log(S) + (r + \frac{1}{2}\sigma^2)(T_1 - t)}{\sigma\sqrt{T_1 - t}},$$

and the *argument* of the cumulative normal N becomes

$$\begin{aligned} & \frac{\log(S) + (r + \frac{1}{2}\sigma^2)(T_1 - t) - \sigma w\sqrt{T_1 - t} - \log(E_2) + (r + \frac{1}{2}\sigma^2)(T_2 - T_1)}{\sigma\sqrt{T_2 - T_1}} \\ = & \frac{\log(S/E_2) + (r + \frac{1}{2}\sigma^2)(T_2 - t) - \sigma\sqrt{T_1 - t}\, w}{\sigma\sqrt{T_2 - T_1}}, \end{aligned}$$

so the expression for I_{11} is given by

$$I_{11} \propto S e^{r(T_1 - t)} \int_{-\infty}^{h_+} N\left(\frac{l_+ - \sigma\sqrt{T_1 - t}\, w}{\sigma\sqrt{T_2 - T_1}}\right) e^{-\frac{w^2}{2}} dw \sigma\sqrt{T_1 - t},$$

where we have defined l_+ to be

$$l_+ \equiv \log(S/E_2) + (r + \frac{1}{2}\sigma^2)(T_2 - t).$$

To use the integral identity derived here we write this as

$$I_{11} \propto S\sqrt{2\pi(T_1 - t)} e^{r(T_1 - t)} \int_{-\infty}^{h_+} \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}}\right) N\left(\frac{l_+ - \sigma w\sqrt{T_1 - t}}{\sigma\sqrt{T_2 - T_1}}\right) dw.$$

We have to pick k_+ and ρ so that they satisfy

$$\frac{l_+ - \sigma w\sqrt{T_1 - t}}{\sigma\sqrt{T_2 - T_1}} = \frac{k_+ - \rho w}{\sqrt{1 - \rho^2}}.$$

Solving as before for ρ we again conclude that ρ is given by Equation 115 and k_+ given by

$$k_+ = \frac{l_+}{\sigma\sqrt{T_2 - T_1}},$$

so that using these two results we have then

$$\begin{aligned} I_{11} & \propto S\sqrt{2\pi(T_1 - t)} e^{r(T_1 - t)} \int_{-\infty}^{h_+} \hat{g}(w) N\left(\frac{k_+ - \rho w}{\sqrt{1 - \rho^2}}\right) dw \\ & = S\sqrt{2\pi(T_1 - t)} e^{r(T_1 - t)} M(h_+, k_+; \rho). \end{aligned}$$

When we include the constant of proportionality in the expression for I_{11} we obtain

$$
\begin{aligned}
I_{11} &= SM(h_+, k_+; \rho) && (119)\\
h_+ &= \frac{\log(S/S^*) + (r + \frac{1}{2}\sigma^2)(T_1 - t)}{\sigma\sqrt{T_1 - t}} \quad \text{and} && (120)\\
k_+ &= \frac{\log(S/E_2) + (r + \frac{1}{2}\sigma^2)(T_2 - t)}{\sigma\sqrt{T_2 - t}}\,. && (121)
\end{aligned}
$$

With all of these we finally summarized the functional form for $V(S,t)$ from Equations 110, 116, 117, 118, 119, 120, and 121. We have found that

$$
\begin{aligned}
V(S,t) &= -E_1 e^{-r(T_1-t)} N(h_-)\\
&- E_2 e^{-r(T_2-t)} M(h_-, k_-; \rho) + SM(h_+, k_+; \rho) \quad \text{with} && (122)\\
h_\pm(S; S^*) &= \frac{\log(S/S^*) + (r \pm \frac{1}{2}\sigma^2)(T_1 - t)}{\sigma\sqrt{T_1 - t}} \quad \text{and} && (123)\\
k_\pm(S; E_2) &= \frac{\log(S/E_2) + (r \pm \frac{1}{2}\sigma^2)(T_2 - t)}{\sigma\sqrt{T_2 - t}} \quad \text{and} && (124)\\
\rho &= \sqrt{\frac{T_1 - t}{T_2 - t}}\,. && (125)
\end{aligned}
$$

The back of the book has $\log(S/E_1)$ (note this is E_1) for the logarithmic term in the expression for $k_\pm$. I believe this is a typo. Please contact me if this is not so and there is an error in the above derivation.

Exercise 4 (evaluating regular chooser options)

The analysis for a regular chooser option follows the same procedure as that for a compound option done in Exercise 3 above but with a payoff at the time T_1 given by either a call or a put, whichever is more valuable at that time. That is, our payoff $\Lambda(S)$ is given by

$$
\Lambda(S) = \max(C(S, T_1) - E_1, P(S, T_1) - E_1, 0)\,.
$$

This option can be evaluated as in Exercise 3 by using Equation 107. In this case it is worthwhile to divide the integration region into *three* regions in terms of S. The three regions are $(0, S^-)$ where the put is more valuable,

(S^-, S^+) where neither option is in the money, and (S^+, ∞) where the call is more valuable. The values of S^- and S^+ are defined such that

$$P(S^-, T_1) = E_1 \,,$$

and

$$C(S^+, T_1) = E_1 \,.$$

Rather than explicitly carry the fraction

$$\frac{e^{-r(T_1-t)}}{\sigma\sqrt{2\pi(T_1-t)}} \,,$$

found in the expression for $V(S,t)$ in Equation 107, we will temporally ignore it in our derivations below and then add it back into the final expressions. Thus we will be working at evaluating $\mathcal{I} \equiv \left(\frac{\sigma\sqrt{2\pi(T_1-t)}}{e^{-r(T_1-t)}}\right) V(S,t)$. Splitting the integration region into the two non-zero regions we find this equals

$$\begin{aligned} \mathcal{I} &= \int_0^{S^-} (P(S,T_1) - E_1) e^{-\left(\log(S'/S) - (r-\frac{1}{2}\sigma^2)(T_1-t)\right)^2 / 2\sigma^2 (T_1-t)} \frac{dS'}{S'} \quad (126) \\ &+ \int_{S^+}^{\infty} (C(S,T_1) - E_1) e^{-\left(\log(S'/S) - (r-\frac{1}{2}\sigma^2)(T_1-t)\right)^2 / 2\sigma^2 (T_1-t)} \frac{dS'}{S'} \,, \quad (127) \end{aligned}$$

as the payoff, $\Lambda(S)$, is zero for the integral with limits S^- to S^+. We have already evaluated integrals like Equation 127 in Exercise 3. Thus to complete this exercise we now need to evaluate integrals like Equation 126. Equation 126 can be split into two parts given by

$$\begin{aligned} I_1 &= \int_0^{S^-} P(S,T_1) e^{-\left(\log(S'/S) - (r-\frac{1}{2}\sigma^2)(T_1-t)\right)^2 / 2\sigma^2 (T_1-t)} \frac{dS'}{S'} \\ I_2 &= -E_1 \int_0^{S^-} e^{-\left(\log(S'/S) - (r-\frac{1}{2}\sigma^2)(T_1-t)\right)^2 / 2\sigma^2 (T_1-t)} \frac{dS'}{S'} \,. \end{aligned}$$

We will attempt to evaluate I_2 first, since it is easier to do. We let

$$v = \frac{\log(\frac{S'}{S}) - (r - \frac{1}{2}\sigma^2)(T_1 - t)}{\sigma\sqrt{T_1 - t}} \quad \text{so} \quad dv = \frac{1}{\sigma\sqrt{T_1 - t}} \frac{dS'}{S'} \,,$$

and we get

$$
\begin{aligned}
I_2 &= -E_1(\sigma\sqrt{T_1-t})\int_{-\infty}^{v(S^-)} e^{-\frac{v^2}{2}}dv \\
&= -E_1\sigma\sqrt{2\pi(T_1-t)}\,N\left(\frac{\log(\frac{S^-}{S})-(r-\frac{1}{2}\sigma^2)(T_1-t)}{\sigma\sqrt{T_1-t}}\right) \\
&= -E_1\sigma\sqrt{2\pi(T_1-t)}\,N\left(-h_-(S;S^-)\right)\,, \qquad (128)
\end{aligned}
$$

using the definition from Equation 123. We will now evaluate I_1. From the expression for $P(S,T_1)$ derived in Chapter 5 we have

$$
I_1 = \int_0^{S^-}\left(E_2e^{-r(T_2-T_1)}N(-d_2(S',T_1)) - S'N(-d_1(S',T_1))\right)e^{-\frac{\left(\log(S'/S)-(r-\frac{1}{2}\sigma^2)(T_1-t)\right)^2}{2\sigma^2(T_1-t)}}\frac{dS'}{S'}\,.
$$

To evaluate this integral we find two subproblems to integrate

$$
\begin{aligned}
I_{11} &= E_2e^{-r(T_2-T_1)}\int_0^{S^-} N(-d_2(S',T_1))e^{-\left(\log(S'/S)-(r-\frac{1}{2}\sigma^2)(T_1-t)\right)^2/2\sigma^2(T_1-t)}\frac{dS'}{S'} \\
I_{12} &= -\int_0^{S^-} N(-d_1(S',T_1))e^{-\left(\log(S'/S)-(r-\frac{1}{2}\sigma^2)(T_1-t)\right)^2/2\sigma^2(T_1-t)}dS'\,.
\end{aligned}
$$

For the integral I_{11} using the definition of d_2 we have

$$
\begin{aligned}
I_{11} &= E_2e^{-r(T_2-T_1)} \\
&\times \int_0^{S^-} N\left(\frac{-\log(S'/E_2)-(r-\frac{1}{2}\sigma^2)(T_2-T_1)}{\sigma\sqrt{T_2-T_1}}\right)e^{-\frac{\left(\log(S'/S)-(r-\frac{1}{2}\sigma^2)(T_1-t)\right)^2}{2\sigma^2(T_1-t)}}\frac{dS'}{S'}\,.
\end{aligned}
$$

To evaluate this we take

$$
\begin{aligned}
w &= \frac{\log(S'/S)-(r-\frac{1}{2}\sigma^2)(T_1-t)}{\sigma\sqrt{T_1-t}} \quad \text{so that} \\
dw &= \frac{dS'}{S'\sigma\sqrt{T_1-t}} \quad \text{and} \\
\log(S') &= \log(S)+(r-\frac{1}{2}\sigma^2)(T_1-t)+\sigma\sqrt{T_1-t}\,w\,. \qquad (129)
\end{aligned}
$$

With these expressions the integral for I_{11} can be written

$$
\begin{aligned}
I_{11} &= E_2e^{-r(T_2-T_1)}\int_{-\infty}^{w(S^-)} \\
&\times N\left(\frac{-\log(S'(w))+\log(E_2)-(r-\frac{1}{2}\sigma^2)(T_2-T_1)}{\sigma\sqrt{T_2-T_1}}\right)e^{-\frac{w^2}{2}}\sigma\sqrt{T_1-t}\,dw\,,
\end{aligned}
$$

Where we recall that S' is a function of w defined by Equation 129 we have written $\log(S'(w))$ to denote this fact. Using Equation 129 we find that the argument of the cumulative normal in terms of w becomes

$$\begin{aligned}
& \frac{-\log(S) - (r - \frac{1}{2}\sigma^2)(T_1 - t) - \sigma\sqrt{T_1 - t}\,w + \log(E_2) - (r - \frac{1}{2}\sigma^2)(T_2 - T_1)}{\sigma\sqrt{T_2 - T_1}} \\
= \; & \frac{-\log(S/E_2) - (r - \frac{1}{2}\sigma^2)(T_2 - t) - \sigma\sqrt{T_1 - t}\,w}{\sigma\sqrt{T_2 - T_1}} \,.
\end{aligned}$$

Thus we find for I_{11}

$$\begin{aligned}
I_{11} &= E_2 e^{-r(T_2 - T_1)} \sigma\sqrt{T_1 - t} \\
&\times \int_{-\infty}^{w(S^-)} N\left(\frac{-\log(S/E_2) - (r - \frac{1}{2}\sigma^2)(T_2 - t) - \sigma\sqrt{T_1 - t}\,w}{\sigma\sqrt{T_2 - T_1}} \right) e^{-\frac{w^2}{2}} dw \,.
\end{aligned}$$

To use our integral identity in Equation 111 we need to find values of k and ρ such that

$$\frac{k - \rho w}{\sqrt{1 - \rho^2}} = \frac{-\log(S/E_2) - (r - \frac{1}{2}\sigma^2)(T_2 - t) - \sigma\sqrt{T_1 - t}\,w}{\sigma\sqrt{T_2 - T_1}} \,.$$

As before, equating the coefficients of w on both sides of this expression requires that $\rho = \sqrt{\frac{T_1 - t}{T_2 - t}}$, and then k is required to be

$$k = \frac{-\log(S/E_2) - (r - \frac{1}{2}\sigma^2)(T_2 - t)}{\sigma\sqrt{T_2 - t}} = -k_-(S; E_2) \,,$$

using the definition of k_- from Equation 124. Thus we finally have

$$\begin{aligned}
I_{11} &= E_2 e^{-r(T_2 - T_1)} \sigma\sqrt{T_1 - t} \int_{-\infty}^{w(S^-)} N\left(\frac{k - \rho w}{\sqrt{1 - \rho^2}} \right) e^{-\frac{w^2}{2}} dw \\
&= E_2 e^{-r(T_2 - T_1)} \sigma\sqrt{2\pi(T_1 - t)} \int_{-\infty}^{w(S^-)} N\left(\frac{k - \rho w}{\sqrt{1 - \rho^2}} \right) \hat{g}(w) dw \\
&= E_2 e^{-r(T_2 - T_1)} \sigma\sqrt{2\pi(T_1 - t)} M(-h_-(S; S^-), -k_-(S; E_2); \rho) \,, \quad (130)
\end{aligned}$$

when we use the definition of h_- in Equation 123 for $w(S^-)$. The remaining integral to evaluate is I_{12} where we have using the definition of d_1 that

$$I_{12} = -\int_0^{S^-} N\left(\frac{-\log(S'/E_2) - (r + \frac{1}{2}\sigma^2)(T_2 - T_1)}{\sigma\sqrt{T_2 - T_1}} \right) e^{-\frac{\left(\log(S'/S) - (r - \frac{1}{2}\sigma^2)(T_1 - t)\right)^2}{2\sigma^2(T_1 - t)}} dS' \,.$$

As we have previously done to evaluate integrals of this type we let

$$
\begin{aligned}
w &= \frac{\log(S'/S) - (r - \frac{1}{2}\sigma^2)(T_1 - t)}{\sigma\sqrt{T_1 - t}} \quad \text{so} \\
dw &= \frac{dS'}{S'\sigma\sqrt{T_1 - t}} \quad \text{and} \\
\log(S') &= \log(S) + (r - \frac{1}{2}\sigma^2)(T_1 - t) + \sigma\sqrt{T_1 - t}\, w \\
S' &= S e^{(r - \frac{1}{2}\sigma^2)(T_1 - t) + \sigma\sqrt{T_1 - t} w} \quad \text{so} \\
dw &= \frac{e^{-(r - \frac{1}{2}\sigma^2)(T_1 - t) - \sigma\sqrt{T_1 - t}\, w}}{S\sigma\sqrt{T_1 - t}} dS' \,.
\end{aligned}
$$

Thus with this substitution I_{12} will now become

$$
\begin{aligned}
I_{12} &= -S\sigma\sqrt{T_1 - t}\, e^{(r - \frac{1}{2}\sigma^2)(T_1 - t)} \\
&\times \int_{-\infty}^{w(S^-)} N\left(\frac{-\log(S'(w)/E_2) - (r + \frac{1}{2}\sigma^2)(T_2 - T_1)}{\sigma\sqrt{T_2 - T_1}} \right) e^{-\frac{w^2}{2}} e^{\sigma\sqrt{T_1 - t} w} \,.
\end{aligned}
$$

Considering the argument of the exponential we can complete the square to get the expression

$$
-\frac{1}{2}\left(w - \sigma\sqrt{T_1 - t} \right)^2 + \frac{1}{2}\sigma^2 (T_1 - t) \,,
$$

so with this simplification we get for I_{12}

$$
\begin{aligned}
I_{12} &= -S\sigma\sqrt{T_1 - t} e^{(r - \frac{1}{2}\sigma^2)(T_1 - t)} e^{\frac{1}{2}\sigma^2 (T_1 - t)} \qquad (131) \\
&\times \int_{-\infty}^{w(S^-)} N\left(\frac{-\log(S'(w)/E_2) - (r + \frac{1}{2}\sigma^2)(T_2 - T_1)}{\sigma\sqrt{T_2 - T_1}} \right) e^{-\frac{1}{2}(w - \sigma\sqrt{T_1 - t})^2} dw \,.
\end{aligned}
$$

To evaluate this we will *add* the value of $\sigma\sqrt{T_1 - t}$ to the integration variable w but before we can do this we will need to evaluate the the argument of the cumulative normal in terms of w. We find

$$
\begin{aligned}
& \frac{-\log(S) - (r - \frac{1}{2}\sigma^2)(T_1 - t) - \sigma\sqrt{T_1 - t}\, w + \log(E_2) - (r + \frac{1}{2}\sigma^2)(T_2 - T_1)}{\sigma\sqrt{T_2 - T_1}} \\
= \; & \frac{-\log(S/E_2) - (r - \frac{1}{2}\sigma^2)(T_1 - t) - (r + \frac{1}{2}\sigma^2)(T_2 - T_1) - \sigma\sqrt{T_1 - t}\, w}{\sigma\sqrt{T_2 - T_1}} \,,
\end{aligned}
$$

so when we add the expression $\sigma\sqrt{T_1 - t}$ to the value of w this argument becomes

$$\frac{-\log(S/E_2) - (r + \frac{1}{2}\sigma^2)(T_2 - t) - \sigma\sqrt{T_1 - t}\, w}{\sigma\sqrt{T_2 - T_1}},$$

and the integral for I_{12} becomes

$$\begin{aligned} I_{12} &= -S\sigma\sqrt{T_1 - t}e^{r(T_1-t)} \\ &\times \int_{-\infty}^{w(S^-)-\sigma\sqrt{T_1-t}} N\left(\frac{-\log(S/E_2) - (r + \frac{1}{2}\sigma^2)(T_2 - t) - \sigma\sqrt{T_1 - t}\, w}{\sigma\sqrt{T_2 - T_1}}\right) e^{-\frac{1}{2}w^2} dw\,. \end{aligned}$$

When we evaluate the upper limit of the integral we see that this equals

$$w(S^-) - \sigma\sqrt{T_1 - t} = \frac{\log(S^-/S) - (r + \frac{1}{2}\sigma^2)(T_1 - t)}{\sigma\sqrt{T_1 - t}} = -h_+(S; S^-)\,.$$

To use our integral identity in Equation 111 we need to find values of k and ρ such that

$$\frac{k - \rho w}{\sqrt{1 - \rho^2}} = \frac{-\log(S/E_2) - (r + \frac{1}{2}\sigma^2)(T_2 - t) - \sigma\sqrt{T_1 - t}\, w}{\sigma\sqrt{T_2 - T_1}}\,.$$

As before, equating the coefficients of w on both sides of this expression requires that $\rho = \sqrt{\frac{T_1 - t}{T_2 - t}}$, and then k is required to be

$$k = \frac{-\log(S/E_2) - (r + \frac{1}{2}\sigma^2)(T_2 - t)}{\sigma\sqrt{T_2 - t}} = -k_+(S; E_2)\,,$$

when we use the definition in Equation 124. Thus we finally have

$$\begin{aligned} I_{12} &= -S\sigma\sqrt{T_1 - t}e^{r(T_1-t)} \int_{-\infty}^{w(S^-)-\sigma\sqrt{T_1-t}} N\left(\frac{k - \rho w}{\sqrt{1 - \rho^2}}\right) e^{-\frac{w^2}{2}} dw \\ &= -S\sigma\sqrt{2\pi(T_1 - t)}e^{r(T_1-t)} \int_{-\infty}^{w(S^-)-\sigma\sqrt{T_1-t}} N\left(\frac{k - \rho w}{\sqrt{1 - \rho^2}}\right) \hat{g}(w) dw \\ &= -S\sigma\sqrt{2\pi(T_1 - t)}e^{r(T_1-t)} M\left(-h_+(S; S^-), -k_+(S; E_2); \rho\right)\,. \end{aligned}$$

With all of these parts we can multiply by the expression $\frac{e^{-r(T_1-t)}}{\sigma\sqrt{2\pi(T_1-t)}}$ to finally get $V(S,t)$ for a chooser option and find

$$\begin{aligned}
V(S,t) &= E_2 e^{-r(T_2-t)} M(-h_-(S;S^-), -k_-(S;E_2);\rho) \\
&- E_2 e^{-r(T_2-t)} M(h_-(S;S^+), k_-(S;S^+);\rho) \\
&- SM(-h_+(S;S^-), -k_+(S;E_2);\rho) \\
&+ SM(h_+(S;S^+), k_+(S;S^+);\rho) \\
&- E_1 e^{-r(T_1-t)} N(-h_-(S;S^-)) \\
&- E_1 e^{-r(T_1-t)} N(h_-(S;S^+)) \,. \qquad (132)
\end{aligned}$$

Here we have used the definitions of $h_\pm$ and $k_\pm$ from Equations 123 and 124.

Exercise 5 (chooser options when $T_1 = T_2$)

If $T_1 = T_2$ then a chooser option gives the holder the right to buy a call or a put at the price E_1, which then immediately expires. Thus the payoff for this instrument is

$$\begin{aligned}
\Lambda(S) &= \max(C(S,T_1) - E_1, P(S,T_1) - E_1, 0) \\
&= \max(\max(S - E_2, 0) - E_1, \max(E_2 - S, 0) - E_1, 0) \,.
\end{aligned}$$

To observe what this payoff looks like we can specify values for E_1 and E_2 and plot this as a function of S. When we take $E_1 =$ and $E_2 =$ we get the plot shown in Figure 12. This plot looks like a *strangle*.

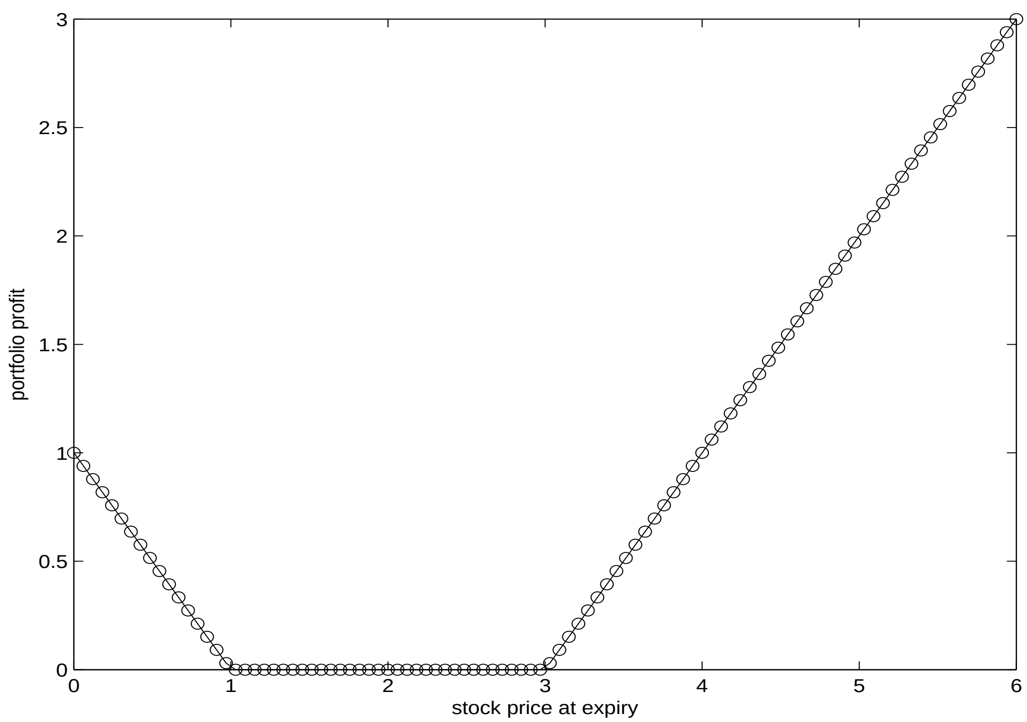

Figure 12: The payoff function for a chooser option when $T_1 = T_2$, for some select values of E_1 and E_2. This is effectively a *strangle* option.

Chapter 12 (Barrier Options)

Notes on the Text

Knock-out Options

From the fact that our boundary conditions for large stock prices does not change from that of a vanilla call when considering a knock-out option $V(s,t) \sim S$ as $S \to \infty$ the change of variables suggested in the text requires

$$\begin{aligned} Ee^{\alpha x+\beta\tau}u(x,\tau) &\sim Ee^{x} \quad \text{as} \quad x \to +\infty \\ u(x,\tau) &\sim e^{(1-\alpha)x-\beta\tau} \quad \text{as} \quad x \to +\infty\,, \end{aligned} \tag{133}$$

which is the books equation 12.3.

To further understand and motivate the expression for the initial condition on $u(x,\tau)$ for a knock-out option derived under the method of images in the text it is instructive to reason as follows. We want to solve $u_\tau = u_{xx}$ with $u(x,0) = u_0(x)$ for $x \geq x_0$ with the zero boundary conditions on x_0 of $u(x_0,\tau) = 0$. To begin we translate the boundary condition the location x_0 to the origin. We can do this by defining a function v as $v(x,\tau) = u(x+x_0,\tau)$. Then with this definition we see that $v(x,\tau)$ satisfies $v_\tau = v_{xx}$ on $0 < x < \infty$ with a boundary condition at $x = 0$ as

$$v(0,\tau) = u(x_0,\tau) = 0\,. \tag{134}$$

With this definition of v we see that v's initial condition becomes

$$v(x,0) = u(x+x_0,0) = u_0(x+x_0) \quad \text{for} \quad 0 < x < +\infty\,. \tag{135}$$

Thus we have reduced our problem to that of solving for the function $v(x,\tau)$ over $0 < x < \infty$.

To solve the equation for $v(x,\tau)$ on $0 < x < +\infty$ we will solve a related problem for another function $w(x,\tau)$ specified on the *entire* real line $-\infty < x < +\infty$ and obtain the desired solution to $v(x,\tau)$ by simply restricting our solution for w to the region $0 < x < \infty$. To derive the problem we should pose for $w(x,\tau)$ observe that $v(x,\tau)$ could be represented as a "symmetric difference" of another function like

$$v(x,\tau) = w(x,\tau) - w(-x,\tau)\,, \tag{136}$$

and still satisfy its required boundary condition at 0 since

$$v(0,\tau) = w(0,\tau) - w(0,\tau) = 0\,, \tag{137}$$

for all τ. The known initial condition on $v(x,\tau)$ requires that

$$v(x,0) = w(x,0) - w(-x,0) \tag{138}$$

equals $u_0(x+x_0)$. Now there are many ways we can specify $w(x,0)$ such that the above expression is true. The simplest may be to just *define* $w(x,0)$ as

$$w(x,0) = \begin{cases} u_0(x+x_0) & x > 0 \\ 0 & x < 0 \end{cases} . \tag{139}$$

With this definition for $w(x,0)$ we see that if $x > 0$ then Equation 138 becomes

$$v(x,0) = u_0(x+x_0) - 0 = u_0(x+x_0)\,,$$

as required by Equation 135.

In summary then, the procedural steps we take to solve for $u(x,\tau)$ are to first solve for $w(x,\tau)$ where $w_\tau = w_{xx}$ on $-\infty < x < +\infty$ with an initial condition given by Equation 139. Once $w(x,\tau)$ is known, we calculate $v(x,\tau)$ using $w(x,\tau) - w(-x,\tau)$. The function of interest $u(x,\tau)$ is obtained by considering $v(x,\tau)$ on $x > 0$.

We can now attempt to undo these transformation and derive an explicit initial condition for $u(x,0)$ for $-\infty < x < \infty$. From the initial conditions on $w(x,\tau)$ given by Equation 139 we see that the initial conditions for $v(x,\tau)$ are given by

$$v(x,0) = w(x,0) - w(-x,0) = \begin{cases} u_0(x+x_0) & x > 0 \\ -u_0(-x+x_0) & x < 0 \end{cases} . \tag{140}$$

Next, we shift our boundary condition from the origin in v back to x_0 in u with the linear transformation $u(x,\tau) = v(x-x_0,\tau)$. Doing this the initial conditions for $u(x,\tau)$ becomes

$$\begin{aligned} u(x,0) &= \begin{cases} u_0(x) & x - x_0 > 0 \\ -u_0(-(x+x_0)+x_0) & x - x_0 < 0 \end{cases} \\ &= \begin{cases} u_0(x) & x > x_0 \\ -u_0(-x+2x_0) & x < x_0 \end{cases} . \end{aligned} \tag{141}$$

Thus using the specific functional form for the payoff of a transformed call $u_0(x) = \max(e^{\frac{1}{2}(k+1)x} - e^{\frac{1}{2}(k-1)x}, 0)$ our initial condition for $u(x,\tau)$ over the entire real line is

$$
\begin{aligned}
u(x,0) &= \begin{cases} \max(e^{\frac{1}{2}(k+1)x} - e^{\frac{1}{2}(k-1)x}, 0) & x > x_0 \\ -\max(e^{\frac{1}{2}(k+1)(-x+2x_0)} - e^{\frac{1}{2}(k-1)(-x+2x_0)}, 0) & x < x_0 \end{cases} \\
&= \begin{cases} \max(e^{\frac{1}{2}(k+1)x} - e^{\frac{1}{2}(k-1)x}, 0) & x > x_0 \\ -\max(e^{(k+1)(x_0-\frac{1}{2}x)} - e^{(k-1)(x_0-\frac{1}{2}x_0)}, 0) & x < x_0 \end{cases}, \qquad (142)
\end{aligned}
$$

which is the result shown in the book but not provided with any explanation of how it was arrived at.

Motivated by how we can split the initial condition on $u(x,\tau)$ into two parts as in Equation 142, we might hypothesis the solution to the knock-out barrier option problem as

$$V(S,t) = Ee^{\alpha x+\beta\tau}(u_1(x,\tau) + u_2(x,\tau)),$$

where $u_2(x,\tau)$ is the solution to the diffusion equation $u_\tau = u_{xx}$ with the antisymmetric initial condition given by

$$u_2(x,0) = -u_0(2x_0 - x). \qquad (143)$$

Because $u_2(x,\tau)$ must satisfy the diffusion equation with the initial conditions given in Equation 143 *and* the fact that the diffusion equation is invariant to negation of the dependent variable u, negation of the spatial variable x, and translation of the variable x, the solution to $u_2(x,\tau)$ is given by performing the required translations and scaling of $u_1(x,\tau)$. We have

$$
\begin{aligned}
u_2(x,\tau) &= -u_1(2x_0 - x, \tau) \\
&= -e^{-\alpha(2x_0-x)-\beta\tau}C(\eta,t)/E,
\end{aligned}
$$

where we have used the fact that $u_1(x,\tau) = e^{-\alpha x-\beta\tau}C(S,t)/E$. In the above solution for u_2 we define η as the transformation of S when x undergoes the translation $x \to 2x_0 - x$. Since $x = \ln(\frac{S}{E})$ and $x_0 = \ln(\frac{X}{E})$ we see that the expression $2x_0 - x$ is given by

$$2x_0 - x = \ln(\frac{X^2}{E^2}) - \ln(\frac{S}{E}) = \ln(\frac{X^2}{ES}),$$

and thus

$$\eta = Ee^{2x_0-x} = E\left(\frac{X^2}{ES}\right) = \frac{X^2}{S}.$$

Using this, we have that u_2 becomes

$$\begin{aligned} u_2(x,\tau) &= -e^{-\alpha(2x_0-x)-\beta\tau}C\left(\frac{X^2}{S},t\right)/E \\ &= -e^{-\alpha(2\ln(X/E)-\ln(S/E))-\beta\tau}C\left(\frac{X^2}{S},t\right)/E\,, \end{aligned}$$

the expression claimed in the book. Thus our entire option value $V(S,t)$ for a knock-out option then becomes

$$\begin{aligned} V(S,t) &= Ee^{\alpha x+\beta\tau}u_1(x,\tau) + Ee^{\alpha x+\beta\tau}\left(-e^{-\alpha(2x_0-x)-\beta\tau}C\left(\frac{X^2}{S},t\right)/E\right) \\ &= Ee^{\alpha x+\beta\tau}u_1(x,\tau) - e^{2\alpha(x-x_0)}C\left(\frac{X^2}{S},t\right)\,. \end{aligned}$$

Since $x - x_0 = \ln(S/X)$ the above becomes

$$\begin{aligned} V(S,t) &= C(S,t) - \left(\frac{S}{X}\right)^{2\alpha}C\left(\frac{X^2}{S},t\right) \\ &= C(S,t) - \left(\frac{S}{X}\right)^{-(k-1)}C\left(\frac{X^2}{S},t\right)\,, \end{aligned} \tag{144}$$

which is the result in the text.

Note that using the above expression for $V(S,t)$ we have $V(X,t) = C(X,t) - C(X,t) = 0$ as required. To show that at expiration we have the required payoff we evaluate $V(S,T)$. We find

$$\begin{aligned} V(S,T) &= C(S,T) - \left(\frac{S}{X}\right)^{-(k-1)}C\left(\frac{X^2}{S},T\right) \\ &= \max(S-E,0) - \left(\frac{S}{X}\right)^{-(k-1)}\max\left(\frac{X^2}{S}-E,0\right)\,. \end{aligned} \tag{145}$$

We know that $S > X$ or otherwise the option is worthless. This inequality on S and X implies the following sequence of inequalities

$$\frac{X}{S} < 1 \Rightarrow \frac{X^2}{S} < X \Rightarrow \frac{X^2}{S} - E < X - E\,. \tag{146}$$

Since the strike must always be greater than the knock-out boundary we know that $E > X$, equivalently that $X - E < 0$ which when we use the last inequality in Equation 146 we conclude that $\frac{X^2}{S} - E < 0$, so that and the second max in Equation 145 has the value zero. Thus our option has a terminal condition given by $V(S,T) = \max(S-E,0)$ as expected.

Exercise Solutions

Exercise 1 (the boundary condition for an out-option)

The boundary condition would becomes $V(X,t) = Z$, that is a constant value of Z (the rebate) at the strike value of X.

Exercise 2 (the boundary condition for an in-option)

The final condition for the knock-in option with a rebate Z when the barrier is never crossed would be given by $V(S,T) = Z$.

Exercise 3 (explicit formulas for European barrier options)

For each of the requested options mentioned in this problem much of the work in deriving analytic expressions for these options was done in the book. The exception to this statement occurs when one considers the addition of rebates to each option. Fortunately, the modifications to the option pricing problem needed to incorporate rebates are simple to understand and implement in the partial differential equation framework espoused in detail in this book.

To add rebates to each of the requested options we proceed as follows. Using the solution to the option pricing problem *without* rebates as a first solution, we derive a *second* solution to the Black-Scholes equation with boundary or initial conditions that represent the rebate. Any boundaries or initial conditions for this second solution that are are not specified by the rebate are taken to be *zero*. Once we have solved this second equation because the Black-Scholes equation is linear the solution to the total option pricing problem is given by the sum of the first and second solution. This procedure will be demonstrated in the option pricing problems below.

A European style down-and-out call: We have from the discussion in the book that without rebates the solution to the down-and-out call $V_1(S,t)$ is given by

$$V_1(S,t) = C(S,t) - \left(\frac{S}{X}\right)^{-(k+1)} C\left(\frac{X^2}{S}, t\right) . \tag{147}$$

This solution has the final and boundary conditions corresponding to no rebate given by

$$V_1(S,T) = \max(S - E, 0) , \quad V_1(X,t) = 0 , \quad V_1(S,t) \sim S \quad S \to \infty .$$

To incorporate a rebate if the stock crosses the barrier X we will solve for a second function $V_2(S,t)$ that satisfies the Black-Scholes equation and has final and boundary conditions given by

$$V_2(S,T) = 0\,, \quad V_2(X,t) = Z\,, \quad V_2(S,t) \sim 0 \quad S \to \infty\,. \tag{148}$$

Then the function $V(S,t) \equiv V_1(S,t) + V_2(S,t)$ will be the desired solution. We now proceed to solve for $V_2(S,t)$. We begin by transforming this problem to the diffusion equation and solving this diffusion equation. Using the transformation implied by Equation 133. The three final and boundary conditions above transform to

$$\begin{aligned} u_2(x,0) &= 0 \\ u_2(x_0,\tau) &= e^{z_0-\alpha x_0}e^{-\beta\tau} \quad \text{for} \quad \tau > 0 \\ u_2(x,\tau) &\sim 0 \quad x \to \infty\,, \end{aligned}$$

where we have defined $x_0 \equiv \ln(\frac{X}{E})$ and $z_0 \equiv \ln(\frac{Z}{E})$. Thus we see that the rebate translates into the problem of solving the diffusion equation on a semi-infinite domain $x_0 < x < +\infty$, with a time-dependent Dirichlet boundary condition and zero initial conditions.

We can translate the above problem to one where the initial condition is at $x = 0$ using the standard technique of defining another function $\tilde{u}_2(x,\tau)$ related to u_2 as $\tilde{u}_2(x,\tau) = u_2(x + x_0, \tau)$. Then the problem defined in terms of the domain of $\tilde{u}_2$ is over the domain $0 < x < +\infty$. Once we have solved for the function $\tilde{u}_2(x,\tau)$ we can get $u_2(x,\tau)$ from $u_2(x,\tau) = \tilde{u}_2(x + x_0, \tau)$.

This problem, as posed for $\tilde{u}_2(x,\tau)$, could be solved numerically or analytically via methods presented in [2, 5]. Unfortunately, the later approach results in an integral expression involving the boundary value $e^{z_0-\alpha x_0}e^{-\beta\tau}$ and the *time* integral of the Gaussian heat kernel which I was unable to evaluate analytically.

Note: If anyone is able to evaluate this integral analytically or knows how to represent the down-and-out call barrier option solution with rebates analytically please email me. The reference [7] seemed to have an explicit formulation of the rebate problem but the copy of that article I had access to was of a very poor quality and I was unable to understand their expression.
A European style down-and-in call: From the discussion in the text a down-and-**in** call plus a down-and-**out** call equals a normal vanilla call. Thus if V is the value of the down-and-in option we wish to value then using

Equation 147 we can easily determine it as

$$
\begin{aligned}
V(S,t) &= C(S,t) - \left(C(S,t) - \left(\frac{S}{X}\right)^{-(k-1)} C\left(\frac{X^2}{S}, t\right)\right) \\
&= \left(\frac{S}{X}\right)^{-(k-1)} C\left(\frac{X^2}{S}, t\right) . \qquad (149)
\end{aligned}
$$

When rebates are included in the problem formulation, following the discussion presented for the down-and-out call with rebates we would need to compute a second solution $V_2(S,t)$ that has the following financial final and boundary conditions

$$
V_2(S,T) = Z\,, \quad V_2(X,t) = 0\,, \quad V(S,t) \sim 0 \quad S \to \infty\,.
$$

Then the function $V(S,t) \equiv V_1(S,t) + V_2(S,t)$ will be the desired solution, where $V_1(S,t)$ is the solution presented for a down-and-in call in Equation 149 earlier. To solve for $V_2(S,t)$ we can transform the Black-Scholes equation into the diffusion equation for a function $u_2(x,\tau)$ that will have initial and boundary conditions given by

$$
\begin{aligned}
u_2(x,0) &= e^{z_0 - \alpha x} \\
u_2(x_0,\tau) &= 0 \\
u_2(x,\tau) &\sim 0 \quad x \to \infty\,,
\end{aligned}
$$

where z_0 and x_0 were defined earlier and the domain of x is $[x_0,\infty)$. We next move the boundary condition at x_0 to the point $x = 0$ by defining another function $\tilde{u}_2(x,\tau)$ as $\tilde{u}_2(x,\tau) = u_2(x + x_0, \tau)$ to get a transformed initial condition on $\tilde{u}_2(x,\tau)$ of

$$
\tilde{u}_2(x,0) = e^{z_0 - \alpha(x + x_0)}\,,
$$

and now the domain of x for the function $\tilde{u}_2(x,\tau)$ is $[0,\infty)$. In summary then after all of these transformation to find the function $\tilde{u}_2(x,\tau)$ we are looking for a solution to the heat equation $\tilde{u}_{2\tau} = \tilde{u}_{2xx}$ on a semi-infinite domain $[0,\infty)$ with boundary and initial conditions as specified above. The analytic solution to this problem [5] is given by

$$
\tilde{u}_2(x,\tau) = \int_0^\infty \tilde{u}_2(\xi,0) G_1(x,\xi,t)\, d\xi\,,
$$

where $G_1(x, \xi, t)$ is the *Green's function of the first kind* for the semi-infinite domain and has the following form

$$G_1(x, \xi, t - \tau) = \frac{1}{\sqrt{4\pi(t - \tau)}} \left(e^{-\frac{(x-\xi)^2}{4(t-\tau)}} - e^{-\frac{(x+\xi)^2}{4(t-\tau)}} \right) . \qquad (150)$$

The integrals needed to evaluate $\tilde{u}_2(x, \tau)$ can then be converted into ones involving the cumulative normal function $N(\cdot)$, to finish the evaluation of the solution $\tilde{u}_2(x, \tau)$.

A European style up-and-out call: If we want to consider up-and-out calls, then we assume that the value of this option is zero if our asset price ever reaches the value boundary value X, as a boundary condition on $V(S, t)$ this requires that $V(X, t) = 0$. The financial final condition transforms into an initial conditions on $u(x, \tau)$ as before:

$$u(x, 0) = \max(e^{\frac{1}{2}(k+1)x} - e^{\frac{1}{2}(k-1)x}, 0) ,$$

on the domain of $-\infty < x \leq x_0$. Finally, our option is worthless if $S = 0$ so we have that $V(0, t) = 0$ or $u(-\infty, t) = 0$. To solve this problem we recognized it as a semi-infinite heat equation on the domain $(-\infty, x_0]$ and given initial conditions. To solve this problem using what we have discussed above we need to

- Shift the boundary condition at $x = x_0$ to the origin $x = 0$.
- Flip the domain to make it $[0, \infty)$.

After these transformations we can solve this problem in the same way as is done in the solution for the European down-and-out call. A rebate would change the boundary condition from $V(X, t) = 0$ to $V(X, t) = Z$ and would require a second solution to the heat/diffusion equation to be added by superposition.

A European style up-and-in call: This option can be valuated by using barrier option parity in that the value of an up-and-in call plus the value of an up-and-out call must equal a vanilla call.

A European style barrier puts: Expressions for European barrier option puts could be derived in a similar manner to their call counter parts.

Exercise 5 (pricing the double knockout call or put)

For a double knock out call the financial description requires boundary conditions on V given by $V(X_1, t) = V(X_2, t) = 0$. Note that in the transformed domain defined by

$$V(S,t) = Ee^{\alpha x + \beta \tau} u(x, \tau) \quad \text{where} \quad u_\tau = u_{xx}\,, \tag{151}$$

the above boundary conditions require that $u(x, \tau)$ satisfy

$$u(x_1, \tau) = 0 \quad \text{and} \quad u(x_2, \tau) = 0$$

for all τ. Here x_i are the transformed barriers $\log(\frac{X_i}{E})$. Thus we see that $u(x, \tau)$ now solves a *boundary value problem*. To satisfy this we should expand our unknown u in a sinusoidal Fourier series. To do this we first translate this problem to a new problem where our left endpoint at the origin rather than x_1. We can do this by defining a new function v as

$$v(x, \tau) = u(x + x_1, \tau)\,.$$

Once this is done the above boundary conditions on $u(x, \tau)$ translate into boundary conditions on $v(x, \tau)$ such that

$$v(0, \tau) = 0 \quad \text{and} \quad v(x_2 - x_1, \tau) = 0\,.$$

Defining $L \equiv x_2 - x_1$ we see that $v(x, \tau)$ will satisfy the above boundary conditions if it has a Fourier sinusoidal series decomposition of the following form

$$v(x, \tau) = \sum_{n=1}^{\infty} A_n(\tau) \sin\left(\frac{n\pi}{L} x\right)\,, \tag{152}$$

with $A_n(\tau)$ unknown functions to be determined from the initial conditions on $v(x, 0)$. From the above equation we have that

$$v(0, \tau) = 0 \quad \text{and} \quad v(x_2 - x_1, \tau) = 0\,,$$

as required. Translating back to the unknown $u(x, \tau)$ gives

$$u(x, \tau) = v(x - x_1, \tau) = \sum_{n=1}^{\infty} A_n(\tau) \sin\left(\frac{n\pi}{x_2 - x_1}(x - x_1)\right)\,. \tag{153}$$

For this functional form for $u(x,\tau)$ to satisfy the partial differential equation $u_\tau = u_{xx}$ requires that each component term $A_n(\tau)\sin\left(\frac{n\pi}{x_2-x_1}(x-x_1)\right)$ satisfy the following ordinary differential equation

$$\frac{A_n(\tau)}{d\tau} = -\left(\frac{n\pi}{x_2-x_1}\right)^2 A_n(\tau) \quad \text{for} \quad n \geq 1\,.$$

Solving this we see that the function $A_n(\tau)$ then solves

$$A_n(\tau) = A_n(0)e^{-\left(\frac{n\pi}{x_2-x_1}\right)^2\tau}\,. \tag{154}$$

At this point we have to specify the values of $A_n(0)$. Their value will depend on the type of option we are pricing. If we assume that we are pricing a *call* option we then initial conditions on $u(x,0)$ that require that $A_n(0)$ satisfy

$$u(x,0) = \max(e^{\frac{1}{2}(k+1)x} - e^{\frac{1}{2}(k-1)x}, 0) = \sum_{n=1}^{\infty} A_n(0)\sin\left(\frac{n\pi}{x_2-x_1}(x-x_1)\right)\,.$$

With a similar expression if we are pricing a double knockout put. For a general initial condition of the form $u_0(x)$, the coefficients of $A_n(0)$ are given by the standard Fourier coefficient integrals

$$A_n(0) = \frac{2}{x_2-x_1}\int_{x_1}^{x_2} u_0(x)\sin\left(\frac{n\pi}{x_2-x_1}(x-x_1)\right)dx\,. \tag{155}$$

Given the pieces represented by Equations 153, 154, and 155 we have the full solution for $u(x,\tau)$ from which we get the full solution for $V(S,t)$ using Equation 151. Note that this is the same result obtained in the book [8].

Exercise 8 (the equation that $U(S,t) = S^\alpha V(a/S,t)$ satisfies)

Note: There is a typo in this problems formulation. The correct differential equation that $U(S,t)$ should satisfy should have a coefficient of the non-differentiated U term given by

$$(1-\alpha)(r + \frac{1}{2}\alpha\sigma^2)\,,$$

rather than the term $(1-\alpha)(r+\frac{1}{2}\sigma^2)$ which is what is printed in the book and has no α multiplying $\frac{1}{2}\sigma^2$. In this problem we verify that the differential operator with the correction above above does vanish when applied to $U(S,t)$. At the end of this problem we describe how this type was discovered.

To begin this problem we let $U(S,t) = S^\alpha V(a/S,t)$, then $U(S,t)$ has t and S derivatives given by

$$\begin{aligned}
U_t &= S^\alpha V_t(a/S,t) \\
U_S &= \alpha S^{\alpha-1} V(a/S,t) + S^\alpha V_\xi(\xi,t)\left(-\frac{a}{S^2}\right) \\
&= \alpha S^{\alpha-1} V(\xi,t) - a S^{\alpha-2} V_\xi(\xi,t)\,. \qquad (156)
\end{aligned}$$

Where we have defined $\xi = \frac{a}{S}$. Using the fact that $(uv)'' = u''v + 2u'v' + uv''$, we can compute the needed second derivative U_{SS} as follows

$$U_{SS} = \alpha(\alpha-1)S^{\alpha-2}V(\xi,t) + 2\alpha S^{\alpha-1}V_\xi(\xi,t)\left(-\frac{a}{S^2}\right) + S^\alpha V_{SS}\,.$$

Now to finish the evaluation of the last term we need to compute the second derivative of V with respect to S. Using the fact that

$$V_S = \frac{\partial V}{\partial \xi}\frac{\partial \xi}{\partial S} = \frac{\partial V(\xi,t)}{\partial \xi}\left(-\frac{a}{S^2}\right)\,,$$

we find V_{SS} is given by

$$\begin{aligned}
V_{SS} &= \frac{\partial V_S}{\partial S} = \frac{\partial}{\partial S}\left(-\frac{a}{S^2}\right)\frac{\partial V}{\partial \xi} + \left(-\frac{a}{S^2}\right)\frac{\partial^2 V}{\partial \xi^2}\frac{\partial \xi}{\partial S} \\
&= \frac{2a}{S^3}V_\xi - \frac{a}{S^2}V_{\xi\xi}\left(-\frac{a}{S^2}\right) \\
&= \frac{2a}{S^3}V_\xi + \frac{a^2}{S^4}V_{\xi\xi}\,.
\end{aligned}$$

With this expression we can put it into U_{SS} to find that U_{SS} then becomes

$$U_{SS} = \alpha(\alpha-1)S^{\alpha-2}V - 2\alpha a S^{\alpha-3}V_\xi + 2aS^{\alpha-3}V_\xi + a^2 S^{\alpha-4}V_{\xi\xi}\,.$$

Eventually we will put $U(S,t)$ into the provided differential equation $\mathcal{L}U$ of

$$\mathcal{L}U \equiv \frac{\partial U}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 U}{\partial S^2} - (r + (\alpha-1)\sigma^2)S\frac{\partial U}{\partial S} - (1-\alpha)(r + \frac{1}{2}\alpha\sigma^2)U\,, \qquad (157)$$

and to facilitate this we first evaluate the expressions $S^2 U_{SS}$ and SU_S. Using the above expression for U_{SS} we find

$$S^2 U_{SS} = \alpha(\alpha-1)S^\alpha V - 2a(\alpha-1)S^{\alpha-1}V_\xi + a^2 S^{\alpha-2}V_{\xi\xi}\,,$$

and using Equation 156 we find the product SU_S given by

$$SU_S = \alpha S^\alpha V - a S^{\alpha-1} V_\xi \,.$$

Putting both of these expressions in the proposed linear partial differential equation $\mathcal{L}U$ Equation 157 then becomes

$$\begin{aligned}
\mathcal{L}U &= S^\alpha V_t + \frac{1}{2}\sigma^2 \left(\alpha(\alpha-1)S^\alpha V - 2a(\alpha-1)S^{\alpha-1}V_\xi + a^2 S^{\alpha-2} V_{\xi\xi}\right) \\
&- (r + (\alpha-1)\sigma^2)(\alpha S^\alpha V - a S^{\alpha-1} V_\xi) - (1-\alpha)(r + \frac{1}{2}\alpha\sigma^2) S^\alpha V \\
&= S^\alpha V_t - r S^\alpha V + a r S^{\alpha-1} V_\xi + \frac{1}{2}\sigma^2 a^2 S^{\alpha-2} V_{\xi\xi} \,,
\end{aligned}$$

when we group each term based on the derivatives of V. Since $\xi = a/S$ we have that the above becomes

$$\mathcal{L}U = S^\alpha V_t - r S^\alpha V + \xi S^\alpha V_\xi + \frac{1}{2}\xi^2 \sigma^2 S^\alpha V_{\xi\xi} \,.$$

Since $V(\xi, t)$ satisfies the Black-Scholes equation we know that

$$V_t + \frac{1}{2}\sigma^2 \xi^2 V_{\xi\xi} + r\xi V_\xi - rV = 0 \,, \qquad (158)$$

showing that the derived expression for $\mathcal{L}U$ vanishes as we were to show. We can *show* that the above expression for $\mathcal{L}U$ is correct by working in a different direction. We begin with the fact that V is a solution to the Black-Scholes equation and from $U(S,t) = S^\alpha V(a/S, t)$ solve for V explicitly. Letting $\xi = a/S$, we have

$$V(\xi, t) = S^{-\alpha} U(S, t) = \frac{\xi^\alpha}{a^\alpha} U(a/\xi, t) \,.$$

This expression is then put into Equation 158 to derive an equation for U. This equation matches Equation 157.

Now if $\alpha = 1 - \frac{r}{\frac{1}{2}\sigma^2}$ then $(\alpha-1)\sigma^2 = -2r$ and $r + (\alpha-1)\sigma^2 = -r$ so that the equation for U in Equation 157 becomes

$$U_t + \frac{1}{2}\sigma^2 S^2 U_{SS} + r S U_S - rU = 0 \,,$$

or the Black-Scholes equation again!

Chapter 13 (A Unifying Framework for Path-dependent Options)

Additional Notes on the Text

In this section of the notes we derive the pricing partial differential equation for an option who's payoff depends on the variable $I \equiv \int_0^T f(S(\tau), \tau) d\tau$. Ito's lemma applied to a function V when V is given by $V = V(S, I, t)$ gives

$$dV = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial I} dI + \frac{\partial V}{\partial S} dS + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} dS^2 .$$

Here dS is given by the expression for geometric Brownian motion we have seen before of

$$dS = \mu S dt + \sigma S dZ ,$$

from which we recall the first order (in dt) approximation of dS^2 of

$$dS^2 \approx \sigma^2 S^2 dZ^2 = \sigma^2 S^2 dt .$$

Here the differential of I as argued in the book is given by $dI = f(S, t) dt$. Using both of these we find dV becomes

$$\begin{aligned} dV &= \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial I} f(S,t) dt + \frac{\partial V}{\partial S} (\mu S dt + \sigma S dZ) + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \sigma^2 S^2 dt \\ &= \sigma S \frac{\partial V}{\partial S} dZ + \left(\frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial S^2} + \mu S \frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} + f(S,t) \frac{\partial V}{\partial I} \right) dt , \end{aligned}$$

which is the same as equation 13.4 in the book and provides an expression for dV when the option depends on an integral term I. Note that we have explicitly listed the stochastic term of $\sigma S \frac{\partial V}{\partial S} dZ$ first.

To obtain the partial differential equation that V must satisfy as before we construct a risk free portfolio Π that is long a single option V and short some amount, say Δ, of the underlying stochastic stock at price S in the usual way as

$$\Pi = V - \Delta S ,$$

here we have explicitly assumed that if we are long the option V then we will need to be short the stock so that the stochastic component of Π will vanish. The differential of the expression Π (over a very short time dt, under which we can assume that the value of Δ is constant) is $d\Pi = dV - \Delta dS$. in $d\Pi$ we can evaluate the expression for dV using Ito's lemma calculated above to derive

$$
\begin{aligned}
d\Pi &= \sigma S \frac{\partial V}{\partial S} dX + \left(\frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial S^2} + \mu S \frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} + f(S,t)\frac{\partial V}{\partial I} \right) dt \\
&- \Delta(\mu S dt + \sigma S dX) \\
&= \left(\sigma S \frac{\partial V}{\partial S} - \Delta \sigma S \right) dX \\
&+ \left(\frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial S^2} + \mu S \frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} + f(S,t)\frac{\partial V}{\partial I} - \Delta \mu S \right) dt \,.
\end{aligned}
$$

From this expression which we see that if we take $\Delta = \frac{\partial V}{\partial S}$ the stochastic term above will vanish and our portfolio evolves *deterministically*. After doing this the term $\mu S \frac{\partial V}{\partial S}$ in the deterministic portion vanishes and we find we are left with a non-stochastic differential for Π given by

$$
d\Pi = \left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} + f(S,t)\frac{\partial V}{\partial I} \right) dt \,.
$$

By the no arbitrage relationship this deterministic portfolio must grow at $r\Pi dt$ or the same return one would obtain on a bank deposit. Thus we see that the partial differential equation for $V(S,I,t)$ satisfies

$$
\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} + f(S,t)\frac{\partial V}{\partial I} = rV - rS\frac{\partial V}{\partial S} \,,
$$

or putting everything on the left-hand-side we obtain

$$
\frac{\partial V}{\partial t} + f(S,t)\frac{\partial V}{\partial I} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \,, \tag{159}
$$

the same as equation 13.5 in the book.

Exercise Solutions

Exercise 1 (jump conditions using delta functions)

We can translate the discretely sampled running sum of $\sum_{i=1}^{N} s(t_i)$ into a continuous integral representation $\int_0^T S(\tau)d\tau$, by using Dirac delta functions shifted by the sample times t_i as

$$\int_0^t \left(\sum_{i=1}^{N} S(\tau)\delta(\tau - t_i) \right) d\tau \, .$$

Note that this is equivalent to the an expression of the form $\int_0^t f(S(\tau), \tau) d\tau$ if we take the function f to be

$$f(S, \tau) = S \sum_{i=1}^{N} \delta(\tau - t_i) \, .$$

Chapter 14 (Asian Options)

Additional Notes on the Text

Similarity reductions in valuating Asian options

In this subsection we will derive the partial differential equation that the value of an Asian option must satisfy if the payoff function is of the form $S^\alpha F(I/S, t)$. In that case we guess that the Asian option will have a value V of the form $V(S, I, t) = S^\alpha H(R, t)$ with $R = \frac{I}{S}$ and for some yet to be determined function $H(R, t)$. We will use this functional form in the partial differential equation for Asian options presented in the book to find an equation for the function $H(R, t)$. To do this we first compute the the required derivatives of $V(S, I, t)$ for the Black-Scholes equation for a continuously sampled arithmetic Asian option which is the books equation 14.2 or

$$\frac{\partial V}{\partial t} + S\frac{\partial V}{\partial I} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0\,. \tag{160}$$

We find partial derivatives of $V(S, I, t)$ when $V = S^\alpha H(I/S, t)$ given by

$$\begin{aligned}
\frac{\partial V}{\partial t} &= S^\alpha \frac{\partial H}{\partial t} && (161)\\
\frac{\partial V}{\partial S} &= \alpha S^{\alpha-1} H + S^\alpha \frac{\partial H}{\partial R}\frac{\partial R}{\partial S} \\
&= \alpha S^{\alpha-1} H + S^\alpha \frac{\partial H}{\partial R}\left(-\frac{I}{S^2}\right) \\
&= \alpha S^{\alpha-1} H - S^{\alpha-2} I \frac{\partial H}{\partial R} && (162)\\
\frac{\partial^2 V}{\partial S^2} &= \alpha(\alpha-1) S^{\alpha-2} H + \alpha S^{\alpha-1} \frac{\partial H}{\partial R}\left(-\frac{I}{S^2}\right) \\
&- (\alpha-2) S^{\alpha-3} I \frac{\partial H}{\partial R} - S^{\alpha-2} I \frac{\partial^2 H}{\partial R^2}\left(-\frac{I}{S^2}\right) \\
&= \alpha(\alpha-1) S^{\alpha-2} H - \alpha I S^{\alpha-3} \frac{\partial H}{\partial R} \\
&- (\alpha-2) I S^{\alpha-3} \frac{\partial H}{\partial R} + S^{\alpha-4} I^2 \frac{\partial^2 H}{\partial R^2} \\
&= \alpha(\alpha-1) S^{\alpha-2} H - 2(\alpha-1) I S^{\alpha-3} \frac{\partial H}{\partial R} + S^{\alpha-4} I^2 \frac{\partial^2 H}{\partial R^2} && (163)\\
\frac{\partial V}{\partial I} &= S^\alpha \frac{\partial H}{\partial R}\frac{\partial R}{\partial I} = S^{\alpha-1}\frac{\partial H}{\partial R}\,. && (164)
\end{aligned}$$

When we put these expressions into the Equation 160 we obtain the following

$$\begin{aligned}
0 &= S^\alpha \frac{\partial H}{\partial t} + S^\alpha \frac{\partial H}{\partial R} \\
&+ \frac{1}{2}\sigma^2 S^2 \left(\alpha(\alpha-1) S^{\alpha-2} H - 2(\alpha-1) I S^{\alpha-3} \frac{\partial H}{\partial R} + S^{\alpha-4} I^2 \frac{\partial^2 H}{\partial R^2}\right) \\
&+ rS\left(\alpha S^{\alpha-1} H - S^{\alpha-2} I \frac{\partial H}{\partial R}\right) - rS^\alpha H\,.
\end{aligned}$$

Expanding everything and then dividing by S^α we find

$$\frac{\partial H}{\partial t} + \frac{\partial H}{\partial R} + \frac{1}{2}\sigma^2\alpha(\alpha-1)H - \sigma^2(\alpha-1)R\frac{\partial H}{\partial R} + \frac{1}{2}\sigma^2 R^2 \frac{\partial^2 H}{\partial R^2} + \alpha r H - rR\frac{\partial H}{\partial R} - rH = 0\,.$$

Grouping all similar derivatives together we obtain

$$\begin{aligned} 0 &= \frac{\partial H}{\partial t} + (1 - \sigma^2(\alpha - 1)R - rR)\frac{\partial H}{\partial R} + \frac{1}{2}\sigma^2 R^2 \frac{\partial^2 H}{\partial R^2} \\ &+ \left(\frac{1}{2}\sigma^2\alpha + r\right)(\alpha - 1)H\,, \end{aligned} \tag{165}$$

which is equation 14.4 in the book.

Valuating continuously sampled arithmetic Asian options

In this subsection we use the similarity solution presented in Equation to derive the specific valuation equation for continuously sampled arithmetic Asian options. Since for continuous arithmetic sampling $R = \frac{I}{S} = \frac{1}{S}\int_0^t S(\tau)d\tau$, our payoff can be written as

$$\Lambda(S, I, t) = \max\left(S - \frac{1}{t}\int_0^t S(\tau)d\tau, 0\right) = S\max\left(1 - \frac{I}{tS}, 0\right) = S^\alpha F(\frac{I}{S}, t)\,,$$

if we take $\alpha = 1$ and the function F to be $F(\frac{I}{S}, t) = \max(1 - \frac{I}{St}, 0)$ or

$$F(R, t) = \max(1 - \frac{R}{t}, 0)\,.$$

Thus our option $V(S, I, t)$ may have the financial form $V(S, R, t) = SH(R, t)$ so that Equation becomes

$$\frac{\partial H}{\partial t} + \frac{1}{2}\sigma^2 R^2 \frac{\partial^2 H}{\partial R^2} + (1 - rR)\frac{\partial H}{\partial R} = 0\,, \tag{166}$$

which is equation 14.7 in the book

Derivations of the boundary conditions for European options

In this subsection we comment on the technical point discussing the boundary conditions for European continuously sampled average strike Asian options. Since $R = \frac{1}{S}\int_0^t S(\tau)d\tau = \frac{I}{S}$. From the discussion in Section 13.2 "Time Integrals of the Random Walk" the expression

$$I' = \int_0^t f(S(\tau), \tau)d\tau\,,$$

has a differential of $dI' = f(S,t)dt = Sdt$, when $f(S,t) = S$. From this expression we see that there is *no* random component in the differential of I'. Thus for $R = \frac{I}{S}$ we have, using Ito's Lemma that (since dS will have a random component) that

$$\begin{aligned} dR &= \frac{\partial R}{\partial I} dI + \frac{\partial R}{\partial S} dS + \frac{1}{2} \frac{\partial^2 R}{\partial S^2} dS^2 \\ &= \frac{1}{S} S dt - \frac{I}{S^2} dS + \frac{1}{2} \left(\frac{2I}{S^3} \right) dS^2 \\ &= dt - R \frac{dS}{S} + R \frac{dS^2}{S^2} \,. \end{aligned}$$

Now since we assume that S satisfies geometric Brownian motion or $\frac{dS}{S} = \mu dt + \sigma dX$, we have that

$$\frac{dS^2}{S^2} = \mu^2 dt^2 + 2\mu\sigma dt dX + \sigma^2 dX^2 \approx \sigma^2 dt \,,$$

to first order. Thus we see that dR becomes

$$\begin{aligned} dR &= dt - R(\mu dt + \sigma dX) + R(\sigma^2 dt) \\ &= -\sigma R dX + (1 + (\sigma^2 - \mu) R) dt \,, \end{aligned}$$

which is the result in the book.

Derivations on put-call parity for the European average strikes

In this subsection we derive put-call parity relationships for European average strike options. To begin consider the payoff at expiration, $\Pi(T)$, on the suggested portfolio of one European average strike call held long and one European average put held short

$$\Pi(T) = S \max\left(1 - \frac{R}{T}, 0\right) - S \max\left(\frac{R}{T} - 1, 0\right) = S - \frac{RS}{T} \,.$$

Now consider what type of instrument will provide a payoff given by the second term $-\frac{RS}{T}$. Since the continuously sampled average strike option has a value that can be expressed as $V(S,R,t) = SH(R,t)$, where $H(R,t)$ is a solution to Equation 166. At expiration we would like to construct a continuously sampled average strike option that has a payoff given by $-\frac{RS}{T}$.

This means that when $t = T$ the function $H(R, t)$ must have a final condition that satisfies $SH(R, T) = -\frac{RS}{T}$ or

$$H(R, T) = -\frac{R}{T}. \tag{167}$$

With this final condition the solution we seek for $H(R, t)$ must also satisfy the standard boundary conditions of

$$H(\infty, t) = 0 \quad \text{and} \quad \frac{\partial H}{\partial t}(0, t) + \frac{\partial H}{\partial R}(0, t) = 0.$$

To find a functional form for $H(R, t)$ that will satisfy these requirements we propose a $H(R, t)$ of the specific form given by

$$H(R, t) = a(t) + b(t)R. \tag{168}$$

Note that for the expression for the payoff in Equation 167 to hold true at $t = T$ when $H(R, t)$ is as Equation 168 we need

$$H(R, T) = a(T) + b(T)R = -\frac{R}{T},$$

which requires that we take $a(T) = 0$ and $b(T) = -\frac{1}{T}$. Now to put the proposed functional form for $H(R, t)$ from Equation 168 into Equation 166 requires we compute

$$\begin{aligned} \frac{\partial H}{\partial t} &= a'(t) + b'(t)R \\ \frac{\partial H}{\partial R} &= b(t) \\ \frac{\partial^2 H}{\partial R^2} &= 0, \end{aligned}$$

so that Equation 166 evaluates to

$$a'(t) + b'(t)R + (1 - rR)b(t) = 0,$$

or grouping powers of R we get

$$a'(t) + b(t) + (b'(t) - rb(t))R = 0. \tag{169}$$

Equating the powers R on both sides of this equation we find for the coefficient of the first power of R to vanish that

$$b'(t) = rb(t) \quad \text{so} \quad b(t) = C_0 e^{rt} \,.$$

When we then require that $b(T) = -\frac{1}{T}$ we find

$$-\frac{1}{T} = C_0 e^{rT} \quad \text{or} \quad C_0 = -\frac{1}{T} e^{-rT} \,,$$

which means that the function $b(t)$ is given by

$$b(t) = -\frac{1}{T} e^{-r(T-t)} \,. \tag{170}$$

Using this result and equating the constant terms in Equation 169 requires that the function $a(t)$ must satisfy

$$a'(t) = -b(t) = \frac{1}{T} e^{-r(T-t)} \,.$$

On integrating this we find $a(t)$ given by

$$a(t) = \frac{1}{rT} e^{-r(T-t)} + C_1 \,.$$

To evaluate C_1 we recall that $a(T) = 0$ requires that $C_1 = -\frac{1}{rT}$ so that the functional form for $a(t)$ is given by

$$a(t) = -\frac{1}{rT}(1 - e^{-r(T-t)}) \,. \tag{171}$$

Thus we can conclude that the put-call parity relationship for average strike options is given by

$$\begin{aligned}
C - P &= S + V(S, R, t) \\
&= S + SH(R, t) \\
&= S + S(a(t) + b(t)R) \\
&= S - S\left(\frac{1}{rT}(1 - e^{-r(T-t)}) + \frac{1}{T} e^{-r(T-t)} R\right) \\
&= S - \frac{S}{rT}(1 - e^{-r(T-t)}) - \frac{S}{T} e^{-r(T-t)} \left(\frac{I}{S}\right) \\
&= S - \frac{S}{rT}(1 - e^{-r(T-t)}) - \frac{1}{T} e^{-r(T-t)} \int_0^t S(\tau) d\tau \,,
\end{aligned}$$

which is the expression in the book.

Notes on the derivations of average rate options

In this subsection of these notes we show how to perform similarity reductions and find explicit solutions to average rate options where the averaging is performed geometrically. As suggested in the text for an average rate options we seek a solution of the form $V(S, I, t) = F(y, t)$ with y given by

$$y = \frac{I + (T - t)\log(S)}{T} \,.$$

Note this newly introduced variable y depends on all of the dependent variables: I, S, and t. To use this change of variable in the partial differential equation satisfied by our option price $V(S, I, t)$ of

$$\frac{\partial V}{\partial t} + \log(S)\frac{\partial V}{\partial I} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \,. \tag{172}$$

We need to evaluate several derivative. First

$$\begin{aligned}
\frac{\partial V}{\partial t} &= \frac{\partial F}{\partial t} + \frac{\partial y}{\partial t}\frac{\partial F}{\partial y} \\
&= \frac{\partial F}{\partial t} - \frac{\log(S)}{T}\frac{\partial F}{\partial y} \,,
\end{aligned}$$

Next the derivative with respect I and S

$$\begin{aligned}
\frac{\partial V}{\partial I} &= \frac{\partial F}{\partial y}\frac{\partial y}{\partial I} = \frac{1}{T}\frac{\partial F}{\partial y} \\
\frac{\partial V}{\partial S} &= \frac{\partial F}{\partial y}\frac{\partial y}{\partial S} = \frac{T - t}{T}\frac{1}{S}\frac{\partial F}{\partial y} \\
\frac{\partial^2 V}{\partial S^2} &= \frac{\partial}{\partial S}\left(\frac{T - t}{T}\frac{1}{S}\right)\frac{\partial F}{\partial y} + \left(\frac{T - t}{T}\right)\frac{1}{S}\frac{\partial^2 F}{\partial y^2}\frac{\partial y}{\partial S} \\
&= -\frac{T - t}{T}\frac{1}{S^2}\frac{\partial F}{\partial y} + \left(\frac{T - t}{T}\frac{1}{S}\right)^2 \frac{\partial^2 F}{\partial y^2} \,.
\end{aligned}$$

When we put these expressions into Equation 172 we obtain

$$\frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 \left(-\frac{T - t}{T}\frac{\partial F}{\partial y} + \left(\frac{T - t}{T}\right)^2 \frac{\partial^2 F}{\partial y^2} \right) + r\left(\frac{T - t}{T}\frac{\partial F}{\partial y}\right) - rF = 0 \,.$$

Grouping everything by the derivatives of F we find the above equation is equivalent to

$$\frac{\partial F}{\partial t} + \left(-\frac{1}{2}\sigma^2 + r\right)\left(\frac{T-t}{T}\right)\frac{\partial F}{\partial y} + \frac{1}{2}\sigma^2\left(\frac{T-t}{T}\right)^2\frac{\partial^2 F}{\partial y^2} - rF = 0\,.$$

which is the books equation 14.12.

Exercise Solutions

Exercise 1 (perpetual options)

This problem is very similar to that of Exercise 6 in Chapter 3 on Page 23.

Exercise 2 (geometric average rate options measured discretely)

Warning: I'm not sure this problem is correct or is representative of what was requested from the text. If anyone has any suggestions as to ways to improve upon what I have here or how to take these results further please email me.
When the geometric average is measured directly we have a payoff at $t = T$ given by

$$\Lambda(T) = \max(\frac{I}{T} - E, 0)\,, \tag{173}$$

and so this option at $t = T$ is valued as $V(S, I, T) = \Lambda(T)$. Following the discussion in the book on discretely sampled averages, to solve this problem we work backwards from $t = T$ to $t = 0$ and impose the jump conditions in $V(S, I, t)$ across each discretely measured time point t_i, treating $I = \sum_{i=1}^{j(t)} \log(S(t_i))$ as a constant (which it is) between the time measurement of the stock price S. The jump conditions we need to apply to $V(S, I, t)$ in this case across the points t_i is

$$V(S, I, t_i^-) = V(S, I + \log(S), t_i^+)\,.$$

Since the payoff at the final time $t = T$ in Equation 173 above is independent of S we might consider looking for a solution to Equation 172 where V is independent of S or the solution to the equation

$$\frac{\partial V}{\partial t} + \log(S)\frac{\partial V}{\partial I} - rV = 0\,.$$

Exercise 3 (continuously sampled arithmetic rate call options)

Warning: I'm not sure this problem is correct or is representative of what was requested from the text. If someone has any suggestions as to how I could improve upon this problem please email me.
If we know for certain that the European arithmetic average rate option will expire in the money then the payoff is given by $\frac{I}{T} - E$ and the option price must satisfy the Black-Scholes equation given by Equation 160 above. Looking for a solution to this equation that is *linear* in I and S we hypothesis the following functional form form $V(S, I, t)$

$$V(S, I, t) = A(t)S + B(t)I + C(t)\,.$$

Then when we put this into Equation 160 we get

$$A'(t)S + B'(t)I + C'(t) + SB(t) + rSA(t) - r(A(t)S + B(t)I + C(t)) = 0\,.$$

Grouping powers of I and S we get

$$S(A'(t) + B(t)) + I(B'(t) - rB(t)) + (C'(t) - rC(t)) = 0\,.$$

Equating the coefficient of I to zero we get that $B(t)$ is given by

$$B(t) = B_0 e^{-r(T-t)}\,,$$

for some constant B_0. Equating the coefficient of S to zero we get that $A(t)$ must satisfy

$$A'(t) + B_0 e^{-r(T-t)} = 0\,,$$

or

$$A(t) = -\frac{B_0}{r} e^{-r(T-t)} + A_0\,,$$

for some constant A_0. Finally equating the constant term equal to zero we see that the function $C(t)$ must equal

$$C(t) = C_0 e^{-r(T-t)}\,,$$

for another constant C_0. Thus the solution to $V(S, I, t)$ in terms of all of these constants is given by

$$V(S, I, t) = S\left(A_0 - \frac{B_0}{r} e^{-r(T-t)}\right) + IB_0 e^{-r(T-t)} + C_0 e^{-r(T-t)}\,.$$

Setting $t = T$ in the above and equating to the known value of the payoff there gives

$$V(S,I,T) = S\left(A_0 - \frac{B_0}{r}\right) + IB_0 + C_0 = \frac{I}{T} - E\,.$$

Now A_0, B_0, and C_0 cannot be functions of S or I. Thus these constants must satisfy

$$\begin{aligned} A_0 - \frac{B_0}{r} &= 0 \\ B_0 &= \frac{1}{T} \\ C_0 &= -E\,. \end{aligned}$$

From which we see that $A_0 = \frac{1}{rT}$ and our full solution for $V(S,I,t)$ becomes

$$V(S,I,t) = \frac{S}{rT}\left(1 - e^{-r(T-t)}\right) + \frac{I}{T}e^{-r(T-t)} - Ee^{-r(T-t)}\,. \tag{174}$$

Exercise 5 (the average strike foreign exchange option)

Note: This problem is not entirely finished.

A foreign exchange option has a valuation $V(S,t)$ that satisfies the Black-Scholes equation but with a dividend or Equation 29, where D_0 is the foreign exchange rate. Since the average strike call foreign exchange rate option has a payoff that looks very similar to that of a average strike call option where in that case we had a payoff given by $S\max\left(1 - \frac{1}{St}\int_0^t S(\tau)d\tau, 0\right)$. In the case of the average strike foreign exchange option the difference is that there is no S factor in front of the $\max(\cdot)$ function in the payoff.

Following the derivation of the similarity reduction result for average strike options, we would modify Equation 29 to include a $S\frac{\partial V}{\partial I}$ term to get

$$\frac{\partial V}{\partial t} + S\frac{\partial V}{\partial I} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + (r - D_0)S\frac{\partial V}{\partial S} - rV = 0\,, \tag{175}$$

and then seek a solution to this equation of the form $V(S,I,t) = H(R,t)$ where $R = \frac{I}{S}$ and I is defined as $\int_0^t S(\tau)d\tau$. To finish this problem we would need compute the differential equation satisfied by $H(R,t)$ when we put $V(S,I,t) = H(R,t)$ into Equation 175, as presented on Page 181 for average strike call options.

Exercise 6 (the jump conditions expressed in the function $H(R, t)$)

The jump conditions for discretely sampled arithmetic average strike options are given by

$$V(S, I, t_i^-) = V(S, I + S, t_i^+)\,, \tag{176}$$

where t_i is the time at which the discrete sample occurs, and $I = \int_0^t f(S(\tau), \tau) d\tau$ and S is the spot price at the time $t = t_i$. Then since the option price has a similarity reduction given by $V(S, I, t) = SH(\frac{I}{S}, t)$ by writing both sides of Equation 176 in this way gives

$$SH\left(\frac{I}{S}, t_i^-\right) = SH\left(\frac{I + S}{S}, t_i^+\right)\,,$$

or using the definition that $R = \frac{I}{S}$ we get

$$H(R, t_i^-) = H(R + 1, t_i^+)\,,$$

as the jump condition for the function $H(R, t)$.

Chapter 15 (Lookback Options)

Additional Notes on the Text

Notes on the derivation of the PDE for lookback options

In this subsection of these notes we will derive the expression for the differential for J_n. From the definition of J_n when we increment t by dt that we have

$$
\begin{aligned}
\left(\int_0^{t+dt} S(\tau)^n d\tau\right)^{1/n} &= \left(\int_0^t S(\tau)^n d\tau + S(t)^n dt\right)^{1/n} \\
&\approx J_n + \frac{1}{n}\left(\int_0^t S(\tau)^n d\tau + S(t)^n dt\right)^{\frac{1}{n}-1}\Bigg|_{dt=0} S(t)^n dt \\
&= J_n + \frac{1}{n}(I_n)^{\frac{1-n}{n}} S^n dt\,.
\end{aligned}
$$

Since $J_n = (I_n)^{1/n}$ we get $I_n = J_n^n$ and the above becomes

$$
dJ_n = \frac{1}{n}(J_n)^{1-n} S^n dt = \frac{1}{n}\frac{S^n}{{J_n}^{n-1}}dt\,,
$$

or the books equation 15.2.

The portfolio when our option depends on J_n

As we have done many times in this book lets consider the differential of a portfolio that is long one share of a lookback option P and short Δ shares of the stock of value S. We find that this portfolio changes by

$$
\begin{aligned}
d\Pi &= dP - \Delta dS \\
&= \frac{\partial P}{\partial t}dt + \frac{\partial P}{\partial S}dS + \frac{1}{2}\frac{\partial^2 P}{\partial S^2}dS^2 + \frac{\partial P}{\partial J_n}dJ_n - \frac{\partial P}{\partial S}dS \\
&= \frac{\partial P}{\partial t}dt + \frac{1}{2}\frac{\partial^2 P}{\partial S^2}(S^2\sigma^2 dt) + \frac{\partial P}{\partial J_n}\left(\frac{1}{n}\frac{S^n}{J_n^{n-1}}dt\right)\,,
\end{aligned}
$$

when we recall that for geometric Brownian motion we have $dS^2 = S^2\sigma^2 dt$ in the limit where $dt \to 0$. This is the books equation 15.3.

Similarity reductions for European lookback options

In this section of these notes we derive the similarity partial differential equation that the option price of a European lookback put option must satisfy. We begin with the simplification of the functional for of $P(S, J, t)$ by assuming that the function $P(S, J, t)$ takes the form

$$P(S, J, t) = JW(\xi, t)\,,$$

where $\xi = \frac{S}{J}$. We the need to put the expression for $P(S, J, t)$ into the Black-Scholes equation and derive a partial differential equation for $W(\xi, t)$. Thus we evaluate

$$\begin{aligned}
\frac{\partial P}{\partial t} &= J\frac{\partial W}{\partial t} \\
\frac{\partial P}{\partial S} &= J\frac{\partial W}{\partial \xi}\frac{\partial \xi}{\partial S} = J\frac{\partial W}{\partial \xi}\left(\frac{1}{J}\right) = \frac{\partial W}{\partial \xi} \\
\frac{\partial^2 P}{\partial S^2} &= \frac{\partial^2 W}{\partial \xi^2}\frac{\partial \xi}{\partial S} = \frac{1}{J}\frac{\partial^2 W}{\partial \xi^2}\,.
\end{aligned}$$

Then with these expression placed in the Black-Scholes equation we get

$$\frac{\partial W}{\partial t} + \frac{1}{2}\sigma^2\xi^2\frac{\partial W}{\partial \xi^2} + r\xi\frac{\partial W}{\partial \xi} - rW = 0\,,$$

which is the equation for W presented in the book. We now map the boundary and final conditions for $P(S, J, t)$ into equivalent conditions on $W(\xi, t)$. First, the boundary condition when $S = 0$ (and correspondingly $\xi = 0$) is

$$P(0, J, t) = Je^{-r(T-t)} = JW(0, t) \quad \text{or} \quad W(0, t) = e^{-r(T-t)}\,.$$

Second, the final condition on $P(S, J, t)$ becomes

$$P(S, J, T) = \max(J - S, 0) = J\max(1 - \frac{S}{J}, 0) = JW(\xi, T)\,,$$

which requires that $W(\xi, T) = \max(1-\xi, 0)$. Finally, the boundary condition on $P(S, J, t)$ at $S = J$ of

$$\frac{\partial P}{\partial J}(S, J, t) = 0\,,$$

is transformed using the identity

$$\frac{\partial}{\partial J} = \left(\frac{\partial \xi}{\partial J}\right)\frac{\partial}{\partial \xi} = -\frac{S}{J^2}\frac{\partial}{\partial \xi} - \frac{\xi}{J}\frac{\partial}{\partial \xi}\,.$$

Using that derivative identity the J derivative of P transforms as

$$\begin{aligned}\frac{\partial P}{\partial J} &= \frac{\partial}{\partial J}(JW(\xi,t)) = W + J\frac{\partial W}{\partial J}(\xi,t) = W + J\left(-\frac{\xi}{J}\frac{\partial W}{\partial J}\right) \\ &= W - \xi\frac{\partial W}{\partial \xi} = 0\,.\end{aligned}$$

So that when $S = J$ we have $\xi = 1$ and this boundary condition becomes

$$\frac{\partial W}{\partial \xi} = W \quad \text{when} \quad \xi = 1\,.$$

Chapter 16 (Options with Transaction Costs)

Exercise Solutions

Exercise 1 (more general transaction costs)

As discussed in the book a transaction cost proportional to the value of the assets trade would be $\kappa_3 S|\nu|$, where ν is the number of shares bought (if $\nu > 0$) or sold (if $\nu < 0$). In the same way a fixed cost for each transaction would be represented by the constant κ_1. A cost proportional to the number of assets traded i.e. ν would be represented by $\kappa_2|\nu|$. Thus including a cost that is proportional to the value of the assets (as considered in the book) our total cost would be

$$\kappa_1 + \kappa_2|\nu| + \kappa_3 S|\nu| \,.$$

This cost would have to be paid out and would reduce the value of our portfolio. Following the book we still have to leading order that the number of shares traded ν should be given by

$$\nu \approx \frac{\partial^2 V}{\partial S^2}\sigma S \phi \sqrt{\delta t}\,,$$

so that our expected cost becomes

$$\mathcal{E}[\kappa_1 + \kappa_2|\nu| + \kappa_3 S|\nu|] = \kappa_1 + \kappa_2\sqrt{\frac{2}{\pi}}\sigma S\left|\frac{\partial^2 V}{\partial S^2}\right|\sqrt{\delta t} + \kappa_3\sqrt{\frac{2}{\pi}}\sigma S^2\left|\frac{\partial^2 V}{\partial S^2}\right|\sqrt{\delta t}\,,$$

where again we have used the fact that $\mathcal{E}[|\phi|] = \sqrt{\frac{2}{\pi}}$, when $\phi \sim \mathcal{N}(0,1)$. Thus our expected change in the value of our portfolio $\mathcal{E}[\delta\Pi]$ is given by

$$\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} - \frac{\kappa_1}{\sqrt{\delta t}} - \kappa_2\sqrt{\frac{2}{\pi\delta t}}\sigma S\left|\frac{\partial^2 V}{\partial S^2}\right| - \kappa_3\sqrt{\frac{2}{\pi\delta t}}\sigma S^2\left|\frac{\partial^2 V}{\partial S^2}\right|\right)\delta t\,.$$

Equating this to $r\left(V - S\frac{\partial V}{\partial S}\right)\delta t$ if we expect our portfolio to earn the risk free rate we find the equation for the value of the option given by

$$\begin{aligned}\frac{\partial V}{\partial t} &+ \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\\ &- \frac{\kappa_1}{\sqrt{\delta t}} - \kappa_2\sigma S\sqrt{\frac{2}{\pi\delta t}}\left|\frac{\partial^2 V}{\partial S^2}\right| - \kappa_3\sigma S^2\sqrt{\frac{2}{\pi\delta t}}\left|\frac{\partial^2 V}{\partial S^2}\right|\\ &+ rS\frac{\partial V}{\partial t} - rV = 0\,. \qquad (177)\end{aligned}$$

Exercise 2 (transaction costs can result in negative option prices)

For a simple example where the transaction costs may result in a negative option prices consider the case where $\kappa_2 = \kappa_3 = 0$ in Equation 177 which gives

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \frac{\kappa_1}{\sqrt{\delta t}} + rS\frac{\partial V}{\partial t} - rV = 0\,.$$

That solutions to this equation can be negative can be seen by observing that the constant $-\frac{\kappa_1}{r\sqrt{\delta t}}$ is a negative solution to the above equation.

Exercise 3 (the diffusion equation with a forcing of $\left|\frac{\partial^2 u}{\partial x^2}\right|$)

To solve this problem lets attempt a similarity solution of the form $u(x,t) = t^{-1/2}h(x/t^{1/2})$. On Page 43 of these notes we derived the similarity differential equation with the substitution $u(x,t) = t^\alpha f(x/t^\beta)$. This problem is the same as that one with the constants $\alpha = -1/2$ and $\beta = 1/2$ and results in the following equation

$$-\frac{1}{2}h(\xi) - \frac{1}{2}\xi h'(\xi) = h''(\xi) + \lambda\,|h''(\xi)|\;,$$

where we have defined $\xi \equiv \frac{x}{t^{1/2}}$. We would then solve this equation for $h(\xi)$ to find a solution to the original equation.

Exercise 4 (solving a nonlinear equation with linear equations)

If we assume that $K \equiv \frac{\kappa}{\sigma\sqrt{\delta t}}$ is small (perhaps due to a small transaction cost κ value) we can attempt to Taylor expand the solution to equation 16.4 in the book. That is, we assume that the solution of

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \sqrt{\frac{2}{\pi}}K\sigma^2 S^2 \left|\frac{\partial^2 V}{\partial S^2}\right| + rS\frac{\partial V}{\partial t} - rV = 0\,, \tag{178}$$

has a form

$$V(S,t) = V_0(S,t) + KV_1(S,t) + O(K^2)\,.$$

Putting this expression into Equation 178 gives

$$\begin{aligned} & \frac{\partial V_0}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_0}{\partial S^2} + rS\frac{\partial V_0}{\partial t} - rV_0 \\ + \quad & K\left[\frac{\partial V_1}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_1}{\partial S^2} - \sigma^2 S^2 \sqrt{\frac{2}{\pi}}\left|\frac{\partial^2 V_0}{\partial S^2}\right| + rS\frac{\partial V_1}{\partial t} - rV_1\right] + O(K^2) = 0\,. \end{aligned}$$

Thus the $O(0)$ (for K^0 or a constant term) equation for $V_0(S,t)$ is the Black-Scholes equation or

$$\frac{\partial V_0}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_0}{\partial S^2} + rS\frac{\partial V_0}{\partial t} - rV_0 = 0\,.$$

The $O(1)$ equation (for K) is a forced Black-Scholes equation given by

$$\frac{\partial V_1}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_1}{\partial S^2} + rS\frac{\partial V_1}{\partial t} - rV_1 = \sigma^2 S^2 \sqrt{\frac{2}{\pi}}\left|\frac{\partial^2 V_0}{\partial S^2}\right|\,.$$

Thus the nonlinear equation becomes two (or more) iterated linear equations.

Chapter 17 (Interest Rate Derivatives)

Additional Notes on the Text

Bond pricing with known interest rates and discretely paid coupons

We will use the bond pricing equation, or the books equation 17.2 given by

$$V(t) = e^{-\int_t^T r(\tau)d\tau} \left(Z + \int_t^T K(t') e^{\int_{t'}^T r(\tau)d\tau} dt' \right) , \tag{179}$$

in the case when our time-dependent coupon payment function $K(t)$ is of the form $K(t) = K_c \delta(t - t_c)$. To do this we need to evaluate the above inner integral. We find this integral given by

$$\int_t^T K_c \delta(t' - t_c) e^{\int_{t'}^T r(\tau)d\tau} dt' = \left\{ \begin{array}{cc} 0 & t > t_c \\ K_c e^{\int {t_c}^T r(\tau)d\tau} & t < t_c \end{array} \right. .$$

This later expression can be written more compactly using the Heaviside function $\mathcal{H}(\cdot)$ as

$$\mathcal{H}(-(t - t_c)) K_c e^{\int {t_c}^T r(\tau)d\tau} ,$$

as is given in the book. If we have N discrete coupons denoted by K_i which are paid at the discrete times t_i where $t < t_1 < t_2 < \cdots < t_N < T$, then the differential equation for the bonds value V

$$\frac{dV}{dt} + K(t) = r(t)V , \tag{180}$$

in this case becomes

$$\frac{dV}{dt} + \sum_{i=1}^N K_k \delta(t - t_i) = r(t)V .$$

When we put this into Equation 179 we again need to evaluate the integral over $K(\cdot)$ which in this case is given by

$$\int_t^T K(t') e^{\int_{t'}^T r(\tau)d\tau} dt' = \int_t^T \sum_{i=1}^N K_i \delta(t' - t_i) e^{\int_{t'}^T r(\tau)d\tau} dt' = \sum_{i=1}^N K_i e^{\int_{t_i}^T r(\tau)d\tau} .$$

Thus the total bond price $V(t)$ under N discrete coupons from Equation 179 becomes

$$V(t) = e^{-\int_t^T r(\tau)d\tau} \left(Z + \sum_{i=1}^{N} K_i e^{\int_{t_i}^T r(\tau)d\tau} \right) .$$

This analysis did not use the Heaviside function since we assumed all coupons happen *after* the current time t.

Notes on the yield curve

I think there is a typo in the book when its states the definition of the yield curve $Y(t;T)$. It should be defined as

$$Y(t;T) = -\frac{\log(V(t;T)/V(T;T))}{T-t} = -\frac{\log(V(t;T)/Z)}{T-t} . \tag{181}$$

Notes on the bond pricing equation

In this section we derive the bond pricing equation in the case when the bond pays a coupon $K(r,t)$. This is very similar to the equivalent derivation presented in the book. In the case where our bond pays coupons then as in the section "bond pricing with known interest rates" we need to add a $K(r,t)$ term to the dt increment of dV to get

$$dV = \left(\frac{\partial V}{\partial t} + K(r,t) \right) dt + \frac{\partial V}{\partial r} dr + \frac{1}{2} w^2 \frac{\partial^2 V}{\partial r^2} dt . \tag{182}$$

Using this expression for dV and following the logic in the section "the bond pricing equation" we consider the change in the mixed portfolio $\Pi = V_1 - \Delta V_2$ as

$$\begin{aligned} d\Pi &= \left(\frac{\partial V_1}{\partial t} + K \right) dt + \frac{\partial V_1}{\partial r} dr + \frac{1}{2} w^2 \frac{\partial^2 V_1}{\partial r^2} dt \\ &- \Delta \left[\left(\frac{\partial V_2}{\partial t} + K \right) dt + \frac{\partial V_2}{\partial r} dr + \frac{1}{2} w^2 \frac{\partial^2 V_2}{\partial r^2} dt \right] . \end{aligned}$$

Here we have explicitly assumed that $K(r,t)$ is paid by both bonds. If we take Δ given by

$$\Delta = \frac{\frac{\partial V_1}{\partial r}}{\frac{\partial V_2}{\partial r}} ,$$

and then the dr terms in $d\Pi$ vanish, and we get

$$d\Pi = \left[\left(\frac{\partial V_1}{\partial t} + K\right) + \frac{1}{2}w^2\frac{\partial^2 V_1}{\partial r^2} - \Delta\left[\left(\frac{\partial V_2}{\partial t} + K\right) + \frac{1}{2}w^2\frac{\partial^2 V_2}{\partial r^2}\right]\right] dt\,.$$

Which to avoid an arbitrage opportunity we set equal to

$$r\Pi dt = r(V_1 - \Delta V_2)dt\,.$$

Thus when we do this and place V_1 on the left-hand-side and V_2 on the right-hand-side

$$\frac{\partial V_1}{\partial t} + K + \frac{1}{2}w^2\frac{\partial^2 V_1}{\partial r^2} - rV_1 = \Delta\left[\frac{\partial V_2}{\partial t} + K + \frac{1}{2}w^2\frac{\partial^2 V_2}{\partial r^2} - rV_2\right].$$

Next we divide by $\frac{\partial V_1}{\partial r}$ on both sides to get

$$\frac{1}{\frac{\partial V_1}{\partial r}}\left(\frac{\partial V_1}{\partial t} + K + \frac{1}{2}w^2\frac{\partial^2 V_1}{\partial r^2} - rV_1\right) = \frac{1}{\frac{\partial V_2}{\partial r}}\left(\frac{\partial V_2}{\partial t} + K + \frac{1}{2}w^2\frac{\partial^2 V_2}{\partial r^2} - rV_2\right).$$

Setting each side equal to $a(r,t)$ which we take equal to $w(r,t)\lambda(r,t) - u(r,t)$ we see that both V_1 and V_2 must satisfy

$$\frac{\partial V}{\partial t} + K(r,t) + \frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} + (u - \lambda w)\frac{\partial V}{\partial r} - rV = 0\,, \qquad (183)$$

which is the desired equation.

Notes on the market price of risk

The differential of our bond price V when V depends on a stochastic interest rate, r, such that $dr = w(r,t)dX + u(r,t)dt$ and a deterministic time variable, t, using Ito's lemma to leading order is given by

$$\begin{aligned} dV &= \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial t}dt + \frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2}dt \\ &= w\frac{\partial V}{\partial r}dX + \left(\frac{\partial V}{\partial t} + u\frac{\partial V}{\partial r} + \frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2}\right) dt\,. \end{aligned}$$

From the zero coupon bond pricing equation derived in the text we have that

$$\frac{\partial V}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} + u\frac{\partial V}{\partial r} = \lambda w\frac{\partial V}{\partial r} + rV\,, \qquad (184)$$

from which when we put this into the coefficient of dt above we find that the value of dV computed above becomes

$$dV = w\frac{\partial V}{\partial r}dX + \left(w\lambda\frac{\partial V}{\partial r} + rV\right)dt\,.$$

Transforming this expression some we find

$$dV - rVdt = w\frac{\partial V}{\partial r}(dX + \lambda dt)\,, \tag{185}$$

which is the books equation 17.12.

Notes on the market price of risk for assets

Considering the expressions derived when considering bond pricing and options pricing we see that each instrument has a deterministic time variable t and one random variable. For bonds the random variable is r while for options it is S the underlying stock price. The differentials of each of these random variables are assumed to satisfy

$$\begin{aligned} dr &= wdX + udt \\ dS &= \sigma SdX + \mu Sdt\,. \end{aligned}$$

Thus the correspondence between the two random variables is

$$\begin{aligned} w &= \sigma S \\ u &= \mu S\,. \end{aligned}$$

So by constructing an option portfolio $\Pi = V_1 - \Delta V_2$ in the same way that we did the bond portfolio would give an expression like Equation 184 but with the substitutions for w and u namely

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial r^2} + (\mu - \lambda_S\sigma S)\frac{\partial V}{\partial r} - rV = 0\,,$$

or the books equation 17.13.

Notes on the solutions of the bond pricing equations

When our stochastic interest rate r satisfies

$$dr = w(r,t)dX + u(r,t)dt\,,$$

with

$$\begin{aligned} w(r,t) &= \sqrt{\alpha(t)r - \beta(t)} && (186)\\ u(r,t) &= -\gamma(t)r + \eta(t) + \lambda(r,t)\sqrt{\alpha(t)r - \beta(t)}\,, && (187) \end{aligned}$$

Then we make the vary simple verification that that $u - \lambda w$ in Equation 183 is independent of $\lambda(r,t)$ since

$$u - \lambda w = -\gamma r + \eta + \lambda w - \lambda w = -\gamma r + \eta\,. \tag{188}$$

independent of λ. Now lets consider solutions to Equation 183 of the following form

$$V(r,t) = Ze^{A(t;T) - rB(t;T)}\,. \tag{189}$$

To put this form into Equation 183 we need the following derivatives

$$\begin{aligned} \frac{\partial V}{\partial t} &= \left(\frac{\partial A}{\partial t} - r\frac{\partial B}{\partial t}\right)V(t;T)\\ \frac{\partial V}{\partial r} &= -B(t;T)V(t;T)\\ \frac{\partial^2 V}{\partial r^2} &= B^2(t;T)V(t;T)\,. \end{aligned}$$

When we put this into Equation 183 and divide by V we get

$$\frac{\partial A}{\partial t} - r\frac{\partial B}{\partial t} + \frac{1}{2}w^2B^2 - (u - \lambda w)B - r = 0\,. \tag{190}$$

which is the books equation 17.17. If we take $\frac{\partial}{\partial r}$ of this expression we find

$$-\frac{\partial B}{\partial t} + \frac{1}{2}B^2\frac{\partial(w^2)}{\partial r} - B\frac{\partial(u - \lambda w)}{\partial r} - 1 = 0\,.$$

Taking another derivative with respect to r we get

$$\frac{1}{2}B^2\frac{\partial^2(w^2)}{\partial r^2} - B\frac{\partial^2}{\partial r^2}(u - \lambda w) = 0\,.$$

When we divide this by B we get

$$\frac{1}{2}B\frac{\partial^2(w^2)}{\partial r^2} - \frac{\partial^2}{\partial r^2}(u - \lambda w) = 0\,.$$

Since B is a function of T and $u - \lambda w$ is not a function of T by changing the value of T the left-hand-side of the above would change values unless

$$\frac{\partial^2(w^2)}{\partial r^2} = 0\,. \tag{191}$$

In this case we would then also have

$$\frac{\partial^2(u - \lambda w)}{\partial r^2} = 0\,. \tag{192}$$

Thus these last to expressions show that w and $u - \lambda w$ must have the functional forms given by Equations 186 and 187.

From Equation 188 we find that $u - \lambda w = \eta - r\gamma$ using this and putting Equations 186 and 187 into Equation 190 we get

$$\frac{\partial A}{\partial t} - r\frac{\partial B}{\partial t} - \frac{1}{2}(\alpha r - \beta)B^2 - (-\gamma r + \eta)B - r = 0\,.$$

Equating power of r gives

$$\frac{\partial A}{\partial t} - \frac{1}{2}\beta B^2 - \eta B \quad = \quad 0 \quad \text{for } O(r^0) \tag{193}$$

$$\frac{\partial B}{\partial t} + \frac{1}{2}\alpha B^2 - \gamma B + 1 \quad = \quad 0 \quad \text{for } O(r^1)\,. \tag{194}$$

The final condition for $V(t;T)$ when $t = T$ is that $V(t,T) = Z$ or

$$Ze^{A(T;T) - rB(T;T)} = Z\,,$$

or

$$A(T;T) = B(T;T) = 0\,.$$

Notes on the analysis for constant parameters

In this subsection of these notes we derive the analytic expressions for $A(t;T)$ and $B(t;T)$ found in the expression $Z(r,t;T) = e^{A(t;T) - rB(t;T)}$ for the pricing of a zero coupon bond when the parameters in the dynamics of the stochastic interest rate r

$$dr = (\eta - \gamma r)dt + \sqrt{\alpha r - \beta}dX \,, \tag{195}$$

namely α, β, γ, and η are are *constant*. There is a lot of algebra in the notes that follow, but having all of the computations in one place can make the verification of the results easier. If desired this section can be skipped on first reading. From the discussion in the the book for the differential equation for B we have

$$\frac{dB}{dt} = \frac{1}{2}\alpha B^2 + \gamma B - 1 = \frac{1}{2}\alpha\left(B^2 + \frac{2\gamma}{\alpha}B - \frac{2}{\alpha}\right) .$$

To begin to integrate this equation lets find the two roots, $r_{1,2}$, of the quadratic in B on the right-hand-side of the above. Using the quadratic equation we find

$$r_{1,2} = \frac{-\frac{2\gamma}{\alpha} \pm \sqrt{\frac{4\gamma^2}{\alpha^2} + 4\left(\frac{2}{\alpha}\right)}}{2} = -\frac{\gamma}{\alpha} \pm \sqrt{\frac{\gamma^2}{\alpha^2} + \frac{2}{\alpha}} = \frac{-\gamma \pm \sqrt{\gamma^2 + 2\alpha}}{\alpha} .$$

Thus we see that

$$\begin{aligned} B^2 + \frac{2\gamma}{\alpha}B - \frac{2}{\alpha} &= (B - r_1)(B - r_2) \\ &= \left(B + \frac{\gamma - \sqrt{\gamma^2 + 2\alpha}}{\alpha}\right)\left(B + \frac{\gamma + \sqrt{\gamma^2 + 2\alpha}}{\alpha}\right) . \end{aligned}$$

If we now introduce a and b so that $a = \frac{-\gamma + \sqrt{\gamma^2 + 2\alpha}}{\alpha}$ and $b = \frac{\gamma + \sqrt{\gamma^2 + 2\alpha}}{\alpha}$, we can write the above as $(B - a)(B + b)$. Before continuing from the definitions of a and b above we have the following simple relationships (which will be used later) for expressions involving a and b

$$ab = \frac{2}{\alpha} \tag{196}$$

$$-a + b = \frac{2\gamma}{\alpha} \tag{197}$$

$$a + b = \frac{2\sqrt{\gamma^2 + 2\alpha}}{\alpha} = \frac{2\psi_1}{\alpha} \,, \tag{198}$$

where we have defined ψ_1 as $\psi_1 = \sqrt{\gamma^2 + 2\alpha}$. Back to the main development, using the above quadratic factorization we have the differential equation we want to solve written as

$$\frac{dB}{dt} = \frac{1}{2}\alpha(B - a)(B + b)\,,$$

or

$$\frac{dB}{(B - a)(B + b)} = \frac{1}{2}\alpha dt\,.$$

Integrating both sides from T (where $B(T; T) = 0$) to t (where $B = B(t; T)$) we have

$$\int_0^{B(t;T)} \frac{dB'}{(B' - a)(B' + b)} = \frac{1}{2}\alpha \int_T^t dt = \frac{1}{2}\alpha(t - T)\,.$$

To evaluate the left-hand-side of this expression we performing a partial fraction decomposition of the given fraction. We find

$$\frac{1}{(B' - a)(B' + b)} = \frac{1}{a + b}\left(\frac{1}{B' - a}\right) - \frac{1}{a + b}\left(\frac{1}{B' + b}\right)\,.$$

Using this we can evaluate the integral over B' to get

$$\begin{aligned}\int_0^{B(t;T)} \frac{dB'}{(B' - a)(B' + b)} &= \frac{1}{a + b}\ln\left(\frac{B - a}{-a}\right) - \frac{1}{a + b}\ln\left(\frac{B + b}{b}\right) \\ &= \frac{1}{a + b}\ln\left(\frac{b(a - B)}{a(B + b)}\right)\,.\end{aligned}$$

Setting this equal to $\frac{1}{2}\alpha(t - T)$ we have

$$\ln\left(\frac{b(a - B)}{a(B + b)}\right) = \frac{a + b}{2}\alpha(t - T)\,.$$

Using Equation 198 we see that $\left(\frac{a+b}{2}\right)\alpha = \psi_1$. Thus we have

$$\ln\left(\frac{B + b}{a - B}\right) = \psi_1(T - t) + \ln\left(\frac{b}{a}\right)\,,$$

or

$$\frac{B + b}{a - B} = \frac{b}{a}e^{\psi_1(T - t)}\,,$$

or

$$B + b = \left(\frac{b}{a}\right) e^{\psi_1(T-t)}(a - B) = be^{\psi_1(T-t)} - \frac{b}{a}e^{\psi_1(T-t)}B\,,$$

or finally solving for B we get

$$B = \frac{b(e^{\psi_1(T-t)} - 1)}{\left(\frac{b}{a}e^{\psi_1(T-t)} + 1\right)} = \frac{e^{\psi_1(T-t)} - 1}{\frac{1}{a}(e^{\psi_1(T-t)} - 1) + \frac{1}{a} + \frac{1}{b}}\,.$$

To simplify this further first consider

$$\begin{aligned}
\frac{1}{a} &= \frac{\alpha}{-\gamma + \sqrt{\gamma^2 + 2\alpha}} \left(\frac{-\gamma - \sqrt{\gamma^2 + 2\alpha}}{-\gamma - \sqrt{\gamma^2 + 2\alpha}}\right) \\
&= \frac{\alpha(-\gamma - \sqrt{\gamma^2 + 2\alpha})}{\gamma^2 - (\gamma^2 + 2\alpha)} = \frac{\gamma + \sqrt{\gamma^2 + 2\alpha}}{2} = \frac{\gamma + \psi_1}{2} = \frac{b\alpha}{2}\,,
\end{aligned} \tag{199}$$

and second from Equations 196 and 198 that

$$\frac{1}{a} + \frac{1}{b} = \frac{a + b}{ab} = \frac{2\psi_1}{\alpha}\left(\frac{\alpha}{2}\right) = \psi_1\,.$$

Thus using these two expressions we get

$$B(t;T) = \frac{2(e^{\psi_1(T-t)} - 1)}{(\gamma + \psi_1)(e^{\psi_1(T-t)} - 1) + 2\psi_1}\,, \tag{200}$$

the expression in the book for $B(t;T)$. To find the solution for $A(t;T)$ we using the time differential equation for A and B to derive

$$\begin{aligned}
\frac{dA}{dB} &= \frac{\eta B - \frac{1}{2}\beta B^2}{\frac{1}{2}\alpha B^2 + \gamma B - 1} = -\frac{\beta}{\alpha}\left(\frac{B^2 - \frac{2\eta}{\beta}B}{B^2 + \frac{2\gamma}{\alpha}B - \frac{2}{\alpha}}\right) \\
&= -\frac{\beta}{\alpha}\left(\frac{B^2 + \frac{2\gamma}{\alpha}B - \frac{2}{\alpha} - \frac{2\gamma}{\alpha}B + \frac{2}{\alpha} - \frac{2\eta}{\beta}B}{B^2 + \frac{2\gamma}{\alpha}B - \frac{2}{\alpha}}\right) \\
&= -\frac{\beta}{\alpha}\left(1 - \frac{2\left(\frac{\gamma}{\alpha} + \frac{\eta}{\beta}\right)B - \frac{2}{\alpha}}{B^2 + \frac{2\gamma}{\alpha}B - \frac{2}{\alpha}}\right)\,.
\end{aligned}$$

In the above we have added and subtracted the same quantity in the numerator so that we have a proper rational function of B. We next factor the denominator in the fraction above as we have done earlier, to get

$$\frac{dA}{dB} = -\frac{\beta}{\alpha}\left(1 - \frac{2\left(\frac{\gamma}{\alpha}+\frac{\eta}{\beta}\right)B - \frac{2}{\alpha}}{(B-a)(B+b)}\right) .$$

To integrate the right-hand-side of this expression we will need to apply partial fractions to the fraction that remains. That is we seek coefficients $\mathcal{A}$ and $\mathcal{B}$ such that

$$\frac{2\left(\frac{\gamma}{\alpha}+\frac{\eta}{\beta}\right)B - \frac{2}{\alpha}}{(B-a)(B+b)} = \mathcal{A}\left(\frac{1}{B-a}\right) + \mathcal{B}\left(\frac{1}{B+b}\right) .$$

To find $\mathcal{A}$, multiply both sides of the above by $B-a$ and let $B=a$ to get

$$\mathcal{A} = \frac{1}{a+b}\left(2\left(\frac{\gamma}{\alpha}+\frac{\eta}{\beta}\right)a - \frac{2}{\alpha}\right) = \frac{2}{a+b}\left(\left(\frac{\gamma}{\alpha}+\frac{\eta}{\beta}\right)a - \frac{1}{\alpha}\right) .$$

To find $\mathcal{B}$ multiply both sides by $B+b$ and let $B=-b$ to get

$$\mathcal{B} = \frac{1}{-(a+b)}\left(2\left(\frac{\gamma}{\alpha}+\frac{\eta}{\beta}\right)(-b) - \frac{2}{\alpha}\right) = \frac{2}{a+b}\left(\left(\frac{\gamma}{\alpha}+\frac{\eta}{\beta}\right)b + \frac{1}{\alpha}\right) .$$

Thus we have shown that

$$\frac{dA}{dB} = -\frac{\beta}{\alpha}\left(1 - \frac{\mathcal{A}}{B-a} - \frac{\mathcal{B}}{B+b}\right) ,$$

which we can integrate from T to t (since $A(T;T)=0$) we get

$$A = -\frac{\beta}{\alpha}B + \frac{\beta}{\alpha}\mathcal{A}\ln\left(\frac{B-a}{-a}\right) + \frac{\beta}{\alpha}\mathcal{B}\ln\left(\frac{B+b}{b}\right) .$$

We have almost shown the result in the book. To fully derive that result consider the following manipulations of the coefficient of $\ln\left(\frac{a-B}{a}\right)$.

$$\begin{aligned}
\frac{\beta}{\alpha}\mathcal{A} &= \frac{2\beta}{(a+b)\alpha}\left(\left(\frac{\gamma}{\alpha}+\frac{\eta}{\beta}\right)a - \frac{1}{\alpha}\right) = \frac{2}{\alpha}\frac{a}{(a+b)}\left(\eta - \frac{\beta}{\alpha}\left(\frac{1}{a}-\gamma\right)\right) \\
&= \frac{2}{\alpha}\frac{a}{(a+b)}\left(\eta - \frac{a\beta}{2}\left[\left(\frac{2}{a\alpha}\right)\left(\frac{1}{a}-\gamma\right)\right]\right) .
\end{aligned}$$

Consider now the expression $\frac{2}{a\alpha}\left(\frac{1}{a}-\gamma\right)$. Using Equation 199 we can write $\frac{1}{a}$ as $\frac{\gamma+\psi_1}{2}$ to get

$$\frac{2}{a\alpha}\left(\frac{1}{a}-\gamma\right)=\frac{2}{a\alpha}\left(\frac{-\gamma+\psi_1}{2}\right)=\frac{-\gamma+\psi_1}{a\alpha}\,.$$

Since $a=\frac{\gamma+\psi_1}{2}$ the above expression evaluates to 1. Thus we have shown that

$$\frac{\beta}{\alpha}\mathcal{A}=\frac{2}{\alpha}\left(\frac{a}{a+b}\right)\left(\eta-\frac{a\beta}{2}\right)=\frac{2}{\alpha}a\psi_2\,,$$

where we have defined ψ_2 as

$$\psi_2=\frac{\eta-a\beta/2}{a+b}\,. \tag{201}$$

Next we consider the coefficient of $\ln\left(\frac{b+B}{b}\right)$ given by

$$\begin{aligned}\frac{\beta}{\alpha}\mathcal{B} &= \frac{2\beta}{(a+b)\alpha}\left(\left(\frac{\gamma}{\alpha}+\frac{\eta}{\beta}\right)b+\frac{1}{\alpha}\right)\\ &= \frac{2}{\alpha}b\frac{\beta}{a+b}\left(\frac{\gamma}{\alpha}+\frac{\eta}{\beta}+\frac{1}{b\alpha}\right)\,.\end{aligned}$$

Introduce ψ_2 into this last expression by solving Equation 201 for η to get that

$$\frac{\eta}{\beta}=\frac{a+b}{\beta}\psi_2+\frac{a}{2}\,.$$

Thus we get that

$$\begin{aligned}\frac{\beta}{\alpha}\mathcal{B} &= \frac{2}{\alpha}b\frac{\beta}{a+b}\left(\frac{\gamma}{\alpha}+\frac{a+b}{\beta}\psi_2+\frac{a}{2}+\frac{1}{b\alpha}\right)\\ &= \frac{2}{\alpha}b\frac{\beta}{a+b}\left(\psi_2+\frac{\beta}{a+b}\left(\frac{\gamma}{\alpha}+\frac{a}{2}+\frac{1}{b\alpha}\right)\right)\,.\end{aligned}$$

Next we use Equation 199 we can obtain that $\frac{1}{b}=\frac{\alpha a}{2}$, and thus $\frac{1}{b\alpha}=\frac{a}{2}$, and

$$\frac{a}{2}+\frac{1}{b\alpha}=a\,.$$

After this we now have

$$\frac{\beta}{\alpha}\mathcal{B}=\frac{2}{\alpha}b\left(\psi_2+\frac{\beta}{a+b}\left(\frac{\gamma}{\alpha}+a\right)\right)\,.$$

We find that the inner most expression is given by

$$\frac{\gamma}{\alpha} + a = \frac{\gamma}{\alpha} + \frac{-\gamma + \psi_1}{\alpha} = \frac{\psi_1}{\alpha},$$

and

$$a + b = \frac{2\psi_1}{\alpha},$$

so that

$$\frac{1}{a+b}\left(\frac{\gamma}{a} + a\right) = \frac{\alpha}{2\psi_1}\frac{\psi_1}{\alpha} = \frac{1}{2}.$$

Thus we have shown that

$$\frac{\beta}{\alpha}\mathcal{B} = \frac{2}{\alpha}b\left(\psi_2 + \frac{\beta}{2}\right).$$

Combining many of these previous results we finally get the desired expression for $A(t;T)$ in terms of $B = B(t;T)$

$$A(t;T) = -\frac{\beta}{\alpha}B + \frac{2}{\alpha}a\ln\left(\frac{a-B}{a}\right) + \frac{2}{\alpha}b\left(\psi_2 + \frac{\beta}{2}\right)\ln\left(\frac{b+B}{b}\right), \quad (202)$$

which is the books equation 17.22.

We now seek to determine the long time to maturity asymptotics of the yield curve. From the definition of the yield curve

$$Y = \frac{-A(t;T) + rB(t;T)}{T-t} = \frac{-A(\tau)}{\tau} + \frac{rB(\tau)}{\tau}.$$

When $\tau \to \infty$ by using Equation 200 we get $B \to \frac{2}{\gamma+\Psi_1}$ and so the second term $\frac{rB(\tau)}{\tau} \to 0$ in this limit. From the differential equation for A

$$\frac{dA}{dt} = \eta B - \frac{1}{2}\beta B^2,$$

when we consider large τ from the known limit of B as $\tau \to \infty$ the right-hand-side of the above expression has the limit

$$\begin{aligned} \eta B - \frac{1}{2}\beta B^2 &\to \eta\left(\frac{2}{\gamma+\psi_1}\right) - \frac{1}{2}\beta\left(\frac{4}{(\gamma+\psi_1)^2}\right) \\ &= \frac{2}{(\gamma+\psi_1)^2}\left(\eta(\gamma+\psi_1) - \beta\right), \end{aligned}$$

which is independent of t. Thus integrating $\int_T^t$ on both sides of the differential equation for A gives

$$A(t;T) = \frac{2}{(\gamma+\psi_1)^2}\left(\eta(\gamma+\psi_1)-\beta\right)(t-T)\,,$$

which when when negate and divide by $T-t=\tau$ is the asymptotic expression for Y presented in the book.

Notes on the volatility of the yield curve slope

From the form for $V(t;T)$ given in Equation 189 or $V(t;T) = Ze^{A(t;T)-rB(t;T)}$ we have

$$Y(t;T) = \frac{-(A(t;T)-rB(t;T))}{T-t} = \frac{-A(t;T)+rB(t;T)}{T-t}\,.$$

To evaluate the functional form of $A(t;T)$ and $B(t;T)$ for times t near T we do a Taylor series expansion of the functions A and B as

$$\begin{aligned} A(t;T) &= A_1(T-t) + A_2(T-t)^2 + \cdots \\ B(t;T) &= B_1(T-t) + B_2(T-t)^2 + \cdots\,, \end{aligned}$$

where there are no constant terms A_0 and B_0 since $A(T;T) = B(T;T) = 0$. We will put these expressions into Equations 193 and 194 assuming that the parameters α, β, γ, are independent of t while the variable η can be time dependent however. Under the above Taylor series the left and right-hand-side of Equation 194 given by

$$\begin{aligned} \frac{\partial B}{\partial t} &= -B_1 - 2B_2(T-t) - 3B_3(T-t)^2 - 4B_4(T-t)^3 + \cdots \\ \frac{1}{2}\alpha B^2 + \gamma B - 1 &= \frac{1}{2}\alpha(B_1^2(T-t)^2 + 2B_1B_2(T-t)^3 + \cdots) \\ &+ \gamma(B_1(T-t) + B_2(T-t)^2 + \cdots) \\ &- 1\,. \end{aligned}$$

Setting the powers of $T-t$ equal in the expressions for $\frac{\partial B}{\partial t}$ and $\frac{1}{2}\alpha B^2 + \gamma B - 1$ gives the following for B_1, B_2, and B_3

$$\begin{aligned} -B_1 &= -1 \Rightarrow B_1 = 1 \\ -2B_2 &= \gamma B_1 \Rightarrow B_2 = -\frac{\gamma}{2} \\ -3B_3 &= \frac{1}{2}\alpha B_1^2 + \gamma B_2 \Rightarrow B_3 = \frac{1}{6}(\gamma^2 - \alpha)\,. \end{aligned}$$

Next under the above Taylor series approximations for $A(t;T)$ and $B(t;T)$ We find the left and right-hand-side of Equation 193 given by

$$\begin{aligned}
\frac{\partial A}{\partial t} &= -A_1 - 2A_2(T-t) - 3A_3(T-t)^2 - 4A_4(T-t)^3 + \cdots \\
\eta(t)B + \frac{1}{2}\beta B^2 &= \eta(t)(B_1(T-t) + B_2(T-t)^2 + B_3(T-t)^3 + \cdots) \\
&+ \frac{1}{2}\beta(B_1^2(T-t)^2 + 2B_1B_2(T-t)^3 + \cdots)
\end{aligned}$$

Setting powers of $T-t$ equal in the expressions for $\frac{\partial A}{\partial t}$ and $\eta(t)B + \frac{1}{2}\beta B^2$ gives the following for A_1, A_2, and A_3

$$\begin{aligned}
-A_1 &= 0 \Rightarrow A_1 = 0 \\
-2A_2 &= \eta(t)B_1 \Rightarrow A_2 = -\frac{\eta(t)}{2} \\
-3A_3 &= \eta(t)\eta(t)B_2 + \frac{1}{2}\beta B_1^2 \Rightarrow A_3 = \frac{1}{6}(\eta(t)\gamma - \beta) .
\end{aligned}$$

Thus with the above coefficients for $B(t;T)$ and $A(t;T)$ we have

$$B(t;T) = (T-t) - \frac{\gamma}{2}(T-t)^2 + \frac{1}{6}(\gamma^2 - \alpha)(T-t)^3 + \cdots \quad (203)$$

$$A(t;T) = -\frac{\eta(t)}{2}(T-t)^2 + \frac{1}{6}(\eta(t)\gamma - \beta)(T-t)^3 + \cdots . \quad (204)$$

We are now able to approximate the yield curve Y as

$$\begin{aligned}
Y(t;T) &= \frac{-A(t;T) + rB(t;T)}{T-t} \\
&= \frac{\frac{\eta(t)}{2}(T-t)^2 + r(T-t) - \frac{r\gamma}{2}(T-t)^2 + \cdots}{T-t} \\
&= r + \frac{1}{2}\left(\eta(t) - r\gamma\right)(T-t) + \cdots , \quad (205)
\end{aligned}$$

if we assume that when $t \approx T$ we have $\eta(t) \approx \eta(0)$ we have the expression in the book for the yield curve close to maturity. We also see that the slope s of the yield curve $s = \frac{dY}{dt}$ is given by

$$s = \frac{1}{2}(\gamma r - \eta(0)) . \quad (206)$$

From this last equation we see that $ds_t = -\frac{1}{2}\gamma dr_t$ and by multiplying both sides of that expression by dr_t, and taking expectations we get

$$E[ds_t dr_t] = -\frac{1}{2}\gamma E[dr_t dr_t] \, ,$$

or

$$\text{cov}(ds, dr) = -\frac{1}{2}\gamma \text{var}(dr) \, ,$$

so solving for γ gives

$$\gamma = -\frac{2\text{cov}(ds, dr)}{\text{var}(dr)} \, .$$

which in words states that γ is given by minus twice the covariance of ds and dr divided by the variance of dr.

Notes on the whole yield curve

We will now derive an integral equation for $\eta(\cdot)$ so that the our resulting model will fit the terms structure exactly. To do that we integrate $\frac{\partial A}{\partial t} = \eta(t)B + \frac{1}{2}\beta B^2$ from t to T and use $A(T;T) = 0$ we get

$$A(t;T) = \int_t^T \eta(s)B(s;T)ds + \frac{1}{2}\beta \int_t^T B^2(s;T)ds \, , \tag{207}$$

which differs from the book in that the book has the negative of the above. We can also write the function B as $B(s;T) = B(T-s)$ if desired. As discussed in the book we assume we *know* the yield curve $Y^*(T)$ at some point t^*. Knowing the yield curve at t^* is the same as knowing A at t^* since our model assumes that $Y(t^*;T) = \frac{-A+r^*B}{T-t^*}$, so

$$A(t^*;T) = r^*B(T-t^*) - (T-t^*)Y(t^*;T) \, ,$$

which when we put this into the left-hand-side of Equation 207 and solve for the integral $\int_{t^*}^T \eta(s)B(T-s)ds$ gives

$$\begin{aligned} \int_{t^*}^T \eta^*(s)B(T-s)ds &= r^*B(T-t^*) - (T-t^*)Y(t^*;T) \\ &- \frac{1}{2}\beta^* \int_{t^*}^T B^2(T-s)ds \, , \end{aligned} \tag{208}$$

which is the books equation 17.26.

Notes on the extended Vasicek Model of Hull & White

In the extended Vasicek Model of Hull & White we have $\alpha = 0$ and $\beta < 0$. Then from Equation 194 the differential equation for $B(t;T)$ becomes

$$\frac{\partial B}{\partial t} = \gamma^* B - 1\,,$$

which when coupled with the final condition of $B(T;T) = 0$ results in the solution for B given by

$$B(t;T) = B(T-t) = \frac{1}{\gamma^*}(1 - e^{-\gamma^*(T-t)})\,. \tag{209}$$

One can also get this result by taking the limiting case where $\alpha \to 0$ of the constant parameter expressions. For example, from the definitions of ψ_1 and $B(t;T)$ above we find

$$\begin{aligned} \psi_1 &= \sqrt{\gamma^2 + 2\alpha} \to \gamma \\ B(t;T) &= \frac{2(e^{\psi_1(T-t)} - 1)}{(\gamma + \psi_1)(e^{\psi_1(T-t)} - 1) + 2\psi_1} \to \frac{2(e^{\gamma(T-t)} - 1)}{2\gamma(e^{\gamma(T-t)} - 1) + 2\gamma} \\ &= \frac{1}{\gamma}(1 - e^{-\gamma(T-t)})\,. \end{aligned}$$

With this expression for $B(t;T)$ using Equation 208 the integral equation for $\eta^*(\cdot)$ becomes

$$\begin{aligned} \frac{1}{\gamma^*}\int_{t^*}^{T} \eta^*(s)(1 - e^{-\gamma^*(T-s)})ds &= \frac{r^*}{\gamma^*}(1 - e^{-\gamma^*(T-t^*)}) - Y^*(T - t^*) \\ &- \frac{1}{2}\beta^* \int_{t^*}^{T} \frac{1}{\gamma^{*2}}(1 - e^{-\gamma^*(T-s)})^2 ds\,, \end{aligned}$$

or multiplying by γ^* to solve for $\int_{t^*}^{T} \eta^*(s)(1 - e^{-\gamma^*(T-s)})ds$ we have

$$\int_{t^*}^{T} \eta^*(s)(1 - e^{-\gamma^*(T-s)})ds = \gamma^* F^*(T)\,, \tag{210}$$

where the function $F^*(T)$ is defined as

$$F^*(T) = \frac{r^*}{\gamma^*}(1 - e^{-\gamma^*(T-t^*)}) - Y^*(T - t^*) - \frac{1}{2}\frac{\beta^*}{\gamma^{*2}}\int_{t^*}^{T}(1 - e^{-\gamma^*(T-s)})^2 ds\,. \tag{211}$$

Note that we can explicitly evaluate this function by evaluating the integration if needed. Taking the derivative with respect to T of both sides of Equation 210 this expression gives

$$\eta^*(T)(1-1) + \int_{t^*}^{T} \eta^*(s)(\gamma^* e^{-\gamma^*(T-s)})ds = \gamma^* F^{*\prime}(T)\,,$$

or

$$\int_{t^*}^{T} \eta^*(s) e^{-\gamma^*(T-s)} ds = F^{*\prime}(T)\,. \tag{212}$$

Taking another derivative with respect to T of this expression gives

$$\eta^*(T) - \gamma^* \int_{t^*}^{T} \eta^*(s) e^{-\gamma^*(T-s)} ds = F^{*\prime\prime}(T)\,. \tag{213}$$

Putting Equation 212 into Equation 213 gives

$$\eta^*(T) - \gamma^* F^{*\prime}(T) = F^{*\prime\prime}(T)\,,$$

or solving for $\eta^*(T)$

$$\eta^*(T) = F^{*\prime\prime}(T) + \gamma^* F^{*\prime}(T)\,. \tag{214}$$

We can use the expression for $F^*(T)$ defined in Equation 211 to evaluate the right-hand-side of the above expression. From Equation 211 we have the derivative with respect to T given by

$$\begin{aligned} F^{*\prime}(T) &= \frac{r^*}{\gamma^*}(\gamma^* e^{-\gamma^*(T-t^*)}) - Y^{*\prime}(T-t^*) - \frac{1}{2}\frac{\beta^*}{\gamma^{*2}} \int_{t^*}^{T} 2(1-e^{-\gamma^*(T-s)})(\gamma^* e^{-\gamma^*(T-s)})ds \\ &= r^* e^{-\gamma^*(T-t^*)} - Y^{*\prime}(T-t^*) - \frac{\beta^*}{\gamma^*} \int_{t^*}^{T} (e^{-\gamma^*(T-s)} - e^{-2\gamma^*(T-s)})ds\,. \end{aligned}$$

The integral above can be performed explicitly. We find

$$\begin{aligned} \int_{t^*}^{T} (e^{-\gamma^*(T-s)} - e^{-2\gamma^*(T-s)})ds &= \left.\frac{e^{-\gamma^*(T-s)}}{\gamma^*}\right|_{t^*}^{T} - \left.\frac{e^{-2\gamma^*(T-s)}}{2\gamma^*}\right|_{t^*}^{T} \\ &= \frac{1}{\gamma^*}\left[\frac{1}{2} - e^{-\gamma^*(T-t^*)} + \frac{1}{2}e^{-2\gamma^*(T-t^*)}\right]\,. \end{aligned}$$

With this expression we can now write $F^{*\prime}(T)$ without any integrals as

$$F^{*\prime}(T) = r^* e^{-\gamma^*(T-t^*)} - Y^{*\prime}(T-t^*) - \frac{\beta^*}{\gamma^{*2}}\left[\frac{1}{2} - e^{-\gamma^*(T-t^*)} + \frac{1}{2}e^{-2\gamma^*(T-t^*)}\right]\,.$$

The second derivative with respect to T of $F^*(T)$ is then

$$F^{*\prime\prime}(T) = -r^*\gamma^* e^{-\gamma^*(T-t^*)} - Y^{*\prime\prime}(T-t^*) - \frac{\beta^*}{\gamma^*}(e^{-\gamma^*(T-t^*)} - e^{-2\gamma^*(T-t^*)}) .$$

Using these two expressions in Equation 214 we find that $\eta^*(T)$ is given by

$$\eta^*(T) = -Y^{*\prime\prime}(T-t^*) - \gamma^* Y^{*\prime}(T-t^*) - \frac{\beta^*}{2\gamma^*}\left[1 - e^{-\gamma^*(T-t^*)}\right] . \quad (215)$$

when we cancel common terms. The above expression agrees with a similar one in [9]. Note that the expression in the book has a term proportional to $T - t^*$ which I think is a typo. We next use Equation 193 to evaluate $A(t;T)$. That equation in this case is given by

$$\frac{\partial A}{\partial t} = \eta^*(t)B - \frac{\beta^*}{2}B^2 .$$

Integrating this from T to t since $A(T;T)$ is zero we get

$$A(t;T) = -\int_t^T \eta^*(s)B(s;T)ds - \frac{\beta^*}{2}\int_T^t B(s;T)^2 ds .$$

Since for this model we know $B(t;T)$ from Equation 209 we have

$$B(s;T)^2 = \frac{1}{\gamma^{*2}}(1 - 2e^{-\gamma^*(T-s)} + e^{-2\gamma^*(T-s)}) ,$$

This second integral in $A(t;T)$ is therefore proportional to

$$\begin{aligned}
\int_T^t B(s;T)^2 ds &= \frac{1}{\gamma^{*2}}\left[(t-T) - \frac{2}{\gamma^*}\, e^{-\gamma^*(T-s)}\Big|_T^t + \frac{1}{2\gamma^*}\, e^{-2\gamma^*(T-s)}\Big|_T^t\right] \\
&= \frac{1}{\gamma^{*2}}\left[-(T-t) - \frac{2}{\gamma^*}(e^{-\gamma^*(T-t)} - 1) + \frac{1}{2\gamma^*}(e^{-2\gamma^*(T-t)} - 1)\right] \\
&= \frac{1}{\gamma^{*2}}\left[-(T-t) - \frac{2}{\gamma^*}e^{-\gamma^*(T-t)} + \frac{1}{2\gamma^*}e^{-2\gamma^*(T-t)} + \frac{3}{2\gamma^*}\right] .
\end{aligned}$$

Thus at this point for $A(t;T)$ we have

$$\begin{aligned}
A(t;T) &= -\int_t^T \eta^*(s)B(s;T)ds \\
&+ \frac{\beta^*}{2\gamma^{*2}}\left(T - t + \frac{2}{\gamma^*}e^{-\gamma^*(T-t)} - \frac{1}{2\gamma^*}e^{-2\gamma^*(T-t)} - \frac{3}{2\gamma^*}\right) . \quad (216)
\end{aligned}$$

We will now need to evaluate (the negative of)

$$\mathcal{I} \equiv \int_t^T \eta^*(s)B(s;T)ds\,,$$

We will use the form of $\eta^*(t)$ given by Equation 215 and keep most of the expressions directly in terms of $B(t;T)$. When we use Equation 215 we see that this integral is really three terms

$$\begin{aligned}
\mathcal{I} &= -\int_t^T Y^{*\prime\prime}(s-t^*)B(s;T)ds \\
&- \gamma^* \int_t^T Y^{*\prime}(s-t^*)B(s;T)ds \\
&- \frac{\beta^*}{2\gamma^*}\int_t^T (1-e^{-2\gamma^*(s-t^*)})B(s;T)ds\,.
\end{aligned}$$

Lets denote these three terms by $\mathcal{I}_1$, $\mathcal{I}_2$, and $\mathcal{I}_3$. Lets integrate the first term, $\mathcal{I}_1$, by parts where we find

$$\begin{aligned}
\mathcal{I}_1 &\equiv -\int_t^T Y^{*\prime\prime}(s-t^*)B(s;T)ds \\
&= -B(s;T)Y^{*\prime}(s-t^*)\Big|_t^T + \int_t^T Y^{*\prime}(s-t^*)\frac{\partial}{\partial s}B(s;T)ds\,.
\end{aligned}$$

Now since $B(T;T)=0$ the first term in the above expression becomes

$$B(t;T)Y^{*\prime}(t-t^*)\,.$$

To evaluate the second expression note that due to the functional form of $B(t;T)$ (an exponential) we can easily take its derivative and relate it back to the functional form of $B(t;T)$ itself. For example we have

$$\begin{aligned}
\frac{\partial}{\partial s}B(s;T) &= \frac{\partial}{\partial s}\left(\frac{1}{\gamma^*}(1-e^{-\gamma^*(T-s)})\right) \\
&= -e^{-\gamma^*(T-s)} = \gamma^* B(s;T) - 1\,,
\end{aligned}$$

which we now use in the second integral to get two terms

$$\gamma^*\int_t^T Y^{*\prime}(s-t^*)B(s;T)ds - \int_t^T Y^{*\prime}(s-t^*)ds\,.$$

The first of these two terms exactly *cancels* the integral term $\mathcal{I}_2$ above while the second term is easily integrated. Thus, due to this cancellation, at this point we have for $\mathcal{I}_1 + \mathcal{I}_2$ the following

$$\mathcal{I}_1 + \mathcal{I}_2 = B(t;T)Y^{*\prime}(t-t^*) - Y^*(T-t^*) + Y^*(t-t^*)\,.$$

We now need to evaluate $\mathcal{I}_3$. In great detail, we find

$$\begin{aligned}
\mathcal{I}_3 &= \frac{\beta^*}{2\gamma^{*2}} \int_t^T (1 - e^{-2\gamma^*(s-t^*)})(1 - e^{-\gamma^*(T-s)})ds \\
&= \frac{\beta^*}{2\gamma^2} \int_t^T (1 - e^{-\gamma^*(T-s)} - e^{-2\gamma^*(s-t^*)} + e^{-\gamma^*(s+T-2t^*)})ds \\
&= \frac{\beta^*}{2\gamma^2} \left[T - t - \frac{1}{\gamma^*}\, e^{-\gamma^*(T-s)}\Big|_t^T + \frac{1}{2\gamma^*}\, e^{-2\gamma^*(s-t^*)}\Big|_t^T - \frac{1}{\gamma^*}\, e^{-\gamma^*(s+T-2t^*)}\Big|_t^T \right],
\end{aligned}$$

which evaluate to (dropping the coefficient $\frac{\beta^*}{2\gamma^2}$)

$$T-t-\frac{1}{\gamma^*}(1-e^{-\gamma^*(T-t)})+\frac{1}{2\gamma^*}(e^{-2\gamma^*(T-t^*)}-e^{-2\gamma^*(t-t^*)})-\frac{1}{\gamma^*}(e^{-\gamma^*(2T-2t^*)}-e^{-\gamma^*(t+T+2t^*)})\,.$$

With this expression we have now completely evaluated $\mathcal{I}$. To get the full expression for $A(t;T)$ we need to negate $\mathcal{I}$ and add it to the appropriate part of $A(t;T)$ namely the exponential terms. The sum we need to evaluate then is

$$\begin{aligned}
&- \frac{\beta^*}{2\gamma^2} \left[T - t - \frac{1}{\gamma^*}(1 - e^{-\gamma^*(T-t)}) + \frac{1}{2\gamma^*}(e^{-2\gamma^*(T-t^*)} - e^{-2\gamma^*(t-t^*)}) \right. \\
&- \left. \frac{1}{\gamma^*}(e^{-\gamma^*(2T-2t^*)} - e^{-\gamma^*(t+T+2t^*)}) \right] \\
&+ \frac{\beta^*}{2\gamma^2} \left[T - t + \frac{2}{\gamma^*} e^{-\gamma^*(T-t)} - \frac{1}{2\gamma^*} e^{-2\gamma^*(T-t)} - \frac{3}{2\gamma^*} \right] \\
&= -\frac{\beta^*}{4\gamma^{*3}} \left[1 - 2e^{-\gamma^*(T-t)} - e^{-2\gamma^*(T-t^*)} \right. \\
&+ \left. e^{-2\gamma^*(T-t)} - e^{-2\gamma^*(t-t^*)} + 2e^{-\gamma^*(t+T-2t^*)} \right]. \qquad (217)
\end{aligned}$$

This expression can be factored without the leading factor of $-\frac{\beta^*}{4\gamma^{*3}}$ into an expression of the following form

$$\left(e^{-\gamma^*(T-t^*)} - e^{-\gamma^*(t-t^*)}\right)^2 (e^{2\gamma^*(t-t^*)} - 1)\,.$$

To show this we expand the above to get

$$(e^{-2\gamma^*(T-t^*)} - 2e^{-\gamma^*(T+t-2t^*)} + e^{-2\gamma^*(t-t^*)})(e^{2\gamma^*(t-t^*)} - 1)\,,$$

or

$$e^{-2\gamma^*(T-t)} - 2e^{-\gamma^*(T-t)} + 1 - e^{-2\gamma^*(T-t^*)} - e^{-2\gamma^*(t-t^*)} + 2e^{-\gamma^*(T+t-2t^*)}\,.$$

This later expression matches term for term the expression in Equation 217, thus remembering to negate the expression $\mathcal{I}_1 + \mathcal{I}_2$ we have found that under the extended Vasicek model of Hull & White that the expression for $A(t;T)$ is given by

$$\begin{aligned} A(t;T) &= Y^*(T-t^*) - Y^*(t-t^*) - B(t;T)Y^{*\prime}(t-t^*) \\ &- \frac{\beta^*}{4\gamma^{*3}}\left(e^{-\gamma^*(T-t^*)} - e^{-\gamma^*(t-t^*)}\right)^2 (e^{2\gamma^*(t-t^*} - 1)\,. \end{aligned} \quad (218)$$

Exercise Solutions

Exercise 1 (the expression for $A(t;T)$)

See the notes on Page 213 specifically Equation 218 for this derivation.

Exercise 2 (the local analysis of the bond pricing equation)

Warning: I was not sure how to do this problem. If anyone knows please contact me.

Exercise 3 (the bond pricing equation with coupons)

See the notes on the section "the bond pricing equation" on Page 199 that end with Equation 183.

Exercise 4 (swap pricing with interest payments at discrete times)

In this case the coupon function, $K(r,t)$, which in a swap contract was $r - r^*$ will become a sum of delta functions at the appropriate discrete times, and thus becomes

$$K(r,t) = (r - r^*)\sum_i \delta(t - t_i)\,.$$

For a single swap exchange as we have seen before these discrete payments will give jump conditions on V of the form

$$V(r, t_i^-) = V(r, t_i^+) + (r - r^*) \,.$$

Exercise 6 (Taylor series of the zero-coupon bond about T)

As suggested by the book for times t close to T let V be

$$V(r,t;T) = a(r) + b(r)(T-t) + c(r)(T-t)^2 + d(r)(T-t)^3 + \cdots \,, \quad (219)$$

or the Taylor series expansion about the maturity time T of bond with value V. Recall that such a bond has the boundary condition of $V(r,T;T) = Z$. Then under the above expansion this boundary condition means that $a(r) = Z$ i.e. $a(\cdot)$ has no r dependence. Then to put the above expression into the zero-coupon stochastic bond pricing equation of

$$\frac{\partial V}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 V}{\partial r^2} + (u - \lambda w)\frac{\partial V}{\partial r} - rV = 0 \,,$$

we need to evaluate several partial derivatives of V. We find

$$\begin{aligned}
\frac{\partial V}{\partial t} &= -b(r) - 2c(r)(T-t) - 3d(r)(T-t)^2 + \cdots \\
\frac{\partial V}{\partial r} &= b'(r)(T-t) + c'(r)(T-t)^2 + d'(r)(T-t)^3 + \cdots \\
\frac{\partial^2 V}{\partial r^2} &= b''(r)(T-t) + c''(r)(T-t)^2 + d''(r)(T-t)^3 + \cdots \,.
\end{aligned}$$

Then we can put these expressions into the zero-coupon bond pricing equation and group terms by powers of $T-t$ to find

$$\begin{aligned}
& (-b(r) - rZ) \\
+ \; & (-2c(r) + \frac{1}{2} w^2 b''(r) + (u - \lambda w) b'(r) - r b(r))(T-t) \\
+ \; & (-3d(r) + \frac{1}{2} w^2 c''(r) + (u - \lambda w) c'(r) - r c(r))(T-t)^2 + \cdots = 0 \,.
\end{aligned}$$

Thus we see that $b(r) = -Zr$. The equation for the $T-t$ power then gives since $b'(r) = -Z$ and $b''(r) = 0$

$$-2c(r) - (u - \lambda w)Z + Zr^2 = 0 \,,$$

or

$$c(r) = \frac{Z}{2}(r^2 - u + \lambda w)\,.$$

Thus we find that near maturity we have

$$V(r,t;T) \approx Z - Zr(T-t) + \frac{Z}{2}(r^2 - u + \lambda w)(T-t)^2 + \cdots\,.$$

Exercise 7 (Taylor series of an interest rate swap about T)

This is the same as in exercise 6 but with a differential equation given by

$$\frac{\partial V}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} + (u - \lambda w)\frac{\partial V}{\partial r} - rV + r - r^* = 0\,, \qquad (220)$$

and a final condition on V of $V(r,T;T) = 0$. In this case the swaps value is $ZV(r,t)$. With the same Taylor series representation for $V(r,t;T)$ as in Equation 219 we first see that $V(r,T;T) = 0$ so that $a(r) = 0$. Then putting the given expression into Equation 220 we get

$$\begin{aligned} & (-b(r) + r - r^*) \\ + \;& (-2c(r) + \frac{1}{2}w^2 b''(r) + (u - \lambda w)b'(r) - rb(r))(T-t) \\ + \;& (-3d(r) + \frac{1}{2}w^2 c''(r) + (u - \lambda w)c'(r) - rc(r))(T-t)^2 + \cdots = 0\,. \end{aligned}$$

Thus we see that $b(r) = r - r^*$. The equation for the $T - t$ power then gives since so that $b'(r) = 1$ and $b''(r) = 0$

$$-2c(r) + u - \lambda w - r(r - r^*) = 0\,,$$

or

$$c(r) = \frac{1}{2}(u - \lambda w - r(r - r^*))\,.$$

Thus we find that $V(r,t;T)$ is given by

$$V(r,t;T) = (r - r^*)(T-t) + \frac{1}{2}(u - \lambda w - r(r - r^*))(T-t)^2 + \cdots\,,$$

for times near maturity.

Exercise 11 (the Cox-Ingersoll and Ross interest rate model)

When $u - \lambda w = ar^2$ and $w = br^{3/2}$ then Equation 183 is

$$\frac{\partial V}{\partial t} + \frac{b^2}{2} r^3 \frac{\partial^2 V}{\partial r^2} + ar^2 \frac{\partial V}{\partial r} - rV = 0\,,$$

for the zero-coupon bond pricing equation.

Chapter 18 (Convertible Bonds)

Additional Notes on the Text

Convertible Bonds with Random Interest Rate

In this section of these notes we derive the **convertible bond pricing equation**. Since *both* the stock of value S and the interest rate r are random we need two instruments to hedge our convertible bond with. Thus we consider a portfolio long one convertible bond (with a maturity date T_1), short Δ_2 convertible bonds with maturity date T_2, and short Δ_1 shares of stock. This portfolio will have values given by

$$\Pi = V_1 - \Delta_2 V_2 - \Delta_1 S \,,$$

where V_1 and V_2 are *both* convertible bonds. Then using Ito's lemma for the *two* stochastic parameters r and S we have that Π changes as

$$\begin{aligned} d\Pi &= \frac{\partial V_1}{\partial t} dt + \frac{\partial V_1}{\partial S} dS + \frac{\partial V_1}{\partial r} dr \\ &+ \frac{1}{2}\left(\sigma^2 S^2 \frac{\partial^2 V_1}{\partial S^2} + 2\rho\sigma S w \frac{\partial V_1}{\partial S \partial r} + w^2 \frac{\partial^2 V_1}{\partial r^2}\right) dt \\ &- \Delta_2 \left(\frac{\partial V_2}{\partial t} dt + \frac{\partial V_2}{\partial S} dS + \frac{\partial V_2}{\partial r} dr + \frac{1}{2}\left(\sigma^2 S^2 \frac{\partial^2 V_2}{\partial S^2} + 2\rho\sigma S w \frac{\partial V_2}{\partial S \partial r} + w^2 \frac{\partial^2 V_2}{\partial r^2}\right) dt\right) \\ &- \Delta_1 dS \,. \end{aligned}$$

Group everything by dt, dS, and dr to get that $d\Pi$ looks like

$$\begin{aligned} d\Pi &= \left[\frac{\partial V_1}{\partial t} + \frac{1}{2}\left(\sigma^2 S^2 \frac{\partial^2 V_1}{\partial S^2} + 2\rho\sigma S w \frac{\partial V_1}{\partial S \partial r} + w^2 \frac{\partial^2 V_1}{\partial r^2}\right)\right. \\ &- \left.\Delta_2 \left(\frac{\partial V_2}{\partial t} + \frac{1}{2}\left(\sigma^2 S^2 \frac{\partial^2 V_2}{\partial S^2} + 2\rho\sigma S w \frac{\partial V_2}{\partial S \partial r} + w^2 \frac{\partial^2 V_2}{\partial r^2}\right)\right)\right] dt \\ &+ \left[\frac{\partial V_1}{\partial S} - \Delta_2 \frac{\partial V_2}{\partial t} - \Delta_1\right] dS \\ &+ \left[\frac{\partial V_1}{\partial r} - \Delta_2 \frac{\partial V_2}{\partial r}\right] dr \,. \end{aligned}$$

We now pick the hedge values Δ_1 and Δ_2 such that the above portfolio is deterministic. That is the coefficients of dS and dr vanish. From the above we see that the coefficient of dr will be equal to zero if we take

$$\Delta_2 = \frac{\frac{\partial V_1}{\partial r}}{\frac{\partial V_2}{\partial r}} \,,$$

and the coefficient of dS will be zero if we take

$$\Delta_1 = \frac{\partial V_1}{\partial S} - \Delta_2 \frac{\partial V_2}{\partial S} \,.$$

Once we have done this by setting $d\Pi = r\Pi dt$ (to avoid arbitrage) land dividing by dt we get the following equation

$$\begin{aligned} \frac{\partial V_1}{\partial t} &+ \frac{1}{2}\left(\sigma^2 S^2 \frac{\partial^2 V_1}{\partial S^2} + 2\rho\sigma S w \frac{\partial V_1}{\partial S \partial r} + w^2 \frac{\partial^2 V_1}{\partial r^2}\right) \\ &- \Delta_2 \left[\frac{\partial V_2}{\partial t} + \frac{1}{2}\left(\sigma^2 S^2 \frac{\partial^2 V_2}{\partial S^2} + 2\rho\sigma S w \frac{\partial V_2}{\partial S \partial r} + w^2 \frac{\partial^2 V_2}{\partial r^2}\right)\right] \\ &= rV_1 - r\Delta_2 V_2 - r\Delta_1 S \\ &= rV_1 - r\Delta_2 V_2 - rS\frac{\partial V_1}{\partial S} + rS\Delta_2 \frac{\partial V_2}{\partial S} \,. \end{aligned}$$

Putting terms that depend on T_1 on one side and terms that depend on T_2 on another side to get

$$\begin{aligned} \frac{\partial V_1}{\partial t} &+ \frac{1}{2}\left(\sigma^2 S^2 \frac{\partial^2 V_1}{\partial S^2} + 2\rho\sigma S w \frac{\partial V_1}{\partial S \partial r} + w^2 \frac{\partial^2 V_1}{\partial r^2}\right) - rV_1 + rS\frac{\partial V_1}{\partial S} \\ &= \Delta_2 \left[\frac{\partial V_2}{\partial t} + \frac{1}{2}\left(\sigma^2 S^2 \frac{\partial^2 V_2}{\partial S^2} + 2\rho\sigma S w \frac{\partial V_2}{\partial S \partial r} + w^2 \frac{\partial^2 V_2}{\partial r^2}\right) - rV_2 + rS\frac{\partial V_2}{\partial S}\right] \,, \end{aligned}$$

or dividing by $\frac{\partial V_1}{\partial r}$ we get

$$\begin{aligned} &\frac{1}{\frac{\partial V_1}{\partial r}}\left(\frac{\partial V_1}{\partial t} + \frac{1}{2}\left(\sigma^2 S^2 \frac{\partial^2 V_1}{\partial S^2} + 2\rho\sigma S w \frac{\partial V_1}{\partial S \partial r} + w^2 \frac{\partial^2 V_1}{\partial r^2}\right) - rV_1 + rS\frac{\partial V_1}{\partial S}\right) \\ = &\frac{1}{\frac{\partial V_2}{\partial r}}\left(\frac{\partial V_2}{\partial t} + \frac{1}{2}\left(\sigma^2 S^2 \frac{\partial^2 V_2}{\partial S^2} + 2\rho\sigma S w \frac{\partial V_2}{\partial S \partial r} + w^2 \frac{\partial^2 V_2}{\partial r^2}\right) - rV_2 + rS\frac{\partial V_2}{\partial S}\right) \,. \end{aligned}$$

Setting each side equal to a separation constant $a(r, S, t)$ which we can write as $w(r, S, t)\lambda(r, S, t) - u(r, S, t)$ we get for both V_1 and V_2 the following

$$\begin{aligned} \frac{\partial V}{\partial t} &+ \frac{1}{2}\left(\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + 2\rho\sigma S w \frac{\partial V}{\partial S \partial r} + w^2 \frac{\partial^2 V}{\partial r^2}\right) \\ &- rV + rS\frac{\partial V}{\partial S} + (u - \lambda w)\frac{\partial V}{\partial r} = 0 \,, \end{aligned} \tag{221}$$

which is the convertible bond pricing equation.

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