Solutions to Problems from: A Primer On Integral Equations of the First Kind by G. Milton Wing

John Weatherwax

Chapter 1 (An Introduction to the Basic Problem)

Problem 1 (examples of integral equations)

There are a great number of integral equations in the physical sciences. Some examples of are

Problem 2 ()

Greens functions are defined as

$$\mathcal{D}(G(r;r_0)) = \delta(r - r_0) \tag{1}$$

with $G(r;r_0)$ the Greens functions and $\mathcal{D}$ is a linear differential operator. The solution to

$$\mathcal{D}(y(r)) = g(r) \tag{2}$$

is given by

$$y = \int G(r;r_0)g(r_0)dr_0 \tag{3}$$

Thus in the setting of Fredholm integral equations of the first kind, $G(r;r_0)$ is the *kernel* of the integral operator.

Chapter 2

Problem 1

From the discussion in that problem we will assume that the probability of a particle reaching $x + \Delta$ is given by $p(x)p(\Delta)$. Thus these two events (going from 0 to x and from x to $x + \Delta$) are considered independent.

It is reasonable to approximate the probability that a particle survives while traveling a small distance Δ in an absorbing media by the expression

$p(\Delta) = (1 - k\Delta)$. Combining these two hypothesis we have

$$p(x + \Delta) = p(x)p(\Delta) = p(x)(1 - k\Delta) \tag{4}$$

This can be manipulate into

$$\frac{p(x + \Delta) - p(x)}{\Delta} = -kp(x) \tag{5}$$

The left hand side of which can be recognized as begin a discrete approximation to the derivative. Taking the limit as $\Delta \to \infty$ we see that we have $dp/dx = -kp$ the solution of which is given by

$$p(x) = pe^{-kx} \tag{6}$$

Comparing this expression that given on page 9 of the textbook we recognize that $k = \sigma$.

Problem 2

Defining $f(E)$ as a particle density (as a function of energy E) means that $f(E)dE$ are the *number* of particles with energy between E and $E + dE$. Thus the fraction of particles that make it to x are given by the number of particles between E and $E + dE$ times the probability that they make it to x or,

$$dN(E) = e^{-\sigma(E)x} f(E)dE\,. \tag{7}$$

To compute the total number of particle that make it to x (or $g(x)$) we integrate this expression obtaining

$$g(x) = \int_{E_{\min}}^{E_{\max}} e^{-\sigma(E)x} f(E)dE \tag{8}$$

Problem 3

This can basically be seen as follows

FIGURE GOES HERE!!!

In the region denoted by the asterisk we see that effectively the function $\sigma = \sigma(E)$ is not invertible. This means that given σ_0 such that $\sigma_L < \sigma_0 < \sigma_R$ no unique $E = \sigma^{-1}(\sigma_0)$ exists. Thus

$$g(x) = \int_{E_{\min}}^{E_{\max}} e^{-\sigma(E)x} f(E)dE \tag{9}$$

can not have a unique inverse $f(\cdot)$. To see this assume that given $g(\cdot)$ a unique $f(\cdot)$ existed, then any $\hat{f}(E)$ that has its functional values exchanged on the subintervals A, B, and C will produce the same output function $g(\cdot)$. As an example consider $f_1(E)$, $f_2(E)$, and $f_3(E)$ as follows. All 3 of these functions have the same integral when integrated against $e^{-\sigma(E)x}$ with respect to E.

Problem 4

Since σ (the absorption cross section of the material) is a function of the density which in tern is a function of the radius to the center of the sphere. We expect σ to to be a function of r only. Consider the variable z to be centered directly along the center of the sphere and we desire the intensity along this one dimensional axis. This is we are not solving the problem when the z-axis if off set by some amount relative to the axis of the sphere. Then looking from the sided the projection of the sphere is

INSERT FIGURE HERE!!!

with each cell absorbing an amount $\sigma(r(s))$. Thus $dA(z) = \sigma(r(s))ds$ and $z^2 + s^2 = r^2$, so $r = \sqrt{z^2 + s^2}$ and we have

$$dA(z) = \sigma(\sqrt{z^2 + s^2})ds \tag{10}$$

Therefore $A(z)$ is given by

$$A(z) = 2\int_{s=0}^{\sqrt{R^2-z^2}} \sigma(\sqrt{z^2 + s^2})ds \tag{11}$$

is the total scattering cross section. It follows that the intensity is given by $I(z) = e^{A(z)}$ and is given by Eq. 2.5 in the text.

Problem 5

To transform

$$I(z) = e^{-2\int_0^{\sqrt{R^2-z^2}} \sigma(\sqrt{z^2+s^2})ds} \quad \text{for} \quad 0 \le z \le R \tag{12}$$

into the desired form we will perform a series of variable transformations. The first of which will be

$$y = z^2 + s^2 \quad \text{with} \tag{13}$$

$$dy = 2sds \quad \text{or} \tag{14}$$

$$ds = \frac{dy}{2\sqrt{y - z^2}} \tag{15}$$

With this substitution the limits of $s = 0$ and $s = \sqrt{R^2 - z^2}$ become $y = z^2$ and $y = R^2$ respectively so the expression for $I(z)$ becomes

$$\begin{aligned} I(z) &= e^{-2\frac{1}{2}\int_{z^2}^{R^2} \sigma(\sqrt{y})\frac{dy}{\sqrt{y-z^2}}} && (16) \\ &= e^{-\int_{z^2}^{R^2} \frac{\sigma(\sqrt{y})}{\sqrt{y-z^2}}dy} && (17) \\ &= e^{\int_{R^2}^{z^2} \frac{\sigma(\sqrt{y})}{\sqrt{y-z^2}}dy} && (18) \\ &= e^{\int_{R^2}^{z^2} \frac{\sigma(\sqrt{y})}{\sqrt{-z^2+y}}dy} && (19) \end{aligned}$$

As our second substitution let $v = -y$, so $dv = -dy$ and the above integral becomes

$$I(z) = \exp\{-\int_{-R^2}^{-z^2} \frac{\sigma(\sqrt{-v})dv}{\sqrt{-z^2 - v}}\} \qquad (20)$$

Continuing our transformations, we define a transformation on x of $x = -z^2$. This gives in the above

$$\begin{aligned} I &= \exp\{-\int_{-R^2}^{x} \frac{\sigma(\sqrt{-v})}{\sqrt{x-v}}dv\} && (21) \\ &= \exp\{-\int_{0}^{x+R^2} \frac{\sigma(\sqrt{R^2-v})}{R^2+x-v}dv\} && (22) \end{aligned}$$

Where for the last step we have used the simple identity

$$\int_a^b f(x)dx = \int_{a+c}^{b+c} f(x-c)dx$$

Finally, letting $x_2 = R^2 + x$ and the above becomes

$$I = \exp\{-\int_0^{x_2} \frac{\sigma(\sqrt{R^2-v})}{\sqrt{x_2 - v}}dv\} \qquad (23)$$

which is the expression desired.

Problem 6

Now $g(x) = \int_0^x k(x-y)f(y)dy$ and it can be shown that the Laplace transform of $g(x)$ (i.e. $G(s)$) is related to the Laplace transform of $f(y)$ and $k(y)$ (i.e. $K(s)$ and $F(s)$) with

$$G(s) = K(s)F(s) \qquad (24)$$

Then given $G(s)$ formally the source $f(x)$ is given by

$$f(x) = \mathcal{L}^{-1}\{\frac{G(s)}{K(s)}\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yx}\frac{g(x)}{k(x)}dx \tag{25}$$

with $\mathcal{L}^{-1}$ the inverse Laplace transform. From the Post and Widder formula (equation 2.26 in the book) and the discussion on page 8, this can be related to to the kth derivative of $\frac{g(x)}{k(x)}$ which if not known exactly will certainly suffer from numerical difficulties.

Problem 7

If a measurement of the energy g is available at x it must be the superposition of released neutrons from all points y on the rod. Since the number of released neutrons in a small interval dy is given by $f(y)dy$ the number reaching point x is given by (assuming a constant absorption cross section σ) as

$$e^{-\sigma|x-y|}f(y)dy \tag{26}$$

Since we desire the total sum of all such contributions from sources at $f(y)dy$ we sum (integrate) to obtain the total strength $g(x)$

$$g(x) = \int_0^b e^{-\sigma|x-y|}f(y)dy \tag{27}$$

or the desired equation.

Problem 8

We can express $g(x)$ as follows

$$g(x) = \int_0^b e^{-\sigma|x-y|}f(y)dy = \int_0^x e^{-\sigma(x-y)}f(y)dy + \int_x^b e^{\sigma(x-y)}f(y)dy \tag{28}$$

This second expression explicitly removes the absolute values signs from the exponents in the integrand by restricting the range of integration to regions where the sign of that expression is known. From this expression we have all analytic functions and we can evaluate the derivative of $g(x)$. Taking the first derivative we obtain

$$g'(x) = e^{-\sigma(x-x)}f(x) + \int_0^x (-\sigma)e^{-\sigma(x-y)} - f(x) + \int_x^b \sigma e^{\sigma(x-y)}f(y)dy = f(x) \tag{29}$$

Canceling the function $f(x)$ and taking the second derivative gives

$$g''(x) = -\sigma f(x) + (-\sigma)^2 \int_0^x e^{-\sigma(x-y)} f(y)dy - \sigma f(x) + \sigma^2 \int_x^b e^{\sigma(x-y)} f(y)dy \quad (30)$$

$$= -2\sigma f(y) + \sigma^2 \left(\int_0^x e^{-\sigma(x-y)} f(y)dy + \int_x^b e^{\sigma(x-y)} f(y)dy \right) \quad (31)$$

Since the terms in the parenthesis is $g(x)$ the differential equation it satisfies is

$$-2\sigma f(x) = g''(x) - \sigma^2 g(x) \quad (32)$$

so $f(x)$ is given by

$$f(x) = \frac{1}{2}\sigma g(x) - \frac{1}{2\sigma} g''(x) \quad (33)$$

Problem 9

Using the results of section 2.6 with

$$\log(I(\rho,\psi)) = I_0(\rho) + \sum_{k=1}^{\infty} I_{1,k}(\rho)\cos(k\psi) + I_{2,k}(\rho)\sin(k\psi) \quad (34)$$

with

$$I_{i,k}(\rho) = 2\int_\rho^R T_k(\frac{\rho}{r}) \frac{r}{\sqrt{r^2-\rho^2}} \sigma_{i,k}(r)dr \quad (35)$$

for $i = 1, 2$ and $k = 0, 1, 2, \ldots$. Note that this is the correct denominator for this problem, see equation 2.8 in the book. Now if the objects cross section is independent of angle ψ we have

$$\frac{dI(\rho,\psi)}{d\psi} = 0 \quad (36)$$

which implies that

$$I_{i,k} = 0 \quad (37)$$

Now for $I_0(\rho)$ we have

$$I_0(\rho) = 2\int_\rho^R T_0(\frac{\rho}{r}) \frac{r}{\sqrt{r^2-\rho^2}} \sigma_0(r)dr = \log(I(\rho)) \quad (38)$$

since $T_0(x) = 1$ and $\sigma_0(r) = \sigma(r)$ we obtain

$$\log(I(\rho)) = 2\int_\rho^R \frac{r\sigma(r)}{\sqrt{r^2-\rho^2}} dr \quad (39)$$

The following sequence of transformations transforms this integral into the desired form. First let

$$r = \sqrt{\rho^2 + s^2} \quad \text{with} \tag{40}$$

$$dr = \frac{sds}{\sqrt{\rho^2 + s^2}} \tag{41}$$

here s is the new integration variable. With this definition the argument of the square root above then becomes

$$r^2 - \rho^2 = \rho^2 + s^2 - \rho^2 = s^2$$

and the endpoints of the integration $r = \rho$ and $r = R$ become $s = 0$ and $s = \sqrt{R^2 - \rho^2}$ respectively. After transformation the new integral is

$$\log(I(\rho)) = 2 \int_0^{\sqrt{R^2-\rho^2}} \sigma(\sqrt{\rho^2 + s^2}) ds \tag{42}$$

This is identical to equation 2.5 from the book with the exception the the sign of equation 2.5 is negative.

Problem 10

Equation 2.2 b in the text expresses the fact that if

$$g(y) = \int_0^\infty e^{-xy} f(y) dy \tag{43}$$

then

$$f(y) = \lim_{k \to \infty} \{ \frac{(-1)^k}{k!} g^{(k)} \left(\frac{k}{y}\right) \left(\frac{k}{y}\right)^{k+1} \} \tag{44}$$

Now we have the following derivatives of the given $g(y)$

$$g(y) = y^{-2} \tag{45}$$

$$g^{(1)}(y) = -2y^{-3} \tag{46}$$

$$g^{(2)}(y) = (-1)^2\, 3!\, y^{-4} \tag{47}$$

$$g^{(3)}(y) = (-1)^3\, 4!\, y^{-5} \tag{48}$$

By induction in general we have

$$g^{(k)}(y) = (-1)^k\, (k+1)!\, y^{-(k+2)} \tag{49}$$

Evaluating $g^{(k)}(\cdot)$ at k/y we obtain

$$g^{(k)}\left(\frac{k}{y}\right) = (-1)^k\,(k+1)!\,\left(\frac{y}{k}\right)^{k+2} \tag{50}$$

So for this function the Post-Widder formula would give an inverse Laplace transform of

$$\begin{aligned} f(y) &= \lim_{k\to\infty}\{\frac{(-1)^k}{k!}(-1)^k\,(k+1)!\,\left(\frac{y}{k}\right)^{k+2}\left(\frac{k}{y}\right)^{k+1}\} && (51)\\ &= \lim_{k\to\infty}\{(k+1)\frac{y}{k}\} = y && (52)\end{aligned}$$

As the second example if $g(y) = y^{-1}$ we have (using induction to compute the derivative in general)

$$\begin{aligned} g(y) &= y^{-1} && (53)\\ g^{(1)}(y) &= -1y^{-2} && (54)\\ g^{(2)}(y) &= (-1)^2\,2!\,y^{-3} && (55)\\ g^{(3)}(y) &= (-1)^3\,3!\,y^{-4} && (56)\end{aligned}$$

In general we have

$$g^{(k)}(y) = (-1)^k\,k!\,y^{-(k+1)} \tag{57}$$

so

$$f(y) = \lim_{k\to\infty}\{\frac{(-1)^k}{k!}(-1)^k\,k!\left(\frac{y}{k}\right)^{k+1}\left(\frac{k}{y}\right)^{k+1}\} = 1 \tag{58}$$

For the general case we calculate

$$\begin{aligned} g(y) &= y^{-p} \qquad \text{for} \qquad p \geq 1 && (59)\\ g^{(1)}(y) &= -py^{-p-1} && (60)\\ g^{(2)}(y) &= -p(-p-1)y^{-p-2} && (61)\\ g^{(3)}(y) &= (-1)^3 p(p+1)(p+2)y^{-p-3} && (62)\\ &\vdots && (63)\\ g^{(k)}(y) &= (-1)^k p(p+1)(p+2)\ldots(p+k-1)y^{-p-k} && (64)\end{aligned}$$

so the Post-Widder formula then gives

$$f(y) = \lim_{k\to\infty}\{\frac{(-1)^k}{k!}(-1)^k p(p+1)\ldots(p+k-1)\left(\frac{y}{k}\right)^{p+k}\left(\frac{k}{y}\right)^{k+1}\} \tag{65}$$

or

$$y^{p-1} \lim_{k\to\infty} \frac{p(p+1)(p+2)\dots(p+k-1)}{k!k^{p-1}} \tag{66}$$

To evaluate this, consider the expression we are taking the limit of. It can be seen to be

$$\frac{p(p+1)(p+2)\dots(p+k-1)}{k!k^{p-1}} \tag{67}$$

which equals

$$\frac{(p-1)!p(p+1)(p+2)\dots(p+k-1)}{(p-1)!\,k!k^{p-1}} \tag{68}$$

or

$$\frac{1}{(p-1)!}\left[\frac{(k+p-1)!}{k!k^{p-1}}\right] \tag{69}$$

or

$$\frac{1}{(p-1)!}\left[\frac{(k+1)(k+2)\dots(k+p-2)(k+p-1)}{k^{p-1}}\right] \tag{70}$$

or

$$\frac{1}{(p-1)!}\left[\left(1+\frac{1}{k}\right)\left(1+\frac{2}{k}\right)\left(1+\frac{3}{k}\right)\dots\left(1+\frac{p-2}{k}\right)\left(1+\frac{p-1}{k}\right)\right] \tag{71}$$

Now since for *fixed* p each term in the above brackets goes to one we finally obtain the final expression for $f(y)$ of

$$f(y) = \frac{y^{p-1}}{(p-1)!} \tag{72}$$

Problem 11

Equation 2.14 is

$$N(t) = N(0)e^{-\lambda t} + \int_0^t (r(y)+s(y))e^{-\lambda(t-y)}dy \tag{73}$$

First let $N(t)$ be the number of large fish and $n(t)$ be the number of fry (small fish) in the lake/pond. Then the differential equation satisfied by N and n are given by

$$\frac{dn(t)}{dt} = s(t) \tag{74}$$

$$\frac{dN(t)}{dt} = r(t)n(t) - \lambda N(t) \tag{75}$$

With $N(0)$ known. This system of differential equations express the fact that the maturation of small fish to large fish occurs at a rate $r(t)$ and that the mortality rate of large fish is λ. The solution to the first equation is given by

$$n(t) = n(0) + \int_0^t s(t')dt' \quad (76)$$

which put into the second equation gives

$$\frac{dN(t)}{dt} = r(t)\left[n(0) + \int_0^t s(t')dt'\right] - \lambda N(t) \quad (77)$$

or

$$\frac{dN(t)}{dt} + \lambda N(t) = r(t)\left[n(0) + \int_0^t s(t')dt'\right]. \quad (78)$$

Since the number of fry fish is initially zero $n(0) = 0$ the above equation simplifies to

$$\frac{dN}{dt} + \lambda N(t) = r(t)S(t). \quad (79)$$

Where we have defined $S(t) = \int_0^t s(t')dt'$. Multiplying both sides by (the integrating factor for this equation) of $e^{\lambda t}$ we obtain

$$\frac{d}{dt}(e^{\lambda t}N(t)) = r(t)S(t)e^{\lambda t}. \quad (80)$$

Which can be integrated yielding

$$e^{\lambda t}N(t) = \int_0^t r(t')S(t')e^{\lambda t'}dt' + N(0). \quad (81)$$

I think that this is a better method for evaluating these population biology questions.

As an alternative model to the above, the model that Wing is considering looks like the following (from Eq. 2.14 in the book):

$$\frac{dN}{dt} = -\lambda N(t) + r(t) + s(t). \quad (82)$$

In the above expression we have the following terms

- $-\lambda N(t)$ is the instantaneous number of deaths of the large fish, i.e. λ is the morality rate for large fish.
- $r(t)$ is the reproduction/propagation rate for large fish.

- $s(t)$ is the introduction of fry fish (assumed instantaneously to mature into large fish).

The function $r(t)$ is something like the reproduction rate of the fish species under consideration and is ultimately of great interest in determining how strongly these fish will breed.

Part (b): This is the situation described earlier. Adding, a mortality rate to the small fish of λ_S (with corresponding mortality rate of λ_L for the large fish) we obtain

$$\begin{aligned}\frac{dN_S(t)}{dt} &= -\lambda_S N_S(t) + s(t) - \beta_{SL} N_S(t) + r_{LS} N_L(t) \quad (83)\\ \frac{dN_L(t)}{dt} &= -\lambda_L N_L(t) + \beta_{SL} N_S(t)\,. \quad (84)\end{aligned}$$

In the above, we have defined β_{SL} to be the fraction of the small fish that become large in one timestep. We are also assuming that $r_{LS}(t)$ is the reproduction rate (rate at which small fish are produced from large fish).

Problem 12

Equation 2.22 in the book is

$$T\sqrt{2g} = \int_0^y \frac{H'(\eta)d\eta}{\sqrt{y-\eta}}\,. \quad (85)$$

If we can write this equation in the form recognized by Able i.e. like

$$J(x) = \int_0^x \frac{\Sigma(y)}{\sqrt{x-y}}dy \quad \text{for} \quad 0 \le x \le b \quad (86)$$

then this Volterra equation of the first kind has an explicit form for its solution given by

$$\Sigma(y) = \frac{1}{\pi}\frac{d}{dy}\int_0^y \frac{J(x)}{\sqrt{y-x}}dx = \frac{1}{\pi}\left\{\frac{J(0)}{\sqrt{y}} + \int_0^y (y-x)^{-1/2} J'(x)dx\right\}\,. \quad (87)$$

In the case consided here since T is independent of x and y the above simplifies to

$$H'(\eta) = \frac{1}{\pi}\left\{\frac{T\sqrt{2g}}{\sqrt{\eta}}\right\} \quad (88)$$

since

$$\frac{d}{dx}(T\sqrt{2g}) = 0 \quad (89)$$

so

$$H'(y) = \frac{T\sqrt{2g}}{\pi\sqrt{y}} \tag{90}$$

and since from the definition of $H(y)$ we have

$$H'(y)^2 = \left(\frac{ds}{dy}\right)^2 = \frac{dx^2 + dy^2}{dy^2} = 1 + \left(\frac{dx}{dy}\right)^2 \tag{91}$$

we get

$$1 + \left(\frac{dx}{dy}\right)^2 = \frac{2T^2 g}{\pi^2 y} \tag{92}$$

or

$$\frac{dx}{dy} = \pm\sqrt{\frac{2T^2 g}{\pi^2 y} - 1}\,. \tag{93}$$

Defining the constant C_1 as $C_1 = \frac{2T^2 g}{\pi^2}$ we obtain

$$dx = \pm\sqrt{\frac{C_1}{y} - 1}\,dy \tag{94}$$

Which can be found in many integral tables and equals...

Problem 13

Part (a): Equation 2.22 is Able's integral equation given by

$$J(x) = \int_0^x \frac{\Sigma(y)}{\sqrt{x-y}}dy \quad \text{for} \quad 0 \le x \le b \tag{95}$$

then this Volterra equation of the first kind has an explicit solution of

$$\Sigma(y) = \frac{1}{\pi}\frac{d}{dy}\int_0^y \frac{J(x)}{\sqrt{y-x}}dx = \frac{1}{\pi}\left\{\frac{J(0)}{\sqrt{y}} + \int_0^y (y-x)^{-1/2} J'(x)dx\right\} \tag{96}$$

If T is no longer constant but depends on y then we obtain

$$H'(\eta) = \frac{1}{\pi}\left\{\frac{\tilde{T}(0)\sqrt{2g}}{\sqrt{\eta}} + \int_0^\eta (\eta - x)^{-1/2}\tilde{T}'(x)dx\right\}\,. \tag{97}$$

Part (b): If $\tilde{T}(y) = T + \epsilon(y)$ the above becomes

$$\begin{aligned} H'(y) &= \frac{1}{\pi}\left\{\frac{(T+\epsilon(0))\sqrt{2g}}{\sqrt{y}} + \int_0^y (y-x)^{-1/2}\epsilon'(x)dx\right\} && (98)\\ &= \frac{1}{\pi}\left\{\frac{T\sqrt{2g}}{\sqrt{y}} + \frac{\epsilon(0)\sqrt{2g}}{\sqrt{y}} + \int_0^y (y-x)^{-1/2}\epsilon'(x)dx\right\} && (99)\end{aligned}$$

This represents a perturbed version of that from problem 12.

Problem 14

$$\tilde{g}(x) = \int_{-x}^{+x} \frac{T_{2n}(\frac{y}{x})\tilde{f}(y)}{\sqrt{x^2 - y^2}} dy \tag{100}$$

Assume that $\tilde{f}(y)$ solves this problem and consider $\tilde{f}(y) + Cy^k$. If Cy^k is in the nullity of this integral equation i.e. is mapped to the zero element by the following operator

$$\mathcal{K}(\cdot) = \int_{-x}^{+x} \frac{T_{2n}(\frac{y}{x})(\cdot)}{\sqrt{x^2 - y^2}} dy \tag{101}$$

then the above equation does not have a unique solution. Consider the operator $mathcalK$ operating on y^k. As such we obtain,

$$\int_{-x}^{+x} \frac{T_{2n}(\frac{x}{y})y^k dy}{\sqrt{x^2 - y^2}} . \tag{102}$$

To evaluate this integral, let $v = \frac{y}{x}$ then $dv = \frac{dy}{x}$ and the above integral becomes

$$\int_{-1}^{+1} \frac{T_{2n}(v)x^k v^k x dv}{\sqrt{x^2 - x^2 v^2}} = x^k \int_{-1}^{+1} \frac{T_{2n}(v)v^k dv}{\sqrt{1 - v^2}} = 0 \quad \text{if} \quad k < 2n \tag{103}$$

by the orthogonality of the Chebyshev polynomials.

Chapter 3

Problem 1

Part (a) If $h(x) = \frac{\sin(x)}{x}$ with $-\pi \leq x \leq \pi$, then the sup norm is given by

$$||h||_\infty = \text{Sup}|h(x)| = \text{Sup}|\frac{\sin(x)}{x}| \tag{104}$$

with the understanding that the supremum is taken over the range $-\pi \leq x \leq \pi$. From elementary calculus we have $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$, and this supremum is given by 1.

The one norm of $h(x)$ is defined by

$$||h||_1 = \int_{-\pi}^{\pi} |h(x)|dx = \int_{-\pi}^{\pi} |\frac{\sin(x)}{x}|dx = 2\int_0^{\pi} |\frac{\sin(x)}{x}|dx \tag{105}$$

This norm does not exist since this integral does not exist do to the $O(1/x)$ singularity at $x = 0$.

The two norm if h is given by

$$||h||_2 = \int_{-\pi}^{\pi} h(x)^2 dx = \int_{-\pi}^{\pi} (\frac{\sin(x)}{x})^2 dx = 2\int_0^{\pi} (\frac{\sin(x)}{x})^2 dx \tag{106}$$

Again this norm does not exist since the integrand has a non-integrable singularity at $x = 0$.

Part (b) If $h(x) = \frac{\cos(x)}{\sqrt{x}}$ with $0 \le x \le 2\pi$, then the sup norm is given by

$$||h||_\infty = \text{Sup}|h(x)| = \text{Sup}|\frac{\cos(x)}{\sqrt{x}}| \tag{107}$$

This norm does not exist since the function is unbounded at $x = 0$. The one norm of $h(x)$ is defined by

$$||h||_1 = \int_0^{2\pi} |h(x)|dx = \int_0^{2\pi} |\frac{\cos(x)}{\sqrt{x}}|dx \tag{108}$$

This integral is difficult to evaluate analytically, but we can upper bound it (and the corresponding norm on $h(x)$) as follows

$$\int_0^{2\pi} |\frac{\cos(x)}{\sqrt{x}}|dx < \int_0^{2\pi} \frac{1}{\sqrt{x}} dx = 2x^{1/2}|_0^{2\pi} = 4\sqrt{2\pi} \tag{109}$$

Now the two norm if h is given by

$$||h||_2 = \int_0^{2\pi} h(x)^2 dx = \int_0^{2\pi} (\frac{\cos(x)}{\sqrt{x}})^2 dx = \int_0^{2\pi} (\frac{\cos(x)^2}{x})dx \tag{110}$$

This norm does not exist since the integrand has a non-integrable singularity at $x = 0$.

Problem 6