IDENTIFICATION AND CHARACTERIZATION OF A MOBILE SOURCE IN A GENERAL PARABOLIC DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS*

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Abstract. We discuss an inverse source problem for a general parabolic differential equation in $\mathbb{R}^n \times \mathbb{R}_+$ with constant coefficients and a source whose strength and support may vary with time. We demonstrate that a knowledge of the solution on any bounded open set $\mathcal{M}$ in $\mathbb{R}^n$ located away from the source for any fixed time $T \geq 0$ determines the so-called "carrier support" [as originally defined in the article, Notions of support for far fields, J. Sylvester, Inverse Problems, pp. 1273-1288, vol. 22, 2006] (a nontrivial subset of the support of the true source) at that coincident time. Additionally, we provide a reconstruction algorithm which can locate the time-varying position of the carrier support of the assumed unknown source with extremely few discrete (possibly nonuniform) measurements taken on such an open set over a wide range of regularity classes of the source. Lastly, we provide a few numerical examples which illustrate the efficacy and robustness of this location and tracking method.

Key words. Inverse problems, diffusion, partial differential equations

AMS subject classifications. 49N30, 58J35, 35K10

1. Introduction. Remote sensing endeavors, especially those in connection with modern defense and industrial quality control applications, have significantly evolved in recent years in both their technologies and the realistic scenarios they address. In particular, the release of toxic or impure substances into an environment of interest, urban or otherwise, by either intentional or unintentional means has become an outstanding contemporary problem of considerable and immediate importance and consequence. Recently, many promising technologies, such as those presented in [12, 7, 4, 6, 14, 13], have been created which can remotely detect the presence of foreign materials in a region of interest and estimate their concentration as a function of position and time, provided one has ample knowledge of the diffusion field on either large spatial measurment sets, or over long periods of time, or both. Clearly, however, we can never hope to simultaneously monitor vast regions of space in many real world settings, nor can we tolerate the need for long periods of meaurement time, or indefinite ones for that matter. Hence, there is an immediate need to possess the capability to quickly detect and determine the location(s) and output strength(s) of life-threatening, or otherwise destructive, sources with extremely limited measurements in space and time of such diffusing substances.

An immediate extension of the work developed in [10] and more recently in [16] is made in this article that efficiently treats the problem of determining the location of a (potentially mobile) source in a generalized advection-diffusion environment with exteremely limited real world resources. For instance, the technology lends itself extremely well to the problem of the airborne release of a life-threatening substance above ground or in an underground subway system. Specifically, in n-dimensional

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space, if the concentration of the diffusing substance can be measured on a small, coarsely sampled or highly distributed, n-dimensional array of sensors, at a single snap-shot in time, then we can robustly, and expeditiously, determine the location of the source with tremendous accuracy in the presence of considerable measurment noise. Hence, with a collection of such time snap-shots of measurement data, we further show that we are able to locate the time-varying position of a moving source and track its current location. Such a capability is of tremendous value to time-sustained source release problems across a variety of scenarios.

We begin the analytical treatment of this problem by considering the general nonhomogeneous second order parabolic partial differential equation

$$(\partial_t - L_x)u(x,t) = f(x,t), \quad u(x,0) = 0, \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}_+, \quad n \geq 1 \tag{1.1}$$

where we define the elliptic operator L_x as

$$L_x := \sum_{i,j=1}^{n} a_{i,j}\partial_{x_i}\partial_{x_j} + \sum_{j=1}^{n} b_j \partial_{x_j} + c,$$

which governs such things as the molecular diffusion of gases, or particulates, generated by the autonomous source f throughout $x \in \mathbb{R}^n$ and over time $t \in \mathbb{R}_+$. For the purposes of clarity we will limit ourselves to the treatment of the case where L_x has constant coefficients. We note; however, that much of the following analysis and framework suits the more general case of coefficients which at least vary with position. This is in fact the aim of future work.

In the treatment to follow we will assume that the source may be decomposed into the product of a temporally dependent function $s \geq 0$ with a potentially spatially dependent and mobile one $g \geq 0$, such that

$$f(x,t) = g(x - \gamma(t))s(t) \geq 0 \quad \forall\ (x,t) \in \mathbb{R}^n \times \mathbb{R}_+$$

where, a priori, g is assumed to be compactly supported for each time t within the domain $B_R(p)$, i.e., the ball of radius R and center p, and $\gamma : [0,T] \to \mathbb{R}^n$, $T \geq 0$. Moreover, we assume that the structure of the strength function s is such that s identically vanishes for values of $t < 0$ and takes on the form of regular, or possibly singular distribution, for values of $t \geq 0$, e.g., the Dirac-delta distribution or the Heaviside function.

Characterization of the source amounts to determining the (possibly time-varying) carrier support – which we will define in detail to follow, and for more detail we refer to [16] – of the source f, which we will assume to be strictly positive for this article. The concept of the carrier support generalizes that of the so-called scattering support which was originally defined and analyzed in much detail in [10]. In short, for any differential operator, such as $P = \partial_t - L_x$, which admits the concept of the unique continuation principle (UCP)[1] the carrier support of the measured field u on $\mathcal{M}$ at time T in the differential equation $Pu = f$, where f is compactly supported, is that subset of the support of f such that there exists an equivalent source $\tilde{f} = \tilde{g}\tilde{s}$ (with $\tilde{f}$ residing in the same regularity class as f, and where $\tilde{f} \neq f$ in the sense of distributions) in the sense that $Pu = \tilde{f}$ everywhere on the complement of the support of f. We summarize this concept with the following definition.

[1]Suppose $Pu = 0$ in some domain $\mathcal{V}$. Then, if u restricted to an open subset $\mathcal{M} \subset \mathcal{V}$ vanishes, the UCP implies that u vanishes throughout the larger domain $\mathcal{V}$.

Definition 1.1. *(Carrier Support) Let P be a differential operator which admits the UCP such that $Pu = f$ on $\mathcal{V}$ and* $\operatorname{supp} f \subset \Omega \subsetneq \mathcal{V}$*, and let P possess the fundamental solution E_P. Then, for some open set $\mathcal{M} \subset \mathcal{V}$, where $\overline{\mathcal{M}} \cap \overline{\Omega} = \emptyset$, and $\tau \leq T$,*

$$\operatorname{carr\,supp} u(\cdot, T)|_{\mathcal{M}} := \bigcap_{E_P * \tilde{f} = u(\cdot, T)|_{\mathcal{M}}} \operatorname{ch\,supp} \tilde{f}(\cdot, \tau), \quad 0 \leq \tau \leq T,$$

where ch denotes the convex hull.

In summary, this definition states that the carrier support of the solution u restricted to the set $\mathcal{M}$ is that common set over all possible sets such that there exists a compactly supported source away from $\mathcal{M}$ such that u on $\mathcal{M}$ may be generated by such a candidate source $\tilde{f}$, i.e., $P\tilde{u} = \tilde{f}$ on $\mathcal{V}$ and $\tilde{u}$ agrees with u on $\mathcal{V} \backslash \operatorname{ch\,supp} f$.

We should also remark that this definition implies that there exists the possibility that the source generating the data u on $\mathcal{M}$ could have existed previous to time T, i.e., $T \geq \tau$. This means that the solution as observed on $\mathcal{M}$ is that of a now-extinct source that last existed at time τ whose support was last on $\operatorname{supp} \tilde{g}(\cdot, \tau)$. This may be summarized through the following (identity) example. Let $H_\tau(t)$ denote the Heaviside function[2], and suppose for $0 \leq \tau_1 < \tau_2$,

$$f(x,t) = \delta_{\gamma(t)}(x) \left(H_{\tau_1}(t) - H_{\tau_2}(t)\right)$$

moves along the trajectory $\gamma : [0, T] \to \mathbb{R}^n$. Then,

$$\operatorname{carr\,supp} u(\cdot, t)|_{\mathcal{M}} = \begin{cases} \emptyset, & t < \tau_1 \\ \operatorname{supp} \delta_{\gamma(t)}(x), & \tau_1 \leq t \leq \tau_2 \\ \operatorname{supp} \delta_{\gamma(\tau_2)}(x), & t \geq \tau_2 \end{cases}.$$

It should be mentioned that this is a very desirable phenomenon, in that detection of the location of an impulse-like source released at time $\tau \geq 0$ can be made with the observed data u on $\mathcal{M}$ at time $t \geq \tau$. This amounts to the ability to track mobile time-sustained sources and determine the point of detonation of impulse-like ones; both of which are invaluable capabilities for a variety of modern and future endeavors across many disciplines and industries.

We note that even though – as we shall come to prove in the following section – we can determine the current location, or that final one when the source's strength was last nonzero, with a snapshot of the current diffusion field on $\mathcal{M}$, we cannot reconstruct the entirety of γ over all times subsequent to t. In particular this means we may analytically extend the solution u of the homogeneous equation $(\partial_t - L_x)u = 0$ back to time t^* at which minimal time u was the solution of a nonhomogeneous equation of the form $(\partial_t - L_x)u = \tilde{f} \neq 0$. Again, we shall revisit this concept in further detail and provide all the necessary technical arguments which support this notion in the following section, and in the proof of our main result (Theorem 3.2) in section 3.

[2]For a test function $\phi \in C^\infty(\mathbb{R}^n \times \mathbb{R})$, for $\delta_{\gamma(t)} H_\tau \in \mathcal{E}'(\mathbb{R}^n) \otimes \mathcal{D}'(\mathbb{R})$, we define the action of the distributional pairing

$$\langle \delta_{\gamma(t)} H_\tau, \phi \rangle = \int_\tau^\infty \int_{\mathbb{R}_x} \phi(x,t) \delta(x - \gamma(t)) dx dt.$$

Several results have been presented to date which can determine the actual support of the source and its strength function s as a function of time, see for example [12, 7, 4, 6]. The fundamental difference in these methods and that to follow in this article is that we only require knowledge of the scalar diffusion field at single snapshots in time on some (possibly small) open subset of the ambient space $\mathbb{R}^n$ which we will assume to be disjointly located from the source in question. Since we require far less information than that required in these previous works, we might expect to fail to fully characterize the source in all its attributes. We shall come to see that this is indeed the case, and that what we can estimate with this limited information is that of the carrier support of the instantaneous support of the true source f at each point $T \geq 0$ in time. We accomplish this by following a strategy similar in nature to the one presented in the articles [10, 11]. Essentially, we employ a unique continuation-like strategy for the assumed positive solution which with the Picard Theorem, gives us a way to uniquely determine the carrier support at any time T. We will describe these ideas in much further detail in what follows in a subsequent section.

We again stress the importance of having measurements on some limited finite domain located away from the source, and which need not surround the source, which serves such applications as source-release problems in complex urban environments, and atmospheric or reservoir problems. More importantly, if the source is moving, we wish to determine its trajectory, and current location, for purposes of perhaps its rapid neutralization, i.e., the source is emitting toxic or impure substances into a domain of interest.

We will make several assumptions on the nature of the coefficients appearing in L_x and of that of the bounded open set $\mathcal{M} \subset \mathbb{R}^n$ where will assume to know the scalar values of the time-varying field $u(x,t)$, for instance we require $\mathcal{M}$ to have a smooth boundary for the purposes of maintaining well-behaved norms of the solution on such sets of interest. Additionally, we will define a few function spaces of interest in which our solutions of the main problem will uniquely exist.

Remark 1. *In the analysis to follow in the upcoming section, we will assume that the following conditions hold:*

- $\partial_t - L_x$ *is parabolic on* $\mathbb{R}^n \times \mathbb{R}$, *i.e.,*

$$\sum_{i,j=1}^{n} a_{i,j}\xi_i\xi_j > 0, \quad \mathbb{R} \ni \xi_i, \xi_j \neq 0$$

- $c \leq 0$
- *the matrix of coefficients* $a_{i,j}$ *is positive definite and invertible*

Remark 2. *Additionally, for* $-n/2 < \sigma_1 \leq 0$, *and* $-1/2 < \sigma_2 \leq 0$, *we define the function spaces*[3]

$$\overset{o}{H}{}^{\sigma_1}_{+}(\overline{\Omega}) = \{g \in H^{\sigma_1}(\mathbb{R}^n) \,:\, g \geq 0, \quad \operatorname{supp} g \subset \overline{\Omega}\},$$

where H^{σ} *is the usual Sobolev space of regularity* $\sigma \in \mathbb{R}$, *and similarly*

$$H^{\sigma_1}_{+}(\mathbb{R}) = \{g \in H^{\sigma_1}(\mathbb{R}) \,:\, g \geq 0\}.$$

Using these conventions we define the positive space of sources, i.e., distributions,

$$f \in \overset{o}{H}{}^{\sigma_1}_{+}(\overline{\Omega}) \bigotimes \overset{o}{H}{}^{\sigma_2}_{+}([0,T])$$

[3] We wish to include singular temporal and spatial distributions such as the Dirac-delta distribution in the larger collection of positive sources, hence we are interested in taking $0 \geq \sigma_1 > -n/2$, and $0 \geq \sigma_2 > -1/2$.

and the space of restricted solutions

$$u|_{\mathcal{M}} \in L^2_+\left([0,T], L^2_+(\mathcal{M})\right)$$

where

$$L^2_+(\mathbb{R}^n) = \{g \in L^2(\mathbb{R}^n) \; : \; g \geq 0\}.$$

In order to minimize symbolic clutter, and ease notation a bit, we make the following identification

$$X_f^{\sigma_1,\sigma_2}(\Omega, T) = \overset{o}{H}{}^{\sigma_1}_+(\overline{\Omega}) \bigotimes \overset{o}{H}{}^{\sigma_2}_+([0,T])$$

and note that its dual space admits the representation

$$\left(X_f^{\sigma_1,\sigma_2}\right)' = H_+^{-\sigma_1}(\mathbb{R}^n) \bigotimes H_+^{-\sigma_2}(\mathbb{R}_+)$$

for any finite T and bounded $\mathcal{M}$. Finally, we note that since $0 \leq -\sigma_1 < n/2$ and $0 \leq -\sigma_2 < 1/2$, by (complex) interpolation, i.e., see for instance page 277 of [17]

$$[L^2(\mathbb{R}^n), H^k(\mathbb{R}^n)]_\theta = H^{k\theta}(\mathbb{R}^n), \quad k = 0, 1, 2, \cdots, \quad 0 \leq \theta \leq 1$$

we have the inclusion

$$H_+^n(\mathbb{R}^n) \bigotimes H_+^1(\mathbb{R}_+) \subset \left(X_f^{\sigma_1,\sigma_2}\right)' \subset L_+^2(\mathbb{R}^n) \bigotimes L_+^2(\mathbb{R}_+).$$

This last fact will be important for us in characterizing the behavior of the solution in the following section on the forward problem.

The plan of the remainder of the paper is as follows. In section two we discuss the forward problem. Namely, we describe how given a well-defined source f generates the solution u and develop the appropriate mapping characterizations between the two. In the following third section, we focus on the inverse source problem, which again, is to locate the support of f from measurements of the diffusion field u taken on some open region which is disjoint and distant from the assumed unknown source f. In this section we both prove uniqueness results for the time-varying reconstruction of the carrier support of f as well as developing a viable reconstruction method which estimates it with little, sparse and possibly nonuniformly sampled data. Section four considers a few numerical examples which robustly illustrate the simplicity and effectiveness of this reconstruction algorithm for a spatially stationary, impulse-like point-source release in one dimension and a time-sustained, moving point source in a convective two dimensional environment.

2. The Forward Problem. We begin this section by noting that the solution of (1.1) is well known and may be constructed with the aid of a fundamental solution which we will call Z, see [5, 3] for the original details of this parametrix-based method. Spefically, let $A = \det a_{i,j}$ and $a^{i,j}$ be the determinant and matrix inverse of the positive definite matrix a respectively. Then,

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n_y} Z(x,y,t,\tau) f(y,\tau) dy d\tau, \tag{2.1}$$

where

$$Z(x,y,t,\tau) = W(x,y,t,\tau) + \int_{\tau}^{t}\int_{\mathbb{R}^n_z} W(x,z,t,s)\Phi(z,y,s,\tau)dzds$$

and where we define

$$W(x,y,t,\tau) = [(4\pi(t-\tau))^n A]^{-1/2}\exp\left(-\sum_{i,j=1}^{n}\frac{a^{i,j}(x_i-y_i)(x_j-y_j)}{4(t-\tau)}\right)$$

and lastly require that Φ satisfy

$$\Phi(x,y,t,\tau) = (L_x-\partial_t)W(x,y,t,\tau) + \int_{\tau}^{t}\int_{\mathbb{R}^n_z}(L_x-\partial_t)W(x,z,t,s)\Phi(z,y,s,\tau)dzds.$$

For convenience we will denote the action of Z on f as simply $\mathcal{Z}f$. Furthermore, we will denote the restriction of $\mathcal{Z}$ to observations on $\mathcal{M}$ and limited to sources f having compact support on Ω as $\mathcal{Z}|_{(\mathcal{M},\Omega)}$ in what is to follow. Additionally, we note that we will write $Z(x-y,t-\tau)$ in the place of $Z(x,y,t,\tau)$ when it appears in the context of the kernel of the convolution integral which maps the source f to the solution u.

We recall some important properties established in [5]. We use them to prove the following boundedness and denseness result.

Proposition 2.1. *(Local Boundedness of the Solution on $\mathcal{M}$) Given the assumptions detailed earlier in remarks 1 and 2, then for each $T \geq 0$, $\mathcal{Z} : X_f^{\sigma_1,\sigma_2}(\Omega,T) \to L^2_{+,loc}(\mathbb{R}^n\backslash\overline{\Omega})$ and has dense range in the latter space.*

Proof. First, for $x \in \mathbb{R}^n\backslash\overline{\Omega}$,

$$\begin{aligned}
|u(\cdot,T)|^2 &= \left|\int_0^T\int_\Omega Z(x-y,T-\tau)f(y,\tau)dyd\tau\right|^2\\
&= \left|\langle Z,f\rangle_{L^2(\Omega\times[0,T])}\right|^2\\
&\leq \|Z(x-\cdot,T-\cdot)\|^2_{\left(X_f^{\sigma_1,\sigma_2}\right)'}\|f\|^2_{X_f^{\sigma_1,\sigma_2}(\Omega,T)}\\
&\leq \|Z(x-\cdot,T-\cdot)\|^2_{\left(X_f^{n/2,1/2}\right)'}\|f\|^2_{X_f^{\sigma_1,\sigma_2}(\Omega,T)}\\
&\leq \|Z(x-\cdot,T-\cdot)\|^2_{\left(X_f^{n,1}\right)'}\|f\|^2_{X_f^{\sigma_1\cdot\sigma_2}(\Omega,T)}.
\end{aligned}$$

Then, according to the Malgrange-Ehrenpreis theorem, the remainder of the proof follows immediately from the fact that $Z(x-\cdot,T-\cdot)$ is smooth on $\mathcal{M}$, i.e., that

$$\int_K \|Z(x-\cdot,T-\cdot)\|^2_{\left(X_f^{n,1}\right)'}dx < \infty$$

for all compact subsets K in $\mathbb{R}^n\backslash\overline{\Omega}$.

We now employ some supporting facts of interest discussed and proven in chapter 1, and theorems 1 and 15, of [5] which help us to establish the claim that $\mathcal{Z}$ as a map from $X_f^{\sigma_1,\sigma_2}(\Omega,T)$ into $L^2_{+,loc}(\mathbb{R}^n\backslash\overline{\Omega})$ has dense range in $L^2_{+,loc}(\mathbb{R}^n\backslash\overline{\Omega})$. We first note, as given on page 28 of [5], that Z and its adjoint are related through the identity

$$Z(x,y,t,\tau) = Z^*(y,x,\tau,t), \quad t > \tau$$

We also remark that, again under the assumptions made in remark 1, Z is a positive kernel, in the sense that the action of Z on any non-negative source f must be greater than or equal to zero. Next, we examine the homogeneous integral equation for the unknown function $v \in L^2_{+,loc}(\mathbb{R}^n \backslash \overline{\Omega})$,

$$\left(\mathcal{Z}^* v\right)(x,t) = 0. \tag{2.2}$$

Since v is non-negative, in addition to the action $\mathcal{Z}^*$ on any non-negative function in its domain, then it follows that equation 2.2 admits only the trivial solution $v = 0$. Hence, according to proposition 2.3 on page 46 of [9], $\mathcal{Z}$ as a map from $X_f^{\sigma_1,\sigma_2}(\Omega, T)$ to $L^2_{+,loc}(\mathbb{R}^n \backslash \overline{\Omega})$ has dense range in $L^2_{+,loc}(\mathbb{R}^n \backslash \overline{\Omega})$.
$\square$

We now consider a unique continuation principle for general parabolic differential equations. Friedman [5] has shown, for $p_0 = (x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}_+$ and

$$\begin{aligned} N(p_0) = \{(x,t) \in \mathbb{R}^n \times \mathbb{R}_+ \ : \ 0 \leq t \leq t_0, \\ \text{and the cylinder } \{x\} \times [0,t] \text{ centered at } p_0 \text{ is simply connected as } t \text{ increases}\}, \end{aligned}$$

that

Proposition 2.2. *If $(\partial_t - L_x)u \leq 0$ ($(\partial_t - L_x)u \geq 0$) in $\mathbb{R}^n \times \mathbb{R}_+$ and if u has a positive maximum (negative minimum) which is attained at $p_0 = (x_0, t_0)$, then $u(p) = u(p_0)$ for all $p \in N(p_0)$.*

This proposition implies that should u vanish on any open, simply connected subset of $\mathbb{R}^n$ for any $T \geq 0$, then u must vanish everywhere such that u remains a (homogeneous) solution of $(\partial_t - L_x)u = 0$. This property is essential in the range characterization to follow and to our estimation of the carrier support of $u(\cdot, T)|_{\mathcal{M}}$.

We now turn our attention to a few fundamental properties of the restriction of $\mathcal{Z}$ to the sets $\mathcal{M}$ and Ω, which we call $\mathcal{Z}|_{(\mathcal{M},\Omega)}$, and note that it is especially important for our inverse problem of determining the location of the unknown source f given measurements away from, and not necessarily surrounding, it. In what follows, $\mathcal{R}(\mathcal{Z}|_{(\mathcal{M},\Omega)})$ denotes the range of $\mathcal{Z}|_{(\mathcal{M},\Omega)}$.

Proposition 2.3. *Let $\Omega_{1,2} \in \mathbb{R}^n$ be two convex open sets whose closures have empty intersection, and suppose $\overline{\mathcal{M}} \cap (\overline{\Omega}_1 \cup \overline{\Omega}_2) = \emptyset$. Then,*

$$\mathcal{R}\left(\mathcal{Z}|_{(\mathcal{M},\Omega_1)}\right) \cap \mathcal{R}\left(\mathcal{Z}|_{(\mathcal{M},\Omega_2)}\right) = \{0\}.$$

Proof. Let $\overline{\Omega}_1 \cap \overline{\Omega}_2 = \emptyset$ and let

$$\begin{aligned} (\partial_t - L_x)u_1 = f_1, \quad \text{supp} f_1(\cdot, t) \subset \Omega_1 \\ (\partial_t - L_x)u_2 = f_2, \quad \text{supp} f_2(\cdot, t) \subset \Omega_2 \end{aligned}$$

such that u_1 and u_2 do not vanish on $\mathcal{M}$. Next, let $v = u_1 - u_2$. Then, v satisfies

$$\begin{aligned} (\partial_t - L_x)v = f_1, \quad \text{on } \Omega_1 \\ (L_x - \partial_t)v = f_2, \quad \text{on } \Omega_2 \end{aligned}$$

so that $v = u_1$ on Ω_1 and $v = -u_2$ on Ω_2. Hence, $u_{1,2}$ vanish on $\Omega_{2,1}$, respectively. Then, by proposition 2.2, u_1 and u_2 must also vanish on $\mathbb{R}^n \backslash \Omega_{2,1}$. Hence, $u_{1,2} \equiv 0$ on $\mathcal{M}$. This is a contradiction. $\square$

We now consider the remainder of the fundamental properties of the restricted mapping $\mathcal{Z}|_{(\mathcal{M},\Omega)}$.

PROPOSITION 2.4. *Let $\mathcal{M}$ and Ω be open subsets of $\mathbb{R}^n$ such that $\Omega \supset supp\ f(\cdot,t)$ for all time $t \in [0,T]$ and assume $\overline{\mathcal{M}} \cap \overline{\Omega} = \emptyset$. Then, for all $T \geq 0$, $\mathcal{Z}|_{(\mathcal{M},\Omega)} : X_f^{\sigma_1,\sigma_2}(\Omega,T) \to L^2_+(\mathcal{M})$ is a compact linear map and has dense range in the latter space.*

Proof. Let $f \in X_f^{\sigma_1,\sigma_2}(\Omega,T)$. Since $Z(x-\cdot,t-\cdot) \in C_+^\infty(\mathbb{R}^n\backslash\overline{\Omega}\times[0,T])$ by the Malgrange-Ehrenpreis theorem, for each $T \geq 0$ we have

$$\begin{aligned}
\|(\mathcal{Z}|_{(\mathcal{M},\Omega)}f)(\cdot,T)\|^2_{L^2_+(\mathcal{M})} &= \int_{\mathcal{M}} \left| \int_0^T \int_\Omega Z(x-y,T-\tau)f(y,\tau)dyd\tau \right|^2 dx \\
&= \int_{\mathcal{M}} \left| \langle Z,f\rangle_{L^2_+(\mathbb{R}^n\times[0,T])} \right|^2 dx \\
&\leq \int_{\mathcal{M}} \|Z(x-\cdot,T-\cdot)\|^2_{\left(X_f^{\sigma_1,\sigma_2}\right)'} \|f\|^2_{X_f^{\sigma_1,\sigma_2}(\Omega,T)} dx \\
&\leq \int_{\mathcal{M}} \|Z(x-\cdot,T-\cdot)\|^2_{\left(X_f^{n/2,1/2}\right)'} \|f\|^2_{X_f^{\sigma_1,\sigma_2}(\Omega,T)} dx \\
&\leq \int_{\mathcal{M}} \|Z(x-\cdot,T-\cdot)\|^2_{\left(X_f^{n,1}\right)'} \|f\|^2_{X_f^{\sigma_1,\sigma_2}(\Omega,T)} dx \\
&\leq C_{1,n,\Omega,\mathcal{M}} \|f\|^2_{X_f^{\sigma_1,\sigma_2}(\Omega,T)},
\end{aligned}$$

where $C_{1,n,\Omega,\mathcal{M}}$ is a constant given by

$$C_{1,n,\Omega,\mathcal{M}} = n\mu(\mathcal{M}) \max_{\mathcal{M}} \|Z(x-\cdot,T-\cdot)\|^2_{\left(X_f^{n,1}\right)'} < \infty.$$

Now, since

$$\|Z(x-\cdot,T-\cdot)f\|^2_{H^n_+(\mathcal{M})} = \|Z(x-\cdot,T-\cdot)f\|^2_{L^2_+(\mathcal{M})} + \|\sum_{k,i=1}^n \partial^k_{x_i} Z(x-\cdot,T-\cdot)f\|^2_{L^2_+(\mathcal{M})},$$

Dirichlet's theorem and the same arguments above imply that

$$\|\sum_{k,i=1}^n \partial^k_{x_i} Z(x-\cdot,T-\cdot)f\|^2_{L^2_+(\mathcal{M})} \leq C_{2,n,\Omega,\mathcal{M}} \|f\|^2_{X_f^{\sigma_1,\sigma_2}(\Omega,T)}$$

where

$$C_{2,n,\Omega,\mathcal{M}} = n\mu(\mathcal{M}) \max_{\substack{\mathcal{M}\\ 1\leq k\leq n}} \|Z(x-\cdot,T-\cdot)\|^2_{\left(X_f^{k,1}\right)'} < \infty.$$

Hence, $\mathcal{Z}$ is bounded between $\overset{o}{H}{}^{\sigma_1}_+(\overline{\Omega}) \bigotimes \overset{o}{H}{}^{\sigma_2}_+([0,T])$ and $H^n_+(\mathcal{M})$. Finally, since the inclusion maps

$$H^n_+(\mathcal{M}) \overset{i}{\hookrightarrow} H^{n-1}_+(\mathcal{M}) \overset{i}{\hookrightarrow} \cdots \overset{i}{\hookrightarrow} H^1_+(\mathcal{M}) \overset{i}{\hookrightarrow} L^2_+(\mathcal{M})$$

are compact, see for example theorem 6.98 in [15], so must it be that $\mathcal{Z}$ is compact from the source space $\overset{o}{H}{}^{\sigma_1}_+(\overline{\Omega}) \bigotimes \overset{o}{H}{}^{\sigma_2}_+([0,T])$ into $L^2_+(\mathcal{M})$ for each T.

Lastly, $\mathcal{Z}|_{(\mathcal{M},\Omega)}$ has dense range in $L^2_+(\mathcal{M})$ since, according to proposition 2.1, the adjoint equation posed on $\mathbb{R}^n \times \mathbb{R}_+$

$$\mathcal{Z}^* v(x,t) = 0$$

implies $v(x,t)$ vanishes throughout $\mathbb{R}^n \times \mathbb{R}$, and again according to proposition 2.3 on page 46 of [9], while linearity of $\mathcal{Z}|_{(\mathcal{M},\Omega)}$ is clear. ∎

Of interest to us in the following section on the inverse problem is the existence of the Hilbert adjoint of this restricted operator. This result follows as a corollary to the previous proposition. That is,

COROLLARY 2.5. *(Hilbert Adjoint) Let $\mathcal{M}$ and Ω be open subsets of $\mathbb{R}^n$ such that $\Omega \supset$ supp $f(\cdot,T)$ and assume $\overline{\mathcal{M}} \cap \overline{\Omega} = \emptyset$. Then, for all $T \in \mathbb{R}_+$, $\mathcal{Z}|^*_{(\mathcal{M},\Omega)} : L^2_+(\mathcal{M}) \to X_f^{\sigma_1,\sigma_2}(\Omega,T)$ exists as a bounded linear map. Moreover,*

$$\left(\mathcal{Z}|^*_{(\mathcal{M},\Omega)} u\right)(x,T) = \int_{\mathcal{M}} Z(z-x,T)u(z,T)dz.$$

Proof. Let $f \in X_f^{\sigma_1,\sigma_2}(\Omega,T)$ and suppose $u(\cdot,T) \in L^2_+(\mathcal{M})$. Since $\mathcal{M}$, Ω and T are all bounded, then each of the spaces on which integrate are sigma finite, and hence we may interchange all orders of integration. We arrive at the formula for $\mathcal{Z}|^*_{(\mathcal{M},\Omega)}$ by noting that,

$$\begin{aligned}
\langle u(\cdot,T)|_{\mathcal{M}}, \mathcal{Z}|_{(\mathcal{M},\Omega)} f\rangle_{L^2_+(\mathcal{M})} &= \int_{\mathcal{M}} u(x,T) \int_0^T \int_{\Omega} Z(x-y,T-\tau) f(y,\tau) dy d\tau dx \\
&= \int_0^T \int_{\Omega} \int_{\mathcal{M}} u(x,T) Z(x-y,T-\tau) dx f(y,\tau) dy d\tau \\
&= \langle \mathcal{Z}|^*_{(\mathcal{M},\Omega)} u(\cdot,T)|_{\mathcal{M}}, f\rangle_{L^2_+(\mathbb{R}^n\times[0,T])}
\end{aligned}$$

Hence,

$$\left(\mathcal{Z}|^*_{(\mathcal{M},\Omega)} u\right)(x,T) = \int_{\mathcal{M}} Z(z-x,T)u(z,T)dz.$$

Similarly, we complete the proof by noting that

$$\begin{aligned}
\left|\langle u(\cdot,T)|_{\mathcal{M}}, \mathcal{Z}|_{(\mathcal{M},\Omega)} f\rangle_{L^2_+(\mathcal{M})}\right|^2 &= \left|\int_{\mathcal{M}} u(x,T) \int_0^T \int_{\Omega} Z(x-y,T-\tau) f(y,\tau) dy d\tau dx\right|^2 \\
&= \left|\int_0^T \int_{\Omega} \int_{\mathcal{M}} u(x,T) Z(x-y,T-\tau) dx f(y,\tau) dy d\tau\right|^2 \\
&= \left|\langle \mathcal{Z}|^*_{(\mathcal{M},\Omega)} u(\cdot,T)|_{\mathcal{M}}, f\rangle_{L^2_+(\mathbb{R}^n\times[0,T])}\right|^2 \\
&\le \|\mathcal{Z}|^*_{(\mathcal{M},\Omega)} u(\cdot,T)|_{\mathcal{M}}\|^2_{\left(X_f^{\sigma_1,n/2}(\Omega,T)\right)'} \|f\|^2_{X_f^{\sigma_1,\sigma_2}(\Omega,T)} \\
&< \infty.
\end{aligned}$$

∎

To ease notation, in what follows $\mathcal{R}\left(\mathcal{Z}|_{X_f^{\sigma_1,\sigma_2}(\Omega,T)}\right)$ denotes the range of $\mathcal{Z}$ restricted to positive sources supported on the set Ω and observations limited to $\mathcal{M}$. Additionally, we will use the shorthand $\mathcal{Z}$ to denote $\mathcal{Z}|_{(\mathcal{M},\Omega)}$. Finally, if Ω is a set in

$\mathbb{R}^n$, then $N_\epsilon(\Omega)$ denotes the union of the set Ω and an neighborhood of its boundary, so that Ω is strictly contained in $N_\epsilon(\Omega)$ for each $\epsilon > 0$.

We are now able to fully characterize the range of $\mathcal{Z}|_{(\mathcal{M},\Omega)}$ acting on distributions in the space $X_f^{\sigma_1,\sigma_2}(\Omega, T)$ for various sets Ω, without specifying their regularity parameter $\sigma = (\sigma_1, \sigma_2) \in \mathbb{R}^2$. This will prove of much use in the numerical implementation and of the forthcoming result. That is we have,

PROPOSITION 2.6. *Let $\Omega_{1,2}$ be bounded convex subsets of $\mathbb{R}^n$ with smooth boundaries Then,*

$$\mathcal{R}(\mathcal{Z}|_{X_f^{\sigma_1,\sigma_2}(\overline{\Omega}_1\cap\overline{\Omega}_2,T)}) \subset \mathcal{R}(\mathcal{Z}|_{X_f^{\sigma_1,\sigma_2}(\overline{\Omega}_1,T)}) \cap \mathcal{R}(\mathcal{Z}|_{X_f^{\sigma_1,\sigma_2}(\overline{\Omega}_2,T)}) \subset \mathcal{R}(\mathcal{Z}|_{X_f^{0,0}(N_\epsilon(\Omega_1\cup\Omega_2),T)}).$$

Proof. Let $t \in [0, T]$ and let $\Omega_{1,2}$ be as stated above. The left lower containment follows from the fact that for $f_{1,2} \in X_f^{\sigma_1,\sigma_2}(\Omega_{1,2}, T)$, then the trivial extension of the form

$$\tilde{f}_1(x,t) = \begin{cases} f_1(x,t), & (x,t) \in \overline{\Omega}_1 \times [0,T] \\ 0, & x \notin \overline{\Omega}_1 \end{cases}$$

ensures the containment.

Next, in the spirit of the proof of Lemma 3.6 in [10] let $f_{1,2} \in X_f^{\sigma_1,\sigma_2}(\Omega_{1,2}, T)$ such that for each T, $\mathcal{Z}f_1 = \mathcal{Z}f_2 = u|_{\mathcal{M}}$. Then, by the unique continuation principle given in proposition 2.2 we have

$$(\mathcal{Z}f_1)(x,T) = (\mathcal{Z}f_2)(x,T), \quad x \in \mathbb{R}^n \backslash (\Omega_1 \cup \Omega_2).$$

Let $\phi \in C^\infty(\mathbb{R}^n \times [0,T])$ be a smooth cut-off function satisfying

$$\phi(x,t) = \begin{cases} 1, & (x,t) \in \mathbb{R}^n \backslash N_\epsilon(\Omega_1 \cap \Omega_2) \times [0,T] \\ 0, & (x,t) \in N_{\epsilon/2}(\Omega_1 \cap \Omega_2) \times [0,T] \end{cases}.$$

Then, for

$$v(x,t) = \begin{cases} \phi(x,t)u_1(x,t), & (x,t) \in \mathbb{R}^n \backslash \Omega_1 \times [0,T] \\ \phi(x,t)u_2(x,t), & (x,t) \in \mathbb{R}^n \backslash \Omega_2 \times [0,T] \\ 0, & (x,t) \in \Omega_1 \cap \Omega_2 \times [0,T] \end{cases}$$

it follows that $v \in C_+^\infty(\mathbb{R}^n \times [0,T])$ and that $(\partial_t - L_x)v = f_3 \in C_+^\infty(\mathbb{R}^n \times [0,T])$ such that

$$(\mathcal{Z}f_3)(x,T) = u_1(x,T) = u_2(x,T), \quad x \notin \Omega_1 \cap \Omega_2.$$

More importantly,

$$(\mathcal{Z}f_3)(x,T) = u_1(x,T) = u_2(x,T), \quad x \in \mathcal{M},$$

where $f_3 \in X_f^{0,0}(N_\epsilon(\Omega_1 \cap \Omega_2), T)$. □

3. The Inverse Source Problem. We present our main result (theorem) concerning the estimation of the time-varying carrier support in this section. The main result presented here owes itself in part to Picard's Theorem. This theorem essentially provides a denumerable representation of a compact linear operator A between two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ in terms of the operator's singular system, as well as a means to assess whether a given element of the second space $\mathcal{H}_2$ is also an element of the closure of the range of A. We take a moment to state the theorem and refer to [1] for its proof and further commentary.

THEOREM 3.1 (Picard). *Let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a compact linear operator from the Hilbert space $\mathcal{H}_1$ into the Hilbert space $\mathcal{H}_2$ with singular system $\{\lambda_n, \varphi_n, \psi_n\}_{n=1}^{\infty}$, i.e.*

$$A\varphi_n = \lambda_n \psi_n$$

and

$$A^*\psi_n = \lambda_n \varphi_n,$$

and let $\langle \cdot, \cdot \rangle$ denote the inner product on $\mathcal{H}_2$. Then, the equation $Af = g$ is solvable if and only if $g \in N(A^*)^\perp$ and

$$\sum_{n=1}^{\infty} \frac{|\langle g, \psi_n \rangle|^2}{\lambda_n^2} < \infty.$$

Moreover, any $\tilde{f}$ of the form

$$\tilde{f} = \sum_{n=1}^{\infty} \frac{\langle g, \psi_n \rangle}{\lambda_n} \varphi_n$$

solves $A\tilde{f} = g$.

Given Picard's theorem, and our previous range characterizations of the operator $\mathcal{Z}|_{(\mathcal{M},\Omega)}$, presented in the previous section, we now have a test which can determine whether the carrier support of the field u on $\mathcal{M}$ at time T is fully within some set of interest Ω by means of testing the convergence of the sum

$$\|\tilde{f}\|^2_{X_f^\sigma(\Omega,T)} = \sum_{n=1}^{\infty} \left| \frac{\langle u(\cdot,T)|_{\mathcal{M}}, \psi_n^{(\sigma)}(\cdot,T) \rangle}{\lambda_n^{(\sigma)}} \right|^2$$

where

$$\tilde{f}(x,t) = (\mathcal{Z}|^*_{(\mathcal{M},\Omega)} \mathcal{Z}|_{(\mathcal{M},\Omega)})^{-1} \mathcal{Z}|^*_{(\mathcal{M},\Omega)} u(\cdot,T)|_{\mathcal{M}}, \quad (x,t) \in \Omega \times [0,T]$$

and where the functions $\psi_n^{(\sigma)}(\cdot,T)$, $\varphi_n^{(\sigma)}(\cdot,T)$ and $\lambda_n^{(\sigma)}$ – which depend on the regularity parameter $\sigma = (\sigma_1, \sigma_2)$ and the sets of interest $\mathcal{M}$ and Ω – are defined through the relationships

$$\left(\mathcal{Z}|^*_{(\mathcal{M},\Omega)} \psi_n^{(\sigma)} \right)(x,T) = \lambda_n^{(\sigma)} \varphi_n^{(\sigma)}(x,t), \quad 0 \le t \le T$$

and

$$\left(\mathcal{Z}|_{(\mathcal{M},\Omega)} \varphi_n^{(\sigma)} \right)(x,T) = \lambda_n^{(\sigma)} \psi_n^{(\sigma)}(x,T).$$

If the sum does not converge, then we are able to conclude that the carrier support of $u(\cdot,T)|_{\mathcal{M}}$ at time T is not fully within the test region Ω.

We formalize this statement with the main theorem of this section.

THEOREM 3.2. *(Carrier Support) Let* $f \in X_f^{\sigma_1,\sigma_2}(\Omega,T)$. *Suppose further that* $\Omega \subset \mathbb{R}^n$ *is bounded with a smooth boundary and let* $\mathcal{Z}|^*_{(\mathcal{M},\Omega)}$ *denote the Hilbert adjoint of* $\mathcal{Z}|_{(\mathcal{M},\Omega)}$ *such that* $\mathcal{Z}|^*_{(\mathcal{M},\Omega)} : L^2_+(\mathcal{M}) \to X_f^{\sigma_1,\sigma_2}(\Omega,T)$. *Suppose further that*

$$\left(\mathcal{Z}|_{(\mathcal{M},\Omega)}\varphi_n^{(\sigma)}\right)(x,T) = \lambda_n^{(\sigma)}\psi_n^{(\sigma)}(x,T)$$

Then, for each fixed T

$$\operatorname{carr\,supp} u(\cdot,T)|_{\mathcal{M}} \subset \Omega \quad \Leftrightarrow \quad \sum_{n=1}^{\infty}\left|\frac{\langle u(\cdot,T)|_{\mathcal{M}}, \psi_n^{(\sigma)}(\cdot,T)\rangle}{\lambda_n^{(\sigma)}}\right|^2 < \infty.$$

Proof. Suppose $\operatorname{carr\,supp} u(\cdot,T)|_{\mathcal{M}} \subset \Omega$. Then, by definition of the carrier support, there exists a source $f \in X_f^{\sigma_1,\sigma_2}(\Omega,T)$ supported on a subset of Ω such that for each T, $(\mathcal{Z}|_{(\mathcal{M},\Omega)}f)(x,T) = u(\cdot,T)|_{\mathcal{M}}$ where $u(\cdot,T)|_{\mathcal{M}} \in \mathcal{R}\left(\mathcal{Z}|_{X_f^{\sigma_1,\sigma_2}(\Omega,T)}\right)$. Since $\mathcal{Z}$ is a compact linear operator between the two Hilbert spaces exhibited in the previous proposition, it admits the representation

$$\mathcal{Z} = \sum_{n=1}^{\infty}\lambda_n^{(\sigma)}\psi_n^{(\sigma)} \otimes \varphi_n^{(\sigma)}$$

such that the action of $\mathcal{Z}$ on f may be written as

$$(\mathcal{Z}f)(x,T) = \sum_{n=1}^{\infty}\lambda_n^{(\sigma)}\langle f, \varphi_n^{(\sigma)}\rangle\psi_n^{(\sigma)}(x,T), \quad x \in \mathcal{M}.$$

Since the left and right eigenfunctions, $\psi_n^{(\sigma)}$ and $\varphi_n^{(\sigma)}$, satisfy

$$\mathcal{Z}^*\psi_n^{(\sigma)} = \lambda_n^{(\sigma)}\varphi_n^{(\sigma)}$$

we note that

$$\begin{aligned}
\langle f, \varphi_n^{(\sigma)}\rangle &= \int_0^T\int_\Omega f(x,\tau)\varphi^{(\sigma)}(x,\tau)dxd\tau \\
&= \langle f, \mathcal{Z}^*\psi_n^{(\sigma)}/\lambda_n^{(\sigma)}\rangle \\
&= \overline{1/\lambda_n^{(\sigma)}}\langle \mathcal{Z}f, \psi_n^{(\sigma)}\rangle \\
&= \overline{1/\lambda_n^{(\sigma)}}\langle u(\cdot,T)|_{\mathcal{M}}, \psi_n^{(\sigma)}(\cdot,T)\rangle.
\end{aligned}$$

Bessel's inequality states,

$$\sum_{n=1}^{\infty}|\langle f, \varphi_n^{(\sigma)}\rangle|^2 \le \|f\|^2_{X_f^{\sigma_1,\sigma_2}(\Omega,T)} < \infty.$$

Hence,

$$\sum_{n=1}^{\infty}\left|\frac{\langle u(\cdot,T)|_{\mathcal{M}}, \psi_n^{(\sigma)}(\cdot,T)\rangle}{\lambda_n^{(\sigma)}}\right|^2 < \infty.$$

Now, suppose $u(\cdot,T)|_{\mathcal{M}} \in \mathcal{R}(\mathcal{Z}|_{X_f^{\sigma_1,\sigma_2}(\Omega,T)})$, where the closure of the latter space is $L^2_+(\mathcal{M})$ which we established in proposition 2.1, and suppose the Picard sum

$$\sum_{n=1}^{\infty} \left| \frac{\langle u(\cdot,T)|_{\mathcal{M}}, \psi_n^{(\sigma)}(\cdot,T)\rangle}{\lambda_n^{(\sigma)}} \right|^2$$

converges. Let f be any source of the form

$$f(x,t) = \sum_{n=1}^{\infty} \frac{\langle u(\cdot,T)|_{\mathcal{M}}, \psi_n^{(\sigma)}(\cdot,T)\rangle}{\lambda_n^{(\sigma)}} \varphi_n^{(\sigma)}(x,t), \quad (x,t) \in \Omega \times [0,T].$$

Then, since each $\varphi_n^{(\sigma)}$ is supported on Ω, any such source will have similar such support. Also, its nontrivial image, a fact from proposition 2.3, under $\mathcal{Z}$ is in $L^2_+(\mathcal{M})$ for each T. Finally,

$$\|\mathcal{Z}f\|^2_{L^2_+(\mathcal{M})} = \sum_{n=1}^{\infty} |\langle u(\cdot,T)|_{\mathcal{M}}, \psi_n^{(\sigma)}(\cdot,T)\rangle|^2 \leq \|u(\cdot,T)\|^2_{L^2_+(\mathcal{M})} < \infty.$$

Hence, $\operatorname{carr\,supp} u(\cdot,T)|_{\mathcal{M}} \subset \Omega$. ▯

Our main result provides us with a reconstruction algorithm which can determine the time-dependent carrier support of the field $u|_{\mathcal{M}}$. We describe this algorithm in the form of the following

COROLLARY 3.3. *Let Ω be an open bounded convex subset of $\mathbb{R}^n$, and $\{\lambda_n^{(\sigma)}, \psi_n^{(\sigma)}, \varphi_n^{(\sigma)}\}$ be the singular system for $\mathcal{Z}|_{(\mathcal{M},\Omega)}$. Then,*

$$\operatorname{carr\,supp} u(\cdot,T)|_{\mathcal{M}} = \bigcap \Omega \quad \textit{such that} \quad \sum_{n=1}^{\infty} \left| \frac{\langle u(\cdot,T)|_{\mathcal{M}}, \psi_n^{(\sigma)}(\cdot,T)\rangle}{\lambda_n^{(\sigma)}} \right|^2 < \infty.$$

Proof. Let Ω be an open bounded convex set and suppose that the infinite series in corollary (3.3) converges. Then, there exists a source $g^{(\Omega)}$, depending on Ω, in $X_f^{\sigma_1,\sigma_2}(\Omega,T)$ such that $\mathcal{Z}|_{(\mathcal{M},\Omega)} g^{(\Omega)} = u(\cdot,T)|_{\mathcal{M}}$. Taking the intersection of the supports of all such g's then yields

$$\bigcap \Omega = \bigcap_{\mathcal{Z}|_{(\mathcal{M},\Omega)} g^{(\Omega)} = u|_{\mathcal{M}}} \operatorname{supp\,ch} g^{(\Omega)} = \operatorname{carr\,supp} u(\cdot,T)|_{\mathcal{M}}.$$

▯

This summability test offers a theoretical and computational basis for the determination and ultimate reconstruction of the carrier support support of $u(\cdot,T)|_{\mathcal{M}}$. Numerically speaking, however, there is some issue of how to actually sum the infinite series. The fact that the operator $\mathcal{Z}|_{(\mathcal{M},\Omega)}$ is compact and smoothing tells us that zero is either an eigenvalue or a point of accumulation. It turns out that zero is a point of accumulation and so the singular values $\lambda_n^{(\sigma)}$ rapidly converge to zero, thereby creating a computational instability problem if we seek to sum the series numerically. Hence, either the summability test needs to be regularized in some manner or we need to pursue another approach which is linked to the infinite series.

4. Numerical Examples. We now consider two numerical examples which demonstrate the ability of the proposed theorem to locate the positions of two localized sources. In the first case, we consider the delta-function impulse source $f(x,t) = \delta_p(x)\delta_0(t) \in \mathcal{E}'(\mathbb{R}) \otimes \mathcal{E}'(\mathbb{R})$. Here, we study the two problems of having knowledge of the scalar diffusion field away from the source with a nontrivial convection field flowing in the downstream sense, i.e., the measurement set is say to the left of the source, while the flow field moves from the left to the right. We examine this problem in the two cases where in the first, we have values of u at some fixed time larger than zero sampled uniformly on our measurement interval, and second, where these values are measured in a nonuniform fashion. In short, in both situations, the minimum of the logarithm of the truncated moving Picard sum is evidently observable and location of the impulse point source is readily accomplished.

In the second case, we study another convective problem, specifically in two-dimensional space $\mathbb{R}^2$. In this case we allow the source to be a sustained one in time – once it has "turned on" – and allow it to move through space along a T^*-period track or orbit $\gamma(t)$, of lifetime T^*, i.e.,

$$f(x,t) = \delta_{\gamma(t)}(x)(H_0(t) - H_{T^*}(t)).$$

For this problem we only present the case of having uniform measurements on $\mathcal{M}$, now in $\mathbb{R}^2$, yet we are able to clearly demonstrate the instantaneous tracking, or location, of the localized source may be done in a very robust fashion, as evidenced by the comparison of the exhibited true and reconstructed source trajectories.

For simplicity in the numerical generation of the data and the numerical implementation of the main results of this article we assume the coefficients in parabolic operator L_x take the form

$$\begin{aligned} &a = I \text{ on } \mathbb{R}^n, \quad n = 1,2 \\ &b = \begin{cases} -1, & \text{on } \mathbb{R}^1 \\ -(1,0), & \text{on } \mathbb{R}^2 \end{cases} \\ &c = 0. \end{aligned}$$

This means our governing equations of interest in the one and two-dimensional cases are

$$\left(\partial_t + \partial_x - \partial_x^2\right) u(x,t) = f(x,t), \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+$$

and

$$\left(\partial_t + \partial_x - \partial_x^2 - \partial_y^2\right) u(x,y,t) = f(x,y,t), \quad (x,y,t) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+.$$

Remark 3. *In the following numerical experiments the variable c will now, instead, denote a coordinate value in either $\mathbb{R}$ or $\mathbb{R}^2$, depending on whether we are addressing the inverse problem on the real line, or on the real plane, which will be evidently clear from the associated context. This coordinate value represents the center, and hence the preferable letter c, of certain test domains of interest (Ω_c), which will be moved around the larger space of interest and will be centered at the various points c to follow.*

Location, and tracking of the moving source, is accomplished by covering the totality of the search space of interest Ω_s with candidate test domains of the forms:

$$\Omega_c^{(1,2)} = \Omega_0^{(1,2)} + \{c_j\}$$

where in $\mathbb{R} \ni c_j$ we have

$$\Omega_0^{(1)} = [0, 1/10]$$

and in $\mathbb{R}^2 \ni c$ we define

$$\Omega_0^{(2)} = [0, 1/10] \times [0, 1/10]$$

and seeking to minimize the objective function

$$J(\Omega_c) = \|\mathcal{Z}|^{\dagger}_{(\mathcal{M},\Omega_c)} u(\cdot, T)|_{\mathcal{M}}\|^2.$$

Here, † denotes the pseudo-inverse.

In summary, when the test set Ω_c fully contains the carrier support of the source, then the objective function should take on small values; while contrarily, when such a test domain does not fully contain the source it should be singular. We acknowledge the phrasing, "small values" is indeed rather ambiguous; however, in the numerical examples to follow, we observe a global minimum value of the objective function when the test domain is exactly centered on the support of the true source. Furthermore, we add that the test domain position centers are taken as

$$\mathbb{R} \ni c_j = \{-10 + j/20\}, \quad j = 0, 1, 2, \cdots, 400$$

and

$$\mathbb{R}^2 \ni c_j = (-2 + j/20, 0 + j/20), \quad j = 0, 1, 2, \cdots, 80.$$

We end this section which discusses the overall strategy pursued in the generation of the numerical evaluation of the results presented in this article with a few words on the simulated data used in this evaluation. In summary, we discretize the standard action of $\mathcal{Z}f$ and employ a Newton-Cotes-type numerical integration scheme to generate its approximation on discretely sampled points of $\mathcal{M}$. More importantly, so as to not perpetrate and inverse crime, we corrupt these approximations with considerable white noise, in the levels of approximately 5%, 10% and 30%.

4.1. Locating a Stationary Impulse Source in $\mathbb{R}$. In this section scalar field is generated by the discretization scheme (via the standard rectangle rule) of the integral

$$u(x, T) = \int_0^T \int_{\mathbb{R}} \frac{e^{-\frac{|x-(T-\tau)-y|^2}{4(T-\tau)}}}{\sqrt{4\pi(T-\tau)}} \delta_p(y)\delta_0(\tau)dyd\tau = \frac{1}{\sqrt{4\pi T}} \int_{\mathbb{R}} e^{-\frac{|x-T-y|^2}{4T}} \delta_p(y)dy \quad (4.1)$$

and again subsequently corrupted by various levels of white noise to avoid committing the standard inverse crime. We examine the data taken over the discretely sampled uniform domain,

$$\mathcal{M}_1 = \{-1.0, -0.8, -0.6, -0.4, -0.2, 0.0, 0.2, 0.4, 0.6, 0.8, 1.0\} \subset \mathbb{R}$$

and the nonuniform domain

$$\mathcal{M}_2 = \{-0.14, -0.178, -.21, -.23, -.49, -0.8, -0.01, 0.62, 0.81, 0.90, 1.0\} \subset \mathbb{R}.$$

We examine three instances of noise-corrupted data, in the amounts 10, 20 and 30 dB – which amounts to 32.0%, 10.0%, and 3.2% relative signal to noise ratio. For

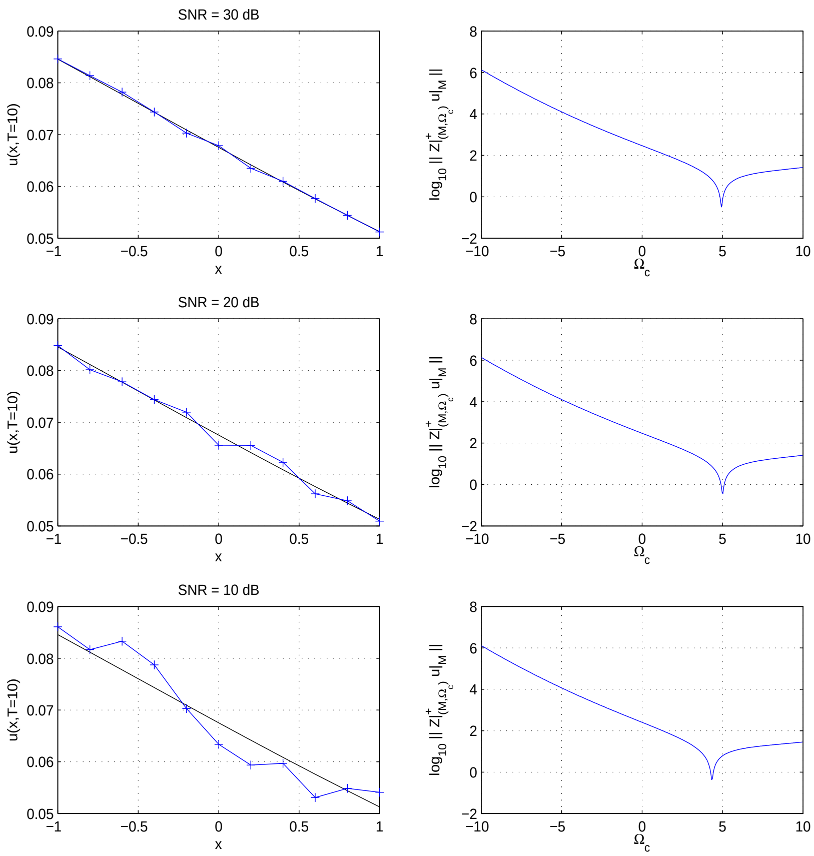

FIG. 4.1. *The exact and measured (noise corrupted) diffusion fields and the norm of the associated truncated Picard test series on a uniform measurement grid.*

relative time $T = 10$ after impulse release, for the discrete samplings taken on $\mathcal{M}_1$ and $\mathcal{M}_2$, we observe that the running truncated Picard series are each minimized at, or very near, to the point of the impulse release, which is $p = 5.0$ in all cases. This observation is very helpful in establishing the fact that the discrete samplings taken on general sets of observation $\mathcal{M}$ are not restricted to such things as equally spaced gridded points. Rather, any collection of discrete points which are coplanar in $\mathbb{R}^n$ will suffice. Moreover, numerical investigations have shown that larger random samplings yield better conditioned systems than for the analogous case of equally separated points, when the number of samplings is the same in both instances. For commercial purposes, this is both necessary and highly advantageous. Figures 4.1 and 4.2 demonstrate the efficacy of locating the localized source, again at $p = 5$, when we have uniform measurements on $\mathcal{M}_1$ and nonuniform ones on $\mathcal{M}_2$.

4.2. Locating a Sustained Moving Source in $\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+$. As in the previous subsection, we begin with the description of the integral which we discretize, and corrupt with ample random noise, that provides our simulated data. Here, we

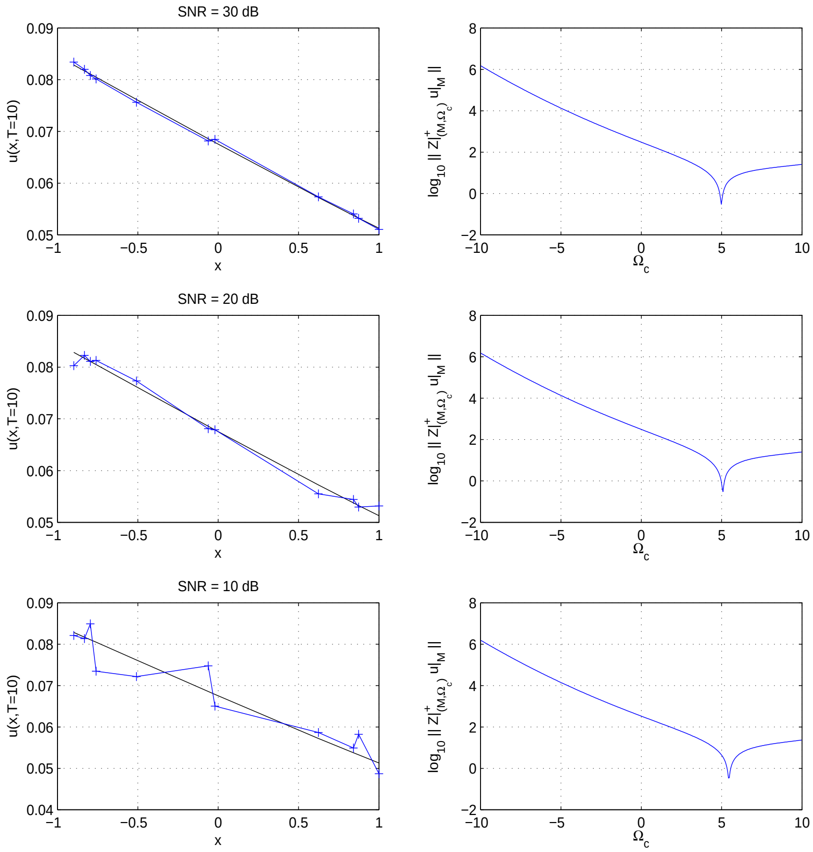

FIG. 4.2. *The exact and measured (noise corrupted) diffusion fields and the norm of the associated truncated Picard test series on a nonuniform (random) measurement grid.*

present the numerical method used to construct the numerical solution to the two-dimensional forward problem. We begin by considering the two-dimensional heat equation with a time dependent source $g((x,y)-\gamma(t))s(t)$ located at $(x_s(t), y_s(t)) = \gamma(t)$ and given by

$$\left(\partial_t + \partial_x - \partial_x^2 - \partial_y^2\right) u(x,y,t) = g((x,y)-\gamma(t))s(t).$$

Following [2] the fundamental solution for the above operator (defined over all space) is given by

$$Z(x-\xi_1, y-\xi_2, t-\tau) = \frac{1}{4\pi(t-\tau)} e^{-\frac{((x-\xi_1)-(t-\tau)+(y-\xi_2))^2}{4(t-\tau)}}.$$

In our two dimensional simulation problem we are interested in solving the inverse source problem for the convective diffusion equation on the half space

$$\Pi_+ = [-\infty, \infty] \times [0, \infty],$$

where we assume there is no flux of the field over the boundary $\{y = 0\}$, i.e., $\partial_y u(x, 0, t) = 0$ for all x and t. This problem then resembles (in one less dimension) the problem of some form of elevated source release in say a large, but local neighborhood of the atmosphere and its contact with the ground. Then, in this case, using the method of images, see [8, 2], we form the Green's function

$$\tilde{Z}(x-\xi_1, y-\xi_2, t-\tau) = Z(x-\xi_1, y-\xi_2, t-\tau) + Z(x-\xi_1, y+\xi_2, t-\tau)$$

which satisfies our boundary condition.

To solve the diffusion equation with a time and spatially dependent source term one can integrate it against such a Green's function to obtain

$$u(x, y, T) = \int_0^T \int_{\mathbb{R}_{\xi_1}} \int_{\mathbb{R}_{\xi_2}} \tilde{Z}(x-\xi_1, y-\xi_2, T-\tau) f(\xi_1, \xi_2, \tau) d\xi_1 d\xi_2 d\tau. \quad (4.2)$$

Due to the the weak singularity at $\tau = T$, the numerical integration requires special treatment. To perform the integration we used a member of the semi-open quadrature rules [18] that do not explicitly evaluate their integrand at the limit of the integration range where the singularity exists. The explicit scheme selected for this investigation is given by (for an integrand $f(x)$ that is singular at the left endpoint x_1)

$$\int_{x_1}^{x_N} f(x)dx = h\left[\frac{23}{12}f_2 + \frac{7}{12}f_3 + f_4 + f_5 + \ldots + f_{N-2} + \frac{13}{12}f_{N-1} + \frac{5}{12}f_N\right]. \quad (4.3)$$

Note that substituting the expression $v = T - \tau$ in equation 4.2 results in the singular limit at the left hand endpoint, to which equation 4.3 can be applied.

We employed the source moving along the figure eight-like lemniscate over one period (of 10 units), which then goes extinct, i.e., in this case

$$f(x, y, t) = \delta_{(\cos\frac{2\pi t}{10}, \sin\frac{4\pi t}{10})}(x, y)(H_0(t) - H_{10}(t)), \quad (x, y) \in \Pi_+.$$

Moreover, our data was then sampled on the grid of points

$$\mathcal{M} = \{0, 1/2, 1\} \times \{0, 1/2, 1\}$$

at snapshots in time corresponding to every 1/10 of unitless time. We demonstrate the outcome of this numerical experiment in figures 4.3, 4.4, 4.5 and 4.6. We use a signal-to-noise level of 25 dB, which again corresponds to 5.6% relative error between the signal strength and that of the additive white noise. The truncation value N which defines the dimension of the singular system used in the truncated Picard series test is chosen so that the singular values obey $\lambda_n^{(0,0)} < 10^{-4}$ for each $n > N$, that is our TSVD tolerance is 0.0001. We used this value as we found the singular values $\lambda_n^{(0,0)}$ rapidly approach zero after this value. Hence, our method of regularization is based on the philosophy of principle component analysis. That is we found the tolerance criterion $\lambda_n^{(0,0)} \geq 10^{-4}$ yielded the principle (dominant) components of the (pseudo-inverted) operator in question.

Figure 4.3 shows the entire ensemble of the reconstructed truncated Picard series over all the test domains at time $T = 1.0$. The colored surface is the value of the logarithm of the truncated series

$$\|\tilde{f}_j\|_{X_f^0(\Omega,T)} = \left(\sum_{n=1}^{N}\left|\frac{\langle u(\cdot,T)|_{\mathcal{M}}, \psi_{n,j}^{(0)}(\cdot,T)\rangle}{\lambda_{n,j}^{(0)}}\right|^2\right)^{1/2}.$$

Figure 4.4 shows the same of objective-type function as the previous one in Figure 4.3, only here we look from beneath to better observe the minimum located near the coordinate pair $(2\pi/10, 4\pi/10)$.

After obtaining the global minimum of the objective function which locates the (potentially mobile) source, we then use smaller moving, time-adaptive test domains of interest to locate the source, knowing with 100% certainty that source and its carrier support reside with each such test domain. Using the most previous estimates of the coordinates of the autonomous source, $(\hat{x}_{j-1}, \hat{y}_{j-1})$, this adaptive search domain is formed as

$$\Omega_j = [\hat{x}_{j-1} - 1/4, \hat{x}_{j-1} + 1/4] \times [\hat{y}_{j-1} - 1/4, \hat{y}_{j-1} + 1/4], \quad j \geq 1.$$

Clearly, we do this to make the computations as efficient as possible, and avoid unnecessary searching. This idea and its results are presented in Figure 4.5 which shows the localized inversions at the time snapshots $T = 2.0$, $T = 4.0$, $T = 6.0$, $T = 8.0$, $T = 10.0$ and $T = 12.0$.

Finally, in Figure 4.6 we show the true track of the source, and its locations at the discrete snapshots in time, along with the estimated or reconstructed track and it's instantaneous estimated positions. We remark that at the time $T = 10.0$, when the source turns-off, i.e., $s(10) = 0$, the carrier support's location remains constant, up to the noise and ill-posedness of the problem. That is, in the event of perfect data and a very well conditioned linear system, the source estimate would remain fixed at the point $(1, 2)$.

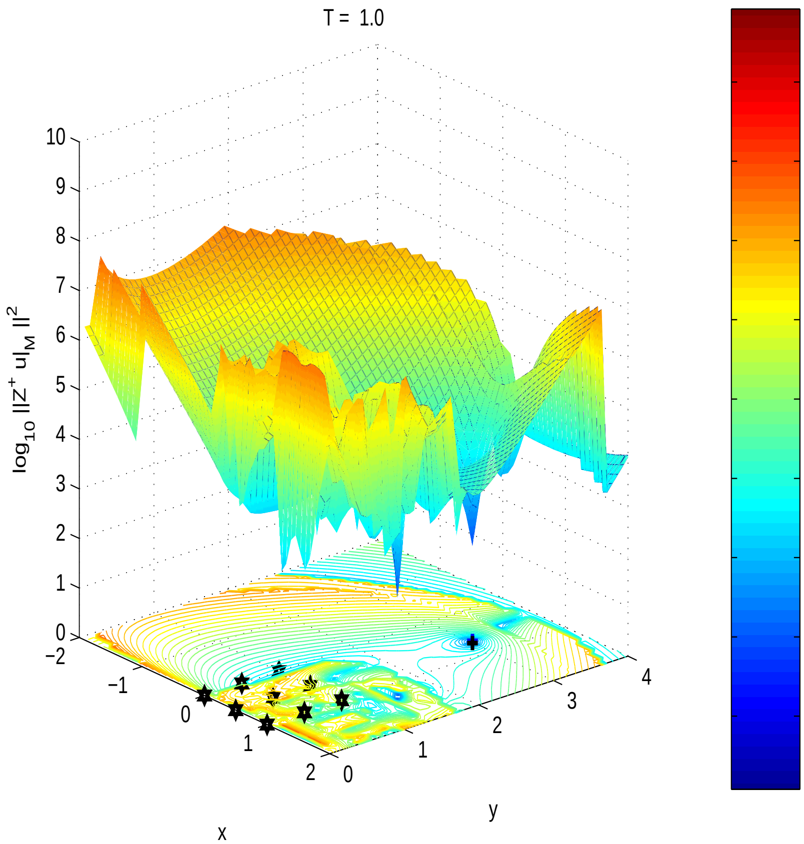

FIG. 4.3. *Numerical truncated Picard tests at the time $T = 1$ for the localized two-dimensional moving source over the fully tessellated domain.*

5. Summary and Conclusions. We have demonstrated that a simple knowledge of the instantaneous scalar field u on any bounded open set $\mathcal{M}$ located away from the (possibly time-varying) support of a source f is sufficient to estimate a nontrivial subset of the actual convex hull of the support of the source which we have called the carrier support of $u|_{\mathcal{M}}$. Additionally, we have provided and examined a viable numerical implementation of this result which can estimate, to essentially arbitrary precision, the trajectory of the carrier support over time, and hence track the moving source in real-time, without a priori assumptions on the regularity of the source. Moreover, we have shown that non-uniform sampling of the bounded and open measurement set $\mathcal{M}$ works as effectively as a uniform one. This result is important as it suggests that a wide collection of point samples distributed over a large domain of interest constitutes a robust methodology to locate and track sources of interest in a variety of applied problems, such as complex convective urban environments or large (aquatic) reservoir-like problems. In each of these, the robust and timely location of the effluent source is critical in nature, and may be accomplished with the few,

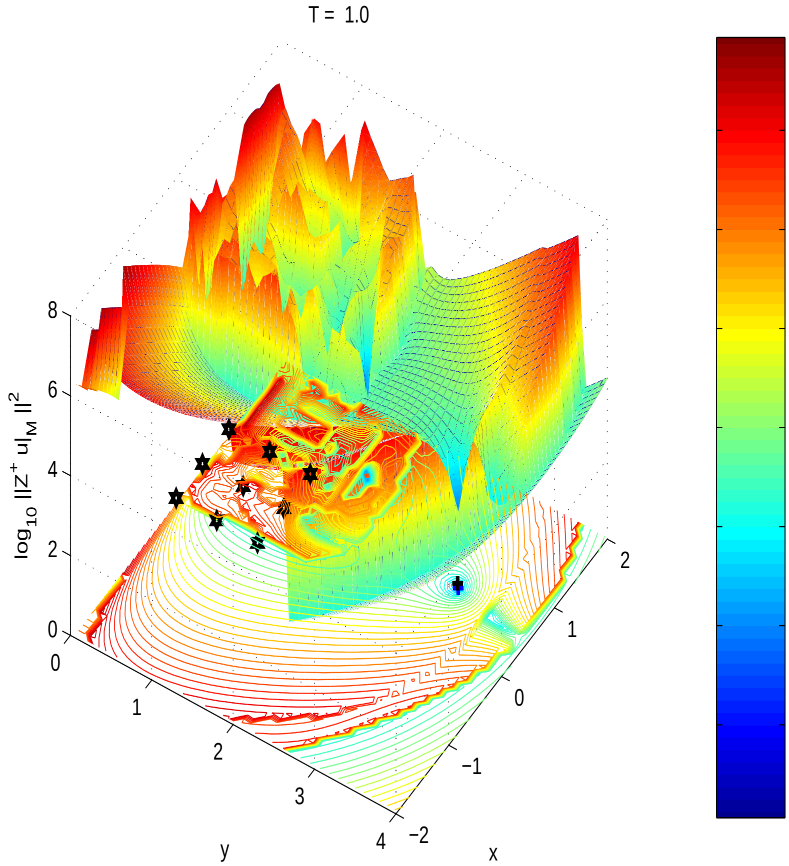

FIG. 4.4. *Inverted view of the former numerical truncated Picard tests at the time $T = 1$ for the localized two-dimensional moving source over the fully tessellated domain.*

sparse, and possibly non-uniformly sampled data assumed known in the analysis in this article.

In brief we mention that the concept of the carrier support does not provide us with a direct method which allows us to estimate the source strength as a function of time, and that this is certainly a significant problem of interest. In some simple cases, such as for constant coefficients of L_x, it may well be the case that by simply examining the local behavior of the objective function near the source, and observing it diminish in size over time, we may conclude that the source is no longer emitting into the system of interest and has become extinct. However, when the coefficients of L_x become more complex, such a simple observation may not persist. In such cases we propose an additional (forthcoming) technique which can estimate the strength of s over the time interval of measurements, $[0, T]$ which is based on a combination of results provided here and some analysis of the Laplace transform of the governing equations of our main problem. Additionally, the current framework developed here only accommodates the location and tracking of a source having but one component. The extension of

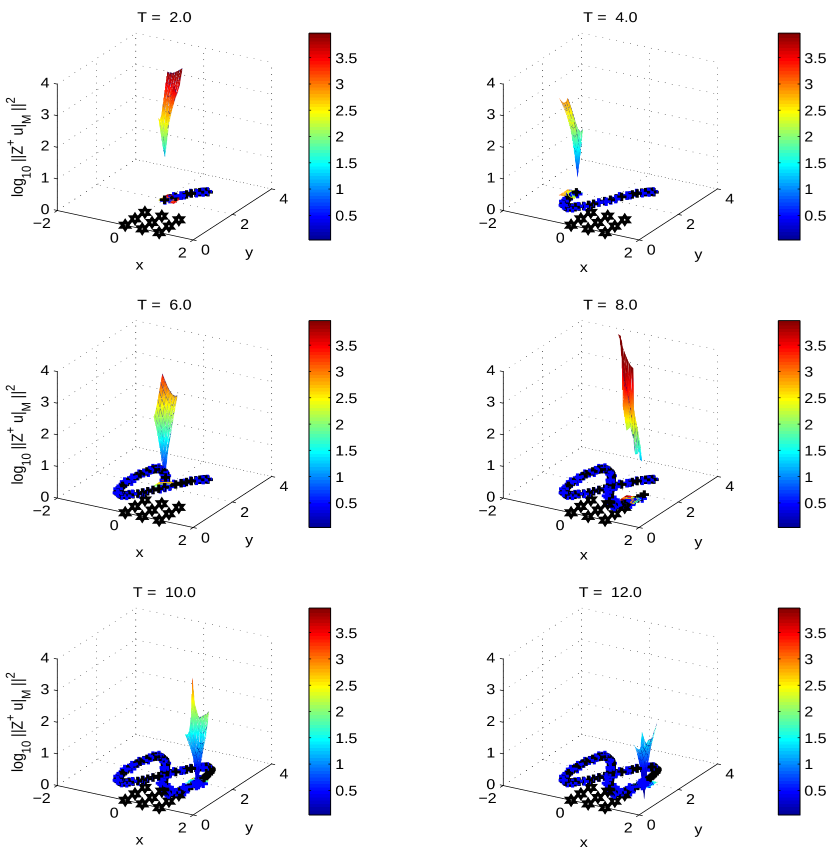

FIG. 4.5. *Localized numerical truncated Picard tests at the time snapshots:* $T = 2.0$*,* $T = 4.0$*,* $T = 6.0$*,* $T = 8.0$*,* $T = 10.0$*, and* $T = 12.0$*.*

the result to the problem of sources having multiple disjoint components is of much interest and is underway at the time of this writing.

Clearly, many future paths of work exists and warrant pursuit. Further among them, a sensitivity analysis of how this method performs when the coefficients are only known to within some specified tolerance of their true values. Additional work consists of treating the problem in a stochastic setting, where these coefficients are not known instaneously as we have assumed throughout this article; rather they are known to possess certain distributional moments and belong to certain distributional families. Again, the possibilities for interesting and value future work are many indeed.

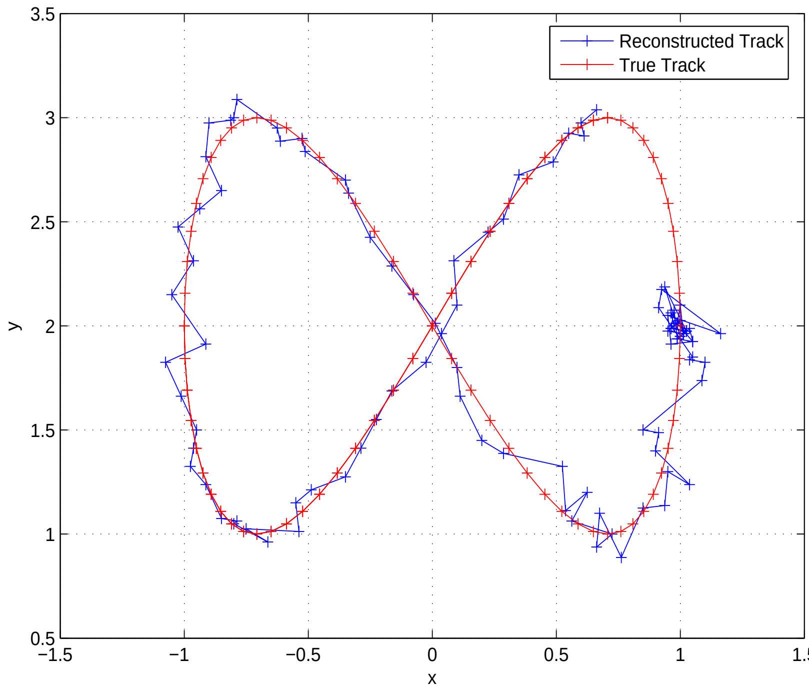

FIG. 4.6. *The discretely sampled true two-dimensional source track and the reconstructed (estimated) source track for the one-period sustained source.*

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