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① $(x_1,x_2)+(y_1,y_2)=(x_1+y_2,x_2+y_1)$

Check each condition in turn. I.

$(x_1,x_2)+(y_1,y_2) = (y_1,y_2)+(x_1,x_2)$

$\parallel$

$(x_1+y_2,x_2+y_1) \neq (y_1+x_2,y_2+x_1)$ so the 1st condition is violated

The 2nd condition will be violated also

3 is satisfied with $(0,0)$

4 is satisfied with $(-x_2,-x_1)$

5 is satisfied

6 is satisfied

7 is satisfied

8 is Not satisfied ~~sati~~. let $5=2+3$ so $c_1=2$ + $c_2=3$

Then $(c_1+c_2)(x) = (5x_1,5x_2)$

$\stackrel{?}{=} 2(x_1,x_2)+3(x_1,x_2) = (2x_1,2x_2)+(3x_1,3x_2)$

$= (2x_1+3x_2,2x_2+3x_1)$ which is Not true

② (1) is satisfied

(2) is satisfied

(3) is satisfied

(4) is satisfied

(5) is Not satisfied

(6) is satisfied

(7) is satisfied

(8) is satisfied

③ (a) cx is Not in the half space so this space is Not a subspace

Of the axioms of ~~linear algebra~~ a vector space

(1) is ~~Not~~ satisfied

(2) is satisfied

(3) is satisfied w/ 0

(4) is Not satisfied

(5) is satisfied

(6) satisfied but each side is Not guaranteed to be in the required half space ($\mathbb{R}^+$) c_1 or $c_2 < 0$

(7) The same comment as in 6 applies

(8) The same comment as in 6 applies

(b) $x + y \equiv xy$

$+ \; cx = x^c$

The unique zero vector is $x \equiv 1$ (in rule # 3)

The unique vector $(-x)$ such that $x + (-x) = 0$ is the reciprocal element of x.

$$(c_1 c_2) x = x^{c_1 c_2}$$

$$c_1(c_2 x) = c_1(x^{c_2}) = (x^{c_2})^{c_1} = x^{c_1 c_2}$$ yes rule # 6 holds

Rule # 7 states that

$$c(x+y) = c(xy) = (xy)^c = x^c \cdot y^c = cx + cy$$ which is true

Rule # 8 states that

$$(c_1 + c_2)x = x^{c_1 + c_2} = x^{c_1} x^{c_2} = c_1 x + c_2 x$$

The zero vector is the unit 1

The vector -2 is the $1/2$

④ Much of this problem is skipped since $A = c\begin{bmatrix}1 & -1\\ 1 & -1\end{bmatrix}$

The smallest subspace containing A are all multiples of the elementary matrix $\begin{bmatrix}1 & -1\\ 1 & -1\end{bmatrix}$

⑤ (a) let the subspace be all multiples of $\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$

(b) Yes because the element

$A + (-1)B = I$ must be in the space

(c)

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(5) (a) Let M consist of all matrices that are multiples of the matrix $\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$.

(b) ~~[illegible]~~ Yes since $1\cdot A + (-1)\cdot B$ must be in this space

[illegible] (c) let the ~~subspace~~ subspace consist of all matrices defined by $a\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} + b\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$

(6) $h(x) = 3(x^2) - 4(5x) = 3x^2 - 20x$

(7) ~~Rule # 7 is no longer true since~~

~~[illegible]~~ ≠ ~~[illegible]~~ ~~The expression~~

~~c(x+y) is interpreted as~~

~~[illegible]~~ ~~f(x) + g(x)~~

≠ ~~the expression~~

cx ~~+ cy is inter~~

Rule # 8 is ~~interpreted~~ ~~as~~ No longer true since

$(c_1+c_2)x$ ≠ ~~f((c_1+c_2)x)~~ is interpreted as

~~f((c_1+c_2)x)~~ $f((c_1+c_2)x)$ ≠

$c_1x + c_2x$ is interpreted as $f(c_1x) + f(c_2x)$

~~[illegible]~~ But in general for arbitrary functions

$$f((c_1+c_2)x) \neq f(c_1x) + f(c_2x)$$

⑧ The 1st rule is $x+y = y+x$ is broken

since $f(g(x)) \neq g(f(x))$ in general

Rule # 2 ~~would~~ ~~be~~ ~~broken~~ ~~[illegible]~~ ~~for the~~ ~~same~~ ~~[illegible]~~ ~~reason~~ is alright.

Rule #3 is ~~[illegible]~~d ~~[illegible]~~ correct with ~~[illegible]~~ the zero vector defined to be x

Rule #4 is ~~[illegible]~~ correct if we define $-x$ to be the inverse of the function $f(\cdot)$ then the rule states that

$f(g(x)) = x$

$f(f^{-1}(x)) = x$ Assuming an inverse of f exists

Rule #7 is broken since

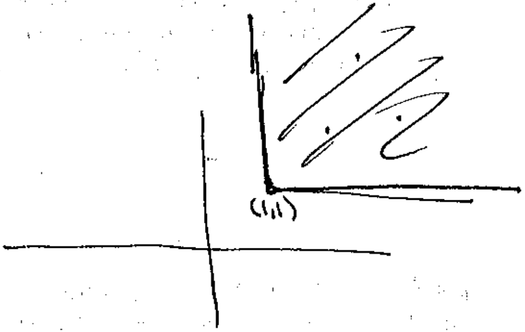
$C(x+y)$ is $cf(g(x))$

& $cx+cy$ is ~~$c(f(x))$~~ $cf(cg(x))$ which are ~~[illegible]~~ Not the same in ~~[illegible]~~. general

Rule # 8 is broken since the left hand side interpreted as $(c_1+c_2)x = (c_1+c_2)f(x)$

While the right hand side is interpreted as

$$c_1 x + c_2 x = c_1 f(c_2 f(x))$$ which are Not equal in general



⑨ ~~[illegible]~~ (a) let the vectors

~~be defined by allowing any x + y ≥ 0~~

~~[illegible]~~

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c\begin{pmatrix} 1 \\ 0 \end{pmatrix} + d\begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad c \geq 0 \ \ \& \ \ d \geq 0$$

Then this set is the upper right corner shown above

~~But~~ ~~[illegible]~~ = ~~[illegible]~~ the sum of any two vectors in the set will also be in the set.

The vector $\frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$ will Not however

(b) let the set consist of the x + y axis (all the points on them). Then for any point x on the axis cx is also on the axis but the point x+y will almost certainly Not be

⑩

(a) Yes

(b) No since $c(b_1,b_2,b_3) = c(1,b_2,b_3)$ is Not in the set if $c = \frac{1}{2}$

~~[illegible]~~ (c) No since if two vectors ~~have~~ ~~$b_1b_2b_3 = 0$~~ x & y are such that $x_1x_2x_3 = 0$ & $y_1y_2y_3 = 0$ there is No gaurentee that ~~x~~ ~~0~~ $x+y$ will have that property. Consider

$$x = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad y = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

(d) Yes this is a subspace

(e) Yes this is a subspace

(f) No since if $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ has this property then

cb should have this property but $cb_1 \le cb_2 \le cb_3$ ~~that~~ might Not be true consider $b = \begin{pmatrix} -100 \\ -10 \\ -1 \end{pmatrix}$ & $c = -1$

Then $b_1 \le b_2 \le b_3$ but ~~$cb_1 \le$~~ $cb_1 \le cb_2 \le cb_3$ is Not true.

⑪ (a) All matrices of the form $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$

$\forall\ a,b \in \mathbb{R}$

(b) All matrices of the form $\begin{bmatrix} a & a \\ 0 & 0 \end{bmatrix}$ $a \in \mathbb{R}$

(c) All matrices of the form $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ or diagonal matrices

⑫ let vector $v_1 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$ + $v_2 = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}$

Then $v_1 + v_2 = \begin{pmatrix} 5 \\ 1 \\ -2 \end{pmatrix}$ but $5 + 1 - 2(-2) = 10 \neq 4$

so the sum is Not on the plane

⑬ ~~This~~ The plane parallel to the previous plane P is

$x + y - 2z = 0$

let vector $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ + $v_2 = \begin{pmatrix} 1 \\ 0 \\ \frac{1}{2} \end{pmatrix}$ which are

both on P_0. Then $v_1 + v_2 = \begin{pmatrix} 2 \\ 1 \\ \frac{3}{2} \end{pmatrix}$

+ checking $2 + 1 - 2(\frac{3}{2}) = 0$ we see it is true

(14) (a) lines, R^2 itself, or (0,0,0)

(b) R^4 itself, hyperplanes of dimension 4 (one constraining equation among 4 variables") that goes through the origin ie ~~eqs~~ constraints ~~[illegible]~~ like the follows

$$ax_1 + bx_2 + cx_3 + dx_4 = 0 \quad *$$

constraints involving 2 equations like the above (going through the origin)

$$\left.\begin{array}{l} ax_1 + bx_2 + cx_3 + dx_4 = 0 \\ Ax_1 + Bx_2 + Cx_3 + Dx_4 = 0 \end{array}\right\} \text{this is effectively a 2D plane}$$

& Constraints involving 3 equations like * & going through the origin, & finally the origin itself.

this is effectively a 1 D line.

(15) (a) line

(b) point (0,0,0)

(c) let x & y ∈ $S \cap T$

Then $x+y \in S \cap T$

& $cx \in S \cap T$ since x & y are both in S & T

which are subspaces & ∴ ~~it in~~ ~~x+y~~ ~~$S \cap T$~~ x+y & cx are both in $S \cap T$

(16) a plane (if the line is in the plane to begin with or all of $\mathbb{R}^3$, ~~since~~

(17) (a) let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ + $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

which are both invertible. $A+B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ which ~~i~~ is Not

thus the set of invertible matrices is Not a subspace

(b) let ~~$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ + $B =$~~

$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ + $B =$ $\begin{bmatrix} 6 & 3 \\ 2 & 1 \end{bmatrix}$

which are both singular

But $A+B =$ $\begin{bmatrix} 7 & 6 \\ 4 & 6 \end{bmatrix}$ which

is Not singular, showing that the set of ~~singular~~ singular matrices is Not a subspace

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(18) (a) True since if A & B are symmetric then

$(A+B)^T = A^T + B^T = A + B$ is symmetric

& $(cA)^T = cA^T = cA$ is symmetric

(b) True since if A & B are ~~skey~~ skew symmetric

then $(A+B)^T = +A^T + B^T = -A - B = -(A+B)$

& A+B is skew symmetric

& if A is skew symmetric then cA is also

since $(cA)^T = cA^T = -cA$

(c) False since if $A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$ which is unsymmetric

& $B = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$ which is unsymmetric

then $A+B = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ should be unsymmetric but it's not

so the set of unsymmetric matrices is not closed under addition & ∴ is Not a subspace.

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(19) If $A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ then the column space is given by

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{pmatrix} x_1 + 2x_2 \\ 0 \\ 0 \end{pmatrix}$$

Which is a line in x the x-axis all combinations of elts on the x-axis

If $B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}$ then the column space of B is

$\begin{pmatrix} x_1 \\ 2x_2 \\ 0 \end{pmatrix}$ or the x + y plane.

If $C = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 0 \end{bmatrix}$ then Cx is given by $\begin{pmatrix} x_1 \\ 2x_1 \\ 0 \end{pmatrix}$

~~or the xy plane~~ or a line in the xy plane.

(20) (a) $\left[\begin{array}{ccc|c} 1 & 4 & 2 & b_1 \\ 2 & 8 & 4 & b_2 \\ -1 & -4 & -2 & b_3 \end{array}\right]$

let $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

Then $E_{21}\left[\begin{array}{ccc|c} 1 & 4 & 2 & b_1 \\ 2 & 8 & 4 & b_2 \\ -1 & -4 & -2 & b_3 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 4 & 2 & b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 + b_1 \end{array}\right]$

So $b_2 = 2b_1$ + $b_1 = -b_3$

(b) $$\begin{bmatrix}1&4\\2&9\\-1&-4\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$

let $E_{21} = \begin{bmatrix}1&0&0\\-2&1&0\\0&0&1\end{bmatrix}$ + $E_{31} = \begin{bmatrix}1&0&0\\0&1&0\\1&0&1\end{bmatrix}$

Then ~~$E_{21}\begin{bmatrix}1&4\\2&9\\-4&-4\end{bmatrix} = \begin{bmatrix}1&4\\0&1\\0&0\end{bmatrix}$~~

$$E_{31}E_{21}\left[\begin{array}{cc|c}1&4&b_1\\2&9&b_2\\-1&-4&b_3\end{array}\right] = \left[\begin{array}{cc|c}1&4&b_1\\0&1&b_2-2b_1\\0&0&b_1+b_3\end{array}\right]$$

= Requiring $b_1 + b_3 = 0$ or $b_1 = -b_3$

(21) A combinations of the columns of B_A $+C$ are also a combination of the columns of A. These two matrices have the same column span.

(22) For the 1st system

$$\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$

For any values of b ~~to~~ the system will have a solution

For the 2nd system

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 0 & 1 & 1 & b_2 \\ 0 & 0 & 0 & b_3 \end{array}\right]$$ we must have ~~[illegible]~~ ~~$b_3=0$~~ $b_3=0$

For the 3rd system

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 0 & 0 & 1 & b_2 \\ 0 & 0 & 1 & b_3 \end{array}\right]$$ ~~Any values of $b_1, b_2, + b_3$ will have a solution~~

~~23~~ $$\left[\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 0 & 0 & 1 & b_2 \\ 0 & 0 & 0 & b_3-b_2 \end{array}\right]$$ so $b_2=b_3$

(23) Unless b is a combination of the ~~[illegible]~~ previous columns of A

if $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ & with $b=\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ has a larger column space

but if $A=\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ + $b=\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ the column space doesn't change

Because b can be written as a linear combination of the column of A + ∴ Adds No new information to the column space

(24) The column space of AB is contained in ~~the~~ ~~column~~ + possibly equal to the column space of A.

If $A=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ~~then~~ + $B=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ which is of smaller ~~[illegible]~~ dimension than the original column space of A.

(25) If ~~$z = y+$~~ $z = x + y$ is a solution to $Az = b + b^*$
If $b + b^*$ are in the column space of A then so is $b + b^*$.

(26) If A is any 5×5 invertible matrix has $\mathbb{R}^5$ as its column space. Since $Ax = b$ always has a solution then A is invertible.

(27) (a) ~~If $b \notin C(A)$~~
~~Then the sum of two~~ ~~Yes this is true~~

(b) True

(c) True

(d) False the matrix I adds a full set of pivots (linear ~~[illegible]~~ independent rows)

let $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ w/ $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

A has a column space of the zero vector +

$A - I = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ has all of $\mathbb{R}^2$ as its column space

(a) False let $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ then $x_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ + $x_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

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are each not in the column space but $x_1 + x_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is in the column space. Thus the set of vectors ~~in the column~~ not in the column space is Not a ~~[illegible]~~ subspace.

㉘ $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$ or $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

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① ~~The matrices (a) + (b) ARE in ordinary echelon Form~~.

For the metric (a)

$$A = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix}$$

let $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Then $E_{21}A = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix}$

let $E_{33} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ then $E_{33}E_{21}A = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \equiv U$

Which has pivot variables x_1 + x_3
+ Free variables in $x_2, x_4,$ + x_5

For the matrix (b)

$A = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 8 & 8 \end{bmatrix}$ let $E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

Then $E_{32}A = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{bmatrix} = U$

Then the free variables are x_3 +
the pivot variables are $x_1 + x_2$.

(2) Since the ordinary echelon form for the matrix in (a) is

$$U = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We find a special solution ~~to each free variable~~ that corresponds to each free variable by assigning ones to each free variable in turn. + then solve for the ~~other~~ pivot variables

For example since the free variables are $x_2, x_4, + x_5$

let $\cancel{x_2=0}$ $x_2=1$, $x_4=0$, $x_5=0$ then our system is

$$\begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ 1 \\ x_3 \\ 0 \\ 0 \end{bmatrix} = 0$$

$$\rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$$

Giving $x_3 = 0$ + $x_1 = -2$

So our special solution is given by ~~-2 0~~ $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Let $x_2=0$, $\cancel{x_3=0}$, $x_4=1$, $x_5=0$

Then our special solution is

$$\begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ 0 \\ x_3 \\ 1 \\ 0 \end{bmatrix} = 0$$

Then $\begin{bmatrix}1&2\\0&1\\0&0\end{bmatrix}\begin{bmatrix}x_1\\x_3\end{bmatrix} = \begin{bmatrix}-4\\-2\end{bmatrix}$

The $x_3 = -2$ $\quad x_1 + 2(-2) = -4$

$\Rightarrow x_1 = 0$

Then the 2nd special solution is given by $\begin{bmatrix}0\\0\\-2\\1\\0\end{bmatrix}$

Our final special solution is obtained by setting

$x_2 = 0,\ x_4 = 0,\ x_5 = 1$

Then our system is

$\begin{bmatrix}1&2&2&4&6\\0&0&1&2&3\\0&0&0&0&0\end{bmatrix}\begin{bmatrix}x_1\\0\\x_3\\0\\1\end{bmatrix} = 0$ which reduces to

$\begin{bmatrix}1&2\\0&1\\0&0\end{bmatrix}\begin{bmatrix}x_1\\x_3\end{bmatrix} = \begin{bmatrix}-6\\-3\end{bmatrix}$ So $x_3 = -3$

$+ x_1 = -6 - 2(-3) = -6 + 6 = 0$

giving $\begin{bmatrix}0\\0\\-3\\0\\1\end{bmatrix}$

Lets check: let $N = \begin{bmatrix}-2&0&0\\1&0&0\\0&-2&-3\\0&1&0\\0&0&1\end{bmatrix}$ then

$$AN = \begin{bmatrix}1&2&2&4&6\\0&0&1&2&3\\0&0&0&0&0\end{bmatrix}\begin{bmatrix}-2&0&0\\1&0&0\\0&-2&-3\\0&1&0\\0&0&1\end{bmatrix} = \begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$$ ✓

For the matrix in (b) we have $U=\begin{bmatrix}2&4&2\\0&4&4\\0&0&0\end{bmatrix}$

then the pivot variables are $x_1 + x_2$,
while the free variables of x_3.

Setting $x_3 = 1$ we obtain the system

$$\begin{bmatrix}2&4\\0&4\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}-2\\-4\end{bmatrix}$$

So $x_2 = -1$ + $x_1 = \frac{-2-(4)(4)}{2} = \frac{-2+4}{2} = 1$

Giving a special solution of $\begin{bmatrix}1\\-1\\1\end{bmatrix}$

③ From problem #2 we have 3 special solutions

$$v_1 = \begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix},\ v_2 = \begin{bmatrix}0\\0\\-2\\1\\0\end{bmatrix}\ +\ v_3 = \begin{bmatrix}0\\0\\-3\\0\\1\end{bmatrix}$$

Then any solution to $Ax = 0$ can be expressed as a linear combination of these special solutions. The nullspace of A contains the vector $x=0$ only when its ~~[illegible]~~ there are no free variables or there exist n pivot variables

④ The reduced echelon form R has ones in the pivot columns of U.
For problem #1 (a) we ~~[illegible]~~ have

$$U = \begin{bmatrix}1&2&2&4&6\\0&0&1&2&3\\0&0&0&0&0\end{bmatrix}$$

Then let $E_{13} = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Then $E_{13}U = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \equiv R$

The null space of R is equal to the null space of U since row opporations don't change the null space

For [illegible] Question #1 pt (b) our matrix U is given by

$U = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{bmatrix}$ then let $E_{12} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Then $E_{12}U = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{bmatrix}$ let $D = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Then $DE_{12}U = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

⑤ For pt (a) we have that

$A = \begin{bmatrix} -1 & 3 & 5 \\ -2 & 6 & 10 \end{bmatrix}$ then

Let $E_{21} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$ we get that

$E_{21}A = \begin{bmatrix} -1 & 3 & 5 \\ 0 & 0 & 0 \end{bmatrix}$

Then since $E_{21}^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ we have that

$$A = E_{21}^{-1} U = \underbrace{\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}}_{\equiv L} \begin{bmatrix} -1 & 3 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

For pt (b) we have that

$$A = \begin{bmatrix} -1 & 3 & 5 \\ -2 & 6 & 7 \end{bmatrix}$$

Then E_{21} is the same &

$$E_{21} A = \begin{bmatrix} -1 & 3 & 5 \\ 0 & 0 & -3 \end{bmatrix}$$

so that $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & 5 \\ 0 & 0 & -3 \end{bmatrix}$

⑥ For pt (a) we have since

$U = \begin{bmatrix} -1 & 3 & 5 \\ 0 & 0 & 0 \end{bmatrix}$ we see that x_1 is a pivot variable & $x_2 + x_3$ are ~~are~~ ~~the~~ ~~three~~ ~~[illegible]~~ free variables

Then 2 special solutions can be ~~[illegible]~~ computed by setting $x_2 = 1, x_3 = 0$ & $x_2 = 0, x_3 = 1$ & solving for x_1. In the 1st case we have

$-x_1 + 3 = 0 \Rightarrow x_1 = 3$ giving a special vector of

$$v_1 = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$$

In the 2nd case we have

$$-x_1 + 5 = 0 \quad \text{giving} \quad x_1 = 5$$

Thus the 2nd special vector is given by $v_2 = \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix}$

~~For a mxn matrix the # of pivot~~

Thus all special solutions to $Ax = 0$ are contained in the set

$$c_1 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}$$

For part (b) we have that

$U = \begin{bmatrix} -1 & 3 & 5 \\ 0 & 0 & -3 \end{bmatrix}$ showing that x_1 & x_3 are pivot variables while x_2 is free.

To solve for the ~~single~~ ~~free~~ vector in the null space set $x_2 = 1$ & solve for x_1 & x_3. This gives

$$\begin{bmatrix} -1 & 3 & 5 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ 1 \\ x_3 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} -1 & 5 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$$

This gives $x_3 = 0$ & $x_1 = 3$

So our special solution is given by $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$

For an $m \times n$ matrix the # of free variables plus the number of pivot variables equals n.

⑦ For pt (a) the Null space of A are all ~~vector~~ pts (x,y,z) Such that

$$\begin{pmatrix} 3c_1 + 5c_2 \\ c_1 + \\ c_2 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

or the plane

$$x = 3y + 5z$$

This is a plane in xyz space. This space can also be described as ~~The~~ ~~span~~ ~~of~~ ~~all~~ All possible linear combinations of the two vectors

$$\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}$$

For pt (b) the null space of A are all pts that are multiples of the vector $\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ this is a line in $\mathbb{R}^3$

or
$x = 3c$
$y = c$
$z = 0$
so $x = 3y$ + $z = 0$

⑧ For pt (a) in problem 5 we have

$$U = \begin{bmatrix} -1 & 3 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

let $D = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ then

$$DU = \begin{bmatrix} 1 & -3 & -5 \\ 0 & 0 & 0 \end{bmatrix}$$ which is in reduced row echelon form

The idenity matrix is simply a I giving

$$DU = \begin{bmatrix} \boxed{1} & -3 & -5 \\ 0 & 0 & 0 \end{bmatrix}$$ for the box around the identity.

For part (b) we have that $U = \begin{bmatrix} -1 & 3 & 5 \\ 0 & 0 & -3 \end{bmatrix}$ so that

Defining $D = \begin{bmatrix} -1 & 0 \\ 0 & -1/3 \end{bmatrix}$ we have that

$$DU = \begin{bmatrix} 1 & -3 & -5 \\ 0 & 0 & 1 \end{bmatrix}$$ is ~~the reduced row echelon form~~.

Then let $E_{13} = \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$ we have that

$$E_{13}DU = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$ for our reduced row echelon form.

our box around the identity in the ~~rows~~ matrix R is around the pivot rows + pivot columns is given by

$$\left[\boxed{\begin{matrix} 1 \\ 0 \end{matrix}} \begin{matrix} -3 \\ 0 \end{matrix} \boxed{\begin{matrix} 0 \\ 1 \end{matrix}} \right]$$

⑨ (a) False this depends on what the reduced echelon matrix looks like - consider $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Then the reduced Echelon matrix R is $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

which has x_2 as a free variable

(b) True. An invertible matrix is defined as one that has a complete set of pivots i.e no free variables

(c) True. since The # of free variables plus the # of pivot variables ~~[illegible]~~ equals n in the case of no free variables we have the maximal # of pivot variables of n.

(d) True. If $m \geq n$ then by pt (c) the # of pivot variables must be less than n & this is equivalent to less than m

If $m < n$ Then we ~~must~~ have fewer equations than unknowns & ~~[illegible]~~ it when reduced to reduced Echelon Form we have a maximal set of pivot variables ~~[illegible]~~ we can have at most m, corresponding to the ~~identity~~ ~~in the~~ block identity in the reduced row echelon form in the $m \times m$ position The remaining $n-m$ variables must be free

⑩ (a) This is Not possible since going from A to U involves zeroing elements below the diagonal only. Thus if an element is non zero above the diagonal it will stay so for all ellimination steps.

(b) let ~~A =~~ [illegible] (whats really required is to have 3 linearly independent columns/rows)

Then let ~~E =~~ [illegible] gives

~~EA =~~ [illegible] let $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -1 & -3 \\ -1 & -2 & -2 \end{bmatrix}$

then with $E = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ we have

$EA = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ contining set $E' = \begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Then $E'EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = R = I$

(c) I don't think its possible for this to be true + the reason is as follows. R must have zeros above each of its pivot variables,

What About $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$ which has No zero entries

Then $U = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ which also equals R.

(d) If $A = U = 2R$ then $R = \frac{1}{2}A = \frac{1}{2}U$

so let $R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \frac{1}{2}A = \frac{1}{2}U$

So

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

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(11) (a) $\begin{bmatrix} 0 & 1 & x & 0 & 0 & x & x \\ 0 & 0 & 0 & 1 & 0 & x & x \\ 0 & 0 & 0 & 0 & 1 & x & x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ Here the symbol "1" corresponds

(b) $\begin{bmatrix} 1 & x & 0 & x & x & 0 & 0 \\ 0 & 0 & 1 & x & x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & x \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 0 & 0 & 1 & x & 0 & x \\ 0 & 0 & 0 & 0 & 0 & 1 & x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(12) (a) $R = \begin{bmatrix} 0 & 1 & x & x & x & x & x & x \\ 0 & 0 & 0 & 1 & x & x & x & x \\ 0 & 0 & 0 & 0 & 1 & x & x & x \\ 0 & 0 & 0 & 0 & 0 & 1 & x & x \end{bmatrix}$ this is & the pivot variables are 2, 4, 5, 6

(b) $\cancel{R = \begin{bmatrix} 1 & x & x & x & x & x \\ 0 & 0 & 1 & x & x & x \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}$ For the free variables to be 2, 4, 5, 6 we have

$$\cancel{R = \begin{array}{c} \begin{bmatrix} 1 & x & 0 & x & 0 & x & x & 0 \\ 0 & 0 & 1 & x & 0 & x & x & 0 \\ 0 & 0 & 0 & 0 & 1 & x & x & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \\ \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{matrix} \end{array}}$$

$$R = \begin{bmatrix} 1 & x & 0 & x & x & x & 0 & 0 \\ 0 & 0 & 1 & x & x & x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) $R = \begin{array}{c} \left[\begin{array}{cccccccc} 0 & 1 & x & x & 0 & x & x & x \\ 0 & 0 & 0 & 0 & 1 & x & x & x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{cccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \end{array} \end{array}$

$R = \begin{array}{c} \left[\begin{array}{cccccccc} 0 & 1 & x & 0 & 0 & x & x & x \\ 0 & 0 & 0 & 1 & 0 & x & x & x \\ 0 & 0 & 0 & 0 & 1 & x & x & x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{cccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \end{array} \end{array}$

⑬ x_4 is certainly a free variable + the special solution is

$x = (0 0 0 1 0)$

⑭ Then x_5 is a free variable. The special solution is

$x = (1, 0, 0, 0, -1)$

⑮ $m \times n$ has r pivots the # of special solutions is $n-r$. The null space contains only zero when $r = n$. The column space is $\mathbb{R}^m$ when $r = m$

⑯ When the matrix has 5 pivots. The column space is $\mathbb{R}^5$ when there are 5 pivots. Since $m = n$ the problem #15 the rank must equal $m = n$

⑰ $A = \begin{bmatrix} 1 & -3 & -1 \end{bmatrix}$

$x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

The free variables are y + z

let $y=1$ + ~~z~~ $z=0$ then $x = 3$ giving the 1st special solution of $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$. The 2nd special solution is given by

$y=0, z=1$

Then $x - 1 = 0$ + $x=1$ so we have

$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

⑱ $x - 3y - z = 12$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

⑰ x in the Nullspace of B means that $Bx = 0$

Thus $ABx = A0 = 0$ + thus is in the null space of AB

then $ABx = 0$

The Null space of B is <u>contained</u> in the Null space of AB

~~Let~~ An obvious example when the ~~Null~~ Null space of AB is larger than that of B is when

$B = \begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}$ which has a Null space given by span of $\begin{bmatrix}0\\ 1\end{bmatrix}$

If $A = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$ then $AB = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$

+ has a Null space given by the span of

$\begin{bmatrix}1\\ 0\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}$ which is larger than that of B

⑳ If A is invertible then Nullspace(AB) = Null(B).

If $v \in$ Nullspace(AB) then

$$ABv = 0 \Rightarrow Bv = 0 \quad \text{by multiplying by } A^{-1}.$$

+ ∴ $v \in$ Nullspace(B)

But that L is invertable

㉑ Looks like x_3 + x_4 are free variables since we set $x_3 = 1$ + $x_4 = 0$ + then $x_3 = 0$ + $x_4 = 1$

$x_3 = 1$, $x_4 = 0$

$$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}2\\ 2\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}1 & 0 & -2\\ 0 & 1 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ 1\end{bmatrix} = \bar{0}$$

In the same way when $x_3 = 0$ & $x_4 = 1$

$$\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_4 \end{bmatrix}$$ combining these two we get

$$\begin{bmatrix} 1 & 0 & -2 & -3 \\ 0 & 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

So $A = \begin{bmatrix} 1 & 0 & -2 & -3 \\ 0 & 1 & -2 & 1 \end{bmatrix}$

(22) $x_4 = 1$ & the other variable are solved ~~fo~~ for

$$\begin{bmatrix} 1 & 0 & 0 & \cancel{0} \\ 0 & 1 & 0 & \cancel{0} \\ 0 & 0 & 1 & \cancel{0} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \cancel{x_4} \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 2 \\ \cancel{4} \end{bmatrix} \quad (1)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

So $A = \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$

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㉓

~~Bt~~ 3 equations , rank = 2 , nullity = 1

∴ $m = 3$ + $n = 2+1 = 3$

let $A = \begin{bmatrix} 1 & 0 & a \\ 1 & 3 & b \\ 5 & 1 & c \end{bmatrix}$ for some a, b, c

Then if $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ is to be in the Nullity of A we have

$$A\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & a \\ 1 & 3 & b \\ 5 & 1 & c \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1+2a \\ 1+3+2b \\ 5+1+2c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

∴ $a = -1/2$; $b = -2$; $c = -3$

Thus our matrix A is then

$$A = \begin{bmatrix} 1 & 0 & -1/2 \\ 1 & 3 & -2 \\ 5 & 1 & -3 \end{bmatrix}$$

㉔ ~~# of equations = 3~~ ; ~~rank = 2 , nullity = 2~~

~~$\Rightarrow n = 2+2 = 4$, then~~

let ~~$A = \begin{bmatrix} 1 & 0 & x & a \\ 1 & 1 & y & b \\ 0 & 1 & z & c \end{bmatrix}$~~ ~~for some x, y, z + a, b, c~~

let ~~$A = \begin{bmatrix} 1 & 0 & a \\ 1 & 1 & b \\ 0 & 1 & c \end{bmatrix}$~~ ~~for some a, b, c~~

(24) The # of equations equals 3 + the rank = 2 We are requiring that the Nullspace be of dimension 2 (2 vectors) Thus $m=3$ + $n=4$. But the dimension of the vectors in the Null space is $3 \neq 4$ + thus it is Not possible to find a matrix with such properties

(25) will $A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}$ work?

then if the column space contains $(1,1,1)$ then $m=3$.

If the nullspace is $(1,1,1,1)$ then $n=4$.

$$A \Rightarrow \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & +1 & 0 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

so If $A\underline{x}=0$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = 0$$

$\Rightarrow$ $x-w=0$; $y-w=0$) $z-w=0$ thus

$$\underline{x} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = w \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$ + our matrix A is correct

(26) A key to solving this problem is to recognize that if the column space of A is also its null space then

$$AA = 0$$

since AA represents A acting on each column of A (& produces zero since the column space is a null space)

Thus we need a matrix A such that

$A^2 = 0$ or if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & cb + d^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

This gives 4 equations for a, b, c, & d.

let $a = 1$ then $d = -1$ ~~[illegible]~~ (By element at (1,2)) & we have

$$\begin{bmatrix} 1 + bc & 0 \\ 0 & cb + 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

let $1 + bc = 0$

say $b = 1$ & $c = -1$ then

$$A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

lets check

then A's row space is given by the span of $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

+ its nullity is given by computing the R matrix

$$R = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{giving} \quad \underline{n} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

(27) In a 3x 3 matrix we have $m=3$ + $n=3$

the column space has dimension r + the nullity must have dimension $n-r$ but r can be either 1, 2, or 3

If we consider each in turn

r	$n-r=3-r$
1	2
2	1
3	0

we never have the column space equal to the row space

(28) If $AB = 0$ then the column space of B is contained in the nullity of A for example

$$AB = A\left[b^1 \mid b^2 \mid \cdots \mid b^n\right] = \left[Ab^1 \mid Ab^2 \mid \cdots \mid Ab^n\right] = 0$$

$$\Rightarrow \quad Ab^1 = Ab^2 = \cdots Ab^n = 0.$$

let $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ which has nullity given by the span of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Then consider $B = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ then

$$AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(29) Almost sure to be the identity.

~~$R = \begin{bmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 0 & x \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & x \end{bmatrix}$~~ ~~let~~ ~~$x \neq 0$~~

with a random 4x3 matrix one is ~~almost~~ most likely to end with

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

(30) (a) let $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ then A has $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ as its null space

but $A^T = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ has $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as its null space

(b) $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ then x_2 is a free variable

but $A^T = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

which has No free variables

A similar case happens when $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ then A has x_2 as a free variable but A^T has x_3 as a free variable

(c) let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \to$ ~~[illegible]~~ $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ has x_1 + x_2 as pivot column

$A^T = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 2 \\ 1 & 0 & 0 \end{bmatrix}$ has x_1 + x_3 as pivot columns

㉛ $A = [I \; I]$

Null space for A is $\begin{bmatrix} I \\ -I \end{bmatrix}$

$B = \begin{bmatrix} I & I \\ 0 & 0 \end{bmatrix}$ has Null space of $\begin{bmatrix} I \\ -I \end{bmatrix}$

$C = I$ has Nullspace of 0

㉜ $x = (2,1,0,1)$ is 4 dimensional so $n=4$

the Null space is a single vector so $n-r=1 \Rightarrow 4-r=1$

$\Rightarrow r=3$ so 3 pivots ~~appear~~ appear in U

㉝ We must have $RN=0$ if $N = \begin{bmatrix} 2 & 3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ then

let ~~[illegible]~~ $R =$ ~~[illegible]~~ $[1 \; -2 \; -3]$

nullity is dimension $=2$

+ $n=3$ $\therefore$ $n-r=2 \Rightarrow 3-r=2 \Rightarrow r=1$ we have only

1 non zero row in R

If $N = \begin{bmatrix}0\\0\\1\end{bmatrix}$ the nullity is dimension 1 & $n=3$

so $n-r=1$

$\Rightarrow$ $3-r=1 \Rightarrow r=2$ we have two

Non zero rows in R

$$R = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\end{bmatrix}$$

If $N=[\ ]$ I'll assume this means $N = \emptyset \begin{bmatrix}0\\0\\0\end{bmatrix}$ the

nullity is of dimension 0 & $n=3$ $\Rightarrow$ $n-r=0$

$\Rightarrow$ $r=3$ & we have 3 Non zero row in R

$$R = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

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(34)

(a) $R = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(b) 1x3 $\cdots$ $\vdots$

~~2~~ 9

$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \end{bmatrix},$

$\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}$

They ar all in Reduced Row echelon form.

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①
~~(a) ~~Ans~~ ~~Yes~~ True~~
~~(b) No~~
~~(c) True~~
~~(d) False~~

② The 3x5 matrix considered is

$$A = \begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 4 & -3 \\ 1 & 3 & 1 & 6 & -4 \end{bmatrix}$$

let $E_{31} \doteq \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ then

$$E_{31}A = \begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 1 & 4 & -3 \end{bmatrix}$$

$$E_{33}E_{31}A = \begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{w/} \quad E_{33} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

so $r = 2$

Then $A = E_{31}^{-1}E_{33}^{-1}R$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Delete the $m-r = 3-2 = 1$ ~~rows~~ ~~then~~ zero rows from R

& the last $m-r$ columns from E

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 4 & -3 \end{bmatrix}$$

Check:

$$\begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 4 & -3 \\ 1 & 3 & 1 & 6 & -4 \end{bmatrix} \quad \text{which is } A$$

③ (a) $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & -2 & -4 & -6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(c) $A = \begin{bmatrix} (-1)^1 & 1 & -1 & 1 \\ (-1)^1 & 1 & -1 & 1 \\ (-1)^1 & 1 & -1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 \end{bmatrix}.$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

④ $$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 3 \\ 2 & 4 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 4 & 6 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \equiv R$$

Then $B = \begin{bmatrix} A & A \end{bmatrix} \Rightarrow \begin{bmatrix} R & R \end{bmatrix}$

& $C = \begin{bmatrix} A & A \\ A & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} R & R \\ 0 & 0 \end{bmatrix}$

⑤ $R = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$

Then B is the $r \times r$ identity matrix

$C = D = 0$

& A is a $r \times (n-r)$ matrix & zeros, since it it was not we would make pivots variables from.

The Null space $N = \cancel{\begin{bmatrix} I \\ -A? \end{bmatrix}} \begin{bmatrix} I \\ 0 \end{bmatrix}$

⑥ R_1 $\quad A_1 = E_1 R_1$

R_2 $\quad A_2 = E_2 R_2$

Then $A_1 + A_2 = E_1 R_1 + E_2 R_2 \underset{\substack{\uparrow \\ \text{if}}}{=} E(R_1 + R_2)$

Require $E = E_1 + E_2$ then this will be true

Thus the ~~e~~ elementary elimination matrices must Add

→ $E_1 =$ $\left[\begin{matrix} \cdot \\ \cdot \end{matrix}\right.$.

Don't follow how to do this ?

⑦ ~~A is mxn then with r pivot columns we will have n-r free variables.~~ A + A^T both have rank r + ∴ A^T must have r pivot columns

let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Then A has pivot variables at x_1 + x_3.

But $A^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ has $x_1 + x_2$ as pivot variables

⑧ If $R = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Then [illegible] [illegible], the pivot ~~column~~ variables are $x_1 + x_2$ while the free variables are $x_3 + x_4$. let $x_3 = 1 + x_4 = 0$ + solve $R\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0$ we get

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \end{bmatrix} \Rightarrow \begin{matrix} x_1 = -2 \\ x_2 = -4 \end{matrix}$$

So $\underline{X} = \begin{bmatrix} -2 \\ -4 \\ 1 \\ 0 \end{bmatrix}$ sim let ~~x_4~~ $x_3 = 0$ + $x_4 = 1$ to get

$$\underline{X} = \begin{bmatrix} -3 \\ -5 \\ 0 \\ 1 \end{bmatrix}$$

Now $y^T R = 0 \iff R^T y = 0$

Then $R^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

So the null space is $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

For $R = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Then $x_1 + x_3$ are free variables while x_2 is a pivot variable

to find the null space let $x_1 = 1$ + $x_3 = 0$ + solve for x_2

This gives an element in the null space of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Let $x_1 = 0$ + $x_3 = 1$ + solve for x_2 to get

$$\begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ x_2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \quad x_2 = -2$$

So the 2nd element of the null space is $\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$

For $y^T R = 0$ consider $R^T = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Then the two vectors in the null space are

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

⑨ $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

So we have a rank of 2 thus we remove $m-r = 2-2 = 0$ row + $n-r = 3-2 = 1$ column, to produce a $r \times r$ invertible matrix. We should remove the free column/variable giving.

$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$ Adding back in the inverse of ellementary matrices that produce the row reduced echelon form gives

$$E^{-1}\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$$

For $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ or rank is $r = 1$

~~So there is a $m-r = 2-1 = 1$ by $n-r = 3$~~

So we should remove $m-r = 2-1 = 1$ row + $n-r = 3-1 = 2$ columns to produce an invertible matrix pick the two free variables, i.e. x_2 + x_3 giving

$[1]$

For $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ we have a rank of 2

So we should remove $m-r = 3-2 = 1$ row

& $n-r = 3-2 = 1$ column. Giving the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

⑩ ~~For~~ Each of the pivot columns can not be constructed from earlier columns, thus by extracting just these r columns No combination of earlier columns can produce any of them. P will have rank r.

⑪ The rank of P^T is $\underline{r}$ also

If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 4 & 7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Thus $P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ so $P^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

let $S^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ only the pivot colums of P^T

So $S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

But we should work this problem on the original matrix not the row reduced one so

$P = \begin{bmatrix}1&2\\2&4\\2&4\end{bmatrix}$ $P^T = \begin{bmatrix}1&2&2\\2&4&4\end{bmatrix}$ which has its 1st 2 columns

as pivot columns let $S^T = \begin{bmatrix}1&2\\2&4\end{bmatrix}$ then $S = \begin{bmatrix}1&2\\2&4\end{bmatrix}$

⑫ (a) If $b_j = \sum_{k=1}^{j-1} \alpha_k b_k$

Then $Ab_j = \sum_{k=1}^{j-1} \alpha_k A b_k = \sum_{k=1}^{j-1} \alpha_k (Ab)_k$ or the same combination of the columns of

AB

(b) Thus AB cannot have any **new** pivot columns (i.e. ones that are not linear combinations of the previous columns since AB cannot produce any non pivot columns than A has) thus

rank(AB) $\leq$ rank(B)

Find A_1 + A_2 such that rank(A_1B) $\equiv$ rank(B)

+ rank(A_2B) $\leq$ rank(B)

Taking $A_1 = I$ gives a matrix such that

rank(A_1B) = rank(B). in general any invertible A_1 will work

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to pg ~~15~~16

For A_2 let $A_2 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ ~~then~~ w/ $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Then $A_2 B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ which has rank 1

so $\text{rank}(A_2 B) = 1 \leq \text{rank}(B) = 2$

For the given B we can still pick $A_1 = I$

+ $A_2 = \underline{0}$.

~~⑬ rank$(B^T A^T) \leq$ rank(A^T). just by replacing A w/ B^T~~

~~+ B w/ A^T in~~

~~the~~

~~rank (AB) ≤ rank (B). How do we show~~

~~rank(AB) ≤ rank(A)? By rank using the fact that~~

~~rank(A^T) rank(A^T) = rank(A)~~

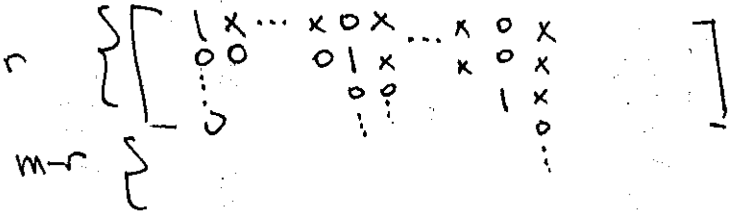
~~+ $B^T A^T = (AB)^T$~~

~~+ rank$((AB)^T)$ = rank(AB) so replacing both both we get~~

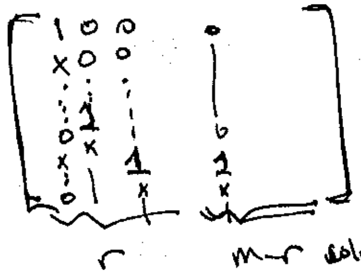
~~rank rank rank(AB) ≤ rank(A) the requested result~~

⑱ So A is $m \times n$

the R is $m \times n$ with ~~[illegible]~~ $m-r$ zeros at the bottom rows



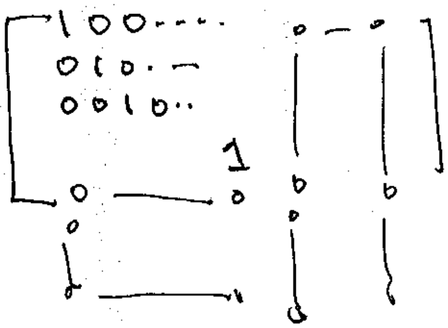
Then $R^T =$



is a matrix of size $n \times m$ with the last ~~last~~ $m-r$ columns zeros.

When we perform rref on R^T we ~~so~~ will get the following

$R^T \to$



$$= \begin{array}{c|c|c} & \leftarrow r \rightarrow & \leftarrow m-r \rightarrow \\ \hline \updownarrow r & I & 0 \\ \hline \updownarrow n-r & 0 & 0 \end{array}$$

with each submatrix of dimension

I is $r \times r$; 0 in the $(1,2)$ block position of $r \times (m-r)$

0 in the $(2,1)$ block position of $(n-r) \times (r)$ & the 0 of position $(2,2)$ ~~$(n-r) \times m-r$~~ $(n-r) \times (m-r)$

Then $(\text{rref}(R'))'$

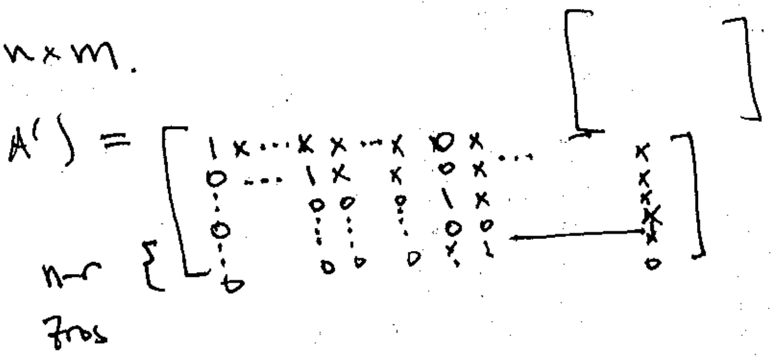
is rref' a matrix

$$\begin{array}{c|c|c} & r & n-r \\ \hline r & I & 0 \\ \hline m-r & 0 & 0 \end{array} \quad m\times n$$

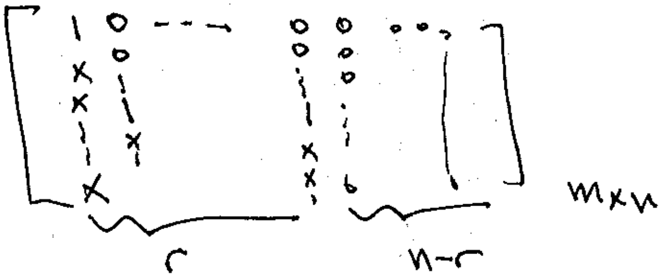
or a matrix with the same size as the original but ~~with the ident~~ of all zeros except for the identity in the upper left corner

(19) A' is of size $n \times m$.

then rref(A') =



Then rref(A')' =



so rref(rref(A')') = $\left[\begin{array}{c|c} I & 0 \\ \hline 0 & 0 \end{array}\right]$

w/ I of size $r \times r$

0 in the (1,2) position

is of size $r \times (n-r)$, the ~~[illegible]~~ block zero in the (2,1) position is of size $(m-r) \times r$ & the ~~[illegible]~~ block zero in the position (2,2) is ~~given by~~ of dimension $(m-r) \times (n-r)$.

Which is the same result as in problem 18.

For one line the rank of A & R are equivalent, thus Y & Z are equivalent, because the form is completely determined by the rank itself which is the same between A & A^T.

(20) $[R \; E] = \text{rref}([A \; I])$

$[S \; F] = \text{rref}([R' \; I])$

So ~~[illegible]~~E $EA = R \Rightarrow R^T = A^T E^T$

& $FR' = S$

E is of size ~~mxn~~ $n \times m$ $\qquad A_{m \times n}$

& F is of size $m \times n$ $\qquad R'_{n \times m}$

So $S = FR^T = F(A^T E^T)$

Then $Z = S^T = EAF^T$ ~~is the relationship between Z~~

So to relate Z to the original R we have

$$A = \cancel{E^{1}} E^{-1} R$$

giving ~~AB~~ $Z = E(E^{-1}R)F^{T} = RF^{T}$

which is a combination of the columns of R.

Problem 2, Section 3.4 Type In 2

~~Now we need 2 special solutions to find to $Ax=0$~~

let $x_2 = 1$ + $x_4 = 0$ + we find that x_1 + x_3 must solve

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = -\begin{bmatrix} 3 \\ 0 \end{bmatrix} \Rightarrow x_1 = -3 \,;\, x_3 = 0$$

let $x_2 = 0$ + $x_4 = 1$ + we find that x_1 + x_3 must solve

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix} \Rightarrow x_1 = 0 \,,\, x_3 = -2 \,.$$

Thus our total solution x to this problem is given by

$$x = \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

We can check this with

$$Ax = \begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{bmatrix}\left(\begin{bmatrix} 1/2 \\ 0 \\ 1/2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 1 \\ 1+2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -3+3 \\ -6+6 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2+2 \\ -8+8 \\ -4+4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

③ consider
$$\begin{bmatrix} 1 & 2 & -2 & b_1 \\ 2 & 5 & -4 & b_2 \\ 4 & 9 & -8 & b_3 \end{bmatrix}$$

Then reduction of this system to reduced row echelon form gives

$$\longrightarrow \begin{bmatrix} 1 & 2 & -2 & b_1 \\ 0 & 1 & 0 & b_2 - 2b_1 \\ 0 & 1 & 0 & b_3 - 4b_1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -2 & b_1 - 2(b_2 - 2b_1) \\ 0 & 1 & 0 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 - 4b_1 - b_2 + 2b_1 \end{bmatrix}$$

) Thus so that the 3rd row is consistent we see that

$$b_3 - 4b_1 - b_2 + 2b_1 = 0$$

or $-2b_1 - b_2 + b_3 = 0$ is required for this system to be solveable. If this condition holds true we have a rank of 2 + a null space of dimension 1.
Solving for a particular solution x_p we set the free variables equal to 0 + solve for the pivot variables. Performing this gives

$$x_1 = b_1 - 2(b_2 - 2b_1) = b_1 - 2b_2 + 4b_1 = 5b_1 - 2b_2$$

+ $x_2 = -2b_1 + b_2$ + $x_3 = 0$

Then to find a basis for the null space set $x_3 = 1$ + solve for x_1 + x_2. We obtain

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$ giving $x_1 = 2$ + $x_2 = 0$

We can check this solution with

$$X = x_p + x_n = \begin{bmatrix} 5b_1 - 2b_2 \\ -2b_1 + b_2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

lets check this solution

$$\begin{bmatrix} 1 & 2 & -2 \\ 2 & 5 & -4 \\ 4 & 9 & -8 \end{bmatrix} \begin{bmatrix} 5b_1 - 2b_2 \\ -2b_1 + b_2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 & 2 & -2 \\ 2 & 5 & -4 \\ 4 & 9 & -8 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5b_1 - 2b_2 + 2(-2b_1 + b_2) \\ 2(5b_1 - 2b_2) + 5(-2b_1 + b_2) \\ 4(5b_1 - 2b_2) + 9(-2b_1 + b_2) \end{bmatrix} + x_3 \begin{bmatrix} 2-2 \\ 4-4 \\ 8-8 \end{bmatrix}$$

$$= \begin{bmatrix} b_1 \\ b_2 \\ 20b_1 - 8b_2 - 18b_1 + 9b_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \underbrace{2b_1 + b_2}_{b_3 \text{ for consistency}} \end{bmatrix}$$

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④ For the 1st problem

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 2 & 5 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

From the augmented matrix

$$A = \begin{bmatrix} 1 & 2 & b_1 \\ 2 & 4 & b_2 \\ 2 & 5 & b_3 \\ 3 & 9 & b_4 \end{bmatrix}$$

then perform row reduction on it obtaining

$$\Rightarrow \begin{bmatrix} 1 & 2 & b_1 \\ 0 & 0 & b_2 - 2b_1 \\ 0 & 1 & b_3 - 2b_1 \\ 0 & 3 & b_4 - 3b_1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & b_1 \\ 0 & 1 & b_3 - 2b_1 \\ 0 & 3 & b_4 - 3b_1 \\ 0 & 0 & b_2 - 2b_1 \end{bmatrix}$$

so $b_2 - 2b_1 = 0$

$$\Rightarrow \begin{bmatrix} 1 & 0 & b_1 - 2(b_3 - 2b_1) \\ 0 & 1 & b_3 - 2b_1 \\ 0 & 0 & b_4 - 3b_1 - 3(b_3 - 2b_1) \\ 0 & 0 & b_2 - 2b_1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 5b_1 - 2b_3 \\ 0 & 1 & -2b_1 + b_3 \\ 0 & 0 & b_4 - 3b_1 - 3b_3 + 6b_1 \\ 0 & 0 & -2b_1 + b_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 5b_1 - 2b_3 \\ 0 & 1 & -2b_1 + b_3 \\ 0 & 0 & 3b_1 - 3b_3 + b_4 \\ 0 & 0 & -2b_1 + b_2 \end{bmatrix}$$

So in addition we must also have $3b_1 - 3b_3 + b_4 = 0$. From the above the rank of this matrix is 2 & since $n=2$ the dimension of the null space is $n-r = 2-2 = 0$. Thus the only solution is

$$x_1 = 5b_1 - 2b_3$$
$$x_2 = -2b_1 + b_3$$

For the 2nd system ~~given~~ again form the augmented matrix

$$\begin{bmatrix} 1 & 2 & 3 & b_1 \\ 2 & 4 & 6 & b_2 \\ 2 & 5 & 7 & b_3 \\ 3 & 9 & 12 & b_4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \\ 0 & 1 & 1 & b_3 - 2b_1 \\ 0 & 3 & 3 & b_4 - 3b_1 \end{bmatrix}$$

exchanging row 2 & row 4 we have

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 0 & 1 & 1 & b_3 - 2b_1 \\ 0 & 3 & 3 & b_4 - 3b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 & b_1 - 2b_3 + 4b_1 \\ 0 & 1 & 1 & b_3 - 2b_1 \\ 0 & 0 & 0 & b_4 - 3b_1 - 3(b_3 - 2b_1) \\ 0 & 0 & 0 & b_2 - 2b_1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 & \cancel{3b_1 - 2b_3}\ 5b_1 - 2b_3 \\ 0 & 1 & 1 & -2b_1 + b_3 \\ 0 & 0 & 0 & \cancel{-b_1 - b_3 + b_4} \\ 0 & 0 & 0 & b_2 - 2b_1 \end{bmatrix} \leftarrow 3b_1 - 3b_3 + b_4$$

Thus ~~cee~~ for a solution we must have

$b_2 - 2b_1 = 0$ + ~~$-b_1 - b_3 + b_4$~~ $= 0$

$ 3b_1 - 3b_3 + b_4$

Thus since $n = 3$ + the rank is $r = 2$ so the dimension of the null space is $n - r = 3 - 2 = 1$.

~~The~~ Setting the free ~~variable~~ variable $x_3 = 0$ we get one particular solution of

$x_1 =$ ~~$3b_1 - b_3$~~ $5b_1 - 2b_3$

+ $x_2 = -$ ~~$2b_1 + b_3$~~ $-2b_1 + b_3$

+ a vector in the null space is given by assigning ~~a~~ the free variable to be 1 + then we obtain

$x_1 =$ ~~2~~ -1

$x_2 = -1$

so $\underline{x} = x_3$ ~~[illegible]~~ $x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

Lets check this

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 5 & 7 \\ 3 & 9 & 12 \end{bmatrix} \left(\begin{bmatrix} 5b_1 - 2b_3 \\ -2b_1 + b_3 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 5b_1 - \cancel{2b_3} - 4b_1 + \cancel{2b_3} \\ 10b_1 - \cancel{4b_3} - 8b_1 + \cancel{4b_3} \\ \cancel{10b_1} - 4b_3 - \cancel{10b_1} + 5b_3 \\ 15b_1 - 6b_3 - 18b_1 + 9b_3 \end{bmatrix} + x_3 \begin{bmatrix} -1 - 2 + 3 \\ -2 - 4 + 6 \\ -2 - 5 + 7 \\ -3 - 9 + 12 \end{bmatrix}$$

$$= \begin{bmatrix} b_1 \\ 2b_1 \\ b_3 \\ -3b_1 + 3b_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \quad \checkmark$$

⑤ Forming the augmented matrix we have

$$[A|b] = \left[\begin{array}{ccc|c} 1 & 3 & 1 & b_1 \\ 3 & 8 & 2 & b_2 \\ 2 & 4 & 0 & b_3 \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} 1 & 3 & 1 & b_1 \\ 0 & -1 & -1 & b_2 - 3b_1 \\ 0 & -2 & -2 & b_3 - 2b_1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 & 1 & b_1 \\ 0 & 1 & 1 & -b_2 + 3b_1 \\ 0 & -2 & -2 & b_3 - 2b_1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -2 & b_1 - 3(-b_2 + 3b_1) \\ 0 & 1 & 1 & -b_2 + 3b_1 \\ 0 & 0 & 0 & b_3 - 2b_1 - 2b_2 + 6b_1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -2 & -8b_1 + 3b_2 \\ 0 & 1 & 1 & 3b_1 - b_2 \\ 0 & 0 & 0 & 4b_1 - 2b_2 + b_3 \end{bmatrix}$$

This matrix has rank 2 & a null space of dimension equal to 1

to have a solution we must have $4b_1 - 2b_2 + b_3 = 0$

& the particular solution ~~if~~ ~~given~~ ~~by~~ can be obtained by setting the pivot variable equal to zero & solving for x_1 & x_2

Thus gives $x_1 = -8b_1 + 3b_2$

+ $x_2 = 3b_1 - b_2$

~~The~~ A basis for the null space can be obtained by setting $x_3 = 1$

+ solving for x_1 + x_2. we get

$$\underline{X} = x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

we can check w/

$$\begin{bmatrix} 1 & 3 & 1 \\ 3 & 8 & 2 \\ 2 & 4 & 0 \end{bmatrix} \left(\begin{bmatrix} -8b_1 + 3b_2 \\ 3b_1 - b_2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} -8b_1 + 3\cancel{b_2} + 9b_1 - 3\cancel{b_2} \\ -24\cancel{b_1} + 9b_2 + 24\cancel{b_1} - 8b_2 \\ -16b_1 + 6b_2 + 12b_1 - 4b_2 \end{bmatrix} + x_3 \begin{bmatrix} 2 - 3 + 1 \\ 6 - 8 + 2 \\ 4 - 4 \end{bmatrix}$$

$$= \begin{bmatrix} b_1 \\ b_2 \\ -4b_1 + 2b_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \checkmark$$

-4

The combination of the rows that gives zero is ~~the~~ a vector such that $y^T A = 0$

or $A^T y = 0$ so y is in the null space of A^T.

~~But from the row ellimination a basis for the~~

One could do Gauss Jordan ellimination on A^T to derive the null space of A^T or consider that

$4b_1 - 2b_2 + b_3 = 0$ so ~~this~~ the combination

$\begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$ of the rows gives the zero vector

The last $m-r$ rows of E give a basis for the left null space

ⓑ (a) let $A = \left[\begin{array}{ccc|c} 1 & 2 & 1 & b_1 \\ 2 & 6 & 3 & b_2 \\ 0 & 2 & 5 & b_3 \end{array}\right]$

Then $\begin{bmatrix} 1 & 2 & 1 & b_1 \\ 0 & 2 & 1 & b_2-2b_1 \\ 0 & 2 & 5 & b_3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 1 & b_1 \\ 0 & 1 & 1/2 & 1/2(b_2-2b_1) \\ 0 & 2 & 5 & b_3 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & b_1-b_2+2b_1 \\ 0 & 1 & 1/2 & 1/2(b_2-2b_1) \\ 0 & 0 & 4 & b_3-b_2+2b_1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 3b_1-b_2\cancel{+2b_1} \\ 0 & 1 & 1/2 & 1/2(b_2-2b_1) \\ 0 & 0 & 1 & \frac{1}{4}(b_3-b_2+2b_1) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 3b_1-b_2 \\ 0 & 1 & 0 & \frac{1}{2}(b_2-2b_1)-\frac{1}{8}(b_3-b_2+2b_1) \\ 0 & 0 & 1 & \frac{1}{4}(b_3-b_2+2b_1) \end{bmatrix}$$

Which has a complete set of pivots & ∴ any vector is in the column space. The rank is 3 & the only combination of the rows that gives zero are when all rows are multiplied by zeros, i.e. No combination of the rows give zero.

For pt (b) we have for an augmented matrix

$$\begin{bmatrix} 1 & 1 & 1 & b_1 \\ 1 & 2 & 4 & b_2 \\ 2 & 4 & 8 & b_3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 & b_1 \\ 0 & 1 & 3 & b_2-b_1 \\ 0 & 2 & 6 & b_3-2b_1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -2 & b_1+b_1-b_2 \\ 0 & 1 & 3 & b_2-b_1 \\ 0 & 0 & 0 & b_3-\cancel{2b_1}-2b_2+\cancel{2b_1} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -2 & 2b_1-b_2 \\ 0 & 1 & 3 & b_2-b_1 \\ 0 & 0 & 0 & b_3-2b_2 \end{bmatrix}$$

Thus for a solution we must have $b_3-2b_2=0$ to be in the column span

so vectors of the form $\begin{pmatrix} b_1 \\ b_2 \\ 2b_2 \end{pmatrix}$.

Combinations of the rows that give zero are

$(0,-2,1)$

⑦ $x_p = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}$

$x_n = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

let x_3 be the pivot variable

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

So the system is

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Then this coefficient matrix has rank 2 + the nullity is given by $3-2=1$

let $x_3 = 0$ to get $x_1 = 2$; $x_2 = 4$

to get the null space let $x_3 = 1$ + solve for x_1 + x_2.

⑧ I ~~case~~ 1×3 system (Not all zeros)

is of rank 1 so the nullity then must be of dimension

$3-1=2$ + would require two basis functions

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⑨ (a) let $v = x_1 - x_2$ then $Av = Ax_1 - Ax_2 = b - b = 0$

(b) Another solution to $Ax = 0$ is $c \cdot v$

since $A(cv) = cAv = c \cdot 0 = 0$

Another solution to $Ax = b$ is $x = x_1 + cv$

since $Ax = Ax_1 + cAv = Ax_1 = b$

⑩ ~~(a) x_p must be multiplied by 1 otherwise combinations are not allowed.~~

~~(b) see prob 13 ... from 3rd edition~~ Next few ~~pages~~

~~(c) see prob 13 from the 3rd edition~~

~~(d) see prob 13 from 3rd edition~~

⑪ x_5 is a free variable. ~~Ro~~ Zero vector is not the only solution to $Ax = 0$ since column 5 ~~can~~ (a free variable) can be expressed as a linear combination of earlier columns. If $Ax = b$ has a solution then it has infinitely many solutions

⑫ row 3 has no pivot. Then that row is zero.

Eq $Ux = c$ is only solvable if the 3rd elt in c is zero. The equation $Ax = b$ (might not be solvable)

one must have ~~(E)~~ if CE is the elementary elimination

Matrix that takes A to U the

$Eb = c$ + the appropriate combination ~~[illegible]~~ of b must sum to

zero i.e $b(3) = 0$ to be solvable

(13) Largest rank is 3. Then there is a pivot in every row of U. The solution to $Ax = b$ always exists. The ~~[illegible]~~ ~~null space~~ ~~of~~ ~~A~~ is column space of A linearly ~~degenerate~~ dependent.

An example might be $A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$

(14) Largest possible rank of 6×4 matrix is 4. There is a pivot in every column of U. The solution ~~of~~ $Ax = b$ is unique (if it exists). The null space of A is the zero vector

An example of A is $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

⑮ $A = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 11 & 5 \\ -1 & 2 & 10 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 4 & 0 \\ 0 & 3 & 5 \\ 0 & 6 & 10 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & 5/3 \\ 0 & 0 & 0 \end{bmatrix}$ which has rank 2

$A^T = \begin{bmatrix} 1 & 2 & -1 \\ 4 & 11 & 2 \\ 0 & 5 & 10 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 6 \\ 0 & 5 & 10 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ which has rank 2

For $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & q \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & q-1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & q-2 \end{bmatrix}$

~~If q=2 A is of~~

If $q=2$ A is of rank 2. If $q \neq 2$ A is of rank 3

$A^T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & q \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & q-1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & q-2 \end{bmatrix}$

Which if $q \neq 2$ is of rank 3 & if $q=2$ is of rank 2

⑯ $A = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 5 \\ 0 & -1 & -4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & -4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 4 \end{bmatrix}$

which is of rank 2.

$$A^TA = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 5 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 6 \\ 1 & 1 & 5 \\ 6 & 5 & 26 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 5 \\ 2 & 1 & 6 \\ 6 & 5 & 26 \end{bmatrix}$$

$\Rightarrow \begin{bmatrix} 1 & 1 & 5 \\ 0 & -1 & -4 \\ 0 & -1 & -4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 5 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$ which is of rank 2

$$AA^T = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} 2+25 & 1+5 \\ 6 & 1+1 \end{bmatrix} = \begin{bmatrix} 27 & 6 \\ 6 & 2 \end{bmatrix}$$

which since $27 \cdot 2 - 12 \neq 0$ is ~~of rank 2~~ invertible & ∴ of rank 2

For $A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & -2 \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ which is of rank 2

$$A^TA = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4+1+1 & 1+2 \\ 1+2 & 1+4 \end{bmatrix}$$

$= \begin{bmatrix} 6 & 3 \\ 3 & 5 \end{bmatrix}$ which since $6 \cdot 5 - 3^2 \neq 0$ is invertible & $\therefore$ of rank 2

Now $AA^T = \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 2 & 3 \\ 2 & 3 & 5 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 1 & 1/2 & 1/2 \\ 2 & 2 & 3 \\ 2 & 3 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1/2 & 1/2 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1/2 & 1/2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Which is of rank 2

⑰ $A = \begin{bmatrix} 3 & 4 & 1 & 0 \\ 6 & 5 & 2 & 1 \end{bmatrix}$

let $E_{21} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$ then

$E_{21}A = \begin{bmatrix} 3 & 4 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{bmatrix} \equiv U$

Then $L = E_{21}^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

So $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 2 & 0 & 3 \\ 0 & 6 & 5 & 4 \end{bmatrix}$$

Then let $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Then $E_{21}A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & -2 & 3 \\ 0 & 6 & 5 & 4 \end{bmatrix}$

let $E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$ then

~~[illegible]~~ $E_{32}E_{21}A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & -2 & 3 \\ 0 & 0 & 11 & -5 \end{bmatrix} = U$

So $A = E_{21}^{-1}E_{32}^{-1}U$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & -2 & 3 \\ 0 & 0 & 11 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & -2 & 3 \\ 0 & 0 & 11 & -5 \end{bmatrix}$$

⑱ (a) $x + y + z = 4$

$A = [1 \quad 1 \quad 1]$ which has rank$=1$ & ∴ a Nullity of dimension $3-1=2$. The free variables are y & z so a particular solution can be obtained by assigning $y=0,\ z=0$ to get $x=4$ & ∴ $x_p = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$. To find the Null space

let ~~$x_2 = 1$ &~~ $y=1$ & $z=0$ & solve for x

to get $x+1+0=0 \Rightarrow x=-1$

giving $v_1 = y\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

let $y=0$ & $z=1$ to get

$x+0+1=0 \Rightarrow x=-1$

giving $v_2 = z\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Thus our solution vector v is

$$v = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

(b) For the 2nd system or ~~[illegible]~~ $A = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & -1 & 1 & 4 \end{array}\right]$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & -2 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

which is ~~of~~ of rank $=2$. Thus z is a free variable. Letting $z=0$ we have a particular solution of $x=4$ + $y=0$. The letting $z=1$ we have a null space solution of $x=-1$ + $y=0$. Giving a total solution of

$$\underline{v} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

⑲ The # of solutions depends on the rank of the column space or the rank of the matrix A. ~~If~~ To have ∞ solutions we must have a null space of dimension of least 1 so $n-r \geq 1$ to begin with. A different right hand side will not change this. If $m > r$ we have zero rows in the reduced row echelon form for A + thus constraints that must be satisfied on the right hand side. If these constraints are not satisfied no solution results

⑳ $A = \begin{bmatrix} 6 & 4 & 2 \\ -3 & -2 & -1 \\ 9 & 6 & q \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2/3 & 1/3 \\ -3 & -2 & -1 \\ 9 & 6 & q \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 2/3 & 1/3 \\ 0 & 0 & 0 \\ 0 & 6-\frac{2}{3}(9) & q-\frac{1}{3}(9) \end{bmatrix} = \begin{bmatrix} 1 & 2/3 & 1/3 \\ 0 & 0 & q-3 \\ 0 & 0 & 0 \end{bmatrix}$

Which ~~has~~ if $q \neq 3$ has rank 2 or if $q = 3$ is of rank 1.

For $B = \begin{bmatrix} 3 & 1 & 3 \\ 9 & 2 & q \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1/3 & 1 \\ 9 & 2 & q \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1/3 & 1 \\ 0 & 2-\frac{q}{3} & [illegible] \end{bmatrix}$

Then [illegible] if $2 - \frac{q}{3} = 0 \Rightarrow q = 6$ B is of rank 1.

if $q \neq 6$ B is of rank 2

㉑ $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & & \\ 4 & & \end{bmatrix}$ Can only have one unique row/column so the others ~~not~~ must be multiples of the 1st

$\Rightarrow A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ 4 & 8 & 16 \end{bmatrix}$

$B = \begin{bmatrix} 2 & & \\ 1 & & \\ 2 & 6 & -3 \end{bmatrix}$ Again can only have multiples of a given row or column so constructing rows 1+2 from 3 we have

$$B = \begin{bmatrix} 2 & 6 & -3 \\ 1 & 3 & -3/2 \\ 2 & 6 & -3 \end{bmatrix}$$

$M = \begin{bmatrix} a & b \\ c & \cdot \end{bmatrix}$ Follows the same pattern

$$\cancel{A} \rightarrow \begin{bmatrix} a & b \\ c & \frac{c}{a} \cdot b \end{bmatrix}$$

(22) The column ~~space~~ space is a line in $\mathbb{R}^m$.

The Nullspace is a plane in $\mathbb{R}^n$. The column space of A^T = row space of A is a line in $\mathbb{R}^n$.

(23) $A = uv^T$ for u column v the multiples of this column to get A

$$A = \begin{bmatrix} 3 & 6 & 6 \\ 1 & 2 & 2 \\ 4 & 8 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \end{bmatrix}$$

For $A = \begin{bmatrix} 2 & 2 & 6 & 4 \\ -1 & -1 & -3 & -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 & 2 \end{bmatrix}$

㉔ A is rank if the 2nd row of U is zero.

let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

$4+12=16$
$2+6$
$4+12$
$8+24=32$

㉕ $AB = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 2 & 1 & 4 \\ 3 & 3/2 & 6 \end{bmatrix} = \begin{bmatrix} 8 & 4 & 16 \\ 16 & 8 & 32 \end{bmatrix}$

which is rank 1

$$AM = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 & b \\ c & bc \end{bmatrix} = \begin{bmatrix} 1+2c & b+2bc \\ 2+4c & 2b+4bc \end{bmatrix}$$

$$= \begin{bmatrix} 1+2c & b+2bc \\ 2(1+2c) & 2b(1+2c) \end{bmatrix}$$ which is rank 1,

㉖ $(u \cdot v^T)(w z^T) = u(v^T w)z^T = (v^T w) \cdot u z^T$

times the number $v^T w$. This has rank one unless

$v^T w = 0$

㉗ ~~(a) $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ b_2 \end{bmatrix}$~~ ~~if $b_2 = 2$ we have~~ a

(a) see next pg

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

has $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ in the null space & ∴ always has an ∞ # of solutions

(a) out

$$\cancel{\begin{bmatrix}1&0\\0&1\\\Phi&\Phi\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}}$$

Consider $\begin{bmatrix}1&0\\0&1\\1&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$

then we require ~~b_1~~ $x+y=b_3$ or $b_1+b_2=b_3$ ~~[illegible]~~ a solution, if this is true then the solution is $x=b_1$ & $y=b_2$ ~~if~~ & only one solution, if not we have no solutions

(c) $\begin{bmatrix}1&0&0\\0&1&0\\\Phi&\Phi&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$

Again ~~#~~ ~~[illegible]~~ we must have

$x+y=b_3$ but $x=b_1$ & $y=b_2$ so we have

with the constraint that $b_1+b_2=b_3$ if true then we have

an ∞ # of solutions (Null space is not empty) if Not

No solution

(d) $A=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$

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㉘ (a) To have no solution we must have a contradictory equation. Thus $r < m$ so that one row in the reduced row echelon form ~~R~~ is all zero (the corresponding component in b are will then not be zero). ~~Then~~ n

(b) $r < n$ since ~~there~~ ~~columns~~ we must have ~~element~~ a non zero null space i.e. more columns (free variables) than pivots (the rank of A)

(c) Again as in part (a) we must have $r < m$ ~~&~~ ~~some~~ ~~b's will have an opposite component such that~~ In this case if E is the elimination matrix that turns A into R. i.e. $EA = R$ ~~or~~ when $r < m$ will have zero rows. Then $Ax = b$ requires

$$EAx = Eb$$

+ then ~~to~~ the zero rows of EAx must have the corresponding zero elements of Eb, thus ~~requires~~ restricting the types of b's that solutions exist for

(d) If $m = n = r$ then we have one solution for every b.

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(29) $$U = \begin{bmatrix}2&4&4\\0&3&6\\0&0&0\end{bmatrix} \to \begin{bmatrix}1&2&2\\0&1&2\\0&0&0\end{bmatrix} \to \begin{bmatrix}1&0&-2\\0&1&2\\0&0&0\end{bmatrix}$$

For $$U = \begin{bmatrix}2&4&4\\0&3&6\\0&0&5\end{bmatrix} \to \begin{bmatrix}1&2&2\\0&1&2\\0&0&5\end{bmatrix} \to \begin{bmatrix}1&0&-2\\0&1&2\\0&0&1\end{bmatrix}$$

$$\to \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

(30) Since U is square each pivot has pivot magnitude of 1 & zeros above and below: i.e. it is the identity

(31) $$\begin{bmatrix}U & 0\end{bmatrix} = \begin{bmatrix}1&2&3&0\\0&0&4&0\end{bmatrix} \Rightarrow \begin{bmatrix}1&2&0&0\\0&0&1&0\end{bmatrix}$$

set the free variable $x_2 = 1$ to obtain the null solution

so $x_3 = 0$

$x_1 = -2$

so $$x_n = x_2\begin{bmatrix}-2\\1\\0\end{bmatrix}$$

to find x_p perform Gauss Jordan ellimination we have $\begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 4 & 8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

So setting the x_2 variable $x_2 = 0$ then $x_1 = -1$ + $x_3 = 2$

so $\underline{x} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

㉜ $\begin{bmatrix} U & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 6 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

So x_1 + x_3 are pivot variables

x_2, ~~[illegible]~~ ca ~~[illegible]~~ is a free variables

the null solution ~~[illegible]~~ pick $x_2 = 1$ + solve for x_1 + x_3

$x_1 = 0$ + $x_3 = 0$ giving $\underline{x}_n = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

The particular solution is given by

$\begin{bmatrix} U & c \end{bmatrix} = \begin{bmatrix} 3 & 0 & 6 & 9 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 5 \end{bmatrix}$

But from the last row No solution can exist

(33) $Ax = b$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 2 & 3 & 2 \\ 1 & 3 & 2 & 0 & 5 \\ 2 & 0 & 4 & 9 & 10 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 2 & 3 & 2 \\ 0 & 3 & 0 & -3 & 3 \\ 0 & 0 & 0 & 3 & 6 \end{bmatrix}$$ Gauss elimination

$$\Rightarrow \begin{bmatrix} 1 & 0 & 2 & 3 & 2 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 2 & 0 & -4 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$ Gauss-Jordan elimination

The pivot variables are x_1, x_2, + x_4 with x_3 a free variable.

The particular solution is given by taking $x_3 = 0$ + we get

$x_1 = -4$; $x_2 = 3$; $x_4 = 2$ giving

$$x_p = \begin{bmatrix} -4 \\ 3 \\ 0 \\ 2 \end{bmatrix}$$

For the ~~homogeneous~~ homogeneous solution set the free variable to 1 & solve for the pivot variables

i.e. $x_1 = -2$, $x_2 = 0$, $x_3 = 1$, $x_4 = 0$

giving $$x_n = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

(34) The only solution of $Ax = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is $x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

So A must be 2×3 + the requirement that

$Ax = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ requires that $A = \begin{bmatrix} x & 1 \\ x & 2 \\ x & 3 \end{bmatrix}$

~~which has rank~~ to have 1 solution the rank must equal n the # of variables so that the null space is zero. This can be obtained if we fill A w/ $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 0 & 3 \end{bmatrix}$

For B the only solution to

$Bx = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is $x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ so B is ~~3x2~~ 2×3

Thus ~~we~~ ~~so~~ to have only one solution we must have a zero null space so $n - r = 0 = 3 - r = 0 \rightarrow r = 3$ but this is not possible since we ~~would~~ would have to see 3 ~~orthogonal~~ ~~columns~~ ~~that~~ ~~linear~~

$$\begin{bmatrix} x & x & x \\ x & x & x \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

~~independent~~

3 linearly independent vectors which is not possible since they are elements of $\mathbb{R}^2$. (+ only two linearly independent vectors can exist. So this is not possible.

35 let the concatenated system be

$[A|b_1|b_2]$ & we get

$$\cancel{\begin{bmatrix}1&3&1&1&1\\1&2&3&3&0\\2&4&6&6&0\\1&1&5&8&0\end{bmatrix}} \qquad A = \begin{bmatrix}1&3&1\\1&2&3\\2&4&6\\1&1&5\end{bmatrix}$$

let $\cancel{E_{21}A = \begin{bmatrix}1&0&0&0\\1&1&0&0\\0&0&1&0\\0\end{bmatrix}}$

$$\rightarrow \begin{bmatrix}1&3&1\\0&-1&2\\0&-2&4\\0&-2&4\end{bmatrix} \Leftrightarrow \quad \text{When } E_1 = \begin{bmatrix}1&0&0&0\\-1&1&0&0\\-2&0&1&0\\-1&0&0&1\end{bmatrix}$$

$$\rightarrow \begin{bmatrix}1&3&1\\0&1&-2\\0&-2&4\\0&\underset{-2}{\cancel{0}}&\underset{4}{\cancel{0}}\end{bmatrix} \quad \text{w/ } E_2 = \begin{bmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

$$\rightarrow \begin{bmatrix}1&3&1\\0&1&-2\\0&0&0\\0&0&0\end{bmatrix} \quad \text{w/ } E_3 = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&2&1&0\\0&2&0&1\end{bmatrix}$$

$\Leftrightarrow$ so $E_3 E_2 E_1 A = L$

$\Rightarrow$ $A = E_1^{-1} E_2^{-1} E_3^{-1} L$

$$\therefore\ A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{bmatrix} U$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} U$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} U$$

Then $LUx = b$

let $Ly = b$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 6 \\ 5 \end{bmatrix}$$

$\Rightarrow\ y_1 = 1\ ;\ -1y_2 = 3-1 = y_2 = -2\ )$

$$y_3 = 6 - 2y_1 - 2y_2 = 6 - 2(1) - 2(-2)$$
$$= 4 + 4 = 8$$

$$y_4 = 5 - y_1 - 2y_2 = 5 - 1 - 2(-2) = 5 - 1 + 4 = 8$$

Then solve for

$$Vx = y$$

$$\Rightarrow \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 8 \\ 8 \end{bmatrix}$$

which can have No solution since the last two rows are Not zero. So we have no solution.

For the 2nd set of b's, we have

$Ly = b$ is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$y_1 = 1$; $1 - y_2 = 0 \Rightarrow y_2 = 1$; $2 + 2 + y_3 = 0 \Rightarrow y_3 = -4$;

$1 + 2 + y_4 = 0 \Rightarrow y_4 = -3$

So $y = \begin{bmatrix} 1 \\ 1 \\ -4 \\ -3 \end{bmatrix}$

then $Vx = y$ is

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -4 \\ -3 \end{bmatrix}$$

~~Again~~, ~~no~~ ~~solution~~ ~~exists~~

③⑥ $Ax = A\begin{bmatrix}1\\0\end{bmatrix} + cA\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\3\end{bmatrix}$

So $A = \begin{bmatrix}1 & x\\3 & x\end{bmatrix}$ to have a nullspace of $\begin{bmatrix}0\\1\end{bmatrix}$

we want have $A = \begin{bmatrix}1 & 0\\3 & 0\end{bmatrix}$

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① To show $v_1, v_2,$ + v_3 are independent ~~bet~~ we can construct a matrix with $v_1, v_2,$ + v_3 as its column + show that this matrix has 3 pivots. Such a matrix looks like

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

+ has 3 pivots $\therefore$ $v_1, v_2, + v_3$ are independent

To show that $v_1, v_2, v_3,$ + v_4 are dependent we have to show that a matrix with these vectors as columns will have at least one free variable.

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix} \Rightarrow \text{[illegible]} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

Which has x_4 as a free variable + $\therefore$ $v_1, v_2, v_3,$ + v_4 cannot be linearly independent.

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② We can have at maximum 4 linearly independent vectors from the span of vectors in $\mathbb{R}^4$.

$v_1, v_2, + v_3$ are linearly independent by observation. To find the linearly independent vectors we can form the matrix of concatenated column vectors + perform row reduction

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = R$$

$\Leftrightarrow$ which we see has rank $\underline{3}$ + pivot variables in the $x_1, x_2, + x_3$ variables or the columns of A (column 1, 2 + 3) are linearly independent. ~~The el elementary elimination matrices~~

used to obtain R are $E_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$E_{32} =$ [illegible] $\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$$E_{43} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

So we have the

$$E_{43}E_{32}E_{21}A = R$$

& $E_{43}E_{32}E_{21} = E$ so $A^T E^T = R^T$

So $E^{-1} = E_{21}^{-1} E_{32}^{-1} E_{43}^{-1}$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

So

$$\cancel{A =} E^T R =$$

Thus $R^T =$

To determine the combination of the vectors that result in the zero vector we are looking for the null space of A. setting each free variable equal to 1, in turn we have

$$x_4=1,\ x_5=0,\ x_6=0 \Rightarrow x_1=1,\ x_2=-1,\ x_3=0$$

giving $\underline{X} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

If $x_5=1,\ x_4=0,\ x_6=0 \rightarrow x_1 = \cancel{0}\ 1,\ x_2=0,\ x_3=-1$

$$\underline{X} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

If $x_6=1,\ x_4=0,\ x_5=0 \rightarrow x_1=0,\ x_2=1,\ x_3=-1$

giving

$$\underline{X} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

To determine the invariant subspace that All 6 columns must be a member of we are looking for y's such that

$$Y^T A = 0$$

or $A^T y = 0$, thus we are looking for the null space for A.

Since $A^T = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$ ✓ we have (reducing A^T) that

$$\Rightarrow \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} ✓ \Rightarrow \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} ✓$$

$$\Rightarrow \cancel{\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}} \quad \cancel{\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & & \end{bmatrix}}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which has ~~an~~ x_4 as a free variable + an element of the null space given by

$\underline{x}_n = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

σ

So ~~this~~ ~~span~~ ~~is~~ each of the v_i vectors is in the hyperplane

$$x_n^T v_i = 0 = [1\ 1\ 1\ 1]\, v_i = 0$$

③ For the columns of σ to be ~~dep~~ independent we would need to be able to apply elementary row operations to ~~#~~ σ to produce R. This is possible if we can ÷ by a, d + f requiring that $a \neq 0$, $d \neq 0$ + $f \neq 0$ if any of these cases is ~~incorrect~~ false σ does not have linearly independent columns

④ The only solution to $\sigma x = 0$ can be obtained by back substitution + gives $x=0$. It is unique since every step is ~~[illegible]~~ ~~invertable~~ reversible + involves ~~invertabl~~ ~~opport~~ unique operations

Since the only linear combinations of the columns of σ that gives zero is the zero vector the columns are linearly independent.

⑤ (a) Consider the Nullspace of $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -7 \\ 0 & -1 & -5 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 7/5 \\ 0 & 1 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 3-\frac{7}{5}(2) \\ 0 & 1 & \frac{7}{5} \\ 0 & 0 & \underbrace{5-\frac{7}{5}}_{\neq 0} \end{bmatrix}$$

Which since this matrix has rank = 3, it is invertible & ∴ the columns are linearly independent.

(b) Consider ~~A~~= the null space of $A = \begin{bmatrix} 1 & 2 & -3 \\ -3 & 1 & 2 \\ 2 & -3 & 1 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -3 \\ 0 & 7 & -7 \\ 0 & -7 & 7 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & -3 \\ 0 & 7 & -7 \\ 0 & 0 & 0 \end{bmatrix}$$ which has rank = 2

& ∴ the 3 vectors are linearly dependent

⑥ let $U = \begin{bmatrix} 2 & 3 & 4 & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ From the U given

the free variable is x_3 & ∴ 3 independent columns are v_1, v_2, v_4, another obvious choice is v_1, v_2, v_3, another choice is can be obtained from v_2, v_3 + v_4 since together $v_2 + v_3$ span the dimension of $x_1 + x_2$.

Since A is simply $E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix}$ times

U the ~~same~~ same columns from U chosen are linearly independent in A.

⑦ Consider ~~[illegible]~~ $1\cancel{v_2 + 1v_2} + 1v_3$

$= \cancel{w_2} - \cancel{w_3} + \cancel{w_1} - \cancel{w_3} + w_1 - \cancel{w_2} =$

$v_1 + v_3 \cancel{= w_2 - w_3 + w_1 - v_2 = -w_3 + w_1}\ v_2$

$= \cancel{-v_2}$

---

consider $v_1 - v_2 + v_3 = \cancel{w_2} - w_3 - \cancel{w_1} + w_3 + \cancel{w_1} - \cancel{w_2}$

$= 0$

Since a non zero linear combination of these vectors gives zero, they are linearly dependent

⑧ consider $c_1v_1 + c_2v_2 + c_3v_3 = 0$ or in terms of $w_1, w_2, \& w_3$ we have

$$c_1w_2 + c_1w_3 + c_2w_1 + c_2w_3 + c_3w_1 + c_3w_2 = 0$$

$$\Rightarrow \quad (c_2 + c_3)w_1 + (c_1 + c_3)w_2 + (c_2 + c_1)w_3 = 0$$

Since $w_1, w_2, \& w_3$ are linearly independent the only sum that gives the zero vector are when

All coefficients are zero, or

$$c_2 + c_3 = 0$$

$$c_1 + c_3 = 0$$

$$c_2 + c_1 = 0$$

This system has only the solution $c_1 = c_2 = c_3 = 0$ & ∴ the vectors $v_1, v_2,$ & v_3 are independent.

⑨

(a) These four vectors are dependent because the dimension of the space $\mathbb{R}^3$ is 3.

(b) Two vectors will be dependent if one is a multiple of the other

(c) The vectors v_1 & $\underline{0}$ are dependent because given $c \neq 0$

$0 \cdot v_1 + c\,\underline{0} = 0$ a non zero linear combination gives zero

⑩ Consider the matrix which represents the planes null space

$$A = [\,1 \quad 2 \quad -3 \quad -1\,]$$

The plane $\overset{\text{as a constraint}}{}$ will reduce the dimension of the space $\mathbb{R}^4$ (which is 4) by 1 (the rank of the matrix A) thus $4-1=3$ linearly independent vectors in this space are possible. To find them consider the ~~[illegible]~~ ~~reduced~~ ~~echelon~~ ~~form~~ ~~for~~ null space of A the free variables are x_2, x_3, + x_4 so the Null space is spanned by

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

⑪ (a) Since $(1,1,-1)$ + $(-1,-1,1)$ are ~~[illegible]~~ ~~linear~~ ~~combinations~~ ~~of~~ multiples of each other the subset of $\mathbb{R}^3$ is a line in $\mathbb{R}^3$.

(b) Since $(0,1,1)$ + $(1,1,0)$ are linearly independent in $\mathbb{R}^3$ they span a plane

(c) Since we have 2 pivots we effectively have only two degrees of freedom in the columns of this matrix + thus the span of ~~this~~ these columns is a plane in $\mathbb{R}^3$.

(d) This is Not a subspace since all linear combinations of vectors with this property won't still have this property. But if we extend it to all of $\mathbb{R}^3$, then these vectors will span that space

⑫ When there is a solution to $Ax=b$. Vector c is in the row space of A when there is a solution to

$y^T A = c$.

The row space is the span of the columns of A^T which always contains the zero vector

⑬ Since U is ~~obtained~~ obtained from A by elementary row operations the row spaces must be the same.

~~So~~ The dimensions of the given spaces are:

dim (column space of A) = dim(column space of U) = 2

row space of A = row space of U which has dimension 2

⑭ (a) Pick $\underline{x} = (0,0,0,0)$ then S has dimension 0

(b) Pick $\underline{x} = (1,1,1,1)$ then S has dimension 1

(c) Pick $\underline{x} =$ ~~$(1,1,0,0)$~~ $(1,1,-1,-1)$ because all permutations are $\perp$ to $(1,1,1,1)$

(d) Pick $\underline{x} = (1,0,0,0)$ then S has dimension 4

⑮ let $x = v+w$

& $y = v-w$

Then $v = \frac{1}{2}(x+y) = \frac{1}{2}(v+w+v-w)$

& $w = \frac{1}{2}(x-y) = \frac{1}{2}(v+w-(v-w))$

The two pairs of vectors span the same subspace. They are a basis for the space when v + w are a basis for the original space

⑯ $v_1, v_2, \dots, v_n$ are linearly independent they span a space of dimension n. They are a basis for that space. It columns of a m by n matrix then $m \geq n$.

⑰ ~~(a)~~ (a) $(1,1,1,1)$

(b) So we are looking for the Null space of

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$$ or the span of

$$\mathbf{x} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

(c) So we are looking for the Null space of

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -1 \end{bmatrix}$$

~~S~~ which is

$$\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

(d) The column space is spanned by $\begin{bmatrix}1\\0\end{bmatrix} \begin{bmatrix}0\\1\end{bmatrix}$

The ~~FEC~~ Null space is spanned by

$$\begin{bmatrix}-1\\0\\1\\0\\0\end{bmatrix} \quad \begin{bmatrix}0\\-1\\0\\1\\0\end{bmatrix} \quad \begin{bmatrix}-1\\0\\0\\0\\1\end{bmatrix}$$

⑱ All invertible transformations on the basis $\begin{bmatrix}1\\0\end{bmatrix}$ & $\begin{bmatrix}0\\1\end{bmatrix}$ will also be a basis, examples of such are $\begin{bmatrix}1\\1\end{bmatrix}$ & $\begin{bmatrix}-1\\-2\end{bmatrix}$

$\begin{bmatrix}2\\0\end{bmatrix}, \begin{bmatrix}0\\3\end{bmatrix}$, & $\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}1\\3\end{bmatrix}$

For two different basis for the row spaces again an invertible ~~[illegible]~~ appli linear transformation applied to the rows yields another valid basis, Examples of such transformation are $E_1 = \begin{bmatrix}1 & 2\\2 & 1\end{bmatrix}$ & $E_2 = \begin{bmatrix}3 & 0\\-1 & 1\end{bmatrix}$

Then two additional basis ~~are~~ (equivalent basis for the rows of V are)

$$E_1 V = \begin{bmatrix}1 & 2\\2 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 1 & 0 & 1\\0 & 1 & 0 & 1 & 0\end{bmatrix} = \begin{bmatrix}1 & 2 & 1 & 2 & 1\\2 & 1 & 2 & 1 & 2\end{bmatrix}$$

$$\& \; E_2 V = \begin{bmatrix}3 & 0\\-1 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 1 & 0 & 1\\0 & 1 & 0 & 1 & 0\end{bmatrix} = \begin{bmatrix}3 & 0 & 3 & 0 & 3\\-1 & 1 & -1 & 1 & -1\end{bmatrix}$$

(19) (a) Vectors might not span $\mathbb{R}^4$.

(b) Vectors are not linearly independent

(c) Any 4 of these vectors might be a basis for $\mathbb{R}^4$

(20) If they are linearly independent, ~~they~~ ~~span~~ ~~$\mathbb{R}$~~ the rank of A is A n. If they span $\mathbb{R}^m$ ~~there~~ ~~must~~ ~~be~~ there must be at least m linearly independent columns & the rank must be ~~at~~ ~~least~~ m. If they are a basis for $\mathbb{R}^m$, then there are exactly m of them & the matrix is square & invertible.

(21) To find a basis for the plane $x-2y+3z=0$ we are looking for the null space for the operator $A=[1\ -2\ 3]$ or the span of

$$\begin{bmatrix}2\\1\\0\end{bmatrix} + \begin{bmatrix}-3\\0\\1\end{bmatrix}$$

To find a basis for this plane & the intersection with the xy plane consider the Augmented system ~~with~~ ~~Ab~~ = [with the eq $z=0$ or

$$A = \begin{bmatrix}1 & -2 & 3\\0 & 0 & 1\end{bmatrix}$$ with

~~For~~ has reduced row echelon form of

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ & a Null space spanned by

$$\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

To find a basis of vectors ⊥ to the plane we are looking for y's such that $y^T x = 0$ where x is a point on the plane. Reading the coefficients from the linear constraint we have a basis for all vectors ⊥ to this plane of

$$\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$

㉒ (a) $Ax = 0$ has only the solution $x = 0$ since A is invertible, equivalently has 5 pivots, the columns are independent

(b) $Ax = b$ is solvable since A is ~~inverted~~ invertible & the columns span $\mathbb{R}^5$

㉓ (a) True, the dimension of S is 5 & the dimension of $\mathbb{R}^6$ is 6 requiring one more vector to be added to the basis of S

(b) ~~True~~ False

consider S given by the ~~span~~ column space of $A=\begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Which never has a component of the x_6 variable.

Then let a basis of $\mathbb{R}^6$ be the column span of

$$A' = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$ which occur after removing

one vector will have a span that includes the x_6 variable

(24) From U we see that ~~column~~ x_1 + x_2 are pivot variables + ∴ a basis for the column span of A is given by the span of $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ + $\begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$

While a basis for the column span of U is given by the span of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ + $\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$.

Basis for the row span are given by ~~the~~ ~~sp~~

$\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ + $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ ~~equivalently~~

The basis for the two Null spaces are given by letting $x_3 = 1$ + solving for x_1 + x_2 or the span of $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

~~$x_1 = -3(-1) - 2(1)$~~

$= \cancel{3-2}$

(25)

~~$Ax = 0$ combo of columns~~

~~$x^T A^T = 0$ comb of rows of A^T = comb of columns of A~~

(a)

(a) ~~True the~~ ~~False~~ ~~consider~~ ~~True, the rank of A is the # of linearly independent columns & to have A have linearly dependent columns we must have $r < n$ if this requires $m \geq r$ (if $m = r$ the columns may be linearly independent)~~

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

False ~~consider~~ in general we need $n > m$ with rank $r = m$. For example consider $A \rightarrow$ then the columns are ~~[illegible]~~ dependent (because one column is all zeros) but the rows are linearly independent

(b) False consider $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ Then the span of the rows is the x axis while the span of the columns is the line $y = x$

(c) True each has dimension ~~[illegible]~~ equal to the rank of the matrix

(d) False the basis is the smallest ~~space~~ ~~that~~ number of vectors that span the space. Consider

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$ Then a basis is given by the 1st 3 columns (the 4th is not needed)

(26) For $A = \begin{bmatrix} 1 & 2 & 5 & 0 & 5 \\ 0 & 0 & c & 2 & 2 \\ 0 & 0 & 0 & d & 2 \end{bmatrix}$ By inspection, if $c = 0$ + $d = 2$ A will have rank 2.

$\Leftrightarrow$ For $B = \begin{bmatrix} c & d \\ d & c \end{bmatrix}$ ~~A~~ to have rank 2

it must be invertible so $c^2 - d^2 \neq 0$ + $c^2 \neq d^2$

$\Rightarrow$ $c \neq \pm d$.

(27) (a) let ~~$B_1 =$~~ $B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$; $B_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

+ let $B_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) let $B_1 \neq$ [illegible], B_2, + B_3 be the same as before

+ let $B_1 = \begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}$ + $B_5 = \begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix}$ +

$B_6 = \begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}$

(c) let ~~B_1, B_2~~ $B_1 = \begin{bmatrix}0&-1&0\\1&0&0\\0&0&0\end{bmatrix}$, $B_2 = \begin{bmatrix}0&0&-1\\0&0&0\\1&0&0\end{bmatrix}$ &

$B_3 = \begin{bmatrix}0&0&0\\0&0&-1\\0&1&0\end{bmatrix}$

㉘ ~~$V_1 = \begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}$, $V_2 = \begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$, $V_3 = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ (Nt~~

~~$V_4 = \begin{bmatrix}1&0&0\\0&0&1\\0&0&0\end{bmatrix}$, $V_5 = \begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}$, $V_6 = \begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}$~~

~~others? $V_7 =$~~

1st only one pivot

$V_1 = \begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}$, $V_2 = \xcancel{\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}}$, $V_3 = \xcancel{\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}}$ Not valid (1,1) = 0

two pivots

No valid
(1,1) = 0

$V_4 = \begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$, $V_5 = \begin{bmatrix}1&0&0\\0&0&1\\0&0&0\end{bmatrix}$, $V_6 = \xcancel{\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}}$ Not valid (1,1) = 0

What about $U_7 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$? This is Not in Row echelon form

since the elt (1,1) ~~should~~ ~~be~~ must be Non zero.

? Not an

(29) If the columns add to zero

$$A = \begin{bmatrix} x & x & x \\ x & x & x \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} =$$

~~Let~~ $A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$ then $A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0$ is equivalent

to

$$\begin{aligned} a+b+c &= 0 \\ d+e+f &= 0 \end{aligned}$$

Since eq is represented by the Null space of the matrix $[1 \;\; 1 \;\; 1]$ [illegible].

Which has a basis of $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Thus I get a basis consisting of $A_1 = \cancel{\begin{bmatrix} -1 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix}}$, $A_2 = \cancel{\begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}}$

$+ \cancel{A_3 =} \cancel{\begin{bmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}} + \cancel{A_4} = \cancel{\begin{bmatrix} -1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix}}$

$$A_1 = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + A_4 = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

For the rows of A to add to zero we must have

$$\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = 0 \quad \text{or}$$

$a + d = 0$

$b + e = 0$

$e + f = 0$

So picking a specific pairing like a+d we have a Null space of $a + d = 0$ or the span of $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Thus a basis for this subspace is given by the span of

$$B_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}, + \quad B_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

(30) From example 4 in section 2.7 consider the linear combination

$$I - P_{31} = \cancel{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \quad \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

then $I - P_{31} - P_{32} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$

then $I - P_{31} - P_{32} + P_{32}P_{21} + P_{21}P_{32} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

constructing the sum in this way is difficult, consider

$$aI + bP_{21} + cP_{32}P_{21} + dP_{31} + eP_{32} + fP_{31}P_{32} = 0$$

$$\Rightarrow \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} + \begin{bmatrix} 0 & b & 0 \\ b & 0 & 0 \\ 0 & 0 & b \end{bmatrix} + \begin{bmatrix} 0 & c & 0 \\ 0 & 0 & c \\ c & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & d \\ 0 & d & 0 \\ d & 0 & 0 \end{bmatrix} +$$

$$\begin{bmatrix} e & 0 & 0 \\ 0 & 0 & e \\ 0 & e & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & f \\ f & 0 & 0 \\ 0 & f & 0 \end{bmatrix} = \begin{bmatrix} a+e & b+c & d+f \\ b+f & a+d & c+e \\ c+d & e+f & a+b \end{bmatrix} = \underline{\underline{0}}$$

or $a+e=0$ & $a+b=0$

$b+c=0$

$d+f=0$

$b+f=0$

$a+d=0$

$c+e=0$

$c+d=0$

$e+f=0$

This is equivalent to the Nullspace of the following operator

a b c d e f

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Using the matlab code elim we obtain the following row reduced echelon form of

$$\Rightarrow \cancel{\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 \end{bmatrix}} \Rightarrow \cancel{\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & & & & \end{bmatrix}}$$

Finish...

(31) (a) All 3x3 matrices

(b) Upper triangular matrices

(c) All ~~3~~ · ~~contai~~ matrices w/ zeros on the ~~off~~ diagonal & a constant value on the diagonal

(32) Basis such that $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = 0$ Thus we are looking for ~~[illegible]~~ two vectors that span the plane $2x + y + z = 0$ i.e. the Null space of $A = [2 \ 1 \ 1]$ this is the span of $\cancel{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}}$ & $\cancel{\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}}$ Thus $\begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix}$

or the span of $\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

Thus let a basis be

$$A_1 = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 2 & 0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 2 \end{bmatrix} \ \& \ A_4 = \begin{bmatrix} -1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

㉝ (a) $y_h = C$

(b) pick $y_p = 3x$

(c) $y = y_p + y_h = 3x + C$

㉞ $y(0) =$ ~~Acos~~ $A + B + C = 0$

Then the Null space of this space is $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Thus ~~A~~ our allowed combination is given by

$y(x) =$ ~~$-C_1 \cos x$~~ · ~~$C_1 \cos 2x$~~ $\begin{bmatrix} \cancel{(-C_1 - C_2)\cos x} \\ \cancel{(C_1)\cos 2x} \\ \cancel{(C_2)\cos 3x} \end{bmatrix}$

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = C_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + C_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Then $y(x) = \begin{bmatrix} A \\ B \\ C \end{bmatrix}^T \begin{bmatrix} \cos x \\ \cos 2x \\ \cos 3x \end{bmatrix} = \left(\begin{bmatrix} -C_1 \\ C_1 \\ 0 \end{bmatrix} + \begin{bmatrix} -C_2 \\ 0 \\ C_2 \end{bmatrix} \right)^T \begin{bmatrix} \cos x \\ \cos 2x \\ \cos 3x \end{bmatrix}$

$$= \begin{bmatrix} -C_1 - C_2 \\ C_1 \\ C_2 \end{bmatrix}^T \begin{bmatrix} \cos x \\ \cos 2x \\ \cos 3x \end{bmatrix} = C_1(-\cos x + \cos 2x) + C_2(-\cos x + \cos 3x)$$

Thus a basis is given by

$y_1(x) = -\cos x + \cos 2x$

& $y_2(x) = -\cos x + \cos 3x$

(35) (a) $\frac{dy}{dx} - 2y = 0$ has solution given by

$y_h(x) = Ae^{2x}$

& ∴ by the span of e^{2x}

(b) $\frac{dy}{dx} - \frac{y}{x} = 0$ has solutions given by

$\frac{dy}{y} = \frac{dx}{x}$

$\Rightarrow$ ~~$e^y = e^x + C$~~ $\ln y = \ln x + C_1 = \ln x + \ln C_2 = \ln(C_2 x)$

$\Rightarrow$ ~~$y = \log(e^x + C)$~~ $\Rightarrow Y = C_2 X$

Thus a basis for the solutions is given by the function x

(36) Of dimension 1 consider y_1, y_2, y_3 given by

$2x, 3x, 5x$

of dimension 2 consider

$2x+1, 3x+5, 5x+3$

of dimension 3 we have

$1, 2x, 3x^2$

(37) let our basis equal $1, x, x^2, x^3$

A basis of this subspace requires that $p(1) = 0$

which restricts the originally free 4 coefficients a, b, c, d in the span of the functions $1, x, x^2, + x^3$ requiring that

$$a + b + c + d = 0$$

This constraint has a null space given by the span of

$$\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

+ thus the ~~set~~ function $(-c_1 - c_2 - c_3) 1 + (c_1) x$

$+ c_2 x^2 + c_3 x^3$ will satisfy the requirement $p(1) = 0$ ∀ $c_1, c_2, + c_3$

or $p(x) = c_1(-1 + x) + c_2(-1 + x^2) + c_3(-1 + x^3)$

Thus the functions

$-1 + x, -1 + x^2, -1 + x^3$ span the space of all 3rd degree polynomials

with value 0 at $x = 1$

(38) When require that

$a+c+d=0$ we are restricting ~~all~~ all possible 4 vectors by one linear constraint, thus will have a dimension of 3. Looking for a basis for the null space for the operator $A = [1 \; 0 \; 1 \; 1]$ we see that $b, c, + d$ are free variables + ∴ the nullity ~~[illegible]~~ ~~[illegible]~~ is spanned by

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

For the space "T" we consider the operator A given by

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

from which we see that ~~a~~ b + d are free variables + the null space is spanned by ~~[illegible]~~ = $(b=1, d=0)$ + $(b=0, d=1)$ the vectors

$$\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

The intersection will have dimension $4-1-2=1$

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① (a) The dimension of the column space ~~[illegible]~~ ~~[illegible]~~ C(A) is 5

The dimension of the row space $C(A^T)$ is 5

The dimension of the Null space N(A) is of ~~[illegible]~~ ~~[illegible]~~ dimension $9-5=4$

The dimension of the left Nullspace $N(A^T)$ is of dimension $7-5=2$

(b) Its column space is $\mathbb{R}^3$

Its left Nullspace is ~~[illegible]~~ spanned by $\begin{pmatrix}\cancel{0}\\ \cancel{0}\\ 0\\ 1\end{pmatrix}$

~~Since its row space must have dimension 3 +~~

~~∴ the left Nullspace has dimension $4-3=1$. By~~ ~~doing~~ ~~reducing~~

~~the matrix A into its row reduced echelon form~~

we ~~will produce a row of~~ ~~four~~ ~~zeros~~ ~~showing~~

~~The test~~ ~~$m-r=$~~

The dimension of the left Nullspace is $3-3=0$ so

the only element in the left Null space is ~~zero~~. the zero vector

② For $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix}$ reducing to row echelon form

gives $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix}$ by $E_{21} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

which has rank = 1 + ∴ a basis for ~~[illegible]~~ C(A) can be obtained

from the 1st column of A or $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

A basis for $C(A^T)$ is given by the 1st row in R or

$\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$, A basis for the null space of A is given

by assigning ones to the free variables & solving for the pivot variables

since x_2 + x_3 are free variables we have for a basis for

the null space is given by

$$\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$$

A basis for the left nullspace is given by the last $m - r = 2 - 1 = 1$

rows of E_{21} or $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

For $B = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & 8 \end{bmatrix}$ we have reducing the following

$\to \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 0 \end{bmatrix}$ with $E_{21} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

Thus B is of rank 2, It has a basis for the

column space given by $\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \end{bmatrix}$

A basis for the row space given by

$\begin{bmatrix}1\\2\\4\end{bmatrix} + \begin{bmatrix}0\\1\\0\end{bmatrix}$ or equivalently $\begin{bmatrix}1\\2\\4\end{bmatrix} + \begin{bmatrix}2\\5\\8\end{bmatrix}$

A basis for the Null space is given by assigning x_3 (the free variable to 1) + solving for $x_1 + x_2$ the pivot variables

giving $\begin{bmatrix}1&2&4\\0&1&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\1\end{bmatrix} \overset{0}{=} \begin{bmatrix}1&2\\0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}-4\\0\end{bmatrix}$

$\Rightarrow x_2 = 0 \Rightarrow x_1 = -4$ so a basis is

$\begin{bmatrix}-4\\0\\1\end{bmatrix}$

~~Further reducing R by with $E_{12} = \begin{bmatrix}1&-2\\0&1\end{bmatrix}$ we get~~

~~$E_{12}\begin{bmatrix}1&2&4\\0&1&0\end{bmatrix} = \begin{bmatrix}1&0&4\\0&1&0\end{bmatrix}$~~

Then ~~$E_{12}E_{21}\begin{bmatrix}1&2&4\\2&5&8\end{bmatrix} = \begin{bmatrix}1&0&4\\0&1&0\end{bmatrix}$~~

$\Rightarrow$ ~~$\begin{bmatrix}1&2&4\\2&5&8\end{bmatrix} = E_{21}^{-1}E_{12}^{-1}\begin{bmatrix}1&0&4\\0&1&0\end{bmatrix}$~~

Since $m=2$ + $r=2$ the dimension of the left nullspace is given by $m-r=0$.

③ Since A is given by ~~[illegible]~~ ~~E~~ as ER, we can read the rank of A from R, which we see is 2. ~~Thus~~ a ~~basis~~ with pivot variables given by x_2 + x_4.

~~[illegible]~~ ~~two~~ Thus a basis for the column space is given by

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$$

A basis for the row space is given by the span of the 1st two rows of A or R.

i.e.
$$\begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}$$

A basis for the null space is given by setting $x_1=1$, $x_3=0$, $x_5=0$ + solving for x_2 + x_4, this gives

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$ + by setting $x_1=0$, $x_3=1$, $x_5=0$

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giving $\begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ + for $x_1 = 0, x_3 = 0, x_5 = 1$

giving $\begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} \cancel{0} & \cancel{1} & 2 & \cancel{3} & 4 \\ \cancel{0} & \cancel{0} & 0 & \cancel{1} & 2 \end{bmatrix} \begin{bmatrix} 0 \\ x_2 \\ 0 \\ x_4 \\ 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} -3 \\ -1 \end{pmatrix}$$

$x_4 = -1$; $x_2 = -3 + 3 = 0$

A basis for the left null space can be obtained by the last $m=3$ minus $r=2$ or 1 rows of E^{-1}

~~This~~ is given by with

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{so} \quad E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & +1 \end{bmatrix}$$

So the last row of E^{-1} is given by ~~than~~

The ~~[illegible]~~ vectors check. $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

Check:

$$\begin{bmatrix} 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ Yes

④ (a) $\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ has the two given vectors as a column space & since the row space is $\mathbb{R}^2$ both $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$

(b) so rank = 1 & dim of nullity = 1

∴ r + dim nullity = n

⇒ n = 2

But the # of components in both the column space vector & the null space vector is $3 \neq 2$

Thus this is not possible

(c) dim null space = 1 + dim left null space

$n - r = 1 + (m - r)$ Must be held [illegible]

let rank = 1 & m = 1 & n = 2

so pick $A = \begin{bmatrix} 1 & 2 \end{bmatrix}$ Then $m = 1$, rank = 1, $n = 2$

dim nullity = $2 - 1 = 1$

dim left null space = $1 - 1 = 0$ & everything is satisfied

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(d) $[1\ 3]\begin{bmatrix}3 & 1\\ a & b\end{bmatrix} = \begin{bmatrix}3+3a & 1+3b\end{bmatrix} = \underline{0}$

$\Rightarrow$ $a = -1$ & $b = -1/3$ so the matrix

$A = \begin{bmatrix}3 & 1\\ -1 & -1/3\end{bmatrix}$ satisfies the ~~[illegible]~~ required conditions

(e) ~~Let~~ ~~A=~~ ~~[illegible]~~ ~~Then~~

$\cancel{\begin{bmatrix}1 & \\ 2 & 2\end{bmatrix}}$ If row space = column space $m=n$

Then dim of null space = $n-r$

& dim of ~~[illegible]~~ left nullspace = $n-r$

~~Let~~ ~~$n=m=2$~~ ~~& $r=1$~~ & ~~they~~ these two spaces have equal dimension

& don't contain the linearly independent

~~[illegible]~~ rows equivalently columns

$\cancel{\begin{bmatrix}1 & 0\\ 2 & 0\end{bmatrix}}$

~~What this is not possible~~

⑤ let $A = \begin{bmatrix}1 & 1 & 0\\ 2 & 1 & 0\end{bmatrix}$

For B to have V as its null space we must have

& ~~let $B = \begin{bmatrix}a & b & c\\ d & e & f\end{bmatrix}$~~ .

Then $B\begin{bmatrix}1\\1\\1\end{bmatrix} = 0$ & $B\begin{bmatrix}2\\1\\0\end{bmatrix} = 0$

which imposes two constraints on B.

let $B = [1 \ a \ b]$ then the 1st condition requires

that $B\begin{bmatrix}1\\1\\1\end{bmatrix} = 1+a+b = 0$

& the second that $B\begin{bmatrix}2\\1\\0\end{bmatrix} = 2+a = 0 \Rightarrow a=-2$

Then $b = -(1+a) = -(1-2) = 1$

so $B = [1 \ -2 \ 1]$

(b) ~~Since~~ Now A has rank 2, $m=3$, & $n=4$

The dimension of its column space is 2
of its row space is 2
& its null space is $n-r=2$
& its left nullspace is $m-r = 3-2=1$

To find basis for each of the spaces we simply ~~[illegible]~~ need to find enough linearly independent vectors.

For the column space pick $\begin{bmatrix}3\\0\\1\end{bmatrix}$ & $\begin{bmatrix}3\\0\\0\end{bmatrix}$

For the row space pick $\begin{bmatrix}0\\3\\3\\3\end{bmatrix}$ & $\begin{bmatrix}0\\1\\0\\1\end{bmatrix}$

For the null space pick $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$ + $\begin{bmatrix}0\\-1\\0\\1\end{bmatrix}$

For the left null space pick $\begin{bmatrix}0\\1\\0\end{bmatrix}$

For B we have $r=1$, $m=3$, $n=1$, then

The dimension of the column space is 1
" " " row space is 1
" " " Null space is 0
" " left null space is $m-r = 2$

A basis for the column space is given by $\begin{bmatrix}1\\4\\5\end{bmatrix}$

A " " " row space is given by $[1]$

A " " " null space is given by 0
" " " left nullspace is given by $\cancel{\begin{bmatrix}-1\\-1\\1\end{bmatrix}} + \cancel{\begin{bmatrix}5\\0\\1\end{bmatrix}}$

$\begin{bmatrix}-4\\1\\0\end{bmatrix} + \begin{bmatrix}-5\\0\\1\end{bmatrix}$

⑦ For A we have $m=n=r=3$ then

The dimension of the column space is 3 + has a basis given by

$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, + \begin{bmatrix}0\\0\\1\end{bmatrix}$

The dimension of the row space is also 3 + has the same basis

The dimension of the null space is 0 + contains only the zero vector

" " " " left nullspace is 0 + " " " " ".

For B we have $m = 3, n = 6, r = 3$

so the dim of the column space is 3 + has the same basis choice,

The dim of the row space is still 3 ~~but~~ + has a basis given by $[1\ 0\ 0\ 1\ 0\ 0]^T, [0\ 1\ 0\ 0\ 1\ 0]^T, [0\ 0\ 1\ 0\ 0\ 1]^T$

The dim of the null space is given by $6-3=3$ + a basis can be obtained by from

$$[1\ 0\ 0\ -1\ 0\ 0]^T,\ [0\ 1\ 0\ 0\ -1\ 0]^T,$$

$$[0\ 0\ 1\ 0\ 0\ -1]^T$$

The dim of the left nullspace is given by $m-r = 3-3 = 0$ + has $\underline{0}$ the zero vector as its only element.

⑧ ~~So~~ For A $m = 3, n = 3+2 = 5, + r = 3$

So the $\dim C(A) = 3$

$\dim C(A^T) = 3$

$\dim(N(A)) = n - r = 5 - 3 = 2$

$\dim(N(A^T)) = m - r = 0$

For B $m = 3+2 = 5, n = 3+3 = 6, r = 3$

So $\dim C(A) = 3$

$\dim C(A^T) = 3$

$\dim N(A) =$ ~~to~~ $n - r = 3$

$\dim N(A^T) = m - r = 5 - 3 = 2$

For C $m = 3, n = 2, + r = 0$

So $\dim C(A) = 0$

$\dim C(A^T) = 0$

$\dim N(A) = n - r = 2$

$\dim N(A^T) = m - r = 3$

⑨

(~~a~~) ~~The rank~~ ~~1st~~ ~~1st~~ ~~dimensions~~ First the equivalence of the ranks

The rank of A alone is equivalent to the rank of $B \equiv \begin{bmatrix} A \\ A \end{bmatrix}$

Because we can simply subtract each row of A from the corresponding "new" row in the concatenated matrix effectively we are applying the elementary transformation matrix

$$E = \begin{bmatrix} I & 0 \\ -I & I \end{bmatrix}$$ to the concatenated matrix $\begin{bmatrix} A \\ A \end{bmatrix}$

to produce $\begin{bmatrix} A \\ 0 \end{bmatrix}$.

Now for $C \equiv \begin{bmatrix} A & A \\ A & A \end{bmatrix}$ we can again multiply by the E above obtaining

$$EC = \begin{bmatrix} I & 0 \\ -I & 0 \end{bmatrix} \begin{bmatrix} A & A \\ A & A \end{bmatrix} = \begin{bmatrix} A & A \\ 0 & 0 \end{bmatrix}$$

continuing to perform row operations on the top half of this matrix we obtain

$\begin{bmatrix} R & R \\ 0 & 0 \end{bmatrix}$ where R is the reduced row echelon matrix for A. Since this has the same rank as R the composite matrix has the same rank as the original.

Now ~~the~~ ~~dimension~~ for pt (a)

If A is $m \times n$ then B is $2m \times n$.

A & B have the same (1) row space
(2) nullity

For pt (b)

If A is $m\times n$ then B is $2m\times n$ + C is $2m\times 2n$

Then B + C have the same

(1) column space

(2) left null space

⑩ $m=3$ + $n=3$ with random entries it is very likely that the matrix will be non singular so its rank $=3$

+ $\dim C(A)=3$
$\dim C(A^T)=3$
$\dim N(A)=0$
$\dim N(A^T)=0$

If A is 3×5, $m=3$ + $n=5$ it is more likely that

$\dim C(A)=3$
$\dim C(A^T)=3$
$\dim N(A) = m-r = 2$
$\dim N(A^T) = m-r = 3-3=0$

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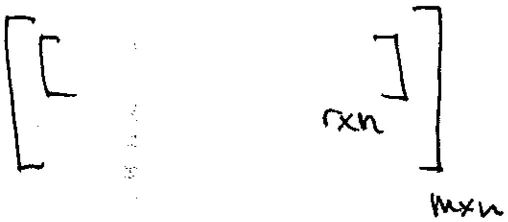
(11) It ∃ a RHS with no solution then when we perform

(a) our elementary row operations on A we are left with a row of zeros in R (or U) that does not have corresponding zero elements in Eb. Thus $r < m$ (since ~~we~~ we must have a row of zeros as always ~~[illegible]~~ · $r \leq n$,



~~[illegible]~~

(b) Because letting ~~E~~ ~~[illegible]~~ y be composed of r zeros ~~stacked~~ ~~atop~~ stacked atop vectors with ones in each component, i.e. in case

~~Then $y^T R = 0$~~ ~~$R = EA$~~

& ~~$y^T EA = 0$~~

$r = 2$ & $m = 4$ consider the vectors

$$Y_1 = \begin{bmatrix}0\\0\\1\\0\end{bmatrix} \quad \& \quad Y_2\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

Then $Y_1^T R = 0$ & $Y_2^T R = 0$

So

$y^T(EA) = 0$ or $(E^T y)^T A = 0$

$\therefore$ $E^T y$ is a Nonzero vector in the left nullspace.

~~Alternatl~~ Alternatively if the left nullspace is Non empty it must have a non zero vector. Since the left nullspaces dimension is given by $m-r$ ~~we~~ ~~know~~ which we know is greater than zero we have the existence of a Non zero element.

⑫ ~~$\begin{bmatrix}1&0&1\\1&2&0\end{bmatrix}$~~ $\rightarrow$ ~~$\begin{bmatrix}1&0&1\\0&2&-1\end{bmatrix}$~~ $\rightarrow$ ~~$\begin{bmatrix}1&0&1\\0&1&0\end{bmatrix}$~~

~~let~~ A = ~~$\begin{bmatrix}1&0&1\\0&2&0\\1&0&1\end{bmatrix}$~~ then ~~[1 0 1] + [1 2 0] are its~~ basis ~~for its row + column~~ space

consider $A = \begin{bmatrix}1&1\\0&2\\1&0\end{bmatrix}\begin{bmatrix}1&0&1\\1&2&0\end{bmatrix} = \begin{bmatrix}2&2&1\\2&4&0\\1&0&1\end{bmatrix}$

If $(1,0,1)$ + $(1,2,0)$ is a basis for the row space then $\dim C(A^T) = 2 = r$

To also be a basis for the nullspace means $n-r = 2$

~~The~~ ~~r=~~ $\Rightarrow$ $n=4$. But these vectors are in $\mathbb{R}^3$ x resulting in a contradiction

⑬ (a) False, consider

$$\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$$

Then the row space is spanned by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ + the column space by $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ which are different

(b) True $-A$ is a trivial linear transformation of A + as such cannot alter the subspaces

(c) If A + B share the same 4 spaces then $E_1 A = R$ + $E_2 B = R$

Think A + ~~B~~ or B are related by an invertible transformation

i.e. $A = E_1^{-1} E_2^{-1} B$.

Pick $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ + $B = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ then the subspaces are the same but A is not a multiple of B

⑭ The rank of A is 3 & a basis for the column space is not given by $\left[\cancel{\begin{bmatrix}1\\0\\0\end{bmatrix}}, \cancel{\begin{bmatrix}2\\1\\0\end{bmatrix}}, \cancel{\begin{bmatrix}3\\2\\1\end{bmatrix}}\right]$

~~of dimension~~ 3. But is given by

~~A basis~~ $\begin{bmatrix}1&0&0\\6&1&0\\9&8&1\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\6\\9\end{bmatrix}$

& $\begin{bmatrix}1&0&0\\6&1&0\\9&8&1\end{bmatrix}\begin{bmatrix}2\\1\\0\end{bmatrix} = 2\begin{bmatrix}1\\6\\9\end{bmatrix} + \begin{bmatrix}0\\1\\8\end{bmatrix} = \cdots$

& last $\begin{bmatrix}1&0&0\\6&1&0\\9&8&1\end{bmatrix}\begin{bmatrix}3\\2\\1\end{bmatrix} = \cdots$

↳ in addition basis can be ~~obtained~~ taken from the standard basis

equivalently since the 3x3 block composing ~~of~~ the 1st 3 pivots of U is invertible

A basis for the row space of dimension 3 is given by

$(1\ 2\ 3\ 4),\ (0\ 1\ 2\ 3)\ (0, 0, 1\ 2)$

The null space has dimension $4-3=1$ & has a basis given by setting $x_4 = 1$ & solving for $x_1, x_2,$ & x_3 in

$Ux = 0$

or $\begin{bmatrix}1&2&3\\0&1&2\\0&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = -\begin{bmatrix}4\\3\\2\end{bmatrix}$

$\Rightarrow x_3 = -2 \quad x_2 - 4 = -3 \Rightarrow x_2 = 1$ & $x_1 = -4 - 2(1) - 3(-2)$

$= -4 - 2 + 6 = 0$

or the vector $\begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix}$

The left null space has dimension $m - r = 3 - 3 = 0$ & ∴ only consists of the zero vector

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⑮ The row space + the left null space will not change

If $v=(1,2,3,4)$ is in the column space of the original matrix the vector in the column space of the new matrix is

$(2,1,3,4)$

⑯ If $v = (1,2,3)$ was a row of A then when we Multiplied by ~~A~~ v this row would give the product of

$$\begin{bmatrix}1&2&3\\ x&x&x\\ \vdots&\vdots&\vdots\\ x&x&x\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}1^2+2^2+3^2\\ x\\ \vdots\\ x\end{bmatrix} = \begin{bmatrix}14\\ x\\ \vdots\\ x\end{bmatrix}$$

. which cannot equal zero.

⑰ For $A = \begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}$ we have rank $= 2$

The ~~Row~~ ~~Sp~~ column space is all vectors $\begin{pmatrix}x\\y\\0\end{pmatrix}$

The row space is all vectors $\begin{pmatrix}0\\y\\z\end{pmatrix}$

The null space is all vectors $\begin{pmatrix}x\\0\\0\end{pmatrix}$

The left null space is all vectors $\begin{pmatrix}0\\0\\z\end{pmatrix}$

For $I+A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

The rank is 3 the row space is given by all vectors $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

The column space is given by all vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

The left null space + the null space contain only the zero vector

⑱ $[A\ b] = \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 4 & 5 & 6 & b_2 \\ 7 & 8 & 9 & b_3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 0 & -3 & -6 & b_2-4b_1 \\ 0 & -6 & -12 & b_3-7b_1 \end{bmatrix}$ with $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -7 & 0 & 1 \end{bmatrix}$

$-14+8=-6$

$-21+9=-12$

$\Rightarrow \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 0 & -3 & -6 & b_2-4b_1 \\ 0 & 0 & -3 & b_3-7b_1-2(b_2-4b_1) \end{bmatrix}$ with $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 0 & -3 & -6 & b_2-4b_1 \\ 0 & 0 & -3 & b_3-2b_2+b_1 \end{bmatrix}$

The combination of the rows that produce the zero row is given by +1 times row 1, −2 times row two, +1 times row 3

Thus the vector $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ is in the null space ~~as~~ of A^T.

A vector in the null space is given by $\underline{x} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

(obtained by setting $x_3 = 1$ + solving for x_1 + x_2)

$x_1 + 2(-2) + 3(1) = 0 \Rightarrow x_1 = 4-3 = 1$

Which is the ~~same thing~~ same vector space as the left null space

⑲ pt (a)

$$\begin{bmatrix} 1 & 2 & b_1 \\ 3 & 4 & b_2 \\ 4 & 6 & b_3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & b_1 \\ 0 & -2 & b_2-3b_1 \\ 0 & -2 & b_3-4b_1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & b_1 \\ 0 & -2 & b_2-3b_1 \\ 0 & 0 & b_3-4b_1-b_2+3b_1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & b_1 \\ 0 & -2 & b_2-3b_1 \\ 0 & 0 & b_3-b_2-b_1 \end{bmatrix}$$

Thus the vector $(-1,-1,1)$ is in the left nullspace which has dimension $m-r=3-2=1$ ✓

For (b)

$$\begin{bmatrix} 1 & 2 & b_1 \\ 2 & 3 & b_2 \\ 2 & 4 & b_3 \\ 2 & 5 & b_4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & b_1 \\ 0 & -1 & b_2-2b_1 \\ 0 & 0 & b_3-2b_1 \\ 0 & 1 & b_4-2b_1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & b_1 \\ 0 & -1 & b_2-2b_1 \\ 0 & 0 & b_3-2b_1 \\ 0 & 0 & b_4-2b_1+b_2-2b_1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & b_1 \\ 0 & -1 & b_2-2b_1 \\ 0 & 0 & b_3-2b_1 \\ 0 & 0 & b_4+b_2-4b_1 \end{bmatrix}$$

Thus the vectors in the left nullspace are given by

$$\begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -4 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

which has dimension $m-r=4-2=2$ ✓

⑳

(a) We must have $Ux=0$ which has 2 pivot variables $x_1 + x_3$ + free variables $x_2 + x_4$. To find the null space we ~~get~~. set $x_2=1, x_4=0$ + solve for $x_1 + x_3$

Thus we get $\begin{bmatrix} -1/2 \\ 1 \\ -3 \\ 0 \end{bmatrix}$

$4x_1 + 2 = 0 \Rightarrow x_1 = -1/2$

+ setting $x_2=0$ + $x_4=1$ + solving for $x_1 + x_3$ we have

$\begin{bmatrix} 1/4 \\ 0 \\ -3 \\ 1 \end{bmatrix}$ $\quad 4x_1 + 2(0) + 0 + 1 = 0$

$x_3 = -3$

(b) The # of independent solutions of $A^T y$ are

$m - r = 3 - 2 = 1$

(c) The column space is spanned by

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 8 \\ 12 \end{bmatrix} \qquad = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$$

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㉑

(a) The vectors u + w $\qquad A = \begin{bmatrix} u & w \end{bmatrix}\begin{bmatrix} v^T \\ z^T \end{bmatrix}$

(b) The vectors v + z

(c) ~~$v^T w = 0$~~ or ~~$z^T u = 0$~~ ? u + w are multiples of each other or are linearly dependent or v + z are multiples of each other or are linearly dependent.

(d) $u = z = (1,0,0)$ + $v = w = (0,0,1)$

$$uv^T = (1,0,0)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$+ \quad wz^T = (0,0,1)\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So $A = uv^T + wz^T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ which has rank = 2

㉒

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 2 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

1x2 · 2x1

$$= \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 4 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 2 & 2 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 2 & 2 \\ 4 & 1 & 1 \end{bmatrix}$$

(23) A basis for the row space is $(3\ 0\ 3)$ & $(1\ 1\ 2)$ which are independent

A basis for the column space is given by $(1\ 4\ 2)$ & $(2\ 5\ 7)$

which are independent..

A is Not invertible because it is the product of two rank 2

matrices & $\therefore$ $\text{rank}(AB) \leq \text{rank}(B) = 2$

to be invertible $\text{rank}(AB) = 3$ which its not

(24) d is in the span of its rows. Solution is unique

when the left null space contains only the zero vector

(25) (a) A & A^T have the same # of pivots, since they

have the same rank, they have the same # of pivots

(b) False let $A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$

Then $y^T = [-2\ 1]$ is in the left nullspace of A

But $y^T(A^T) = [-2\ \ 1]\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \neq \underline{0}$ & is Not in the

left nullspace of ~~A~~ A^T

(c) False, pick any invertible matrix say of size m×m. (then the row & column

spaces are the entire of $\mathbb{R}^m$. But many invertible matrices

exist such that $A \neq A^T$

For example let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

(d) If ~~$A^T = -A$~~ ~~At~~ ~~$A =$~~ ~~$\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$~~ ~~so~~ True since,

If $A^T = -A$ then the rows of A are the negative columns of A & ∴ have exactly the same ~~span~~ span

㉖ The rows of ~~B~~ C are combinations of the rows of B. The rank of C cannot be greater than the rank of B. So the rows of C^T are the rows of A^T, so the rank of C^T = rank of C ~~&~~ cannot be larger than the rank of A^T = rank of A.

㉗ To have rank 1 the two rows must be multiples of each other & the two columns must be multiples of each other

~~pick~~ ~~$d = \left(\frac{c}{a}\right)$~~ ~~$c = ka$~~ ~~$d = kb$~~

To make the rows multiples of each other assume row 2 is a multiple ~~[illegible]~~ (k) of row 1 or

$$ka = c \quad \& \quad kb = d$$

$$\therefore \quad k = \frac{c}{a} \qquad \therefore \quad d = \frac{c}{a} \cdot b$$

A basis for the row space is then given by the vector ~~[a, b]~~ (a,b) + ~~the~~ ~~null~~ a basis for the Null space is given by $\begin{bmatrix} -\frac{b}{a} \\ 1 \end{bmatrix} \propto \begin{bmatrix} -b \\ a \end{bmatrix}$

(28) The rank of B is 2 + has a basis of the row space given by the 1st two rows in its ~~a~~ representation. ~~A basis for~~ Its left ~~null~~ of null space ~~is given~~ ~~by~~ (~~it~~ must be of dimension 8 - 2 = 6) ~~+ when reduced to Row echelon form looks~~ like

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdots & & & & & 0 \\ 0 & 0 & & & & & & \vdots \\ 0 & & & & & & & \\ 0 & \vdots & & & & & & \\ 0 & 0 & & & & & & 0 \end{bmatrix}$$

~~Thus~~ And a basis for the left Null space is given by

The reduced row echelon matrix looks like

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cdot & & & & & & 0 \\ 0 & & \cdot & & & & & 0 \\ 0 & & & \cdot & & & & 0 \\ 0 & & & & & & & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

with E obtained by $EA = R$ w/

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$