Solutions to the Problems in Numerical Analysis by David Kincaid and Ward Cheney

John Weatherwax

To my family.

Introduction

This is a solution manual for *some* of the problems in the excellent numerical analysis textbook:

Numerical Analysis: Mathematics of Scientific Computing
by David Kincaid and Ward Cheney

This solution manual was prepared form the *first* edition of the textbook.

I'm currently working on finishing more of the problems in this book. In the meantime I'm publishing my partial results for any student who does not want to wait for the full book to be finished.

One of the benefits of this manual is that I use the `R` statistical language to perform any of the needed numerical computations (rather than do them "by-hand"). Thus if you work though this manual you will be learning the `R` language at the same time as you learn statistics. The `R` programming language is one of the most desired skills for anyone who hopes to use data/statistics in their future career. The `R` code can be found at the following location:

`http://waxworksmath.com/Authors/G_M/Kincaid/kincaid.html`

As a final comment, I've worked hard to make these notes as good as I can, but I have no illusions that they are perfect. If you feel that that there is a better way to accomplish or explain an exercise or derivation presented in these notes; or that one or more of the explanations is unclear, incomplete, or misleading, please tell me. If you find an error of any kind – technical, grammatical, typographical, whatever – please tell me that, too. I'll gladly add to the acknowledgments in later printings the name of the first person to bring each problem to my attention.

Chapter 2: Computer Arithmetic

Stable and Unstable Computations; Conditioning

Notes on Numerical Instability

Now if x is single precision then when $x_1 = \frac{1}{3}$ the absolute error in representing x_1 in a computer will be of order 10^{-8}. Note that in the formula used to compute x_{n+1} from x_n and x_{n-1} we are multiplying x_n by $\frac{13}{3}$ at each iteration. Thus the absolute error will grow by a factor of $\frac{13}{3}$ at each iteration.

- This means that x_2 has an absolute error of $\left(\frac{13}{3}\right) 10^{-8}$.
- and x_3 has absolute error $\left(\frac{13}{3}\right)^2 10^{-8}$,
- and x_4 has absolute error $\left(\frac{13}{3}\right)^3 10^{-8}$.

Continuing this logic we see that x_n has absolute error of $\left(\frac{13}{3}\right)^{n-1} 10^{-8}$. To make the absolute error in $x_n = O(10)$ which would make the calculation worthless since the actual answer is $\left(\frac{1}{3}\right)^n$ and we would have a complete loss of all significant digits we would need

$$\left(\frac{13}{3}\right)^{n-1} \times 10^{-8} \approx 10^1 \,,$$

or

$$\left(\frac{13}{3}\right)^{n-1} \approx 10^{+9} \,,$$

or solving this for n we find

$$(n-1) \approx \frac{9 \log 10}{\log\left(\frac{13}{3}\right)} = 14.13268 \,.$$

Thus $n \approx 15.13268$ so it takes around fifteen iterations before we have an absolute error or $O(10)$ (which is what we see in the example in the book).

If we use double precision instead of single precision then absolute error in represengint $x_1 = \frac{1}{3}$ in a computer will be $O(10^{-16})$. This means that the absolute error of iteration n for x_n will be so large as to render the computation "worthless" when

$$\left(\frac{13}{3}\right)^{n-1} \times 10^{-16} \approx 10^1 \,.$$

Solving this for n we find

$$n - 1 \approx \frac{17 \log 10}{\log\left(\frac{13}{3}\right)} = 26.69505 \,.$$

Thus when $n \geq 27.69505$ all precision is lost.

Chapter 9: Numerical Solutions of Partial Differential Equations

First-Order Partial Differential Equations; Characteristic Curves

Problem 1

Part (a): From the given differential equation the method of characteristics would seek functions $x(s)$, $y(s)$, and $u(s)$ such that

$$\frac{dx}{ds} = 1$$
$$\frac{dy}{ds} = x$$
$$\frac{du}{ds} = 0\,,$$

with the initial conditions

$$x(0) = 0$$
$$y(0) = r$$
$$u(0) = f(r)\,.$$

Integrating the differential equation for $x(s)$ gives $x(s) = s$. Integrating the differential equation for $u(s)$ gives $u(s) = u(0) = f(r)$. Using these in the differential equation for $y(s)$ gives

$$\frac{dy}{ds} = s\,,$$

or

$$y(s) = \frac{s^2}{2} + r\,.$$

Now from $x = s$ and $y = \frac{s^2}{2} + r$ we have $r = y - \frac{x^2}{2}$ so that $u = f(r)$ becomes

$$u = f\left(y - \frac{x^2}{2}\right)\,,$$

for the solution $u = u(x, y)$.

Part (b): From the given differential equation the method of characteristics would seek functions

$$\frac{dx}{ds} = 1$$
$$\frac{dy}{ds} = 2u$$
$$\frac{du}{ds} = 0\,,$$

with initial conditions given by

$$
\begin{aligned}
x(0) &= 0 \\
y(0) &= r \\
u(0) &= f(r) \,.
\end{aligned}
$$

Integrating the differential equation for $x(s)$ we find $x(s) = s$. Integrating the differential equation for $u(s)$ gives $u(s) = u(0) = f(r)$. Using these in the differential equation for $y(s)$ gives

$$
\frac{dy}{ds} = 2f(r) \,.
$$

Integrating we get

$$
y = 2f(r)s + C \,,
$$

for C a constant. Using the initial condition $y(0) = r$ gives $C = r$ so

$$
y = 2f(r)s + r \,.
$$

Thus the solution is represented in parameterized form as

$$
\begin{aligned}
x &= s \\
y &= 2f(r)s + r \\
u &= f(r) \,.
\end{aligned}
$$

Part (c): From the given differential equation the method of characteristics would seek functions

$$
\begin{aligned}
\frac{dx}{ds} &= x \\
\frac{dy}{ds} &= 2y \\
\frac{du}{ds} &= 0 \,,
\end{aligned}
$$

with initial conditions given by

$$
\begin{aligned}
x(0) &= 1 \\
y(0) &= r \\
u(0) &= f(r) \,.
\end{aligned}
$$

Integrating these three equations gives

$$
\begin{aligned}
x(s) &= C_1 e^{s} \\
y(s) &= C_2 e^{2s} \\
u(s) &= C_3 \,.
\end{aligned}
$$

Using the initial conditions to evaluate the constants C_1, C_2, and C_3 when $s = 0$ we get

$$
\begin{aligned}
x(s) &= e^{s} \\
y(s) &= r e^{2s} \\
u(s) &= f(r) \,.
\end{aligned}
$$

The first two of these equations combine to give

$$y = rx^2 \quad \text{or} \quad r = \frac{y}{x^2}\,.$$

This means that

$$u = f\left(\frac{y}{x^2}\right)\,,$$

for the solution $u = u(x, y)$.

Problem 2

We are given the differential equation

$$u_x + y u_y = 0\,,$$

with the initial condition of

$$u(18, 3e) = \frac{k\pi}{2}\,.$$

The method of characteristics seeks to find functions $x(s)$, $y(s)$, and $u(s)$ such that

$$\begin{aligned}
\frac{dx}{ds} &= 1 \\
\frac{dy}{ds} &= y \\
\frac{du}{ds} &= 0\,,
\end{aligned}$$

with initial conditions given by

$$\begin{aligned}
x(0) &= 18 \\
y(0) &= 3e \\
u(0) &= \frac{k\pi}{2}\,.
\end{aligned}$$

If we integrate the above and use these initial conditions we get

$$\begin{aligned}
x(s) &= s + 18 \\
y(s) &= 3e^{s+1} \\
u(s) &= \frac{k\pi}{2}\,.
\end{aligned}$$

If we seek to evaluate $u(17, 3)$ then we first need to find s such that

$$\begin{aligned}
s + 18 &= 17 \\
3e^{s+1} &= 3\,.
\end{aligned}$$

Both of these equations are true when $s = -1$. Using the solution for $u(s)$ above we see that $u(-1) = \frac{k\pi}{2}$.

Problem 3

The differential equation in Example 8 is

$$xu_x + yuu_y = xy\,,$$

with the boundary-values of

$$u(x,y) = 2xy \quad \text{when} \quad xy = 3\,.$$

The solution found there was

$$u(x,y) = -1 + \sqrt{43 + 2xy}\,.$$

From this note that

$$\begin{aligned} u_x &= \frac{2y}{2\sqrt{43+2xy}} = \frac{y}{\sqrt{43+2xy}} \\ u_y &= \frac{2x}{2\sqrt{43+2xy}} = \frac{x}{\sqrt{43+2xy}}\,, \end{aligned}$$

so that

$$xu_x + yuu_y = \frac{xy}{\sqrt{43+2xy}} + \frac{xyu}{\sqrt{43+2xy}} = \frac{xy}{\sqrt{43+2xy}}(1+u) = xy\,,$$

as it should to be a solution. Next if $xy = 3$ note that

$$u(x,y) = -1 + \sqrt{43+6} = -1 + 7 = 6 = 2xy\,,$$

again as it must.

Problem 4

The differential equation in Example 6 is

$$6u_x + xu_y = y\,,$$

with the boundary-values of

$$u = e^x \sin(y)\,,$$

on the curve $y = x^3$. The method of characteristics seeks to find functions $x(s)$, $y(s)$, and $u(s)$ such that

$$\begin{aligned} \frac{dx}{ds} &= 6 \\ \frac{dy}{ds} &= x \\ \frac{du}{ds} &= y\,, \end{aligned}$$

with initial conditions given by

$$\begin{aligned} x(0) &= r \\ y(0) &= r^3 \\ u(0) &= e^r \sin(r^3)\,. \end{aligned}$$

Integrating the differential equation for $x(s)$ we find $x(s) = 6s+C$ for C a constant. Applying the initial conditions $x(0) = r$ we find $C = r$ so $x(s) = 6s + r$. Using this in the equation for $y(s)$ we find

$$\frac{dy}{ds} = 6s + r\,.$$

Integrating the differential equation for $y(s)$ we find $y(s) = 3s^2 + rs + D$ for D a constant. Applying the initial conditions for $y(s)$ we find $y(s) = 3s^2 + rs + r^3$. Using these, we have that the differential equation for $u(s)$ is given by

$$\frac{du}{ds} = 3s^2 + rs + r^3\,.$$

Integrating this we get

$$u(s) = s^3 + \frac{rs^2}{2} + r^3 s + E\,,$$

for a constant E. Applying the initial conditions for $u(s)$ gives

$$u(s) = s^3 + \frac{rs^2}{2} + r^3 s + e^r \sin(r^3)\,.$$

Thus the solution is represented in parameterized form as

$$\begin{aligned} x &= 6s + r \\ y &= 3s^2 + rs + r^3 \\ u &= s^3 + \frac{rs^2}{2} + r^3 s + e^r \sin(r^3)\,. \end{aligned}$$

We are told that $(1, 1)$ is on the same characteristic as $(7, 5)$. This means that

$$\begin{aligned} 1 &= 6s + r \\ 1 &= 3s^2 + rs + r^3\,, \end{aligned}$$

for some r and s. Note that $s = 0$ and $r = 1$ satisfy the above system. Now if $(7, 5)$ is on the same characteristic then there needs to be a value for s such that

$$\begin{aligned} 7 &= 6s + 1 \qquad (1) \\ 5 &= 3s^2 + s + 1\,, \qquad (2) \end{aligned}$$

Now Equation 1 is satisfied by $s = 1$ and that this value of s also satisfies Equation 2. The value of u at $(x, y) = (7, 5)$ is then given by

$$u = s^3 + \frac{rs^2}{2} + r^3 s + e^r \sin(r^3)\Big|_{s=1,r=1} = 1 + \frac{1}{2} + 1 + e\sin(1) = \frac{5}{2} + e\sin(1)\,.$$

Problem 5

Equation 7 in the book is

$$u_x + c u_y = 0\,,$$

which can be written as

$$\frac{d}{dx} u(x, y(x)) = 0\,,$$

if

$$\frac{dy}{dx} = c\,.$$

Next equation 15 in the book is

$$u_x + y u_y = 0\,.$$

If we parameterize characteristic in (x, y) as function of a variable s as $x = x(s)$ and $y = y(s)$ then taking the s derivative of u can be expressed as

$$\frac{d}{ds} u(x(s), y(s)) = \frac{dx}{ds} u_x + \frac{dy}{ds} u_y\,.$$

This will equal the given differential equation if $\frac{dx}{ds} = 1$ and $\frac{dy}{ds} = y$.

Problem 6

Example 7 in the book is

$$6u_x + x u_y = y\,,$$

with $u(x, y) = 4$ when $x = y$. The method of characteristics seeks to find functions $x(s)$, $y(s)$, and $u(s)$ such that

$$\begin{aligned}
\frac{dx}{ds} &= 6 \\
\frac{dy}{ds} &= x \\
\frac{du}{ds} &= y\,,
\end{aligned}$$

with initial conditions given by

$$\begin{aligned}
x(0) &= r \\
y(0) &= r \\
u(0) &= 4\,.
\end{aligned}$$

Integrating the differential equation for $x(s)$ and enforcing the initial conditions we find

$$x(s) = 6s + r\,.$$

This means that the differential equation for $y(s)$ is given by

$$\frac{dy}{ds} = 6s + r\,.$$

Integrating and enforcing the initial conditions we find

$$y = 3s^2 + rs + r\,.$$

Finally, the differential equation for $u(s)$ is then

$$u'(s) = 3s^2 + rs + r\,,$$

Integrating and enforcing the initial conditions we find

$$u = s^3 + \frac{1}{2}rs^2 + rs + 4\,.$$

From $x = 6s + r$ we have $r = x - 6s$. If we put that into the expression for $y(s)$ we find

$$y = 3s^2 + s(x - 6s) + (x - 6s) = -3s^2 + (x - 6)s + x\,.$$

Given (x, y) the value of s that satisfies the above must satisfy

$$-3s^2 + (x - 6)s + x - y = 0\,,$$

or

$$s^2 + \left(2 - \frac{x}{3}\right)s + \frac{1}{3}(y - x) = 0\,.$$

Solving this with the quadratic equation gives

$$s = \frac{-\left(2 - \frac{x}{3}\right) \pm \sqrt{\left(2 - \frac{x}{3}\right)^2 - \frac{4}{3}(y - x)}}{2}\,.$$

The fact that we have two signs above (i.e. a $\pm$) results in a different solution when we use the two different values of s (i.e. s_- or s_+) to evaluate $u(s, r)$ above.

The book solved for s in terms of r (rather than r in terms of s like I did here). Solving the equation in that way will still give two roots to the quadratic equation and two different values for $u(x, y)$.

Problem 7

Our differential equation is given by

$$u_x + u_y = u^2\,,$$

with the boundary conditions that

$$u(x, y) = y \quad \text{on} \quad x + y = 0\,.$$

Let the initial conditions be parameterized by r and s where r "selects" the characteristic curve and s "moves" along it so that

$$\begin{aligned} x(s;r) &= r \\ y(s;r) &= -x = -r \\ u(s;r) &= y = -r\,. \end{aligned}$$

Now parameterized by s the total derivative of $u(s)$ is given by

$$\frac{du}{ds}(x(s),y(s)) = \frac{dx}{ds}u_x + \frac{dy}{ds}u_y = u^2\,.$$

To match the given differential equation we must have

$$\begin{aligned} \frac{dx}{ds} &= 1 \\ \frac{dy}{ds} &= 1 \\ \frac{du}{ds} &= u^2\,, \end{aligned}$$

with the initial conditions

$$\begin{aligned} x(0) &= r \\ y(0) &= -r \\ u(0) &= -r\,. \end{aligned}$$

Integrating the differential equation for $x(s)$ and applying the initial conditions we get $x(s) = s + r$. Integrating the differential equation for $y(s)$ and applying the initial conditions we get $y(s) = s - r$. The differential equation for $u(s)$ can be written as

$$\frac{du}{u^2} = ds\,.$$

Integrating both sides gives

$$-\frac{1}{u} = s + C\,,$$

for a constant C. This means that

$$u = -\frac{1}{s+C}\,.$$

The initial conditions on u mean that C must satisfy

$$u(0) = \frac{-1}{C} = -r \quad \text{so} \quad C = \frac{1}{r}\,.$$

Using this we have that $u(s)$ looks like

$$u(s) = \frac{-1}{s+\frac{1}{r}} = \frac{-r}{sr+1}\,.$$

Thus the solution is represented in parameterized form as

$$
\begin{aligned}
x &= s + r \\
y &= s - r \\
u &= -\frac{r}{1 + sr}\,.
\end{aligned}
$$

In the above we can solve for r and s in terms of x and y to get

$$
\begin{aligned}
r &= \frac{1}{2}(x - y) \\
s &= \frac{1}{2}(x + y)\,.
\end{aligned}
$$

Using these in the expression for u above we get

$$
u(x, y) = \frac{-\frac{1}{2}(x - y)}{\frac{1}{2}(x - y)\frac{1}{2}(x + y) + 1} = \frac{2(y - x)}{4 + (x - y)(x + y)}\,.
$$

Lets check that this is a solution. First when $x + y = 0$ we have $x = -y$ and on that boundary $u(x, y)$ above becomes

$$
u(x, y) = \frac{2(2y)}{4} = y\,,
$$

as it should.

Next for the x derivative of u we find

$$
\begin{aligned}
u_x &= \frac{-2}{4 + (x - y)(x + y)} - \frac{2(y - x)}{(4 + (x - y)(x + y))^2}(x + y + x - y) \\
&= \frac{2}{(4 + (x - y)(x + y))^2}\left(-4 - (x - y)(x + y) - (y - x)(x + y) - (y - x)(x - y)\right)\,.
\end{aligned}
$$

For the y derivative of u we find

$$
\begin{aligned}
u_y &= \frac{2}{4 + (x - y)(x + y)} - \frac{2(y - x)}{(4 + (x - y)(x + y))^2}(-(x + y) + (x - y)) \\
&= \frac{2}{(4 + (x - y)(x + y))^2}\left(4 + (x - y)(x + y) + (y - x)(x + y) - (y - x)(x - y)\right)\,.
\end{aligned}
$$

Using these we see that

$$
u_x + u_y = \frac{2}{(4 + (x - y)(x + y))^2}\left[-2(y - x)(x - y)\right] = \frac{4(y - x)^2}{(4 + (x - y)(x + y))^2} = u^2\,,
$$

as it should.

Problem 8

Our differential equation is

$$u_x + 2u_y = u\,,$$

with $u = 1$ when $y = 2x$. Let the characteristics of this problem be parameterized by r and s where r "selects" the characteristic curve and s "moves" along it. In that coordinate system, the differential equation can be described as

$$\begin{aligned} \frac{d}{ds}u(x(s), y(s)) &= u \quad \text{when} \\ \frac{dx}{ds} &= 1 \\ \frac{dy}{ds} &= 2\,, \end{aligned}$$

and the initial conditions can be described as

$$\begin{aligned} x(0) &= r \\ y(0) &= 2r \\ u(0) &= 1\,. \end{aligned}$$

Integrating the differential equation for x and applying the initial condition gives $x = s + r$. Integrating the differential equation for y and applying the initial condition gives $y = 2s+2r$. The differential equation for u is

$$\frac{du}{ds} = u\,,$$

or integrating gives

$$\ln(u) = s + C\,,$$

for some constant C. The initial conditions on u give $u(s = 0) = 1$ so $C = 0$ and

$$\ln(u) = s \quad \text{or} \quad u = e^s\,.$$

From the above we have

$$\begin{aligned} x &= s + r \\ y &= 2(s + r)\,. \end{aligned}$$

Notice that we cannot "invert this" to solve for s and r in terms of x and y. This means that this boundary value problem has no solution.

Problem 9

Our differential equation is

$$uu_x + u_y = 1\,,$$

with $u = r$ on the curve $x = r^2$ and $y = r^2$. Let the characteristic of this problem be parameterized by r and s where r "selects" the characteristic curve and s "moves" along it. In that coordinate system, the differential equation can be described as

$$\frac{dx}{ds} = u$$
$$\frac{dy}{ds} = 1$$
$$\frac{du}{ds} = 1\,.$$

With initial conditions described as

$$x(0) = r^2$$
$$y(0) = 2r$$
$$u(0) = r\,.$$

Integrating the differential equation for y and applying the initial condition gives $y = s + 2r$. Integrating the differential equation for u and applying the initial condition gives $u = s + r$. The differential equation for x then takes the form

$$\frac{dx}{ds} = u = s + r\,,$$

or integrating with respect to s (and applying the initial conditions) gives

$$x(s) = \frac{s^2}{2} + rs + r^2\,.$$

Thus the solution is represented in parameterized form as

$$x = \frac{s^2}{2} + rs + r^2$$
$$y = s + 2r$$
$$u = s + r\,.$$

Problem 10

Our differential equation is

$$u_x + 2u_y = y\,,$$

with $u = r^2$ on the curve $x = \cos(r)$ and $y = \sin(r)$. Let the characteristic of this problem be parameterized by r and s where r "selects" the characteristic curve and s "moves" along it. In that coordinate system, the differential equation can be described as

$$\frac{dx}{ds} = 1$$
$$\frac{dy}{ds} = 2$$
$$\frac{du}{ds} = y\,.$$

With initial conditions described as

$$x(0) = \cos(r)$$
$$y(0) = \sin(r)$$
$$u(0) = r^2 .$$

Integrating the differential equation for x and applying the initial condition gives $x = s + \cos(r)$. Integrating the differential equation for y and applying the initial condition gives $y = 2s + \sin(r)$. The differential equation for u then takes the form

$$\frac{du}{ds} = y = 2s + \sin(r) ,$$

or integrating with respect to s (and applying the initial conditions) gives

$$u = s^2 + s\sin(r) + r^2 .$$

Thus the solution is represented in parameterized form as

$$x = s + \cos(r)$$
$$y = 2s + \sin(r)$$
$$u = s^2 + s\sin(r) + r^2 .$$

Problem 11

Our differential equation is

$$a(x)u_x + b(x)u_y = 0 ,$$

where $a(x)$ and $b(x)$ are functions of x only. Let the characteristic of this problem be parameterized by r and s where r "selects" the characteristic curve and s "moves" along it. In that coordinate system, the differential equation can be described as

$$\frac{dx}{ds} = a(x)$$
$$\frac{dy}{ds} = b(x)$$
$$\frac{du}{ds} = 0 .$$

With initial conditions described as

$$x(0) = x_0$$
$$y(0) = y_0$$
$$u(0) = u_0 .$$

Integrating the differential equation for u and applying the initial condition gives

$$u(s) = u_0 ,$$

Integrating the differential equation for x and y is done by integrating the coupled system

$$\frac{dx}{ds} = a(x)$$
$$\frac{dy}{ds} = b(x)\,,$$

with initial conditions of

$$x(0) = x_0$$
$$y(0) = y_0\,.$$

For all u on the curve (x, y) given by the solution to the above coupled differential equation we will have $u = u_0$ a constant. I don't know of a way to get that curve explicitly.

Quasi-Linear Second-Order Equation; Characteristics

Notes on the Text: The Derivation of the Characteristic Curves

Starting with the equation

$$au_{xx} + bu_{xy} + cu_{yy} + e = 0\,. \tag{3}$$

WWX: Working here.

References

[1] G. Corliss. Which root does the bisection algorithm find? *SIAM Review*, 19(2):325–327, 1977.

[2] W. Ferrar. *A text-book of convergence*. The Clarendon Press, 1938.