Notes On the Book:
Paul Wilmott on Quantitative Finance
by Paul Wilmott

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Introduction

Here you'll find some notes that I wrote up as I worked through this excellent book. I've worked hard to make these notes as good as I can, but I have no illusions that they are perfect. If you feel that that there is a better way to accomplish or explain an exercise or derivation presented in these notes; or that one or more of the explanations is unclear, incomplete, or misleading, please tell me. If you find an error of any kind – technical, grammatical, typographical, whatever – please tell me that, too. I'll gladly add to the acknowledgments in later printings the name of the first person to bring each problem to my attention.

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The Random Behavior of Assets

Notes on time-scaling returns

When we have prices sampled at fixed times, S_i, the discrete model proposed for their returns R_i is

$$R_i = \frac{S_{i+1} - S_i}{S_i} = \text{mean} + \text{standard deviation} \times \phi \,, \tag{1}$$

where ϕ is a random draw from a standard Gaussian distribution (mean zero and variance one). We will call these mean and standard deviation estimates the *measured* estimates, since they explicitly depend on using the measured prices S_i for their estimation. They also correspond to a mean and standard deviation of the returns over the length of time represented by the amount of time between the prices S_{i+1} and S_i or $t_{i+1} - t_i$. The question we then pose is: given the measured parameters mean and standard deviation how do we modify these in the case we are interested in the mean return and the standard deviation or the return for timescales *different* than the measurement time scales. Let μ and σ be the numerical values of these quantities for returns over the *desired* timestep length, which we will take to be $T_{i+1} - T_i$. To express this difference in timescales the book defines δt which is given by

$$\delta t \equiv \frac{t_{i+1} - t_i}{T_{i+1} - T_i} \,. \tag{2}$$

Note that the above fraction must be *dimensionless*, that is if the measurement timescales is in days $t_{i+1} - t_i = 1\,\text{day}$ and the desired timescale is in years $T_{i+1} - T_i = 1\,\text{year}$, then the value of δt should be

$$\delta t = \frac{1\,\text{day}}{1\,\text{year}} = \frac{1\,\text{day}}{252\,\text{day}} \,,$$

since there are 252 trading days in one year. Thus if the desired timescale is over longer a longer amount of time (where $T_{i+1} - T_i > t_{i+1} - t_i$) we expect $\delta t < 1$ and if it is over a shorter amount of time we expect $\delta t > 1$. Given this definition then we have that the mean return we want μ is given by

$$\mu = \text{mean} \left(\frac{T_{i+1} - T_i}{t_{i+1} - t_i} \right) = \frac{\text{mean}}{\delta t} \,. \tag{3}$$

This is the expression that shows how we scale the mean returns from one timescale to another. The standard deviation is scaled in a similar manner. If σ is the standard deviation over the timescale of interest we have

$$\sigma = \frac{\text{standard deviation}}{\delta t^{1/2}} \,. \tag{4}$$

This is the expression that shows how we scale the standard deviation of returns from one timescale to another. Two simple examples will make this clear. Using the data from the book we assume that we measure daily returns (using daily prices) and have

$$\text{mean} = 0.002916$$
$$\text{standard deviation} = 0.024521 \,.$$

Then we want to compute

- The yearly mean and standard deviation of returns. In that case, as we talked about above we have
$$\delta t = \frac{1}{252} = 0.00396\,.$$
Thus using Equations 3 and 4 we get
$$\mu = 0.7348 \quad \text{and} \quad \sigma = 0.389242\,.$$

- The hourly mean and standard deviation of returns. In that case we have
$$\delta t = \frac{1\,\text{day}}{1\,\text{hour}} = \frac{6.5\,\text{hours}}{1\,\text{hours}} = 6.5\,.$$
Then using Equations 3 and 4 we get
$$\mu = 0.0004486 \quad \text{and} \quad \sigma = 0.009617\,.$$

Typically we measure the returns over a daily timescale and then report yearly values for μ and σ. In that case if we want statistics for returns over a shorter time period (less than a year) then δt is what fraction of the longer time length the short time length is. For example, in going from the annual mean rate of return μ_{yearly} and standard deviation σ_{yearly} to a daily rate of return and uncertainty we scale by the appropriate fraction
$$\begin{aligned}\mu_{\text{daily}} &= \left(\frac{1}{252}\right)\mu_{\text{yearly}}\\ \sigma_{\text{daily}} &= \left(\frac{1}{252}\right)^{1/2}\sigma_{\text{yearly}}\,.\end{aligned}$$
Using these scalings, δt in the return model gives us
$$R_i = \frac{S_{i+1}-S_i}{S_i} = \mu\delta t + \sigma\phi\delta t^{1/2}\,, \tag{5}$$
where ϕ is a draw from a standard normal random variable. Solving for S_{i+1} we get
$$S_{i+1} - S_i = \mu S_i \delta t + \sigma S_i \phi \delta t^{1/2}\,. \tag{6}$$

Notes on exponentially weighted volatility estimation

From the definition of the exponentially weighted estimate of σ_i^2 given by
$$\sigma_i^2 = \left(\frac{1-\lambda}{\delta t}\right)\sum_{j=-\infty}^{i}\lambda^{i-j}R_j^2\,, \tag{7}$$
we can write
$$\begin{aligned}\sigma_i^2 &= \left(\frac{1-\lambda}{\delta t}\right)\lambda\sum_{j=-\infty}^{i}\lambda^{i-1-j}R_j^2 = \left(\frac{1-\lambda}{\delta t}\right)\lambda\left[\sum_{j=-\infty}^{i-1}\lambda^{i-1-j}R_j^2 + \lambda^{-1}R_i^2\right]\\ &= \left(\frac{1-\lambda}{\delta t}\right)\lambda\left[\left(\frac{\delta t}{1-\lambda}\right)\sigma_{i-1}^2 + \lambda^{-1}R_i^2\right]\\ &= \lambda\sigma_{i-1}^2 + \left(\frac{1-\lambda}{\delta t}\right)R_i^2\,,\end{aligned} \tag{8}$$

which is a recursive expression for exponentially weighted volatility estimation. If we compute R_i using daily prices and we want σ to be in units of yearly volatility then $\delta t = \frac{1}{252}$. If you want σ to be an estimate of daily volatility then $\delta t = 1$.

Elementary Stochastic Calculus

Notes on the mean square limit

To evaluate the expectation after we expand the square of $\sum_{j=1}^{n}(X(t_j) - X(t_{j-1}))^2 - t$, we need to count how many terms we have in the double sum

$$\sum_{i=1}^{n}\sum_{j=1}^{i-1}(X(t_i) - X(t_{i-1}))^2(X(t_j) - X(t_{j-1}))^2\,.$$

We can do this simply as

$$\sum_{i=1}^{n}\sum_{i=1}^{i-1}1 = \sum_{i=1}^{n}(i-1) = \sum_{i=1}^{n}i - \sum_{i=1}^{n}1 = \frac{1}{2}n(n+1) - n = \frac{1}{2}n(n-1)\,.$$

Notes on functions of stochastic variables and Ito's lemma

From the definition of a stochastic integral we have an expression like

$$W(t) = \int_0^t f(\tau)dX(\tau) = \lim_{n\to\infty}\sum_{j=1}^{n}f(t_{j-1})(X(t_j) - X(t_{j-1}))\,, \tag{9}$$

where $t_j = j\frac{t}{n}$. When this is expressed as a differential relation we have

$$dW = f(t)dX\,.$$

Thus we expect that sums of differences like $X(t_j) - X(t_{j-1})$ seen in Equation 9 play a prominent role in obtaining differential relationships. Given this observation and the fact that we want to evaluate the derivative of $F(X)$ when X is stochastic variable motivates us to consider the following sum of differences (which we denote $\mathcal{S}$) and where the time points t_j are spaced by $h \equiv \frac{\delta x}{n}$

$$\begin{aligned}
\mathcal{S} &= [F(X(t+h)) - F(X(t))] \\
&+ [F(X(t+2h)) - F(X(t+h))] + \\
&+ [F(X(t+3h)) - F(X(t+2h))] + \\
&\vdots \\
&+ [F(X(t+(n-1)h)) - F(X(t+(n-2)h))] \\
&+ [F(X(t+nh)) - F(X(t+(n-1)h))]\,.
\end{aligned}$$

Note that one way to evaluate $\mathcal{S}$ is to note that since it is a telescoping sum that all of the "middle" terms cancel when summed and we are left with

$$\mathcal{S} = F(X(t+nh)) - F(X(t)) = F(X(t+\delta t)) - F(X(t))\,.$$

Another way to evaluate $\mathcal{S}$ is to use Taylor's series to expand each difference in the function $F(\cdot)$ in terms of a difference in terms of the stochastic variable X (shown here for the first) as

$$F(X(t+h)) - F(X(t)) = (X(t+h) - X(t))\frac{dF(X(t))}{dX} + \frac{1}{2}(X(t+h) - X(t))^2\frac{d^2F(X(t))}{dX^2}\,.$$

Then each difference in $\mathcal{S}$ above turns into the sum of two terms and we get

$$\begin{aligned}
\mathcal{S} &= (X(t+h) - X(t))\frac{dF(X(t))}{dX} \\
&+ \frac{1}{2}(X(t+h) - X(t))^2\frac{d^2F(X(t))}{dX^2} \\
&+ (X(t+2h) - X(t+h))\frac{dF(X(t+h))}{dX} \\
&+ \frac{1}{2}(X(t+2h) - X(t+h))^2\frac{d^2F(X(t+h))}{dX^2} \\
&\vdots \\
&+ (X(t+nh) - X(t+(n-1)h))\frac{dF(X(t+(n-1)h))}{dX} \\
&+ \frac{1}{2}(X(t+nh) - X(t+(n-1)h))^2\frac{d^2F(X(t+(n-1)h))}{dX^2} \\
&= \sum_{j=1}^{n}(X(t+jh) - X(t+(j-1)h))\,\frac{dF(X(t+(j-1)h))}{dX} \qquad (10) \\
&+ \frac{1}{2}\sum_{j=1}^{n}(X(t+jh) - X(t+(j-1)h))^2\frac{d^2F(X(t+(j-1)h))}{dX^2}\,. \qquad (11)
\end{aligned}$$

We consider the two summation terms 10 and 11 above. The first sum above (Equation 10) is a discrete approximation to

$$\int_t^{t+\delta t}\frac{dF}{dX}dX\,.$$

For each term in the second sum above (Equation 11), as argued in the text, we evaluate the second derivatives at the left-most end point $X(t)$, so that it comes out of the summation. In addition, the quadratic sum that remains is a discrete approximation in the mean squared sense of

$$\int_t^{t+\delta t}(dX)^2 = \delta t\,.$$

thus we get

$$F(X(t+\delta t)) - F(X(t)) = \int_t^{t+\delta t}\frac{dF}{dX}(X(\tau))dX(\tau) + \frac{1}{2}\int_t^{t+\delta t}\frac{d^2F}{dX^2}(X(t))d\tau\,.$$

Note that the argument of the first integral is evaluated at τ the variable of integration, while the argument of the second integral is evaluated at the left most end point t and is a constant with respect to the variable of integration τ. If we desire to extend this expression to integration lengths t where we cannot just evaluate $\frac{d^2F}{dX^2}$ at the left-hand end point we need to evaluate this expression at τ rather than t. This gives

$$F(X(t)) - F(X(0)) = \int_0^t\frac{dF}{dX}(X(\tau))dX(\tau) + \frac{1}{2}\int_0^t\frac{d^2F}{dX^2}(X(\tau))d\tau\,. \qquad (12)$$

When we write this using the differential equation shorthand we get

$$dF = \frac{dF}{dX} dX + \frac{1}{2} \frac{d^2 F}{dX^2} dt \,, \tag{13}$$

for the stochastic differential equation satisfied by $F(X)$.

Notes on Ito from Taylor

If we have a variable, say S, that changes according to a stochastic dX *and* a continuous term dt as

$$dS = a(S,t)dt + b(S,t)dX \,. \tag{14}$$

If we have a function of S say $V(S)$ then we can derive the expression for dV using *Taylor series* and the heuristic $dX^2 \sim dt$. Performing a two term Taylor expansion of $V(S)$ we have

$$dV = \frac{dV}{dS} dS + \frac{1}{2} \frac{d^2 V}{dS^2} dS^2 \,.$$

Using the heuristics discussed in the book we find

$$\begin{aligned} dS^2 &= (a(S,t)dt + b(S,t)dX)^2 = a(S,t)^2 dt^2 + 2a(S,t)b(S,t)dtdX + b(S,t)^2 dX^2 \\ &= b(S,t)^2 dt \,. \end{aligned} \tag{15}$$

So that with this dV becomes

$$dV = \frac{dV}{dS} dS + \frac{1}{2} b(S,t)^2 \frac{d^2 V}{dS^2} dt \,. \tag{16}$$

We could replace dS with $adt + bdX$ in the above to get an expression in terms of the Brownian increment dX to get

$$dV = \left(a(S,t) \frac{dV}{dS} + \frac{1}{2} b(S,t)^2 \frac{d^2 V}{dS^2} \right) dt + b(S,t) \frac{dV}{dS} dX \,. \tag{17}$$

As another slight generation if $V = V(S,t)$, so that V depends on the deterministic time t as well as the stochastic term S then by using Taylor's series we get

$$dV = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} dS^2$$

as in Equation 15 we have $dS^2 = b^2 dX^2 = b^2 dt$ so dV above becomes

$$dV = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{1}{2} b(S,t)^2 \frac{\partial^2 V}{\partial S^2} \,. \tag{18}$$

Notes on Ito in Higher Dimensions

Now Taylor's series for a function $V = V(S_1, S_2, t)$ of *two* variables S_1 and S_2 (and time t) would have first derivative terms for S_1, S_2 and t, second derivatives terms for S_1 and S_2 and a *cross* derivative term for S_1 and S_2, given by

$$\begin{aligned} dV &= \frac{\partial V}{\partial t}dt \\ &+ \frac{\partial V}{\partial S_1}dS_1 + \frac{1}{2}\frac{\partial^2 V}{\partial S_1^2}dS_1^2 \\ &+ \frac{\partial V}{\partial S_2}dS_2 + \frac{1}{2}\frac{\partial^2 V}{\partial S_2^2}dS_2^2 \\ &+ \frac{\partial^2 V}{\partial S_1 \partial S_2}dS_1 dS_2 \,. \end{aligned}$$

Now as earlier we consider the heuristics $dS_1^2 = b_1^2 dt$, $dS_2^2 = b_2^2 dt$, for the squares of the random terms and

$$dS_1 dS_2 = (a_1 dt + b_1 dX)(a_2 dt + b_2 dX_2) = b_1 b_2 dX_1 dX_2 = b_1 b_2 \rho dt \,,$$

for the cross product. In that case we thus get for dV the following

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S_1}dS_1 + \frac{\partial V}{\partial S_2}dS_2 + \frac{1}{2}b_1^2\frac{\partial^2 V}{\partial {S_1}^2}dt + b_1 b_2 \rho \frac{\partial^2 V}{\partial S_1 \partial S_2}dt + \frac{1}{2}b_2^2\frac{\partial^2 V}{\partial {S_2}^2}dt \,. \tag{19}$$

Pertinent Examples: The Lognormal Random Walk

In this case the differential equation for S satisfies

$$dS = \mu S dt + \sigma S dX \,, \tag{20}$$

and we will use the heuristics that

$$dS^2 = \sigma^2 S^2 dX^2 = \sigma^2 S^2 dt \,.$$

If we consider a function F defined as $F(S) = \log(S)$ then

$$\frac{dF}{dS} = \frac{1}{S} \quad \text{and} \quad \frac{d^2 F}{dS^2} = -\frac{1}{S^2} \,,$$

so using the "Ito from Taylor" idea to compute the differential of this function gives

$$\begin{aligned} d(\log(S)) &= dF = \frac{dF}{dS}dS + \frac{1}{2}\frac{d^2F}{dS^2}dS^2 \\ &= \frac{1}{S}(\mu S dt + \sigma S dX) + \frac{1}{2}\left(-\frac{1}{S^2}\right)\sigma^2 S^2 dt \\ &= \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dX \,. \end{aligned}$$

From the above expression for $d\log(S)$ we can integrate this to get

$$\log(S(t)) - \log(S(0)) = \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma(X(t) - X(0)) \,,$$

or solving for $S(t)$ we get

$$S(t) = S(0) e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma(X(t) - X(0))} \,. \tag{21}$$

Since $X(t)$ is a Gaussian process we can write $X(t) - X(0) = \phi\sqrt{t}$ where ϕ is a random draw from a $\mathcal{N}(0,1)$ distribution.

Pertinent Examples: A Mean Reverting Random Walk

In this case lets consider the stochastic differential equation for r given by

$$dr = (\nu - \gamma r)dt + \sigma dX \,.$$

If we let $W = r - \xi$ where ξ is not yet determined. We then have that

$$\begin{aligned} dW = dr &= (\nu - \gamma r)dt + \sigma dX = (\nu - \gamma(w + \xi))dt + \sigma dX \\ &= (\nu - \gamma\xi - \gamma W)dt + \sigma dX \,. \end{aligned}$$

Thus if we pick ξ such that $\nu - \gamma\xi = 0$ or $\xi = \frac{\nu}{\gamma}$ then

$$W = r - \frac{\nu}{\gamma} \,,$$

and

$$dW = -\gamma W dt + \sigma dX \,. \tag{22}$$

Note the drift term in the above stochastic differential equation is $-\gamma W dt$ and has a W as a factor while the stochastic term σdX does not have a W factor. This is like a partial lognormal random walk and is called a *Ornstein-Uhlenbeck* process. Let $I = e^{\gamma t}$ and consider $d(IW)$. We have

$$\begin{aligned} d(IW) &= IdW + WdI = e^{\gamma t}(-\gamma W dt + \sigma dX) + \gamma W I dt \\ &= \sigma e^{\gamma t} dX \,. \end{aligned}$$

This we can integrate from 0 to t to get

$$IW(t) - IW(0) = \sigma \int_0^t e^{\gamma s} dX(s) \,.$$

Multiply by $\frac{1}{I} = e^{-\gamma t}$ on both sides to get

$$\begin{aligned} W(t) &= e^{-\gamma t} W(0) + \sigma e^{-\gamma t} \int_0^t e^{\gamma s} dX(s) \\ &= e^{-\gamma t} W(0) + \sigma \int_0^t e^{\gamma(s-t)} dX(s) \,. \end{aligned} \tag{23}$$

We can further simplify this by using integration by parts on the last term to get

$$\begin{aligned}\int_0^t e^{\gamma(s-t)} dX(s) &= e^{\gamma(s-t)} X(s)\big|_{s=0}^{t} - \gamma \int_0^t e^{\gamma(s-t)} X(s) ds \\ &= X(t) - e^{-\gamma t} X(0) - \gamma \int_0^t e^{\gamma(s-t)} X(s) ds\,.\end{aligned}$$

Since we assume that $X(0) = 0$ the second term in the above vanishes. Thus we get for $W(t)$ the following

$$W(t) = e^{-\gamma t} W(0) + \sigma \left(X(t) - \gamma \int_0^t e^{\gamma(s-t)} X(s) ds \right) ,$$

and for $r(t)$ we get

$$r(t) = W(t) + \frac{\nu}{\gamma} = \frac{\nu}{\gamma} + e^{-\gamma t} \left(r(0) + \frac{\nu}{\gamma} \right) + \sigma \left(X(t) - \gamma \int_0^t e^{\gamma(s-t)} X(s) ds \right) .$$

Pertinent Examples: Another Mean Reverting Random Walk

In this case r is governed by

$$dr = (\nu - \mu r) dt + \sigma r^{1/2} dX ,$$

then the heuristics we use is $dr^2 = \sigma^2 r dX^2 = \sigma^2 r dt$. When we consider the function F defined as $F = r^{1/2}$ we have that

$$\frac{dF}{dr} = \frac{1}{2} r^{-1/2} \quad \text{and} \quad \frac{d^2F}{dr^2} = -\frac{1}{4} r^{-3/2} .$$

Thus using Ito from Taylor to compute dF gives us

$$\begin{aligned} d(r^{1/2}) &= dF = \frac{dF}{dr} dr + \frac{1}{2} \frac{d^2F}{dr^2} dr^2 \\ &= \frac{1}{2} r^{-1/2} dr + \frac{1}{2} \left(-\frac{1}{4} r^{-3/2} \right) (\sigma^2 r dt) \\ &= \left(\frac{4\nu - \sigma^2}{8F} - \frac{1}{2} \mu F \right) dt + \frac{1}{2} \sigma dX . \end{aligned}$$

Now the stochastic term is constant (a relatively simple expression) but the coefficient of the drift term is more complicated. This leads us to consider if we can find a function $F(r)$ under the same stochastic differential equation for r such that with the help of Ito's lemma has a zero drift term for dF. This function $F(r)$ would have

$$\begin{aligned} d(r^{1/2}) &= dF = \frac{dF}{dr} dr + \frac{1}{2} \frac{d^2F}{dr^2} dr^2 \\ &= \frac{dF}{dr} ((\nu - \mu r) dt + \sigma r^{1/2} dX) + \frac{1}{2} \frac{d^2F}{dr^2} \sigma^2 r dt \\ &= \left((\nu - \mu r) \frac{dF}{dr} + \frac{1}{2} \sigma^2 r \frac{d^2F}{dr^2} \right) dt + \sigma r^{1/2} \frac{dF}{dr} dX . \end{aligned}$$

Setting the coefficient of dt equal to zero gives

$$(\nu - \mu r)\frac{dF}{dr} + \frac{1}{2}\sigma^2 r \frac{d^2F}{dr^2} = 0\,. \tag{24}$$

If we introduce the function $Y(r)$ defined by $Y(r) = \frac{dF}{dr}$ this equation becomes

$$\frac{dY}{dr} = -\frac{2(\nu - \mu r)}{\sigma^2 r} Y = \left(-\frac{2\nu}{\sigma^2 r} + \frac{2\mu}{\sigma^2}\right) dr\,.$$

Integrating both sides gives

$$\log(Y(r)) = -\frac{2\nu}{\sigma^2}\log(r) + \frac{2\mu}{\sigma^2} r + C\,,$$

for a constant C. Thus solving for $Y(r)$ we get

$$Y(r) = A r^{-\frac{2\nu}{\sigma^2}} e^{\frac{2\mu}{\sigma^2} r} \tag{25}$$

The Black-Scholes Model

Notes on solutions to the Black-Scholes equation

In this section we show that S and e^{rt} satisfy the Black-Scholes equation. The Black-Scholes equation is given by

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0\,. \tag{26}$$

For $V(S,t) = AS$, we can calculated each of the required derivative on the left hand side of this expression as follows

$$\begin{aligned} \frac{\partial V}{\partial t} &= 0 \\ \frac{\partial V}{\partial S} &= A \\ \frac{\partial^2 V}{\partial S^2} &= 0\,. \end{aligned}$$

Thus substituting $V = AS$ into the left hand side of the Black-Scholes equation gives

$$0 + 0 + rSA - rAs = 0\,,$$

showing that $V = AS$ is a solution. We note that this solution represents a pure investment in the underlying. Note that also in this case

$$\Delta = \frac{\partial V}{\partial S} = A\,.$$

In the case when $V = Ae^{rt}$ we again evaluate each derivative in tern and find that

$$\begin{aligned} \frac{\partial V}{\partial t} &= rAe^{rt} \\ \frac{\partial V}{\partial S} &= 0 \\ \frac{\partial^2 V}{\partial S^2} &= 0\,, \end{aligned}$$

so placing $V = Ae^{rt}$ into the left hand side of the Black-Scholes equation we obtain

$$rAe^{rt} - rAe^{rt} = 0\,,$$

proving that $V = Ae^{rt}$ is a solution. This solution represents an investment in a fixed interest rate account like a bank account. Note that when $V = Ae^{rt}$ we have $\Delta = 0$.

Notes on options on futures

Here we will transform the Black-Scholes equation in the original variables (t, S) into the new variables (v, F) defined in terms of the original variables by

$$\begin{aligned} v &= t \\ F &= e^{r(T_F - t)} S \,. \end{aligned}$$

Note that the above transformation has an inverse given by

$$\begin{aligned} t &= v \\ S &= F e^{-r(T_F - t)} = F e^{-r(T_F - v)} \,. \end{aligned}$$

Then the derivatives with respect to (t, S) transform as

$$\begin{aligned} \frac{\partial}{\partial t} &= \frac{\partial F}{\partial t}\frac{\partial}{\partial F} + \frac{\partial v}{\partial t}\frac{\partial}{\partial v} = -re^{r(T_F - t)} S \frac{\partial}{\partial F} + \frac{\partial}{\partial v} \\ &= -Fr\frac{\partial}{\partial F} + \frac{\partial}{\partial v} \\ \frac{\partial}{\partial S} &= \frac{\partial F}{\partial S}\frac{\partial}{\partial F} + \frac{\partial v}{\partial S}\frac{\partial}{\partial v} = e^{r(T_F - t)}\frac{\partial}{\partial F} \\ &= e^{r(T_F - v)}\frac{\partial}{\partial F} \\ \frac{\partial^2}{\partial S^2} &= e^{2r(T_F - v)}\frac{\partial^2}{\partial F^2} \end{aligned}$$

Thus we put these two expressions into the Black-Scholes equation for $\mathcal{V} = V(F, v)$ we get

$$-Fr\frac{\partial \mathcal{V}}{\partial F} + \frac{\partial \mathcal{V}}{\partial v} + \frac{1}{2}(\sigma^2 F^2 e^{-2r(T_F - v)}) e^{2r(T_F - v)} \frac{\partial^2 \mathcal{V}}{\partial F^2} + r(F e^{-r(T_F - v)}) e^{r(T_F - v)} \frac{\partial \mathcal{V}}{\partial F} - r\mathcal{V} = 0 \,.$$

When we cancel terms we get

$$\frac{\partial \mathcal{V}}{\partial v} + \frac{1}{2}\sigma^2 F^2 \frac{\partial^2 \mathcal{V}}{\partial F^2} - r\mathcal{V} = 0 \,,$$

the pricing equation for an option on a future.

Partial Differential Equations

Transformations to a constant coefficient diffusion equation

In this section of these notes we verify the transformations needed to change the Black Scholes equation into the diffusion equation. As suggested in the book lets define unitless parameters x, τ, and U in terms of the given financial parameters T, t, S etc. as

$$\begin{aligned}
S &= e^x \Rightarrow x = \log(S) \\
t &= T - \frac{\tau}{\frac{1}{2}\sigma^2} \Rightarrow \tau = \frac{1}{2}\sigma^2(T-t) \\
V(S,t) &= e^{\alpha x + \beta\tau} U(x,t)\,.
\end{aligned}$$

Then with this transformation the derivatives needed in the Black-Scholes equation become

$$\begin{aligned}
\frac{\partial}{\partial t} &= \frac{\partial \tau}{\partial t}\frac{\partial}{\partial \tau} + \frac{\partial x}{\partial t}\frac{\partial}{\partial x} = -\frac{1}{2}\sigma^2 \frac{\partial}{\partial \tau} \\
\frac{\partial}{\partial S} &= \frac{\partial \tau}{\partial S}\frac{\partial}{\partial \tau} + \frac{\partial x}{\partial S}\frac{\partial}{\partial x} = \frac{1}{S}\frac{\partial}{\partial x} = e^{-x}\frac{\partial}{\partial x} \\
\frac{\partial^2}{\partial S^2} &= e^{-x}\frac{\partial}{\partial x}\left(e^{-x}\frac{\partial}{\partial x}\right) = -e^{-2x}\frac{\partial}{\partial x} + e^{-2x}\frac{\partial^2}{\partial x^2}\,.
\end{aligned}$$

Using these the derivatives of V in the Black-Scholes Equation 26 becomes

$$\begin{aligned}
\frac{\partial V}{\partial t} &= -\frac{1}{2}\sigma^2\frac{\partial}{\partial \tau}(e^{\alpha x+\beta\tau}U) = -\frac{1}{2}\sigma^2 e^{\alpha x+\beta\tau}\left(\beta U + \frac{\partial U}{\partial \tau}\right) \\
\frac{\partial V}{\partial S} &= e^{-x}\frac{\partial}{\partial x}(e^{\alpha x+\beta\tau}U) = e^{(\alpha-1)x+\beta\tau}\left(\alpha U + \frac{\partial U}{\partial x}\right) \\
\frac{\partial^2 V}{\partial S^2} &= e^{-x}\frac{\partial}{\partial x}\left[e^{(\alpha-1)x+\beta\tau}\left(\alpha U + \frac{\partial U}{\partial x}\right)\right] \\
&= e^{-x}\left[(\alpha-1)e^{(\alpha-1)x+\beta\tau}\left(\alpha U + \frac{\partial U}{\partial x}\right) + e^{(\alpha-1)x+\beta\tau}\left(\alpha\frac{\partial U}{\partial x} + \frac{\partial^2 U}{\partial x^2}\right)\right] \\
&= e^{(\alpha-2)x+\beta\tau}\left[\alpha(\alpha-1)U + (2\alpha-1)\frac{\partial U}{\partial x} + \frac{\partial^2 U}{\partial x^2}\right]\,.
\end{aligned}$$

When we put these expressions into the Black-Scholes Equation 26 we get

$$\begin{aligned}
-\frac{1}{2}\sigma^2 e^{\alpha x+\beta\tau}\left(\beta U + \frac{\partial U}{\partial \tau}\right) &+ \frac{1}{2}\sigma^2 e^{\alpha x+\beta\tau}\left[\alpha(\alpha-1)U + (2\alpha-1)\frac{\partial U}{\partial x} + \frac{\partial^2 U}{\partial x^2}\right] \\
&+ r e^{\alpha x+\beta\tau}\left[\alpha U + \frac{\partial U}{\partial x}\right] - r e^{\alpha x+\beta\tau}U = 0\,.
\end{aligned}$$

When we cancel the exponential factor, take the time derivative to one side of the equal sign, and group terms we get

$$\frac{1}{2}\sigma^2\frac{\partial U}{\partial \tau} = \left(-\frac{\beta}{2}\sigma^2 + \frac{1}{2}\sigma^2\alpha(\alpha-1) + r\alpha - r\right)U + \left(\frac{1}{2}\sigma^2(2\alpha-1) + r\right)\frac{\partial U}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 U}{\partial x^2}\,.$$

If we take

$$\alpha = -\frac{1}{2}\left(\frac{2r}{\sigma^2} - 1\right), \tag{27}$$

then $2\alpha - 1 = -\frac{2r}{\sigma^2}$ and the coefficient of $\frac{\partial U}{\partial x}$ vanishes. With this value for α lets now look at the coefficient the U term. When we replace α with the above expression and simplify we get

$$-\frac{\beta}{2}\sigma^2 - \frac{1}{8\sigma^2}(2r + \sigma^2)^2 .$$

To make this vanish we must take β given by

$$\beta = -\frac{1}{4\sigma^4}(2r + \sigma^2)^2 = -\frac{1}{4}\left(\frac{2r}{\sigma^2} + 1\right)^2 . \tag{28}$$

Using this value for β and canceling the common factor of $\frac{1}{2}\sigma^2$ we get the following pure diffusion equation for $U(x, \tau)$

$$\frac{\partial U}{\partial \tau} = \frac{\partial^2 U}{\partial x^2},$$

as claimed.

Notes on Similarity Reductions

If we consider the function

$$u(x, t) = \int_0^{x/t^{1/2}} e^{-\frac{1}{4}\xi^2} d\xi , \tag{29}$$

we can show that it satisfies $u_t = u_{xx}$. To do this, we compute the needed derivatives.

$$\begin{aligned}
u_t &= e^{-\frac{1}{4}\frac{x^2}{t}}\left(-\frac{x}{2t^{3/2}}\right) \\
u_x &= e^{-\frac{1}{4}\frac{x^2}{t}}\left(\frac{1}{t^{1/2}}\right) \\
u_{xx} &= \frac{1}{\sqrt{t}}e^{-\frac{1}{4}\frac{x^2}{t}}\left(-\frac{1}{2}\frac{x}{t}\right) = -\frac{x}{2t^{3/2}}e^{-\frac{1}{4}\frac{x^2}{t}} ,
\end{aligned}$$

from which we see directly that $u_t = u_{xx}$ as claimed.

Let us now look for solutions of a particular form motivated by the form of the above expression for $u(x, t)$. Consider $u \equiv t^{-1/2}f(\xi)$ where $\xi \equiv \frac{x}{t^{1/2}}$. Next we put $t^{-1/2}f(\xi)$ into $u_t = u_{xx}$, to see what requirements this imposes on the function $f(\cdot)$. To do this we need to compute u_t and u_{xx}. We find

$$\begin{aligned}
u_t &= -\frac{1}{2}t^{-3/2}f(\xi) + t^{-1/2}f'(\xi)\left(-\frac{1}{2}\frac{x}{t^{3/2}}\right) = -\frac{1}{2}t^{-3/2}f(\xi) - \frac{1}{2}xt^{-2}f'(\xi) \\
u_x &= t^{-1/2}f'(\xi)\left(\frac{1}{t^{1/2}}\right) = t^{-1}f'(\xi) \\
u_{xx} &= t^{-1}f''(\xi)\left(\frac{1}{t^{1/2}}\right) = t^{-3/2}f''(\xi) .
\end{aligned}$$

We then put these expressions into the diffusion equation $u_{xx} = u_t$ we get

$$t^{-3/2} f''(\xi) = -\frac{1}{2} t^{-3/2} f(\xi) - \frac{1}{2} x t^{-2} f'(\xi) \,.$$

Multiply this equation by $t^{3/2}$ and remember that $\xi = \frac{x}{t^{1/2}}$ to get

$$f''(\xi) = -\frac{1}{2} f(\xi) - \frac{1}{2} \xi f'(\xi) \,. \tag{30}$$

Note that the right-hand-side of this equation is $-\frac{1}{2}\frac{d}{d\xi}(\xi f(\xi))$, and so integrating both sides gives

$$f'(\xi) = -\frac{1}{2} \xi f(\xi) + C \,.$$

If we take $C = 0$ (we just want to try and find any solution) we get

$$\frac{f'(\xi)}{f(\xi)} = -\frac{1}{2} \xi \,,$$

or integrating both sides and solving for $f(\xi)$ gives

$$f(\xi) = D e^{-\frac{1}{4}\xi^2} \,.$$

If we take $D = 1$ we have the function $f(\cdot)$ stated in the book. If we next replace ξ with $\frac{x}{t^{1/2}}$ we see that

$$u(x,t) = t^{-1/2} e^{-\frac{1}{4}\frac{x^2}{t}} \,,$$

is a solution to $u_t = u_{xx}$.

the Black-Scholes formula and the 'greeks'

Derivations of the formula for Calls, Puts, and Simple Digitals

In this section of these notes we perform and verify my understanding of the derivations suggested in simplifying the Black-Scholes equation. Defining $V(S,t)$ as $V(S,t) = e^{-r(T-t)}U(S,t)$ we have the time derivative of $U(S,t)$ given by

$$\frac{\partial V}{\partial t} = re^{-r(T-t)}U + e^{-r(T-t)}\frac{\partial U}{\partial t} = rV + e^{-r(T-t)}\frac{\partial U}{\partial t}\,.$$

When we put this into the Black-Scholes Equation 26 we get

$$\frac{\partial U}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 U}{\partial S^2} + rS\frac{\partial U}{\partial S} = 0\,.$$

If we next let $\tau = T - t$ then the change in the time derivative from t to τ will introduce a negative sign and gives

$$\frac{\partial U}{\partial \tau} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 U}{\partial S^2} + rS\frac{\partial U}{\partial S}\,.$$

We next introduce the variable $\xi = \log(S)$ so that $S = e^{\xi}$ and find that the S derivatives transform as

$$\begin{aligned}
\frac{\partial}{\partial S} &= \frac{\partial \xi}{\partial S}\frac{\partial}{\partial \xi} = \frac{1}{S}\frac{\partial}{\partial \xi} = e^{-\xi}\frac{\partial}{\partial \xi} \quad \text{and} \\
\frac{\partial^2}{\partial S^2} &= e^{-\xi}\frac{\partial}{\partial \xi}\left(e^{-\xi}\frac{\partial}{\partial \xi}\right) = e^{-2\xi}\frac{\partial^2}{\partial \xi^2} - e^{-2\xi}\frac{\partial}{\partial \xi}\,.
\end{aligned}$$

With this transformation, the Black-Scholes equation becomes

$$\begin{aligned}
\frac{\partial U}{\partial \tau} &= \frac{1}{2}\sigma^2 e^{2\xi}\left(e^{-2\xi}\frac{\partial^2 U}{\partial \xi^2} - e^{-2\xi}\frac{\partial U}{\partial \xi}\right) + re^{\xi}\left(e^{-\xi}\frac{\partial U}{\partial \xi}\right) \\
&= \frac{1}{2}\sigma^2\frac{\partial^2 U}{\partial \xi^2} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial U}{\partial \xi}\,.
\end{aligned}$$

Which is a partial differential equation with constant coefficients. Next lets perform a change of variables on this equation going from the variables (ξ,τ) to new variables (x,τ') defined as

$$\begin{aligned}
x &= \xi + \left(r - \frac{1}{2}\sigma^2\right)\tau \\
\tau' &= \tau\,,
\end{aligned}$$

so that the inverse of this transformation is given by

$$\begin{aligned}
\xi &= x - \left(r - \frac{1}{2}\sigma^2\right)\tau' \\
\tau &= \tau'\,.
\end{aligned}$$

The derivatives in the old coordinates (ξ, τ) transform to derivatives in the new coordinates (x, τ') using the chain rule as

$$\begin{aligned}
\frac{\partial}{\partial \tau} &= \frac{\partial \tau'}{\partial \tau}\frac{\partial}{\partial \tau'} + \frac{\partial x}{\partial \tau}\frac{\partial}{\partial x} = \frac{\partial}{\partial \tau'} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial}{\partial x} \\
\frac{\partial}{\partial \xi} &= \frac{\partial \tau'}{\partial \xi}\frac{\partial}{\partial \tau'} + \frac{\partial x}{\partial \xi}\frac{\partial}{\partial x} = \frac{\partial}{\partial x}\,,
\end{aligned}$$

and our differential equation in the new variables (x, τ') is given by

$$\frac{\partial U}{\partial \tau'} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial U}{\partial x} = \frac{1}{2}\sigma^2\frac{\partial^2 U}{\partial x^2} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial U}{\partial x}\,,$$

or dropping the prime on τ we get

$$\frac{\partial U}{\partial \tau} = \frac{1}{2}\sigma^2\frac{\partial^2 U}{\partial x^2}\,. \tag{31}$$

To solve this equation lets try a solution for $U(x, \tau)$ of the form

$$U(x, \tau) = \tau^{\alpha} f\left(\frac{x - x'}{\tau^{\beta}}\right)\,. \tag{32}$$

Then to verify Equation 31 we need to evaluate τ and x derivatives of U. Defining η as

$$\eta \equiv \frac{x - x'}{\tau^{\beta}}\,, \tag{33}$$

we find that the derivatives we need of U are

$$\begin{aligned}
U_{\tau} &= \alpha\tau^{\alpha-1}f(\eta) + \tau^{\alpha}f'(\eta)\left(-\beta\left(\frac{x - x'}{\tau^{\beta-1}}\right)\right) = \alpha\tau^{\alpha-1}f(\eta) - \frac{\beta\tau^{\alpha}\eta}{\tau}f'(\eta) \\
U_x &= \frac{\tau^{\alpha}}{\tau^{\beta}}f'(\eta) \\
U_{xx} &= \frac{\tau^{\alpha}}{\tau^{2\beta}}f''(\eta)\,.
\end{aligned}$$

Thus Equation 31 becomes

$$\frac{1}{2}\sigma^2\tau^{\alpha-2\beta}f''(\eta) = \tau^{\alpha-1}\left(\alpha f(\eta) - \beta\eta f'(\eta)\right)\,. \tag{34}$$

Equating the powers of η on both sides gives $\alpha - 2\beta = \alpha - 1$ so that we get

$$\beta = \frac{1}{2}\,.$$

When we require that $\int_{-\infty}^{\infty} U(x, \tau; x')dx$ to be *independent* of τ means that for the functional form for $U(x, \tau)$ we are considering and changing the x integration into one over η means that

$$\int_{-\infty}^{\infty}\tau^{\alpha} f\left(\frac{x - x'}{\tau^{\beta}}\right)dx = \int_{-\infty}^{\infty}\tau^{\alpha+\beta} f(\eta)\,d\eta\,,$$

must be independent of τ. This means that $\alpha + \beta = 0$ or

$$\alpha = -\beta = -\frac{1}{2}.$$

Once we have α and β we can put these into Equation 34 to get an equation very similar to Equation 30 earlier. Following the same algebraic steps following Equation 30 and specifying the constant D so that the function $f(\cdot)$ over $-\infty$ to $+\infty$ integrates to one, we obtain the function form for $f(\eta)$ given by

$$f(\eta) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\eta^2}{2\sigma^2}},$$

When we put $\eta = \frac{x-x'}{\tau^{1/2}}$ into the expression for $W(x,\tau)$ we finally end with

$$W(x,\tau) = \frac{1}{\sqrt{2\pi\tau}\sigma} \exp\left\{ -\frac{(x-x')^2}{2\sigma^2\tau} \right\}.$$

Superimposing fundamental solutions $W(x,\tau)$ for various values of x' weighted by the payoff function Payoff$(\cdot)$, and then transforming back into the original S, t variables gives for the solution $V(S,t)$ of the Black-Scholes equation

$$V(S,t) = \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_0^\infty \text{Payoff}(S') e^{-[\log(S'/S)-(r-\frac{1}{2}\sigma^2)(T-t)]^2/2\sigma^2(T-t)} \frac{dS'}{S'}.$$

When there is a dividend yield D the r in the expression above becomes $r - D$ or

$$V(S,t) = \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_0^\infty \text{Payoff}(S') e^{-[\log(S'/S)-(r-D-\frac{1}{2}\sigma^2)(T-t)]^2/2\sigma^2(T-t)} \frac{dS'}{S'}. \tag{35}$$

From this point onward in these notes we will try to be consistent (in this chapter at least) in that we will always include a dividend yield term D in all of our expressions.

Notes on the BS Formula for a European Call

To value a European Call recall that the payoff function in that case is given by

$$\text{Payoff}(S) = \max(S-E, 0),$$

and so that when we put this expression into Equation 35, perform the required integrations, we get

$$C(S,t) = Se^{-D(T-t)}N(d_1) - Ee^{-r(T-t)}N(d_2) \tag{36}$$

$$d_1 \equiv \frac{\log(S/E) + (r - D + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \tag{37}$$

$$d_2 \equiv \frac{\log(S/E) + (r - D - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \tag{38}$$

$$= d_1 - \sigma\sqrt{T-t} \tag{39}$$

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}\phi^2} d\phi. \tag{40}$$

From the definition of $N(x)$ we can see that

$$N'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \quad \text{and} \quad N''(x) = -xN'(x)\,. \tag{41}$$

With these results the expression for $N'(d_2)$ is given in terms of $N'(d_1)$ by

$$\begin{aligned} N'(d_2) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_2^2} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(d_1 - \sigma\sqrt{T-t})^2} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_1^2}\, e^{d_1\sigma\sqrt{T-t}}\, e^{-\frac{1}{2}\sigma^2(T-t)} \\ &= N'(d_1) e^{d_1\sigma\sqrt{T-t}}\, e^{-\frac{1}{2}\sigma^2(T-t)} = N'(d_1) e^{\log(S/E)+(r-D+\frac{1}{2}\sigma^2)(T-t)}\, e^{-\frac{1}{2}\sigma^2(T-t)} \\ &= \frac{S}{E} N'(d_1) e^{(r-D)(T-t)}\,. \end{aligned} \tag{42}$$

Notes on the BS Formula for a European Put

Since we know the analytical expression for a European call we can use Put-Call parity to derive the analytic expression for a European put. From Put-Call parity we have that

$$C - P = Se^{-D(T-t)} - Ee^{-r(T-t)}\,, \tag{43}$$

which when we put in the known expression for $C(S,t)$ and solve for $P(S,t)$ we find

$$\begin{aligned} P(S,t) &= Se^{-D(T-t)}(N(d_1) - 1) - Ee^{-r(T-t)}(N(d_2) - 1) \\ &= -Se^{-D(T-t)}N(-d_1) + Ee^{-r(T-t)}N(-d_2)\,, \end{aligned} \tag{44}$$

where we have used the fact that

$$N(d) + N(-d) = 1\,. \tag{45}$$

Notes on the BS Formula for a Binary Calls and Puts

If our payoff Payoff(S) is a step function at the strike E i.e. Payoff$(S) = \mathcal{H}(S-E)$, where $\mathcal{H}$ is the Heaviside function, then from the general expression for the evaluation of the option price above in Equation 35 we see that

$$\begin{aligned} V(S,t) &= \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_0^\infty \mathcal{H}(S'-E) e^{-[\log(S'/S)-(r-D-\frac{1}{2}\sigma^2)(T-t)]^2/2\sigma^2(T-t)} \frac{dS'}{S'} \\ &= \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_E^\infty e^{-[\log(S'/S)-(r-D-\frac{1}{2}\sigma^2)(T-t)]^2/2\sigma^2(T-t)} \frac{dS'}{S'}\,. \end{aligned}$$

To evaluate this integral introduce an integration variable v (unrelated to the variable V for option price) such that

$$\begin{aligned} v &= \frac{-\log(S'/S) + (r - D - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \quad \text{and} \\ dv &= \frac{dv}{dS'} dS' = -\frac{1}{\sigma\sqrt{T-t}} \frac{dS'}{S'}\,. \end{aligned}$$

With this our logarithmic differential becomes $\frac{dS'}{S'} = -\sigma\sqrt{T-t}\,dv$ and our integral above transforms to

$$V(S,t) = -\frac{e^{-r(T-t)}}{\sqrt{2\pi}} \int_{d_2}^{-\infty} e^{-v^2/2} dv = e^{-r(T-t)} N(d_2)\,. \tag{46}$$

when we recall the definition of d_2. This is the formula for the value of a **Binary call option**.

If our payoff Payoff(S) is instead a step function that turns *off* at the strike E i.e. Payoff$(S) = \mathcal{H}(E-S)$, where $\mathcal{H}$ is the Heaviside function, then from the general expression for the evaluation of the option price above in Equation 35 we see that

$$\begin{aligned}
V(S,t) &= \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_0^{\infty} \mathcal{H}(E-S') e^{-[\log(S'/S)-(r-D-\frac{1}{2}\sigma^2)(T-t)]^2/2\sigma^2(T-t)} \frac{dS'}{S'} \\
&= \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_0^{E} e^{-[\log(S'/S)-(r-D-\frac{1}{2}\sigma^2)(T-t)]^2/2\sigma^2(T-t)} \frac{dS'}{S'}\,.
\end{aligned}$$

To evaluate this integral we again introduce the integration variable v such that

$$\begin{aligned}
v &= \frac{-\log(S'/S) + (r-D-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \quad \text{and} \\
dv &= \frac{dv}{dS'} dS' = -\frac{1}{\sigma\sqrt{T-t}} \frac{dS'}{S'}\,.
\end{aligned}$$

With this our logarithmic differential becomes $\frac{dS'}{S'} = -\sigma\sqrt{T-t}\,dv$ and our integral above transforms to

$$\begin{aligned}
V(S,t) &= -\frac{e^{-r(T-t)}}{\sqrt{2\pi}} \int_{\infty}^{d_2} e^{-v^2/2} dv = e^{-r(T-t)} \left[\frac{1}{\sqrt{2\pi}} \int_{d_2}^{\infty} e^{-v^2/2} dv \right] \\
&= e^{-r(T-t)} \left[1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_2} e^{-v^2/2} dv \right] \\
&= e^{-r(T-t)} (1 - N(d_2))\,. \qquad (47)
\end{aligned}$$

again using the definition of d_2. This is the formula for the value of a **Binary put option**.

Notes on the derivation of Delta for some common contracts

This section of the book introduces the notation of an options delta which is denoted as the symbol Δ. In this section of these notes we will *derive* all of the given delta expressions presented in the book. To do this it will be helpful to have the S derivative of d_1 and d_2. Using Equations 37 and 38 we see that these are equal and given by

$$\frac{\partial d_1}{\partial S} = \frac{\partial d_2}{\partial S} = \frac{1}{\sigma S\sqrt{T-t}}\,. \tag{48}$$

- The expression for the delta of a European calls given by Equation 36 becomes

$$\begin{aligned}
\Delta &= e^{-D(T-t)}N(d_1) + Se^{-D(T-t)}N'(d_1)\frac{\partial d_1}{\partial S} - Ee^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial S} \\
&= e^{-D(T-t)}N(d_1) + \frac{e^{-D(T-t)}N'(d_1)}{\sigma\sqrt{T-t}} - \frac{Ee^{-r(T-t)}N'(d_2)}{S\sigma\sqrt{T-t}} \\
&= e^{-D(T-t)}N(d_1) + \frac{1}{\sigma\sqrt{2\pi(T-t)}}\left[e^{-D(T-t)}e^{-\frac{1}{2}d_1^2} - \frac{E}{S}e^{-r(T-t)}e^{-\frac{1}{2}d_2^2}\right].
\end{aligned}$$

  when we use the expression for $N'(\cdot)$ given by Equation 41. Lets consider the two terms in brackets above. From the definition of d_1 and d_2 we can replace d_2 with $d_1 - \sigma\sqrt{T-t}$ and then expand the square in the second exponent to get

$$\Delta = e^{-D(T-t)}N(d_1) + \frac{e^{-\frac{1}{2}d_1^2}}{\sigma\sqrt{2\pi(T-t)}}\left[e^{-D(T-t)} - \frac{E}{S}e^{-r(T-t)}e^{-\frac{1}{2}(-2\sigma d_1\sqrt{T-t}+\sigma^2(T-t))}\right].$$

  We now consider the exponent of the third term. Since the product $d_1\sigma\sqrt{T-t}$ equals

$$\log(S/E) + (r + \frac{1}{2}\sigma^2)(T-t),$$

  we get a third term with an exponent of

$$\begin{aligned}
&- r(T-t) + d_1\sigma\sqrt{T-t} + -\frac{1}{2}\sigma^2(T-t) \\
&= -r(T-t) + \log(S/E) + (r + \frac{1}{2}\sigma^2)(T-t) - \frac{1}{2}\sigma^2(T-t) \\
&= \log(S/E).
\end{aligned}$$

  With this the expression the two terms in brackets become

$$e^{-D(T-t)} - \frac{E}{S}e^{-D(T-t)}e^{\log(S/E)} = 0.$$

  Thus we get for the delta of a European call

$$\Delta = e^{-D(T-t)}N(d_1), \qquad (49)$$

  the expression claimed in the book.

- The expression for the delta of a European put can be given by taking the S derivative of Equation 44 or by taking the derivative of the put-call parity relationship

$$C - P = Se^{-D(T-t)} - Ee^{-r(T-t)},$$

  and using the known delta for a European call. Taking the S derivative of this expression we see that

$$\frac{\partial C}{\partial S} - \frac{\partial P}{\partial S} = e^{-D(T-t)},$$

  or

$$\Delta = \frac{\partial P}{\partial S} = \frac{\partial C}{\partial S} - e^{-D(T-t)} = e^{-D(T-t)}(N(d_1) - 1), \qquad (50)$$

  the expression claimed in the book.

- The delta for a binary call is given by taking the S derivative of Equation 46, where we find

$$\begin{aligned}\Delta &= \frac{\partial}{\partial S} e^{-r(T-t)} N(d_2) = e^{-r(T-t)} N'(d_2) \frac{\partial d_2}{\partial S} \\ &= \frac{e^{-r(T-t)} N'(d_2)}{\sigma S \sqrt{T-t}} \,. \end{aligned} \tag{51}$$

- The delta for a binary put is given by taking the S derivative of Equation 47, where we find

$$\Delta = \frac{\partial}{\partial S}(e^{-r(T-t)}(1 - N(d_2))) = -\frac{e^{-r(T-t)} N'(d_2)}{\sigma S \sqrt{T-t}} \,, \tag{52}$$

  the negative of the binary call delta.

Notes on the derivation of Gamma for some common contracts

Here the book introduces the notation of an options gamma which is denoted as the symbol Γ. In this section of these notes we will *derive* the given gamma expressions presented in the book.

- To derive Γ for a European call we take the S derivative of Equation 49. We find

$$\begin{aligned}\Gamma &= \frac{\partial^2 C}{\partial S^2} = \frac{\partial}{\partial S} e^{-D(T-t)} N(d_1) = e^{-D(T-t)} N'(d_1) \frac{\partial d_1}{\partial S} \\ &= \frac{e^{-D(T-t)} N'(d_1)}{\sigma S \sqrt{T-t}} \,. \end{aligned} \tag{53}$$

- To derive Γ for a European put we take the S derivative of Equation 50. We find

$$\begin{aligned}\Gamma &= \frac{\partial^2 P}{\partial S^2} = \frac{\partial}{\partial S}(e^{-D(T-t)}(N(d_1) - 1)) = \frac{\partial^2 C}{\partial S^2} \\ &= \frac{e^{-D(T-t)} N'(d_1)}{\sigma S \sqrt{T-t}} \,, \end{aligned} \tag{54}$$

  the *same* as the Γ for a European call.

- To derive Γ for a binary call using Equation 51 we have

$$\begin{aligned}\Gamma &= \frac{\partial}{\partial S}\left(\frac{e^{-r(T-t)} N'(d_2)}{S\sigma\sqrt{T-t}}\right) = \frac{e^{-r(T-t)} N''(d_2)}{S\sigma\sqrt{T-t}}\left(\frac{1}{\sigma S \sqrt{T-t}}\right) - \frac{e^{-r(T-t)} N'(d_2)}{S^2 \sigma \sqrt{T-t}} \\ &= \frac{e^{-r(T-t)}}{\sigma S^2 \sqrt{T-t}}\left[\frac{N''(d_2)}{\sigma\sqrt{T-t}} - N'(d_2)\right] .\end{aligned}$$

  But we can evaluate $N''(d_2)$ in terms of $N'(d_2)$ using Equation 41 and we get

$$\begin{aligned}\Gamma &= \frac{e^{-r(T-t)}}{\sigma S^2 \sqrt{T-t}}\left[-\frac{d_2}{\sigma\sqrt{T-t}} - 1\right] N'(d_2) = -\frac{e^{-r(T-t)}}{\sigma^2 S^2 (T-t)}\left[d_2 + \sigma\sqrt{T-t}\right] N'(d_2) \\ &= -\frac{e^{-r(T-t)} d_1 N'(d_2)}{\sigma^2 S^2 (T-t)} \,. \end{aligned} \tag{55}$$

- To derive Γ for a binary put recall that since the delta's for a binary put and a binary call are the negatives of each other, the gamma for a binary put must be the negative of the gamma of a binary call and we have

$$\Gamma = \frac{e^{-r(T-t)} d_1 N'(d_2)}{\sigma^2 S^2 (T-t)} \,. \tag{56}$$

Notes on the derivation of Theta for some common contracts

Here the book introduces the notation of an options theta which is denoted as the symbol θ and defined as $\frac{\partial V}{\partial t}$. In this section of these notes we will *derive* the given theta expressions presented in the book. To derive these we will need the Black-Scholes equation with a continuous dividend yield D which is given by

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-D)S\frac{\partial V}{\partial S} - rV = 0 \,. \tag{57}$$

Thus an options theta can be obtained in terms of its value (V), its delta ($\frac{\partial V}{\partial S}$), and its gamma ($\frac{\partial^2 V}{\partial S^2}$) by solving for $\frac{\partial V}{\partial t}$ or

$$\begin{aligned} \theta \equiv \frac{\partial V}{\partial t} &= -\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - (r-D)S\frac{\partial V}{\partial S} + rV \\ &= -\frac{1}{2}\sigma^2 S^2 \Gamma - (r-D)S\Delta + rV \,. \end{aligned} \tag{58}$$

We now have everything we need to calculate θ for some options.

- For a European call we have

$$\begin{aligned} V &= Se^{-D(T-t)}N(d_1) - Ee^{-r(T-t)}N(d_2) \\ \Delta &= e^{-D(T-t)}N(d_1) \\ \Gamma &= \frac{e^{-D(T-t)}N'(d_1)}{\sigma S\sqrt{T-t}} \,, \end{aligned}$$

so Equation 58 above becomes

$$\begin{aligned} \theta &= -\frac{1}{2}\sigma^2 S^2 \left(\frac{e^{-D(T-t)}N'(d_1)}{\sigma S\sqrt{T-t}} \right) - (r-D)Se^{-D(T-t)}N(d_1) \\ &+ r(Se^{-D(T-t)}N(d_1) - Ee^{-r(T-t)}N(d_2)) \\ &= -\frac{\sigma S e^{-D(T-t)}N'(d_1)}{2\sqrt{T-t}} + DSe^{-D(T-t)}N(d_1) - rEe^{-r(T-t)}N(d_2) \,. \end{aligned} \tag{59}$$

- For a European put we have

$$\begin{aligned} V &= -Se^{-D(T-t)}N(-d_1) + Ee^{-r(T-t)}N(-d_2) \\ \Delta &= e^{-D(T-t)}(N(d_1) - 1) = -e^{-D(T-t)}N(-d_1) \\ \Gamma &= \frac{e^{-D(T-t)}N'(d_1)}{\sigma S\sqrt{T-t}} \,, \end{aligned}$$

so Equation 58 above becomes

$$
\begin{aligned}
\theta &= -\frac{1}{2}\sigma^2 S^2 \left(\frac{e^{-D(T-t)} N'(d_1)}{\sigma S \sqrt{T-t}} \right) + (r-D) S e^{-D(T-t)} N(-d_1) \\
&+ r(-S e^{-D(T-t)} N(-d_1) + E e^{-r(T-t)} N(-d_2)) \\
&= -\frac{\sigma S N'(d_1)}{2\sqrt{T-t}} - D S e^{-D(T-t)} N(-d_1) + r E e^{-r(T-t)} N(-d_2) \,. \qquad (60)
\end{aligned}
$$

Since $N'(x)$ is an even function of x we could write the $N'(d_1)$ factor in the first term as $N'(-d_1)$ if desired. This last equation could also be obtained from put-call parity by taking the time derivative of the put-call parity expression 43 and using the theta for a European call.

- For a binary call using what we have derived before we have

$$
\begin{aligned}
\theta &= -\frac{1}{2}\sigma^2 S^2 \left(-\frac{e^{-r(T-t)} d_1 N'(d_2)}{\sigma^2 S^2 (T-t)} \right) - (r-D) S \frac{e^{-r(T-t)} N'(d_2)}{\sigma S \sqrt{T-t}} + r e^{-r(T-t)} N(d_2) \\
&= r e^{-r(T-t)} N(d_2) + e^{-r(T-t)} N'(d_2) \left[\frac{d_1}{2(T-t)} - \frac{r-D}{\sigma\sqrt{T-t}} \right] . \qquad (61)
\end{aligned}
$$

- For a binary put using what we have derived before we have

$$
\begin{aligned}
\theta &= -\frac{1}{2}\sigma^2 S^2 \left(\frac{e^{-r(T-t)} d_1 N'(d_2)}{\sigma^2 S^2 (T-t)} \right) + (r-D) S \frac{e^{-r(T-t)} N'(d_2)}{\sigma S \sqrt{T-t}} + r e^{-r(T-t)} (1 - N(d_2)) \\
&= r e^{-r(T-t)} (1 - N(d_2)) - e^{-r(T-t)} N'(d_2) \left[\frac{d_1}{2(T-t)} - \frac{r-D}{\sigma\sqrt{T-t}} \right] . \qquad (62)
\end{aligned}
$$

Notes on the derivation of Speed for some common contracts

Here the book introduces the notation of an options speed which is defined as $\frac{\partial^3 V}{\partial S^3}$. In this section of these notes we will *derive* the given theta expressions presented in the book.

- For a European call taking the S derivative of Equation 53 and using Equation 48 we get

$$
\frac{\partial^3 V}{\partial S^3} = \frac{e^{-D(T-t)} N''(d_1)}{\sigma^2 S^2 (T-t)} - \frac{e^{-D(T-t) N'(d_1)}}{\sigma S^2 \sqrt{T-t}} \,.
$$

  Using Equation 41 we can write this as

$$
\frac{\partial^3 V}{\partial S^3} = -\frac{e^{-D(T-t)} N'(d_1)}{\sigma^2 S^2 (T-t)} \left[d_1 + \sigma\sqrt{T-t} \right] . \qquad (63)
$$

- Since the gamma for a European put and a European call are the same, the speed of a European put must equal the speed of a European call or Equation 63.

- For a binary call, using Equation 56 we see that we will need to evaluate

$$\frac{\partial N'(d_2)}{\partial S} = N''(d_2)\frac{\partial d_2}{\partial S} = -\frac{N'(d_2)d_2}{\sigma S\sqrt{T-t}}\,, \tag{64}$$

the same equation hold when d_2 is replaced by d_1. Using this we find

$$\begin{aligned}\frac{\partial^3 V}{\partial S^3} &= -\frac{e^{-D(T-t)}N'(d_2)}{\sigma^3 S^3 (T-t)^{3/2}} + \frac{e^{-D(T-t)}d_1 d_2 N'(d_2)}{\sigma^3 S^3 (T-t)^{3/2}} + 2\frac{e^{-D(T-t)}d_1 N'(d_2)}{\sigma^2 S^3 (T-t)} \\ &= -\frac{e^{-D(T-t)}N'(d_2)}{\sigma^2 S^3 (T-t)}\left[\frac{1-d_1 d_2}{\sigma\sqrt{T-t}} - 2d_1\right]\,. \end{aligned}\tag{65}$$

- Since the gamma for a binary put is the negative of the gamma of a binary call the speed of a binary put is the negative of a binary call or the *negative* of Equation 65.

Notes on the derivation of Vega for some common contracts

Here the book introduces the notation of an options vega which is defined as $\frac{\partial V}{\partial \sigma}$. In this section of these notes we will *derive* the given vega expressions presented in the book.

- For a European call taking the σ derivative of Equation 53 we have

$$\frac{\partial V}{\partial \sigma} = Se^{-D(T-t)}N'(d_1)\frac{\partial d_1}{\partial \sigma} - Ee^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial \sigma}\,.$$

From Equation 42 we change the factor $N'(d_2)$ into a factor in terms of $N'(d_1)$ to get

$$\begin{aligned}\frac{\partial V}{\partial \sigma} &= Se^{-D(T-t)}N'(d_1)\frac{\partial d_1}{\partial \sigma} - Ee^{-r(T-t)}\left[\frac{S}{E}e^{(r-D)(T-t)}N'(d_1)\right]\frac{\partial d_2}{\partial \sigma} \\ &= Se^{-D(T-t)}N'(d_1)\left[\frac{\partial d_1}{\partial \sigma} - \frac{\partial d_2}{\partial \sigma}\right]\,.\end{aligned}$$

From the definition of d_1 and d_2 given by Equation 37 and 38 above we find that

$$\frac{\partial d_1}{\partial \sigma} = -\frac{d_1}{\sigma} + \sqrt{T-t} \quad \text{and} \quad \frac{\partial d_2}{\partial \sigma} = -\frac{d_2}{\sigma} - \sqrt{T-t}\,,$$

so their difference is given by

$$\frac{\partial d_1}{\partial \sigma} - \frac{\partial d_2}{\partial \sigma} = \frac{d_2 - d_1}{\sigma} + 2\sqrt{T-t} = -\sqrt{T-t} + 2\sqrt{T-t} = \sqrt{T-t}\,,$$

when we recall that $d_2 = d_1 - \sigma\sqrt{T-t}$. Thus we have

$$\frac{\partial V}{\partial \sigma} = Se^{-D(T-t)}N'(d_1)\sqrt{T-t}\,, \tag{66}$$

as the expression for the vega of a European call.

- For a European put taking the σ derivative of the put-call parody Equation 43 gives

$$\frac{\partial P}{\partial \sigma} = \frac{\partial C}{\partial \sigma} \,,$$

showing that the two contracts have the same value for vega.

- For a binary call we have

$$\begin{aligned} \frac{\partial}{\partial \sigma} \left(e^{-r(T-t)} N(d_2) \right) &= e^{-r(T-t)} N'(d_2) \left[-\sqrt{T-t} - \frac{d_2}{\sigma} \right] \\ &= -e^{-r(T-t)} N'(d_2) \left[\sqrt{T-t} + \frac{d_2}{\sigma} \right] \,. \end{aligned} \tag{67}$$

- For a binary put we have

$$\frac{\partial}{\partial \sigma} \left(e^{-r(T-t)} (1 - N(d_2)) \right) = e^{-r(T-t)} N'(d_2) \left[\sqrt{T-t} + \frac{d_2}{\sigma} \right] \,, \tag{68}$$

the negative of the vega for a binary call.

Notes on the derivation of Rho for some common contracts

Here the book introduces the notation of an options rho which is defined as $\rho = \frac{\partial V}{\partial r}$. In this section of these notes we will *derive* the given rho expressions presented in the book.

- For a European call taking the r derivative of V is given by

$$\frac{\partial V}{\partial r} = S e^{-D(T-t)} N'(d_1) \frac{\partial d_1}{\partial r} + E(T-t) e^{-r(T-t)} N(d_2) - E e^{-r(T-t)} N'(d_2) \frac{\partial d_2}{\partial r} \,.$$

Since $\frac{\partial d_1}{\partial r} = \frac{\sqrt{T-t}}{\sigma} = \frac{\partial d_2}{\partial r}$ so that the above simplifies to

$$\begin{aligned} \frac{\partial V}{\partial r} &= (S e^{-D(T-t)} N'(d_1) - E e^{-r(T-t)} N'(d_2)) \frac{\partial d_1}{\partial r} + E(T-t) e^{-r(T-t)} N(d_2) \\ &= E(T-t) e^{-r(T-t)} N(d_2) \,. \end{aligned} \tag{69}$$

Where we have used Equation 42 to eliminate the first term.

- To compute the rho for a European put we can take the r derivative of the put-call parody relationship Equation 43 to get the rho for a European put is given by

$$\begin{aligned} \frac{\partial P}{\partial r} &= \frac{\partial C}{\partial r} - E(T-t) e^{-r(T-t)} = E(T-t) e^{-r(T-t)} (N(d_2) - 1) \\ &= -E(T-t) e^{-r(T-t)} N(-d_2) \,. \end{aligned} \tag{70}$$

- For a binary call we have

$$\begin{aligned} \frac{\partial}{\partial r} \left(e^{-r(T-t)} N(d_2) \right) &= -(T-t) e^{-r(T-t)} N(d_2) + e^{-r(T-t)} N'(d_2) \frac{\partial d_2}{\partial r} \\ &= -(T-t) e^{-r(T-t)} N(d_2) + \frac{\sqrt{T-t}}{\sigma} e^{-r(T-t)} N'(d_2) \,. \end{aligned} \tag{71}$$

- For a binary put we have

$$\frac{\partial}{\partial r}\left(e^{-r(T-t)}(1-N(d_2))\right) = -(T-t)e^{-r(T-t)}(1-N(d_2)) - \frac{\sqrt{T-t}}{\sigma}e^{-r(T-t)}N'(d_2)\,. \quad (72)$$

Notes on the sensitivity of D for some common contracts

In this section of these notes we derive the given expressions for $\frac{\partial V}{\partial D}$ presented in the book.

- For a European call taking the D derivative of V is given by

$$\frac{\partial V}{\partial D} = -S(T-t)e^{-D(T-t)}N(d_1) + Se^{-D(T-t)}N'(d_1)\frac{\partial d_1}{\partial D} - Ee^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial D}\,.$$

  Use Equation 42 to convert $N'(d_2)$ into a term with $N'(d_1)$ as

$$\begin{aligned}
\frac{\partial V}{\partial D} &= -S(T-t)e^{-D(T-t)}N(d_1) + Se^{-D(T-t)}N'(d_1)\frac{\partial d_1}{\partial D} \\
&- Ee^{-r(T-t)}\left[\frac{S}{E}N'(d_1)e^{(r-D)(T-t)}\right]\frac{\partial d_2}{\partial D} \\
&= -S(T-t)e^{-D(T-t)}N(d_1) + Se^{-D(T-t)}N'(d_1)\left[\frac{\partial d_1}{\partial D} - \frac{\partial d_1}{\partial D}\right]\,.
\end{aligned}$$

  Now

$$\frac{\partial d_1}{\partial D} = -\frac{\sqrt{T-t}}{\sigma} \quad \text{and} \quad \frac{\partial d_2}{\partial D} = \frac{\partial d_1}{\partial D} = -\frac{\sqrt{T-t}}{\sigma}\,,$$

  so $\frac{\partial d_2}{\partial D} - \frac{\partial d_1}{\partial D} = 0$ and we get

$$\frac{\partial V}{\partial D} = -(T-t)Se^{-D(T-t)}N(d_1)\,. \quad (73)$$

- To compute the derivative of a European put with respect to D we can take the D derivative of the put-call parody relationship Equation 43 to get that

$$\begin{aligned}
\frac{\partial P}{\partial D} &= \frac{\partial C}{\partial D} + (T-t)Se^{-D(T-t)} = -(T-t)Se^{-D(T-t)}N(d_1) + (T-t)Se^{-D(T-t)} \\
&= (T-t)Se^{-D(T-t)}N(-d_1)\,. \quad (74)
\end{aligned}$$

- For a binary call we have

$$\begin{aligned}
\frac{\partial}{\partial D}\left(e^{-r(T-t)}N(d_2)\right) &= e^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial D} \\
&= -\frac{\sqrt{(T-t)}}{\sigma}e^{-r(T-t)}N'(d_2)\,. \quad (75)
\end{aligned}$$

- For a binary put we have

$$\frac{\partial}{\partial D}\left(e^{-r(T-t)}(1-N(d_2))\right) = \frac{\sqrt{T-t}}{\sigma}e^{-r(T-t)}N'(d_2)\,. \quad (76)$$

Simple generalizations of the Black-Scholes world

Notes on commodities: incorporating the cost of carry

Given the Black-Scholes equation used for commodities

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r+u)S\frac{\partial V}{\partial S} - rV = 0\,, \tag{77}$$

here u represents the storage costs assumed proportional to the commodities price. If we introduce the variable F defined by $F = Se^{(r+u-y)(T-t)}$, where y is the convenience yield, then we can simplify Equation 77 by performing the change of variables from (S,t) to $(F,\bar{t})$ defined as

$$S = Fe^{-(r+u-y)(T-\bar{t})} \quad \text{and} \quad t = \bar{t}\,,$$

which has an inverse transformation is given by

$$F = Se^{(r+u-y)(T-t)} \quad \text{and} \quad \bar{t} = t\,.$$

Using these we find the derivatives of S, and t transform as

$$\begin{aligned}
\frac{\partial}{\partial t} &= \frac{\partial \bar{t}}{\partial t}\frac{\partial}{\partial \bar{t}} + \frac{\partial F}{\partial t}\frac{\partial}{\partial F} = \frac{\partial}{\partial \bar{t}} - (r+u-y)F\frac{\partial}{\partial F} \\
\frac{\partial}{\partial S} &= \frac{\partial \bar{t}}{\partial S}\frac{\partial}{\partial \bar{t}} + \frac{\partial F}{\partial S}\frac{\partial}{\partial F} = e^{(r+u-y)(T-\bar{t})}\frac{\partial}{\partial F} \quad \text{so} \quad \frac{\partial^2}{\partial S^2} = e^{2(r+u-y)(T-\bar{t})}\frac{\partial^2}{\partial F^2}\,.
\end{aligned}$$

When we put these into Equation 77 and let $V = H(F,t)$ we get

$$\frac{\partial H}{\partial \bar{t}} - (r+u-y)F\frac{\partial H}{\partial F} + \frac{1}{2}\sigma^2 S^2 e^{2(r+u-y)(T-\bar{t})}\frac{\partial^2 H}{\partial F^2} + (r+u)Se^{(r+u-y)(T-\bar{t})}\frac{\partial H}{\partial F} - rH = 0\,,$$

or simplifying and writing everything in terms of $(F,\bar{t})$ we get

$$\frac{\partial H}{\partial \bar{t}} + \frac{1}{2}\sigma^2 F^2 \frac{\partial^2 H}{\partial F^2} + yF\frac{\partial H}{\partial F} - rH = 0\,,$$

which is the equation quoted in the book.

Notes on the time dependent Black-Scholes equation

Consider the Black-Scholes equation where the "parameters" are now time-dependent

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma(t)^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r(t) - D(t))S\frac{\partial V}{\partial S} - r(t)V = 0\,, \tag{78}$$

to simplify this equation we first transform the dependent variable V as $V = e^{-\beta(t)}\overline{V}$ to get

$$-\dot{\beta}(t)e^{-\beta(t)}\overline{V} + e^{-\beta(t)}\frac{\partial \overline{V}}{\partial t} + \frac{1}{2}\sigma(t)^2 S^2 e^{-\beta(t)}\frac{\partial^2 \overline{V}}{\partial S^2} + (r(t) - D(t))Se^{-\beta(t)}\frac{\partial \overline{V}}{\partial S} - r(t)e^{-\beta(t)}\overline{V} = 0\,.$$

Multiply this equation by $e^{\beta(t)}$ and rearrange terms to get

$$\frac{\partial \overline{V}}{\partial t} + \frac{1}{2}\sigma(t)^2 S^2 \frac{\partial^2 \overline{V}}{\partial S^2} + (r(t) - D(t))S\frac{\partial \overline{V}}{\partial S} - (r(t) + \dot{\beta}(t))\overline{V} = 0\,. \tag{79}$$

We can further simplify this by transforming the independent variables S and t as

$$\overline{S} = Se^{\alpha(t)} \quad \text{and} \quad \overline{t} = \gamma(t)\,,$$

under which we see that the derivative of S and t transform as

$$\begin{aligned}
\frac{\partial}{\partial t} &= \frac{\partial \overline{t}}{\partial t}\frac{\partial}{\partial \overline{t}} + \frac{\partial \overline{S}}{\partial t}\frac{\partial}{\partial \overline{S}} = \dot{\gamma}(t)\frac{\partial}{\partial \overline{t}} + \dot{\alpha}(t)Se^{\alpha(t)}\frac{\partial}{\partial \overline{S}} = \dot{\gamma}(t)\frac{\partial}{\partial \overline{t}} + \dot{\alpha}(t)\overline{S}\frac{\partial}{\partial \overline{S}} \\
\frac{\partial}{\partial S} &= \frac{\partial \overline{t}}{\partial S}\frac{\partial}{\partial \overline{t}} + \frac{\partial \overline{S}}{\partial S}\frac{\partial}{\partial \overline{S}} = e^{\alpha(t)}\frac{\partial}{\partial \overline{S}} \quad \text{so} \quad \frac{\partial^2}{\partial S^2} = e^{2\alpha(t)}\frac{\partial^2}{\partial \overline{S}^2}\,.
\end{aligned}$$

When we put these expressions into Equation 79 we get

$$\dot{\gamma}(t)\frac{\partial \overline{V}}{\partial \overline{t}} + \dot{\alpha}(t)\overline{S}\frac{\partial \overline{V}}{\partial \overline{S}} + \frac{1}{2}\sigma(t)^2\overline{S}^2\frac{\partial^2 \overline{V}}{\partial \overline{S}^2} + (r(t) - D(t))\overline{S}\frac{\partial \overline{V}}{\partial \overline{S}} - (r(t) + \dot{\beta}(t))\overline{V} = 0\,,$$

or when we group terms some

$$\dot{\gamma}(t)\frac{\partial \overline{V}}{\partial \overline{t}} + \frac{1}{2}\sigma(t)^2\overline{S}^2\frac{\partial^2 \overline{V}}{\partial \overline{S}^2} + (r(t) - D(t) + \dot{\alpha}(t))\overline{S}\frac{\partial \overline{V}}{\partial \overline{S}} - (r(t) + \dot{\beta}(t))\overline{V} = 0\,. \tag{80}$$

As we have not specified the functions α, β, or γ we will do so now to make the above equation as simple as possible. To make the coefficient of $\overline{V}$ vanish in Equation 80 we take $\beta(t)$ so that it satisfies $\dot{\beta}(t) = -r(t)$ which holds if we take

$$\beta(t) \equiv \int_t^T r(\tau)d\tau\,.$$

To make the coefficient of $\frac{\partial \overline{V}}{\partial \overline{S}}$ vanish in Equation 80 lets pick $\alpha(t)$ in the same way. That is we will define $\alpha(t)$ as

$$\alpha(t) \equiv \int_t^T (r(\tau) - D(\tau))d\tau\,.$$

When we use these two definitions for $\alpha(t)$ and $\beta(t)$ Equation 80 then becomes

$$\dot{\gamma}(t)\frac{\partial \overline{V}}{\partial \overline{t}} + \frac{1}{2}\sigma(t)^2\overline{S}^2\frac{\partial^2 \overline{V}}{\partial \overline{S}^2} = 0\,.$$

Finally we take $\dot{\gamma}(t) = -\sigma(t)^2$ (note the negative sign) or

$$\gamma(t) \equiv \int_t^T \sigma(\tau)^2 d\tau\,,$$

we can make the time dependence of the coefficient $\sigma(t)^2$ cancel. With this we then get

$$\frac{\partial \overline{V}}{\partial \overline{t}} = \frac{1}{2}\overline{S}^2\frac{\partial^2 \overline{V}}{\partial \overline{S}^2}\,. \tag{81}$$

Then if we solve this equation for $\overline{V}(\overline{S}, \overline{t})$ the solution to the original time-dependent Equation 78 is given by reversing the transformations made above. That is given a functional form or a numerical procedure for calculating $\overline{V}(\overline{S}, \overline{t})$ the solution to the original problem is given by

$$V = e^{-\beta(t)}\overline{V}(Se^{\alpha(t)}, \gamma(t))\,.$$

Notes on pricing formula for power options

Assume we know how to value an option with a payoff given by the function $\text{Payoff}(S)$. Then by a simple transformation we can easily value an option that has a payoff given by $\text{Payoff}(S^\alpha)$. Since we have S to the power of α this type of option is known as a **power** option. To show how this is done we perform the transformation of the independent variable given by

$$\mathcal{L} = S^\alpha \,.$$

Then the chain rule gives that S derivative in terms of $\mathcal{L}$ derivatives transform like

$$\begin{aligned}
\frac{\partial}{\partial S} &= \frac{\partial \mathcal{L}}{\partial S}\frac{\partial}{\partial \mathcal{L}} = \alpha S^{\alpha-1}\frac{\partial}{\partial \mathcal{L}} \\
\frac{\partial^2}{\partial S^2} &= \alpha(\alpha-1)S^{\alpha-2}\frac{\partial}{\partial \mathcal{L}} + \alpha S^{\alpha-1}\frac{\partial}{\partial S}\left(\frac{\partial}{\partial \mathcal{L}}\right) \\
&= \alpha(\alpha-1)S^{\alpha-2}\frac{\partial}{\partial \mathcal{L}} + \alpha S^{\alpha-1}\left(\alpha S^{\alpha-1}\frac{\partial}{\partial \mathcal{L}}\right)\left(\frac{\partial}{\partial \mathcal{L}}\right) \\
&= \alpha(\alpha-1)S^{\alpha-2}\frac{\partial}{\partial \mathcal{L}} + \alpha^2 S^{2(\alpha-1)}\frac{\partial^2}{\partial \mathcal{L}^2} \,.
\end{aligned}$$

Using the fact that $S = \mathcal{L}^{1/\alpha}$ we get

$$\frac{\partial}{\partial S} = \alpha \mathcal{L}^{\frac{\alpha-1}{\alpha}}\frac{\partial}{\partial \mathcal{L}} \quad \text{and} \quad \frac{\partial^2}{\partial S^2} = \alpha(\alpha-1)\mathcal{L}^{\frac{\alpha-2}{\alpha}}\frac{\partial}{\partial \mathcal{L}} + \alpha^2 \mathcal{L}^{\frac{2(\alpha-1)}{\alpha}}\frac{\partial^2}{\partial \mathcal{L}^2} \,.$$

Then the Black-Scholes Equation 57 in terms of $\mathcal{L}$ becomes

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 \mathcal{L}^{\frac{2}{\alpha}}\left[\alpha(\alpha-1)\mathcal{L}^{\frac{\alpha-2}{\alpha}}\frac{\partial V}{\partial \mathcal{L}} + \alpha^2\mathcal{L}^{\frac{2(\alpha-1)}{\alpha}}\frac{\partial^2 V}{\partial \mathcal{L}^2}\right] + (r-D)\mathcal{L}^{\frac{1}{\alpha}}\left[\alpha\mathcal{L}^{\frac{\alpha-1}{\alpha}}\frac{\partial V}{\partial \mathcal{L}}\right] - rV = 0 \,.$$

When we simplify this some we get

$$\frac{\partial V}{\partial t} + \frac{1}{2}\alpha^2\sigma^2\mathcal{L}^2\frac{\partial^2 V}{\partial \mathcal{L}^2} + \alpha\left[(r-D) + \frac{1}{2}\sigma^2(\alpha-1)\right]\mathcal{L}\frac{\partial V}{\partial \mathcal{L}} - rV = 0 \,. \tag{82}$$

Now since we can solve the original problem with a payoff of $\text{Payoff}(S)$ with parameter inputs σ, r, D, the solution used to value the power option, is given by simply taking $S \to S^\alpha$ and changing the definitions of the (σ, r, D) parameters. For example, solving Equation 57 with

$$\sigma \to \alpha\sigma \,, \quad \text{and} \quad D \to -(\alpha-1)\left(r - \frac{\alpha}{\alpha-1}D + \frac{\alpha}{2}\sigma^2\right) \,,$$

is equivalent to solving Equation 82.

Notes on pricing the log contract

The log-contract has a payoff given by $V(S,T) = \log\left(\frac{S}{E}\right)$. If we take $V(S,t)$ of the form

$$V(S,t) = a(t) + b(t)\log\left(\frac{S}{E}\right) \,. \tag{83}$$

Then the final condition of $V(S,t)$ when $t = T$ with the above expression for $V(S,t)$ implies final conditions on $a(t)$ and $b(t)$ of $a(T) = 0$ and $b(T) = 1$. From the above expression for $V(S,t)$ we find derivatives given by

$$
\begin{aligned}
\frac{\partial V}{\partial t} &= \dot{a}(t) + \dot{b}(t)\log\left(\frac{S}{E}\right) \\
\frac{\partial V}{\partial S} &= \frac{b(t)}{S} \quad \text{and} \quad \frac{\partial^2 V}{\partial S^2} = -\frac{b(t)}{S^2}\,.
\end{aligned}
$$

When we put these expressions into the Black-Scholes Equation 57 we get

$$
\dot{a} + \dot{b}\log\left(\frac{S}{E}\right) + \frac{1}{2}\sigma^2 S^2\left(-\frac{b}{S^2}\right) + (r - D)S\left(\frac{b}{S}\right) - r\left(a + b\log\left(\frac{S}{E}\right)\right) = 0\,,
$$

or when we simplify some

$$
\dot{a} - \left(\frac{1}{2}\sigma^2 - (r - D)\right)b - ra + \left(\dot{b} - rb\right)\log\left(\frac{S}{E}\right) = 0\,.
$$

Equating to zero the coefficient of $\log\left(\frac{S}{E}\right)$, solving with the final condition $b(T) = 1$ gives

$$
b(t) = e^{-r(T-t)}\,.
$$

We then find that $a(t)$ satisfies

$$
\dot{a} - ra = \left(\frac{1}{2}\sigma^2 - r + D\right)b = \left(\frac{1}{2}\sigma^2 - r + D\right)e^{-r(T-t)}\,.
$$

Which is an ordinary differential equation like 311. Then in this case here we have that the integrating factor $\mu(t)$ is given by $\mu(t) = e^{-rt}$ and $a(t)$ is

$$
\begin{aligned}
a(t) &= Ce^{rt} + e^{rt}\int e^{-rt'}\left(\frac{1}{2}\sigma^2 - r + D\right)e^{-r(T-t')}dt' \\
&= Ce^{rt} + e^{r(t-T)}\left(\frac{1}{2}\sigma^2 - r + D\right)t\,.
\end{aligned}
$$

Then we must pick the constant C such that $a(T) = 0$ or

$$
Ce^{rT} + \left(\frac{1}{2}\sigma^2 - r + D\right)T = 0 \quad \text{so} \quad C = -\left(\frac{1}{2}\sigma^2 - r + D\right)Te^{-rT}\,.
$$

With this $a(t)$ is given by

$$
\begin{aligned}
a(t) &= -\left(\frac{1}{2}\sigma^2 - r + D\right)Te^{-r(T-t)} + \left(\frac{1}{2}\sigma^2 - r + D\right)te^{-r(T-t)} \\
&= \left(r - D - \frac{1}{2}\sigma^2\right)(T - t)e^{-r(T-t)}\,,
\end{aligned}
$$

which is the solution given in the book.

We can explicitly evaluate the option value when the payoff is of a log contract type. Using the general option integration evaluation framework expressed by Equation 35 when we consider a payoff function given by

$$\text{Payoff}(S) = \max\left(\log\left(\frac{S}{E}\right), 0\right),$$

the integrand vanishes unless $\frac{S}{E} > 1$ or $S > E$. This means that we need to evaluate

$$\begin{aligned}
V &= \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_E^{+\infty} e^{-\left[\log(S'/S)+(r-\frac{1}{2}\sigma^2)(T-t)\right]^2/2\sigma^2(T-t)} \log\left(\frac{S'}{E}\right) \frac{dS'}{S'} \\
&= \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_E^{+\infty} e^{-\left[\log(S'/S)+(r-\frac{1}{2}\sigma^2)(T-t)\right]^2/2\sigma^2(T-t)} \log(S') \frac{dS'}{S'} \\
&- \log(E) \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_E^{+\infty} e^{-\left[\log(S'/S)+(r-\frac{1}{2}\sigma^2)(T-t)\right]^2/2\sigma^2(T-t)} \frac{dS'}{S'}.
\end{aligned}$$

When we split the original integrand into two parts using $\log\left(\frac{S'}{E}\right) = \log(S') - \log(E)$. This second integral is the value of a binary call option, see the discussion on Page 20, and is given by Equation 46 multiplied by $-\log(E)$ or

$$-\log(E)e^{-r(T-t)}N(d_2). \tag{84}$$

To evaluate the first integral we let

$$\begin{aligned}
v &= \frac{-\log(S'/S) + (r - D - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \quad \text{so} \\
\log(S') &= \log(S) + (r - D - \frac{1}{2}\sigma^2)(T-t) - \sigma\sqrt{T-t}\, v \quad \text{and} \\
dv &= -\frac{1}{\sigma\sqrt{T-t}} \frac{dS'}{S'}.
\end{aligned}$$

Now the end points of the integral transform to $v(E) = d_2$ and $v(+\infty) = -\infty$, so when we change the order the the integration and negate the integrand this first integral becomes

$$\begin{aligned}
& \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \left[\sigma\sqrt{T-t} \int_{-\infty}^{d_2} e^{-\frac{v^2}{2}} \left(\log(S) + (r - D - \frac{1}{2}\sigma^2)(T-t) - \sigma\sqrt{T-t}v\right) dv\right] \\
= & \frac{e^{-r(T-t)}}{\sqrt{2\pi}} \left[\left(\log(S) + (r - D - \frac{1}{2}\sigma^2)(T-t)\right)\sqrt{2\pi}N(d_2) - \sigma\sqrt{T-t} \int_{-\infty}^{d_2} v e^{-\frac{v^2}{2}} dv\right].
\end{aligned}$$

The first term above is given by

$$\left(\log(S) + (r - D - \frac{1}{2}\sigma^2)(T-t)\right) N(d_2)e^{-r(T-t)}, \tag{85}$$

while the second term above is given by

$$\begin{aligned}
-\frac{\sigma\sqrt{T-t}e^{-r(T-t)}}{\sqrt{2\pi}} \left[-e^{-\frac{v^2}{2}}\right|_{-\infty}^{d_2} &= -\frac{\sigma\sqrt{T-t}e^{-r(T-t)}}{\sqrt{2\pi}} \left(-e^{-\frac{{d_2}^2}{2}}\right) \\
&= \sigma\sqrt{T-t}N'(d_2)e^{-r(T-t)}.
\end{aligned} \tag{86}$$

Then combining the two parts given by Equations 85, and 84 into one expression and then adding the part given by Equation 84 we get that the value of our log contract, V, is given by

$$V = \left(\log\left(\frac{S}{E}\right) + (r - D - \frac{1}{2}\sigma^2)(T - t) \right) N(d_2) e^{-r(T-t)} + \sigma\sqrt{T - t} N'(d_2) e^{-r(T-t)} \,. \quad (87)$$

the same expression as in the book.

Early exercise and American options

Notes on the perpetual American put

In this section we perform the derivation of the value of a perpetual American put. The Black-Scholes equation is given by

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0\,.$$

If our solution depends on S only i.e $V = V(S)$ when we put this solution into the Black-Scholes equation we find

$$\frac{1}{2}\sigma^2 S^2 \frac{d^2 V}{dS^2} + rS\frac{dV}{dS} - rV = 0\,.$$

This is a Euler differential equation and has solution given by $V(S) = S^p$ for some p. Taking the S derivative of this ansatz gives

$$\begin{aligned}
\frac{dV}{dS} &= pS^{p-1} \\
\frac{d^2V}{dS^2} &= p(p-1)S^{p-2}\,,
\end{aligned}$$

which we can put into the equation above to obtain

$$\frac{1}{2}\sigma^2 S^2 p(p-1)S^{p-2} + rSpS^{p-1} - rS^p = 0\,.$$

Factoring S^p from the above equation we see that p must satisfy

$$\frac{1}{2}\sigma^2 p(p-1) + rp - r = 0\,, \tag{88}$$

or grouping powers of p we find that p solves

$$\frac{\sigma^2}{2}p^2 + \left(r - \frac{\sigma^2}{2}\right)p - r = 0\,.$$

Solving for p using the quadratic equation we find that two possible values for p are given by

$$p = \frac{-\left(r - \frac{\sigma^2}{2}\right) \pm \sqrt{(r - \frac{\sigma^2}{2})^2 + 4\left(\frac{\sigma^2}{2}\right)r}}{2\left(\frac{\sigma^2}{2}\right)} = \frac{\left(\frac{\sigma^2}{2} - r\right) \pm \left(r + \frac{\sigma^2}{2}\right)}{\sigma^2}\,.$$

Taking the plus sign in the above we find that one value for p is

$$p_+ = 1\,,$$

while if we take the minus sign we find that another value for p is

$$p_- = -\frac{2r}{\sigma^2}\,.$$

Thus the complete solution in this case is given by

$$V(S) = AS + BS^{-\frac{2r}{\sigma^2}},$$

for two arbitrary constants A and B. If $S \to +\infty$ then our puts value must go to zero i.e. $V(S) \to 0$ so we must take $A = 0$. If we assume that there is a maximum stock price S^* at which we will exercise our option then at S^* we must have our option value equal the payoff at that price or $V(S^*) = E - S^*$. In terms of B this means that

$$BS^{*-2r/\sigma^2} = E - S^* \quad \text{so} \quad B = (E - S^*)S^{*2r/\sigma^2}.$$

With this we have that $V(S)$ for a perpetual American put is given by

$$V(S) = (E - S^*)\left(\frac{S}{S^*}\right)^{2r/\sigma^2}. \tag{89}$$

As argued in the book we pick S^* to make the expression for $V(S)$ *as large as possible*. If we write $V(S)$ as $V(S) = S^{2r/\sigma^2}[(E - S^*)S^{*2r/\sigma^2}]$ so that the leading expression is not a function of S^* by setting the first derivative of this expression with respect to S^* equal to zero we get

$$\frac{\partial V}{\partial S^*} = S^{2r/\sigma^2}\left[-S^{*2r/\sigma^2} + \frac{2r}{\sigma^2}S^{*2r/\sigma^2-1}(E - S^*)\right] = 0.$$

When we solve for S^* we get

$$S^* = \frac{E}{1 + \sigma^2/2r}. \tag{90}$$

With this value of S^* we have

$$E - S^* = \frac{\sigma^2/2r}{1 + \sigma^2/2r}E, \tag{91}$$

and B is then given by

$$B = \left(\frac{(\sigma^2/2r)E}{1 + \sigma^2/2r}\right)\frac{E^{2r/\sigma^2}}{(1 + \sigma^2/2r)^{2r/\sigma^2}} = \frac{\sigma^2}{2r}\left(\frac{E}{1 + \frac{\sigma^2}{2r}}\right)^{\frac{2r}{\sigma^2}+1}.$$

With this value of B we find that the value $V(S)$ given by

$$V(S) = \begin{cases} \frac{\sigma^2}{2r}\left(\frac{E}{1+\frac{\sigma^2}{2r}}\right)^{\frac{2r}{\sigma^2}+1} S^{-\frac{2r}{\sigma^2}} & S > S^* \\ E - S & S < S^* \end{cases}, \tag{92}$$

where S^* is given by Equation 90. This function is plotted in Figure 1. The book then argues that at the point $S = S^*$ the slope of the function $(E - S^*)\left(\frac{S}{S^*}\right)^{-\frac{2r}{\sigma^2}}$ is tangent to the payoff function $E - S$. To show that we consider the difference of these two functions

$$D = (E - S^*)\left(\frac{S}{S^*}\right)^{-\frac{2r}{\sigma^2}} - (E - S),$$

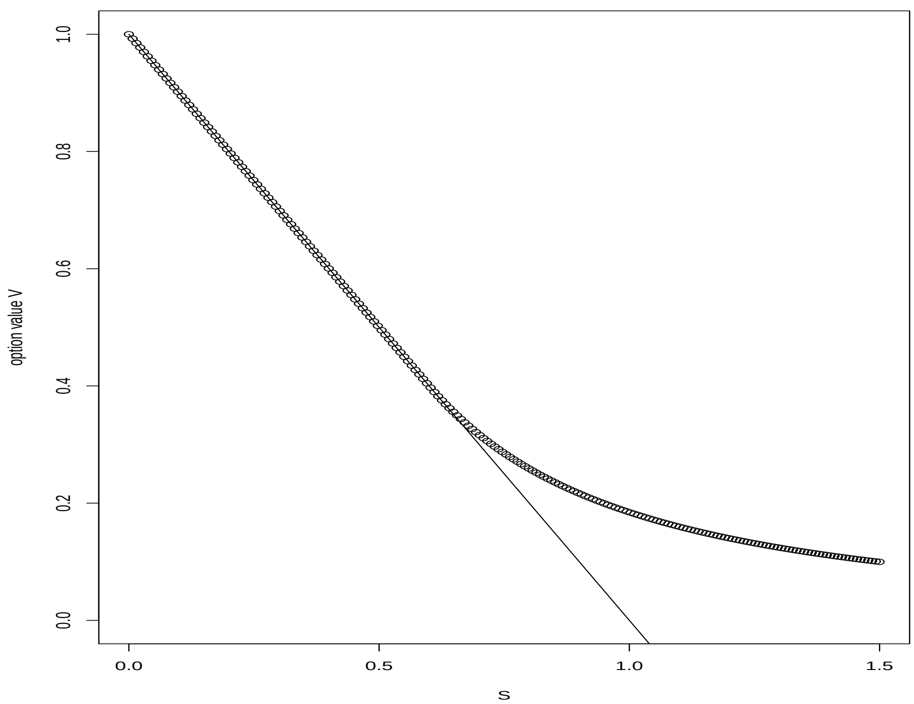

Figure 1: A plot of $V(S)$ given by Equation 92 for a perpetual American put for $E = 1$ and $\frac{\sigma^2}{2r} = 0.666$.

and take the S derivative of this expression evaluated at $S = S^*$. We find

$$\frac{\partial D}{\partial S} = \frac{E - S^*}{S^*}(E - S^*)S^{*\frac{2r}{\sigma^2}}\left(-\frac{2r}{\sigma^2}S^{-\frac{2r}{\sigma^2}-1}\right) + 1\,.$$

If we let $S = S^*$ we get

$$(E - S^*)S^{*\frac{2r}{\sigma^2}}\left(-\frac{2r}{\sigma^2}S^{*-\frac{2r}{\sigma^2}-1}\right) + 1\,,$$

or

$$-\frac{2r}{\sigma^2}(E - S^*)S^{*-1} + 1\,.$$

Since $E - S^*$ is given by Equation 91 this becomes

$$-\frac{E}{1 + \sigma^2/2r}\cdot\frac{1 + \sigma^2/2r}{E} + 1 = 0\,,$$

as claimed.

The book then argues that if we consider cases where the power function above meets the payoff function $E - S$ at a point where it is *not* tangent an arbitrage opportunity results. In the book's case (b), since $V(S) < \max(E - S, 0)$ we can create an arbitrage opportunity by buying the option for $-V(S)$ and immediately exercise the option to get a profit of $E - S$. The total profit of this strategy would then be

$$-V(S) + E - S > 0\,,$$

and we have a guaranteed profit in this case.

The perpetual American put (with a continuous dividend yield)

The solution to the dividend modified Black-Scholes equation in this case is given by

$$V(S) = AS^{\alpha^+} + BS^{\alpha^-}\,, \tag{93}$$

where $\alpha^- < 0 < \alpha^+$. Then since again as $S \to +\infty$ we expect that the put value $V(S) \to 0$, which means that $A = 0$ and we have $V(S) = BS^{\alpha^-}$. To match the payoff at some value S^* means that

$$BS^{*\alpha^-} = E - S^* \quad \text{or} \quad B = \frac{E - S^*}{S^{*\alpha^-}}\,. \tag{94}$$

With this expression we get that $V(S)$ is given by

$$V(S) = (E - S^*)\left(\frac{S}{S^*}\right)^{\alpha^-}\,.$$

As before we look for the value of S^* that makes this expression a maximum. Writing $V(S)$ as $V(S) = S^{\alpha^-}(E - S^*)S^{*-\alpha^-}$, to find the value of S^* that makes this expression a maximum we need solve

$$\frac{\partial V}{\partial S^*} = S^{\alpha^-}\left[-S^{*-\alpha^-} - \alpha^-(E - S^*)S^{*-\alpha^- -1}\right] = 0\,.$$

Multiply by $S^{*\alpha^- +1}$ and solve for S^* to get

$$S^* = \frac{E\alpha^-}{-1+\alpha^-} = \frac{E}{1-1/\alpha^-}\,. \tag{95}$$

With the above expression for S^* we have that $E - S^*$ is given by

$$E - S^* = -E\left(\frac{1/\alpha^-}{1-1/\alpha^-}\right)\,.$$

Using this we can compute B as

$$B = -\frac{1}{\alpha^-}\left(\frac{E}{1-1/\alpha^-}\right)^{1-\alpha^-}\,.$$

Thus we finally get that $V(S)$ is given by

$$V(S) = \begin{cases} -\frac{1}{\alpha^-}\left(\frac{E}{1-1/\alpha^-}\right)^{1-\alpha^-} S^{\alpha^-} & \text{when} \quad S > \frac{E}{1-1/\alpha^-} \\ E - S & \text{when} \quad S < \frac{E}{1-1/\alpha^-} \end{cases}\,,$$

for the valuation of a perpetual American put with dividends.

Notes on the perpetual American call

For a perpetual American call we must have $V \to 0$ as $S \to 0$ so the coefficient of B in Equation 93 must be zero, to give $V(S) = AS^{\alpha^+}$. Fitting our function V to the payoff at S^* gives

$$AS^{*\alpha^+} = S^* - E \quad \Rightarrow \quad A = \frac{S^* - E}{S^{*\alpha^+}}\,,$$

and the functional form for $V(S)$ then looks like

$$V(S) = \left(\frac{S^* - E}{S^{*\alpha^+}}\right) S^{\alpha^+} = (S^* - E)\left(\frac{S}{S^*}\right)^{\alpha^+}\,.$$

To find the explicit value of S^* we choose it to make $V(S)$ as large as possible. We write V as $V(S) = S^{\alpha^+}(S^* - E)S^{*-\alpha^+}$, then to maximize V we take the S^* derivative of V, set the result equal to zero, and solve for S^*. This equation is then

$$\frac{\partial V}{\partial S^*} = S^{\alpha^+}\left[S^{*-\alpha^+} - \alpha^+(S^* - E)S^{*-\alpha^+-1}\right] = 0\,.$$

If we divide by $S^{*-\alpha^+-1}$ we get $S^* + \alpha^+(S^* - E) = 0$ or

$$S^* = \frac{-E\alpha^+}{1-\alpha^+} = E\frac{1}{1-\frac{1}{\alpha^+}} \quad \text{and} \quad S^* - E = E\frac{1/\alpha^+}{1-1/\alpha^+}\,. \tag{96}$$

Then we have for A

$$A = E\frac{1/\alpha^+}{1-1/\alpha^+}\left(\frac{(1-1/\alpha^+)^{\alpha^+}}{E^{\alpha^+}}\right) = \frac{1}{\alpha^+}\left(\frac{E}{1-1/\alpha^+}\right)^{1-\alpha^+} \tag{97}$$

Probability Density Functions and First Exit Times

We begin with the assumption that our random variable y evolves according to

$$dy = A(y,t)dt + B(y,t)dX \,. \tag{98}$$

From which we can evaluate the expectation and the variance of this process over the time interval dt. We have that $E[dy] = A(y,t)dt$ and

$$E[dy^2] = E[A^2(y,t)dt^2 + 2A(y,t)B(y,t)dtdX + B^2(y,t)dX^2] \,.$$

Then using the rule-of-thumb that $E[dX] \sim dt^{1/2}$ in terms of dt the above becomes

$$E[dy^2] = A^2(y,t)dt^2 + 2A(y,t)B(y,t)dt^{3/2} + B^2(y,t)dt \,.$$

From this we can compute the variance of y as

$$\begin{aligned} \mathrm{Var}[dy] &= E[dy^2] - E[dy]^2 \\ &= 2A(y,t)B(y,t)dt^{3/2} + B^2(y,t)dt \approx B^2(y,t)dt \,, \end{aligned}$$

to leading order.

Notes on the trinomial random walk

We now discuss how we can pick $\phi^+(y,t)$ and $\phi^-(y,t)$ such that we match the mean and variance between the discrete model and the continuous processes over a small time step δt. The mean change of δy

$$\phi^+\delta y + (1 - \phi^+ - \phi^-)0 + \phi^-(-\delta y) = (\phi^+ - \phi^-)\delta y \,.$$

The mean change in δy^2 is

$$\phi^+\delta y^2 + (1 - \phi^+ - \phi^-)0^2 + \phi^-\delta y^2 = (\phi^+ - \phi^-)\delta y^2$$

Since $\mathrm{Var}[\delta y] = E[\delta y^2] - E[\delta y]^2$ we get

$$\begin{aligned} \mathrm{Var}[\delta y] &= (\phi^+ + \phi^-)\delta y^2 - (\phi^+ - \phi^-)^2\delta y^2 \\ &= (\phi^+ + \phi^- - (\phi^+ - \phi^-))\delta y^2 \,. \end{aligned}$$

From the mean and variance of the process dy computed above, when we match the mean and variance we get two conditions

$$(\phi^+ - \phi^-)\delta y = A(y,t)\delta t \tag{99}$$

$$(\phi^+ + \phi^- - (\phi^+ - \phi^-)^2)\delta y^2 = B^2(y,t)\delta t \tag{100}$$

Putting Equation 99 into Equation 100 gives

$$\left(\phi^+ + \phi^- - A^2\frac{\delta t^2}{\delta y^2}\right)\delta y^2 = B^2\delta t \,,$$

or

$$(\phi^+ + \phi^-)\delta y^2 = B^2\delta t + A^2\delta t^2 \approx B^2\delta t\,, \tag{101}$$

when we drop the $A^2\delta t^2$ term. Adding Equations 99 to 101 we get

$$2\phi^+ = A\frac{\delta t}{\delta y} + B^2\frac{\delta t}{\delta y^2} \quad \text{or} \quad \phi^+ = \frac{1}{2}\frac{\delta t}{\delta y^2}(B^2 + A\delta y)\,. \tag{102}$$

Next using Equation 99 to solve for ϕ^- in terms of ϕ^+ we have

$$\begin{aligned}
\phi^- &= \phi^+ - A\frac{\delta t}{\delta y} = \frac{1}{2}\left(\frac{\delta t}{\delta y^2}\right)(B^2 + A\delta y) - A\frac{\delta t}{\delta y} \\
&= \frac{1}{2}\left(\frac{\delta t}{\delta y^2}\right)(B^2 - A\delta y)\,.
\end{aligned} \tag{103}$$

Notes on the forward equation

In this derivation, the point (y,t) is the starting location and (y',t') is the ending location. We have the discrete density update equation

$$\begin{aligned}
p(y,t;y',t') &= \phi^-(y'+\delta y,t'-\delta t)p(y,t;y'+\delta y,t'-\delta t) \\
&+ (1-\phi^-(y',t'-\delta t) - \phi^+(y',t'-\delta t))p(y,t;y',t'-\delta t) \\
&+ \phi^+(y'-\delta y,t'-\delta t)p(y,t;y'-\delta y,t'-\delta t)\,.
\end{aligned} \tag{104}$$

The Taylor series we need to expand this are

$$\begin{aligned}
p(y,t;y'+\delta y,t'-\delta t) &= p(y,t;y',t') + \frac{\partial p}{\partial y'}\delta y - \frac{\partial p}{\partial t'}\delta t + \frac{1}{2}\frac{\partial^2 p}{\partial y'^2}\delta y^2 + \cdots \\
p(y,t;y',t'-\delta t) &= p(y,t;y',t') - \frac{\partial p}{\partial t'}\delta t + \cdots \\
p(y,t;y'-\delta y,t'-\delta t) &= p(y,t;y',t') - \frac{\partial p}{\partial y'}\delta y - \frac{\partial p}{\partial t'}\delta t + \frac{1}{2}\frac{\partial^2 p}{\partial y'^2}\delta y^2 + \cdots \\
\phi^-(y'+\delta y,t'-\delta t) &= \phi^-(y',t') + \frac{\partial \phi^-}{\partial y'}\delta y - \frac{\partial \phi^-}{\partial t'}\delta t + \frac{1}{2}\frac{\partial^2 \phi^-}{\partial y'^2}\delta y^2 + \cdots \\
\phi^+(y'-\delta y,t'-\delta t) &= \phi^+(y',t') + \frac{\partial \phi^+}{\partial y'}\delta y - \frac{\partial \phi^+}{\partial t'}\delta t + \frac{1}{2}\frac{\partial^2 \phi^+}{\partial y'^2}\delta y^2 + \cdots \\
\phi^-(y',t'-\delta t) &= \phi^-(y',t') - \frac{\partial \phi^-}{\partial t'}\delta t + \cdots \\
\phi^+(y',t'-\delta t) &= \phi^+(y',t') - \frac{\partial \phi^+}{\partial t'}\delta t + \cdots\,.
\end{aligned}$$

We put these expressions into the right-hand-side of Equation 104 in the MATHEMATICA file `forward_equation_derivation.nb` and use $\delta y \sim \delta t^{1/2}$ we get (using p to denote $p(y,t;y',t')$ we have

$$\begin{aligned}
p = p &+ \left[\frac{\partial \phi^-}{\partial y'}p - \frac{\partial \phi^+}{\partial y'}p + \frac{\partial p}{\partial y'}\phi^- - \frac{\partial p}{\partial y'}\phi^+\right]\delta t^{1/2} \\
&+ \left[-\frac{\partial p}{\partial t'} + \frac{\partial p}{\partial y'}\frac{\partial \phi^-}{\partial y'} + \frac{\partial p}{\partial y'}\frac{\partial \phi^+}{\partial y'} + \frac{1}{2}\left(\frac{\partial^2 \phi^-}{\partial y'^2}p + \frac{\partial^2 \phi^+}{\partial y'^2}p + \phi^-\frac{\partial^2 p}{\partial y'^2} + \phi^+\frac{\partial^2 p}{\partial y'^2}\right)\right]\delta t + O(\delta t^{3/2})\,,
\end{aligned}$$

or

$$
\begin{aligned}
0 &= \left[\frac{\partial(\phi^- p)}{\partial y'} - \frac{\partial(\phi^+ p)}{\partial y'}\right]\delta t^{1/2} \\
&+ \left[-\frac{\partial p}{\partial t'} + \frac{\partial p}{\partial y'}\frac{\partial}{\partial y'}(\phi^- + \phi^+) + \frac{1}{2}\left(p\frac{\partial^2}{\partial y'^2}(\phi^- + \phi^+) + (\phi^- + \phi^+)\frac{\partial^2 p}{\partial y'^2}\right)\right]\delta t + O(\delta t^{3/2}) \\
&= \frac{\partial}{\partial y'}\left[p(\phi^- - \phi^+)\right]\delta t^{1/2} \\
&+ \left[-\frac{\partial p}{\partial t'} + \frac{1}{2}\left(p\frac{\partial^2}{\partial y'^2}(\phi^- + \phi^+) + 2\frac{\partial p}{\partial y'}\frac{\partial}{\partial y'}(\phi^- + \phi^+) + \frac{\partial^2 p}{\partial y'^2}(\phi^- + \phi^+)\right)\right]\delta t + O(\delta t^{3/2})\,.
\end{aligned}
$$

From Equations 102 and 103 evaluated at (y', t') we have that

$$
\phi^- - \phi^+ = -\frac{\delta t}{\delta y^2}A(y', t')\delta y = -\frac{\delta t}{\delta y}A(y', t') = -\delta t^{1/2}A(y', t')\,,
$$

and

$$
\phi^- + \phi^+ = \frac{\delta t}{\delta y^2}B(y', t')^2 = B(y', t')^2\,,
$$

so the above becomes

$$
0 = -\frac{\partial}{\partial y'}(pA)\delta t + \left[-\frac{\partial p}{\partial t'} + \frac{1}{2}\left(p\frac{\partial^2 B^2}{\partial y'^2} + 2\frac{\partial p}{\partial y'}\frac{\partial B^2}{\partial y'} + \frac{\partial^2 p}{\partial y'^2}B^2\right)\right]\delta t + O(\delta t^{3/2})\,.
$$

Then to leading order for p we get

$$
\frac{\partial p}{\partial t'} = -\frac{\partial pA(y', t')}{\partial y'} + \frac{1}{2}\frac{\partial^2 pB(y', t')^2}{\partial y'^2}\,, \tag{105}
$$

which is the forward Kolmogorov or Fokker-Planck equation.

Notes on expected first exit times

If the function $u(y, t)$ is the **expected first exit time** then u must satisfy the differential equation

$$
\frac{\partial u}{\partial t} + \frac{1}{2}B(y, t)^2\frac{\partial^2 u}{\partial y^2} + A(y, t)\frac{\partial u}{\partial y} = -1\,. \tag{106}
$$

For the logarithmic asset $y = S$ and since $dS = \mu S dt + \sigma S dX$ we have $A = \mu S$ and $B = \sigma S$, then Equation 107 we must solve is given by

$$
\frac{1}{2}\sigma^2 S^2\frac{d^2 u}{dS^2} + \mu S\frac{du}{dS} = -1\,, \tag{107}
$$

with boundary conditions $u(S_0) = u(S_1) = 0$. Writing this as

$$
\frac{d}{dS}\left(\frac{du}{dS}\right) + \frac{2\mu}{\sigma^2}\frac{1}{S}\frac{du}{dS} = -\frac{2}{\sigma^2 S^2}\,,
$$

we can use the results from the appendix on Page 154, to solve for $\frac{du}{dS}$. We first find an integrating factor given by

$$\exp\left(\int \frac{2\mu}{\sigma^2}\frac{dS'}{S'}\right) = \exp\left(\log\left(S^{\frac{2\mu}{\sigma^2}}\right)\right) = S^{\frac{2\mu}{\sigma^2}} = S^{\kappa}\,.$$

Where we have defined κ as $\kappa \equiv \frac{2\mu}{\sigma^2}$. With this we have that $\frac{du}{dS}$ given by

$$\begin{aligned}
\frac{du}{dS} &= \frac{1}{S^{\kappa}}\left[\int S'^{\kappa}\left(-\frac{2}{\sigma^2 S'^2}\right)dS' + C\right] \\
&= -\frac{2}{\sigma^2(\kappa-1)}S^{-1} + CS^{-\kappa}\,.
\end{aligned}$$

Integrating this expression once more to get $u(S)$ we find

$$u(S) = \frac{2}{\sigma^2(1-\kappa)}\log(S) + \frac{S^{1-\kappa}}{1-\kappa}C + D\,. \tag{108}$$

We now need to pick C and D to match the boundary conditions $u(S_0) = u(S_1) = 0$. This means we need to solve

$$\frac{2}{\sigma^2(1-\kappa)}\log(S_0) + \frac{S_0^{1-\kappa}}{1-\kappa}C + D = 0 \tag{109}$$

$$\frac{2}{\sigma^2(1-\kappa)}\log(S_1) + \frac{S_1^{1-\kappa}}{1-\kappa}C + D = 0\,. \tag{110}$$

If we take the negative of the first equation and add this to the second equation we find that C is given by

$$C = -\frac{2}{\sigma^2}\frac{\log(S_1/S_0)}{S_1^{1-\kappa} - S_0^{1-\kappa}}\,.$$

When we put this into Equation 109 and then solve for D we get

$$D = -\frac{2}{\sigma^2(1-\kappa)}\log(S_0) + \frac{2}{\sigma^2(1-\kappa)}\frac{S_0^{1-\kappa}}{S_1^{1-\kappa} - S_0^{1-\kappa}}\log(S_1/S_0)\,.$$

Then with C and D specified we find $u(S)$ given by

$$\begin{aligned}
u(S) &= \frac{2}{\sigma^2(1-\kappa)}\left(\log(S/S_0) + \frac{-S^{1-\kappa} + S_0^{1-\kappa}}{S_1^{1-\kappa} - S_0^{1-\kappa}}\log(S_1/S_0)\right) \\
&= \frac{1}{\frac{1}{2}\sigma^2 - \mu}\left(\log(S/S_0) - \frac{1-(S/S_0)^{1-2\mu/\sigma^2}}{1-(S_1/S_0)^{1-2\mu/\sigma^2}}\log(S_1/S_0)\right)\,.
\end{aligned} \tag{111}$$

Notes on expectations and Black-Scholes

Recall the backwards Kolmogorov equation for the random variable y that satisfies the stochastic differential equation $dy = A(y,t)dt + B(y,t)dX$ or

$$\frac{\partial p}{\partial t} + \frac{1}{2}B^2(y,t)\frac{\partial^2 p}{\partial y^2} + A(y,t)\frac{\partial p}{\partial y} = 0\,. \tag{112}$$

When the variable y is the stock price S and satisfies geometric Brownian motion with $dS = \mu S dt + \sigma S dX$ we have $A = \mu S$ and $B = \sigma S$ so that the the backwards Kolmogorov equation in this case becomes

$$\frac{\partial p}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 p}{\partial S^2} + \mu S \frac{\partial p}{\partial S} = 0\,. \tag{113}$$

If we have a payoff at $t = T$ depending on the final value of S given by $F(S)$, then we can calculate the expected payoff by solving the backwards Kolmogorov equation with a final condition given by this payoff. Call this function $p_F(S,t)$, so that $p_F(S,T) = F(S)$. Consider the discounted value of $p_F(S,t)$ or

$$e^{-r(T-t)} p_F(S,t)\,.$$

If we call this function $V(S,t)$ then $p_F(S,t) = e^{r(T-t)}V(S,t)$ and derivative of p_F in terms of V become

$$\begin{aligned} \frac{\partial p_F}{\partial S} &= e^{r(T-t)}\frac{\partial V}{\partial S} \quad \text{and} \quad \frac{\partial^2 p_F}{\partial S^2} = e^{r(T-t)}\frac{\partial^2 V}{\partial S^2} \\ \frac{\partial p_F}{\partial t} &= -re^{-r(T-t)}V + e^{r(T-t)}\frac{\partial V}{\partial t}\,. \end{aligned}$$

When we put this into the backwards Kolmogorov equation we get

$$-re^{-r(T-t)}V + e^{r(T-t)}\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 e^{r(T-t)}\frac{\partial^2 V}{\partial S^2} + \mu S\frac{\partial V}{\partial S} = 0\,,$$

or

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \mu S\frac{\partial V}{\partial S} - rV = 0\,.$$

This is just like the Black-Scholes equation when the coefficient of $\frac{\partial V}{\partial S}$ is rS rather than μS.

Notes on a common misconception of a call options delta

In this section we consider how to compute the probability that a stock will be above the strike E at the time T given that at an earlier time t it has value S. To solve this problem we will need to solve the forward Kolmogorov equation with initial conditions given by a delta function. This solution to this differential equation is quoted in the book and is given by

$$p(S,t;S',t') = \frac{1}{\sigma S'\sqrt{2\pi(t'-t)}} e^{-\frac{(\log(S/S') + (\mu - \frac{1}{2}\sigma^2)(t'-t))^2}{2\sigma^2(t'-t)}}\,. \tag{114}$$

To evaluate the total probability that the stock ends above E at the time $t' = T$ is given by

$$\int_{S'=E}^{\infty} p(S,t;S',T)dS' = \frac{1}{\sigma\sqrt{2\pi(T-t)}}\int_{S'=E}^{\infty}\frac{1}{S'}e^{\frac{(\log(S/S') + (\mu - \frac{1}{2}\sigma^2)(T-t))^2}{2\sigma^2(T-t)}} dS'\,.$$

To integrate this let

$$\begin{aligned} v &= \frac{\log(S/S') + (\mu - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \quad \text{so} \\ dv &= -\frac{1}{S'}\frac{1}{\sigma\sqrt{T-t}}dS'\,, \end{aligned}$$

and we get

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\log(S/E)+(\mu-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}} e^{-\frac{1}{2}v^2} dv = N\left(\frac{\log(S/E)+(\mu-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right).$$

The book defines the expression in the argument to $N(\cdot)$ as d_1'. This differs from d_1 given by Equation 37 in two ways. One way is that the r in the coefficient of $T-t$ in d_1 is replaced with a μ in the above expression. A second way is that the *sign* of $\frac{1}{2}\sigma^2$ in the above expression is negative where in the expression for d_1 it is positive.

Multi-asset options

Notes on exchange options

The payoff of the exchange option considered can be written

$$
\begin{aligned}
V(S_1, S_2, T) &= \max(q_1 S_1 - q_2 S_2, 0) = \max\left(q_1 S_2\left(\frac{S_1}{S_2} - \frac{q_2}{q_1}\right), 0\right) \\
&= q_1 S_2 \max\left(\xi - \frac{q_2}{q_1}, 0\right) ,
\end{aligned}
$$

with $\xi \equiv \frac{S_1}{S_2}$. Thus in terms of the function H defined with $V(S_1, S_2, t) = q_1 S_2 H(\xi, t)$ we have our final condition given by

$$
H(\xi, T) = \max\left(\xi - \frac{q_2}{q_1}, 0\right) .
$$

If we now consider the derivatives of V with respect to S_1 and S_2 in terms of the new variable ξ. Recalling that the definition of the partial derivative of a function with respect to S_1 means that we keep the other variables (like S_2 and t) constant we find that

$$
\begin{aligned}
\frac{\partial}{\partial S_1} &= \frac{\partial \xi}{\partial S_1}\frac{\partial}{\partial \xi} = \frac{1}{S_2}\frac{\partial}{\partial \xi} && (115) \\
\frac{\partial}{\partial S_2} &= \frac{\partial \xi}{\partial S_2}\frac{\partial}{\partial \xi} = -\frac{S_1}{S_2^2}\frac{\partial}{\partial \xi} = -\frac{\xi}{S_2}\frac{\partial}{\partial \xi} && (116) \\
\frac{\partial^2}{\partial {S_1}^2} &= \frac{\partial}{\partial S_1}\left(\frac{1}{S_2}\frac{\partial}{\partial \xi}\right) = \frac{1}{S_2}\frac{\partial}{\partial S_1}\left(\frac{\partial}{\partial \xi}\right) = \frac{1}{S_1^2}\frac{\partial}{\partial \xi}\left(\frac{\partial}{\partial \xi}\right) = \frac{1}{S_2^2}\frac{\partial^2}{\partial \xi^2} && (117) \\
\frac{\partial^2}{\partial {S_2}^2} &= \frac{\partial}{\partial S_2}\left(-\frac{S_1}{S_2^2}\frac{\partial}{\partial \xi}\right) = -S_1\frac{\partial}{\partial S_2}\left(\frac{1}{S_2^2}\frac{\partial}{\partial \xi}\right) \\
&= -S_1\left[-\frac{2}{S_2^3}\frac{\partial}{\partial \xi} + \frac{1}{S_2^2}\frac{\partial}{\partial S_2}\left(\frac{\partial}{\partial \xi}\right)\right] = -S_1\left[-\frac{2}{S_2^3}\frac{\partial}{\partial \xi} + \frac{1}{S_2^2}\left(-\frac{\xi}{S_2}\frac{\partial^2}{\partial \xi^2}\right)\right] \\
&= \frac{2\xi}{S_2^2}\frac{\partial}{\partial \xi} + \frac{\xi^2}{S_2^2}\frac{\partial^2}{\partial \xi^2} && (118) \\
\frac{\partial^2}{\partial S_1 \partial S_2} &= \frac{\partial}{\partial S_1}\left(-\frac{\xi}{S_2}\frac{\partial}{\partial \xi}\right) = -\frac{1}{S_2}\frac{\partial}{\partial S_1}\left(\xi\frac{\partial}{\partial \xi}\right) \\
&= -\frac{1}{S_2^2}\frac{\partial}{\partial \xi}\left(\xi\frac{\partial}{\partial \xi}\right) = -\frac{1}{S_2^2}\left[\frac{\partial}{\partial \xi} + \xi\frac{\partial^2}{\partial \xi^2}\right] . && (119)
\end{aligned}
$$

These give the formulas for the transformation of the S_1 and S_2 derivatives into derivatives with respect to ξ. In the Black-Scholes equation we have S_1 and S_2 derivatives of the function V which we need to convert into derivatives of the function H. We find given how V and H

are related and the above relationships that

$$\frac{\partial V}{\partial S_1} = q_1 S_2 \frac{\partial H}{\partial S_1} = q_1 \frac{S_2}{S_2}\frac{\partial H}{\partial \xi} = q_1 \frac{\partial H}{\partial \xi} \tag{120}$$

$$\frac{\partial^2 V}{\partial {S_1}^2} = \frac{\partial}{\partial S_1}\left(q_1 \frac{\partial H}{\partial \xi}\right) = \frac{q_1}{S_2}\frac{\partial^2 H}{\partial \xi^2} \tag{121}$$

$$\frac{\partial V}{\partial S_2} = q_1 \frac{\partial}{\partial S_2}(S_2 H) = q_1\left[H + S_2 \frac{\partial H}{\partial S_2}\right] = q_1\left[H - \xi\frac{\partial H}{\partial \xi}\right] \tag{122}$$

$$\begin{aligned}\frac{\partial^2 V}{\partial {S_2}^2} &= \frac{\partial}{\partial S_2}\left[q_1\left(H - \xi\frac{\partial H}{\partial \xi}\right)\right] = -q_1\frac{\xi}{S_2}\left[\frac{\partial H}{\partial \xi} - \frac{\partial H}{\partial \xi} - \xi\frac{\partial^2 H}{\partial \xi^2}\right] \\ &= \frac{q_1 \xi^2}{S_2}\frac{\partial^2 H}{\partial \xi^2}\end{aligned} \tag{123}$$

$$\frac{\partial^2 V}{\partial S_1 \partial S_2} = \frac{\partial}{\partial S_1}\left(\frac{\partial V}{\partial S_2}\right) = q_1 \frac{\partial}{\partial S_1}\left(H - \xi\frac{\partial H}{\partial \xi}\right) \tag{124}$$

$$= \frac{q_1}{S_2}\left(\frac{\partial H}{\partial \xi} - \frac{\partial H}{\partial \xi} - \xi\frac{\partial^2 H}{\partial \xi^2}\right) = -\frac{q_1 \xi}{S_2}\frac{\partial^2 H}{\partial \xi^2}\,. \tag{125}$$

It is these derivatives we put into the two dimensional Black-Scholes equation

$$\begin{aligned}&\frac{\partial V}{\partial t} + \frac{1}{2}\left[\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial {S_1}^2} + 2\sigma_1\sigma_2\rho_{12} S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} + \sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial {S_2}^2}\right] \\ &\quad + (r - D_1) S_1 \frac{\partial V}{\partial S_1} + (r - D_2) S_2 \frac{\partial V}{\partial S_2} - rV = 0\,,\end{aligned}$$

to get

$$\begin{aligned}&q_1 S_2 \frac{\partial H}{\partial t} + \frac{1}{2}\sigma_1^2 S_1^2 \frac{q_1}{S_2}\frac{\partial^2 H}{\partial \xi^2} - \sigma_1\sigma_2\rho_{12} S_1 S_2 \frac{q_1 \xi}{S_2}\frac{\partial^2 H}{\partial \xi^2} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{q_1 \xi^2}{S_2}\frac{\partial^2 H}{\partial \xi^2} \\ &\quad + q_1 (r - D_1) S_1 \frac{\partial H}{\partial \xi} + q_1 (r - D_2) S_2 \left[H - \xi\frac{\partial H}{\partial \xi}\right] - r q_1 S_2 H = 0\,.\end{aligned}$$

If we divide by $q_1 S_2$ and simplify some we get

$$\frac{\partial H}{\partial t} + \frac{1}{2}(\sigma_1^2 - 2\sigma_1\sigma_2\rho_{12} + \sigma_2^2)\xi^2 \frac{\partial^2 H}{\partial \xi^2} + (D_2 - D_1)\xi\frac{\partial H}{\partial \xi} - D_2 H = 0\,.$$

If we define the coefficient in front of $\xi^2 \frac{\partial^2 H}{\partial \xi^2}$ as $\frac{1}{2}\sigma'^2$ then the above equation is exactly the Black-Scholes equation with simple a parameter change

$$r \to D_2\,, \quad D \to D_1\,, \quad \sigma \to \sigma'\,, \quad \text{and} \quad E \to \frac{q_2}{q_1}\,.$$

Then we know the solution for $H(\xi, t)$ and it is given by Equation 36 or

$$H(\xi, t) = \xi e^{-D_1(T-t)} N(d_1') - \frac{q_2}{q_1} e^{-D_2(T-t)} N(d_2')\,,$$

with d_1' and d_2' given by

$$\begin{aligned}d_1' &= \frac{\log\left(\frac{q_1}{q_2}\frac{S_1}{S_2}\right) + (D_2 - D_1 + \frac{1}{2}\sigma'^2)(T-t)}{\sigma'\sqrt{T-t}} \\ d_2' &= d_1' - \sigma'\sqrt{T-t}\,,\end{aligned}$$

Then with this expression for $H(\xi, t)$ we have that V is given by

$$
\begin{aligned}
V(S_1, S_2, t) &= q_1 S_2 H(\xi, t) \\
&= q_1 S_1 e^{-D_1(T-t)} N(d_1') - q_2 S_2 e^{-D_2(T-t)} N(d_2'),
\end{aligned}
$$

Now using Equation 120 and Equation 122 the hedged portfolio Π for this option is given by

$$
\Pi = V - \Delta_1 S_1 - \Delta_2 S_2 = q_1 S_2 H - q_1 S \frac{\partial H}{\partial \xi} - \left(q_1 H - q_1 \xi \frac{\partial H}{\partial \xi} \right) S_2 = 0 .
$$

Notes on quantos

We can consider the differential of the product $S_N S_\$$ using the multidimensional version of Ito's lemma. We have $d(S_N S_\$)$ given by

$$
\begin{aligned}
& \left(\frac{\partial}{\partial t}(S_N S_\$) + \frac{1}{2} \left[\sigma_N^2 S_N^2 \frac{\partial^2}{\partial {S_N}^2}(S_N S_\$) + 2\sigma_N \sigma_\$ \, \rho S_N S_\$ \frac{\partial^2}{\partial S_N \partial S_\$}(S_N S_\$) + \sigma_\$^2 S_\$^2 \frac{\partial^2}{\partial {S_\$}^2}(S_N S_\$) \right] \right) dt \\
+ \quad & \frac{\partial}{\partial S_N}(S_N S_\$) dS_N + \frac{\partial}{\partial S_\$}(S_N S_\$) dS_\$ \\
= \quad & \sigma_N \sigma_\$ \, \rho S_N S_\$ dt + S_\$ dS_N + S_N dS_\$,
\end{aligned}
$$

which is the expression used in the book for the change in the value of the quanto's portfolio.

How to Delta Hedge

Warning: In this chapter I attempted to duplicate several of the results stated in the chapter. In a couple of places I found results that differed from what the book presented. I checked my algebra several times and couldn't find any mistakes. These differences maybe due to typos or errors in my algebra. If anyone gets a different results from the book or these notes (or the same) please contact me.

Notes on what if actual and implied volatility are different

Consider an at the money (ATM) straddle or a portfolio that is long one call option and one long one put option with the same strike and time till expiration so $V = C + P$. If we are close to expiration then from the two expressions given in the book earlier

$$\begin{aligned}
C &\approx 0.4 S e^{-D(T-t)} \sigma \sqrt{T-t} \\
P &\approx 0.4 S e^{-D(T-t)} \sigma \sqrt{T-t} \,.
\end{aligned}$$

So if we add these two contracts

$$V = C + P \approx 0.8 S e^{-D(T-t)} \sigma \sqrt{T-t} \,.$$

Since $\sqrt{\frac{2}{\pi}} \approx 0.8$ when $D = 0$ this is the result considered in the book. If the *true* volatility σ is larger than the *implied* volatility $\tilde{\sigma}$ then the profit from this straddle will be proportional to the difference between σ and $\tilde{\sigma}$ or

$$\text{PnL} \approx 0.8 S (\sigma - \tilde{\sigma}) \sqrt{T-t} \,,$$

when $D = 0$.

Notes on hedging with actual volatility σ

In this section we further document and provide more extensive derivations of many of the results in the book. We start with the portfolio *before* the time step dt which in net is given by

$$V^i - \Delta^a S + (-V^i + \Delta^a S) = 0 \,. \tag{126}$$

Then after a discrete amount of time, each "term" representing the option, the stock position, and any left over cash, in the above expression changes according to

$$V^i + dV^i + (-\Delta^a (S + dS)) + \left[(-V^i + \Delta^a S)(1 + r dt) - \Delta^a D S dt \right] \,.$$

When we recall Equation 126 the above becomes

$$\text{PnL}_{t,t+dt} \equiv dV^i - \Delta^a dS + (-V^i + \Delta^a S) r dt - \Delta^a D S dt \,, \tag{127}$$

which is the expression in the book. If the option had been priced with the actual volatility σ then there must be no profit (or loss) from this transaction and the expression above will be zero. This is expressed by replacing "i" with "a" in the above to get

$$dV^a - \Delta^a dS - r(V^a - \Delta^a S)dt - \Delta^a DSdt = 0\,.$$

We can subtract this last expression (which has value zero) from the previous one to get

$$dV^i - dV^a + r(-V^i dt) + rV^a dt = dV^i - dV^a - r(V^i - V^a)dt = e^{rt} d(e^{-rt}(V^i - V^a))\,.$$

This is the profit we obtain over the time t to $t + dt$.

The total profit is the integral of the infinitesimal profit $e^{rt_0} d(e^{-rt}(V^i - V^a))$ or

$$\begin{aligned} e^{rt_0} \int_{t_0}^{T} d(e^{-rt}(V^i - V^a)) &= e^{rt_0} \left[e^{-rt}(V^i - V^a) \right|_{t_0}^{T} \\ &= e^{rt_0} \left[e^{-rT}(V^i(T) - V^a(T)) - e^{-rt_0}(V^i(t_0) - V^a(t_0)) \right] \\ &= -(V^i(t_0) - V^a(t_0)) = V^a - V^i\,, \end{aligned}$$

which shows that there is a guaranteed profit for this strategy. We can also consider the one time step mark-to-market profit and loss in Equation 127 using Ito's lemma. For the term dV^i we have

$$dV^i = \theta^i dt + \Delta^i dS + \frac{1}{2}\sigma^2 S^2 \Gamma^i dt\,,$$

to get

$$\mathrm{PnL}_{t,t+dt} = \theta^i dt + \Delta^i dS + \frac{1}{2}\sigma^2 S^2 \Gamma^i dt - \Delta^a dS - r(V^i - \Delta^a S)dt - \Delta^a DSdt\,.$$

Since our stock price changes according to $dS = \mu S dt + \sigma S dX$ the above becomes

$$\theta^i dt + (\Delta^i - \Delta^a)\mu S dt + \frac{1}{2}\sigma^2 S^2 \Gamma^i dt + \sigma S dX(\Delta^i - \Delta^a) - r(V^i - \Delta^a S)dt - \Delta^a DSdt\,.$$

Recall the Black-Scholes equation

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + (r - D)SV_S - rV = 0\,,$$

with $\sigma \to \tilde{\sigma}$ and written in terms of "the greeks" of

$$\theta^i = -\frac{1}{2}\tilde{\sigma}^2 S^2 \Gamma^i - (r - D)S\Delta^i + rV^i\,. \tag{128}$$

Using this we can replace θ^i in the expression for $\mathrm{PnL}_{t,t+dt}$ above to get

$$\begin{aligned} \mathrm{PnL}_{t,t+dt} &= \frac{1}{2}(\sigma^2 - \tilde{\sigma}^2)S^2\Gamma^i dt + (\Delta^i - \Delta^a)\mu S dt + (\Delta^i - \Delta^a)\sigma S dX \\ &+ \Delta^a r S dt - \Delta^a DSdt - rS\Delta^i dt + DS\Delta^i dt \\ &= \frac{1}{2}(\sigma^2 - \tilde{\sigma}^2)S^2\Gamma^i dt + (\Delta^i - \Delta^a)\mu S dt + (\Delta^i - \Delta^a)\sigma S dX \\ &- (\Delta^i - \Delta^a) r S dt + (\Delta^i - \Delta^a) SDdt \\ &= \frac{1}{2}(\sigma^2 - \tilde{\sigma}^2)S^2\Gamma^i dt + (\Delta^i - \Delta^a)\left[(\mu - r + D)Sdt + \sigma S dX\right]\,. \end{aligned}$$

Notes on hedging with implied volatility $\tilde{\sigma}$

When we hedge with implied volatility $\tilde{\sigma}$ the total portfolio initially is the same as given by Equation 127 but with $\Delta^a \to \Delta^i$ or

$$\text{PnL}_{t,t+dt} = dV^i - \Delta^i dS - r(V^i - \Delta^i S)dt - \Delta^i DSdt\,.$$

We again use Ito's lemma to replace dV^i with an expression in terms of Θ^i, Δ^i, Γ^i to get

$$\text{PnL}_{t,t+dt} = \theta^i dt + \frac{1}{2}\sigma^2 S^2 \Gamma^i dt + \Delta^i dS - \Delta^i dS - r(V^i - \Delta^i S)dt - \Delta^i DSdt\,,$$

or grouping terms

$$\text{PnL}_{t,t+dt} = \theta^i dt + \frac{1}{2}\sigma^2 S^2 \Gamma^i dt - r(V^i - \Delta^i S)dt - \Delta^i DSdt\,.$$

Now use Black-Scholes with $\sigma \to \tilde{\sigma}$ or Equation 128 we get

$$-\frac{1}{2}\tilde{\sigma}^2 S^2 \Gamma^i dt - (r - D)S\Delta^i + rV^i + \frac{1}{2}\sigma^2 S^2 \Gamma^i dt - r(V^i - \Delta^i S)dt - \Delta^i DSdt\,,$$

which simplifies to

$$\text{PnL}_{t,t+dt} = \frac{1}{2}(\sigma^2 - \tilde{\sigma}^2)S^2\Gamma^i dt\,. \tag{129}$$

The present value of this profit is $e^{-r(t-t_0)}$ multiplied by the above, so the total profit is

$$\int_{t_0}^{T} e^{-r(t-t_0)}\frac{1}{2}(\sigma^2 - \tilde{\sigma}^2)S^2\Gamma^i dt = \frac{1}{2}(\sigma^2 - \tilde{\sigma}^2)\int_{t_0}^{T} e^{-r(t-t_0)}S^2\Gamma^i dt\,.$$

As explained in the book if you hedge with a volatility σ_h then it can be shown that the principal value of the profit is given by

$$V(S,t;\sigma_h) - V(S,t;\tilde{\sigma}) + \frac{1}{2}(\sigma^2 - \sigma_h^2)\int_{t_0}^{T} e^{-r(t-t_0)}S^2\Gamma^h dt\,. \tag{130}$$

It would seem to be a good idea to hedge with a volatility σ_h that maximized this expression.

The expected profit after hedging using implied volatility

For the differential equation

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + \mu S\frac{\partial P}{\partial S} + \frac{1}{2}(\sigma^2 - \tilde{\sigma}^2)e^{-r(t-t_0)}S^2\Gamma^i\frac{\partial P}{\partial I} = 0\,, \tag{131}$$

with the final condition $P(S,I,T) = I$ when we consider a solution of the form

$$P(S,I,t) = I + H(S,t)\,,$$

we have derivatives of P in terms of H given by

$$\frac{\partial P}{\partial t} = \frac{\partial H}{\partial t}\,, \quad \frac{\partial P}{\partial S} = \frac{\partial H}{\partial S}\,, \quad \frac{\partial^2 P}{\partial S^2} = \frac{\partial^2 H}{\partial S^2}\,, \quad \frac{\partial P}{\partial I} = 1\,.$$

So that the differential equation for H satisfies

$$\frac{\partial H}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 H}{\partial S^2} + \mu S \frac{\partial H}{\partial S} + \frac{1}{2}(\sigma^2 - \tilde{\sigma}^2)e^{-r(t-t_0)} S^2 \Gamma^i = 0\,. \tag{132}$$

First consider the source term $\frac{1}{2}(\sigma^2 - \tilde{\sigma}^2)e^{-r(t-t_0)} S^2 \Gamma^i$ recall that when we hedge with implied volatility $\tilde{\sigma}$, the expression for Γ^i is given by Equation 53 (with $D = 0$) or

$$\Gamma^i = \frac{\partial^2 V}{\partial S^2} = \frac{N'(d_1)}{\tilde{\sigma} S \sqrt{T-t}}\,,$$

and using the facts

$$\begin{aligned} N'(d_1) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_1^2} \\ d_2 &= \frac{\log(S/E) + (r - \frac{1}{2}\tilde{\sigma}^2)(T-t)}{\tilde{\sigma}\sqrt{T-t}} \\ d_1 &= d_2 + \tilde{\sigma}\sqrt{T-t}\,, \end{aligned}$$

we first have that

$$N'(d_1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(d_2 + \tilde{\sigma}\sqrt{T-t})^2} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(d_2^2 + 2\tilde{\sigma}\sqrt{T-t}d_2 + \tilde{\sigma}^2(T-t))} = \frac{1}{\sqrt{2\pi}} e^{-\tilde{\sigma}\sqrt{T-t}d_2} e^{-\frac{1}{2}\tilde{\sigma}^2(T-t)} e^{-\frac{1}{2}d_2^2}\,.$$

Now from the definition of d_2 we see that

$$-\tilde{\sigma}\sqrt{T-t}\, d_2 = -\log(S/E) - (r - \frac{1}{2}\tilde{\sigma}^2)(T-t)\,,$$

and thus

$$\begin{aligned} N'(d_1) &= \frac{1}{\sqrt{2\pi}} \exp\left\{ -\log(S/E) - (r - \frac{1}{2}\tilde{\sigma}^2)(T-t) - \frac{1}{2}\tilde{\sigma}^2(T-t) \right\} e^{-\frac{1}{2}d_2^2} \\ &= \frac{1}{\sqrt{2\pi}} \exp\left\{ -\log(S/E) - r(T-t) \right\} e^{-\frac{1}{2}d_2^2} = \frac{E}{S\sqrt{2\pi}} e^{-r(T-t)} e^{-\frac{1}{2}d_2^2}\,. \end{aligned}$$

With this the source term becomes

$$\begin{aligned} \frac{1}{2}(\sigma^2 - \tilde{\sigma}^2)e^{-r(t-t_0)} S^2 \Gamma^i &= \frac{\frac{1}{2}(\sigma^2 - \tilde{\sigma}^2)e^{-r(t-t_0)} S}{\tilde{\sigma}\sqrt{T-t}} \left(\frac{E}{S\sqrt{2\pi}} e^{-r(T-t)} e^{-\frac{1}{2}d_2^2} \right) \\ &= \frac{E(\sigma^2 - \tilde{\sigma}^2)}{2\tilde{\sigma}\sqrt{2\pi(T-t)}} e^{-r(T-t_0)} e^{-\frac{1}{2}d_2^2}\,, \end{aligned} \tag{133}$$

the same expression quoted in the book. We now perform the change of variables from (S, t) to (x, τ) where x and τ are defined by

$$x = \ln\left(\frac{S}{E}\right) + \frac{2}{\sigma^2}(\mu - \frac{1}{2}\sigma^2)\tau = \ln\left(\frac{S}{E}\right) + (\mu - \frac{1}{2}\sigma^2)(T-t) \tag{134}$$

$$\tau = \frac{\sigma^2}{2}(T-t)\,. \tag{135}$$

Note that this is different than what the text of the chapter says but matches what the appendix to this chapter states. In this case we have that our t and S derivatives transform as

$$\begin{aligned}
\frac{\partial}{\partial t} &= \frac{\partial x}{\partial t}\frac{\partial}{\partial x} + \frac{\partial \tau}{\partial t}\frac{\partial}{\partial \tau} = -\left(\mu - \frac{1}{2}\sigma^2\right)\frac{\partial}{\partial x} - \frac{\sigma^2}{2}\frac{\partial}{\partial \tau} \\
\frac{\partial}{\partial S} &= \frac{\partial x}{\partial S}\frac{\partial}{\partial x} + \frac{\partial \tau}{\partial S}\frac{\partial}{\partial \tau} = \frac{1}{S}\frac{\partial}{\partial x} \\
\frac{\partial^2}{\partial S^2} &= \frac{\partial}{\partial S}\left(\frac{1}{S}\frac{\partial}{\partial x}\right) = -\frac{1}{S^2}\frac{\partial}{\partial x} + \frac{\partial}{\partial S}\left(\frac{\partial}{\partial x}\right) = -\frac{1}{S^2}\frac{\partial}{\partial x} + \frac{1}{S^2}\frac{\partial^2}{\partial x^2}\,.
\end{aligned}$$

With this transformation of the independent variables we find that the operator given in Equation 132 or

$$\frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + \mu S \frac{\partial}{\partial S}\,,$$

becomes

$$\begin{aligned}
\frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + \mu S \frac{\partial}{\partial S} &= -\left(\mu - \frac{1}{2}\sigma^2\right)\frac{\partial}{\partial x} - \frac{\sigma^2}{2}\frac{\partial}{\partial \tau} - \frac{1}{2}\sigma^2\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2} + \mu\frac{\partial}{\partial x} \\
&= \frac{\sigma^2}{2}\left(-\frac{\partial}{\partial \tau} + \frac{\partial^2}{\partial x^2}\right)\,, \qquad (136)
\end{aligned}$$

as claimed in the appendix. If we perform this change of variables in Equation 132, when we define the change of coordinate function as $h(x,\tau) = H(S,t)$ we get

$$\frac{1}{2}\sigma^2\left(-\frac{\partial h}{\partial \tau} + \frac{\partial^2 h}{\partial x^2}\right) + \frac{1}{2}(\sigma^2 - \tilde{\sigma}^2)e^{-r(t-t_0)}S^2\Gamma^i = 0\,.$$

Solving for $\frac{\partial h}{\partial \tau}$ we get

$$\frac{\partial h}{\partial \tau} = \frac{\partial^2 h}{\partial x^2} + \left(\frac{\sigma^2 - \tilde{\sigma}^2}{\sigma^2}\right)e^{-r(t-t_0)}S^2\Gamma^i\,. \qquad (137)$$

Note that the book calls the function $H(S,t)$ above as the function $F(S,t)$ and calls $h(x,\tau)$ as the function $w(x,\tau)$ in the appendix. In this last expression we want to use Equation 133 with $\sqrt{T-t} = \frac{\sqrt{2\tau}}{\sigma}$ we get

$$\frac{\partial h}{\partial \tau} = \frac{\partial^2 h}{\partial x^2} + \left(\frac{E(\sigma^2 - \tilde{\sigma}^2)}{2\tilde{\sigma}\sigma\sqrt{\pi\tau}}\right)e^{-r(T-t_0)}e^{-\frac{1}{2}d_2^2}\,. \qquad (138)$$

Warning: I get a factor of 2 in the denominator of the forcing term in Equation 138 that the book does not have. I have checked this algebra several times and can't seem to find any errors. If anyone sees anything wrong with my derivation please let me know (or if it is correct). When we express d_2 in terms of the variables x and τ we find

$$\begin{aligned}
d_2 &= \frac{\log(S/E) + (r - \frac{1}{2}\tilde{\sigma}^2)(T-t)}{\tilde{\sigma}\sqrt{T-t}} \\
&= \frac{\sigma}{\tilde{\sigma}}\left(\frac{x - \frac{2}{\sigma^2}(\mu - \frac{1}{2}\sigma^2)\tau + \frac{2}{\sigma^2}(r - \frac{1}{2}\tilde{\sigma}^2)\tau}{\sqrt{2\tau}}\right) = d_2(x,\tau)\,. \qquad (139)
\end{aligned}$$

If we denote the forcing function in Equation 138 by $f(x,\tau)$ then recalling that the solution $h(x,\tau)$ to $\frac{\partial h}{\partial \tau} = \frac{\partial^2 h}{\partial x^2} + f(x,\tau)$ that is initially zero when $\tau = 0$ and has zero at plus and minus infinity in x is given by

$$h(x,\tau) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \int_0^{\tau} \frac{f(x',\tau')}{\sqrt{\tau-\tau'}} e^{-\frac{(x-x')^2}{4(\tau-\tau')}} d\tau' dx' \,. \tag{140}$$

Using this, when we put in what we know the expression for $f(x,\tau)$ is we get for the solution to Equation 138 is

$$h(x,\tau) = \left(\frac{1}{2\sqrt{\pi}}\right) \frac{E(\sigma^2 - \tilde{\sigma}^2)e^{-r(T-t_0)}}{2\tilde{\sigma}\sigma\sqrt{\pi}} \int_{-\infty}^{\infty} \int_0^{\tau} \frac{1}{\sqrt{\tau'}} \frac{1}{\sqrt{\tau-\tau'}} K(x,\tau,x',\tau') d\tau' dx' \,, \tag{141}$$

where

$$K(x,\tau,x',\tau') = \exp\left\{ -\frac{(x-x')^2}{4(\tau-\tau')} - \frac{\sigma^2}{4\tilde{\sigma}^2\tau'} \left(x' - \frac{2}{\sigma^2}(\mu - \frac{1}{2}\sigma^2)\tau' + \frac{2}{\sigma^2}(r - \frac{1}{2}\tilde{\sigma}^2)\tau' \right)^2 \right\} .$$

To evaluate the integral in Equation 141 above needed to evaluate $h(x,\tau)$ we want to expand the argument of the exponential in $K(x,\tau,x',\tau')$ to get a quadratic polynomial in x'. We then take this polynomial and complete the square to write it in the form $-a(x'+b)^2 + c$, where a, b, and c can possibly be functions of the variables t and τ'. We do this algebra in the Mathematica file `expectation_single_stock.nb`. In that script we find that the value of a and c are given by

$$a(\tau,\tau') = \frac{1}{4(\tau-\tau')} + \frac{\sigma^2}{4\tilde{\sigma}^2\tau'} \tag{142}$$

$$c(x,\tau,\tau') = -\frac{\sigma^2(x - m(\tau'))^2}{4(\sigma^2(\tau-\tau') + \tilde{\sigma}^2\tau')} \quad \text{with} \tag{143}$$

$$m(\tau') = \frac{2}{\sigma^2}(\mu - \frac{1}{2}\sigma^2)\tau' - \frac{2}{\sigma^2}(r - \frac{1}{2}\tilde{\sigma}^2)\tau' \,. \tag{144}$$

The value of b is computed in the Mathematica workbook but is not needed since it plays no roll in the final expression for $h(x,\tau)$. With some simplifications these expressions agree with the ones in the book. When we put this factorization into Equation 141 and exchange the order of the τ' and x' integration we get

$$\begin{aligned} h(x,\tau) &= \left(\frac{1}{2\sqrt{\pi}}\right) \frac{E(\sigma^2 - \tilde{\sigma}^2)e^{-r(T-t_0)}}{2\tilde{\sigma}\sigma\sqrt{\pi}} \int_0^{\tau} \frac{1}{\sqrt{\tau'}} \frac{1}{\sqrt{\tau-\tau'}} \int_{-\infty}^{\infty} \exp(-a(x'+b)^2 + c) dx' d\tau' \\ &= \left(\frac{1}{2\sqrt{\pi}}\right) \frac{E(\sigma^2 - \tilde{\sigma}^2)e^{-r(T-t_0)}}{2\tilde{\sigma}\sigma\sqrt{\pi}} \int_0^{\tau} \frac{1}{\sqrt{\tau'}} \frac{1}{\sqrt{\tau-\tau'}} \exp(c) \left(\int_{-\infty}^{\infty} e^{-ax'^2} dx' \right) d\tau' \,. \end{aligned}$$

Since

$$\int_{-\infty}^{\infty} e^{-ax'^2} dx' = \sqrt{\frac{\pi}{a}} \,, \tag{145}$$

we have that the above is given by

$$h(x,\tau) = \frac{1}{4\sqrt{\pi}} \frac{E(\sigma^2 - \tilde{\sigma}^2)e^{-r(T-t_0)}}{\tilde{\sigma}\sigma} \int_0^{\tau} \frac{1}{\sqrt{\tau'}} \frac{1}{\sqrt{\tau-\tau'}} \frac{\exp(c)}{\sqrt{a}} d\tau' \,. \tag{146}$$

Note that both a and c are functions of τ' and cannot be taken out of the integral. We now make the change of variable from τ' to s where s is defined in terms of τ' as

$$s = t + \frac{2}{\sigma^2}\tau' \,.$$

We also replace all τ's and x's with their equivalent expressions in terms of t and S using Equations 134 and 135. To do this we need to transform several pieces in the above integral namely

$$\begin{aligned}
\tau' &= \frac{\sigma^2}{2}(s-t) \\
\tau - \tau' &= \frac{\sigma^2}{2}(T-s) \\
a &= \frac{1}{2\sigma^2\tilde{\sigma}^2}\left(\frac{\tilde{\sigma}^2(s-t)+\sigma^2(T-s)}{(s-t)(T-s)}\right) \\
c &= -\frac{\left[\ln\left(\frac{S}{E}\right)+(\mu-\frac{1}{2}\sigma^2)(T-s)+(r-\frac{1}{2}\tilde{\sigma}^2)(s-t)\right]^2}{2(\sigma^2(T-s)+\tilde{\sigma}^2(s-t))} \\
\tau'(\tau-\tau')a &= \frac{\sigma^2}{8\tilde{\sigma}^2}(\tilde{\sigma}^2(s-t)+\sigma^2(T-s)) \,.
\end{aligned}$$

With all of these expressions in terms of S and t we finally find that $H(S,t)$ is given by

$$\begin{aligned}
H(S,t) &= \frac{E(\sigma^2-\tilde{\sigma}^2)e^{-r(T-t_0)}}{\sigma^2\sqrt{2\pi}}\int_t^T \frac{1}{\sqrt{\tilde{\sigma}^2(s-t)+\sigma^2(T-s)}} \\
&\times \exp\left(-\frac{\left[\ln\left(\frac{S}{E}\right)+(\mu-\frac{1}{2}\sigma^2)(T-s)+(r-\frac{1}{2}\tilde{\sigma}^2)(s-t)\right]^2}{2(\sigma^2(T-s)+\tilde{\sigma}^2(s-t))}\right) ds \,. \qquad (147)
\end{aligned}$$

This can be evaluated at a particular time t_0 and stock price S_0 as desired. **Warning:** This result is different from the one presented in the text. One difference is that I have a $\frac{1}{\sigma^2}$ in front of everything. Another difference is the factor of 2 (from before). The largest difference, however, is that the coefficients of $T-s$ and $s-t$ in the square root and in the denominator of the fraction in the exponent are exchanged. I've checked this result several times and have not found anything wrong with it. If anyone sees anything wrong with this (or agrees with it) please contact me.

The variance of profit after hedging using implied volatility

To evaluate the *variance* of the profit when we hedge with implied volatility we need to calculate the solution to Equation 131 with the final condition now given by $P(S,I,T) = I^2$. To solve this we consider a solution of the form

$$V(S,t,I) = I^2 + 2IH(S,t) + G(S,t) \,,$$

then we have the following derivatives

$$
\begin{aligned}
\frac{\partial V}{\partial t} &= 2I\frac{\partial H}{\partial t} + \frac{\partial G}{\partial t} \\
\frac{\partial V}{\partial S} &= 2I\frac{\partial H}{\partial S} + \frac{\partial G}{\partial S} \\
\frac{\partial^2 V}{\partial S^2} &= 2I\frac{\partial^2 H}{\partial S^2} + \frac{\partial^2 G}{\partial S^2} \\
\frac{\partial V}{\partial I} &= 2I + 2H\,.
\end{aligned}
$$

When we put these into Equation 131 we get

$$
\begin{aligned}
2I \quad & \left[\frac{\partial H}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 H}{\partial S^2} + \mu S\frac{\partial H}{\partial S} + \frac{1}{2}(\sigma^2 - \tilde{\sigma}^2)e^{-r(t-t_0)}S^2\Gamma^i\right] \\
+ \quad & \left[\frac{\partial G}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 G}{\partial S^2} + \mu S\frac{\partial G}{\partial S} + \frac{1}{2}(\sigma^2 - \tilde{\sigma}^2)e^{-r(t-t_0)}S^2\Gamma^i H\right] = 0\,.
\end{aligned}
$$

If we equate the coefficients of powers of I to zero we get the two differential equations presented in the book for $H(S,t)$ and $G(S,t)$. Earlier in this section we have shown how to solve the H equation and thus we know its explicit solution given by Equation 147. To solve the equation for G we will perform the same change of variables as in Equations 134 and 135. We will represent $G(S,t)$ in the (x,τ) coordinates as the function $g(x,\tau)$. Performing the desired change of variables we find the equation for $g(x,\tau)$ given by

$$
\frac{\sigma^2}{2}\left(-\frac{\partial g}{\partial \tau} + \frac{\partial^2 g}{\partial x^2}\right) + \frac{1}{2}(\sigma^2 - \tilde{\sigma}^2)e^{-r(t-t_0)}S^2\Gamma^i h(x,\tau) = 0\,,
$$

where the function $h(x,\tau)$ given by Equation 146. Note that the book calls $g(x,\tau)$ above $w(x,\tau)$. Using the same transformations that lead to Equation 138, plus the expression for $h(x,\tau)$ given by Equation 146 we get

$$
\begin{aligned}
\frac{\partial g}{\partial \tau} = \frac{\partial^2 g}{\partial x^2} &+ \left(\frac{E(\sigma^2 - \tilde{\sigma}^2)}{2\sigma\tilde{\sigma}\sqrt{\pi\tau}}\right) e^{-r(T-t_0)} e^{-\frac{1}{2}d_2(x,\tau)^2} \\
&\times \frac{1}{4\sqrt{\pi}}\left(\frac{E(\sigma^2 - \tilde{\sigma}^2)e^{-r(T-t_0)}}{\tilde{\sigma}\sigma}\right)\int_0^\tau \frac{1}{\sqrt{\tau'}}\frac{1}{\sqrt{\tau-\tau'}}\frac{\exp(c(x,\tau,\tau'))}{\sqrt{a(\tau,\tau')}}d\tau'\,.
\end{aligned}
$$

When we simplify the coefficient of the integral, the equation we need to solve for $g(x,\tau)$ is

$$
\frac{\partial g}{\partial \tau} = \frac{\partial^2 g}{\partial x^2} + \left(\frac{E^2(\sigma^2 - \tilde{\sigma}^2)^2 e^{-2r(T-t_0)}}{8\pi\tilde{\sigma}^2\sigma^2}\right)\frac{e^{-\frac{1}{2}d_2(x,\tau)^2}}{\sqrt{\tau}}\int_0^\tau \frac{1}{\sqrt{\tau'}}\frac{1}{\sqrt{\tau-\tau'}}\frac{\exp(c(x,\tau,\tau'))}{\sqrt{a(\tau,\tau')}}d\tau'\,.
$$

Note that with the above notation we are very explicit in what variables the functions d_2, a, and c depend on. For example, we write $d_2(x,\tau)$ to indicate that d_2 is a function of x and τ. This will help when we need to treat the integral above as a forcing term and need to integrate over its independent variables. To do that we will need to replace x with x', τ with τ', and τ' with τ''. Using the known solution to this type of partial differential equation given by 140 we get for $g(x,\tau)$ the following

$$
\begin{aligned}
g(x,\tau) = & \left(\frac{E^2(\sigma^2 - \tilde{\sigma}^2)^2 e^{-2r(T-t_0)}}{16\pi^{3/2}\tilde{\sigma}^2\sigma^2}\right) \\
& \times \int_{-\infty}^{\infty}\int_0^\tau \frac{1}{\sqrt{\tau-\tau'}}\left(\frac{e^{-\frac{1}{2}d_2(x',\tau')^2}}{\sqrt{\tau'}}\int_0^{\tau'}\frac{1}{\sqrt{\tau''}}\frac{1}{\sqrt{\tau'-\tau''}}\frac{\exp(c(x',\tau',\tau''))}{\sqrt{a(\tau',\tau'')}}d\tau''\right)e^{-\frac{1}{4}\frac{(x-x')^2}{\tau-\tau'}}d\tau' dx'\,.
\end{aligned}
$$

The integral in the above expression is the following triple integral

$$\int_{-\infty}^{\infty}\int_0^{\tau}\int_0^{\tau'} \frac{1}{\sqrt{\tau'}\sqrt{\tau''}\sqrt{\tau-\tau'}\sqrt{\tau'-\tau''}\sqrt{a(\tau',\tau'')}}$$
$$\times \quad \exp\left\{-\frac{1}{2}d_2(x',\tau')^2 - \frac{1}{4}\frac{(x-x')^2}{\tau-\tau'} + c(x',\tau',\tau'')\right\} d\tau'' d\tau' dx' .$$

Again we perform the x' integration by completing the square of the argument of the exponent with respect to the variable x'. Given what we know about the functional form of $d_2(x',\tau')$ and $c(x',\tau',\tau'')$ via Equations 139 and 143 we can write the argument of the exponential as $-d(x'+f)^2+\tilde{g}$ where d, f, and $\tilde{g}$ are potential functions of τ, τ', and τ''. In the Mathematica file `variance_single_stock.nb` we find

$$d(\tau,\tau',\tau'') = \frac{\sigma^2}{4\tilde{\sigma}^2\tau'} + \frac{1}{4(\tau-\tau')} + \frac{\sigma^2}{4(\sigma^2(\tau'-\tau'')+\tilde{\sigma}^2\tau'')} , \tag{148}$$

and the function $\tilde{g}(x,\tau,\tau',\tau'')$ is given in the Mathematica file. This last expression for d agrees with the one from the book. Also in that same Mathematica file I verified that the expression for $\tilde{g}$ derived there agrees with the one given in the book. Once we have these expressions for d and $\tilde{g}$ determined, performing the x' integral by recalling Equation 145 we get

$$g(x,\tau) = \left(\frac{E^2(\sigma^2-\tilde{\sigma}^2)^2 e^{-2r(T-t_0)}}{16\pi^{3/2}\tilde{\sigma}^2\sigma^2}\right)$$
$$\times \int_0^{\tau}\int_0^{\tau'} \frac{\exp(\tilde{g}(x,\tau,\tau',\tau''))}{\sqrt{\tau'}\sqrt{\tau''}\sqrt{\tau-\tau'}\sqrt{\tau'-\tau''}\sqrt{a(\tau',\tau'')}}\sqrt{\frac{\pi}{d(\tau,\tau',\tau'')}} d\tau'' d\tau' .$$

To simplify this we transform the variable τ'' in the inner most integral to the variable u defined as

$$\tau'' = \frac{\sigma^2}{2}(T-u) .$$

In that case the differential transforms as $d\tau'' = -\frac{\sigma^2}{2}du$ and the point $\tau''=0$ becomes $u=T$ and the point $\tau''=\tau'$ becomes the point $u = T - \frac{2}{\sigma^2}\tau'$ and the double integral above becomes

$$\frac{\sigma\sqrt{\pi}}{\sqrt{2}}\int_0^{\tau}\int_{T-\frac{2}{\sigma^2}\tau'}^{T} \frac{\exp\left(\tilde{g}\left(x,\tau,\tau',\frac{\sigma^2}{2}(T-u)\right)\right)}{\sqrt{\tau'}\sqrt{T-u}\sqrt{\tau-\tau'}\sqrt{\tau'-\frac{\sigma^2}{2}(T-u)}\sqrt{a\left(\tau',\frac{\sigma^2}{2}(T-u)\right)d\left(\tau,\tau',\frac{\sigma^2}{2}(T-u)\right)}} du d\tau' .$$

We now change the integrand from τ' to the variable s defined as

$$\tau' = \frac{\sigma^2}{2}(T-s) .$$

In that case the differential transforms as $d\tau' = -\frac{\sigma^2}{2}ds$ and the point $\tau'=0$ becomes $s=T$ and the point $\tau'=\tau$ becomes the point $s = T - \frac{2}{\sigma^2}\tau$. With this change of variable the leading coefficient and the limits of the integrals become

$$\frac{\sigma\sqrt{\pi}}{\sqrt{2}}\int_{T-\frac{2}{\sigma^2}\tau}^{T}\int_s^{T} \Box \, du ds ,$$

while the integrand, $\square$, becomes

$$\frac{\exp\left(\tilde{g}\left(x,\tau,\frac{\sigma^2}{2}(T-s),\frac{\sigma^2}{2}(T-u)\right)\right)}{\sqrt{T-s}\sqrt{T-u}\sqrt{\tau-\frac{\sigma^2}{2}(T-s)}\sqrt{u-s}\sqrt{a\left(\frac{\sigma^2}{2}(T-s),\frac{\sigma^2}{2}(T-u)\right)d\left(\tau,\frac{\sigma^2}{2}(T-s),\frac{\sigma^2}{2}(T-u)\right)}}.$$

We now want to convert the solution $g(x,\tau)$ found above above into the variables (S,t) or $G(S,t)$. To this end we again use Equations 134 and 135 and find

$$\begin{aligned}
G(S,t) = &\left(\frac{E^2(\sigma^2-\tilde{\sigma}^2)^2e^{-2r(T-t_0)}}{16\pi\tilde{\sigma}^2\sigma^2}\right) \\
&\times \int_t^T\int_s^T \exp\left(\tilde{g}\left(\ln\left(\frac{S}{E}\right)+(\mu-\frac{1}{2}\sigma^2)(T-t),\frac{\sigma^2}{2}(T-t),\frac{\sigma^2}{2}(T-s),\frac{\sigma^2}{2}(T-u)\right)\right) \\
&\times \frac{1}{\sqrt{T-s}\sqrt{T-u}\sqrt{s-t}\sqrt{u-s}} \\
&\times \frac{1}{\sqrt{a\left(\frac{\sigma^2}{2}(T-s),\frac{\sigma^2}{2}(T-u)\right)d\left(\frac{\sigma^2}{2}(T-t),\frac{\sigma^2}{2}(T-s),\frac{\sigma^2}{2}(T-u)\right)}}.
\end{aligned} \tag{149}$$

In this expression the function $\tilde{g}(x,\tau,\tau',\tau'')$ is given in the Mathematica file, the function $a(\tau',\tau'')$ is given by Equation 142 and $d(\tau,\tau',\tau'')$ is given by Equation 148.

Warning: This result is similar in form but somewhat different than the result the book has. I have not had time to compare the two versions and see if there are differences. If anyone sees anything wrong with what I have done above (or think it is correct) please contact me.

Fixed-Income Products and Analysis

Notes on bootstraping with discrete data

To compute the yield y_2 we use

$$Z_2^M = e^{-y_1(T_1-t)} e^{-y_2(T_2-t)} = Z_1^M e^{-y_2(T_2-t)} ,$$

so that y_2 is given by

$$y_2 = -\frac{\log(Z_2^M/Z_1^M)}{T_2 - T_1} .$$

This expression is not implemented in the excel example given in the book. Instead the above expression is written as

$$\begin{aligned} y_2 &= \frac{-\log(Z_2^M) + \log(Z_1^M)}{T_2 - T_1} \\ &= \frac{-\frac{\log(Z_2^M)}{T_2-t}(T_2-t) + \frac{\log(Z_1^M)}{T_1-t}(T_1-t)}{(T_2-t)-(T_1-t)} . \end{aligned}$$

We can replace 2 with i and 1 with $i-1$ to get the general recurrence relationship. The spreadsheet is calculating $-\frac{\log(Z_i^M)}{T_i-t}$ in column `C` and T_i-t in column `A`. So the above expression above for y_i becomes

$$\frac{C(i)A(i) - C(i-1)A(i-1)}{A(i) - A(i-1)} ,$$

which is the formula given in the book.

Swaps

Notes on bootstraping

Since at inception the swap has no value to either party then as argued in the book we have

$$\tau r_s \sum_{i=1}^{N} Z(t;T_i) - 1 + Z(t;T_N) = 0 \,.$$

Solving for r_s gives

$$r_s = \frac{1 - Z(t;T_N)}{\tau \sum_{i=1}^{N} Z(t;T_i)} \tag{150}$$

which is the quoted swap rate. If we assume that $r_s(T)$ is known from the yield curve we can take $N = 1$ in the above expression to get

$$r_s(T_1) = \frac{1 - Z(t;T_1)}{\tau Z(t;T_1)} \,.$$

When we solve that expression for $Z(t;T_1)$ we get

$$Z(t;T_1) = \frac{1}{1 + \tau r_s(T_1)} \,.$$

We now consider the case where we assume we have $Z(t;T_i)$ for $1 \le i \le j$ and want to determine $Z(t;T_{j+1})$. If we have $Z(t;T_j)$ for $1 \le i \le j$ then from Equation 150 with $N = j + 1$ we have

$$r_s(T_{j+1}) = \frac{1 - Z(t;T_{j+1})}{\tau \left(\sum_{i=1}^{j+1} Z(t;T_i) \right)} \,.$$

If we solve for $Z(t;T_{j+1})$ in this expression we get

$$Z(t;T_{j+1}) = \frac{1 - r_s(T_{j+1})\tau \sum_{i=1}^{j} Z(t;T_i)}{1 + r_s(T_{j+1})\tau} \quad \text{for} \quad j = 1, 2, \cdots \,.$$

The Binomial Model

Notes on statistics from the binomial random walk

For these examples we first show that if we take the parameters u, v, and p given by

$$u = 1 + \sigma\sqrt{\delta t} \tag{151}$$

$$v = 1 - \sigma\sqrt{\delta t} \tag{152}$$

$$p = \frac{1}{2} + \frac{\mu\sqrt{\delta t}}{2\sigma}\,, \tag{153}$$

then we get the same average asset price change. The expected asset price after one time step is given by the standard expression for expectations and we have

$$\begin{aligned}
puS + (1-p)vS &= \left(\frac{1}{2} + \frac{\mu\sqrt{\delta t}}{2\sigma}\right)(1+\sigma\sqrt{\delta t})S + \left(\frac{1}{2} - \frac{\mu\sqrt{\delta t}}{2\sigma}\right)(1-\sigma\sqrt{\delta t})S \\
&= \left[\frac{1}{2} + \frac{\sigma}{2}\sqrt{\delta t} + \frac{\mu\sqrt{\delta t}}{2\sigma} + \frac{\mu\delta t}{2}\right]S + \left[\frac{1}{2} - \frac{\sigma}{2}\sqrt{\delta t} - \frac{\mu\sqrt{\delta t}}{2\sigma} + \frac{\mu\delta t}{2}\right]S \\
&= (1+\mu\delta t)S\,.
\end{aligned}$$

This gives the expected change in the asset of $\mu S\delta t$ which has the expected return of $\mu\delta t$ as we expect for a geometric random walk. The expected value in the *change* in asset price using the above is then $\mu\delta t S$.

Next if denote the change in the asset price by ΔS (this is not the same Δ introduced later in this chapter), we want to determine the variance of ΔS. Since the asset can go up with probability p to a change in asset value of $uS - S = (u-1)S$ or down with probability $1-p$ to a change in asset value of $vS - S = (v-1)S$ if it goes down which happens with probability $1-p$. Using these expressions and the definition of the variance we find

$$\begin{aligned}
\mathrm{Var}[\Delta S] &= E[(\Delta S - E[\Delta S])^2] \\
&= p[(u-1)S - \mu\delta t S]^2 + (1-p)[(v-1)S - \mu\delta t S]^2 \\
&= p(\sigma\sqrt{\delta t} - \mu\delta t)^2 S^2 + (1-p)(-\sigma\sqrt{\delta t} - \mu\delta t)^2 S^2 \\
&= p(\sigma^2\delta t - 2\mu\sigma\delta t^{3/2} + \mu^2\delta t^2)S^2 + (1-p)(\sigma^2\delta t + 2\mu\sigma\delta t^{3/2} + \mu^2\delta t^2)S^2 \\
&= (\sigma^2\delta t + \mu^2\delta t^2 + (1-2p)(2\mu\sigma\delta t^{3/2}))S^2\,.
\end{aligned}$$

But from what we know about the expressions for p we have that

$$1 - 2p = -\frac{\mu\sqrt{\delta t}}{\sigma}\,,$$

and thus the above becomes

$$\begin{aligned}
\mathrm{Var}[\Delta S] &= \left(\sigma^2\delta t + \mu^2\delta t^2 + -\frac{\mu\sqrt{\delta t}}{\sigma}(2\mu\sigma\delta t^{3/2})\right)S^2 \\
&= S^2(\sigma^2\delta t - \mu^2\delta t^2)\,.
\end{aligned}$$

As $\delta t \to 0$ this becomes $S^2\sigma^2\delta t$ so that $\text{Var}[\Delta S] \approx S^2\sigma^2\delta t$ and thus $\text{Var}\left[\frac{\Delta S}{S}\right] \approx \sigma^2\delta t$ so we see that the standard deviation of returns is approximately $\sigma\sqrt{\delta t}$.

Notes on the value of an option

With the derivation of the hedging amount Δ need to make the final value of our portfolio equal independent of what value the stock takes

$$\Delta = \frac{V^+ - V^-}{uS - vS} = \frac{V^+ - V^-}{(u-v)S}\,, \tag{154}$$

there are two ways we can evaluate the final portfolio value. The first where we assume that the stock goes up gives

$$V^+ - \Delta uS = V^+ - u\left(\frac{V^+ - V^-}{u-v}\right) = \frac{-vV^+ + uV^-}{u-v}\,, \tag{155}$$

and the second wen we assume that the stock goes down is

$$V^- - \Delta vS = V^- - v\left(\frac{V^+ - V^-}{u-v}\right) = \frac{-vV^+ + uV^-}{u-v}\,. \tag{156}$$

which are the same as they must be. With the definition of Δ given by Equation 154 we find that the original portfolio Π has a value given by

$$\Pi = V - \Delta S = V - \frac{V^+ - V^-}{u-v}\,. \tag{157}$$

Asserting that the new portfolio is given by $\Pi + \delta\Pi = \Pi + r\Pi\delta t = (1 + r\delta t)\Pi$ and setting this equal to Equation 156 gives

$$(1 + r\delta)\Pi = \frac{-vV^+ + uV^-}{u-v}\,.$$

But putting in what we know about Π via Equation 157 into the left-hand-side of the above gives

$$(1 + r\delta t)\left(V - \frac{V^+ - V^-}{u-v}\right) = V^- - v\left(\frac{V^+ - V^-}{u-v}\right)\,.$$

Solving for $(1 + r\delta t)V$ in the above we have

$$(1 + r\delta t)V = (1 + r\delta t)\left(\frac{V^+ - V^-}{u-v}\right) + \frac{uV^- - vV^+}{u-v}\,. \tag{158}$$

If we write the right-hand-side as linear function of the two prices V^- and V^+ we get

$$\begin{aligned}
(1 + r\delta t)V &= \left(\frac{1 + r\delta t}{u-v} - \frac{v}{u-v}\right)V^+ + \left(-\frac{1 + r\delta t}{u-v} + \frac{u}{u-v}\right)V^- \\
&= \left(\frac{1 + r\delta t - v}{u-v}\right)V^+ + \left(\frac{-1 - r\delta t + u}{u-v}\right)V^-\,.
\end{aligned}$$

Recalling what we know about u and v we see that $u - v = 2\sigma\sqrt{\delta t}$ and the above equals

$$
\begin{aligned}
(1 + r\delta t)V &= \left(\frac{1 + r\delta t - (1 - \sigma\sqrt{\delta t})}{2\sigma\sqrt{\delta t}}\right) V^+ + \left(\frac{-1 - r\delta t + 1 + \sigma\sqrt{\delta t})}{2\sigma\sqrt{\delta t}}\right) V^- \\
&= \left(\frac{1}{2} + \frac{r\sqrt{\delta t}}{2\sigma}\right) V^+ + \left(\frac{1}{2} - \frac{r\sqrt{\delta t}}{2\sigma}\right) V^- \qquad (159) \\
&= p'V^+ + (1 - p')V^- \,, \qquad (160)
\end{aligned}
$$

were we have taken

$$
p' \equiv \frac{1}{2} + \frac{r\sqrt{\delta t}}{2\sigma} \,. \qquad (161)
$$

Notes on the continuous time limit

Consider the small time step expansion or Taylor series of the values for V^+ and V^-. We have for the value of u and v we have been considering that

$$
\begin{aligned}
V^+ &= V(uS, t + \delta t) = V((1 + \sigma\sqrt{\delta t})S, t + \delta t) = V(S + \sigma S\sqrt{\delta t}, t + \delta t) \\
&= V(S, t) + V_S \sigma\sqrt{\delta t} S + \frac{1}{2} V_{SS} \sigma^2 \delta t S^2 + V_t \delta t + \cdots \\
V^- &= V(vS, t + \delta t) = V((1 - \sigma\sqrt{\delta t})S, t + \delta t) = V(S - \sigma S\sqrt{\delta t}, t + \delta t) \\
&= V(S, t) - V_S \sigma\sqrt{\delta t} S + \frac{1}{2} V_{SS} \sigma^2 \delta t S^2 + V_t \delta t + \cdots \,.
\end{aligned}
$$

Then under these approximations Equation 154 can now be written in the limit of small time step $\delta t \to 0$ as

$$
\Delta = \frac{V^+ - V^-}{(u - v)S} \approx \frac{2V_S \sigma\sqrt{\delta t} S}{2\sigma\sqrt{\delta t} S} = V_S \,.
$$

In the same way Equation 159 under this limit becomes

$$
\begin{aligned}
(1 + r\delta t)V &= \left(\frac{1}{2} + \frac{r\sqrt{\delta t}}{2\sigma}\right) (V + \sigma\sqrt{\delta t} S V_S + \frac{1}{2}\sigma^2 \delta t S^2 V_{SS} + V_t \delta t) \\
&+ \left(\frac{1}{2} - \frac{r\sqrt{\delta t}}{2\sigma}\right) (V - \sigma\sqrt{\delta t} S V_S + \frac{1}{2}\sigma^2 \delta t S^2 V_{SS} + V_t \delta t) \\
&= V + \frac{1}{2}\sigma^2 \delta t S^2 V_{SS} + V_t \delta t + \frac{r\sqrt{\delta t}}{2\sigma} 2\sigma\sqrt{\delta t} S V_S \,.
\end{aligned}
$$

When we simplify the above we get

$$
V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV = 0 \,.
$$

Notes on another parametrization

We start with a geometric random walk in continuous time which is given by

$$dS = \mu S dt + \sigma S dX .$$

This equation has been seen before and has a solution given by $S(t)$ given by Equation 21. We need to pick the parameters u, v, and p such that we have the same discrete expectation after the time step δt as the continuous process or $E[S(\delta t)]$. That means that

$$puS + (1-p)vS = E[S(\delta t)] = E\left[S e^{\left(\mu - \frac{1}{2}\sigma^2\right)\delta t + \sigma\phi\sqrt{\delta t}}\right] = S e^{\left(\mu - \frac{1}{2}\sigma^2\right)\delta t} E\left[e^{\sigma\phi\sqrt{\delta t}}\right] .$$

We next need to evaluate the expectation over the Gaussian random variable ϕ.

$$E\left[e^{\sigma\phi\sqrt{\delta t}}\right] = \int_{-\infty}^{\infty} e^{\sigma\phi\sqrt{\delta t}} \left(\frac{e^{-\frac{1}{2}\phi^2}}{\sqrt{2\pi}}\right) d\phi .$$

To evaluate this integral we group everything into the same exponent and complete the square by writing the exponent as

$$-\frac{1}{2}\phi^2 + \sigma\sqrt{\delta t}\phi = -\frac{1}{2}(\phi - \sigma\sqrt{\delta t})^2 + \frac{1}{2}\sigma^2\delta t .$$

With this substitution the above expectation becomes

$$E\left[e^{\sigma\phi\sqrt{\delta t}}\right] = \frac{1}{\sqrt{2\pi}} e^{\frac{1}{2}\sigma^2\delta t} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}(\phi - \sigma\sqrt{\delta t})^2} d\phi = e^{\frac{1}{2}\sigma^2\delta t} .$$

Using this the total expectation we want to evaluate is given by multiplying this by $S e^{\left(\mu - \frac{1}{2}\sigma^2\right)\delta t}$ to get

$$puS + (1-p)vS = S e^{\mu\delta t} ,$$

which if we divide by S gives

$$pu + (1-p)v = e^{\mu\delta t} ,$$

or solving for p we have

$$p = \frac{e^{\mu\delta t} - v}{u - v} , \tag{162}$$

the expression we were to show.

Next we want to make our continuous process have the same *variance* at the time δt that our discrete binomial process has. To do that we first evaluate the expectation of $S(\delta t)^2$ for our continuous process and then use the fact that $\mathrm{Var}[S(\delta t)] = E[S(\delta t)^2] - E[S(\delta t)]^2$. We find

$$E[S(\delta t)^2] = E\left[S^2 e^{2\left(\mu - \frac{1}{2}\sigma^2\right)\delta t + 2\sigma\phi\sqrt{\delta t}}\right] = S^2 e^{2\left(\mu - \frac{1}{2}\sigma^2\right)\delta t} E\left[e^{2\sigma\phi\sqrt{\delta t}}\right] ,$$

which is the same type of expectation we have evaluated above by taking $\sigma \to 2\sigma$ and we see that it evaluates to

$$S^2 e^{(2\mu + \sigma^2)\delta t} .$$

After having computed expected value of $S(\delta t)$ and of $S(\delta t)^2$ above the variance of the continuous random walk is

$$\begin{aligned}
\mathrm{Var}[S(\delta t)] &= E[S(\delta t)^2] - E[S(\delta t)]^2 \\
&= e^{(2\mu+\sigma^2)\delta t} S^2 - e^{2\mu\delta t} S^2 \\
&= S^2 e^{2\mu\delta t}(e^{\sigma^2\delta t} - 1)\,. \qquad (163)
\end{aligned}$$

To compute the variance of $S(\delta t)$ under the discrete binomial process requires the calculation of $E[S(\delta t)^2]$. Using the definition of our binomial tree we find this expectation given by

$$\begin{aligned}
E[S(\delta t)^2] &= p(uS)^2 + (1-p)(vS)^2 \\
&= (pu^2 + (1-p)v^2)S^2\,. \qquad (164)
\end{aligned}$$

Using this, the discrete variance is then given by

$$\begin{aligned}
\mathrm{Var}[S(\delta t)] &= (pu^2 + (1-p)v^2)S^2 - S^2 e^{2\mu\delta t} \\
&= S^2(pu^2 + (1-p)v^2 - e^{2\mu\delta t})\,. \qquad (165)
\end{aligned}$$

Setting this expression equal to the continuous variance expression in Equation 163 requires

$$pu^2 + (1-p)v^2 - e^{2\mu\delta t} = e^{\sigma^2\delta t + 2\mu\delta t} - e^{2\mu\delta t}\,.$$

This then becomes

$$pu^2 + (1-p)v^2 = e^{(2\mu+\sigma^2)\delta t}\,, \qquad (166)$$

which is the equation in the book. Solving this equation for p we get

$$p = \frac{e^{(2\mu+\sigma^2)\delta t} - v^2}{u^2 - v^2}\,. \qquad (167)$$

Dividing Equation 162 by Equation 167 we obtain

$$\left(\frac{e^{\mu\delta t} - v}{u - v}\right)\left(\frac{u^2 - v^2}{e^{(2\mu+\sigma^2)\delta t} - v^2}\right) = 1\,.$$

This allows us to solve for $u + v$ and we find

$$u + v = \frac{e^{(2\mu+\sigma^2)\delta t} - v^2}{e^{\mu\delta t} - v}\,,$$

When we multiply by $e^{\mu\delta t} - v$ on both side this expression becomes

$$(u+v)e^{\mu\delta t} - uv - v^2 = e^{(2\mu+\sigma^2)\delta t} - v^2\,,$$

or

$$(u+v)e^{\mu\delta t} - uv = e^{(2\mu+\sigma^2)\delta t}\,.$$

If we now enforce the constraint the the up returns u and the down returns v are equal in magnitude that is we take $u = \frac{1}{v}$ and obtain (after multiplying by $ve^{-\mu\delta t}$) the equation

$$v^2 - (e^{-\mu\delta t} + e^{(\mu+\sigma^2)\delta t})v + 1 = 0\,.$$

If we define a variable A as

$$A = \frac{1}{2}\left(e^{-\mu\delta t} + e^{(\mu+\sigma^2)\delta t}\right) , \quad (168)$$

we obtain a quadratic equation for v given by

$$v^2 - 2Av + 1 = 0 . \quad (169)$$

This has solutions given by the quadratic formula. We find

$$v = A \pm \sqrt{A^2 - 1} . \quad (170)$$

Note that u since it equals $1/v$ is then given by

$$\begin{aligned} u &= \frac{1}{v} = \frac{1}{A \pm \sqrt{A^2-1}} = \left(\frac{1}{A \pm \sqrt{A^2-1}}\right)\left(\frac{A \mp \sqrt{A^2-1}}{A \mp \sqrt{A^2-1}}\right) \\ &= A \mp \sqrt{A^2-1} . \end{aligned} \quad (171)$$

Since $v < u$ we must take the negative sign in the expression 170 for v and the positive sign for u in the expression 171. This is the solution for u and v given in the book. If we consider the case where δt is small we can Taylor expand the expressions for u, v, and p. We do that in the Mathematica file `chap_15_algebra.nb` where we find that

$$\begin{aligned} u &\approx 1 + \sigma\delta t^{1/2} + \frac{1}{2}\sigma^2\delta t + O(\delta t^{3/2}) \\ v &\approx 1 - \sigma\delta t^{1/2} + \frac{1}{2}\sigma^2\delta t + O(\delta t^{3/2}) \\ p &\approx \frac{1}{2} + \frac{(\mu - \frac{1}{2}\sigma^2)}{2\sigma}\delta t^{1/2} + O(\delta t^{3/2}) . \end{aligned}$$

How accurate is the normal approximation?

Notes on the empirical probability of a 20% SPX fall

In this section the empirical probability is just the number of times this event happened (once) divided by the the number of times we could have had that event happen $24 \times 252 = 6048$, this fraction $\frac{1}{6048} = 0.00016534$ as claimed.

Notes on the theoretical probability of a 20% SPX fall

If we estimate the daily volatility to be $\sigma_{\text{daily}} = 0.0106$, then the annualized volatility is given by

$$\sigma^2_{\text{yearly}} = 252\sigma^2_{\text{daily}} = 0.02831\,,$$

which when we take the square root gives $\sigma_{\text{yearly}} = 0.16826$ the quoted 16.8% annual volatility. Then to compute the theoretical probability that a random sample from a normal with zero mean and standard deviation of 0.0106 is less than -0.2 we need to evaluate the cumulative normal distribution at the point -0.2 or in `R` notation we need to evaluate `pnorm(-0.2,mean=0,sd=0.0106)`. When we evaluate this expression using `R` we get $1.046805\,10^{-79}$.

Investment Lessons from Blackjack and Gambling

Notes on the Kelly Criterion

Assume that we can bet on the ith outcome of an experiment that if we bet an amount A will pay out $\phi_i A$. Here ϕ_i is negative if we "loose" and we have to pay money while ϕ_i is positive if we "win" and we get to collect money. If we start with a portfolio valued at V_0 and we bet a fraction f of it after the bet will will have

$$V_0 + \phi_i(fV_0) = V_0(1 + f\phi_i)\,.$$

If we place M bets sequentially betting a fraction f of our current portfolio each time, then the total portfolio value after these M bets starting with V_0 is given by by

$$V(f) = V_0 \prod_{i=1}^{M}(1 + f\phi_i)\,. \tag{172}$$

This is a function of the fraction to bet f. We evaluate the above expression for two values of f in Figure 2. For small values of f the function $V(f)$ grows very slowly, while for larger values of f the total portfolio value $V(f)$ can reach 0 before the time step equals M. We expect that there must be an optimal f such that we maximize some function of $V(f)$ after M time steps.

Taking logarithms and then the expectation leads us to consider evaluating $E[\log(1 + \phi f)]$ (I have dropped the experiment index i on ϕ), which we do by first expanding $\log(1+\phi f)$ in a Taylor series and then taking the expectations. Recall that the Taylor series of $\log(1 + x)$ is given by

$$\log(1 + x) = \sum_{k=1}^{\infty} \frac{1}{k}(-1)^{k+1}x^k = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots \tag{173}$$

From this we have

$$\log(1 + \phi f) = \phi f - \frac{1}{2}\phi^2 f^2 + \cdots\,.$$

Since $E[\phi] = \mu$ and $E[\phi^2] = \sigma^2 + E[\phi]^2 = \sigma^2 + \mu^2$, the expectation of this expression is

$$E[\log(1 + \phi f)] \approx \mu f - \frac{1}{2}(\sigma^2 + \mu^2)f^2\,. \tag{174}$$

To maximize $E[\log(1 + \phi f)]$ with respect to f, we take the derivative with respect to f, set the result equal to zero, and solving for f to get

$$\mu - (\sigma^2 + \mu^2)f = 0 \quad \text{or} \quad f = \frac{\mu}{\sigma^2 + \mu^2}\,.$$

Using this value we find that the maximum expectation is given by

$$\max_f(E[\log(1 + \phi f)]) = \frac{\mu^2}{\sigma^2 + \mu^2} - \frac{1}{2}\frac{\mu^2}{\sigma^2 + \mu^2} = \frac{1}{2}\left(\frac{\mu^2}{\sigma^2 + \mu^2}\right)\,.$$

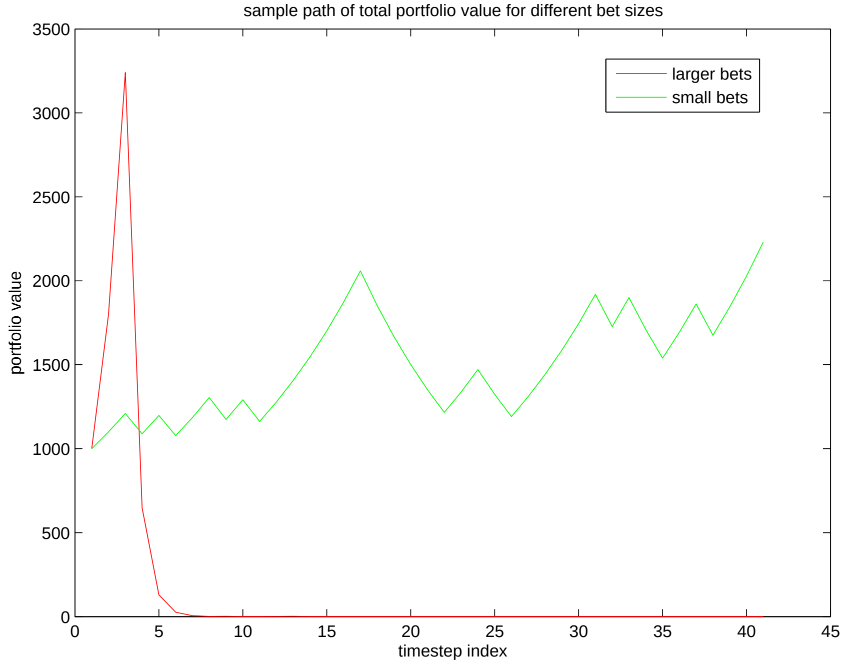

Figure 2: Two sample paths when we bet a large $f = 0.8$ or a small $f = 0.1$ fraction of our wealth at each time step. Note that the probabilities are set such that we should win in the long run and you see the small bet curve slowly increase. The large bet curve takes a serious loss which shrinks the portfolio to zero and we cannot play any more.

Notes on arbitrage in horse betting

We will place bets on all N horses such that the total bet made is one unit i.e. $\sum_{i=1}^{N} w_i = 1$. Then if horse j wins we will win back an amount $(q_j + 1)w_j$, where q_j is set by the bookies. This is after having paid 1 to place all N bets (the other $N - 1$ bets return 0). Thus we will have an arbitrage opportunity if no matter which horse wins we make a positive profit if

$$(q_j + 1)w_j - 1 > 0 \quad \text{or} \quad w_j > \frac{1}{q_j + 1},$$

for all j. Summing both sides of the above for all bets gives

$$\sum_{i=1}^{N} w_i > \sum_{i=1}^{N} \frac{1}{q_i + 1}.$$

Since $\sum_{i=1}^{N} w_i = 1$ this means *for* arbitrage that $1 > \sum_{i=1}^{N} \frac{1}{q_i+1}$ must hold. If the opposite inequality holds then there is *no* arbitrage.

Notes on how to bet

When we bet w_i on horse i for $i = 1, 2, \ldots, N$ then we have an initial outlay of money given by $\sum_{i=1}^{N} w_i = 1$ dollars. If we assume that horse i will win with probability p_i where due to our bet on that horse we will win an amount $W = w_i(q_i + 1)$ so that the average amount won will thus be

$$E[W] = \sum_{i=1}^{N} p_i w_i (q_i + 1) \,.$$

Here W is a discrete random variable with probability distribution of p_i. Since we have to outlay an amount 1 to place all of these N bets the expected total amount denoted by m, we will have after these N bets is given by

$$m \equiv \left(\sum_{i=1}^{N} p_i w_i (q_i + 1) \right) - 1 \,.$$

The variance of these wins is

$$\sigma^2 = E[(W - \bar{W})^2] = \sum_{i=1}^{N} p_i [w_i(q_i + 1) - (m + 1)]^2 \,,$$

since the expected value of W is $\sum_{i=1}^{N} p_i w_i (q_i + 1) = m - 1$. We now consider some procedures that would yield "optimal" betting solutions for the example odds q_i and probabilities p_i given.

- In the case where we want to maximize the expected winnings or m we will pick the index j that makes $p_i(q_i + 1)$ the largest and bet all our money on that horse. In the MATLAB script `max_expected_return.m` we verify that this strategy has $m = 0.4$ and $\sigma = 2.80$.

- In the case where we would want to minimize the standard deviation. In the MATLAB script `min_std_dev.m` we verify that this strategy produces has $m = -0.6216$ and $\sigma = 5.6064\, 10^{-5}$ with a weight vector of

  ```
  0.0631, 0.0541, 0.1892, 0.1892, 0.1261, 0.1261, 0.1261, 0.1261
  ```

  This matches quite well with what the book obtained. It would perhaps be better to use the logistic transformation (see the next item) in performing this optimization.

- In this case we want to maximize the return *divided* by the standard deviation. To do this and still use the unbounded MATLAB solver we will first map the infinite range $(-\infty, +\infty)$ to the range $(0, 1)$. To do this we will use the logistic transformation, in that we obtain the needed weights w_i from variables with an infinite range x_i using

  $$w_i = \frac{e^{x_i}}{1 + \sum_{l=1}^{N-1} e^{x_l}} \quad \text{for} \quad i = 1, 2, \cdots, N-1$$
  $$w_N = 1 - \sum_{l=1}^{N-1} w_l = \frac{1}{1 + \sum_{l=1}^{N-1} e^{x_l}} \,.$$

  Then we can optimize over the x_i's. In the MATLAB script `max_adj_return.m` we find $m = 0.3082$ and $\sigma = 1.6352$ and weights given by

```
0.4590, 0.5410, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000
```

These results match quite will thoese found in the book.

Portfolio Management

Notes on diversification

In this section we assume we are considering a portfolio, denoted by Π, of N stocks. We will let w_i be the number of *shares* of the ith security valued at a per share price of S_i. Then the portfolio's value at this time is given by

$$\Pi = \sum_{i=1}^{N} w_i S_i \,. \tag{175}$$

If each stock S_i undergoes a return of R_i then their value changes to $S_i(1+R_i)$ and this new portfolio has a value $\Pi + \delta\Pi$ give by

$$\Pi + \delta\Pi = \sum_{i=1}^{N} w_i S_i (1 + R_i) \,.$$

Thus the portfolio change, $\delta\Pi$, is given by $\sum_{i=1}^{N} w_i S_i R_i$ and the return on our portfolio over this time interval is given by

$$\frac{\delta\Pi}{\Pi} = \frac{1}{\Pi}\sum_{i=1}^{N} w_i S_i R_i = \sum_{i=1}^{N}\left(\frac{w_i S_i}{\Pi}\right) R_i = \sum_{i=1}^{N} W_i R_i \,, \tag{176}$$

where we have defined the portfolio weights W_i as

$$W_i \equiv \frac{w_i S_i}{\Pi} = \frac{w_i S_i}{\sum_{l=1}^{N} w_l S_l} \,, \tag{177}$$

and we have used Equation 175 to replace the symbol Π with a summation. Note that these weights W_i, by construction, sum to one. The mean portfolio return (if our time frame for the individual returns R_i is T) is computed as

$$\mu_\Pi = \frac{1}{T} E\left[\frac{\delta\Pi}{\Pi}\right] = \frac{1}{T}\sum_{i=1}^{N} W_i E[R_i] = \frac{1}{T}\sum_{i=1}^{N} W_i T \mu_i = \sum_{i=1}^{N} W_i \mu_i \,, \tag{178}$$

while the variance of our portfolio's returns over T are computed is

$$\sigma_\Pi^2 = \frac{1}{T}\mathrm{Var}\left[\frac{\delta\Pi}{\Pi}\right] = \sum_{i=1}^{N}\sum_{j=1}^{N} W_i W_j \rho_{ij} \sigma_i \sigma_j \,. \tag{179}$$

Notes on return risk characteristics in a portfolio

As a simple example of the meaning of the terms "the efficient frontier", consider the case where we have just two stocks say A and B, where we own a portfolio fraction W of the first and correspondingly a portfolio fraction $1 - W$ of the other. Then the average return on this two stock portfolio from Equation 178 is given by

$$\mu_\Pi = W\mu_A + (1 - W)\mu_B \,,$$

and the variance of this portfolio from Equation 179 is

$$\sigma_\Pi^2 = W^2\sigma_A^2 + 2W(1 - W)\rho\sigma_A\sigma_E + (1 - W)^2\sigma_E^2 \,.$$

We can plot the points $(\sigma_\Pi(W), \mu_\Pi(W))$ as a function of W for various portfolio weights i.e. $0 \le W \le 1$. This gives us the set of possible values for (σ_Π, μ_Π) we could obtain corresponding to different values of W. In the two stock case given a desired (and fixed) value for the portfolio return, μ_Π, from the formula above gives an explicit value for W which then explicitly determines the value of σ_Π. In the general case with N instruments there are many settings for W_i which could give the same desired portfolio mean return μ_Π. We would most naturally want to select the weights that give a mean value of μ_Π while minimizing the value of σ_Π. This leads to the efficient frontier in the general case. That is, we want to determine the values for W_i such that $\sum_{i=1}^N W_i = 1$ and that the value of μ_Π given by

$$\mu_\Pi = \sum_{i=1}^N W_i\mu_i \,,$$

is *fixed*. Subject to these two constraints we want to then *minimize* the value of σ_Π given by Equation 179. Note is seems we could also frame the asset allocation problem as to specify the values of W_i such that they sum to 1 and that maximize the Sharpe ratio

$$\frac{\mu_\Pi}{\sigma_\Pi} \,.$$

Notes on the Capital Asset Pricing Model (CAPM)

Under the Capital Asset Pricing Model (CAPM) the ith stocks return is modeled as

$$R_i = \alpha_i + \beta_i R_M + \epsilon_i \,, \tag{180}$$

and its variance σ_i^2, is given by

$$\sigma_i^2 = \beta_i^2\sigma_M^2 + e_i^2 \,. \tag{181}$$

Here R_M is the "market" return and σ_M^2 is the market variance over the same return time frame. With these returns we see that the expected return for the ith stock is given by

$$\mu_i = \alpha_i + \beta_i\mu_M \,,$$

and our expected portfolio return by Equation 178 is then given by

$$\mu_\Pi = \sum_{i=1}^N W_i\alpha_i + \left(\sum_{i=1}^N W_i\beta_i\right)\mu_M \,.$$

Here μ_M is the expected market return. We can define the portfolio values of α and β as

$$\alpha_\Pi = \sum_{i=1}^{N} W_i \alpha_i \quad \text{and} \quad \beta_\Pi = \sum_{i=1}^{N} W_i \beta_i \,,$$

so that $\mu_\Pi = \alpha_\Pi + \beta_\Pi \mu_M$. The variance of our portfolio's return can be computed in the CAPM framework also. We find from the expression for $\frac{\delta\Pi}{\Pi}$ of

$$\frac{\delta\Pi}{\Pi} = \sum_{i=1}^{N} W_i R_i \,,$$

that [3]

$$\text{Var}\left[\frac{\delta\Pi}{\Pi}\right] = \sum_{i=1}^{N}\sum_{j=1}^{N} W_i W_j \text{Cov}(R_i, R_j) \,.$$

Thus we need to compute $\text{Cov}(R_i, R_j)$. We find

$$\begin{aligned}
\text{Cov}(R_i, R_j) &= E[(R_i - \bar{R}_i)(R_j - \bar{R}_j)] \\
&= E[(\beta_i(R_M - \mu_M) + \epsilon_i)(\beta_j(R_M - \mu_M) + \epsilon_j)] \\
&= \beta_i \beta_j E[(R_M - \mu_M)^2] + E[\epsilon_i \epsilon_j] \\
&= \beta_i \beta_j \sigma_M^2 + \delta_{ij} e_i^2 \,,
\end{aligned}$$

where δ_{ij} is the Kronecker delta. Thus using that we find

$$\sigma_\Pi^2 = \left(\sum_{i=1}^{N}\sum_{j=1}^{N} W_i W_j \beta_i \beta_j\right)\sigma_M^2 + \sum_{i=1}^{N} W_i^2 e_i^2 \,.$$

If we assume that $W_i \sim \frac{1}{N}$ then the first term above dominates since

$$\begin{aligned}
O\left(\sum_{i=1}^{N} W_i^2 e_i^2\right) &\approx O\left(\frac{1}{N^2}\sum_{i=1}^{N} 1\right) = O\left(\frac{1}{N}\right) \\
O\left(\sum_{i=1}^{N}\sum_{j=1}^{N} W_i W_j \beta_i \beta_j \sigma_M^2\right) &\approx O\left(\frac{1}{N^2} N^2\right) = O\left(1\right) \,.
\end{aligned}$$

Value at Risk

Notes on VaR for a single asset

Recall that the variance of return of our stock over a time length of δt is given by

$$\mathrm{Var}\left[\frac{\delta S}{S}\right] = \sigma^2 \delta t\,.$$

The variance of δS over this time is then given by

$$\mathrm{Var}[\delta S] = S^2\sigma^2\delta t\,.$$

The standard deviation of δS over this time is the square root of the above and we get

$$\mathrm{Std}\left[\delta S\right] = S\sigma\delta t^{1/2}\,.$$

To compute the probability that a stock moves less than some number of standard deviations from is starting value of S we introduce the cumulative normal distribution function $\alpha(\cdot)$ where for example if we desire a confidence of 99% we take $c = 0.99$ and find that $\alpha(1-c) = \alpha(0.01) = -2.326$. Thus with a probablity of 0.01% we will loose at least the amount

$$-2.326 S\sigma\delta t^{1/2}\,.$$

This is the potential loss of *one* share of the stock. If we actually hold Δ shares of the stock we must multiply the above number by Δ. If we want to evaluate δt to be 1 week where σ^2 is the *annualized* variance then $\delta t = \frac{1}{52}$ since there are 52 weeks in one year.

If there is a drift to the distribution of returns then we assume that the distribution of returns $\frac{\delta S}{S}$ is normal with a mean of $\mu\delta t$ and a variance of $\sigma^2\delta t$. Then the lower limit of the confidence interval of $\frac{\delta S}{S}$ is given by

$$\mu\delta t - \alpha(1-c)\sigma\delta t^{1/2}\,.$$

The lower limit of δS over this length of time δt is then given by multiplying the above by S and if we have Δ shares we get

$$\mathrm{VaR} = \Delta S(\mu\delta t - \alpha(1-c)\sigma\sqrt{\delta t})\,.$$

Notes on the Delta-Gamma approximation

If we assume that the change in asset price S over the time δt is given by

$$\delta S = \mu S\delta t + \sigma S\delta t^{1/2}\phi\,,$$

where ϕ is a random draw from a standard normal. Then the square of δS is given by

$$\delta S^2 = \mu^2 S^2\delta t^2 + 2\mu\sigma S^2\phi\delta t^{3/2} + \sigma^2 S^2\phi^2\delta t = \sigma^2 S^2\phi^2\delta t + O(\delta t^{3/2})\,.$$

Then performing a Taylor expansion of $V(S,t)$ our portfolio V changes during δt as

$$\begin{aligned}
\delta V &= \frac{\partial V}{\partial S}(\mu S\delta t + \sigma S\delta t^{1/2}\phi) + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2\phi^2\delta t + \frac{\partial V}{\partial t}\delta t \\
&= \frac{\partial V}{\partial S}\sigma S\phi\delta t^{1/2} + \left(\frac{\partial V}{\partial S}\mu S + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2\phi^2 + \frac{\partial V}{\partial t}\right)\delta t + \cdots \\
&= \Delta\sigma S\phi\delta t^{1/2} + \delta t\left(\Delta\mu S + \frac{1}{2}\Gamma\sigma^2 S^2\phi^2 + \theta\right) + \cdots .
\end{aligned} \tag{182}$$

We see that δV is a quadratic in ϕ where $\phi \sim N(0,1)$. We can write δV as a function of ϕ by completing the square to find

$$\begin{aligned}
\delta V &= \frac{1}{2}\Gamma\sigma^2 S^2\delta t\left(\phi^2 + \frac{2\Delta}{\Gamma\sigma S\delta t^{1/2}}\phi\right) + \delta t(\Delta\mu S + \theta) \\
&= \frac{1}{2}\Gamma\sigma^2 S^2\delta t\left(\phi^2 + \frac{2\Delta}{\Gamma\sigma S\delta t^{1/2}}\phi + \frac{\Delta^2}{\Gamma^2\sigma^2 S^2\delta t}\right) - \frac{\Delta^2}{2\Gamma} + \delta t(\Delta\mu S + \theta) \\
&= \frac{1}{2}\Gamma\sigma^2 S^2\delta t\left(\phi + \frac{\Delta}{\Gamma\sigma S\delta t^{1/2}}\right)^2 - \frac{\Delta^2}{2\Gamma} + \delta t(\Delta\mu S + \theta)
\end{aligned}$$

Note that the last three terms are nonrandom and thus is not subject to risk. Depending on the *sign* of Γ the change in δV will be larger or small than this deterministic number. We see that

$$\delta V \geq -\frac{\Delta^2}{2\Gamma} + \delta t(\Delta\mu S + \theta) \quad \text{if} \quad \Gamma > 0\,.$$

Thus in this case we have a *lower bound* on our potential loss δV. While if $\Gamma < 0$ we find

$$\delta V \leq -\frac{\Delta^2}{2\Gamma} + \delta t(\Delta\mu S + \theta) \quad \text{if} \quad \Gamma < 0\,.$$

and there is no lower bound. It stands to reason that all things being equal one desires portfolios that have a positive Γ.

Chapter 23: Barrier Options

That $S^{\alpha}V_{\mathrm{BS}}(X/S,t)$ is also a solution to Black-Scholes equation

As discussed we now show that the expression

$$\tilde{V}(S,t) \equiv S^{1-\frac{2r}{\sigma^2}} V_{\mathrm{BS}}\left(\frac{X}{S}, t\right) ,$$

where V_{BS} is a solution to the Black-Scholes equation is also a solution to the Black-Scholes equation. To do this, consider the needed derivatives (using the substitution $\xi \equiv \frac{X}{S}$ when needed)

$$\begin{aligned}
\tilde{V}_t &= S^{1-\frac{2r}{\sigma^2}} \frac{\partial V_{\mathrm{BS}}}{\partial t} \\
\tilde{V}_S &= \left(1 - \frac{2r}{\sigma^2}\right) S^{-\frac{2r}{\sigma^2}} V_{\mathrm{BS}} + S^{1-\frac{2r}{\sigma^2}} \frac{\partial V_{\mathrm{BS}}}{\partial \xi} \cdot \left(-\frac{X}{S^2}\right) \\
&= \left(1 - \frac{2r}{\sigma^2}\right) S^{-\frac{2r}{\sigma^2}} V_{\mathrm{BS}} - X S^{-1-\frac{2r}{\sigma^2}} \frac{\partial V_{\mathrm{BS}}}{\partial \xi} \\
\tilde{V}_{SS} &= \left(1 - \frac{2r}{\sigma^2}\right)\left(-\frac{2r}{\sigma^2}\right) S^{-1-\frac{2r}{\sigma^2}} V_{\mathrm{BS}} + \left(1 - \frac{2r}{\sigma^2}\right) S^{-\frac{2r}{\sigma^2}} \frac{\partial V_{\mathrm{BS}}}{\partial \xi} \cdot \left(-\frac{X}{S^2}\right) \\
&- X\left(-1 - \frac{2r}{\sigma^2}\right) S^{-2-\frac{2r}{\sigma^2}} \frac{\partial V_{\mathrm{BS}}}{\partial \xi} - X S^{-1-\frac{2r}{\sigma^2}} \frac{\partial^2 V_{\mathrm{BS}}}{\partial \xi^2} \cdot \left(-\frac{X}{S^2}\right) \\
&= -\frac{2r}{\sigma^2}\left(1 - \frac{2r}{\sigma^2}\right) S^{-1-\frac{2r}{\sigma^2}} V_{\mathrm{BS}} + \frac{4Xr}{\sigma^2} S^{-2-\frac{2r}{\sigma^2}} \frac{\partial V_{\mathrm{BS}}}{\partial \xi} + X^2 S^{-3-\frac{2r}{\sigma^2}} \frac{\partial^2 V_{\mathrm{BS}}}{\partial \xi^2}
\end{aligned}$$

we will put this into the left-hand-side of the Black-Scholes Equation 26 to get

$$\begin{aligned}
& S^{1-\frac{2r}{\sigma^2}} \frac{\partial V_{\mathrm{BS}}}{\partial t} \\
+ \quad & \frac{1}{2}\sigma^2 \left[-\frac{2r}{\sigma^2}\left(1 - \frac{2r}{\sigma^2}\right) S^{1-\frac{2r}{\sigma^2}} V_{\mathrm{BS}} + \frac{4Xr}{\sigma^2} S^{-\frac{2r}{\sigma^2}} \frac{\partial V_{\mathrm{BS}}}{\partial \xi} + X^2 S^{-1-\frac{2r}{\sigma^2}} \frac{\partial^2 V_{\mathrm{BS}}}{\partial \xi^2} \right] \\
+ \quad & r\left[\left(1 - \frac{2r}{\sigma^2}\right) S^{1-\frac{2r}{\sigma^2}} V_{\mathrm{BS}} - X S^{-\frac{2r}{\sigma^2}} \frac{\partial V_{\mathrm{BS}}}{\partial \xi} \right] - r S^{1-\frac{2r}{\sigma^2}} V_{\mathrm{BS}} .
\end{aligned}$$

Number each term above with a V_{BS} in it using the numbers $1 \ldots 7$. Note that if we add the terms numbered 2 and 5 we get zero. Combining the rest of the terms we get

$$S^{1-\frac{2r}{\sigma^2}} \left[\frac{\partial V_{\mathrm{BS}}}{\partial t} + \frac{1}{2}\sigma^2 X^2 S^2 \frac{\partial^2 V_{\mathrm{BS}}}{\partial \xi^2} + r X S \frac{\partial V_{\mathrm{BS}}}{\partial \xi} - r V_{\mathrm{BS}} \right]$$

When we replace XS with ξ we see that the above is the Black-Scholes equation in the variables ξ and t and thus vanishes showing that the original expression $S^{1-\frac{2r}{\sigma^2}} V_{\mathrm{BS}}\left(\frac{X}{S}, t\right)$ is a solution to the Black-Scholes equation. Note that the solution to the down and out call given by

$$V(S,t) = V_{\mathrm{BS}}(S,t) - \left(\frac{S}{S_d}\right)^{1-\frac{2r}{\sigma^2}} V_{\mathrm{BS}}\left(\frac{S_d^2}{S}, t\right) , \tag{183}$$

satisfies the correct final conditions. Since we must have $S_d < S$ we see that $\frac{S_d}{S} < 1$ thus

$$\frac{S_d^2}{S} < S_d < E\,,$$

where we have assumed that S_d is below E. Using this we get when $t = T$ that the expression $V_{\rm BS}\left(\frac{S_d^2}{S}, t\right)$ takes the value

$$V_{\rm BS}\left(\frac{S_d^2}{S}, T\right) = \max\left(\frac{S_d^2}{S} - E, 0\right) = 0\,,$$

and we have shown that the final value for a down-and-out call option equals $V_{\rm BS}(S,T)$ as it should.

Notes on Hedging Barrier Options

Here we duplicate some of the statements made in the book. For example, we take $D = 0$, $r = 0$, and $S = S_d$ in Equation 36 we find that C, d_1, and d_2 are given by

$$\begin{aligned}
C(S_d, t) &= S_d N(d_1) - E N(d_2) \\
d_1 &= \frac{\log(S_d/E) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} \\
d_2 &= \frac{\log(S_d/E) - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\,.
\end{aligned}$$

While in Equation 44 with strike take to be $\frac{S_d^2}{E}$ (again $D = r = 0$) we have

$$\begin{aligned}
P(S_d, t) &= -S_d N(-\tilde{d}_1) + \frac{S_d^2}{E} N(-\tilde{d}_2) \\
\tilde{d}_1 &= \frac{\log\left(\frac{S_d E}{S_d^2}\right) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} = -d_2 \\
\tilde{d}_2 &= \frac{\log\left(\frac{E}{S_d}\right) - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} = -d_1\,.
\end{aligned}$$

Thus using these last expressions for $\tilde{d}_1$ and $\tilde{d}_2$ in the formula for $P(S_d, t)$ we see that

$$P(S_d, t) = -S_d N(d_2) + \frac{S_d^2}{E} N(d_1) = \left(\frac{S_d}{E}\right)(S_d N(d_1) - E N(d_2)) = \frac{S_d}{E} C(S_d, t)\,.$$

This last equation says that we need to hedge each call with $\frac{E}{S_d}$ puts (note the reciprocal) and we are perfectly hedged at $S = S_d$ and when $r = 0$.

Chapter 25 (Asian Options)

Notes on discretely sampled averages

For a discretely sampled geometric average we have

$$\begin{aligned} A_i &= \exp\left(\frac{1}{i}\sum_{k=1}^{i}\log(S(t_k))\right) = \exp\left(\frac{1}{i}\sum_{k=1}^{i-1}\log(S(t_k)) + \frac{1}{i}\log(S(t_i))\right) \\ &= \exp\left(\frac{i-1}{i}\log(A_{i-1}) + \frac{1}{i}\log(S(t_i))\right) . \end{aligned}$$

Thus continuity of the option price across the sampling date is given by

$$V(S, A, t_i^-) = V\left(S, \exp\left(\frac{i-1}{i}\log(A) + \frac{1}{i}\log(S)\right), t_i^+\right) .$$

Notes on exponentially weighted and other averages

For the given expression of I note that we can write it as

$$I = \lambda\int_{-\infty}^{t} e^{-\lambda(t-\tau)}s(\tau)d\tau = e^{-\lambda t}\lambda\int_{-\infty}^{t} e^{\lambda\tau}s(\tau)d\tau ,$$

from which we can more easily compute the differential of this expression dI. We find

$$\begin{aligned} dI &= -\lambda e^{-\lambda t}\left(\lambda\int_{-\infty}^{t} e^{\lambda\tau}s(\tau)d\tau\right)dt + e^{-\lambda t}\lambda(e^{\lambda t}S)dt \\ &= -\lambda I dt + \lambda S dt \\ &= \lambda(S-I)dt , \end{aligned}$$

the claimed expression. Note that with this type of averaging by following the discussion from earlier in the book we have a partial differential equation for the option value V given by

$$\frac{\partial V}{\partial t} + \lambda(S-I)\frac{\partial V}{\partial I} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 .$$

In the discrete case for the jump conditions we need the expression I_i given recursively by $I_i = F(S(t_i), I_{i-1}, i)$. In this case, this is computed as

$$\begin{aligned} I_i &\equiv \lambda\sum_{k=-\infty}^{i} e^{-\lambda(t_i-t_k)}S(t_k) = \lambda\left[\sum_{k=-\infty}^{i-1} e^{-\lambda(t_i-t_k)}S(t_k) + S(t_i)\right] \\ &= \lambda\left[e^{-\lambda(t_i-t_{i-1})}\sum_{k=-\infty}^{i-1} e^{-\lambda(t_{i-1}-t_k)}S(t_k) + S(t_i)\right] \\ &= \lambda\left[e^{-\lambda(t_i-t_{i-1})}\left(\frac{I_{i-1}}{\lambda}\right) + S(t_i)\right] \\ &= e^{-\lambda(t_i-t_{i-1})}I_{i-1} + \lambda S(t_i) . \end{aligned}$$

We can derive the jump conditions for V using this expression in the following way

$$V(S,I,t_i^-) = V(S,e^{-\lambda(t_i^+ - t_{i-1})}I + \lambda S, t_i^+)\,.$$

Notes on similarity reductions

For the reduction discussed in the book we introduce the two variables I and R given by

$$I \equiv \int_0^t S(\tau)d\tau \quad \text{and} \quad R \equiv \frac{S}{\int_0^t S(\tau)d\tau} = \frac{S}{I}\,.$$

With these two variables, we can write the running payoff of a continuously sampled arithmetic strike option as

$$\max\left(S - \frac{1}{t}\int_0^t S(\tau)d\tau, 0\right) = \max\left(RI - \frac{I}{t}, 0\right) = I\max\left(R - \frac{1}{t}, 0\right)\,.$$

Based on the fact that in the payoff we can factor I outside of another function (namely $\max\left(R - \frac{1}{t}, 0\right)$) we propose an option price V of the following form

$$V = I\,W(R,t)\,,$$

with again $R = \frac{S}{I}$. Then to find the differential equation that W satisfies we need to compute the derivatives of V, needed by the Black-Scholes equation, but in terms of derivatives of W. We find

$$\begin{aligned}
\frac{\partial V}{\partial t} &= I\frac{\partial W}{\partial t} \\
\frac{\partial V}{\partial S} &= I\frac{\partial W}{\partial R}\left(\frac{1}{I}\right) = \frac{\partial W}{\partial R} \\
\frac{\partial^2 V}{\partial S^2} &= \frac{\partial}{\partial S}\left(\frac{\partial W}{\partial R}\right) = \frac{\partial}{\partial R}\left(\frac{\partial W}{\partial R}\right)\left(\frac{\partial R}{\partial S}\right) = \frac{1}{I}\frac{\partial^2 W}{\partial R^2} \\
\frac{\partial V}{\partial I} &= W + I\frac{\partial W}{\partial R}\frac{\partial R}{\partial I} = W + I\frac{\partial W}{\partial R}\left(-\frac{S}{I^2}\right) = W - R\frac{\partial W}{\partial R}\,.
\end{aligned}$$

Therefore with these derivatives the Black-Scholes equation becomes (in terms of W)

$$I\frac{\partial W}{\partial t} + RI\left(W - R\frac{\partial W}{\partial R}\right) + \frac{1}{2}\sigma^2\frac{I^2R^2}{I}\frac{\partial^2 W}{\partial R^2} + rIR\frac{\partial W}{\partial R} - rIW = 0\,.$$

The second term above comes from the $S\frac{\partial V}{\partial I}$ term needed to included the state variable I. If we divide by I we get

$$\frac{\partial W}{\partial t} + R\left(W - R\frac{\partial W}{\partial R}\right) + \frac{1}{2}\sigma^2 R^2\frac{\partial^2 W}{\partial R^2} + rR\frac{\partial W}{\partial R} - rW = 0\,,$$

or changing the order of various terms and grouping we finally have

$$\frac{\partial W}{\partial t} + \frac{1}{2}\sigma^2 R^2\frac{\partial^2 W}{\partial R^2} + R(r-R)\frac{\partial W}{\partial R} - (r-R)W = 0\,, \tag{184}$$

the same as in the book.

Put-call parity for the European average strike option

In this subsection we derive put-call parity relationships for European average strike options. To begin consider the payoff at expiration, $\Pi(T)$, on the suggested portfolio of one European average strike call held long and one European average put held short

$$\Pi(T) = I \max\left(1 - \frac{1}{T}, 0\right) - I \max\left(\frac{1}{T} - 1, 0\right) = S - \frac{I}{T}.$$

Now consider what type of instrument will provide a payoff given by the second term $-\frac{I}{T}$. Since the continuously sampled average strike option has a value that can be expressed as $V(S, I, t) = IW(R, t)$, where $W(R, t)$ is a solution to Equation 184. At expiration we would like to construct a continuously sampled average strike option that has a payoff given by $-\frac{I}{T}$. This means that when $t = T$ the function $W(R, t)$ must have a final condition that satisfies $IW(R, T) = -\frac{I}{T}$ or

$$W(R, T) = -\frac{1}{T}. \tag{185}$$

With this final condition the solution we seek for $W(R, t)$ must also satisfy the standard boundary conditions of

$$W(0, t) = 0 \quad \text{and} \quad W(R, t) \sim R \quad \text{as} \quad R \to \infty.$$

To find a functional form for $W(R, t)$ that will satisfy these requirements we propose a $W(R, t)$ of the specific form given by

$$W(R, t) = a(t)R + b(t). \tag{186}$$

Note that for the expression for the payoff in Equation 185 to hold true at $t = T$ when $W(R, t)$ is as Equation 186 we need

$$W(R, T) = a(T)R + b(T) = -\frac{1}{T},$$

which requires that we take $a(T) = 0$ and $b(T) = -\frac{1}{T}$. Now to put this proposed functional form for $W(R, t)$ from Equation 186 into Equation 184 requires we compute

$$\begin{aligned}
\frac{\partial W}{\partial t} &= a'(t)R + b'(t) \\
\frac{\partial W}{\partial R} &= a(t) \\
\frac{\partial^2 W}{\partial R^2} &= 0,
\end{aligned}$$

so that Equation 184 evaluates to

$$a'(t)R + b'(t) + R(r - R)a(t) - (r - R)(b(t) + a(t)R) = 0,$$

or grouping powers of R and simplifying we get

$$(a'(t) + b(t))R + (b'(t) - rb(t)) = 0. \tag{187}$$

Equating the powers R on both sides of this equation we find for the coefficient of the zero power of R to vanish requires that

$$b'(t) = rb(t) \quad \text{so} \quad b(t) = C_0 e^{rt} \,.$$

When we then require that $b(T) = -\frac{1}{T}$ we find

$$-\frac{1}{T} = C_0 e^{rT} \quad \text{or} \quad C_0 = -\frac{1}{T} e^{-rT} \,,$$

which means that the function $b(t)$ is given by

$$b(t) = -\frac{1}{T} e^{-r(T-t)} \,. \tag{188}$$

Using this result and equating the constant terms in Equation 187 requires that the function $a(t)$ must satisfy

$$a'(t) = -b(t) = \frac{1}{T} e^{-r(T-t)} \,.$$

On integrating this we find $a(t)$ given by

$$a(t) = \frac{1}{rT} e^{-r(T-t)} + C_1 \,.$$

To evaluate C_1 we recall that $a(T) = 0$ requires that $C_1 = -\frac{1}{rT}$ so that the functional form for $a(t)$ is given by

$$a(t) = -\frac{1}{rT}(1 - e^{-r(T-t)}) \,. \tag{189}$$

Using these we conclude that the put-call parity relationship for average strike options is

$$\begin{aligned}
C - P &= S + V(S, I, t) = S + IW(R, t) \\
&= S + I(b(t) + a(t)R) \\
&= S + I\left(-\frac{1}{T} e^{-r(T-t)}\right) + IR\left(-\frac{1}{rT}(1 - e^{-r(T-t)})\right) \\
&= S - \frac{S}{rT}(1 - e^{-r(T-t)}) - \frac{1}{T} e^{-r(T-t)} \int_0^t S(\tau) d\tau \,,
\end{aligned} \tag{190}$$

using $IR = S$.

Notes on the square root of three rule

In this section we simply compute the value of the variable $\bar{\sigma}_G^2$ when σ is a constant. We compute

$$\bar{\sigma}_G^2 = \frac{1}{T} \int_0^T \sigma(t)^2 \left(\frac{T-t}{T}\right)^2 dt = \frac{\sigma^2}{T} \int_0^T \left(\frac{T-t}{T}\right)^2 dt \,.$$

Let $v = \frac{T-t}{T}$ so that $dv = -\frac{dt}{T}$ and the above becomes

$$\bar{\sigma}_G^2 = -\sigma^2 \int_1^0 v^2 dv = \sigma^2 \left(\frac{v^3}{3}\right|_0^1 = \frac{\sigma^2}{3} \,.$$

Thus $\bar{\sigma}_G = \frac{\sigma}{\sqrt{3}}$.

Chapter 26 (Lookback Options)

Notes on similarity reductions for lookback options

To motivate the similarity transformation we are going to make, we first note that we can write the payoff for the lookback strike put as

$$\max(M - S, 0) = M \max\left(1 - \frac{S}{M}, 0\right),$$

which is of the form of the variable M times another function (of the ratio $\frac{S}{M}$). This motivates us to look for a lookback options solution of the functional form

$$M^\alpha W\left(\frac{S}{M}\right),$$

for some (yet to be determined) power α and function W. If we define $\xi \equiv \frac{S}{M}$ we can change the Black-Scholes equation in V into an equation for $W = W(\xi, t)$. To do this we need the following derivatives. We have

$$\begin{aligned}
\frac{\partial V}{\partial t} &= M^\alpha \frac{\partial W}{\partial t} \\
\frac{\partial V}{\partial S} &= M^\alpha \frac{\partial W}{\partial \xi}\frac{\partial \xi}{\partial S} = M^\alpha \frac{\partial W}{\partial \xi}\left(\frac{1}{M}\right) = M^{\alpha-1}\frac{\partial W}{\partial \xi} \\
\frac{\partial^2 V}{\partial S^2} &= \frac{\partial}{\partial S}\left(M^{\alpha-1}\frac{\partial W}{\partial \xi}\right) = M^{\alpha-1}\frac{\partial^2 W}{\partial \xi^2}\cdot\frac{1}{M} = M^{\alpha-2}\frac{\partial^2 W}{\partial \xi^2}.
\end{aligned}$$

Then the Black-Scholes equation for V (using $S = M\xi$) becomes

$$M^\alpha \frac{\partial W}{\partial t} + \frac{1}{2}\sigma^2 M^2 \xi^2 \left(M^{\alpha-2}\frac{\partial^2 W}{\partial \xi^2}\right) + r(M\xi)M^{\alpha-1}\frac{\partial W}{\partial \xi} - rM^\alpha W = 0.$$

If we divide this by M^α we get

$$\frac{\partial W}{\partial t} + \frac{1}{2}\sigma^2 \xi^2 \frac{\partial^2 W}{\partial \xi^2} + r\xi \frac{\partial W}{\partial \xi} - rW = 0.$$

With our final condition that $V(S, M, T) = M^\alpha W(\xi, T)$. The boundary condition of $\frac{\partial V}{\partial M} = 0$ on $S = M$ becomes

$$\frac{\partial V}{\partial M} = \frac{\partial}{\partial M}(M^\alpha W(\xi, t)) = \alpha M^{\alpha-1} W(\xi, t) + M^\alpha \frac{\partial W}{\partial \xi}\cdot\frac{\partial \xi}{\partial M}.$$

Since

$$\frac{\partial \xi}{\partial M} = \frac{\partial}{\partial M}\left(\frac{S}{M}\right) = -\frac{S}{M^2} = -\frac{\xi}{M},$$

we get that

$$\frac{\partial V}{\partial M} = \alpha M^{\alpha-1} W - M^{\alpha-1}\xi \frac{\partial W}{\partial \xi},$$

If we set this equal to zero and then divide by $M^{\alpha-1}$ we get

$$\xi \frac{\partial W}{\partial \xi} - \alpha W = 0 \,.$$

On $S = M$ we have $\xi = 1$ and we get

$$\frac{\partial W}{\partial \xi} - \alpha W = 0 \quad \text{on} \quad \xi = 1 \,,$$

the equation given in the book.

Chapter 27 (Derivatives and Stochastic Control)

Notes on Passport Options

From the book we have that our portfolio has an evolution given by

$$d\pi = r(\pi - qS)dt + qdS\,.$$

If our stock had the standard dynamics $dS = \mu S dt + \sigma S dX$ then we can write $d\pi$ as

$$d\pi = r(\pi - qS)dt + q(\mu S dt + \sigma S dX) = (r(\pi - qS) + q\mu S)dt + q\sigma S dX\,. \tag{191}$$

This explicitly shows the coefficient of the dX term which is needed in taking derivatives with respect to π using Ito's Lemma. If we define our hedged portfolio in the normal way as long one option and short some amount of stock as $\Pi = V - \Delta S$ since V is a function of S, π, and t we find that

$$\begin{aligned}
d\Pi &= \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt + \frac{\partial V}{\partial S} dS + \frac{\partial V}{\partial \pi} d\pi + \frac{1}{2} q^2 S^2 \sigma^2 \frac{\partial^2 V}{\partial \pi^2} dt + (qS\sigma)(S\sigma) \frac{\partial^2 V}{\partial \pi \partial S} dt - \Delta dS \\
&= \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + q\sigma^2 S^2 \frac{\partial^2 V}{\partial \pi \partial S} + \frac{1}{2} q^2 S^2 \sigma^2 \frac{\partial^2 V}{\partial \pi^2}\right) dt + \frac{\partial V}{\partial S} dS + \frac{\partial V}{\partial \pi} d\pi - \Delta dS \\
&= \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + q\sigma^2 S^2 \frac{\partial^2 V}{\partial \pi \partial S} + \frac{1}{2} q^2 S^2 \sigma^2 \frac{\partial^2 V}{\partial \pi^2} + r(\pi - qS)\frac{\partial V}{\partial \pi}\right) dt \\
&+ \frac{\partial V}{\partial S} dS + q\frac{\partial V}{\partial \pi} dS - \Delta dS\,,
\end{aligned}$$

where in the last equation we have used the fact that $d\pi = r(\pi - qS)dt + qdS$. To make the coefficient of the random term dS equal to zero we must enforce

$$\frac{\partial V}{\partial S} + q\frac{\partial V}{\partial \pi} - \Delta = 0\,,$$

or solving for Δ that

$$\Delta = \frac{\partial V}{\partial S} + q\frac{\partial V}{\partial \pi}\,. \tag{192}$$

Under this condition $d\Pi$ is deterministic and we set $d\Pi = r\Pi dt$ to find

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + q\sigma^2 S^2 \frac{\partial^2 V}{\partial \pi \partial S} + \frac{1}{2} q^2 S^2 \sigma^2 \frac{\partial^2 V}{\partial \pi^2} + r(\pi - qS)\frac{\partial V}{\partial \pi} - rV + rS\Delta = 0\,,$$

or since we know an expression for Δ via Equation 192 we have

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + q\sigma^2 S^2 \frac{\partial^2 V}{\partial \pi \partial S} + \frac{1}{2} q^2 S^2 \sigma^2 \frac{\partial^2 V}{\partial \pi^2} + rS\frac{\partial V}{\partial S} + r\pi\frac{\partial V}{\partial \pi} - rV = 0\,. \tag{193}$$

The final condition on our passport option is $V(S, \pi, T) = \max(\pi, 0)$.

To more easily solve this equation, we seek a similarity solution of the form $V(S, \pi, t) = SH(\xi, t)$ with $\xi = \frac{\pi}{S}$. To derive the differential equation for $H(\xi, t)$ we need to take derivatives of the given functional form for V. We find

$$
\begin{aligned}
\frac{\partial V}{\partial t} &= S\frac{\partial H}{\partial t} \\
\frac{\partial V}{\partial S} &= H + S\frac{\partial H}{\partial \xi}\left(-\frac{\pi}{S^2}\right) = H - \xi\frac{\partial H}{\partial \xi} \\
\frac{\partial^2 V}{\partial S^2} &= \frac{\partial}{\partial S}\left(H - \xi\frac{\partial H}{\partial \xi}\right) = \frac{\partial}{\partial \xi}\left(H - \xi\frac{\partial H}{\partial \xi}\right)\left(\frac{\partial \xi}{\partial S}\right) \\
&= \left(\frac{\partial H}{\partial \xi} - \frac{\partial H}{\partial \xi} - \xi\frac{\partial^2 H}{\partial \xi^2}\right)\left(-\frac{\pi}{S^2}\right) = \frac{\xi^2}{S}\frac{\partial^2 H}{\partial \xi^2} \\
\frac{\partial V}{\partial \pi} &= S\frac{\partial H}{\partial \xi}\cdot\frac{1}{S} = \frac{\partial H}{\partial \xi} \\
\frac{\partial^2 V}{\partial \pi^2} &= \frac{\partial}{\partial \pi}\left(\frac{\partial H}{\partial \xi}\right) = \frac{1}{S}\frac{\partial^2 H}{\partial \xi^2} \\
\frac{\partial^2 V}{\partial S \partial \pi} &= \frac{\partial}{\partial S}\left(\frac{\partial H}{\partial \xi}\right) = \frac{\partial}{\partial \xi}\left(\frac{\partial H}{\partial \xi}\right)\frac{\partial \xi}{\partial S} = \frac{\partial^2 H}{\partial \xi^2}\left(-\frac{\pi}{S^2}\right) = -\frac{\xi}{S}\frac{\partial^2 H}{\partial \xi^2}\,.
\end{aligned}
$$

Using these derivatives in Equation 193 we find

$$
\begin{aligned}
& S\frac{\partial H}{\partial t} + \frac{1}{2}\sigma^2 S^2\left(\frac{\xi^2}{S}\frac{\partial^2 H}{\partial \xi^2}\right) + q\sigma^2 S^2\left(-\frac{\xi}{S^2}\frac{\partial^2 H}{\partial \xi^2}\right) + \frac{1}{2}q^2 S^2\sigma^2\left(\frac{1}{S}\frac{\partial^2 H}{\partial \xi^2}\right) \\
+ \quad & rS\left(H - \xi\frac{\partial H}{\partial \xi}\right) + r\pi\left(\frac{\partial H}{\partial \xi}\right) - rSH = 0\,.
\end{aligned}
$$

If we divide by S and simplify we get

$$
\frac{\partial H}{\partial t} + \left(\frac{1}{2}\sigma^2\xi^2 - q\sigma^2\xi + \frac{1}{2}q^2\sigma^2\right)\frac{\partial^2 H}{\partial \xi^2} = 0\,.
$$

or when we simplify the coefficient of $\frac{\partial^2 H}{\partial \xi^2}$

$$
\frac{\partial H}{\partial t} + \frac{1}{2}\sigma^2(\xi - q)^2\frac{\partial^2 H}{\partial \xi^2} = 0\,. \tag{194}
$$

The payoff transforms as follows

$$
V(S, \pi, T) = \max(\pi, 0) = SH(\xi, T)\,,
$$

when we divide by S we get

$$
H(\xi, T) = \max(\xi, 0)\,,
$$

for the simplified similarity solution final condition.

We now write the optimal strategy in terms of H. We find

$$
\begin{aligned}
\max_{|q|\le 1}\left(q\sigma^2 S^2\frac{\partial^2 V}{\partial S\partial \pi} + \frac{1}{2}q^2\sigma^2 S^2\frac{\partial^2 V}{\partial \pi^2}\right) &= \max_{|q|\le 1}\left(q\sigma^2 S^2\left(-\frac{\xi}{S}\frac{\partial^2 H}{\partial \xi^2}\right) + \frac{1}{2}q^2\sigma^2 S^2\left(\frac{1}{S}\frac{\partial^2 H}{\partial \xi^2}\right)\right) \\
&= \max_{|q|\le 1}\left(\left(-q\xi + \frac{1}{2}q^2\right)\frac{\partial^2 H}{\partial \xi^2}\right) \\
&= \max_{|q|\le 1}\left(\left(\frac{1}{2}(q^2 - 2q\xi + \xi^2) - \frac{1}{2}\xi^2\right)\frac{\partial^2 H}{\partial \xi^2}\right) \\
&= \max_{|q|\le 1}\left((q - \xi)^2\frac{\partial^2 H}{\partial \xi^2}\right)\,,
\end{aligned}
$$

since $\frac{1}{2}$ is just a scaling and ξ is a constant in the above maximization. If the sign of $\frac{\partial^2 H}{\partial \xi^2}$ is positive we can factor that expression out of the maximization above to find the optimal strategy to be

$$q = \begin{cases} -1 & \text{when} \quad \xi > 0 \\ +1 & \text{when} \quad \xi < 0 \end{cases} .$$

Assuming the optimal control q specified above we find the coefficient of $\frac{\partial^2 H}{\partial \xi^2}$ in Equation 194 to be given by

$$\begin{aligned} (\xi - q)^2 &= (\xi + 1)^2 \quad \text{when} \quad \xi > 0 \\ (\xi - q)^2 &= (\xi - 1)^2 = (-\xi + 1)^2 \quad \text{when} \quad \xi < 0 \,. \end{aligned}$$

In either case we can write the above as $(|\xi| + 1)^2$ for all ξ and we get

$$\frac{\partial H}{\partial t} + \frac{1}{2}\sigma^2(|\xi| + 1)^2 \frac{\partial^2 H}{\partial \xi^2} = 0 \,,$$

the same as in the book.

Chapter 28 (Miscellaneous Exotics)

Notes on Forward-Start Options

Following the book, we look for a similarity solution of the form $V(S, \mathcal{S}, t) = \mathcal{S}H(\xi, t)$ with $\xi = \frac{S}{\mathcal{S}}$ and $\mathcal{S} = S(T_1)$ the asset price at the time T_1. To derive the differential equation for $H(\xi, t)$ we need to take derivatives of the given functional form for V. We find

$$
\begin{aligned}
\frac{\partial V}{\partial t} &= \mathcal{S}\frac{\partial H}{\partial t} \\
\frac{\partial V}{\partial S} &= \mathcal{S}\left(\frac{1}{\mathcal{S}}\frac{\partial H}{\partial \xi}\right) = \frac{\partial H}{\partial \xi} \\
\frac{\partial^2 V}{\partial S^2} &= \frac{1}{\mathcal{S}}\frac{\partial^2 H}{\partial \xi^2}
\end{aligned}
$$

Using these derivatives in the Black-Scholes equation we get

$$
\mathcal{S}\frac{\partial H}{\partial t} + \frac{1}{2}\sigma^2\mathcal{S}^2\xi^2\left(\frac{1}{\mathcal{S}}\frac{\partial^2 H}{\partial \xi^2}\right) + r(\mathcal{S}\xi)\left(\frac{\partial H}{\partial \xi}\right) - r\mathcal{S}H = 0\,.
$$

If we divide by $\mathcal{S}$ and simplify we get

$$
\frac{\partial H}{\partial t} + \frac{1}{2}\sigma^2\xi^2\frac{\partial^2 H}{\partial \xi^2} + r\xi\frac{\partial H}{\partial \xi} - rH = 0\,.
$$

The boundary conditions transform as follows

$$
V(S, \mathcal{S}, T) = \max(S - \mathcal{S}, 0) = \mathcal{S}\max(\xi - 1, 0)SH(\xi, T)\,.
$$

Since the left-hand-side is equal to $\mathcal{S}H(\xi, T)$ we get that $H(\xi, T) = \max(\xi - 1, 0)$ in agreement with the book.

Notes on the Volatility Option

With one state variable $\mathcal{S}_i = S(t_{i-1})$ we can write the other state variable I_i as follows

$$
\begin{aligned}
I_i &= \sqrt{\frac{1}{\delta t(i-1)}\sum_{j=1}^{i}\log\left(\frac{S(t_j)}{S(t_{j-1})}\right)^2} \\
&= \sqrt{\frac{1}{\delta t(i-1)}\sum_{j=1}^{i-1}\log\left(\frac{S(t_j)}{S(t_{j-1})}\right)^2 + \frac{1}{\delta t(i-1)}\log\left(\frac{S(t_i)}{S(t_{i-1})}\right)^2} \\
&= \sqrt{\frac{1}{\delta t(i-1)}I_{i-1}^2(\delta t(i-2)) + \frac{1}{\delta t(i-1)}\log\left(\frac{S(t_i)}{S(t_{i-1})}\right)^2} \\
&= \sqrt{\left(\frac{i-2}{i-1}\right)I_{i-1}^2 + \frac{1}{\delta t(i-1)}\log\left(\frac{S(t_i)}{\mathcal{S}_i}\right)^2}\,,
\end{aligned}
$$

note the state $\mathcal{S}_i$ value in the last term inside the square root above. In the continuous case the definition of I is given by

$$I = \sqrt{\frac{1}{t}\int_0^t \sigma(S,\tau)d\tau}\,.$$

From this the differential of I or dI is given by

$$\begin{aligned} dI &= \frac{1}{2}\left(\frac{1}{t}\int_0^t \sigma(S,\tau)d\tau\right)^{-1/2}\left[\frac{\sigma(S,t)^2}{t} - \frac{1}{t^2}\int_0^t \sigma^2(S,\tau)d\tau\right]dt \\ &= \frac{1}{2I}\left(\frac{\sigma(S,t)^2}{t} - \frac{I^2}{t}\right)dt = \left(\frac{\sigma(S,t)^2 - I^2}{2tI}\right)dt\,. \end{aligned}$$

Chapter 30: One-Factor Interest Rate Models

Notes on the Text

Notes on the bond pricing equation for the general model

In this section we derive the bond pricing equation in the case when the bond pays a coupon $K(r,t)$. This is very similar to the equivalent derivation presented in the book. In the case where our bond pays coupons we need to add a $K(r,t)$ term to the dt increment of dV to get

$$dV = \left(\frac{\partial V}{\partial t} + K(r,t)\right) dt + \frac{\partial V}{\partial r} dr + \frac{1}{2} w^2 \frac{\partial^2 V}{\partial r^2} dt \,. \tag{195}$$

Using this expression for dV we consider the change in the mixed portfolio $\Pi = V_1 - \Delta V_2$ as

$$\begin{aligned} d\Pi &= \left(\frac{\partial V_1}{\partial t} + K\right) dt + \frac{\partial V_1}{\partial r} dr + \frac{1}{2} w^2 \frac{\partial^2 V_1}{\partial r^2} dt \\ &- \Delta \left[\left(\frac{\partial V_2}{\partial t} + K\right) dt + \frac{\partial V_2}{\partial r} dr + \frac{1}{2} w^2 \frac{\partial^2 V_2}{\partial r^2} dt \right] \,. \end{aligned}$$

Here we have explicitly assumed that $K(r,t)$ is paid by both bonds. If we take Δ given by

$$\Delta = \frac{\frac{\partial V_1}{\partial r}}{\frac{\partial V_2}{\partial r}} \,,$$

and then the dr terms in $d\Pi$ vanish, and we get

$$d\Pi = \left[\left(\frac{\partial V_1}{\partial t} + K\right) + \frac{1}{2} w^2 \frac{\partial^2 V_1}{\partial r^2} - \Delta \left[\left(\frac{\partial V_2}{\partial t} + K\right) + \frac{1}{2} w^2 \frac{\partial^2 V_2}{\partial r^2} \right] \right] dt \,.$$

Which to avoid an arbitrage opportunity we set equal to

$$r\Pi dt = r(V_1 - \Delta V_2) dt \,.$$

Thus when we do this and place V_1 on the left-hand-side and V_2 on the right-hand-side

$$\frac{\partial V_1}{\partial t} + K + \frac{1}{2} w^2 \frac{\partial^2 V_1}{\partial r^2} - rV_1 = \Delta \left[\frac{\partial V_2}{\partial t} + K + \frac{1}{2} w^2 \frac{\partial^2 V_2}{\partial r^2} - rV_2 \right] \,.$$

Next we divide by $\frac{\partial V_1}{\partial r}$ on both sides to get

$$\frac{1}{\frac{\partial V_1}{\partial r}} \left(\frac{\partial V_1}{\partial t} + K + \frac{1}{2} w^2 \frac{\partial^2 V_1}{\partial r^2} - rV_1 \right) = \frac{1}{\frac{\partial V_2}{\partial r}} \left(\frac{\partial V_2}{\partial t} + K + \frac{1}{2} w^2 \frac{\partial^2 V_2}{\partial r^2} - rV_2 \right) \,.$$

Setting each side equal to $a(r,t)$ which we take equal to $w(r,t)\lambda(r,t) - u(r,t)$ we see that both V_1 and V_2 must satisfy

$$\frac{\partial V}{\partial t} + K(r,t) + \frac{1}{2} w^2 \frac{\partial^2 V}{\partial r^2} + (u - \lambda w) \frac{\partial V}{\partial r} - rV = 0 \,, \tag{196}$$

which is the desired equation.

Notes on what is the market price of risk

The differential of our bond price V when V depends on a stochastic interest rate, r, such that $dr = w(r,t)dX + u(r,t)dt$ and a deterministic time variable, t, using Ito's lemma to leading order is given by

$$\begin{aligned} dV &= \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial t}dt + \frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2}dt \\ &= w\frac{\partial V}{\partial r}dX + \left(\frac{\partial V}{\partial t} + u\frac{\partial V}{\partial r} + \frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2}\right)dt\,. \end{aligned}$$

From the zero coupon bond pricing equation derived in the text we have that

$$\frac{\partial V}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} + u\frac{\partial V}{\partial r} = \lambda w\frac{\partial V}{\partial r} + rV\,, \tag{197}$$

from which when we put this into the coefficient of dt above we find that the value of dV computed above becomes

$$dV = w\frac{\partial V}{\partial r}dX + \left(w\lambda\frac{\partial V}{\partial r} + rV\right)dt\,.$$

Transforming this expression some we find

$$dV - rVdt = w\frac{\partial V}{\partial r}(dX + \lambda dt)\,, \tag{198}$$

which is the books equation 30.5.

Notes on tractable models and the solutions of the bond pricing equations

When our stochastic interest rate r satisfies

$$dr = w(r,t)dX + u(r,t)dt\,,$$

with

$$w(r,t) = \sqrt{\alpha(t)r + \beta(t)} \tag{199}$$

$$u(r,t) = -\gamma(t)r + \eta(t) + \lambda(r,t)\sqrt{\alpha(t)r + \beta(t)}\,, \tag{200}$$

Then we make the vary simple verification that that $u - \lambda w$ in Equation 196 is independent of $\lambda(r,t)$ since

$$u - \lambda w = -\gamma r + \eta + \lambda w - \lambda w = -\gamma r + \eta\,. \tag{201}$$

independent of λ. Now lets consider solutions to Equation 196 of the following form

$$Z(r,t) = e^{A(t;T) - rB(t;T)}\,. \tag{202}$$

To put this form into Equation 196 we need the following derivatives

$$
\begin{aligned}
\frac{\partial Z}{\partial t} &= \left(\frac{\partial A}{\partial t} - r\frac{\partial B}{\partial t}\right) Z(t;T) \\
\frac{\partial Z}{\partial r} &= -B(t;T)Z(t;T) \\
\frac{\partial^2 Z}{\partial r^2} &= B^2(t;T)Z(t;T)\,.
\end{aligned}
$$

When we put this into Equation 196 and divide by Z we get

$$
\frac{\partial A}{\partial t} - r\frac{\partial B}{\partial t} + \frac{1}{2}w^2B^2 - (u - \lambda w)B - r = 0\,. \tag{203}
$$

which is the books equation 30.9. If we take $\frac{\partial}{\partial r}$ of this expression we find

$$
-\frac{\partial B}{\partial t} + \frac{1}{2}B^2\frac{\partial(w^2)}{\partial r} - B\frac{\partial(u - \lambda w)}{\partial r} - 1 = 0\,.
$$

Taking another derivative with respect to r we get

$$
\frac{1}{2}B^2\frac{\partial^2(w^2)}{\partial r^2} - B\frac{\partial^2}{\partial r^2}(u - \lambda w) = 0\,.
$$

When we divide this by B we get

$$
\frac{1}{2}B\frac{\partial^2(w^2)}{\partial r^2} - \frac{\partial^2}{\partial r^2}(u - \lambda w) = 0\,.
$$

Since B is a function of T and $u - \lambda w$ is not a function of T by changing the value of T the left-hand-side of the above would change values unless

$$
\frac{\partial^2(w^2)}{\partial r^2} = 0\,. \tag{204}
$$

In this case we would then also have

$$
\frac{\partial^2(u - \lambda w)}{\partial r^2} = 0\,. \tag{205}
$$

Thus these last to expressions show that w and $u - \lambda w$ must have the functional forms given by Equations 199 and 200.

From Equation 201 we find that $u - \lambda w = \eta - r\gamma$ by using this and putting Equations 199 and 200 into Equation 203 we get

$$
\frac{\partial A}{\partial t} - r\frac{\partial B}{\partial t} + \frac{1}{2}(\alpha r + \beta)B^2 - (-\gamma r + \eta)B - r = 0\,.
$$

Equating power of r on the left-hand-side to the zero on the right-hand-side gives

$$
\begin{aligned}
\frac{\partial A}{\partial t} + \frac{1}{2}\beta(t)B^2 - \eta(t)B &= 0 \quad \text{for } O(r^0) && (206) \\
\frac{\partial B}{\partial t} - \frac{1}{2}\alpha(t)B^2 - \gamma(t)B + 1 &= 0 \quad \text{for } O(r^1)\,, && (207)
\end{aligned}
$$

both of which match the equations given in the book. Note that we have explicitly used function notation for α, β, γ, and η to emphasize the fact that that in general these expressions can be functions of t. The final condition for $Z(t;T)$ when $t = T$ is that $Z(t,T) = 1$ or

$$e^{A(T;T)-rB(T;T)} = 1\,,$$

or again equating powers of r gives

$$A(T;T) = B(T;T) = 0\,.$$

The bond pricing equation with constant parameters

In this subsection of these notes we derive the analytic expressions for $A(t;T)$ and $B(t;T)$ found in the expression $Z(r,t;T) = e^{A(t;T)-rB(t;T)}$ for the pricing of a zero coupon bond when the parameters in the dynamics of the stochastic interest rate r

$$dr = (\eta - \gamma r)dt + \sqrt{\alpha r + \beta}dX\,, \tag{208}$$

namely α, β, γ, and η are are *constant*. There is a lot of algebra in the notes that follow, but having all of the computations in one place can make the verification of the results easier. If desired this section can be skipped on first reading. From the discussion in the the book for the differential equation for B we have

$$\frac{dB}{dt} = \frac{1}{2}\alpha B^2 + \gamma B - 1 = \frac{1}{2}\alpha\left(B^2 + \frac{2\gamma}{\alpha}B - \frac{2}{\alpha}\right)\,. \tag{209}$$

To begin to integrate this equation lets find the two roots, $r_{1,2}$, of the quadratic in B on the right-hand-side of the above. Using the quadratic equation we find

$$r_{1,2} = \frac{-\frac{2\gamma}{\alpha} \pm \sqrt{\frac{4\gamma^2}{\alpha^2} + 4\left(\frac{2}{\alpha}\right)}}{2} = -\frac{\gamma}{\alpha} \pm \sqrt{\frac{\gamma^2}{\alpha^2} + \frac{2}{\alpha}} = \frac{-\gamma \pm \sqrt{\gamma^2 + 2\alpha}}{\alpha}\,.$$

Thus we see that

$$\begin{aligned} B^2 + \frac{2\gamma}{\alpha}B - \frac{2}{\alpha} &= (B - r_1)(B - r_2) \\ &= \left(B + \frac{\gamma - \sqrt{\gamma^2 + 2\alpha}}{\alpha}\right)\left(B + \frac{\gamma + \sqrt{\gamma^2 + 2\alpha}}{\alpha}\right)\,. \end{aligned}$$

If we now introduce a and b so that $a = \frac{-\gamma + \sqrt{\gamma^2+2\alpha}}{\alpha}$ and $b = \frac{\gamma + \sqrt{\gamma^2+2\alpha}}{\alpha}$, we can write the above as $(B - a)(B + b)$. Before continuing from the definitions of a and b above we have the following simple relationships (which will be used later) for expressions involving a and b

$$ab = \frac{2}{\alpha} \tag{210}$$

$$-a + b = \frac{2\gamma}{\alpha} \tag{211}$$

$$a + b = \frac{2\sqrt{\gamma^2 + 2a}}{\alpha} = \frac{2\psi_1}{\alpha}\,, \tag{212}$$

where we have defined ψ_1 as $\psi_1 = \sqrt{\gamma^2 + 2\alpha}$. Back to the main development, using the above quadratic factorization we have the differential equation we want to solve written as

$$\frac{dB}{dt} = \frac{1}{2}\alpha(B - a)(B + b)\,,$$

or

$$\frac{dB}{(B-a)(B+b)} = \frac{1}{2}\alpha dt\,.$$

Integrating both sides from T (where $B(T;T) = 0$) to t (where $B = B(t;T)$) we have

$$\int_0^{B(t;T)} \frac{dB'}{(B'-a)(B'+b)} = \frac{1}{2}\alpha \int_T^t dt = \frac{1}{2}\alpha(t - T)\,.$$

To evaluate the left-hand-side of this expression we performing a partial fraction decomposition of the given fraction. We find

$$\frac{1}{(B'-a)(B'+b)} = \frac{1}{a+b}\left(\frac{1}{B'-a}\right) - \frac{1}{a+b}\left(\frac{1}{B'+b}\right)\,.$$

Using this we can evaluate the integral over B' to get

$$\begin{aligned}
\int_0^{B(t;T)} \frac{dB'}{(B'-a)(B'+b)} &= \frac{1}{a+b}\ln\left(\frac{B-a}{-a}\right) - \frac{1}{a+b}\ln\left(\frac{B+b}{b}\right) \\
&= \frac{1}{a+b}\ln\left(\frac{b(a-B)}{a(B+b)}\right)\,.
\end{aligned}$$

Setting this equal to $\frac{1}{2}\alpha(t - T)$ we have

$$\ln\left(\frac{b(a-B)}{a(B+b)}\right) = \frac{a+b}{2}\alpha(t-T)\,.$$

Using Equation 212 we see that $\left(\frac{a+b}{2}\right)\alpha = \psi_1$. Thus we have

$$\ln\left(\frac{B+b}{a-B}\right) = \psi_1(T-t) + \ln\left(\frac{b}{a}\right)\,,$$

or

$$\frac{B+b}{a-B} = \frac{b}{a}e^{\psi_1(T-t)}\,,$$

or

$$B + b = \left(\frac{b}{a}\right)e^{\psi_1(T-t)}(a - B) = be^{\psi_1(T-t)} - \frac{b}{a}e^{\psi_1(T-t)}B\,,$$

or finally solving for B we get

$$B = \frac{b(e^{\psi_1(T-t)} - 1)}{\left(\frac{b}{a}e^{\psi_1(T-t)} + 1\right)} = \frac{e^{\psi_1(T-t)} - 1}{\frac{1}{a}(e^{\psi_1(T-t)} - 1) + \frac{1}{a} + \frac{1}{b}}\,.$$

To simplify this further first consider

$$\begin{aligned}
\frac{1}{a} &= \frac{\alpha}{-\gamma + \sqrt{\gamma^2 + 2\alpha}}\left(\frac{-\gamma - \sqrt{\gamma^2+2\alpha}}{-\gamma - \sqrt{\gamma^2+2\alpha}}\right) \\
&= \frac{\alpha(-\gamma - \sqrt{\gamma^2+2\alpha})}{\gamma^2 - (\gamma^2 + 2\alpha)} = \frac{\gamma + \sqrt{\gamma^2+2\alpha}}{2} = \frac{\gamma + \psi_1}{2} = \frac{b\alpha}{2}\,,
\end{aligned} \tag{213}$$

and second from Equations 210 and 212 that

$$\frac{1}{a}+\frac{1}{b}=\frac{a+b}{ab}=\frac{2\psi_1}{\alpha}\left(\frac{\alpha}{2}\right)=\psi_1\,.$$

Thus using these two expressions we get

$$B(t;T)=\frac{2(e^{\psi_1(T-t)}-1)}{(\gamma+\psi_1)(e^{\psi_1(T-t)}-1)+2\psi_1}\,, \tag{214}$$

the expression in the book for $B(t;T)$. To find the solution for $A(t;T)$ we using the time differential equation for A and B to derive

$$\begin{aligned}
\frac{dA}{dB} &= \frac{\eta B-\frac{1}{2}\beta B^2}{\frac{1}{2}\alpha B^2+\gamma B-1}=-\frac{\beta}{\alpha}\left(\frac{B^2-\frac{2\eta}{\beta}B}{B^2+\frac{2\gamma}{\alpha}B-\frac{2}{\alpha}}\right) \\
&= -\frac{\beta}{\alpha}\left(\frac{B^2+\frac{2\gamma}{\alpha}B-\frac{2}{\alpha}-\frac{2\gamma}{\alpha}B+\frac{2}{\alpha}-\frac{2\eta}{\beta}B}{B^2+\frac{2\gamma}{\alpha}B-\frac{2}{\alpha}}\right) \\
&= -\frac{\beta}{\alpha}\left(1-\frac{2\left(\frac{\gamma}{\alpha}+\frac{\eta}{\beta}\right)B-\frac{2}{\alpha}}{B^2+\frac{2\gamma}{\alpha}B-\frac{2}{\alpha}}\right)\,.
\end{aligned} \tag{215}$$

In the above we have added and subtracted the same quantity in the numerator so that we have a proper rational function of B. We next factor the denominator in the fraction above as we have done earlier, to get

$$\frac{dA}{dB}=-\frac{\beta}{\alpha}\left(1-\frac{2\left(\frac{\gamma}{\alpha}+\frac{\eta}{\beta}\right)B-\frac{2}{\alpha}}{(B-a)(B+b)}\right)\,.$$

To integrate the right-hand-side of this expression we will need to apply partial fractions to the fraction that remains. That is we seek coefficients $\mathcal{A}$ and $\mathcal{B}$ such that

$$\frac{2\left(\frac{\gamma}{\alpha}+\frac{\eta}{\beta}\right)B-\frac{2}{\alpha}}{(B-a)(B+b)}=\mathcal{A}\left(\frac{1}{B-a}\right)+\mathcal{B}\left(\frac{1}{B+b}\right)\,.$$

To find $\mathcal{A}$, multiply both sides of the above by $B-a$ and let $B=a$ to get

$$\mathcal{A}=\frac{1}{a+b}\left(2\left(\frac{\gamma}{\alpha}+\frac{\eta}{\beta}\right)a-\frac{2}{\alpha}\right)=\frac{2}{a+b}\left(\left(\frac{\gamma}{\alpha}+\frac{\eta}{\beta}\right)a-\frac{1}{\alpha}\right)\,.$$

To find $\mathcal{B}$ multiply both sides by $B+b$ and let $B=-b$ to get

$$\mathcal{B}=\frac{1}{-(a+b)}\left(2\left(\frac{\gamma}{\alpha}+\frac{\eta}{\beta}\right)(-b)-\frac{2}{\alpha}\right)=\frac{2}{a+b}\left(\left(\frac{\gamma}{\alpha}+\frac{\eta}{\beta}\right)b+\frac{1}{\alpha}\right)\,.$$

Thus we have shown that

$$\frac{dA}{dB}=-\frac{\beta}{\alpha}\left(1-\frac{\mathcal{A}}{B-a}-\frac{\mathcal{B}}{B+b}\right)\,,$$

which we can integrate from T to t (since $A(T;T) = 0$) we get

$$A = -\frac{\beta}{\alpha}B + \frac{\beta}{\alpha}\mathcal{A}\ln\left(\frac{B-a}{-a}\right) + \frac{\beta}{\alpha}\mathcal{B}\ln\left(\frac{B+b}{b}\right).$$

We have almost shown the result in the book. To fully derive that result consider the following manipulations of the coefficient of $\ln\left(\frac{a-B}{a}\right)$.

$$\begin{aligned}
\frac{\beta}{\alpha}\mathcal{A} &= \frac{2\beta}{(a+b)\alpha}\left(\left(\frac{\gamma}{\alpha}+\frac{\eta}{\beta}\right)a - \frac{1}{\alpha}\right) = \frac{2}{\alpha}\frac{a}{(a+b)}\left(\eta - \frac{\beta}{\alpha}\left(\frac{1}{a}-\gamma\right)\right) \\
&= \frac{2}{\alpha}\frac{a}{(a+b)}\left(\eta - \frac{a\beta}{2}\left[\left(\frac{2}{a\alpha}\right)\left(\frac{1}{a}-\gamma\right)\right]\right).
\end{aligned}$$

Consider now the expression $\frac{2}{a\alpha}\left(\frac{1}{a}-\gamma\right)$. Using Equation 213 we can write $\frac{1}{a}$ as $\frac{\gamma+\psi_1}{2}$ to get

$$\frac{2}{a\alpha}\left(\frac{1}{a}-\gamma\right) = \frac{2}{a\alpha}\left(\frac{-\gamma+\psi_1}{2}\right) = \frac{-\gamma+\psi_1}{a\alpha}.$$

Since $a = \frac{\gamma+\psi_1}{2}$ the above expression evaluates to 1. Thus we have shown that

$$\frac{\beta}{\alpha}\mathcal{A} = \frac{2}{\alpha}\left(\frac{a}{a+b}\right)\left(\eta - \frac{a\beta}{2}\right) = \frac{2}{\alpha}a\psi_2,$$

where we have defined ψ_2 as

$$\psi_2 = \frac{\eta - a\beta/2}{a+b}. \tag{216}$$

Next we consider the coefficient of $\ln\left(\frac{b+B}{b}\right)$ given by

$$\begin{aligned}
\frac{\beta}{\alpha}\mathcal{B} &= \frac{2\beta}{(a+b)\alpha}\left(\left(\frac{\gamma}{\alpha}+\frac{\eta}{\beta}\right)b + \frac{1}{\alpha}\right) \\
&= \frac{2}{\alpha}b\frac{\beta}{a+b}\left(\frac{\gamma}{\alpha}+\frac{\eta}{\beta}+\frac{1}{b\alpha}\right).
\end{aligned}$$

Introduce ψ_2 into this last expression by solving Equation 216 for η to get that

$$\frac{\eta}{\beta} = \frac{a+b}{\beta}\psi_2 + \frac{a}{2}.$$

Thus we get that

$$\frac{\beta}{\alpha}\mathcal{B} = \frac{2}{\alpha}b\frac{\beta}{a+b}\left(\frac{\gamma}{\alpha}+\frac{a+b}{\beta}\psi_2+\frac{a}{2}+\frac{1}{b\alpha}\right) = \frac{2}{\alpha}b\left(\psi_2 + \frac{\beta}{a+b}\left(\frac{\gamma}{\alpha}+\frac{a}{2}+\frac{1}{b\alpha}\right)\right).$$

Next we use Equation 213 we can obtain that $\frac{1}{b} = \frac{\alpha a}{2}$, and thus $\frac{1}{b\alpha} = \frac{a}{2}$, and

$$\frac{a}{2} + \frac{1}{b\alpha} = a.$$

After this we now have

$$\frac{\beta}{\alpha}\mathcal{B} = \frac{2}{\alpha}b\left(\psi_2 + \frac{\beta}{a+b}\left(\frac{\gamma}{\alpha}+a\right)\right).$$

We find that the inner most expression is given by

$$\frac{\gamma}{\alpha} + a = \frac{\gamma}{\alpha} + \frac{-\gamma + \psi_1}{\alpha} = \frac{\psi_1}{\alpha},$$

and

$$a + b = \frac{2\psi_1}{\alpha},$$

so that

$$\frac{1}{a+b}\left(\frac{\gamma}{a} + a\right) = \frac{\alpha}{2\psi_1}\frac{\psi_1}{\alpha} = \frac{1}{2}.$$

Thus we have shown that

$$\frac{\beta}{\alpha}\mathcal{B} = \frac{2}{\alpha}b\left(\psi_2 + \frac{\beta}{2}\right).$$

Combining many of these previous results we finally get the desired expression for $A(t;T)$ in terms of $B = B(t;T)$

$$A(t;T) = -\frac{\beta}{\alpha}B + \frac{2}{\alpha}a\ln\left(\frac{a-B}{a}\right) + \frac{2}{\alpha}b\left(\psi_2 + \frac{\beta}{2}\right)\ln\left(\frac{b+B}{b}\right), \tag{217}$$

which is the books equation 30.15.

We now seek to determine the long time to maturity asymptotics of the yield curve. From the definition of the yield curve

$$Y = \frac{-A(t;T) + rB(t;T)}{T-t} = \frac{-A(\tau)}{\tau} + \frac{rB(\tau)}{\tau}.$$

When $\tau \to \infty$ by using Equation 214 we get $B \to \frac{2}{\gamma+\Psi_1}$ and so the second term $\frac{rB(\tau)}{\tau} \to 0$ in this limit. From the differential equation for A

$$\frac{dA}{dt} = \eta B - \frac{1}{2}\beta B^2,$$

when we consider large τ from the known limit of B as $\tau \to \infty$ the right-hand-side of the above expression has the limit

$$\begin{aligned}\eta B - \frac{1}{2}\beta B^2 \quad &\to \quad \eta\left(\frac{2}{\gamma+\psi_1}\right) - \frac{1}{2}\beta\left(\frac{4}{(\gamma+\psi_1)^2}\right) \\ &= \quad \frac{2}{(\gamma+\psi_1)^2}\left(\eta(\gamma+\psi_1) - \beta\right),\end{aligned}$$

which is independent of t. Thus integrating $\int_T^t$ on both sides of the differential equation for A gives

$$A(t;T) = \frac{2}{(\gamma+\psi_1)^2}\left(\eta(\gamma+\psi_1) - \beta\right)(t-T),$$

which when we negate and divide by $T - t = \tau$ is the asymptotic expression presented in the book.

Recall from Chapter 10 on Page 40 that for a stochastic process that has dynamics given by

$$dy = A(y,t)dt + B(y,t)dX,$$

The Fokker-Planck or forward Kolmogorov equation is

$$\frac{\partial P}{\partial t'} = \frac{1}{2}\frac{\partial^2}{\partial y'^2}(B(y',t')^2 P) - \frac{\partial}{\partial y'}(A(y',t')P) \tag{218}$$

which for the spot rate r dynamics considered in this chapter of

$$dr = (u - \lambda w)dt + w dX\,,$$

means that we take $A(r,t) = u(r,t) - \lambda(r,t)w(r,t)$ and $B(r,t) = w(r,t)$ to get the Fokker-Planck equation of

$$\frac{\partial P}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial r^2}(w^2 P) - \frac{\partial}{\partial r}((u - \lambda w)P)\,. \tag{219}$$

In steady-state $\frac{\partial P}{\partial t} \to 0$ and our notation changes to $P = P_\infty$ to give

$$\frac{1}{2}\frac{d^2}{dr^2}(w^2 P_\infty) = \frac{d}{dr}((u - \lambda w)P_\infty)\,.$$

Under the constant parameter model from Equation 208 this becomes

$$\frac{1}{2}\frac{d^2}{dr^2}((\alpha r + \beta)P_\infty) = \frac{d}{dr}((\eta - \gamma r)P_\infty)\,.$$

Multiplying by two and integrating both sides of the above we get

$$\frac{d}{dr}((\alpha r + \beta)P_\infty) = 2(\eta - \gamma r)P_\infty + C_1\,,$$

for some constant C_1. Using the product rule to take the derivative of the expression on the left-hand-side gives

$$\alpha P_\infty + (\alpha r + \beta)\frac{dP_\infty}{dr} = 2(\eta - \gamma r)P_\infty + C_1\,,$$

or

$$(\alpha r + \beta)\frac{dP_\infty}{dr} + (\alpha - 2\eta + 2\gamma r)P_\infty = C_1\,,$$

or on dividing both sides by $\alpha r + \beta$ gives

$$\frac{dP_\infty}{dr} + \frac{\alpha - 2\eta + 2\gamma r}{\alpha r + \beta}P_\infty = \frac{C_1}{\alpha r + \beta}\,. \tag{220}$$

This is a non-homogeneous differential equation for P_∞ and the total solution is given by the sum of a homogeneous solution and a particular solution. We first solve the homogeneous problem where we take the right-hand-side equal to zero. Writing this expression as

$$\begin{aligned}
\frac{dP_\infty}{P_\infty} &= \left(\frac{2\eta - 2\gamma r - \alpha}{\alpha r + \beta}\right) dr \\
&= -\frac{2\gamma}{\alpha}\left(\frac{r + \frac{\alpha}{2\gamma} - \frac{\eta}{\gamma}}{r + \frac{\beta}{\alpha}}\right) dr = -\frac{2\gamma}{\alpha}\left(\frac{r + \frac{\beta}{\alpha} - \frac{\beta}{\alpha} + \frac{\alpha}{2\gamma} - \frac{\eta}{\gamma}}{r + \frac{\beta}{\alpha}}\right) dr \\
&= -\frac{2\gamma}{\alpha}\left(1 - \left(\frac{\beta}{\alpha} - \frac{\alpha}{2\gamma} + \frac{\eta}{\gamma}\right)\left(r + \frac{\beta}{\alpha}\right)^{-1}\right) dr\,.
\end{aligned}$$

We can integrate both sides to get

$$\ln(|p_\infty|) = -\frac{2\gamma}{\alpha}\left(r - \left(\frac{\beta}{\alpha} - \frac{\alpha}{2\gamma} + \frac{\eta}{\gamma}\right)\ln\left(r + \frac{\beta}{\alpha}\right)\right) + C_2\,,$$

for some constant C_2. On solving for $p_\infty(r)$ we have

$$p_\infty(r) = e^{C_2} e^{-\frac{2\gamma}{\alpha} r} \exp\left\{\frac{2\gamma}{\alpha}\left(\frac{\beta}{\alpha} - \frac{\alpha}{2\gamma} + \frac{\eta}{\gamma}\right)\ln\left(r + \frac{\beta}{\alpha}\right)\right\}$$

Note that the coefficient of the logarithm in the exponential can be written

$$\frac{2\gamma}{\alpha}\left(\frac{\beta}{\alpha} - \frac{\alpha}{2\gamma} + \frac{\eta}{\gamma}\right) = \frac{2\gamma\beta}{\alpha^2} - 1 + \frac{2\eta}{\alpha}\,.$$

Motivated by the expression above, define k to be

$$k = \frac{2\eta}{\alpha} + \frac{2\beta\gamma}{\alpha^2}\,. \tag{221}$$

Then we get for $p_\infty(r)$

$$\begin{aligned} p_\infty(r) &= e^{C_2} e^{-\frac{2\gamma}{\alpha} r}\left(r + \frac{\beta}{\alpha}\right)^{k-1} = e^{C_2} e^{-\frac{2\gamma}{\alpha}\left(r+\frac{\beta}{\alpha}\right)} e^{\frac{2\gamma}{\alpha}\left(\frac{\beta}{\alpha}\right)}\left(r + \frac{\beta}{\alpha}\right)^{k-1} \\ &= D\left(r + \frac{\beta}{\alpha}\right)^{k-1} e^{-\frac{2\gamma}{\alpha}\left(r+\frac{\beta}{\alpha}\right)}\,, \end{aligned}$$

for some new constant D. Having found the homogeneous solution we next need to go and find the inhomogeneous solution. In general to do that one would introduce an unknown function $v(r)$ and put $v(r)\left(r + \frac{\beta}{\alpha}\right)^{k-1} e^{-\frac{2\gamma}{\alpha}\left(r+\frac{\beta}{\alpha}\right)}$ into the differential Equation 220 to derive an equation for $v(r)$. When one does that one gets

$$\frac{dv}{dr} = \frac{C_1}{\alpha}\left(r + \frac{\beta}{\alpha}\right)^{-k+2} e^{\frac{2\gamma}{\alpha}\left(r+\frac{\beta}{\alpha}\right)}\,.$$

If we don't take $C_1 = 0$ then this solution will have an exponentially growing component as $r \to \infty$. Thus we take $C_1 = 0$. To finish this formulation we must have the integral of $p_\infty(r)$ evaluate to 1 or

$$I = \int_{-\beta/\alpha}^{\infty} D\left(r + \frac{\beta}{\alpha}\right)^{k-1} e^{-\frac{2\gamma}{\alpha}\left(r+\frac{\beta}{\alpha}\right)} dr = 1\,.$$

To evaluate this let $u = r + \frac{\beta}{\alpha}$ then this integral becomes

$$D\int_0^\infty u^{k-1} e^{-\frac{2\gamma}{\alpha} u} du\,.$$

Let $v = \frac{2\gamma}{\alpha} u$ and $dv = \frac{2\gamma}{\alpha} du$ and the above becomes

$$\begin{aligned} I &= D\int_0^\infty \left(\frac{\alpha}{2\gamma}\right)^{k-1} v^{k-1} e^{-v} dv \left(\frac{\alpha}{2\gamma}\right) \\ &= D\left(\frac{\alpha}{2\gamma}\right)^{k}\int_0^\infty v^{k-1} e^{v} dv = D\left(\frac{\alpha}{2\gamma}\right)^{k}\Gamma(k) = 1\,. \end{aligned}$$

So solving for D we thus have

$$D = \left(\frac{2\gamma}{\alpha}\right)^k \frac{1}{\Gamma(k)}\,.$$

Thus with this explicit value of D we have our final expression for $p_\infty(r)$ of

$$p_\infty(r) = \left(\frac{2\gamma}{\alpha}\right)^k \frac{1}{\Gamma(k)} \left(r + \frac{\beta}{\alpha}\right)^{k-1} e^{-\frac{2\gamma}{\alpha}\left(r+\frac{\beta}{\alpha}\right)} \quad \text{for} \quad r \geq -\frac{\beta}{\alpha}\,. \tag{222}$$

The mean of this distribution can now be computed

$$\int_{-\beta/\alpha}^{\infty} r p_\infty(r) dr = \int_{-\beta/\alpha}^{\infty} \left(r + \frac{\beta}{\alpha}\right) p_\infty(r) dr - \frac{\beta}{\alpha} \int_{-\beta/\alpha}^{\infty} p_\infty(r) dr\,.$$

Now by normalization the second term above is given by $-\frac{\beta}{\alpha}$, while the first term is given by

$$\begin{aligned}
\frac{1}{\Gamma(k)} \left(\frac{2\gamma}{\alpha}\right)^k \int_{-\beta/\alpha}^{\infty} \left(r + \frac{\beta}{\alpha}\right)^k e^{-\frac{2\gamma}{\alpha}\left(r+\frac{\beta}{\alpha}\right)} dr &= \frac{1}{\Gamma(k)} \left(\frac{2\gamma}{\alpha}\right)^k \int_0^\infty v^k e^{-\frac{2\gamma}{\alpha} v} dv \\
&= \frac{1}{\Gamma(k)} \left(\frac{2\gamma}{\alpha}\right)^k \left(\frac{\alpha}{2\gamma}\right)^{k+1} \int_0^\infty u^k e^{-u} du \\
&= \frac{1}{\Gamma(k)} \left(\frac{\alpha}{2\gamma}\right) \Gamma(k+1) = \frac{\alpha k}{2r}\,.
\end{aligned}$$

Combining with $-\frac{\beta}{\alpha}$ we have a mean of

$$\frac{\alpha k}{2r} - \frac{\beta}{\alpha}\,,$$

as claimed in the text.

Notes on named models: Vasicek

We are told that the Vasicek model is a special case of Equation 208 with $\alpha = 0$ and $\beta > 0$. In that case the stochastic interest rate r satisfies

$$dr = (\eta - \gamma r)dt + \beta^{1/2} dX\,. \tag{223}$$

The solution for the pricing of the general zero-coupon bond is still given by $e^{A(t;T) - rB(t;T)}$ with A and B modified since we are now considering the case $\alpha = 0$. To derive the expressions for $A(t;T)$ and $B(t;T)$ in this case we could simplify the general solutions found in Equations 214 and 217 by taking the limit as $\alpha \to 0$. For the function $B(t;T)$ given by Equation 214 this seems to be a tractable approach since the limits of ψ_1 as $\alpha \to 0$ is simply γ and we get

$$B(t;T) = \frac{2(e^{\gamma(T-t)} - 1)}{2\gamma(e^{\gamma(T-t)} - 1) + 2\gamma} = \frac{1}{\gamma}(1 - e^{-\gamma(T-t)})\,, \tag{224}$$

the same as in the book. For $A(t;T)$, however, there seem to be a great number of indeterminate limits that need to be simplify to compute the expression for A when $\alpha = 0$, and thus it is preferable to go back to the original derivation of A starting with the assumption that $\alpha = 0$. Starting with Equation 215 in that case we get

$$\frac{dA}{dB} = \frac{\eta B - \frac{1}{2}\beta B^2}{\gamma B - 1} = -\frac{1}{2}\frac{\beta}{\gamma}\left(\frac{B^2 - \frac{2\eta}{\beta}B}{B - \frac{1}{\gamma}}\right).$$

Performing long division on this fraction of B shows that we can write

$$\frac{B^2 - \frac{2\eta}{\beta}B}{B - \frac{1}{\gamma}} = B + \left(\frac{1}{\gamma} - \frac{2\eta}{\beta}\right) + \frac{\frac{1}{\gamma}\left(\frac{1}{\gamma} - \frac{2\eta}{\beta}\right)}{B - \frac{1}{\gamma}},$$

which is more easily integrated as a function of B. Thus we have shown that

$$\frac{dA}{dB} = -\frac{1}{2}\frac{\beta}{\gamma}\left(B + \left(\frac{1}{\gamma} - \frac{2\eta}{\beta}\right) + \frac{1}{\gamma}\left(\frac{1}{\gamma} - \frac{2\eta}{\beta}\right)\frac{1}{B - \frac{1}{\gamma}}\right),$$

Integrating this expression from T to t and using the fact that $A(T;T) = B(T;T) = 0$ gives

$$\begin{aligned}
A(t;T) &= -\frac{1}{2}\frac{\beta}{\gamma}\left(\frac{B^2}{2} + \left(\frac{1}{\gamma} - \frac{2\eta}{\beta}\right)B + \frac{1}{\gamma}\left(\frac{1}{\gamma} - \frac{2\eta}{\beta}\right)\ln\left(\frac{B - \frac{1}{\gamma}}{-\frac{1}{\gamma}}\right)\right) \\
&= -\frac{\beta}{4\gamma}B^2 - \frac{1}{2}\frac{\beta}{\gamma}\left(\frac{1}{\gamma} - \frac{2\eta}{\beta}\right)B - \frac{\beta}{2\gamma^2}\left(\frac{1}{\gamma} - \frac{2\eta}{\beta}\right)\ln(1 - \gamma B) \\
&= -\frac{\beta}{4\gamma}B^2 + \frac{1}{\gamma^2}\left(-\frac{\beta}{2} + \gamma\eta\right)B + \frac{1}{\gamma^3}\left(-\frac{\beta}{2} + \gamma\eta\right)\ln\left(e^{-\gamma(T-t)}\right) \\
&= -\frac{\beta}{4\gamma}B^2 + \left(\gamma\eta - \frac{\beta}{2}\right)\left[\frac{1}{\gamma^2}B + \frac{1}{\gamma^3}(-\gamma)(T-t)\right] \\
&= -\frac{\beta}{4\gamma}B^2 + \frac{1}{\gamma^2}(B - T + t)\left(\gamma\eta - \frac{\beta}{2}\right),
\end{aligned} \tag{225}$$

which is the expression in the book. To determine the probability density for r we could again take the limit $\alpha \to 0$ of the general expression Equation 222 but this again involves several indeterminate limits and proceeding that way is more difficult than simply deriving the expression for $p_\infty(r)$ starting from the assumption that $\alpha = 0$. In that case the steady-state Fokker-Planck Equation 219 becomes

$$\frac{d^2}{dr^2}(\beta P_\infty) = 2\frac{d}{dr}((\eta - \gamma r)P_\infty).$$

Integrating both sides gives

$$\beta\frac{dP_\infty}{dr} = 2(\eta - \gamma r)P_\infty + C_1,$$

for some constant C_1. Thus we have

$$\frac{dP_\infty}{dr} - \frac{2}{\beta}(\eta - \gamma r)P_\infty = \frac{C_1}{\beta}.$$

If we solve the homogeneous part of this equation first we need to consider

$$\frac{dP_\infty}{P_\infty} = \frac{2}{\beta}(\eta - \gamma r)dr\,,$$

or integrating both sides gives

$$\ln(P_\infty) = \frac{2}{\beta}\eta r - \frac{\gamma}{\beta}r^2 + C_2\,.$$

On solving for P_∞ we find

$$\begin{aligned} P_\infty &= e^{C_2}\exp\left\{-\frac{\gamma}{\beta}\left(r^2 - \frac{2\eta}{\gamma}r\right)\right\} = C_2\exp\left\{-\frac{\gamma}{\beta}\left(r^2 - \frac{2\eta}{\gamma}r + \frac{\eta^2}{\gamma^2} - \frac{\eta^2}{\gamma^2}\right)\right\} \\ &= C_2\exp\left\{-\frac{\gamma}{\beta}\left(r - \frac{\eta}{\gamma}\right)^2 + \frac{\eta^2}{\beta\gamma}\right\} = D\exp\left\{-\frac{\gamma}{\beta}\left(r - \frac{\eta}{\gamma}\right)^2\right\}\,. \end{aligned}$$

Note that in the Vasicek model since $\beta > 0$ and $\alpha \to 0$ our lower limit of r becomes in this case $-\frac{\beta}{\alpha} \to -\infty$. This logic is used in the to make this expression a valid density since we must require that P_∞ integrate to one. This requires that D must satisfy

$$\begin{aligned} \int_{-\infty}^{\infty} De^{-\frac{\gamma}{\beta}\left(r-\frac{\eta}{\gamma}\right)^2}dr &= D\int_{-\infty}^{\infty} e^{-\frac{\gamma}{\beta}r^2}dr = 2D\int_0^{\infty} e^{-\frac{\gamma}{\beta}r^2}dr \\ &= 2D\sqrt{\frac{\beta}{\gamma}}\int_0^{\infty} e^{-v^2}dv = 2D\sqrt{\frac{\beta}{\gamma}}\frac{\sqrt{\pi}}{2} = D\sqrt{\frac{\beta\pi}{\gamma}} = 1\,, \end{aligned}$$

or

$$D = \sqrt{\frac{\gamma}{\beta\pi}}\,.$$

Thus our density is given by

$$P_\infty(r) = \sqrt{\frac{\gamma}{\beta\pi}}e^{-\frac{\gamma}{\beta}\left(r-\frac{\eta}{\gamma}\right)^2}\,, \tag{226}$$

as claimed in the book. This density has a mean that can be easily calculated as

$$\int_{-\infty}^{\infty} rP_\infty(r)dr = \int_{-\infty}^{\infty}\left(r - \frac{\eta}{\gamma}\right)P_\infty(r)dr + \frac{\eta}{\gamma}\int_{-\infty}^{\infty} P_\infty(r)dr\,.$$

Since P_∞ is a normalized probability density this second integral is $\frac{\eta}{\gamma}$. The first integral becomes

$$\sqrt{\frac{\gamma}{\beta\pi}}\int_{-\infty}^{\infty}\left(r - \frac{\eta}{\gamma}\right)e^{-\frac{\gamma}{\beta}\left(r-\frac{\eta}{\gamma}\right)^2}dr = 0\,,$$

by symmetry. Thus the mean of this distribution is $\frac{\eta}{\gamma}$ as claimed in the book.

Notes on named models: Cox, Ingersoll, and Ross

This model takes $\beta = 0$, which from Equation 208 gives

$$dr = (\eta - \gamma r)dt + \sqrt{\alpha r}dX\,,$$

for the differential equation satisfied by r. The value of a zero-coupon bond is still given by $e^{A(t;T)-rB(t;T)}$ where A and B are given by Equations 214 and 217 with $\beta = 0$, in all the dependent relationships. The expression for the steady-state probability density $P_\infty(r)$ is given by Equation 222 again with $\beta = 0$. In that case we have $k = \frac{2\eta}{\alpha}$ so that

$$P_\infty(r) = \left(\frac{2\gamma}{\alpha}\right)^k \frac{1}{\Gamma(k)} r^{k-1} e^{-\frac{2\gamma}{\alpha}r} \quad \text{for} \quad r \geq 0\,.$$

The mean value for of this steady-state probability density is

$$\frac{\alpha k}{2\gamma} = \frac{\alpha}{2\gamma}\frac{2\eta}{\alpha} = \frac{\eta}{\gamma}\,,$$

in agreement with the book.

Notes on named models: Ho & Lee

The Ho & Lee model has $\alpha = \gamma = 0$, $\beta > 0$ and η a function of time i.e. $\eta = \eta(t)$, so from Equation 208 we have

$$dr = \eta(t)dt + \sqrt{\beta}dX\,, \tag{227}$$

as the differential equation for the spot rate r. In this book, it is also common to see the constant c defined as $c = \sqrt{\beta}$. Using Equation 201, with the parameters specified above (namely $\gamma = 0$) the zero-coupon bond pricing Equation 196 under the Ho & Lee model takes the form

$$\frac{\partial Z}{\partial t} + \frac{1}{2}c^2\frac{\partial^2 Z}{\partial r^2} + \eta(t)\frac{\partial Z}{\partial r} - rZ = 0\,. \tag{228}$$

When we consider a solution of this equation of the form $Z(t;T) = e^{A(t;T)-rB(t;T)}$, in Equation 209 we see that $B(t;T)$ satisfies $\frac{dB}{dt} = -1$, which when we integrate from T to t with $B(T;T) = 0$ gives

$$B(t;T) = T - t\,. \tag{229}$$

Then from Equation 206 for $A(t;T)$ we have

$$\frac{dA}{dt} = \eta(t)B - \frac{1}{2}\beta B^2 = \eta(t)(T-t) - \frac{\beta}{2}(T-t)^2\,.$$

Integrating this from T to t gives

$$\begin{aligned} A(t;T) &= \int_T^t \eta(s)(T-s)ds - \frac{1}{2}\beta\int_T^t (T-s)^2 ds \\ &= \frac{\beta}{6}(T-t)^3 - \int_t^T \eta(s)(T-s)ds\,. \end{aligned} \tag{230}$$

Thus combining these two parts the full solution for, $Z(t;T)$, is then given by

$$Z(t;T) = \exp\left\{\frac{1}{6}c^2(T-t)^3 - \int_t^T \eta(s)(T-s)ds - (T-t)r\right\}\,. \tag{231}$$

Notes on Equity and FX Forwards and Futures: Forwards

Consider a portfolio, Π, consisting of one forward contract, of value $V(S,r,t)$, short Δ underlying shares of stock, and short Δ_1 risk-free bonds or

$$\Pi = V(S,r,t) - \Delta S - \Delta_1 Z \,,$$

Ito's lemma for the *two* stochastic parameters r and S on Π gives

$$\begin{aligned} d\Pi &= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial r}dr \\ &+ \frac{1}{2}\left(\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + 2\rho\sigma S w \frac{\partial V}{\partial S \partial r} + w^2 \frac{\partial^2 V}{\partial r^2}\right) dt \\ &- \Delta dS - \Delta_1 \left(\frac{\partial Z}{\partial t}dt + \frac{\partial Z}{\partial r}dr + \frac{1}{2}w^2 \frac{\partial^2 Z}{\partial r^2}dt\right) . \end{aligned}$$

Group everything by the differentials dt, dS, and dr to get

$$\begin{aligned} d\Pi &= \left[\frac{\partial V}{\partial t} + \frac{1}{2}\left(\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + 2\rho\sigma S w \frac{\partial V}{\partial S \partial r} + w^2 \frac{\partial^2 V}{\partial r^2}\right)\right. \\ &- \left.\Delta_1 \left(\frac{\partial Z}{\partial t} + \frac{w^2}{2}\frac{\partial^2 Z}{\partial r^2}\right)\right] dt + \left[\frac{\partial V}{\partial S} - \Delta\right] dS + \left[\frac{\partial V}{\partial r} - \Delta_1 \frac{\partial Z}{\partial r}\right] dr \,. \end{aligned}$$

We now pick Δ and Δ_1 such that the above portfolio is deterministic. The coefficient of dr will be equal to zero if we take

$$\Delta_1 = \frac{\frac{\partial V}{\partial r}}{\frac{\partial Z}{\partial r}} \,.$$

The coefficient of dS will be zero if we take

$$\Delta = \frac{\partial V}{\partial S} \,.$$

Then setting $d\Pi = r\Pi dt$ and dividing by dt we get

$$\begin{aligned} \frac{\partial V}{\partial t} &+ \frac{1}{2}\left(\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + 2\rho\sigma S w \frac{\partial V}{\partial S \partial r} + w^2 \frac{\partial^2 V}{\partial r^2}\right) - \Delta_1 \left[\frac{\partial Z}{\partial t} + \frac{w^2}{2}\frac{\partial^2 Z}{\partial r^2}\right] \\ &= rV - r\Delta S - r\Delta_1 Z = rV - rS\frac{\partial V}{\partial S} - r\Delta_1 Z \,. \end{aligned}$$

Putting terms that depend on V on one side and terms that depend on Z on another to get

$$\begin{aligned} \frac{\partial V}{\partial t} &+ \frac{1}{2}\left(\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + 2\rho\sigma S w \frac{\partial V}{\partial S \partial r} + w^2 \frac{\partial^2 V}{\partial r^2}\right) - rV + rS\frac{\partial V}{\partial S} \\ &= \Delta_1 \left[\frac{\partial Z}{\partial t} + \frac{w^2}{2}\frac{\partial^2 Z}{\partial r^2} - rZ\right] , \end{aligned}$$

or dividing by the numerator of Δ_1 or $\frac{\partial V}{\partial r}$ we get

$$\begin{aligned} &\frac{1}{\frac{\partial V}{\partial r}}\left(\frac{\partial V}{\partial t} + \frac{1}{2}\left(\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + 2\rho\sigma S w \frac{\partial V}{\partial S \partial r} + w^2 \frac{\partial^2 V}{\partial r^2}\right) - rV + rS\frac{\partial V}{\partial S}\right) \\ = &\frac{1}{\frac{\partial Z}{\partial r}}\left(\frac{\partial Z}{\partial t} + \frac{w^2}{2}\frac{\partial^2 Z}{\partial r^2} - rZ\right) , \end{aligned}$$

Since Z satisfies Equation 196 with $K(r,t) = 0$, the right-hand-side of the above is equal to $-(u(r,t)) - w(r,t)\lambda(r,t))$. When we make this substitution and multiply both sides by $\frac{\partial V}{\partial r}$ the equation for V becomes the following

$$
\begin{aligned}
\frac{\partial V}{\partial t} &+ \frac{1}{2}\left(\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + 2\rho\sigma S w \frac{\partial V}{\partial S \partial r} + w^2 \frac{\partial^2 V}{\partial r^2}\right) \\
&- rV + rS\frac{\partial V}{\partial S} + (u - \lambda w)\frac{\partial V}{\partial r} = 0\,, \qquad (232)
\end{aligned}
$$

or the equation in the book. The final condition on our forward contract is $V(S,r,T) = S - \bar{S}$. The book then assert that a solution to this equation the the given boundary condition is given by $V(S,r,t) = S - \bar{S}Z$. To show that this true we can take the required derivatives and see if they satisfy Equation 232. The derivatives of this proposed solution are given by

$$
\begin{aligned}
\frac{\partial V}{\partial t} &= -\bar{S}\frac{\partial Z}{\partial t} \\
\frac{\partial V}{\partial S} &= 1 \\
\frac{\partial^2 V}{\partial S^2} &= 0 \\
\frac{\partial V}{\partial r} &= -\bar{S}\frac{\partial Z}{\partial r} \\
\frac{\partial^2 V}{\partial r \partial S} &= 0 \\
\frac{\partial^2 V}{\partial r^2} &= -\bar{S}\frac{\partial^2 Z}{\partial r^2}\,.
\end{aligned}
$$

Thus using these for the left-hand-side of Equation 232 we get

$$
\begin{aligned}
&- \bar{S}\frac{\partial Z}{\partial t} + 0 + 0 + \frac{1}{2}w^2(-\bar{S})\frac{\partial^2 Z}{\partial r^2} + rS + (u - \lambda w)\left(-\bar{S}\frac{\partial Z}{\partial r}\right) - r(S - \bar{S}Z) \\
&= -\bar{S}\left[\frac{\partial Z}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 Z}{\partial r^2} + (u - \lambda w)\frac{\partial Z}{\partial r} - rZ\right]
\end{aligned}
$$

Since Z satisfies Equation 196 this final expression is zero.

Notes on Equity and FX Forwards and Futures: Futures

The partial differential equation satisfied by the futures price $F(S,r,t)$ is

$$
\begin{aligned}
\frac{\partial F}{\partial t} &+ \frac{1}{2}\left(\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} + 2\rho\sigma S w \frac{\partial F}{\partial S \partial r} + w^2 \frac{\partial^2 F}{\partial r^2}\right) \\
&+ rS\frac{\partial F}{\partial S} + (u - \lambda w)\frac{\partial F}{\partial r} = 0\,, \qquad (233)
\end{aligned}
$$

note that there is no $-rF$ term as there is in the forwards Equation 232. The reason that there is no $-rF$ term in the above expression is that when we equate $d\Pi$ equal to $r\Pi dt$ our portfolio initially has a value of $-\Delta S - \Delta_1 Z$ since a futures contract has no value initially.

Motivated by the quoted forward price of $\frac{S}{Z}$ we will consider an expression for the futures price of the form $F(S,r,t) = \frac{S}{p(r,t)}$. Then the equation that $p(r,t)$ satisfies is given by putting this expression into Equation 232 then the needed derivatives of F in terms of those of p become

$$
\begin{aligned}
\frac{\partial F}{\partial t} &= -\frac{S}{p^2}\frac{\partial p}{\partial t} \\
\frac{\partial F}{\partial S} &= \frac{1}{p} \\
\frac{\partial^2 F}{\partial S^2} &= 0 \\
\frac{\partial F}{\partial r} &= -\frac{S}{p^2}\frac{\partial p}{\partial r} \\
\frac{\partial^2 F}{\partial r \partial S} &= -\frac{S}{p^2}\frac{\partial p}{\partial r} \\
\frac{\partial^2 F}{\partial r^2} &= \frac{2S}{p^3}\left(\frac{\partial p}{\partial r}\right)^2 - \frac{S}{p^2}\frac{\partial^2 p}{\partial r^2}\,.
\end{aligned}
$$

Then using these derivatives we find that the differential equation for $p(r,t)$ given by

$$
-\frac{S}{p^2}\frac{\partial p}{\partial t} - \frac{\rho\sigma S w}{p^2}\frac{\partial p}{\partial r} + \frac{1}{2}w^2\left(\frac{2S}{p^3}\left(\frac{\partial p}{\partial r}\right)^2 - \frac{S}{p^2}\frac{\partial^2 p}{\partial r^2}\right) + \frac{rS}{p} + (u-\lambda w)\left(-\frac{S}{p^2}\frac{\partial p}{\partial r}\right) = 0\,.
$$

Multiply by $-\frac{p^2}{S}$ and changing the order of some terms we get

$$
\frac{\partial p}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 p}{\partial r^2} + (u-\lambda w)\frac{\partial p}{\partial r} - rp\frac{w^2}{p}\left(\frac{\partial p}{\partial r}\right)^2 + \rho\sigma w\frac{\partial p}{\partial r} = 0\,, \tag{234}
$$

which is the same equation as in the book. The final condition on F of $F(S,r,T) = S = \frac{S}{p(r,T)}$ means that the final condition on p is $p(r,T) = 1$.

Chapter 31: Yield Curve Fitting

Notes on the Text

Notes on Ho & Lee

In the Ho & Lee model the risk-neutral spot rate has the process

$$dr = \eta(t)dt + cdX\,,$$

where the theoretical model price for zero-coupon bonds then is given by $Z(r,t;T) = e^{A(t;T)-r(T-t)}$, where

$$A(t;T) = -\int_t^T \eta(s)(T-s)ds + \frac{1}{6}c^2(T-t)^3\,.$$

On today $t = t^*$ we can pick the functional form of $\eta(t)$ to match the *market* prices of traded zero coupon bonds that mature at time T denoted by $Z_M(t^*;T)$. Under the Ho & Lee model this means that

$$Z_M(t^*;T) = e^{A(t^*;T)-r^*(T-t^*)}\,.$$

Using the expression above for $A(t;T)$ this means we need $\eta^*(t)$ to satisfy

$$\int_{t^*}^T \eta^*(s)(T-s)ds = -\log(Z_M(t^*;T)) - r^*(T-t^*) + \frac{1}{6}c^2(T-t^*)^3\,. \tag{235}$$

We can extract $\eta^*(\cdot)$ from this expression by taking T derivatives. The first T derivative of this expression gives

$$\eta^*(T)(0) + \int_{t^*}^T \eta^*(s)ds = -\frac{\partial}{\partial T}\log(Z_M(t^*;T)) - r^* + \frac{1}{2}c^2(T-t^*)^2\,.$$

Note the first term is zero due to evaluating $T-s$ at $s=T$. Another T derivative of this expression gives

$$\eta^*(T) = -\frac{\partial^2}{\partial T^2}\log(Z_M(t^*;T)) + c^2(T-t^*)\,.$$

When we use this expression for the functional form for $\eta^*(t)$ we get for $A(t;T)$

$$A(t;T) = -\int_t^T c^2(s-t^*)(T-s)ds + \int_t^T \frac{\partial^2}{\partial s^2}\log(Z_M(t^*;s))(T-s)ds + \frac{1}{6}c^2(T-t)^3\,.$$

To evaluate the first term in the expression for $A(t;T)$ we write it (without the c^2) as

$$
\begin{aligned}
-\int_t^T (s-t^*)(T-s)ds &= \int_t^T (-s+t^*)(T-s)ds \\
&= \int_t^T (T-s+t^*-T)(T-s)ds \\
&= \int_t^T (T-s)^2 ds - (T-t^*)\int_t^T (T-s)ds \\
&= -\frac{1}{3}\left.(T-s)^3\right|_t^T + \frac{1}{2}\left.(T-t^*)(T-s)^2\right|_t^T \\
&= \frac{1}{3}(T-t)^3 - \frac{1}{2}(T-t^*)(T-t)^2 .
\end{aligned}
$$

Combining this expression (with the c^2 factor) with the last term, $\frac{1}{6}c^2(T-t)^3$, in the expression for $A(t;T)$ we find

$$
\frac{c^2}{3}(T-t)^3 - \frac{c^2}{2}(T-t^*)(T-t)^2 + \frac{1}{6}c^2(T-t)^3 = -\frac{1}{2}c^2(t-t^*)(T-t)^2 .
$$

To evaluate the integral of the derivative or the second term in the expression for $A(t;T)$ we will use integration by parts. We find

$$
\begin{aligned}
\int_t^T \frac{\partial^2}{\partial s^2}\log(Z_M(t^*;s))(T-s)ds &= \left.(T-s)\frac{\partial}{\partial s}\log(Z_M(t^*;s))\right|_t^T \\
&+ \int_t^T \frac{\partial}{\partial s}\log(Z_M(t^*;s))ds \\
&= -(T-t)\frac{\partial}{\partial t}\log(Z_M(t^*;t)) + \log\left(\frac{Z_M(t^*;T)}{Z_M(t^*,t)}\right) .
\end{aligned}
$$

Thus for $A(t;T)$ combining all of these expressions we get

$$
\begin{aligned}
A(t;T) &= \log\left(\frac{Z_M(t^*;T)}{Z_M(t^*;t)}\right) - (T-t)\frac{\partial}{\partial t}\log(Z_M(t^*;t)) \\
&- \frac{1}{2}c^2(t-t^*)(T-t)^2 . \qquad (236)
\end{aligned}
$$

Notes on the extended Vasicek model of Hull and White

The Hull & White extended Vasicek model has the following differential equation for the short rate

$$
dr = (\eta(t) - \gamma r)dt + c dX ,
$$

this expression matches the more general differential equation for r of

$$
dr = (\eta(t) - \gamma(t) r)dt + \sqrt{\alpha(t) r + \beta(t)}dX ,
$$

if we take $\alpha(t) \equiv 0$, the values of γ, and β constant and the value for $c = \beta^{1/2}$. Then the general equation for B is given by Equation 207 but since we are taking $\alpha = 0$ and the value of γ independent of time or in this specific case by

$$\frac{\partial B}{\partial t} = \gamma B - 1 \quad \text{with} \quad B(T;T) = 0\,.$$

This is integrated to give the solution

$$B(t;T) = \frac{1}{\gamma}(1 - e^{-\gamma(T-t)})\,. \tag{237}$$

Using Equation 206 we find $A(t;T)$ is given by

$$\frac{\partial A}{\partial t} = \eta(t)B - \frac{c^2}{2}B^2\,,$$

since $c^2 = \beta$. Integrating this from T to t since $A(T;T)$ is zero we get

$$A(t;T) = -\int_t^T \eta(s)B(s;T)ds - \frac{c^2}{2}\int_T^t B(s;T)^2 dt\,.$$

Since for this model we know $B(t;T)$ from Equation 237 we have

$$B(s;T)^2 = \frac{1}{\gamma^2}(1 - 2e^{-\gamma(T-s)} + e^{-2\gamma(T-s)})\,,$$

This second integral is therefore proportional to

$$\begin{aligned}
\int_T^t B(s;T)^2 ds &= \frac{1}{\gamma^2}\left[(t-T) - \frac{2}{\gamma}\,e^{-\gamma(T-s)}\Big|_T^t + \frac{1}{2\gamma}\,e^{-2\gamma(T-s)}\Big|_T^t\right] \\
&= \frac{1}{\gamma^2}\left[-(T-t) - \frac{2}{\gamma}(e^{-\gamma(T-t)} - 1) + \frac{1}{2\gamma}(e^{-2\gamma(T-t)} - 1)\right] \\
&= \frac{1}{\gamma^2}\left[-(T-t) - \frac{2}{\gamma}e^{-\gamma(T-t)} + \frac{1}{2\gamma}e^{-2\gamma(T-t)} + \frac{3}{2\gamma}\right]\,.
\end{aligned}$$

Thus at this point for $A(t;T)$ we have

$$A(t;T) = -\int_t^T \eta(s)B(s;T)ds + \frac{c^2}{2\gamma^2}\left(T - t + \frac{2}{\gamma}e^{-\gamma(T-t)} - \frac{1}{2\gamma}e^{-2\gamma(T-t)} - \frac{3}{2\gamma}\right)\,. \tag{238}$$

We next impose conditions such that $A(t;T)$ and $B(t;T)$ produce values for zero-coupon bonds that match market prices *today* i.e. when $t = t^*$. Under the assumed model zero-coupon bonds are given by

$$Z_M(t^*;T) = e^{A(t^*;T) - r^* B(t^*;T)}\,.$$

so that by taking logarithms this requires that

$$A(t^*;T) = \log(Z_M(t^*;T)) + r^* B(t^*;T)\,. \tag{239}$$

Since $B(t^*;T)$ is known via Equation 237 (assuming we have a way of calibrating γ) and we know the functional form for $A(t;T)$ this is in fact a condition on the function $\eta(\cdot)$ in the expression for $A(t;T)$. To determine this requirement on the functional form of $\eta(\cdot)$ we first set Equation 238 equal to Equation 239 (with $t = t^*$ and $\eta = \eta^*$) and then take T derivatives to remove the integrals. Doing this and taking the first T derivative we get

$$\begin{aligned}
\frac{\partial}{\partial T}A(t^*;T) &= -\eta^*(T)B(T;T) - \int_{t^*}^{T} \eta^*(s)\frac{\partial}{\partial T}B(s;T)ds \\
&+ \frac{c^2}{2\gamma^2}\left(1 - 2e^{-\gamma(T-t^*)} + e^{-2\gamma(T-t^*)}\right) \\
&= \frac{\partial}{\partial T}\log(Z_M(t^*;T)) + r^*\frac{\partial}{\partial T}B(t^*;T)\,.
\end{aligned}$$

Since $B(T;T) = 0$ and $\frac{\partial}{\partial T}B(t;T) = e^{-\gamma(T-t)}$ we can put these expressions into the above and solve for the term with the integral of $\eta^*(s)$ to get

$$\begin{aligned}
\int_{t^*}^{T} \eta^*(s)e^{-\gamma(T-s)}ds &= -\frac{\partial}{\partial T}\log(Z_M(t^*;T)) - r^*e^{-\gamma(T-t^*)} \\
&+ \frac{c^2}{2\gamma^2}\left(1 - 2e^{-\gamma(T-t^*)} + e^{-2\gamma(T-t^*)}\right)\,. \qquad (240)
\end{aligned}$$

Taking another T derivative of this expression gives

$$\begin{aligned}
\eta^*(T) - \gamma\int_{t^*}^{T} \eta^*(s)e^{-\gamma(T-s)}ds &= -\frac{\partial^2}{\partial T^2}\log(Z_M(t^*;T)) + \gamma r^*e^{-\gamma(T-t^*)} \\
&+ \frac{c^2}{2\gamma^2}(2\gamma e^{-\gamma(T-t^*)} - 2\gamma e^{-2\gamma(T-t^*)})\,.
\end{aligned}$$

Solving for $\eta^*(T)$ by using Equation 240 we get

$$\begin{aligned}
\eta^*(T) &= -\frac{\partial^2}{\partial T^2}\log(Z_M(t^*;T)) + \gamma r^*e^{-\gamma(T-t^*)} + \frac{c^2}{\gamma}(e^{-\gamma(T-t^*)} - e^{-2\gamma(T-t^*)}) \\
&+ \gamma\left[-\frac{\partial}{\partial T}\log(Z_M(t^*;T)) - r^*e^{-\gamma(T-t^*)} + \frac{c^2}{2\gamma^2}(1 - 2e^{-\gamma(T-t^*)} + e^{-2\gamma(T-t^*)})\right]\,,
\end{aligned}$$

on canceling terms and simplifying

$$\eta^*(T) = -\frac{\partial^2}{\partial T^2}\log(Z_M(t^*;T)) - \gamma\frac{\partial}{\partial T}\log(Z_M(t^*;T)) + \frac{c^2}{2\gamma}(1 - e^{-2\gamma(T-t^*)})\,. \qquad (241)$$

To get the functional form for $\eta(t)$ (rather than $\eta(T)$) i.e. as a function of t while it sounds funny in words simply replace T in the above with t.

Given the above form for $\eta^*(\cdot)$, to compute $A(t;T)$ for any time t, we need to use Equation 238 which based in the form for $B(t;T)$ means we will need to evaluate (the negative of)

$$\mathcal{I} \equiv \int_{t}^{T} \eta^*(s)B(s;T)ds\,,$$

since the other terms in Equation 238 are already explicitly given. As a road map of the calculations ahead, we will first evaluate the integral $\mathcal{I}$ (which has several steps) and once we have that result add it to the remaining terms from Equation 238.

As evaluating this integral was the trickiest part of this derivation (for me) since there seemed to be several ways one could try to proceed. We will use the form of $\eta^*(t)$ given by Equation 241 and keep most of the expression directly in terms of $B(t;T)$. When we use Equation 241 we see that this integral is really three terms

$$
\begin{aligned}
\mathcal{I} &= -\int_t^T \frac{\partial^2}{\partial s^2}\log\left(Z_M(t^*;s)\right)B(s;T)ds \\
&- \gamma\int_t^T \frac{\partial}{\partial s}\log\left(Z_M(t^*;s)\right)B(s;T)ds \\
&+ \frac{c^2}{2\gamma}\int_t^T (1-e^{-2\gamma(s-t^*)})B(s;T)ds\,.
\end{aligned}
$$

Lets denote these three terms by $\mathcal{I}_1$, $\mathcal{I}_2$, and $\mathcal{I}_3$. Lets integrate the first term, $\mathcal{I}_1$, by parts where we find

$$
\begin{aligned}
\mathcal{I}_1 &\equiv -\int_t^T \frac{\partial^2}{\partial s^2}\log\left(Z_M(t^*;s)\right)B(s;T)ds \\
&= -B(s;T)\frac{\partial}{\partial s}\log(Z_M(t^*;s))\Big|_t^T + \int_t^T\left(\frac{\partial}{\partial s}\log(Z_M(t^*;s))\right)\frac{\partial}{\partial s}B(s;T)ds\,.
\end{aligned}
$$

Now since $B(T;T)=0$ the first term in the above expression becomes

$$
B(t;T)\frac{\partial}{\partial t}\log(Z_M(t^*;t))\,.
$$

To evaluate the second expression note that due to the functional form of $B(t;T)$ (an exponential) we can easily take its t derivative and relate it back to the functional form of $B(t;T)$ itself. For example we have

$$
\begin{aligned}
\frac{\partial}{\partial s}B(s;T) &= \frac{\partial}{\partial s}\left(\frac{1}{\gamma}(1-e^{-\gamma(T-s)})\right) \\
&= -e^{-\gamma(T-s)} = \gamma B(s;T)-1\,,
\end{aligned}
$$

which we now use in the second integral to get two terms

$$
\gamma\int_t^T\left(\frac{\partial}{\partial s}\log(Z_M(t^*;s))\right)B(s;T)ds - \int_t^T\frac{\partial}{\partial s}\log(Z_M(t^*;s))ds\,.
$$

The first of these two terms exactly *cancels* the integral term $\mathcal{I}_2$ above while the second term is easily integrated. Thus, due to this cancellation, at this point we have for $\mathcal{I}_1+\mathcal{I}_2$ the following

$$
\mathcal{I}_1+\mathcal{I}_2 = B(t;T)\frac{\partial}{\partial t}\log(Z_M(t^*;t)) - \log\left(\frac{Z_M(t^*;T)}{Z_M(t^*;t)}\right)\,.
$$

We now need to evaluate $\mathcal{I}_3$. In great detail, we find

$$
\begin{aligned}
\mathcal{I}_3 &= \frac{c^2}{2\gamma^2}\int_t^T(1-e^{-2\gamma(s-t^*)})(1-e^{-\gamma(T-s)})ds \\
&= \frac{c^2}{2\gamma^2}\int_t^T(1-e^{-\gamma(T-s)}-e^{-2\gamma(s-t^*)}+e^{-\gamma(s+T-2t^*)})ds \\
&= \frac{c^2}{2\gamma^2}\left[T-t-\frac{1}{\gamma}\,e^{-\gamma(T-s)}\Big|_t^T+\frac{1}{2\gamma}\,e^{-2\gamma(s-t^*)}\Big|_t^T-\frac{1}{\gamma}\,e^{-\gamma(s+T-2t^*)}\Big|_t^T\right]\,,
\end{aligned}
$$

which evaluate to (dropping the coefficient $\frac{c^2}{2\gamma^2}$)

$$T - t - \frac{1}{\gamma}(1 - e^{-\gamma(T-t)}) + \frac{1}{2\gamma}(e^{-2\gamma(T-t^*)} - e^{-2\gamma(t-t^*)}) - \frac{1}{\gamma}(e^{-\gamma(2T-2t^*)} - e^{-\gamma(t+T+2t^*)}) \,.$$

With this expression we have now completely evaluated $\mathcal{I}$. To get the full expression for $A(t;T)$ we need to negate $\mathcal{I}$ and add it to the appropriate part of $A(t;T)$ namely the exponential terms. The sum we need to evaluate then is

$$\begin{aligned}
- & \frac{c^2}{2\gamma^2}\left[T - t - \frac{1}{\gamma}(1 - e^{-\gamma(T-t)}) + \frac{1}{2\gamma}(e^{-2\gamma(T-t^*)} - e^{-2\gamma(t-t^*)})\right. \\
- & \left.\frac{1}{\gamma}(e^{-\gamma(2T-2t^*)} - e^{-\gamma(t+T+2t^*)})\right] \\
+ & \frac{c^2}{2\gamma^2}\left[T - t + \frac{2}{\gamma}e^{-\gamma(T-t)} - \frac{1}{2\gamma}e^{-2\gamma(T-t)} - \frac{3}{2\gamma}\right] \\
= & -\frac{c^2}{4\gamma^3}\left[1 - 2e^{-\gamma(T-t)} - e^{-2\gamma(T-t^*)} + e^{-2\gamma(T-t)} - e^{-2\gamma(t-t^*)} + 2e^{-\gamma(t+T-2t^*)}\right] \,. \qquad (242)
\end{aligned}$$

To see if we are finished with our derivation lets see if this expression matches the proposed exponential expressions in $A(t;T)$ presented in the book. This expression, without the leading factor of $-\frac{c^2}{4\gamma^3}$, is

$$\left(e^{-\gamma(T-t^*)} - e^{-\gamma(t-t^*)}\right)^2 (e^{2\gamma(t-t^*)} - 1) \,,$$

or "squaring" the first factor gives

$$(e^{-2\gamma(T-t^*)} - 2e^{-\gamma(T+t-2t^*)} + e^{-2\gamma(t-t^*)})(e^{2\gamma(t-t^*)} - 1) \,,$$

or multiplying each factor together gives

$$e^{-2\gamma(T-t)} - 2e^{-\gamma(T-t)} + 1 - e^{-2\gamma(T-t^*)} - e^{-2\gamma(t-t^*)} + 2e^{-\gamma(T+t-2t^*)} \,.$$

This expression matches term for term the expression in Equation 242, thus remembering to negate the expression $\mathcal{I}_1 + \mathcal{I}_2$ we have found that under the extended Vasicek model of Hull & White that the expression for $A(t;T)$ is given by

$$\begin{aligned}
A(t;T) &= \log\left(\frac{Z_M(t^*;T)}{Z_M(t^*;t)}\right) - B(t;T)\frac{\partial}{\partial t}\log(Z_M(t^*;t)) \\
&- \frac{c^2}{4\gamma^3}\left(e^{-\gamma(T-t^*)} - e^{-\gamma(t-t^*)}\right)^2 (e^{2\gamma(t-t^*} - 1) \qquad (243)
\end{aligned}$$

Notes on Yield-Curve Fitting: Against

As suggested by the book for times t close to T let Z be

$$Z(r,t;T) = 1 + a(r)(T-t) + b(r)(T-t)^2 + c(r)(T-t)^3 + \cdots \,, \qquad (244)$$

or the Taylor series expansion about the maturity time T of bond with value 1. Then to put the above expression into the zero-coupon stochastic bond pricing equation of

$$\frac{\partial Z}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 Z}{\partial r^2} + (u - \lambda w) \frac{\partial Z}{\partial r} - rZ = 0\,,$$

we need to evaluate several partial derivatives of Z. We find

$$\begin{aligned}
\frac{\partial Z}{\partial t} &= -a(r) - 2b(r)(T-t) - 3c(r)(T-t)^2 + \cdots \\
\frac{\partial Z}{\partial r} &= a'(r)(T-t) + b'(r)(T-t)^2 + c'(r)(T-t)^3 + \cdots \\
\frac{\partial^2 Z}{\partial r^2} &= a''(r)(T-t) + b''(r)(T-t)^2 + c''(r)(T-t)^3 + \cdots .
\end{aligned}$$

Then we can put these expressions into the zero-coupon bond pricing equation and group terms by powers of $T - t$ to find

$$\begin{aligned}
& (-a(r) - r) \\
+ \; & (-2b(r) + \frac{1}{2} w^2 a''(r) + (u - \lambda w) a'(r) - r a(r))(T-t) \\
+ \; & (-3c(r) + \frac{1}{2} w^2 c''(r) + (u - \lambda w) b'(r) - r b(r))(T-t)^2 + \cdots = 0\,.
\end{aligned}$$

Thus we see that $a(r) = -r$. The equation for the $T - t$ power then gives since $a'(r) = -1$ and $a''(r) = 0$

$$-2b(r) - (u - \lambda w) + r^2 = 0\,,$$

or

$$b(r) = \frac{1}{2} r^2 - \frac{1}{2}(u - \lambda w)\,.$$

Thus we find that near maturity we have

$$Z(r, t; T) \approx 1 - r(T-t) + \frac{1}{2}(r^2 - u + \lambda w)(T-t)^2 + \cdots . \tag{245}$$

If $Z(r, t; T)$ is given via Equation 244 then using the approximation

$$\log(1 + x) \approx x - \frac{1}{2} x^2 + \frac{1}{3} x^3 + \cdots ,$$

valid when $x \ll 1$ we can evaluate the yield to maturity $-\frac{1}{T-t} \log(Z(r, t; T))$. The log term can be simplified as

$$\begin{aligned}
\log(Z(r, t; T)) &= \log\left(1 + a(T-t) + b(T-t)^2 + c(T-t)^3 + \cdots\right) \\
&= a(T-t) + b(T-t)^2 + c(T-t)^3 + \cdots \\
&- \frac{1}{2}\left(a(T-t) + b(T-t)^2 + c(T-t)^3 + \cdots\right)^2 \\
&+ \frac{1}{3}\left(a(T-t) + b(T-t)^2 + c(T-t)^3 + \cdots\right)^3 \\
&= a(T-t) + b(T-t)^2 + c(T-t)^3 + \cdots \\
&- \frac{1}{2}\left(a^2(T-t)^2 + 2ab(T-t)^3 + \cdots\right) + \frac{1}{3}\left(a^3(T-t)^3 + \cdots\right) \\
&= a(T-t) + \left(b - \frac{1}{2}a^2\right)(T-t)^2 + \left(c - ab + \frac{1}{3}\right)(T-t)^3 + \cdots .
\end{aligned}$$

Thus we have

$$-\frac{1}{T-t}\log(Z(r,t;T)) = -a + \left(\frac{1}{2}a^2 - b\right)(T-t) + \left(ab - c - \frac{1}{3}a^3\right)(T-t)^3 + \cdots$$

Since $-a(r) = r$ the slope of the yield curve is given by

$$\begin{aligned}\frac{\partial}{\partial(T-t)}\left[-\frac{\log(Z(r,t;T))}{T-t}\right] &= \frac{1}{2}a^2 - b \\ &= \frac{1}{2}r^2 - \frac{1}{2}r^2 + \frac{1}{2}(u-\lambda w) = \frac{1}{2}(u-\lambda w)\,, \qquad (246)\end{aligned}$$

and is one half the risk-neutral drift.

Chapter 32: Interest Rate Derivatives

Notes on the Text

Notes on the relationship between a caplet and a bond option

If we get the cashflow payment $\max(r_L - r_c, 0)$ at the time t_i, then by the theory of present value [2], to determine the value of this cashflow at t_{i-1} we need to discount it by the factor

$$\frac{1}{1 + r_L \tau} \, .$$

Here r_L and τ are measured in years (τ is notionally a fraction of a year). Then we have for the present value of this cash flow at the time t_{i-1} denoted by PV_{i-1} the following

$$\begin{aligned}
\mathrm{PV}_{i-1} &= \frac{1}{1 + r_L \tau} \max(r_L - r_c, 0) \\
&= \frac{1}{1 + r_L \tau} \max\left(\frac{r_L \tau + 1 - 1 - r_c \tau}{\tau}, 0 \right) \\
&= \frac{1}{\tau} \max\left(1 - \frac{1 + r_c \tau}{1 + r_L \tau}, 0 \right) \, .
\end{aligned}$$

Warning: this is $\frac{1}{\tau}$ multiplied by the expression presented in the book. Note that the above expression has a payoff that looks like a put option.

Chapter 33 (Convertible Bonds)

Additional Notes on the Text

Convertible Bonds with Random Interest Rate

In this section of these notes we derive the **convertible bond pricing equation**. Since *both* the stock of value S and the interest rate r are random we need two instruments to hedge our convertible bond with. Thus we consider a portfolio long one convertible bond (with a maturity date T_1), short Δ_2 zero-coupon bonds with maturity date T_2, and short Δ_1 shares of stock. This portfolio will have values given by

$$\Pi = V - \Delta_2 Z - \Delta_1 S\,.$$

Then using Ito's lemma for the *two* stochastic parameters r and S we have that Π changes as

$$\begin{aligned}
d\Pi &= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial r}dr \\
&+ \frac{1}{2}\left(\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + 2\rho\sigma S w \frac{\partial V}{\partial S \partial r} + w^2 \frac{\partial^2 V}{\partial r^2}\right) dt \\
&- \Delta_2 \left(\frac{\partial Z}{\partial t}dt + \frac{\partial Z}{\partial r}dr + \frac{1}{2}w^2 \frac{\partial^2 Z}{\partial r^2}dt\right) - \Delta_1 dS\,.
\end{aligned}$$

Group everything by dt, dS, and dr to get that $d\Pi$ looks like

$$\begin{aligned}
d\Pi &= \left[\frac{\partial V}{\partial t} + \frac{1}{2}\left(\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + 2\rho\sigma S w \frac{\partial V}{\partial S \partial r} + w^2 \frac{\partial^2 V}{\partial r^2}\right) - \Delta_2\left(\frac{\partial Z}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 Z}{\partial r^2}\right)\right] dt \\
&+ \left[\frac{\partial V}{\partial S} - \Delta_1\right] dS + \left[\frac{\partial V}{\partial r} - \Delta_2 \frac{\partial Z}{\partial r}\right] dr\,.
\end{aligned}$$

We now pick the hedge values Δ_1 and Δ_2 such that the above portfolio is deterministic. That is the coefficients of dS and dr vanish. From the above we see that the coefficient of dr will be equal to zero if we take

$$\Delta_2 = \frac{\frac{\partial V}{\partial r}}{\frac{\partial Z}{\partial r}}\,,$$

and the coefficient of dS will be zero if we take

$$\Delta_1 = \frac{\partial V}{\partial S}\,.$$

Once we have done this by setting $d\Pi = r\Pi dt$ (to avoid arbitrage) and dividing by dt we get the following equation

$$\begin{aligned}
\frac{\partial V}{\partial t} &+ \frac{1}{2}\left(\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + 2\rho\sigma S w \frac{\partial V}{\partial S \partial r} + w^2 \frac{\partial^2 V}{\partial r^2}\right) - \Delta_2\left[\frac{\partial Z}{\partial t} + \frac{1}{2}\left(w^2 \frac{\partial^2 Z}{\partial r^2}\right)\right] \\
&= rV - r\Delta_2 Z - r\Delta_1 S \\
&= rV - r\Delta_2 Z - rS\frac{\partial V}{\partial S}\,.
\end{aligned}$$

Putting terms that depend on T_1 on one side and terms that depend on T_2 on another side to get

$$\begin{aligned}\frac{\partial V}{\partial t} &+ \frac{1}{2}\left(\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + 2\rho\sigma S w \frac{\partial V}{\partial S \partial r} + w^2 \frac{\partial^2 V}{\partial r^2}\right) - rV + rS\frac{\partial V}{\partial S} \\ &= \Delta_2 \left[\frac{\partial Z}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 Z}{\partial r^2} - rZ\right],\end{aligned}$$

or dividing by $\frac{\partial V}{\partial r}$ we get

$$\begin{aligned}&\frac{1}{\frac{\partial V}{\partial r}}\left(\frac{\partial V}{\partial t} + \frac{1}{2}\left(\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + 2\rho\sigma S w \frac{\partial V}{\partial S \partial r} + w^2 \frac{\partial^2 V}{\partial r^2}\right) - rV + rS\frac{\partial V}{\partial S}\right) \\ = &\frac{1}{\frac{\partial Z}{\partial r}}\left(\frac{\partial Z}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 Z}{\partial r^2} - rZ\right).\end{aligned}$$

Using Equation 196 with $K(r,t) = 0$ we see that the right-hand-side of the above equals $-(u - \lambda w)$. Doing this and simplifying some we get for $V(S,r,t)$ following equation

$$\begin{aligned}\frac{\partial V}{\partial t} &+ \frac{1}{2}\left(\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + 2\rho\sigma S w \frac{\partial V}{\partial S \partial r} + w^2 \frac{\partial^2 V}{\partial r^2}\right) \\ &- rV + rS\frac{\partial V}{\partial S} + (u - \lambda w)\frac{\partial V}{\partial r} = 0,\end{aligned} \tag{247}$$

which is the **convertible bond pricing equation**. Here $\lambda = \lambda(r,S,t)$ is the market price of interest rate risk.

Notes on a Special Model

If we consider the Vasicek short term interest rate model where the differential equation for r is given by Equation 223 we can look for solutions to Equation 247 of the following form

$$V(S,r,t) = g(r,t) H\left(\frac{S}{g(r,t)}, t\right), \tag{248}$$

with $g(r,t) = Z(r,t;T) = e^{A(t;T) - rB(t;T)}$ and $B(t;T)$ given by Equation 224 and $A(t;T)$ given by Equation 225.

In the Mathematica file `a_convertible_bond_pricing_model.nb` we substitute the functional form given in Equation 248 into the convertible bond pricing Equation 247. When we do that the resulting expression involves t and r derivatives of $g(r,t)$. We next replace the t derivative of $g(r,t)$ using Equation 196. When we do this and make the substitution of $\xi = \frac{S}{g(r,t)}$ or $S = \xi g(r,t)$ we get

$$\frac{\partial H}{\partial t} + \frac{\xi^2}{2}\left(\sigma^2 - 2w\rho\sigma \frac{g_r(r,t)}{g(r,t)} + w^2 \frac{g_r(r,t)^2}{g(r,t)^2}\right)\frac{\partial^2 H}{\partial \xi^2} = 0.$$

Since $g(r,t) = e^{A(t;T) - rB(t;T)}$ we have that

$$g_r(r,t) = -B(t;T) g(r,t),$$

and the above becomes

$$\frac{\partial H}{\partial t}+\frac{\xi^2}{2}\left(\sigma^2+2w\rho\sigma B(t;T)+w^2B(t;T)^2\right)\frac{\partial^2 H}{\partial \xi^2}=0\,.$$

In the Vasicek interest rate model

$$dr=(u-\lambda w)dt+wdX=(\eta-\gamma r)dt+\beta^{1/2}dX$$

so $w=\beta^{1/2}$ and the above becomes

$$\frac{\partial H}{\partial t}+\frac{\xi^2}{2}\left(\sigma^2+2B(t;T)\rho\beta^{1/2}\sigma+B(t;T)^2\beta\right)\frac{\partial^2 H}{\partial \xi^2}=0\,,$$

the expression quoted in the book. The value just before maturity of V is $V(S,T^-)=\max(nS,1)$ and so

$$gH(\xi^-,T^-)=\max(n\xi^-g,1)\,,$$

or

$$H(\xi,T^-)=\max(n\xi,1)\,.$$

The no arbitrage constraint of $V\geq nS$ is $g(r,t)H(\xi,t)\geq n\xi g(r,t)$ or

$$H(\xi,t)\geq n\xi\,.$$

Chapter 34 (Mortgage-backed Securities)

Additional Notes on the Text

Notes on monthly payments in the fixed rate mortgage

If the home is worth a nominal present value of 1 we can use the theory of discounting at an interest rate of r_M to determine monthly payment amount x if the mortgage lasts N years. The present value of a constant income stream $(x, x, \cdots, x)$ where x is payed out $12N$ times is given by

$$\sum_{i=1}^{12N} \frac{x}{\left(1+\frac{r_M}{12}\right)^i} = x \sum_{i=1}^{12N} \left(1+\frac{r_M}{12}\right)^{-i} = x\left(1+\frac{r_M}{12}\right)^{-1} \left[\frac{1-\left(1+\frac{r_M}{12}\right)^{-12N}}{1-\left(1+\frac{r_M}{12}\right)^{-1}}\right]$$
$$= x\left[\frac{1-\left(1+\frac{r_M}{12}\right)^{-12N}}{1+\frac{r_M}{12}-1}\right] = \frac{12x}{r_M}\left(1-\left(1+\frac{r_M}{12}\right)^{-12N}\right) .$$

Where we have used the identity

$$\sum_{i=1}^{N} a^i = a\left(\frac{1-a^N}{1-a}\right) . \quad (249)$$

When we solve for x in the above we get

$$x = \frac{r_M/12}{1-\left(1+\frac{r_M}{12}\right)^{-12N}} .$$

Notes on Modeling Prepayment: The PSA Model

If we consider the given discussion of the simple PSA model then we see that the CPR as a function of time t when time is measured in months is given by

$$\text{CPR} = \begin{cases} \prod_{k=1}^{t}\left(1-\frac{0.002}{12}k\right) & 0 \le t \le 30 \\ \text{CPR}(30)\left(1-\frac{0.06}{12}\right)^{t-30} & t \ge 30 \end{cases} .$$

In the python code `plot_psa_model.py` we implement this function and then plot it as a function of year. When we do that we obtain the plot given in Figure 3. This plot matches the one given in the book.

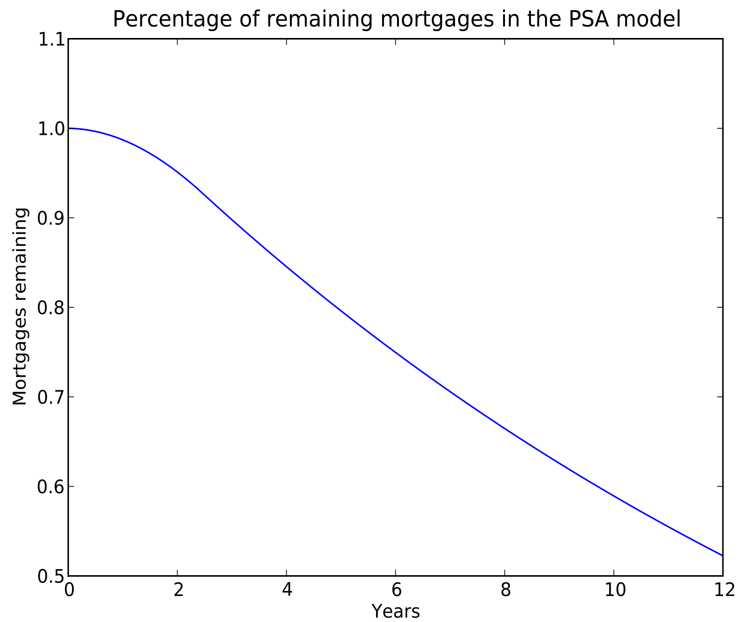

Figure 3: A duplicate of the books figure 34.4.

Notes on Valuing MBS

Using the same discounting logic as above but now in continuous time the present value P of the loan amount which is replicated with a continuous payment stream of x is given by

$$P = x \int_t^T e^{-r_M(\tau - t)} d\tau \,. \tag{250}$$

In a small amount of time dt this expression changes because we must pay the bank that holds our mortgage an amount $r_M P$ but at the same time we make a payment of x. Thus our mortgage amount P changes as

$$dP = (r_M P - x) dt \,.$$

Now consider the case where some fraction, $1 - Q(r, t)$, of the mortgages are payed off prematurely. Note this fraction Q is a function of time since inception of the loan t and the market interest rate r. Then the magnitude of the present value of the income stream represented by $x \int_t^T e^{-r_M(\tau - t)} d\tau$ earlier will change as the fraction of mortgages changes. We then have

$$P = xQ \int_t^T e^{-r_M(\tau - t)} d\tau \,,$$

Using the product rule dP is given by

$$\begin{aligned} dP &= dQ \left(x \int_t^T e^{-r_M(\tau - t)} d\tau \right) + Q\, d \left(x \int_t^T e^{-r_M(\tau - t)} d\tau \right) \\ &= dQ \frac{P}{Q} + (r_M P - xQ) dt \,. \end{aligned}$$

If we model the change in Q as

$$dQ = -a(t)f(r)Qdt\,, \tag{251}$$

then using the above we get

$$dP = (r_M P - xQ - a(t)f(r)P)dt\,, \tag{252}$$

for the differential equation satisfied for P. Note P only has a differential with respect to t. We would next like to find the differential equation for $V(r, P, Q, t)$, using the evolution of dQ given by Equation 251, the evolution of dP given by Equation 252 and our standard model for the spot interest rate $dr = (u - \lambda w)dt + wdX$. With these modeling assumptions, the partial differential equation for $V(r, P, Q, t)$ can be argued by following the discussion in the section "Can I Reverse Engineer a Partial Differential Equation to get the Model and Contract" given in [4]. Thus since r is a stochastic variable we will need to have terms

$$V_t + \frac{1}{2}w^2 V_{rr} + (u - \lambda w)V_r\,.$$

Since Q and P depend only on time t via Equations 251 and 252 they will require terms

$$-a(t)f(r)QV_Q \quad \text{and} \quad (r_M P - xQ - a(t)f(r)P)V_p\,.$$

The contract has the present value V and thus there is a $-rV$ term and finally there are cash flows due to regular payments of the form xQ and due to early prepayments of the form $a(t)f(r)P$ giving two source terms. When we combined all of these expressions we get

$$\begin{aligned} &V_t + \frac{1}{2}w^2 V_{rr} + (u - \lambda w)V_r + (r_M P - xQ - a(t)f(r)P)V_p \\ &- a(t)f(r)QV_Q - rV + (a(t)f(r)P + xQ) = 0\,, \end{aligned} \tag{253}$$

for the equation used to value V. In the special case where r is constant and equal to r_M we have $w = 0$ and $V_r = 0$ so the left-hand-side of the above

$$V_t + (r_M P - xQ - a(t)f(r_M)P)V_p - a(t)f(r_M)QV_Q - r_M V + a(t)f(r_M)P + xQ\,.$$

If we then consider this when $V = P$ we have $V_P = 1$, $V_Q = 0$, and $V_t = 0$ it becomes

$$r_M P - xQ - a(t)f(r_M)P - r_M P + a(t)f(r_M)P + xQ = 0\,,$$

showing that the MBS Equation 253 is satisfied.

Chapter 35 (Multi-Factor Interest Rate Modeling)

Additional Notes on the Text

Notes on the theoretical framework for two factors

Following the discussion in the book we have three equations for the two hedge ratios Δ_1 and Δ_2 given by

$$\frac{\partial Z}{\partial r} - \Delta_1 \frac{\partial Z_1}{\partial r} - \Delta_2 \frac{\partial Z_2}{\partial r} = 0 \quad (254)$$

$$\frac{\partial Z}{\partial l} - \Delta_1 \frac{\partial Z_1}{\partial l} - \Delta_2 \frac{\partial Z_2}{\partial l} = 0 \quad (255)$$

$$\mathcal{L}'(Z) - \Delta_1 \mathcal{L}'(Z_1) - \Delta_2 \mathcal{L}'(Z_2) = 0 \,. \quad (256)$$

Since there are three equations and only two unknowns Δ_1 and Δ_2, the linear system given by all three equations must be over-specified and thus any one of the equations is a linear combination of the other two. Because of this fact there exists values λ_r and λ_l such that $\lambda_r w - u$ times the first row plus $\lambda_l q - p$ times the second row equals the third row. When we write this out and then group everything by the function Z (similar equations hold for Z_1 and Z_2) that

$$\mathcal{L}'(Z) = (\lambda_r w - u)\frac{\partial Z}{\partial r} + (\lambda_l q - p)\frac{\partial Z}{\partial l} \,. \quad (257)$$

When we write out the expression for $\mathcal{L}'(Z)$ and put all terms on one side we get

$$\begin{aligned} &\frac{\partial Z}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 Z}{\partial r^2} + \rho w q \frac{\partial^2 Z}{\partial r \partial l} + \frac{1}{2} q^2 \frac{\partial^2 Z}{\partial l^2} \\ &\quad + (u - \lambda_r w)\frac{\partial Z}{\partial r} + (p - \lambda_l q)\frac{\partial Z}{\partial l} - rZ = 0 \,. \end{aligned} \quad (258)$$

As an additional piece of information, once we have specified that the third row is a linear combination of the other two we can simply compute Δ_1 and Δ_2 by solving Equation 254 and 255. Writing these two equations in the more standard form as

$$\begin{aligned} \frac{\partial Z_1}{\partial r}\Delta_1 + \frac{\partial Z_2}{\partial r}\Delta_2 &= \frac{\partial Z}{\partial r} \\ \frac{\partial Z_1}{\partial l}\Delta_1 + \frac{\partial Z_2}{\partial l}\Delta_2 &= \frac{\partial Z}{\partial l} \,, \end{aligned}$$

we can solve them using Crammer's rule. To use this we need the determinant of the coefficient matrix

$$D \equiv \begin{vmatrix} \frac{\partial Z_1}{\partial r} & \frac{\partial Z_2}{\partial r} \\ \frac{\partial Z_1}{\partial l} & \frac{\partial Z_2}{\partial l} \end{vmatrix} = \frac{\partial Z_1}{\partial r}\frac{\partial Z_2}{\partial l} - \frac{\partial Z_2}{\partial r}\frac{\partial Z_1}{\partial l} \,.$$

With this then computed Δ_1 and Δ_2 are then given by

$$\begin{aligned} \Delta_1 &= \frac{1}{D} \begin{vmatrix} \frac{\partial Z}{\partial r} & \frac{\partial Z_2}{\partial r} \\ \frac{\partial Z}{\partial l} & \frac{\partial Z_2}{\partial l} \end{vmatrix} = \frac{1}{D}\left(\frac{\partial Z}{\partial r}\frac{\partial Z_2}{\partial l} - \frac{\partial Z_2}{\partial r}\frac{\partial Z}{\partial l} \right) \\ \Delta_2 &= \frac{1}{D} \begin{vmatrix} \frac{\partial Z_1}{\partial r} & \frac{\partial Z}{\partial r} \\ \frac{\partial Z_1}{\partial l} & \frac{\partial Z}{\partial l} \end{vmatrix} = \frac{1}{D}\left(\frac{\partial Z_1}{\partial r}\frac{\partial Z}{\partial l} - \frac{\partial Z_1}{\partial l}\frac{\partial Z}{\partial r} \right) \,. \end{aligned}$$

Note I believe the book has a typo in the expression for Δ_2.

Notes on Modeling the Long-term Rate

As discussed in the book a consol bond, C_0, must satisfy the pricing equation

$$\mathcal{L}'(C_0) + 1 = (\lambda_r w - u)\frac{\partial C_0}{\partial r} + (\lambda_l q - p)\frac{\partial C_0}{\partial l}\,. \tag{259}$$

Where the linear operator $\mathcal{L}'$ is given by

$$\begin{aligned}
\mathcal{L}'(Z) &\equiv \mathcal{L}(Z) - rZ & (260)\\
&\equiv \frac{\partial Z}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 Z}{\partial r^2} + \rho w q\frac{\partial^2 Z}{\partial r \partial l} + \frac{1}{2}q^2\frac{\partial^2 Z}{\partial l^2} - rZ\,. & (261)
\end{aligned}$$

The yield l on a consol bond is given by $l = \frac{1}{C_0}$, thus $C_0 = \frac{1}{l}$. With this expression have that $\frac{\partial C_0}{\partial t} = \frac{\partial C_0}{\partial r} = 0$, $\frac{\partial C_0}{\partial l} = -\frac{1}{l^2}$, and $\frac{\partial^2 C_0}{\partial l^2} = \frac{2}{l^3}$, so we find for Equation 259 the expression

$$\frac{1}{2}q^2\left(\frac{2}{l^3}\right) - \frac{r}{l} + 1 = -\frac{\lambda_l q - p}{l^2}\,.$$

When we solve this for $p - \lambda_l q$ we get

$$p - \lambda_l q = \frac{q^2}{l} - rl + l^2\,.$$

Warning: The book has the expression $\frac{q^2}{l^2}$ rather than $\frac{q^2}{l}$ for the first term in the above expression, which I think is a typo. Thus, since we now know an expression for $\lambda_l q - p$ in terms of l we find for Equation 257

$$\mathcal{L}'(Z) = (\lambda_r w - u)\frac{\partial Z}{\partial r} - \left(l^2 - rl + \frac{q^2}{l}\right)\frac{\partial Z}{\partial l}\,,$$

where $\mathcal{L}'(Z)$ is given by Equation 260.

Notes on General Multi-Factor Models: Vasicek

The general pricing equation for a fixed income instrument V given N factors x_i is given by

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\rho_{ij}\sigma_i\sigma_j\frac{\partial^2 V}{\partial x_i \partial x_j} + \sum_{i=1}^{N}(\mu_i - \lambda_i\sigma_i)\frac{\partial V}{\partial x_i} - rV = 0\,. \tag{262}$$

In general all parameters μ_i, σ_i, ρ_{ij} and λ_i can be functions of all factors $\mathbf{x}$ and time t. This most general framework makes the model to difficult to solve analytically and we will postulate some special models that make the model tractable. To further simplify things we consider the multi-factor Vasicek model where the correlations and volatilities are assumed to not be functions of the factors $\mathbf{x}$. Furthermore we take all cross correlations zero, and all model parameters are independent of time t. Thus we have $\sigma_i(t) = c_i$,

$$\mu_i(t) - \lambda_i(t)\sigma_i(t) = a_i - \sum_{j=1}^{N} b_{ij}x_j\,,$$

and $\rho_{ij} = d_{ij}(t) = 0$ if $i \neq j$. Then assuming a solution to Equation 262 of the form

$$V(x,t;T) = \exp\left\{ f_0(t,T) + \sum_{i=1}^{N} f_i(t;T) x_i \right\} , \tag{263}$$

we can evaluate the needed partial derivatives and find alternative differential equations for the functions f_i. From the form above we find

$$V_t = \left(\dot{f}_0 + \sum_{i=1}^{N} \dot{f}_i x_i \right) V \,, \quad V_{x_i} = f_i V \,, \quad \text{and} \quad V_{x_i x_j} = f_i f_j V \,.$$

Putting these in Equation 262, using the stated assumptions, and dividing by V we first get

$$\dot{f}_0 + \sum_{i=1}^{N} \dot{f}_i x_i + \frac{1}{2} \sum_{i=1}^{N} \sigma_i^2 f_i^2 + \sum_{i=1}^{N} \left(a_i - \sum_{j=1}^{N} b_{ij} x_j \right) f_i - g_0(t) - \sum_{i=1}^{N} g_i(t) x_i = 0 \,.$$

Since $\sigma_i = c_i$ and changing the order of the summations over one of the terms gives

$$\dot{f}_0 + \sum_{i=1}^{N} \dot{f}_i x_i + \frac{1}{2} \sum_{i=1}^{N} {c_i}^2 f_i^2 + \sum_{i=1}^{N} a_i f_i - \sum_{j=1}^{N} \left(\sum_{i=1}^{N} b_{ij} f_i \right) x_j - g_0(t) - \sum_{i=1}^{N} g_i(t) x_i = 0 \,.$$

Grouping the terms by each factor x_i gives

$$\begin{aligned} &\dot{f}_0 + \frac{1}{2} \sum_{i=1}^{N} {c_i}^2 f_i^2 + \sum_{i=1}^{N} a_i f_i - g_0(t) = 0 \\ &\dot{f}_i - \sum_{k=1}^{N} b_{ki} f_k - g_i(t) = 0 \quad \text{for} \quad i = 1, 2, \cdots, N \,. \end{aligned}$$

The same equations as in the book.

Chapter 36 (Empirical Behavior of the Spot Interest Rate)

Additional Notes on the Text

Notes on the volatility structure: νr^{β}

From the form for dr taken by many popular models of

$$dr = u(r)dt + \nu r^{\beta} dX \,, \tag{264}$$

we have

$$E[dr^2] = O(dt^2) + O(dt^{3/2}) + \nu^2 r^{2\beta} E[dX^2] = \nu^2 r^{2\beta} dt \,,$$

where we have dropped higher order terms in dt. Thus we have

$$\log(E[dr^2]) = \log(\nu^2) + \log(dt) + 2\beta \log(r) \,.$$

Next we take $dr \approx \delta r$ and $dt \approx \delta t$, i.e. discretizing r and t by putting these variables into buckets and with this data plot $\log(E((\delta r)^2))$ vs. $\log(r)$. By fitting a least square line to that we can find values for the parameters ν and β. Using the figure given in the book as an example, just to understand the procedure, we will attempt to estimate ν. From the given plot of $\log(E((\delta r)^2))$ vs. $\log(r)$ assume we find that the empirical intercept on the $\log(E((\delta r)^2))$ axis is given by β_0. Thus from the above expression we have

$$\log(\nu^2) + \log(\delta t) \approx \beta_0 \,.$$

Since we are considering daily data where, $\delta t = 1$ so that $\log(\delta t) = 0$ and $\log(\nu) = \frac{\beta_0}{2}$ or $\nu = e^{-\frac{\beta_0}{2}}$.

Notes on the drift structure: $u(r)$

For the short rate dynamics

$$dr = u(r)dt + \nu r^{\beta} dX \equiv A(r,t)dt + B(r,t)dX$$

we have $A(r,t) = u(r)$ and $B(r,t) = \nu r^{\beta}$ so the Fokker-Planck equation given by Equation 218 becomes

$$\frac{\partial p}{\partial t} = \frac{1}{2}\nu^2 \frac{\partial^2 (r^{2\beta} p)}{\partial r^2} - \frac{\partial (u(r)p)}{\partial r} \,. \tag{265}$$

As from the previous section we now "know" values for ν and β we want to determine the functional form form $u(r)$. In steady-state the time derivative above vanishes and if we then integrate Equation 265 once we have

$$\frac{1}{2}\nu^2 \frac{d}{dr}(r^{2\beta} p_{\infty}(r)) - u(r) p_{\infty}(r) = C \,.$$

As discussed in Chapter 30 which begins on Page 90 we need to take the constant of integration C equal to zero to have $p_\infty(r)$ represent a valid density. When we do that and solve for $u(r)$ we find

$$
\begin{aligned}
u(r) &= \frac{1}{p_\infty(r)}\left[\frac{1}{2}\nu^2\frac{d}{dr}(r^{2\beta}p_\infty(r))\right] = \frac{1}{p_\infty(r)}\left[\nu^2\beta r^{2\beta-1}p_\infty(r) + \frac{\nu^2}{2}r^{2\beta}\frac{d}{dr}(p_\infty(r))\right] \\
&= \nu^2\beta r^{2\beta-1} + \frac{1}{2}\nu^2 r^{2\beta}\frac{d}{dr}\log(p_\infty(r))\,.
\end{aligned}
$$

This gives an expression for $u(r)$ as a function of the steady-state density $p_\infty(r)$, which hopefully is easier to estimate. Taking a lognormal distribution with location $\log(\bar{r})$ and scale a for $p_\infty(r)$ or

$$
p_\infty(r) = \frac{1}{\sqrt{2\pi}ar}\exp\left\{-\frac{1}{2a^2}\left(\log\left(\frac{r}{\bar{r}}\right)\right)^2\right\}\,.
$$

we can evaluate what this means for $u(r)$. Since

$$
\begin{aligned}
\log(p_\infty(r)) &= -\log(r) - \frac{1}{2a^2}\left(\log\left(\frac{r}{\bar{r}}\right)\right)^2 - \frac{1}{2}\log(2\pi) - \log(a) \quad \text{so} \\
\frac{d}{dr}\log(p_\infty(r)) &= -\frac{1}{r} - \frac{1}{a^2}\log\left(\frac{r}{\bar{r}}\right)\left(\frac{1}{r}\right)\,.
\end{aligned}
$$

Thus we get for $u(r)$ the following

$$
\begin{aligned}
u(r) &= \nu^2\beta r^{2\beta-1} + \frac{\nu^2}{2}r^{2\beta-1}\left[-1 - \frac{1}{a^2}\log\left(\frac{r}{\bar{r}}\right)\right] \\
&= \nu^2 r^{2\beta-1}\left[\beta - \frac{1}{2} - \frac{1}{2a^2}\log\left(\frac{r}{\bar{r}}\right)\right]\,.
\end{aligned}
$$

The book then makes the following statement: "The real spot rate is mean-reverting to 8%". Which I don't understand. I would assume that the value we mean revert to is given by the value of r that $u(r) = 0$. From the above functional form we see that this value is given by

$$
r = \bar{r}e^{a^2(2\beta-1)}\,,
$$

If we take the numbers given in the book for the US LIBOR rate given by $a = 0.4$, $\bar{r} = 0.08$, and $\beta = 1.13$ we get $r = 0.097$, which would make me think that the real spot rate is mean-reverting to 9.7%. If anyone sees anything wrong with my argument above please contact me.

Notes on the slope of the yield curve

The equations from this section are derived earlier in these notes, for example Equations 245 and 246.

Notes on the properties of the forward rate curve on average

Since our short term interest rate dynamics is given by $dr = u(r,t)dt + w(r,t)dX$ when we consider the steady-state Fokker-Plank Equation 218 with $A(r,t) = u(r)$ and $B(r,t) = w(r)$ we have

$$\frac{1}{2}\frac{\partial^2(w^2 p_\infty)}{\partial r^2} = \frac{\partial(u p_\infty)}{\partial r}.$$

We can integrate each side, recalling as on Page 99 that the constant of integration when we do so is zero, and then solve for u to get

$$u = \frac{1}{2p_\infty}\frac{d}{dr}(w^2 p_\infty). \tag{266}$$

Doing the same thing but for the risk neutral density $p^*_\infty(r)$ we have the risk adjusted drift $u - \lambda w$ equal to the similar expression

$$u - \lambda w = \frac{1}{2p^*_\infty}\frac{d}{dr}(w^2 p^*_\infty).$$

In this expression put in what we know for u from Equation 266 to get

$$\frac{1}{2p_\infty}\frac{d}{dr}(w^2 p_\infty) - \lambda w = \frac{1}{2p^*_\infty}\frac{d}{dr}(w^2 p^*_\infty).$$

We next divide both sides by $\frac{w^2}{2}$ to get

$$\frac{1}{w^2 p_\infty}\frac{d}{dr}(w^2 p_\infty) - \frac{2\lambda}{w} = \frac{1}{w^2 p^*_\infty}\frac{d}{dr}(w^2 p^*_\infty).$$

Now integrating both sides we get

$$\log(w^2 p_\infty) - 2\int^r \frac{\lambda(s)}{w(s)}ds = \log(w^2 p^*_\infty),$$

or

$$\log\left(\frac{p_\infty}{p^*_\infty}\right) = 2\int^r \frac{\lambda(s)}{w(s)}ds,$$

or solving for $p^*_\infty(r)$ in terms of $p_\infty(r)$ we find

$$p^*_\infty = p_\infty \exp\left\{-2\int^r \frac{\lambda(s)}{w(s)}ds\right\}. \tag{267}$$

We can also replace the risk adjusted drift term, $u - \lambda w$, in the bond pricing Equation 196 in terms of an expression that depends on the risk-neural density to get

$$\frac{\partial Z}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 Z}{\partial r^2} + \frac{1}{2p^*_\infty}\frac{d}{dr}(w^2 p^*_\infty)\frac{\partial Z}{\partial r} - rZ = 0. \tag{268}$$

Consider the middle two terms which we write as

$$\frac{1}{2p^*_\infty}\left[w^2 p^*_\infty\frac{\partial^2 Z}{\partial r^2} + \frac{d}{dr}(w^2 p^*_\infty)\frac{\partial Z}{\partial r}\right],$$

or using the product rule we have

$$\frac{1}{2p^*_\infty}\frac{\partial}{\partial r}\left[w^2 p^*_\infty \frac{\partial Z}{\partial r}\right] .$$

With this Equation 268 becomes

$$p^*_\infty(r)\left[\frac{\partial Z}{\partial t} - rZ\right] = -\frac{1}{2}\frac{\partial}{\partial r}\left[w^2 p^*_\infty(r)\frac{\partial Z}{\partial r}\right] .$$

But we know $p^*_\infty(r)$ in terms of $p_\infty(r)$ via Equation 267. When we put that expression in and then integrate from $r = 0$ to $r = \infty$ we get

$$\int_0^\infty p_\infty(r) e^{-2\int^r \frac{\lambda(s)}{w(s)} ds}\left(\frac{\partial Z}{\partial t} - rZ\right) dr = 0 .$$

Assuming time homogeneous of the function Z i.e. that $Z(t;T) = Z(\tau)$ where $\tau = T - t$ we would then have $\frac{\partial Z}{\partial t} = -\frac{\partial Z}{\partial \tau}$ and the above integral is given by

$$\int_0^\infty p_\infty(r) e^{-2\int^r \frac{\lambda(s)}{u(s)} ds}\left(\frac{\partial Z}{\partial \tau} - rZ(\tau)\right) dr = 0 .$$

Note that this expression must hold for *all* values of τ.

Chapter 37 (The Heath, Jarrow & Morton and Brace, Gatarek & Musiela Models)

Additional Notes on the Text

Notes on the forward rate equation

In terms of the forward rate curve $F(t;T)$ the price of a zero-coupon bond at time t that pays 1 \$ at time T is given by

$$Z(t,T) = e^{-\int_t^T F(t;s)ds} \,. \tag{269}$$

We then assume that the differential evolution of $Z(t;T)$ is given by

$$dZ(t;T) = \mu(t,T)Z(t;T)dt + \sigma(t,T)Z(t;T)dX \,.$$

Here we are assuming that $T > t$ is fixed and t changes by increasing in the direction of T. When we take $T = t$ we have $Z(t;t) = 1$ (a constant) so we expect $dZ(t;t) = 0$. In addition, when $T = t$ the increment dt must be 0 since there is no further increase in t possible. Thus using those values in the above we get

$$0 = \sigma(t,t)Z(t;t)dX \,,$$

which since $dX \neq 0$ means that

$$\sigma(t,t) = 0 \,.$$

From Equation 269 we can solve for $F(t;T)$ in terms of $Z(t;T)$ to get

$$F(t;T) = -\frac{\partial}{\partial T}\log(Z(t;T)) \,. \tag{270}$$

From this we would evaluate that the differential of $F(t;T)$ as

$$dF(t;T) = -\frac{\partial}{\partial T} d(\log(Z(t;T))) \,, \tag{271}$$

which shows that to evaluate $dF(t;T)$ means that we need to evaluate $d(\log(Z(t;T))$ or the differential of the function of Z, $\log(Z(t;T))$. To evaluate this differential recall that when the differential of a variable G evolves according to

$$dG = A(G,t)dt + B(G,t)dX \,,$$

the differential of a function of G, say $f(G)$, is given by

$$df = \left(A(G,t)\frac{df}{dG} + \frac{1}{2}B^2(G,t)\frac{d^2 f}{dG^2} \right) dt + B(G,t)\frac{df}{dG}dX \,.$$

Using this result with

$$\frac{d\log(Z)}{dZ} = \frac{1}{Z} \quad \text{so} \quad \frac{d^2\log(Z)}{dZ^2} = -\frac{1}{Z^2} \,,$$

we can compute $d(\log(Z(t;T)))$ as

$$
\begin{aligned}
d(\log(Z(t;T))) &= \left(\mu Z\left(\frac{1}{Z}\right) + \frac{1}{2}\sigma^2 Z^2\left(-\frac{1}{Z^2}\right)\right) dt + \left(\sigma Z\left(\frac{1}{Z}\right)\right) dX \\
&= \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dX\,.
\end{aligned}
$$

Where in the above we recall that μ and σ are functions of (t, T). Then from Equation 271 by taking $-\frac{\partial}{\partial T}$ of this expression to get we have $dF(t;T)$ given by

$$
dF(t;T) = \frac{\partial}{\partial T}\left(\frac{1}{2}\sigma^2(t,T) - \mu(t,T)\right) dt - \frac{\partial}{\partial T}\sigma(t,T)dX\,. \tag{272}
$$

Notes on the dynamics of the spot rate process

We now derive the dynamics for the short term interest rate $r(t)$. To do that we first note that $r(t)$ is related to the forward rate curve $F(t, T)$ by setting $T = t$ or $r(t) = F(t, t)$. Let today be t^* and t a time in the future $t > t^*$, then we can write $r(t)$ as

$$
r(t) = F(t,t) = F(t^*,t) + \int_{t^*}^{t} dF(s;t)\,.
$$

We use Equation 272 with $t \to s$ and $T \to t$, so that we can obtain the needed expression $dF(s;t)$ as follows

$$
\begin{aligned}
dF(s;t) &= \frac{\partial}{\partial t}\left(\frac{1}{2}\sigma^2(s,t) - \mu(s,t)\right) ds - \frac{\partial}{\partial t}\sigma(s,t)dX \\
&= \left(\sigma(s,t)\frac{\partial\sigma(s,t)}{\partial t} - \frac{\partial\mu(s,t)}{\partial t}\right) ds - \frac{\partial\sigma(s,t)}{\partial t}dX\,.
\end{aligned}
$$

Putting this into the expression above for $r(t)$ we get

$$
r(t) = F(t^*,t) + \int_{t^*}^{t}\left(\sigma(s,t)\frac{\partial\sigma(s,t)}{\partial t} - \frac{\partial\mu(s,t)}{\partial t}\right) ds - \int_{t^*}^{t}\frac{\partial\sigma(s,t)}{\partial t}dX(s)\,.
$$

Now we have written dX as $dX(s)$ to be the component of randomness that takes place as s moves towards t from below. With this expression we can now compute dr and find

$$
\begin{aligned}
dr = \Bigg\{ &\frac{\partial F}{\partial t}(t^*,t) + \left(\sigma(s,t)\frac{\partial\sigma(s,t)}{\partial t} - \frac{\partial\mu(s,t)}{\partial t}\right|_{s=t} + \int_{t^*}^{t}\frac{\partial}{\partial t}\left(\sigma(s,t)\frac{\partial\sigma(s,t)}{\partial t} - \frac{\partial\mu(s,t)}{\partial t}\right) ds \\
&- \left.\frac{\partial\sigma(s,t)}{\partial t}\right|_{s=t} - \int_{t^*}^{t}\frac{\partial^2\sigma(s,t)}{\partial t^2}dX(s)\Bigg\} dt\,.
\end{aligned}
$$

Here the left round parenthesis before the second term in this expression matchs up with the "evaluation" notation $|_{s=t}$ to its right expressing the fact that everything between these two expressions should be evaluated at $s = t$. Since $\sigma(t,t) = 0$, the second term above vanishes and we get

$$dr = \left\{ \frac{\partial F}{\partial t}(t^*, t) - \left(\frac{\partial \mu(s,t)}{\partial t}\bigg|_{s=t} + \int_{t^*}^{t} \left(\left(\frac{\partial \sigma(s,t)}{\partial t} \right)^2 + \sigma(s,t) \frac{\partial^2 \sigma(s,t)}{\partial t^2} - \frac{\partial^2 \mu(s,t)}{\partial t^2} \right) ds \right. \right.$$
$$\left. - \frac{\partial \sigma(s,t)}{\partial t}\bigg|_{s=t} - \int_{t^*}^{t} \frac{\partial^2 \sigma(s,t)}{\partial t^2} dX(s) \right\} dt \,.$$

Note that the term $\int_{t^*}^{t} \frac{\partial^2 \sigma(s,t)}{\partial t^2} dX(s)$ makes the process for r non Markov.

Notes on the market price of risk

When the portfolio suggested $Z(t;T_1) - \Delta Z(t;T_2)$ is perfectly hedged, then $d\Pi$ must equal $r\Pi$ or

$$d\Pi = Z(t;T_1)\mu(t,T_1) - \Delta z(t;T_2)\mu(t,T_2) = r(t)\Pi \,,$$

or solving for all of the T_1 variables in terms of the T_2 variables

$$Z(t;T_1)\mu(t,T_1) - r(t)Z(t;T_1) = \Delta \left\{ Z(t;T_2)\mu(t;T_2) - r(t)Z(t;T_2) \right\} \,.$$

When we replace Δ with the optimal hedge value given by

$$\Delta = \frac{\sigma(t,T_1)Z(t;T_1)}{\sigma(t,T_2)Z(t;T_2)} \,, \tag{273}$$

and again solving for all of the T_1 variables in terms of the T_2 variables we get

$$\frac{\mu(t;T_1) - r(t)}{\sigma(t;T_1)} = \frac{\mu(t;T_2) - r(t)}{\sigma(t;T_2)} \,.$$

Since the left-hand-side is a function of only T_1 and the right-hand-side is a function of only T_2 to be equal we need to equate these expressions to something that is independent of T say $\lambda(t)$ and get

$$\mu(t,T) = r(t) + \lambda(t)\sigma(t,T) \,.$$

The expression $\lambda(t)$ is the market price of risk.

Notes on real and risk neutral process for F

In this section we derive the risk neutral dynamics of the forward rate curve, using the known risk neutral dynamics for the process for $dZ(t;T)$. To begin, if we take the representation for dF to be given by

$$dF(t;T) = m(t,T)dt + \nu(t,T)dX \,,$$

then by Equation 272 we must have

$$\nu(t,T) = -\frac{\partial \sigma(t,T)}{\partial T} \Rightarrow \sigma(t,T) = -\int_t^T \nu(t,s)ds \,, \tag{274}$$

since $\sigma(t,t) = 0$. In addition $\mu(t,T)$ is given by

$$\frac{\partial}{\partial T}\left(\frac{1}{2}\sigma^2(t,T) - \mu(t,T)\right) = \sigma(t,T)\frac{\partial\sigma(t,T)}{\partial T} - \frac{\partial\mu(t,T)}{\partial T} = \nu(t,T)\int_t^T \nu(t,s)ds - \frac{\partial\mu(t,T)}{\partial T}\,.$$

In the risk-neutral world for dZ we have $\mu(t,T) = r(t)$, so $\frac{\partial\mu}{\partial T} = 0$ and we have

$$m(t,T) = \nu(t,T)\int_t^T \nu(t,s)ds\,, \tag{275}$$

so putting everything together we get

$$dF(t;T) = \nu(t,T)\left(\int_t^T \nu(t,s)ds\right)dt + \nu(t,T)dX\,, \tag{276}$$

for the risk neutral process for F.

Simple one factor example: Ho & Lee

Recall that in the Ho & Lee model we have forward rate dynamics given by $dr = \eta(t)dt + cdX$ with $c = \sqrt{\beta}$. This model is discussed on Page 103 of these notes where we derive the expression for $Z(r,t;T)$. In the Ho & Lee model the function $\eta(t)$ is specified to fit the yield curve at a given time t^*. Mathematically this means that the function $\eta(\cdot)$ must satisfy

$$\begin{aligned}
F(t^*;T) &= -\frac{\partial}{\partial T}\log(Z(r,t^*;T)) \\
&= \frac{\partial}{\partial T}\left((T-t^*)r(t^*) + \int_{t^*}^T \eta(s)(T-s)ds - \frac{1}{6}c^2(T-t^*)^3\right) \\
&= r(t^*) + \int_{t^*}^T \eta(s)ds - \frac{1}{2}c^2(T-t^*)^2\,.
\end{aligned} \tag{277}$$

Thus solving for the integral of η we get

$$\int_{t^*}^T \eta(s)ds = F(t^*;T) + \frac{1}{2}c^2(T-t^*)^2\,.$$

To extract the function η we take $\frac{\partial}{\partial T}$ of this expression and evaluate it at $T = t$ to get

$$\eta(t) = \frac{\partial}{\partial t}F(t^*;t) + c^2(t-t^*)\,. \tag{278}$$

Now that we know $\eta(\cdot)$ as a function we can put it back into Equation 277 with $t^* \to t$ to get $F(t;T)$ at any other time t. When we do that we have

$$\begin{aligned}
dF &= dr(t) - \eta(t)dt + c^2(T-t)dt \\
&= c^2(T-t)dt + cdX\,,
\end{aligned}$$

when we use the dynamics of the spot rate process $dr = \eta(t)dt + cdX$ under the Ho & Lee model.

Notes on the non-infinitesimal short rate

Let $j(t,\tau)$ be an interest rate that is accrued m times per annum. This is related to the continuous rate $\bar{F}(t,\tau)$ via.

$$\left(1+\frac{j(t,\tau)}{m}\right)^m = e^{\bar{F}(t,\tau)} . \tag{279}$$

We will want the $j(t,\tau)$ rate to follow a lognormal model. This means that we take $j(t,\tau)$ to have the typical lognormal form

$$dj(t,\tau) = m_j(t,\tau)dt + \gamma(t,\tau)j(t,\tau)dX .$$

We now solve for $\bar{F}(t,\tau)$ in term of $j(t,\tau)$ and $j(t,\tau)$ in terms of $\bar{F}(t,\tau)$. First taking the logarithm of Equation 279 and solving for $\bar{F}(t,\tau)$ we get

$$\bar{F}(t,\tau) = m\log\left(1+\frac{j(t,\tau)}{m}\right) . \tag{280}$$

Second taking the mth root of Equation 279 and then solving for $j(t,\tau)$ we get

$$1+\frac{j(t,\tau)}{m} = e^{\bar{F}(t,\tau)/m} \Rightarrow j(t,\tau) = m\left(e^{\bar{F}(t,\tau)/m}-1\right) . \tag{281}$$

We can take the differential of the expression Equation 280 via Ito's rule since

$$d\bar{F}(t,\tau) = \left(\frac{d\bar{F}}{dj}m_j(t,\tau)+\frac{1}{2}\gamma^2 j^2\frac{d^2\bar{F}}{dj^2}\right)dt + \left(\gamma j\frac{d\bar{F}}{dj}\right)dX .$$

Since the needed derivatives are given by

$$\begin{aligned}
\frac{d\bar{F}}{dj} &= \frac{1}{1+j/m} = e^{-\bar{F}/m} \\
\frac{d^2\bar{F}}{dj^2} &= -\frac{1}{m(1+j/m)^2} = -\frac{1}{m}e^{-2\bar{F}/m} ,
\end{aligned}$$

we can compute the above differential of $\bar{F}$ to find

$$d\bar{F} = \left(e^{-\bar{F}/m}m_j(t,\tau) - \frac{1}{2m}\gamma(t,\tau)^2 j(t,\tau)^2 e^{-2\bar{F}/m}\right)dt + \left(\gamma m(e^{\bar{F}/m}-1)e^{-\bar{F}/m}\right)dX .$$

Simplifying the coefficient of dX we get

$$m\gamma(t,\tau)\left(1-e^{-\bar{F}/m}\right) ,$$

which is different than the book in that it has a minus sign in the exponential and a different leading sign.

Notes on the Brace, Gatarek, and Musiela Model

The dynamics of the forward rate and zero coupon bonds are given by

$$dF_i = \mu_i F_i dt + \sigma_i F_i dX_i \,, \tag{282}$$

and

$$dZ_i = rZ_i dt + Z_i \sum_{j=1}^{i-1} a_{ij} dX_j \,, \tag{283}$$

respectively. From Ito's lemma applied to the one period discounting expression

$$Z_i = (1 + \tau F_i) Z_{i+1} \,,$$

we have

$$dZ_i = (1 + \tau F_i) dZ_{i+1} + \tau Z_{i+1} dF_i + \tau \sigma_i F_i Z_{i+1} \left(\sum_{j=1}^{i} a_{i+1,j} \rho_{ij} \right) dt \,.$$

Put the expressions for dF_i via Equation 282 and dZ_i via Equation 283 into this to get

$$\begin{aligned} rZ_i dt + Z_i \sum_{j=1}^{i-1} a_{ij} dX_j = (1 + \tau F_i) & \left(rZ_{i+1} dt + Z_{i+1} \sum_{j=1}^{i} a_{i+1,j} dX_j \right) \\ & + \tau Z_{i+1} (\mu_i F_i dt + \sigma_i F_i dX_i) \\ & + \tau \sigma_i F_i Z_{i+1} \left(\sum_{j=1}^{i} a_{i+1,j} \rho_{ij} \right) dt \,. \end{aligned}$$

We will derive recursive relations for a_{ij} by considering the above as three separate equations by equating the coefficients for dX_i, dX_j (for $j \le i-1$) and dt. Equating coefficients for dX_i on each side we get

$$0 = (1 + \tau F_i) Z_{i+1} a_{i+1,i} + \tau Z_{i+1} \sigma_i F_i \,,$$

or solving for $a_{i+1,i}$ we get

$$a_{i+1,i} = -\frac{\sigma_i F_i \tau}{1 + \tau F_i} \,. \tag{284}$$

Equating coefficients of dX_j on both sides when $j = 1, 2, \cdots i-1$ gives

$$Z_i a_{ij} = (1 + \tau F_i) Z_{i+1} a_{i+1,j} \,.$$

Since $Z_i = (1 + \tau F_i) Z_{i+1}$ we obtain

$$a_{ij} = a_{i+1,j} \quad \text{for} \quad j < i \,. \tag{285}$$

Using Equation 285 as many times as needed we can write

$$a_{i+1,j} = a_{i,j} = a_{i-1,j} = a_{i-2,j} = \cdots = a_{j+1,j} \,.$$

Then to evaluate $a_{j+1,j}$ we can use Equation 284 with $i = j$ to conclude that

$$a_{i+1,j} = -\frac{\sigma_j F_j \tau}{1 + \tau F_j} \quad \text{for} \quad j < i \,.$$

Equating coefficients of dt on both sides we get

$$rZ_i = (1 + \tau F_i) r Z_{i+1} + \tau F_{i+1} \mu_i F_i + \tau \sigma_i F_i Z_{i+1} \left(\sum_{j=1}^{i} a_{i+1,j} \rho_{ij} \right) .$$

As $1 + \tau F_i = \frac{Z_i}{Z_{i+1}}$ we have $rZ_i = (1 + \tau F_i) r Z_{i+1}$ and the terms involving r drop out. We can solve for μ_i to get

$$\mu_i = -\sigma_i \sum_{j=1}^{i} a_{i+1,j} \rho_{ij} .$$

Chapter 38 (Fixed Income Term Sheets)

Additional Notes on the Text

Notes on index amortizing rate swaps

We value the amortizing swap between reset dates using the standard one-factor interest rate model where Equation 196 with $K(r,t)=0$. On each reset date t_i we have two affects. The first is that the principal P changes and the second is that interest is paid. These together give an internal boundary condition on $V(r,P,t)$ across t_i of

$$V(r,P,t_i^-) = (r-r_f)P + V(r,g(r)P,t_i^+)\,.$$

At the end of the contract $t=T$ there is a final payment

$$V(r,P,T) = (r-r_f)P\,.$$

If we look for a similarity solution for V of the form $V(r,P,t)=PH(r,t)$, then as Equation 196 is linear and independent of P the differential equation for H is the same as that for V i.e. Equation 196. The internal jump conditions on V in terms of H are

$$PH(r,t_i^-) = (r-r_f)P + g(r)PH(r,t_i^+)\,,$$

or dividing by P

$$H(r,t_i^-) = r-r_f + g(r)H(r,t_i^+)\,. \tag{286}$$

The final condition on V in terms of H is $PH(r,T)=(r-r_f)P$ or

$$H(r,T) = r-r_f\,. \tag{287}$$

As in the book, we numerically solve for the function $H(r,t)$ using the code `IARS3D.m` via finite differences. We will assume an interest rate model given by

$$dr = udt + \mathrm{vol}\, r dX\,,$$

so the function w is $w=\mathrm{vol}\, r$, and vol is a constant. We also have $u-\lambda w = \alpha - \beta r$ with α and β constants. Following the code presented in the book we define the variables r_f as `fixedrate`, T as `exprry`, $t_{i+1}-t_i$ the time between resets as `period`, `NRS` equal to the number of discretizations of the interest rate r variable, `NTS` equal to number of discretization of the time variable. To complete the finite difference formulation we need to discretize the boundary conditions. We do that now. To evaluate $\frac{\partial H}{\partial t}(r=0,t)$ we write the bond pricing differential equation (in H) as

$$\begin{aligned}
\frac{\partial H}{\partial t} &= -\frac{1}{2}w^2\frac{\partial^2 H}{\partial r^2} - (u-\lambda w)\frac{\partial H}{\partial r} - rH \\
&= -\frac{1}{2}\mathrm{vol}^2 r^2\frac{\partial^2 H}{\partial r^2} - (\alpha+\beta r)\frac{\partial H}{\partial r} - rH\,.
\end{aligned}$$

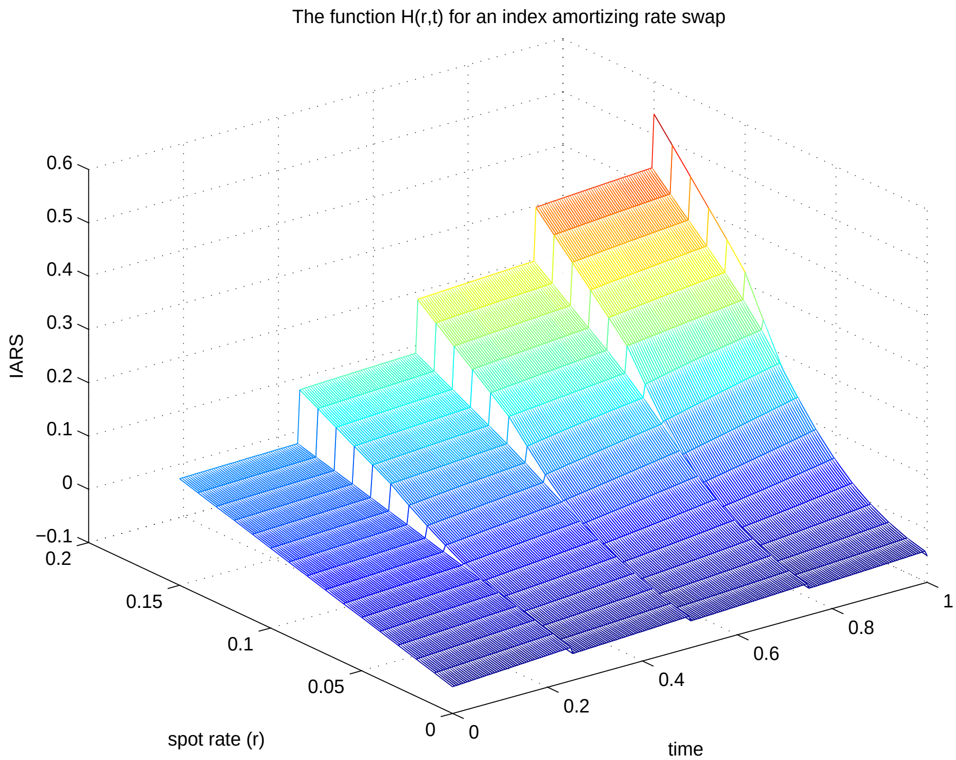

Figure 4: A duplicate of the books figure 38.7.

When $r = 0$ the right-hand-side of this expression becomes $-\alpha \frac{\partial H}{\partial r}$ thus the boundary condition at $r = 0$ is

$$\frac{\partial H}{\partial t}(0, t) = -\alpha \frac{\partial H}{\partial r}(0, t) \,.$$

When we define

$$H_{i,k} = H(i\Delta r, k\Delta t) \,,$$

a finite difference approximation to this boundary condition becomes

$$\frac{H_{0,k+1} - H_{0,k}}{dt} = -\alpha \left(\frac{H_{1,k} - H_{0,k}}{dr} \right) \,,$$

or

$$H_{0,k+1} = H_{0,k} - \alpha \frac{dt}{dr}(H_{1,k} - H_{0,k}) \,.$$

For the boundary condition when $r \to \infty$ we take $\frac{\partial^2 H}{\partial r^2} = 0$ to get

$$H_{NRS,k+1} - 2H_{NRS-1,k+1} + H_{NRS-2,k+1} = 0 \,,$$

which can be solved for $H_{NRS,k+1}$. The jump conditions are done in a similar way. We implement a finite difference scheme for solving this PDE in the MATLAB function `IAR3D.m`. This function calls the amortizing schedule given in the function `amortizing_schedule.m` and is driven with the driving code `IAR3D_driver.m`. When that script is run it produces the plot given in Figure 4. This plot matches quite well the similar plot given in the book.

Chapter 39 (Value of the Firm and the Risk of Default)

Additional Notes on the Text

Notes on Merton's model of equity as a option

In this model we have D, to be the amount of debt that must be payed back at time T and V is the value of the debt now. Consider a portfolio of the current value of the debt V minus some fraction Δ of the share price or

$$\Pi = V - \Delta S\,.$$

This is the portfolio that the bank or debt institution would need to hold to perfectly hedge their risk that the underlying company will not be able to pay back its debts. We consider V as a function of time t and the underlying assets A of the company so that $V = V(t, A)$. Since V is the value of the companies debt at time t and A are the current company assets at the same time we have

$$S = A - V(t, A)\,, \tag{288}$$

so in this model S is a function of t and A also. Then since A evolves according to

$$dA = \mu A dt + \sigma A dX\,, \tag{289}$$

we have $d\Pi$ given by

$$\begin{aligned} d\Pi &= dV - \Delta dS \\ &= \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 A^2 \frac{\partial^2 V}{\partial A^2}\right) dt + \frac{\partial V}{\partial A} dA - \Delta\left[\left(\frac{\partial S}{\partial t} + \frac{1}{2}\sigma^2 A^2 \frac{\partial^2 S}{\partial A^2}\right) dt + \frac{\partial S}{\partial A} dA\right] \\ &= \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 A^2 \frac{\partial^2 V}{\partial A^2} - \Delta\frac{\partial S}{\partial t} - \frac{1}{2}\Delta\sigma^2 A^2 \frac{\partial^2 S}{\partial A^2}\right) dt + \left(\frac{\partial V}{\partial A} - \Delta\frac{\partial S}{\partial A}\right) dA\,. \end{aligned}$$

To make the coefficient of the random term dA equal to zero we pick

$$\Delta = \frac{\frac{\partial V}{\partial A}}{\frac{\partial S}{\partial A}}\,.$$

To avoid arbitrage we set $d\Pi = r\Pi dt$ which gives

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 A^2 \frac{\partial^2 V}{\partial A^2} - rV = \Delta\left[\frac{\partial S}{\partial t} + \frac{1}{2}\sigma^2 A^2 \frac{\partial^2 S}{\partial A^2} - rS\right]\,. \tag{290}$$

Using $S = A - V$ we get

$$\frac{\partial S}{\partial t} = -\frac{\partial V}{\partial t}\,, \quad \frac{\partial S}{\partial A} = 1 - \frac{\partial V}{\partial A}\,, \quad \frac{\partial^2 S}{\partial A^2} = -\frac{\partial^2 V}{\partial A^2}\,, \tag{291}$$

and Δ is given by

$$\Delta = \frac{\frac{\partial V}{\partial A}}{\frac{\partial S}{\partial A}} = \frac{\frac{\partial V}{\partial A}}{1 - \frac{\partial V}{\partial A}}\,.$$

Using these expressions Equation 290 becomes

$$\left(1-\frac{\partial V}{\partial A}\right)\left(\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2A^2\frac{\partial^2V}{\partial A^2}-rV\right)=\frac{\partial V}{\partial A}\left[-\frac{\partial V}{\partial t}-\frac{1}{2}\sigma^2A^2\frac{\partial^2V}{\partial A^2}-rA+rV\right].$$

Canceling terms we get

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2A^2\frac{\partial^2V}{\partial A^2}+rA\frac{\partial V}{\partial A}-rV=0\,,$$

with the final condition of $V(A,T)=\min(D,A)$. If we replace V with expressions in terms of S using Equations 291, we get the same equation as above but with V replaced by S. That is S satisfies exactly the same partial differential equation that V does.

Notes on Merton's model with stochastic interest rates

In this model we introduce a stochastic interest rate r such that

$$dr=u(r,t)dt+w(r,t)dX_1\,,$$

and A still has the differential Equation 289. Now V is a function of A, r, and t and S in terms of V is given by $S=A-V$. Consider the portfolio Π given by

$$\Pi=V(A,r,t)-\Delta S-\Delta' Z(r,t)\,,$$

where hedging is done with the stock with value S and a bond with value Z. The differential of Π is

$$d\Pi=dV-\Delta dS-\Delta' dZ\,.$$

Using $dS=dA-dV$ and Ito's lemma to evaluate dV and dZ we get for $d\Pi$

$$\begin{aligned}
d\Pi &= dV-\Delta(dA-dV)-\Delta' dZ=(1+\Delta)dV-\Delta dA-\Delta' dZ\\
&=(1+\Delta)\left[\frac{\partial V}{\partial t}dt+\frac{\partial V}{\partial A}dA+\frac{\partial V}{\partial r}dt+\frac{1}{2}\left(w^2\frac{\partial^2V}{\partial r^2}+2\rho w\sigma A\frac{\partial^2V}{\partial A\partial r}+\sigma^2A^2\frac{\partial^2V}{\partial A^2}\right)dt\right]\\
&-\Delta dA\\
&-\Delta'\left[\frac{\partial Z}{\partial t}dt+\frac{1}{2}w^2\frac{\partial^2Z}{\partial r^2}dt+\frac{\partial Z}{\partial r}dr\right]\\
&=\left\{(1+\Delta)\left[\frac{\partial V}{\partial t}+\frac{1}{2}\left(w^2\frac{\partial^2V}{\partial r^2}+2\rho w\sigma A\frac{\partial^2V}{\partial A\partial r}+\sigma^2A^2\frac{\partial^2V}{\partial A^2}\right)\right]-\Delta'\left[\frac{\partial Z}{\partial t}+\frac{1}{2}w^2\frac{\partial^2Z}{\partial r^2}\right]\right\}dt\\
&=\left\{(1+\Delta)\frac{\partial V}{\partial A}-\Delta\right\}dA\\
&=\left\{(1+\Delta)\frac{\partial V}{\partial r}-\Delta'\frac{\partial Z}{\partial r}\right\}dr\,.
\end{aligned}$$

If we pick the hedge ratios Δ and Δ' to make both the coefficients of dA and dr equal to zero we must choose

$$(1+\Delta)\frac{\partial V}{\partial A}-\Delta=0\Rightarrow\Delta=\frac{\frac{\partial V}{\partial A}}{1-\frac{\partial V}{\partial A}} \tag{292}$$

$$(1+\Delta)\frac{\partial V}{\partial r}-\Delta'\frac{\partial Z}{\partial r}=0\Rightarrow\Delta'=\frac{\Delta}{\frac{\partial Z}{\partial r}}=\frac{\frac{\partial V}{\partial A}}{(1-\frac{\partial V}{\partial A})\frac{\partial Z}{\partial r}}\,. \tag{293}$$

Once this is done to avoid arbitrage we set the deterministic portfolio change, $d\Pi$, equal to that from a interest bearing account which is $r\Pi dt$. Doing this we get

$$\begin{aligned}
& (1+\Delta)\left[\frac{\partial V}{\partial t}+\frac{1}{2}\left(w^2\frac{\partial^2 V}{\partial r^2}+2\rho w\sigma A\frac{\partial^2 V}{\partial A\partial r}+\sigma^2 A^2\frac{\partial^2 V}{\partial A^2}\right)\right]-\Delta'\left[\frac{\partial Z}{\partial t}+\frac{1}{2}w^2\frac{\partial^2 Z}{\partial r^2}\right] \\
= & \; r[V-\Delta S-\Delta' Z] \\
= & \; r[V-\Delta(A-V)-\Delta' Z] \\
= & \; r[(1+\Delta)V-\Delta A-\Delta' Z]\,.
\end{aligned}$$

When we put the terms that depend on V on the left-hand-side and the terms that depend on Z on the right-hand-side we have

$$\begin{aligned}
& (1+\Delta)\left[\frac{\partial V}{\partial t}+\frac{1}{2}\left(w^2\frac{\partial^2 V}{\partial r^2}+2\rho w\sigma A\frac{\partial^2 V}{\partial A\partial r}+\sigma^2 A^2\frac{\partial^2 V}{\partial A^2}\right)\right]-r\left[(1+\Delta)V-\Delta A\right] \\
= & \; \Delta'\left[\frac{\partial Z}{\partial t}+\frac{1}{2}w^2\frac{\partial^2 Z}{\partial r^2}\right]-r\Delta' Z\,.
\end{aligned}$$

Dividing this by Δ gives

$$\begin{aligned}
& \left(\frac{1}{\Delta}+1\right)\left[\frac{\partial V}{\partial t}+\frac{1}{2}\left(w^2\frac{\partial^2 V}{\partial r^2}+2\rho w\sigma A\frac{\partial^2 V}{\partial A\partial r}+\sigma^2 A^2\frac{\partial^2 V}{\partial A^2}\right)\right]-r\left[\left(\frac{1}{\Delta}+1\right)V-\Delta A\right] \\
= & \; \frac{1}{\frac{\partial Z}{\partial r}}\left\{\left[\frac{\partial Z}{\partial t}+\frac{1}{2}w^2\frac{\partial^2 Z}{\partial r^2}\right]-rZ\right\}\,.
\end{aligned}$$

To further simplify this first, recall that from Equation 196, with $K(r,t)=0$ that the right-hand-side of the above equals $-(u-\lambda w)$. Next from the definition of Δ given in Equation 292 that

$$\frac{1}{\Delta}+1=\frac{1}{\frac{\partial V}{\partial A}}\,.$$

Thus the above equation becomes

$$\frac{1}{\frac{\partial V}{\partial A}}\left[\frac{\partial V}{\partial t}+\frac{1}{2}\left(w^2\frac{\partial^2 V}{\partial r^2}+2\rho w\sigma A\frac{\partial^2 V}{\partial A\partial r}+\sigma^2 A^2\frac{\partial^2 V}{\partial A^2}\right)\right]-r\left[\frac{V}{\frac{\partial V}{\partial A}}-A\right]=-(u-\lambda w)\,.$$

Rearranging this expression some we finally get

$$\frac{\partial V}{\partial t}+\frac{1}{2}\left(w^2\frac{\partial^2 V}{\partial r^2}+2\rho w\sigma A\frac{\partial^2 V}{\partial A\partial r}+\sigma^2 A^2\frac{\partial^2 V}{\partial A^2}\right)+(u-\lambda w)\frac{\partial V}{\partial A}-rV=0\,.$$

This partial differential equation formulation for V is completed with a specification of the final condition on V of

$$V(A,T)=\min(D,A)\,.$$

Notes on Modeling with Measurable Parameters and Variables

We assume that E, the gross annualized earnings, evolves according to geometric Brownian motion $dE=\mu E dt+\sigma E dX$. Here dt is a differential time increment and dX is a stochastic increment. We denote C by the cash in the bank when we immediately invest the profits and is defined in terms of E as

$$C = \int_0^t ((1-k)E(\tau) - E^*)e^{r(t-\tau)}d\tau \,.$$

If we assume that V the value of the company debt is a function of E, t, and C to derive a partial differential equation for V we need to know what is dC, the differential of C. From the above expression, by evaluating the integrand at $\tau = t$ and then taking the partial derivative with respect to t inside the integrand (which just brings down an r) we have

$$\begin{aligned} dC &= ((1-k)E - E^*)dt + r\left(\int_0^t ((1-k)E(\tau) - E^*)e^{r(t-\tau)}d\tau\right) dt \\ &= (((1-k)E - E^*) + rC)dt \,. \end{aligned}$$

Now that we know the dynamic behavior of E and C in that we know the stochastic differential equations that they satisfy i.e.

$$\begin{aligned} dE &= \mu E dt + \sigma E dX \\ dC &= ((1-k)E - E^* + rC)dt \,, \end{aligned}$$

we can follow the discussion in the section "Can I Reverse Engineer a Partial Differential Equation to get the Model and Contract" given in [4] to determine the partial differential equation that $V(E, C, t)$ must satisfy. Namely, we start with a term like

$$\frac{\partial V}{\partial t} \,.$$

Because the stochastic differential equation for E has random term $\sigma E dX$ we will have a second derivative term like

$$\frac{1}{2}(\sigma E)^2 \frac{\partial^2 V}{\partial E^2} \,.$$

Because of the term $\mu E dt$ in dE we have the first derivative term

$$\mu E \frac{\partial V}{\partial E} \,.$$

Because of the term $((1-k)E - E^* + rC)dt$ in dC we have the first derivative term

$$((1-k)E - E^* + rC)\frac{\partial V}{\partial C} \,.$$

Finally due to the no arbitrage condition we have a $-rV$ term. Thus in total we get

$$\frac{\partial V}{\partial t} + \frac{1}{2}(\sigma E)^2 \frac{\partial^2 V}{\partial E^2} + \mu E \frac{\partial V}{\partial E} + ((1-k)E - E^* + rC)\frac{\partial V}{\partial C} + -rV = 0 \,,$$

the same as in the book.

Chapter 40 (Credit Risk)

Additional Notes on the Text

Notes on stochastic risk of default

We start with the value of our risky bond denoted by $V(r,p,t)$ where r is the spot interest rate, p is the instantaneous risk of default (so that in a time of dt we have a probability of default given by pdt) and t is time. We will assume that p and r are stochastic variable that behave as

$$\begin{aligned} dr &= u(r,t)dt + w(r,t)dX_2 & (294)\\ dp &= \gamma(r,p,t)dt + \delta(r,p,t)dX_1\,, & (295)\end{aligned}$$

for two Brownian increments dX_1 and dX_2 and unspecified modeling functions γ, δ, u and w. Note that we would not expect the functions u and w to depend on the value of p. To derive the risk neutral pricing equation for $V(r,p,t)$ consider a portfolio of V hedged with some amount Δ of risk free bonds or

$$\Pi = V(r,p,t) - \Delta Z(r,t)\,.$$

Consider the change in Π in the case of no default (ND) which happens with probability $1-pdt$. Then using Ito's calculus we have

$$\begin{aligned} d\Pi_{\text{ND}} &= \left(\frac{\partial V}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} + \rho\delta w\frac{\partial^2 V}{\partial p\partial r} + \frac{1}{2}\delta^2\frac{\partial^2 V}{\partial p^2}\right)dt + \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial p}dp \\ &- \Delta\left(\left(\frac{\partial Z}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 Z}{\partial r^2}\right)dt + \frac{\partial Z}{\partial r}dr\right)\,. & (296)\end{aligned}$$

We pick the value of the hedge ratio, Δ, to eliminate the dr term thus take

$$\Delta = \frac{\frac{\partial V}{\partial r}}{\frac{\partial Z}{\partial r}}\,.$$

In the case of default (with default or WD), which happens with probability pdt, we loose the entire value of the risky bond and thus

$$d\Pi_{\text{WD}} = -V + O(dt^{1/2})\,. \qquad (297)$$

Then the *expected* change in our portfolio Π over dt is given by

$$\begin{aligned} E[d\Pi] &= d\Pi_{\text{ND}}(1-pdt) + d\Pi_{\text{WD}}\,pdt \\ &= d\Pi_{\text{ND}} - d\Pi_{\text{ND}}\,pdt + d\Pi_{\text{WD}}\,pdt\,.\end{aligned}$$

Now since by our choice of Δ the dr term in $d\Pi_{\text{ND}}$ has vanished so using Equation 296 the product $d\Pi_{\text{ND}}\,pdt$ is

$$\begin{aligned} d\Pi_{\text{ND}}\,pdt &= (\text{stuff})pdt^2 + (\text{stuff})dp\,dt \\ &= (\text{stuff})pdt^2 + (\text{stuff})\gamma dt^2 + (\text{stuff})\delta dt^{3/2}\,,\end{aligned}$$

when we use Equation 295 for dp and the rule of thumb that $E[dX] = dt^{1/2}$. These expressions are all subdominant to dt as $dt \to 0$. Thus we have that to leading order in dt that

$$
\begin{aligned}
E[d\Pi] &= d\Pi_{\mathrm{ND}} + d\Pi_{\mathrm{WD}}\, p dt \\
&= \left(\frac{\partial V}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 V}{\partial r^2} + \rho \delta w \frac{\partial^2 V}{\partial p \partial r} + \frac{1}{2} \delta^2 \frac{\partial^2 V}{\partial p^2} \right) dt \\
&+ \gamma \frac{\partial V}{\partial p} dt - \Delta \left(\frac{\partial Z}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 Z}{\partial r^2} \right) dt \\
&- pV dt\,.
\end{aligned}
$$

Warning: I've replaced the expression dp that multiplies the derivative $\frac{\partial V}{\partial p}$ with γdt but I'm not sure why I can drop the δdX_1 term since dp is governed by Equation 295. If anyone knows why this is an acceptable approximation please let me know.

To avoid arbitrage we set this expression equal to $r\Pi dt$ or

$$
r\,(V - \Delta Z)\, dt\,.
$$

Once we do that we want to put all terms in Z on the left-hand-side and all terms in V on the right-hand-side as

$$
\begin{aligned}
\Delta \left(\frac{\partial Z}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 Z}{\partial r^2} - rZ \right) &= -pV - rV + \gamma \frac{\partial V}{\partial p} \\
&= \frac{\partial V}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 V}{\partial r^2} + \rho \delta w \frac{\partial^2 V}{\partial p \partial r} + \frac{1}{2} \delta^2 \frac{\partial^2 V}{\partial p^2}\,.
\end{aligned}
$$

Recalling the definition of Δ and from the one factor bond pricing Equation 196 with $K = 0$ we have that the left-hand-side of the above is given by

$$
-(u - \lambda w) \frac{\partial V}{\partial r}\,.
$$

Where λ is the market price of interest rate risk. When we combine these two expressions we get the stochastic risky bond pricing equation given by

$$
\frac{\partial V}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 V}{\partial r^2} + \rho \delta w \frac{\partial^2 V}{\partial p \partial r} + \frac{1}{2} \delta^2 \frac{\partial^2 V}{\partial p^2} + (u - \lambda w) \frac{\partial V}{\partial r} + \gamma \frac{\partial V}{\partial p} - (r + p)V = 0\,. \tag{298}
$$

Note from this expression we can immediately get the non-stochastic risky bond pricing equation by taking all $\frac{\partial}{\partial p} = 0$ in the above equation. As a check of our results, if p does not change at all i.e. is a constant then $\gamma = \delta = 0$ and Equation 298 becomes

$$
\frac{\partial V}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 V}{\partial r^2} + (u - \lambda w) \frac{\partial V}{\partial r} - (r + p)V = 0\,.
$$

If we let $V = e^{-p(T-t)} Z(r,t)$ and put this into the above equation we see that it is satisfied and thus when p is constant the solution for V is like a discounted zero risk bond. If we consider the special case where $\rho = 0$ and γ and δ are independent of r then $\gamma = \gamma(p,t)$ and $\delta = \delta(p,t)$ then Equation 298 gives

$$
\frac{\partial V}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 V}{\partial r^2} + \frac{1}{2} \delta(p,t)^2 \frac{\partial^2 V}{\partial p^2} + (u - \lambda w) \frac{\partial V}{\partial r} + \gamma(p,t) \frac{\partial V}{\partial p} - (r + p)V = 0\,.
$$

In this equation we can look for a separable solution of the form $V(r,p,t) = Z(r,t)H(p,t)$. We find the needed derivatives of V given by

$$\begin{aligned}
\frac{\partial V}{\partial t} &= \frac{\partial Z}{\partial t} H + Z\frac{\partial H}{\partial t} \\
\frac{\partial V}{\partial r} &= H\frac{\partial Z}{\partial r} \quad \text{and} \quad \frac{\partial^2 V}{\partial r^2} = H\frac{\partial^2 Z}{\partial r^2} \\
\frac{\partial V}{\partial p} &= Z\frac{\partial H}{\partial p} \quad \text{and} \quad \frac{\partial^2 V}{\partial p^2} = Z\frac{\partial^2 H}{\partial p^2} \,.
\end{aligned}$$

Then using these in the partial differential equation above we find that

$$H\frac{\partial Z}{\partial t} + Z\frac{\partial H}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 Z}{\partial r^2}H + \frac{1}{2}\delta^2\frac{\partial^2 H}{\partial p^2}Z + (u-\lambda w)\frac{\partial Z}{\partial r}H + \gamma\frac{\partial H}{\partial p}Z - (r+p)ZH = 0\,.$$

Lets group the first, third, fifth, and part of the seventh term together on one side to get

$$H\left[\frac{\partial Z}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 Z}{\partial r^2} + (u-\lambda w)\frac{\partial Z}{\partial r} - rZ\right] = Z\left[\frac{\partial H}{\partial t} + \frac{1}{2}\delta^2\frac{\partial^2 H}{\partial p^2} + \gamma\frac{\partial H}{\partial p}Z - pH\right]\,.$$

The left-hand-side of this expression is zero via Equation 196 the pricing equation for a risk less bond and we get that H must satisfy

$$\frac{\partial H}{\partial t} + \frac{1}{2}\delta^2\frac{\partial^2 H}{\partial p^2} + \gamma\frac{\partial H}{\partial p}Z - pH = 0\,.$$

In case we have a positive recovery meaning that if the risky bond defaults we get some payment. We can model this by taking

$$d\Pi_{\text{WD}} = -V + Q\,.$$

Now this $d\Pi_{\text{WD}}$ term enters as a $d\Pi_{\text{WD}}pdt$ the partial differential equation obtained in this case will have a term pQ in the equation.

Notes on hedging the default

In this section we study if we can use another risky bond V_1 to hedge the risk of default. To study that consider the portfolio Π long one risky bond, short Δ risk free bonds, and short Δ_1 risky bonds V_1 given by

$$\Pi = V - \Delta Z - \Delta_1 V_1\,.$$

We assume that $V = V(r,p,t)$ and $V_1 = V_1(r,p,t)$ and we assume that p is a *constant* then when both bonds do *not* default we have that our portfolio changes as

$$\begin{aligned}
d\Pi_{\text{ND}} &= \left(\frac{\partial V}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2}\right)dt + \frac{\partial V}{\partial r}dr \qquad (299) \\
&- \Delta\left[\left(\frac{\partial Z}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 Z}{\partial r^2}\right)dt + \frac{\partial Z}{\partial r}dr\right] - \Delta_1\left[\left(\frac{\partial V_1}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V_1}{\partial r^2}\right)dt + \frac{\partial V_1}{\partial r}dr\right]\,.
\end{aligned}$$

In case both bonds default (we are assuming that they are highly correlated) we have

$$d\Pi_{\text{WD}} = -V - \Delta_1(-V_1) = -V + \Delta_1 V_1 \,.$$

We pick Δ_1 such that $d\Pi_{\text{WD}} = 0$ or in other words to completely eliminate the default risk. This means we take

$$\Delta_1 = \frac{V}{V_1} \,.$$

Next we compute the expected portfolio change in value as

$$E[d\Pi] = d\Pi_{\text{ND}}(1 - pdt) + d\Pi_{\text{WD}}(pdt) = d\Pi_{\text{ND}}(1 - pdt) \,.$$

Since our choice for Δ_1 made the expression $d\Pi_{\text{WD}}$ vanish. As argued earlier the term $d\Pi_{\text{ND}}(pdt)$ is subdominant to $d\Pi_{\text{ND}}$ we can drop its contribution to simply get

$$E[d\Pi] = d\Pi_{\text{ND}} \,.$$

Pick Δ to eliminate dr term in Equation 299 to get

$$\frac{\partial V}{\partial r} - \Delta \frac{\partial Z}{\partial r} - \Delta_1 \frac{\partial V_1}{\partial r} = 0 \,.$$

or

$$\Delta = \frac{\frac{\partial V}{\partial r} - \Delta_1 \frac{\partial V_1}{\partial r}}{\frac{\partial Z}{\partial r}} = \frac{V_1 \frac{\partial V}{\partial r} - V \frac{\partial V_1}{\partial r}}{V_1 \frac{\partial Z}{\partial r}} \,. \tag{300}$$

Thus with these two choices we have

$$E[d\Pi] = d\Pi_{\text{ND}} = \left(\frac{\partial V}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 V}{\partial r^2} - \Delta \left(\frac{\partial Z}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 Z}{\partial r^2} \right) - \Delta_1 \left(\frac{\partial V_1}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 V_1}{\partial r^2} \right) \right) dt \,.$$

To avoid arbitrage we set this equal to $r\Pi dt = r(V - \Delta Z - \Delta_1 V_1)dt$ and get

$$\frac{\partial V}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 V}{\partial r^2} - rV - \Delta \left(\frac{\partial Z}{\partial t} + \frac{1}{2} w^2 \frac{\partial Z}{\partial r} - rZ \right) - \Delta_1 \left(\frac{\partial V_1}{\partial t} + \frac{1}{2} w^2 \frac{\partial V_1}{\partial r} - rV_1 \right) = 0 \,.$$

Since $Z(r,t)$ must satisfy the one factor bond pricing Equation 196 with $K = 0$ we have that

$$\frac{\partial Z}{\partial t} + \frac{1}{2} w^2 \frac{\partial Z}{\partial r} - rZ = -(u - \lambda w) \frac{\partial Z}{\partial r} \,,$$

the expression multiplying Δ when we use the expression for Δ in Equation 300 becomes

$$\frac{\partial V}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 V}{\partial r^2} - rV + \left[\frac{\partial V}{\partial r} - \Delta_1 \frac{\partial V_1}{\partial r} \right] (u - \lambda w) - \Delta_1 \left[\frac{\partial V_1}{\partial t} + \frac{1}{2} w^2 \frac{\partial V_1}{\partial r} - rV_1 \right] = 0 \,.$$

or if we put terms with V on one side of the equation and terms with V_1 on the other side of the equation then we have

$$\frac{\partial V}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 V}{\partial r^2} - rV + (u - \lambda w) \frac{\partial V}{\partial r} = \Delta_1 \left[\frac{\partial V_1}{\partial t} + \frac{1}{2} w^2 \frac{\partial V_1}{\partial r} - rV_1 + (u - \lambda w) \frac{\partial V_1}{\partial r} \right] \,.$$

If we multiply this equation by $\frac{1}{V}$ then the left-hand-side is a function of V while the right-hand-side is a function of V_1. The only way this is possible is if both sides equal a constant. We denote this constant as $\lambda_1(r, p, t)$ to get

$$\frac{\partial V}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 V}{\partial r^2} + (u - \lambda w) \frac{\partial V}{\partial r} - (r + \lambda_1(r, p, t))V = 0 \,.$$

The book seems to then take the expression for the λ_1 constant to be linear in p and writes it as $\lambda_1(r, t)p$.

Notes on the forward equation

Since we have $P(0,T) = e^{TQ}$ if we diagonalize Q as $Q = MDM^{-1}$ then

$$P(0,T) = M(e^{TD})M^{-1}\,.$$

This means that M also diagonalizes $P(0,T)$, thus to compute Q we take the given $P(0,T)$ matrix and diagonalize it as

$$P(0,T) = MD'M^{-1}\,.$$

Then set $D' = e^{TD}$. Since both sides are diagonal matrices we can compute the elements of D by taking logarithms of the elements of D'. Once we have the matrices M and D as above we can form Q as

$$Q = MDM^{-1}\,.$$

Notes on the backwards equation

Consider the expression for $P(t-dt,t')$, which represents a transition from the times $t-dt$ to t'. Break this up into two transitions; one from $t-dt$ to t and the other from t to t'. Then keeping the order of the matrices consistent we have doing these two steps that

$$P(t-dt,t') = P(t-dt,t)P(t,t') = P_{dt}P(t,t') = (I+dtQ)P(t,t')\,,$$

or

$$\frac{P(t,t') - P(t-dt,t')}{dt} = -QP(t,t')\,.$$

In the limit that $dt \to 0$ we get

$$\frac{\partial P(t,t')}{\partial t} = -QP(t,t')\,, \tag{301}$$

the backwards equation.

Chapter 41 (Credit Derivatives)

Additional Notes on the Text

Notes on default only when payment is due

Here V is the price/value of the credit derivative and is a function of p the instantaneous probability of default p and time t so $V = V(p, t)$. Consider a portfolio of V hedged with Δ amount of the risky bonds Z^* or

$$\Pi = V - \Delta Z^* .$$

Here Z^* is the market traded risky bond (that might default). Then consider $d\Pi$ we have

$$d\Pi = dV - \Delta dZ^* .$$

Then from the definition of $p = \frac{\log(Z/Z^*)}{T-t}$ we have that

$$Z^* = Ze^{-p(T-t)} .$$

If we assume that the riskless bond has a value $Z = e^{-r(T-t)}$ then this becomes

$$Z^* = e^{-(r+p)(T-t)} .$$

We further assume that p satisfies the stochastic differential equation

$$dP = \mu dt + \sigma dX .$$

Then by Ito's lemma dV is given by

$$dV = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial p} dp + \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial p^2} dt ,$$

and dZ^* is given by

$$\begin{aligned} dZ^* &= \frac{\partial Z^*}{\partial t} dt + \frac{\partial Z^*}{\partial p} dp + \frac{1}{2}\frac{\partial^2 Z^*}{\partial p^2} dp^2 \\ &= pZ^* dt - (T-t)Z^* dp + \frac{1}{2}(T-t)^2 Z^* \sigma^2 dt . \end{aligned}$$

So using the expressions for dV and dZ^* we find $d\Pi$ given by

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial p^2} \right) dt + \frac{\partial V}{\partial p} dp + \Delta \left(pZ^* + \frac{1}{2}(T-t)^2 Z^* \sigma^2 \right) dt - \Delta(T-t)Z^* dp .$$

If we pick Δ to make the dp terms vanish we need to take

$$\Delta = \frac{\frac{\partial V}{\partial p}}{(T-t)Z^*} = \frac{\frac{\partial V}{\partial p}}{(T-t)Ze^{-p(T-t)}} = \frac{\frac{\partial V}{\partial p}}{(T-t)e^{-(r+p)(T-t)}} , \qquad (302)$$

when we take the risk free bond price Z to be $Z = e^{-r(T-t)}$. We then set $d\Pi$ equal to $r\Pi dt = r(V - \Delta Z^*)dt$ to avoid arbitrage and bring all terms to one side to get

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial p^2} - rV - \Delta \left(pZ^* + \frac{1}{2}\sigma^2 (T-t)^2 Z^* + rZ^* \right) = 0 \,.$$

From the fact that $Z^* = e^{-(r+p)(T-t)}$ and the form for Δ given by Equation 302 we have that the product ΔZ^* is given by

$$\Delta Z^* = \frac{1}{T-t} \frac{\partial V}{\partial p} \,,$$

and we get for the pricing equation for V the following

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial p^2} + \left[\frac{p+r}{T-t} + \frac{1}{2}\sigma^2 (T-t) \right] \frac{\partial V}{\partial p} - rV = 0 \,. \tag{303}$$

Warning: In the above expression I have $\frac{p+r}{T-t}$ rather than $\frac{p}{T-t}$ as the book has. If anyone sees anything wrong with this derivation please let me know.

Chapter 43 (CrashMetrics)

Additional Notes on the Text

Notes on CrashMetrics for one stock

We assume that we have a portfolio Π of options on one stock with underlying price S. Then the change in the portfolio $\delta\Pi$ when the stock changes by δS is given by a function $F(\cdot)$ as

$$\delta\Pi = F(\delta S)\,.$$

We assume that we can expand $\delta\Pi$ in a second order Taylor expansion and find

$$\delta\Pi = \Delta\delta S + \frac{1}{2}\Gamma\delta S^2 + \cdots\,, \tag{304}$$

since $F(0) = 0$. We can then ask the question as to what is the worst change to our portfolio over a fixed time frame. Under the assumption that the change in asset price δS over this fixed time frame is bounded as

$$-\delta S^- < \delta S < \delta S^+\,, \tag{305}$$

the value we want to find is

$$\min_{-\delta S^- < \delta S < \delta S^+} F(\delta S)\,.$$

To minimize this we take the first derivative and set it equal to zero as

$$\frac{d(\delta\Pi)}{d(\delta S)} = \Delta + \Gamma\delta S = 0 \quad \text{or} \quad \delta S = -\frac{\Delta}{\Gamma}\,. \tag{306}$$

Checking the second derivative of $F(\delta S)$ of the given Taylor series approximation we find

$$\frac{d^2(\delta\Pi)}{d(\delta S)^2} = \Delta > 0\,,$$

showing that the δS found above does indeed give a minimum of $F(\delta S)$. Denoting this value by δS_{worst} we find the worst value for $\delta\Pi$ in this case and when δS_{worst} satisfies Equation 305 is given by

$$\delta\Pi_{\text{worst}} = -\frac{\Delta^2}{\Gamma} + \frac{1}{2}\Gamma\left(\frac{\Delta^2}{\Gamma^2}\right) = -\frac{\Delta^2}{2\Gamma}\,.$$

Notes on portfolio optimization and the platinum hedge

We now assume that we wish to hedge the risk to changes in the value of our portfolio by buying some number (λ) of a hedge instrument with assumed first derivative Δ^* and second derivative Γ^*. After the addition of these λ of the hedge contracts the total portfolio has a first order exposure to movements in S of

$$\delta S(\Delta + \lambda\Delta^*)\,,$$

and a second order exposure to movements in S of

$$\frac{1}{2}\delta S^2(\Gamma + \lambda\Gamma^*)\,.$$

When we add in a fixed trading cost of $|\lambda|C$ (here C is in units of *dollars* and represents the bid-offer spread) we have that $\delta\Pi$ becomes

$$\delta\Pi = \delta S(\Delta + \lambda\Delta^*) + \frac{1}{2}\delta S^2(\Gamma + \lambda\Gamma^*) - |\lambda|C\,. \tag{307}$$

It is this expression we now want to consider with respect to the possible values we could select for λ. The book defines the **Platinum Hedge** as the selection of λ that results in the best possible hedge. To determine how this is computed, we imagine that we have already selected a value for λ and then consider the worst possible outcome that could happen to our portfolio. The worst possible outcome is for the market to select a value of δS such that Equation 307 is as negative (small) as possible. We then desire to pick λ such that the *minimum* of Equation 307 is as *large* as possible. This is a bit similar to minmax problems in game theory.

Thus given a value of λ the from the equations derived earlier in this chapter the market will pick the value of δS such that our $\delta\Pi$ is as small as possible. From Equation 306 this value would be

$$\delta S_{\text{worst}} = -\frac{\Delta + \lambda\Delta^*}{\Gamma + \lambda\Gamma^*}\,.$$

If this value for δS is not within the bounds given by Equation 305 we will not actually suffer such a loss and the minimum we suffer will come from one of the end points of the domain (either $-\delta S^-$ or $+\delta S^+$). If this value of δS_{worst} is valid, when we put this into Equation 307 our portfolio suffers a loss of amount

$$\delta\Pi_{\text{worst}} = -\frac{(\Delta + \lambda\Delta^*)^2}{2(\Gamma + \lambda\Gamma^*)} - |\lambda|C\,. \tag{308}$$

The platinum hedge then picks λ such that we *maximize* the expression for $\delta\Pi_{\text{worst}}$, as a function of λ. For the greeks and boundary given in the book of

$$\begin{aligned}
\Delta &= 10\\
\Gamma &= 400\\
\Delta^* &= 0.5\\
\Gamma^* &= 5\\
C &= 0.002\\
-\delta S^- &= -0.05 \quad \text{and} \quad +\delta S^+ = +0.05\,,
\end{aligned}$$

the function $\delta\Pi_{\text{worst}}(\lambda)$ is plotted computed and plotted in the Matlab/Octave scripts `deltaPi_vs_lambda.m` and `plot_deltaPi.m` and is shown as a function of λ in Figure 5.

Notes on the multi-asset/single-index model

To verify our understanding of this section in the python code `crash_coefficient_kappa.py` we estimate a given equities crash coefficient κ. It has a parameter, `num_of_extreme_moves`,

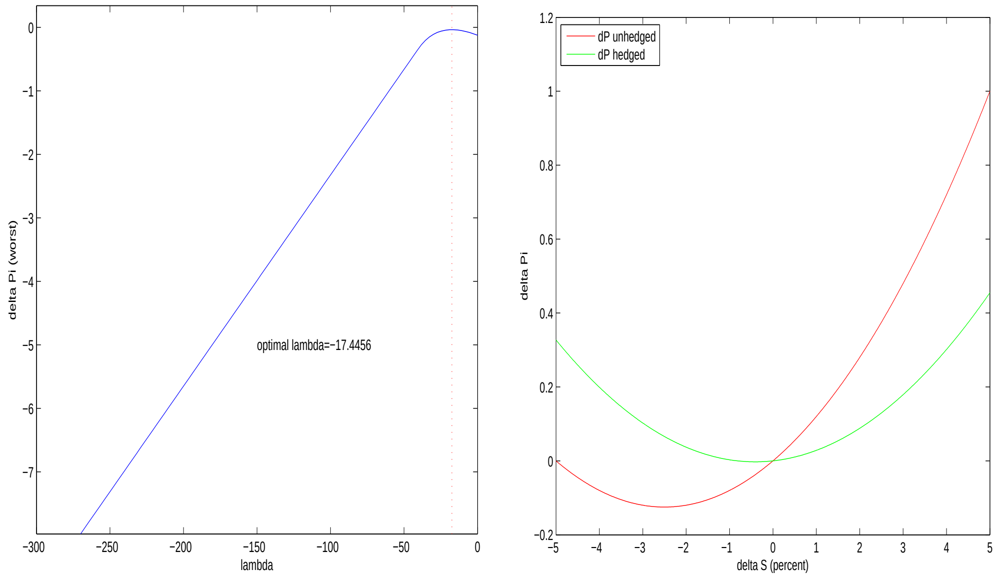

Figure 5: **Left:** A plot of Equation 308 as a function of λ. The λ corresponding to the maximum gives the optimal hedge. **Right:** A plot of the corresponding portfolio change $\delta\Pi$ as a function of δS for an unhedged and a hedged portfolio.

that determines the number of extreme return points to use in estimating κ. Taking the value of this parameter to be a very large number results in using all of the return points and the estimated κ is an estimate of the CAPM's β. An example of the fits this routine produces is shown in Figure 6, where we estimate the CAPM β and the crash coefficient κ for Disney `DIS` against the `SPX` index.

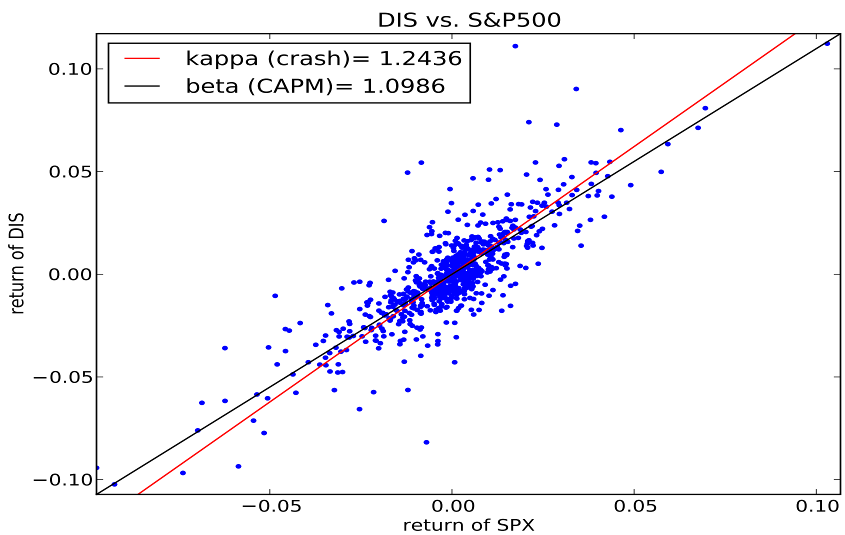

Figure 6: Estimated linear fits $y = \kappa x$ and $y = \beta x$ where κ for the returns of the stock `DIS` vs is the returns of the `SPX` index. Here κ is the crash index and β is the CAPM coefficient. Note that the numerical estimate of κ is *larger* than that of β as one would expect from their definitions.

Chapter 50: Deterministic Volatility Surfaces

Notes on Backing Out the Local Volatility Surfaces

We find $\frac{\partial V}{\partial T}$ given by using the expression for $V(E,T)$

$$
\begin{aligned}
\frac{\partial V}{\partial T} &= -re^{-r(T-t^*)}\int_E^\infty (S-E)p(S^*,t^*;S,T)dS \\
&+ e^{-r(T-t^*)}\int_E^\infty (S-E)\frac{\partial p}{\partial T}dS \\
&= -rV + e^{-r(T-t^*)}\int_E^\infty (S-E)\frac{\partial p}{\partial T}dS\,.
\end{aligned}
$$

Using the Fokker-Plank equation of

$$
\frac{\partial p}{\partial T} = \frac{1}{2}\frac{\partial^2}{\partial S^2}(\sigma^2 S^2 p) - \frac{\partial}{\partial S}(rSp)\,, \tag{309}
$$

for the time derivative (and then integrating by parts) we have

$$
\begin{aligned}
\frac{\partial V}{\partial T} &= -rV + e^{-r(T-t^*)}\int_E^\infty (S-E)\left[\frac{1}{2}\frac{\partial^2}{\partial T^2}(\sigma^2 S^2 p) - \frac{\partial}{\partial S}(rSp)\right]dS \\
&= -rV + e^{-r(T-t^*)}\left[\frac{1}{2}(S-E)\frac{\partial}{\partial S}(\sigma^2 S^2 p)\Big|_E^\infty - \frac{1}{2}\int_E^\infty \frac{\partial}{\partial S}(\sigma^2 S^2 p)dS\right. \\
&- \left.(S-E)rSp|_E^\infty + \int_E^\infty rSpdS\right] \\
&- rV + e^{-r(T-t^*)}\left[-\frac{1}{2}\sigma^2 S^2 p\Big|_E^\infty + \int_E^\infty rSpdS\right] \\
&= -rV + e^{-r(T-t^*)}\left[\frac{1}{2}\sigma^2 E^2 p(S^*,t^*;E,T) + r\int_E^\infty SpdS\right]\,,
\end{aligned}
$$

which is Equation 25.5. Now writing $\int_E^\infty SpdS$ as

$$
\begin{aligned}
\int_E^\infty SpdS &= \int_E^\infty (S-E)pdS + E\int_E^\infty pdS \\
&= Ve^{r(T-t^*)}E\left(-e^{r(T-t^*)}\frac{\partial V}{\partial E}\right)\,,
\end{aligned}
$$

so that

$$
\begin{aligned}
\frac{\partial V}{\partial T} &= -rV + \frac{1}{2}e^{-r(T-t^*)}\sigma^2 E^2 p \\
&+ re^{-r(T-t^*)}\left(Ve^{r(T-t^*)} - Ee^{r(T-t^*)}\frac{\partial V}{\partial E}\right) \\
&= \frac{1}{2}e^{-r(T-t^*)}\sigma^2 E^2 e^{r(T-t^*)}\frac{\partial^2 V}{\partial E^2} - rE\frac{\partial V}{\partial E}\,.
\end{aligned}
$$

or

$$
\frac{\partial V}{\partial T} = \frac{1}{2}\sigma^2 E^2 \frac{\partial^2 V}{\partial E^2} - rE\frac{\partial V}{\partial E}\,. \tag{310}
$$

which is the equation in the book.

Appendix A: Mathematical Notes

Elementary Methods for Solving Differential Equations

Given the inhomogeneous linear first-order differential equation

$$\frac{dy}{dt} + p(t)y = g(t)\,, \tag{311}$$

In this section of these notes we review how to solve this differential equation. See [1], for more details. The solution is given by

$$y(t) = \frac{1}{\mu(t)}\left[\int \mu(t')g(t')dt' + C\right]\,,$$

where C is an arbitrary constant specified to fit the initial condition on $y(t)$ and the "integrating factor" $\mu(t)$ is defined as

$$\mu(t) = \exp\left(\int p(t')dt'\right)\,. \tag{312}$$

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[3] S. Ross. *A First Course in Probability*. Macmillan, 3rd edition, 1988.

[4] P. Wilmott. *Frequently Asked Questions in Quantitative Finance (Wiley Series in Financial Engineering)*. Wiley, January 2007.