Solving Two-Dimensional PDE's with PDETWO

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Supposed one wanted to solve the following system of parital differential equations

$$\begin{aligned}\frac{\partial u_l}{\partial t} \quad = \quad & f_l(t, x, y, u_1, \ldots, u_{\mathrm{NPDE}}, \frac{\partial u_1}{\partial x}, \ldots, \frac{\partial u_{\mathrm{NPDE}}}{\partial x}, \frac{\partial u_1}{\partial y}, \ldots, \frac{\partial u_{\mathrm{NPDE}}}{\partial y}, \\ & \frac{\partial}{\partial x}\left(\mathrm{DH}_{l,1}\frac{\partial u_1}{\partial x}\right), \ldots, \frac{\partial}{\partial x}\left(\mathrm{DH}_{l,\mathrm{NPDE}}\frac{\partial u_{\mathrm{NPDE}}}{\partial x}\right), \\ & \frac{\partial}{\partial y}\left(\mathrm{DV}_{l,1}\frac{\partial u_1}{\partial y}\right), \ldots, \frac{\partial}{\partial y}\left(\mathrm{DV}_{l,\mathrm{NPDE}}\frac{\partial u_{\mathrm{NPDE}}}{\partial y}\right))\end{aligned}$$

for $a_1 < x < b_1$, $a_2 < y < b_2$, $t > t_0$, $l = 1, 2, \ldots, \mathrm{NPDE}$. The horizontal boundary conditions must satisfy

$$\mathrm{AH}_l u_l + \mathrm{BH}_l \frac{\partial u_l}{\partial y} = \mathrm{CH}_l \tag{1}$$

$$\mathrm{AV}_l u_l + \mathrm{BV}_l \frac{\partial u_l}{\partial x} = \mathrm{CV}_l \tag{2}$$

with initial conditions given by

$$u_l(t_0, x, y) = \phi_l(x, y) \quad \text{for} \quad (x, y) \in R, \quad l = 1, 2, \ldots, \mathrm{NPDE} \tag{3}$$

Example 1

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - 6xye^x e^y(xy + x + y - 3) \quad \text{for} \quad 0 \le x \le 1, \quad 0 \le y \le 1 \tag{4}$$

with boundary conditions given by

$$u(t, x, y) = 0 \tag{5}$$

with initial conditions given by the exact solution of

$$u(0, x, y) = 0 \tag{6}$$

It is known that the steady state solution to this problem is given by

$$u(x, y) = 3e^x e^y (x - x^2)(y - y^2) \tag{7}$$

Example 2

$$\frac{\partial u}{\partial t} = -u\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right) + 0.01\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{8}$$

with initial conditions given by the exact solution of

$$u(0, x, y) = (1 + e^{x+y-t})^{-1} \tag{9}$$

Example 3

$$\begin{aligned}
\frac{\partial u_1}{\partial t} &= u_2\left(\frac{\partial}{\partial y}\left(u_2\frac{\partial u_1}{\partial y}\right) + \frac{\partial}{\partial y}\left(u_1\frac{\partial u_2}{\partial y}\right)\right) - 3u_2\frac{\partial u_2}{\partial x} + 4u_1\frac{\partial u_1}{\partial y} - \frac{\partial u_2}{\partial y} \\
\frac{\partial u_2}{\partial t} &= \frac{\partial}{\partial x}\left(u_2\frac{\partial u_1}{\partial x}\right) + \frac{\partial}{\partial x}\left(u_1\frac{\partial u_2}{\partial x}\right) - \frac{\partial u_2}{\partial y}
\end{aligned}$$

with vertical boundary conditions