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go to pg 8

(5) For A we have eigenvalues given by

$$\begin{vmatrix} 1-\lambda & 0 \\ 1 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)^2 = 0 \Rightarrow \lambda = 1$$

For B we have eigenvalues given by

$$\begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)^2 = 0 \Rightarrow \lambda = 1$$

For $A+B$ we have eigenvalues given by

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 1 = 0 \Rightarrow (2-\lambda) = \pm 1$$

$$\Rightarrow \lambda = 2 \mp 1 = 1, 3$$

so the eigenvalues of $A+B$ are Not equal to the eigenvalues of A plus the eigenvalues of B. This would be true if A & B had the same eigenvectors which will happen iff A & B commute.

i.e. $AB = BA$. Checking this test for the matrices given here we have

$$AB = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

& $BA = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ which are Not

equal & consequently A & B don't have the same eigenvectors

(b) Find the null space of $A - \lambda I$

(9) (a) Multiply $Ax = \lambda x$ by A on the left in both sides

$$A^2 x = \lambda A x = \lambda(\lambda x) = \lambda^2 x$$

(b) Multiply by $\frac{1}{\lambda} A^{-1}$ on both sides to get

$$\frac{1}{\lambda} x = A^{-1} x$$

(c) Add $I \cdot x$ on both sides of the eq to get

$$(A + I)x = \lambda x + Ix = (\lambda + 1)x$$

(10) For A the eigenvalues are given by

$$|A - \lambda I| = \begin{vmatrix} .6-\lambda & .2 \\ .4 & .8-\lambda \end{vmatrix} = (.6-\lambda)(.8-\lambda) - .08 = 0$$

or $.48 - 1.4\lambda + \lambda^2 - .08 = 0$

$\Rightarrow \lambda^2 - 1.4\lambda + .4 = 0$ $\lambda = 1$ as it must for all Markov Matrices

So

$$\lambda = \frac{1.4 \pm \sqrt{(1.4)^2 - 4(.4)}}{2} = \frac{1.4 \pm \sqrt{1.96 - 1.6}}{2}$$

$$\begin{array}{r} 1.4 \\ \times\ 1.4 \\ \hline 56 \\ 140 \\ \hline 1.96 \end{array}$$

$$= \frac{1.4 \pm \sqrt{.36}}{2}$$

$$= \frac{1.4 \pm .6}{2} = \frac{2}{2}, \frac{.8}{2} = 1, .4$$

$$= 1, \frac{2}{5}$$

Then the eigenvectors for A are given by (for $\lambda = 1$)

$$A - \lambda I = \begin{bmatrix} .6 - 1 & .2 \\ .4 & .8 - 1 \end{bmatrix} = \begin{bmatrix} -.4 & .2 \\ .4 & -.2 \end{bmatrix}$$

which has a nullspace given by the span of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

For $\lambda = .4$ we have

$$A - \lambda I = \begin{bmatrix} .2 & .2 \\ .4 & .4 \end{bmatrix}$$ which has a null space given by the span of

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

~~For~~ $\cancel{A^\infty}$ $\cancel{A^\infty}$ For A^∞ our eigenvalues are given by

$$\lambda_1 = 1 \quad \& \quad \lambda_2 = \left(\frac{2}{5}\right)^\infty = 0$$

& with the same eigenvectors as A. Now A^∞ is obtained from

$\cancel{A^\infty} =$

the diagonalization of A i.e. $A = S \Lambda S^{-1}$

$$= \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2/5 \end{bmatrix} \frac{1}{(1+2)} \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$$

then $A^{\infty} = S \Lambda^{\infty} S^{-1} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \frac{1}{3} \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$

$$= \frac{1}{3} \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1/3 & 1/3 \\ 2/3 & 2/3 \end{bmatrix}$$

So A^{100} is given by

$$A^{100} = S \Lambda^{100} S^{-1} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1^{100} & 0 \\ 0 & (\frac{2}{5})^{100} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \frac{1}{3}$$

$$= \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & (\frac{2}{5})^{100} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \frac{1}{3}$$

$$= \begin{bmatrix} 1 & -(\frac{2}{5})^{100} \\ 2 & (\frac{2}{5})^{100} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \frac{1}{3}$$

$$= \begin{bmatrix} 1 + 2(\frac{2}{5})^{100} & 1 - (\frac{2}{5})^{100} \\ 2 - 2(\frac{2}{5})^{100} & 2 + (\frac{2}{5})^{100} \end{bmatrix} \frac{1}{3}$$

$$= \begin{bmatrix} 1/3 & 1/3 \\ 2/3 & 2/3 \end{bmatrix} + (\frac{2}{5})^{100} \begin{bmatrix} 2/3 & -1/3 \\ -2/3 & 1/3 \end{bmatrix}$$

which is a very slight perturbation from A^{∞}

(11) P is a block diagonal matrix & as such has eigenvalues given by the eigenvalues of the block matrices. Since $\lambda = 1$ is the eigenvalue of the lower right block & the upper right block is given by

$$\begin{bmatrix} .2 & .4 \\ .4 & .8 \end{bmatrix}$$ which has eigenvalues given by

$$\begin{vmatrix} .2-\lambda & .4 \\ .4 & .8-\lambda \end{vmatrix} = 0 \Rightarrow (.2-\lambda)(.8-\lambda) - .16 = 0$$

$$\Rightarrow \cancel{.16} - .2\lambda - .8\lambda + \lambda^2 - \cancel{.16} = 0$$

$$\lambda^2 - 1.0\lambda = 0 \Rightarrow \lambda = 0 \ \& \ \lambda = +1.$$

So the eigenvectors are given by for $\lambda = 1$

$$\begin{bmatrix} -.8 & .4 & 0 \\ .4 & -.2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1/2 & 0 \\ 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\left\{ \frac{.4}{(-.8)} \quad -\frac{.4}{.8} = -\frac{1}{2} \quad -\frac{.2}{.4} = -\frac{1}{2} \right.$$

So one eigenvector is $\underline{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

& Another is given by $\underline{x} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

For $\lambda = 0$ we have

$$\begin{bmatrix} .2 & .4 & 0 \\ .4 & .8 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

~~Eq~~ So an eigenvector is given by $\underline{x} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

For P^{100} we have the same eigenvectors as for P & the eigenvalues given by $0^{100}, 1^{100} = 0, 1$.

Thus everything for P^{100} is the same as P

If two eigenvectors share the same λ then so do all linear combinations of the eigenvectors. Thus since $v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ + $v_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ share the same eigenvalue of $\lambda = 1$ so will their sum

$$v_1 + v_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$ which has no zero components

Which we can check with $P\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} .2+.8 \\ .4+1.6 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

(12) The rank-one projection matrix is given by $P = uu^T$

$$P = \frac{1}{6}\begin{bmatrix} 1 \\ 1 \\ 3 \\ 5 \end{bmatrix} \frac{1}{6}\begin{bmatrix} 1 & 1 & 3 & 5 \end{bmatrix} = \frac{1}{36}\begin{bmatrix} 1 & 1 & 3 & 5 \\ 1 & 1 & 3 & 5 \\ 3 & 3 & 9 & 15 \\ 5 & 5 & 15 & 25 \end{bmatrix}$$

(a) Now

$$Pu = \frac{1}{36}\begin{bmatrix}1&1&3&5\\1&1&3&5\\3&3&9&15\\5&5&15&25\end{bmatrix}\begin{bmatrix}1\\1\\3\\5\end{bmatrix}\frac{1}{6}$$

$$= \frac{1}{6^3}\begin{bmatrix}1+1+9+25\\1+1+9+25\\3+3+27+75\\5+5+45+125\end{bmatrix}$$

$$= \frac{1}{6^3}\begin{bmatrix}36\\36\\108\\180\end{bmatrix} = \frac{1}{6}\begin{bmatrix}1\\1\\3\\5\end{bmatrix} = u \checkmark$$

Thus u is an eigenvector with eigenvalue $= 1$.

(b) If v is perpendicular to u then $u^Tv = v^Tu = 0$

& $Pv = uu^Tv = u \cdot 0 = 0$ so v is an eigenvector with eigenvalue $\lambda = 0$.

(c) To find 3 independent eigen-vectors of P all with eigenvalues $= 0$ ~~with~~ which means that we need to find 3 vectors $\perp$ to u

Each of these vectors must satisfy

$$\begin{bmatrix}1 & 1 & 3 & 5\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = 0$$

or the 3 vectors that span the nullspace of A defined to be $A = \begin{bmatrix} 1 & 1 & 3 & 5 \end{bmatrix}$. 3 vectors in the null space are given by letting $x_2 = 1, x_3 = 0, x_4 = 0 \Rightarrow x_1 = -1$

$x_2 = 0, x_3 = 1, x_4 = 0 \Rightarrow x_1 = -3$

$x_2 = 0, x_3 = 0, x_4 = 1 \Rightarrow x_1 = -5$

Giving the 3 vectors $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

(13) $\det(Q - \lambda I) = \begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{vmatrix} = 0$

$\Rightarrow (\cos\theta - \lambda)^2 + \sin^2\theta = 0$

$\Rightarrow \lambda^2 - 2\cos\theta\,\lambda + \cos^2\theta + \sin^2\theta = 0$

$\Rightarrow \lambda^2 - 2\cos\theta\,\lambda + 1 = 0$

which gives $\lambda = \dfrac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$

$= \cancel{\dfrac{2\cos\theta \pm \sqrt{\frac{4}{2}(1+\cos(2\theta))}}{2}}$

$= \cancel{2\cos\theta \pm \sqrt{4}}$

$= \cancel{\dfrac{2\cos\theta \pm \sqrt{4\cos^2\theta - \cos^2\theta - \sin^2\theta}}{2}}$

$= \cancel{2\cos\theta \pm \sqrt{3\cos^2\theta - \sin^2\theta}}$

go to 17

So then $\lambda = \cos\theta \pm \sqrt{\cos^2\theta - 1}$

$= \cos\theta \pm i\sin\theta$

To find the eigenvectors we solve $(Q-\lambda I)x = 0$ which is given by

$$(Q-\lambda I) = \begin{bmatrix} \cos\theta - (\cos\theta \pm i\sin\theta) & -\sin\theta \\ \sin\theta & \cos\theta - (\cos\theta \pm i\sin\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \mp i\sin\theta & -\sin\theta \\ \sin\theta & \mp i\sin\theta \end{bmatrix} = \sin\theta \begin{bmatrix} \mp i & -1 \\ 1 & \mp i \end{bmatrix}$$

Which has eigenvectors given $v = \begin{bmatrix} \pm i \\ 1 \end{bmatrix}$ i.e.

$$v_1 = \begin{bmatrix} i \\ 1 \end{bmatrix} \quad \& \quad v_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

Which has a root $\lambda = 1$ obviously + other roots given by.

$$\begin{array}{r|rrrr} & -\lambda^2 & +1 & & \\ \hline \lambda - 1 & -\lambda^3 & \lambda^2 & \lambda & -1 \\ & -\lambda^3 & +\lambda^2 & & \\ \hline & & 0 & \lambda - 1 & \end{array}$$

Thus $-\lambda^3 + \lambda^2 + \lambda - 1 = (\lambda - 1)(-\lambda^2 + 1)$

$$= -(\lambda-1)(\lambda^2 - 1)$$

$$= -(\lambda - 1)^2(\lambda + 1)$$

(15) Consider

$\det(A - \lambda I)$ factored into its n factors

$$\det(A - \lambda I) = \prod_{i=1}^{n} (\lambda_i - \lambda)$$

Then set $\lambda = 0$ to get $\det(A) = \prod_{i=1}^{n} \lambda_i$

(16) If A has $\lambda_1 = 3$ + $\lambda_2 = 4$ then

$$\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)$$

$$= \lambda_1\lambda_2 - (\lambda_1 + \lambda_2)\lambda + \lambda^2$$

So $\det(A - \lambda I) = 12 - (7\lambda) + \lambda^2$

The quadratic formula gives

$$\lambda_1 = \frac{a+d+\sqrt{(a+d)^2-4(ad-bc)}}{2}$$

& $$\lambda_2 = \frac{a+d-\sqrt{(a+d)^2-4(ad-bc)}}{2}$$

then $\lambda_1+\lambda_2 = \frac{2(a+d)}{2} = a+d$ which is the linear term in the determinant expansion i.e. $a+d = \lambda_1+\lambda_2$.

(17) We can ~~of~~ always generate matrices with any given eigenvalues by constructing them from

$$S\begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix} S^{-1}$$ with different choices for the eigenvector spaces S.

For example, pick eigenvectors $= (1,2)$ & $(-1,1)$ then our matrix A is

$$\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}\frac{1}{(1+2)}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 4 & -5 \\ 8 & 5 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 4+10 & 4-5 \\ 8-10 & 8+5 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 14 & -1 \\ -2 & 13 \end{bmatrix}$$

other matrices can be generated in the same manner.

(18) (a) The rank of A cannot be ~~determined~~ from the given information.
For example let A be ~~[illegible]~~ given as follows

~~$A = S\Lambda S^{-1}$ for various S matrices with $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$~~.

~~Then~~ $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ Then A ~~has~~ is diagonal & has eigenvalues as given & A has rank = 2

Also consider A given by

$$A = S\Lambda S^{-1} = \cancel{\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}} S^{-1}$$

& ~~S^{-1} is given by~~ $\cancel{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \\ 0 & 0 & 1 & 1 \end{bmatrix}}$

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -1 & 1 & 0 \\ -1/2 & 3/2 & -1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 2 \\ 0 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -1 & 1 & 0 \\ -1/2 & 3/2 & -1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1-1 & -1+3 & -1 \\ -1 & 3 & -1 \\ -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 & -1 \\ -1 & 3 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

(1b)

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 3 & -2 & -2 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & -2 & 1 & -3 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1/2 & 3/2 & -1/2 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/2 & 1/2 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1/2 & 3/2 & -1/2 \end{array}\right]$$

This matrix has rank 3 as can be seen by the following transformations

$$\begin{bmatrix} 0 & 2 & -1 \\ -1 & 3 & -1 \\ -1 & 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 3 & -1 \\ 0 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & -1 \\ 0 & -2 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & -2 \end{bmatrix}$$ which has rank $= 3$

Thus we have shown two matrices A_1 + A_2 both of which have the same eigen~~vectors~~ values but different rank; showing that rank is not uniquely determined by the eigenvalues.

(b) $|B^T B| = |B^T| \cdot |B| = |B|^2 = (0 \cdot 1 \cdot 2)^2 = 0$

(c) The eigenvalues of $B^T B$ are given by $0^2, 1^2, 2^2 = 0, 1, 4$.

pf: 0 is an eigenvalue since the determinant of $B^T B$ is zero by the above.

$\cancel{|B^T B - \lambda I| = 0}$ ~~let x be an eigenvector of B with eigenvalue~~ $\cancel{\lambda}$. ~~Then~~ $\cancel{Bx = \lambda} x$ so ~~by multiplying by~~ $\cancel{B^T}$ ~~on the left hand side of this equation we obtain.~~

$B^T \cancel{Bx = \lambda B}^T x$

$\cancel{B^T B = B \cdot B^T}$

$\cancel{x^T B^T = \lambda x^T}$

I'm not sure how to show that the other eigenvalues are given by λ^2.

This will certainly be true if B^T & B commute i.e.

$B^T B = B \cdot B^T$ ~~but~~ & i.e. if B & B^T have the same eigenvectors.

~~This is certainly true~~ But I don't know how to show this.

(d) The eigenvalues of $B+I$ are the eigenvalues of B plus 1.
which gives $1, 2, 3$. The eigenvalues of $(B+I)^{-1}$ are the
inverses of the eigenvalues of $B+I$ & are given by $1, \frac{1}{2}, \frac{1}{3}$.

(19) ~~The trace of A~~ let $A = \begin{bmatrix} 0 & 1 \\ c & d \end{bmatrix}$ then the trace of
A must equal $0 + d = d = \lambda_1 + \lambda_2 = 4 + 7 = 11 \Rightarrow d = 11$
& the determinant of A must equal $|A| = -c = \lambda_1 \cdot \lambda_2 = 28$
$\Rightarrow c = -28$ Thus $A = \begin{bmatrix} 0 & 1 \\ -28 & 11 \end{bmatrix}$

(20) Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & b & c \end{bmatrix}$ if the eigenvalues are $-3, 0,$ & 3

then $\text{trace}(A) = 0 + 0 + c = c = \lambda_1 + \lambda_2 + \lambda_3 = 0 \Rightarrow c = 0$

& $\det(A) = \begin{vmatrix} 0 & 1 \\ a & 0 \end{vmatrix} = a = \lambda_1 \cdot \lambda_2 \cdot \lambda_3 = 0$

$\therefore$ ~~$A =$~~ from what we know about A we now have

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & b & 0 \end{bmatrix}$$ computing the characteristic equation ~~we know~~ we have that

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & b & -\lambda \end{vmatrix} = -\lambda \begin{vmatrix} -\lambda & 1 \\ b & -\lambda \end{vmatrix} = -\lambda(\lambda^2 - b)$$

$$= -\lambda^3 + \lambda b = -\lambda^3 + 9\lambda \Rightarrow b = 9$$ & our matrix A

is given by $$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 9 & 0 \end{bmatrix}$$

(21) $\det(A - \lambda I) = \det(A^T - \lambda I)$ since $I^T = I$

let ~~$A =$~~ A be the canonical counter example ~~example that fails~~ for eigenvector / eigenvalue ~~question~~ statement. let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

Then A has eigenvalue / eigenvector pairs given by

$\lambda_1 = 0 \quad x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & $\lambda_2 = 1 \quad x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Then $A^T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ which has ~~the same eigenvalues 0 & 1~~ let

~~eigenvectors~~ ~~eigenvectors~~ ~~given by~~

~~eigenvector / eigenvalue pairs given by~~

$$\cancel{\lambda_1 = 0 \quad x_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \& \quad \lambda_2 = 1 \quad x_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}}$$

---

let $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ & $B = A^T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ the examples

from problem 5 in this section. Then both A & B have $\lambda = 1$ with algebraic multiplicity of two. The eigenvectors of A are given by a basis for the nullspace of

$$A - \lambda I = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$ or the span of ~~which~~ is ~~spanned by~~ $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

The eigenvectors of A^T are given by ~~A^T~~ a basis for the nullspace of

$$A^T - \lambda I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$ ~~which~~ is or the span of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Since these vectors are obviously not ~~spanned by~~ equivalent the eigenvectors of A & A^T are different

(22) $M = \begin{bmatrix} .6 & .8 & .1 \\ .2 & .1 & .4 \\ .2 & .1 & .5 \end{bmatrix}$ ~~which has an eigenvalue equal to 1~~.

Then $M^T \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} .6 & .2 & .2 \\ .8 & .1 & .1 \\ .1 & .4 & .5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ so M^T has an

eigenvalue given by $\lambda = 1$ $\therefore$ M must have an eigenvalue $\lambda = 1$.

Since a 3x3 singular Markov matrix must have 2 eigenvalues equal to 0 + 1.

But ~~also~~ also ~~must~~ since it must have a $\text{Trace}(M) = \frac{1}{2}$ we know that

$0 + 1 + \lambda = \frac{1}{2}$ so $\lambda = -\frac{1}{2}$ is the 3rd eigenvalue. To assemble M

construct it from its eigenvalues by assigning random ~~(unit norm eigenvectors~~) eigenvectors ?

ie $M = S \Lambda S^{-1}$. But we can simplify things some by working with M^T

which have the same eigenvalues + we know that $(1,1,1)$ is an the

eigenvector corresponding to $\lambda = 1$. Thus

$$M^T = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot$$

To compute the inverse of S we ~~see~~ have

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 2 & 0 & -1 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & -2 & 1 & -2 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1/2 & 1 & -1/2 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 1 & 0 & -1/2 & 0 & 1/2 \\ 0 & 0 & 1 & -1/2 & 1 & -1/2 \end{array}\right] \qquad \text{Thus } S^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ -1 & 2 & -1 \end{bmatrix}$$

$$\text{So } M^T = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ -1 & 2 & -1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 1/2 & 0 \\ 1 & 0 & 0 \\ 1 & -1/2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ -1 & 2 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1/2 & 0 & 3/2 \\ 1 & 0 & 1 \\ 3/2 & 0 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1/4 & 0 & 3/4 \\ 1/2 & 0 & 1/2 \\ 3/4 & 0 & 1/4 \end{bmatrix}$$

which gives for $M = \begin{bmatrix} 1/4 & 1/2 & 3/4 \\ 0 & 0 & 0 \\ 3/4 & 1/2 & 1/4 \end{bmatrix}$ which is a valid Markov matrix.

~~This is not a very good Markov matrix since the 2nd column is all zeros.~~ ~~It~~

(23) let $A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$; $A_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$; $A_3 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in general

to determine $a, b, c, + d$

~~$(1-\lambda)^2$~~

since $\lambda_1 = \lambda_2 = 0$ we have that from the trace + determinant identities that

~~$1-2\lambda+\lambda^2-1$~~

~~$-2\lambda+\lambda^2$~~

$$0 = a + d \Rightarrow a = -d$$
$$0 = ad - cb \Rightarrow 0 = -d^2 - cb \Rightarrow d^2 = -cb$$

let $a = 1$, $d = -1$ + $-cb = 1$ so pick $c = -1$ + $b = 1$ to obtain $A_3 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ then checking the determinant of A_3 we have $|A_3 - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ -1 & -1-\lambda \end{vmatrix}$

$$= (1-\lambda)(-1-\lambda) + 1 = 0$$
$$= -(1-\lambda)(1+\lambda) + 1$$
$$= -1 + \lambda^2 + 1 = 0 \Rightarrow \lambda^2 = 0.$$

~~Thus~~ Now $\forall A_i$ well check that

$$A_i^2 = 0$$

For A_1 we have

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

For A_2 we have

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

For A_3 we have

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \checkmark$$

In general when $a = -d$ & $d^2 = -cb$ we have

$$\begin{bmatrix} -d & b \\ c & d \end{bmatrix}\begin{bmatrix} -d & b \\ c & d \end{bmatrix} = \begin{bmatrix} d^2 + bc & -db + bd \\ -cd + dc & cb + d^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(24) We know since A is singular that at least one eigenvalue is given by $\lambda = 0$. ~~Considering the characteristic equation of A we know that~~

$\cancel{|A - \lambda I|} =$ | A corresponding eigenvector is given by any vector x such that

$$\begin{bmatrix} 2 & 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Two such vectors are $\underline{x} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$ & $\underline{x} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

A third eigenvector/eigenvalue combination in the rank 1 case like we have here is $\underline{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ because then

$$Ax = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}(2+2+2) = 6\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\underline{X} = \begin{bmatrix}1\\2\end{bmatrix}$$

So $\underline{X}_7 = \cancel{\begin{bmatrix}1\\2\\1\\1\end{bmatrix}}$ has eigenvalue given by 6.

(25) ~~$Ax = A\left(\sum_i c_i x_i\right) = \sum_i c_i A x_i = \sum_i c_i \lambda_i x_i$~~

~~& $Bx = \sum_i c_i \lambda_i x_i$ by the same logic. Since A & B have the same action ~~perform the~~ same transformation on any vector, they must be the same linear transformation.~~

(26) Consider $|A - \lambda I| = \left| \begin{bmatrix} B & C \\ 0 & D \end{bmatrix} - \begin{bmatrix} \lambda I & 0 \\ 0 & \lambda I \end{bmatrix} \right|$

$$= \left| \begin{bmatrix} B-\lambda I & C \\ 0 & D-\lambda I \end{bmatrix} \right| = \cancel{|B-\lambda I|} \cdot |B-\lambda I| \cdot |D-\lambda I|$$ since the

lower left hand corner of ~~$|A-\lambda I|$~~ $A-\lambda I$ is the zero matrix. But this expression vanishes whenever $|B-\lambda I| = 0$ or $|D-\lambda I| = 0$ which happen when $\lambda = 1, 2$ or $\lambda = 5, 7$ respectively. Thus the eigenvalues of A are given by $1, 2, 5, 7$

(27) For A since $A = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}$ we see that

A is rank 1 with 3 eigenvalues given by 0 (counted according to multiplicity) + 1 eigenvalue given by $\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = 4$. For ~~this pattern we can~~ ~~compute the~~ rank one matrices we can easily compute the eigenvectors since they are given either by the null vectors of the operator

$$\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix} \quad \text{i.e.} \quad \begin{bmatrix}-1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\0\\1\end{bmatrix} \text{ each w/ eigenvalue } 0$$

+ the vector $\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$ with eigenvalue 4.

For C we see that it has rank $= 2$ + thus is not invertible + $\therefore \lambda = 0$ is an eigenvalue. Since rank + nullity $= 4$ ~~nullity is of~~ we know the null space is of dimension two. Two vectors that span this space are given by

$$\begin{bmatrix}1\\0\\-1\\0\end{bmatrix} + \begin{bmatrix}0\\1\\0\\-1\end{bmatrix}$$

Two other vectors with eigenvalues of 2 are given by

$$\begin{bmatrix}1\\0\\1\\0\end{bmatrix} + \begin{bmatrix}0\\1\\0\\1\end{bmatrix}$$

~~(28) Since the eigenvalues of A were given by 0 with algebraic multiplicity 3 & 4 the eigenvalues of $A-I$ are -1 with algebraic multiplicity 3 & 3. If A is a 5×5 matrix of all ones the A has eigenvalues 0 with multiplicity 4 + an single eigenvalue with value 5. $A-I$ will have 4 eigenvalues with value -1 + a single eigenvalue with value 4.~~

~~The determinant of B is given by $(-1)^3(3) = -3$~~

~~The determinant of B (when 5×5) is given by $(-1)^4(4) = 4$~~

(29) For $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$ an upper triangular matrix the eigenvalues can

read off the diagonal & are given by $1, 4, \&\ 6$

For B ~~we have~~ ~~$B^T =$~~ I don't see an obvious way to extract the eigenvalues. computing the characteristic equation we have

$$|B-\lambda I| = \begin{vmatrix} -\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 3 & 0 & -\lambda \end{vmatrix} = -\lambda \begin{vmatrix} 2-\lambda & 0 \\ 0 & -\lambda \end{vmatrix} + 1 \begin{vmatrix} 0 & 2-\lambda \\ 3 & 0 \end{vmatrix}$$

$$= -\lambda(-\lambda(2-\lambda)) - 3(2-\lambda)$$

$$= \lambda^2(2-\lambda) - 6 + 3\lambda$$

$$= -\lambda^3 + 2\lambda^2 + 3\lambda - 6$$

From the expression to the determinant we see that

$\lambda = 2$ ~~is~~ as must be a root of the above cubic equation. Factoring $\lambda - 2$ out of this cubic equation we ~~ob~~ obtain.

$$
\begin{array}{r|l}
 & -\lambda^2 + 3 \\
\lambda - 2 & -\lambda^3 + 2\lambda^2 + 3\lambda - 6 \\
 & -\lambda^3 + 2\lambda^2 \\
\hline
 & 3\lambda - 6 \\
 & 3\lambda - 6 \\
\hline
 & 0
\end{array}
$$

$$\Rightarrow \quad (\lambda-2)(-\lambda^2+3) = -(\lambda-2)(\lambda-\sqrt{3})(\lambda+\sqrt{3})$$

For C we recognize it as a rank 1 matrix like

$$C = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$ which has ~~θ~~ ~~a~~ eigenvalue / eigenvector combinations given by

$$\lambda = 0 + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$+ \quad \lambda = 6 \text{ w/ } x = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

(30) Consider $A\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}a+b\\c+d\end{bmatrix} = (a+b)\begin{bmatrix}1\\1\end{bmatrix}$

$\therefore (1,1)^T$ is an eigenvector of A, with eigenvalue $a+b$.

Since $a+b=c+d$ multiplying by d & subtracting bc from both sides we have

$$ad + bd - bc = cd + d^2 - bc$$

$$ad - bc = -bd + cd + d^2 - bc = -b(d+c) + d(d+c)$$
$$= (d+c)(d-b)$$

$$\cancel{a - c = d - b = -(b-d)}$$

$$|A-\lambda I| = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - bc = 0$$

$$\Rightarrow ad - (a+d)\lambda + \lambda^2 - bc = 0$$

$$\Rightarrow \lambda^2 - (a+d)\lambda + (ad-bc) = 0$$

$$\cancel{\lambda^2 - (a+d)\lambda + (d+c)(d-b) = 0}$$

$$\cancel{\lambda = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(d+c)(d-b)}}{2}}$$

Which has solution given by

$$\lambda = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2} = \frac{(a+d) \pm \sqrt{a^2 + 2ad + d^2 - 4ad + 4bc}}{2}$$

$$= \frac{(a+d) \pm \sqrt{a^2 - 2ad + d^2 + 4bc}}{2}$$

From our one relation ~~exactly~~ among $a, b, c, + d$ replace a ~~one~~ with $a = c+d-b$ to obtain

$$\lambda = \frac{c+2d-b \pm \sqrt{(c+d-b)^2 - 2(c+d-b)d + d^2 + 4bc}}{2}$$

Expanding the expression in the square root we have

$$c^2 + \cancel{2cd} - 2cb + \cancel{d^2} - \cancel{2db} + b^2 - \cancel{2cd} - \cancel{2d^2} + \cancel{2bd} + d^2 + 4bc$$

$$= c^2 + 2bc - \cancel{d^2} + b^2 + \cancel{d^2} = c^2 + 2bc + b^2 = (c+b)^2$$

giving

$$\lambda = \frac{c+2d-b \pm (c+b)}{2} = \begin{cases} \dfrac{2c+2d}{2} = c+d \\[2ex] \dfrac{2d-2b}{2} = d-b \end{cases}$$

The 1st expression $c+d$ is what we found before the 2nd eigenvalue is given by $d-b$. A much easier way to compute this is to recognize that the trace$(A) = \lambda_1 + \lambda_2 = a+b + \lambda_2 = a+d$ so

$$\lambda_2 = d-b.$$

(31) To exchange the 1st two rows & columns of A, ~~set~~ $P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

considering the null space of

$$A - 11 \cdot I = \cancel{\begin{bmatrix} -10 & 2 & 1 \\ 3 & -4 & 3 \\ 4 & 8 & -6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1/5 & -1/10 \\ 3 & -4 & 3 \\ 4 & 8 & -6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1/5 & -1/10 \\ 0 & -17/5 & \\ & & \end{bmatrix}}$$

$$\cancel{\frac{3}{5} - \frac{4 \cdot 5}{5} = \frac{3-20}{5} = \frac{-17}{5}}$$

$$\frac{4}{5} + \frac{40}{5} = \frac{44}{5}$$

$$\frac{4}{10} = \frac{2}{5} - \frac{7(5)}{5}$$

$$\cancel{\frac{3}{5} - \frac{5(5)}{5} = \frac{3-25}{5} = \frac{-22}{5}} \qquad \cancel{\frac{3}{10} + \frac{30}{10}}$$

$$\frac{2}{5} - \frac{35}{5} = -\frac{33}{5}$$

$$= \begin{bmatrix} -10 & 2 & 1 \\ 3 & -5 & 3 \\ 4 & 8 & -7 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1/5 & -1/10 \\ 3 & -5 & 3 \\ 4 & 8 & -7 \end{bmatrix}$$

$$-\frac{\cancel{5}^{1}}{\cancel{22}_{2}}\left(\frac{\cancel{33}^{3}}{\cancel{10}_{2}}\right)$$

$$-\frac{3}{4}$$

$$-\frac{33}{8} \cdot \frac{5}{44} =$$

$$\Rightarrow \begin{bmatrix} 1 & -1/5 & -1/10 \\ 0 & -22/5 & 33/10 \\ 0 & \frac{44}{5} & -33/5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1/5 & -1/10 \\ 0 & 1 & -3/4 \\ 0 & 1 & -3/4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -1/5 & -1/10 \\ 0 & 1 & -3/4 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -1/4 \\ 0 & 1 & -3/4 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\cancel{\frac{1}{5}\left(\frac{-3}{4}\right) - \frac{1}{10} = \frac{-3}{20} - \frac{2}{20} = \frac{-5}{20} = \frac{-1}{4}}$$

which has a null space given by $\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$

For PAP we have

$$PAP - 11 \cdot I = \begin{bmatrix} -5 & 3 & 3 \\ 2 & -10 & 1 \\ 8 & 4 & -7 \end{bmatrix}$$

which is worked the same way but I'm stopping here

32 (a) A basis for the null space is given by ~~0~~. span of u.
A basis for the column space is given by span of $\{v, w\}$

(b) let $x = \frac{1}{3}v + \frac{1}{5}w$. Then $Ax = \frac{1}{3}Av + \frac{1}{5}Aw$

$$= \frac{3}{3}v + \frac{5}{5}w = v + w$$

Then all solutions are given by

$$x = c \cdot u + \frac{1}{3}v + \frac{1}{5}w$$

(c) $Ax = u$ ~~multiply by~~ ~~will have a solution iff~~ will have a solution iff u is in the column space of A. This means that

~~$\Rightarrow$~~ $u \in \text{Span}\{v, w\} \Rightarrow u = c_1 v + c_2 w$

This implies that $u, v, + w$ are linearly dependent in contradiction to the assumed independence of $u, v, + w$.

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(1) To factor $A = S\Lambda S^{-1}$ we 1st compute the eigenvalues & eigenvectors of A

The eigenvalues of A are given by

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ 0 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) = 0 \Rightarrow \lambda = 1 \text{ or } \lambda = 3$$

Then the eigenvectors associated w/ eigenvalue $\lambda = 1$ is given by the nullspace of $A - I$ or

$$\begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix} \text{ which is } \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The eigenvector associated w/ eigenvalue $\lambda = 3$ is given by the nullspace of $A - 3I$ or

$$\begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Thus the Matrix whose columns are given by the eigenvectors is given by

$$S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{so} \quad S^{-1} = \frac{1}{1}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

Thus $$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

which we can easily check as

$$A \stackrel{?}{=} \begin{bmatrix} 1 & 3 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$ which is A.

For $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ ~~This has rank = 1 so we are worried that~~ it ~~may not have enough eigenvectors~~ to be ~~diagonalized. We'll have to see.~~ Computing its eigenvectors we have

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 2 & 2-\lambda \end{vmatrix} = 0 \Rightarrow \cancel{(1-\lambda)(2-\lambda) - 2 = 0}$$

$$(1-\lambda)(2-\lambda) - 2 = 0 \Rightarrow 2 - 3\lambda + \lambda^2 - 2 = 0$$

$\Rightarrow \lambda(\lambda - 3) = 0$ so $\lambda = 0$ or $\lambda = 3$. The eigenvector associated with $\lambda = 0$ is given the nullspace of A or

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The eigenvector associated with $\lambda = 3$ is given by the nullspace of

$A - 3I$ or $\begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix}$ which is spanned by $\begin{bmatrix} 1 \\ +2 \end{bmatrix}$

Thus S our matrix of ~~coefficients~~ eigenvectors is given by

$$S = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \quad \text{with } S^{-1} = \frac{1}{2+1}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$$

So our eigenvector decomposition of A is given by

$$A = S\Lambda S^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2/3 & -1/3 \\ 1/3 & 1/3 \end{bmatrix}$$

Up

② ~~If~~ $\cancel{A = S\Lambda S^{-1}}$ ~~then~~ $\cancel{A^3 = (S\Lambda S^{-1})(S\Lambda S^{-1})(S\Lambda S^{-1}) = S\Lambda^2\Lambda S^{-1} = S\Lambda^3 S^{-1}}$

$\cancel{A^{-1} = (S\Lambda S^{-1})^{-1} = S\Lambda^{-1}S^{-1}}$

③ Then A can be assembled from its eigenvectors + eigenvalues by

$A = S\Lambda S^{-1}$. We have $S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ so $S^{-1} = \frac{1}{1}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

of $\cancel{A \in \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}}$

So $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 2 & 5 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$$

(4) If $A = S\Lambda S^{-1}$ then the eigenvalue matrix for A is Λ. The eigenvalue matrix for $A+2I$ is given by $\Lambda + 2I$. The eigenvector matrix for $A+2I$ is the same as that for A i.e. the matrix S.

Consider $S(\Lambda + 2I)(S^{-1}) = S\Lambda S^{-1} + 2S\cdot S^{-1} = A + 2I$

(5) (a) False A can still have an eigenvalue equal to zero

(b) True the matrix of eigenvectors S has an inverse.

(c) True, S has ~~fort~~ full rank & ∴ is invertible

(d) False since S could have repeated eigenvalues & ∴ possible a non complete set of eigenvectors.

---

(6) Then A is a diagonal matrix since $S = I = S^{-1}$ & ∴

$$A = S\Lambda S^{-1} = \Lambda.$$

If the eigenvector matrix S is triangular then S^{-1} is triangular then A is since $A = S\Lambda S^{-1}$ & left multiplication of S^{-1} by Λ is multiplication of the rows of S^{-1} & ∴ ΛS^{-1} is still triangular. Since S & ΛS^{-1} are both triangular their product is triangular & ∴ A is triangular

(7) If $A = \begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix}$ then A has eigenvectors given by

$$|A - \lambda I| = \begin{vmatrix} 4-\lambda & 0 \\ 1 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (4-\lambda)(2-\lambda) = 0$$

so $\lambda = 2$ or $\lambda = 4$

The ~~eigenvalues~~ eigenvectors for A are given by the nullspace of the matrix $A-\lambda I$

For $\lambda=2$ this ~~matr~~ ~~wat~~ matrix is

$$\begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}$$ which has nullspace given by the span of $\alpha \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

For $\lambda = 4$ this matrix is

$$\begin{bmatrix} 0 & 0 \\ 1 & -2 \end{bmatrix}$$ which has nullspace given by the span ~~of~~ $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Thus All matrices that diagonalize A are given by

$$S = \begin{bmatrix} 0 & 2\beta \\ \alpha & \beta \end{bmatrix} \quad w/ \quad S^{-1} = \frac{1}{(-2\alpha\beta)} \begin{bmatrix} \beta & -2\beta \\ -\alpha & 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2\alpha} & \frac{1}{\alpha} \\ \frac{1}{2\beta} & 0 \end{bmatrix}$$

~~A~~ The matrices that diagonalize A are the same ones that diagonalize A^{-1} so the S + S^{-1} above apply to the diagonalization ~~of~~ A^{-1} also.

(B) We can assemble A from its eigenvectors using

$$A = S\Lambda S^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \left(\frac{-1}{2}\right) \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 & \lambda_2 \\ \lambda_1 & -\lambda_2 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \left(\frac{-1}{2}\right)$$

$$= \left(-\frac{1}{2}\right)\begin{bmatrix} -\lambda_1-\lambda_2 & -\lambda_1+\lambda_2 \\ -\lambda_1+\lambda_2 & -\lambda_1-\lambda_2 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} \lambda_1+\lambda_2 & \lambda_1-\lambda_2 \\ \lambda_1-\lambda_2 & \lambda_1+\lambda_2 \end{bmatrix}$$

(9) If $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

Then $A^2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

+ $A^3 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$

$A^4 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$

Since $F_0 = 0,\ F_1 = 1,\ F_2 = 1, \dots$ we have that if we define

~~Then. since~~ $\cancel{F_n = A^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}}$

$u_n = \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}$ Then $u_{n+1} = \begin{bmatrix} F_{n+2} \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} F_{n+1} + F_n \\ F_{n+1} \end{bmatrix}$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = A u_n$$

with $u_0 = \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Iterating $u_{n+1} = A u_n$ we have that

$$u_n = A^n u_0$$

If we want F_{20} we extract the 2nd component from U_{20}.

Since $U_{20} = A^{20} u_0 = \cancel{\ }$

It will help to have u_0 written in terms ~~of~~ of the eigenvectors of A.

~~this gives~~ doing this gives $u_0 = \dfrac{x_1 - x_2}{\lambda_1 - \lambda_2}$ w/ $x_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}$ + $x_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$

So $U_{20} = \dfrac{(\lambda_1)^{20} x_1 - (\lambda_2)^{20} x_2}{\lambda_1 - \lambda_2}$ since For the Fibonacci matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

we have $\lambda_1 = \dfrac{1+\sqrt{5}}{2}$ + $\lambda_2 = \dfrac{1-\sqrt{5}}{2}$ ~~or~~ ~~vector~~ the value ~~of~~ of F_{20} is

given by $\dfrac{\lambda_1^{20} - \lambda_2^{20}}{\lambda_1 - \lambda_2} = \dfrac{1}{\sqrt{5}}\left[\left(\dfrac{1+\sqrt{5}}{2}\right)^{20} - \left(\dfrac{1-\sqrt{5}}{2}\right)^{20}\right] = \ldots$

(10) $G_{k+2} = \frac{1}{2}(G_k + G_{k+1})$ then defining

$U_k = \begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix}$ we have $U_{k+1} = \begin{bmatrix} G_{k+2} \\ G_{k+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(G_k + G_{k+1}) \\ G_{k+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix}$

So $A = \begin{bmatrix} 1/2 & 1/2 \\ 1 & 0 \end{bmatrix}$

The eigenvalues + eigenvectors of A are given by

$$|A - \lambda I| = \begin{vmatrix} 1/2 - \lambda & 1/2 \\ 1 & -\lambda \end{vmatrix} = -\lambda\left(\tfrac{1}{2} - \lambda\right) - \tfrac{1}{2} = 0$$

$$\Rightarrow \lambda^2 - \tfrac{1}{2}\lambda - \tfrac{1}{2} = 0$$

$$\lambda = \frac{\frac{1}{2} \pm \sqrt{\frac{1}{4} - 4(-\frac{1}{2})}}{2} = \frac{\frac{1}{2} \pm \sqrt{\frac{1}{4} + 2}}{2} = \frac{1}{4} \pm \frac{1}{2}\sqrt{\frac{9}{4}}$$

$$= \frac{1}{4} \pm \frac{1}{2}\cdot\frac{3}{2} = \frac{1 \pm 3}{4} = \begin{cases} -\frac{1}{2} \\ 1 \end{cases}$$

The eigenvectors are given by the nullspace of $A - \lambda I$. For $\lambda = -\frac{1}{2}$ this is $\begin{bmatrix} 1 & \frac{1}{2} \\ 1 & +\frac{1}{2} \end{bmatrix}$ which has a nullspace given by the span of $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$. For $\lambda = 1$ this matrix is

$\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ 1 & -1 \end{bmatrix}$ which has nullspace given by the span of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(b) Then $A^n = S\Lambda^n S^{-1}$ w/ $S = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$ & $S^{-1} = \frac{1}{1+2}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$

$$= \frac{1}{3}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$$

So $A^n = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} (-\frac{1}{2})^n & 0 \\ 0 & 1 \end{bmatrix}\frac{1}{3}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$

$$= \frac{1}{3}\begin{bmatrix} (-\frac{1}{2})^n & 1 \\ -2(-\frac{1}{2})^n & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} (-\frac{1}{2})^n + 2 & -(-\frac{1}{2})^n + 1 \\ -2(-\frac{1}{2})^n + 2 & 2(-\frac{1}{2})^n + 1 \end{bmatrix}$$

From which we see that $A^{\infty} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$

(c) If $G_0 = 0$ & $G_1 = 1$ then $v_0 = \begin{bmatrix} G_1 \\ G_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

So $U_\infty = \cancel{\frac{1}{3}} \; A^{\infty} v_0 = \frac{1}{3}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \frac{2}{3}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Thus the ~~Gibonacci~~ ~~num~~ $G_\infty = \frac{2}{3}$ the Fibonacci numbers approach $2/3$

(11) From the given pieces & the eigenvector decomposition ~~we~~ ~~see that~~ ~~S~~ $A = S \Lambda S^{-1}$ we recognize $S = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}$ then

$$S^{-1} = \frac{1}{\lambda_1 - \lambda_2}\begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}$$

So we have the decomposition of

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\frac{1}{(\lambda_1 - \lambda_2)}\begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}$$

Then $$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^k = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix}\frac{1}{(\lambda_1 - \lambda_2)}\begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}$$

So the requested multiplication is given by

$$S\Lambda^k S^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = S\Lambda^k \frac{1}{(\lambda_1 - \lambda_2)}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= S \frac{1}{(\lambda_1-\lambda_2)} \begin{bmatrix} \lambda_1^k \\ -\lambda_2^k \end{bmatrix} = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \frac{1}{(\lambda_1-\lambda_2)} \begin{bmatrix} \lambda_1^k \\ -\lambda_2^k \end{bmatrix}$$

$$= \frac{1}{\lambda_1-\lambda_2} \begin{bmatrix} \lambda_1^{k+1} - \lambda_2^{k+1} \\ \lambda_1^k - \lambda_2^k \end{bmatrix}$$ which has a second component given by

$$F_k = \frac{\lambda_1^k - \lambda_2^k}{\lambda_1 - \lambda_2}$$

(12) The original equation for the λ's is the characteristic equation given by

$$\lambda^2 - \lambda - 1 = 0$$

Since $\lambda_1 = \frac{1+\sqrt{5}}{2}$ + $\lambda_2 = \frac{1-\sqrt{5}}{2}$ are the ~~gone~~ solutions to this quadratic equation we see that multiply by λ^k this equation can be written as

$$\lambda^{k+2} - \lambda^{k+1} - \lambda^k = 0 \quad \text{or}$$

$$\lambda^{k+2} = \lambda^{k+1} + \lambda^k .$$

Then the linear combination of λ_1^k + λ_2^k must satisfy this ~~so~~ Thus

$$F_k = \frac{\lambda_1^k - \lambda_2^k}{\lambda_1 - \lambda_2}$$ ~~lore~~ ~~the~~ F_k will satisfy this recurrence relation

+ has values $F_0 = 0$ + $F_1 = \frac{\lambda_1 - \lambda_2}{\lambda_1 - \lambda_2} = 1$

(13) So define $v_0 = \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = x_1 + x_2$

Then ~~F~~ ~~A~~ $v_{20} = A^{20} v_0 = A^{20}(x_1 + x_2) = \lambda_1^{20} x_1 + \lambda_2^{20} x_2$

$= \lambda_1^{20} \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} + \lambda_2^{20} \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$ so the second component of this

vector is given by $\lambda_1^{20} + \lambda_2^{20}$

so $F_{20} = \left(\frac{1+\sqrt{5}}{2}\right)^{20} + \left(\frac{1-\sqrt{5}}{2}\right)^{20} =$

(14) $F_0 = 0$, $F_1 = 1$, $F_2 = 1$, $\ldots$, $F_{k+2} = F_k + F_{k+1}$

Prove that F_{3k} is even. ~~From the explicit representation of the~~ ~~E~~ one might be able to do this ~~by~~ ~~the~~ using the explicit Fibonacci representation but it will prob. be easier to prove this by induction. ~~i.e.~~ ~~F~~ Since $F_3 = 2$ we have the starting condition ~~fo~~ of an induction proof true. Then Assuming that F_{3k} is even for $k \leq n$ we desire to show that it is even for $F_{3(n+1)}$. ~~This expression is~~ ~~the~~ ~~Defining~~ ~~$v_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$~~

~~we have $v_{k+1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} v_k$ so evaluating this for $k+1 = 3(n+1)$ we obtain~~

$$\begin{bmatrix} F_{3(n+1)} \\ F_{3(n+1)-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} F_{3n+2} \\ F_{3n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^2 \begin{bmatrix} F_{3n+1} \\ F_{3n} \end{bmatrix}$$

Now $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

$$F_{3(n+1)} = F_{3n+3} = F_{3n+2} + F_{3n+1}$$

$$= F_{3n+1} + F_{3n} + F_{3n+1} = F_{3n} + 2F_{3n+1}.$$

Thus since F_{3n} is even & $2F_{3n+1}$ is even $F_{3(n+1)}$ is even

By induction F_{3n} is even $\forall n$.

(15) (a) True, $\lambda \neq 0$ $\therefore$ A is is invertible

(b) It's possible but not definite if the repeated eigenvalue has enough eigenvectors (Not this is generally true)

(c) It is possible if the ~~eigen~~ the $\lambda = 2$ eigenvalue does not have enough eigen vectors

(16) (a) False the multiple eigenvector could ~~be~~ correspond to a nonzero eigenvalue

(b) This must be true or else if not we would have another distinct eigenvector

(c) This is true. There are not enough eigenvectors to fill the eigenvector matrix S.

(17) For the 1st $A = \begin{bmatrix} 8 & b \\ c & 2 \end{bmatrix}$ ~~we would have~~ $\cancel{|A - \lambda I| = \begin{vmatrix} 8-\lambda & b \\ c & 2-\lambda \end{vmatrix}}$

$= \cancel{(8-\lambda)(2-\lambda) - bc = 16 - 10\lambda + \lambda^2 - bc}$

~~to trade this into~~ Since the $\det(A) = \lambda_1\lambda_2 = 25$ this gives

$16 - bc = 25$

$\Rightarrow bc = 16 - 25 = -9 \quad \therefore$ pick $b = 1$ & $c = -9$ giving $A = \begin{bmatrix} 8 & 1 \\ -9 & 2 \end{bmatrix}$

Then $|A - \lambda I| = \begin{vmatrix} 8-\lambda & 1 \\ -9 & 2-\lambda \end{vmatrix} = (8-\lambda)(2-\lambda) + 9 = 16 - 10\lambda + \lambda^2 + 9$

$= 25 - 10\lambda + \lambda^2 = (\lambda - 5)^2$. An eigenvector for $\lambda = 5$ is given by the nullspace of $A - 5I$ which is $\begin{bmatrix} 3 & 1 \\ -9 & -3 \end{bmatrix}$

or $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$

Which has only one eigenvector ~~gravity~~ as requested.

For $A = \begin{bmatrix} 9 & 4 \\ c & 1 \end{bmatrix}$ we must have $Tr(A) = 10 = \lambda_1 + \lambda_2 = 10$ yes

& $\det(A) = 9 - 4c = 25 \Rightarrow -4c = 16 \Rightarrow c = -4$

Thus $A = \begin{bmatrix} 9 & 4 \\ -4 & 1 \end{bmatrix}$ Then the characteristic equation for A is

given by $|A - \lambda I| = \begin{vmatrix} 9-\lambda & 4 \\ -4 & 1-\lambda \end{vmatrix} = (9-\lambda)(1-\lambda) + 16$

$= 9 - 10\lambda + \lambda^2 + 16 = (\lambda - 5)^2$ as expected. We also

have the eigenvectors for this A given by the nullspace of $A - 5I$

or $\begin{bmatrix} 4 & 4 \\ -4 & -4 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Finally for $A = \begin{bmatrix} 10 & 5 \\ -5 & d \end{bmatrix}$ the determinant requirement gives

$10d + 25 = 25 \Rightarrow d = 0$ so $A = \begin{bmatrix} 10 & 5 \\ -5 & 0 \end{bmatrix}$

Then the characteristic equation for A is given by

$|A - \lambda I| = \begin{vmatrix} 10-\lambda & 5 \\ -5 & -\lambda \end{vmatrix} = \lambda^2 - 10\lambda + 25 = (\lambda - 5)^2$

An the eigenvector is given by the nullspace of

$\cancel{A=\lambda}$ $A-5I = \begin{bmatrix} 5 & 5 \\ -5 & -5 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(18) The rank of $A-3I$ is 1, & ∴ since rank + nullity = 2

the ~~matri~~ nullspace has dimension $2-1=1$ & ∴ there don't exist

a complete set of eigenvectors for the $\lambda=3$ eigenvalue.

If we changed the 1,1 or 2,2 element to 3.01 then the eigenvalues of

$\cancel{A-3I} = \cancel{\begin{bmatrix} 0 & 1 \\ 0 & .01 \end{bmatrix}}$ A are given by 3 & 3.01 so ~~one~~ ~~of~~

since they are different we are guaranteed to have ~~unique~~ independent eigenvectors

& ∴ A is diagonalizable.

(17) If every λ has magnitude less than 1. Since A is Markov matrix

it has an eigenvalue equal to 1 & ∴ will not iterate to zero.

For B it has eigenvalues given by $|B-\lambda I|=0$ or

$$\begin{vmatrix} .6-\lambda & .9 \\ .1 & .6-\lambda \end{vmatrix} = (.6-\lambda)^2 - .09 = 0$$

$$\Rightarrow (.6-\lambda)^2 = \frac{9}{100}$$

$$\Rightarrow \frac{6}{10}-\lambda = \pm\frac{3}{10} \Rightarrow \lambda = \frac{6\pm 3}{10} = .3 \text{ or } .9$$

since $|\lambda_1| < 1$ & $|\lambda_2| < 1$ $A^k \to \underline{0}$ as $k \to \infty$

(20) For A in problem 19 we know since it is a Markov matrix that one eigenvalue is equal to 1. Thus from the trace / determinant formulas

$$\lambda_1 + \lambda_2 = 1.2 \quad \& \quad \lambda_1 \cdot \lambda_2 = .36 - .16 = .2$$

$\Rightarrow$ if $\lambda_1 = 1 \Rightarrow \lambda_2 = .2$, the eigenvector for $\lambda_1 = 1$ is given by the nullspace of $A - I = \begin{bmatrix} -.4 & .4 \\ .4 & -.4 \end{bmatrix}$ or the span of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

For $\lambda_2 = .2$ the eigenvector for $\lambda_2 = .2$ is given by the nullspace of $A - .2I$ or $\begin{bmatrix} .4 & .4 \\ .4 & .4 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Thus our matrix of eigenvectors is given by $S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

with $S^{-1} = \dfrac{1}{-1-1} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$ so we have

$$A = S \Lambda S^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & .2 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

Thus since $\Lambda^k = \begin{bmatrix} 1^k & 0 \\ 0 & .2^k \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & .2^k \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ as $k \to \infty$

the limit of $S \Lambda^k S^{-1}$ is given by

$$\begin{bmatrix} 1 & +1 \\ 1 & -1 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \cancel{\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}} \begin{bmatrix} 1/2 & 1/2 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$ which has the eigenvector corresponding

to the $\lambda = 1$ eigenvalue in the columns

(21) The eigenvalues for B in problem 19 are given by

$\lambda_1 = .3$ & $\lambda_2 = .9$ The corresponding eigenvectors are given by the null space of

$$\begin{bmatrix} .3 & .9 \\ .1 & .3 \end{bmatrix}$$ or the span of $$\begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

For $\lambda_2 = .9$ the equivalent pieces are

$$\begin{bmatrix} -.3 & .9 \\ .1 & -.3 \end{bmatrix}$$ or the span of $$\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Thus to evaluate $B^{10} v_0$ we decompose v_0 into its basis of eigenvectors of B. Doing this in matrix form we have

$$\begin{bmatrix} 3 & 3 & 6 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1^1 & c_1^2 & c_1^3 \\ c_2^1 & c_2^2 & c_2^3 \end{bmatrix}$$

Where I have concatenated the coefficient vectors used to expand each v_0.

For example $\begin{bmatrix} 3 \\ 1 \end{bmatrix} = c_1^1 \begin{bmatrix} -3 \\ 1 \end{bmatrix} + c_2^1 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

Then this matrix of coefficients is given by

$$\begin{bmatrix} c_1^1 & c_1^2 & c_1^3 \\ c_2^1 & c_2^2 & c_2^3 \end{bmatrix} = \frac{1}{(-3-3)} \begin{bmatrix} 1 & -3 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} 3 & 3 & 6 \\ 1 & -1 & 0 \end{bmatrix}$$

$$= \frac{-1}{6} \begin{bmatrix} 0 & 6 & 6 \\ -6 & 0 & -6 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

~~which~~ we see

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} = 1x_2$$

$$\begin{bmatrix} 3 \\ -1 \end{bmatrix} = -x_1$$

$$\begin{bmatrix} 6 \\ 0 \end{bmatrix} = -x_1 + x_2$$

Which could have been obtained by inspection. Thus since $B^{10} = S\Lambda^{10}S^{-1}$

we have since $S = \begin{bmatrix} -3 & 3 \\ 1 & 1 \end{bmatrix}$ $\quad S^{-1} = \frac{-1}{6} \begin{bmatrix} 1 & -3 \\ -1 & -3 \end{bmatrix} = \begin{bmatrix} -1/6 & 1/2 \\ 1/6 & 1/2 \end{bmatrix}$

we then have
$$B^{10} = \begin{bmatrix} -3 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} .3^{10} & 0 \\ 0 & .9^{10} \end{bmatrix} \begin{bmatrix} -1/6 & 1/2 \\ 1/6 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} -3(.3)^{10} & 3(.9)^{10} \\ (.3)^{10} & (.9)^{10} \end{bmatrix} \begin{bmatrix} -1/6 & 1/2 \\ 1/6 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}(.3)^{10} + \frac{1}{2}(.9)^{10} & -\frac{3}{2}(.3)^{10} + \frac{3}{2}(.9)^{10} \\ -\frac{1}{6}(.3)^{10} + \frac{1}{6}(.9)^{10} & \frac{1}{2}(.3)^{10} + \frac{1}{2}(.9)^{10} \end{bmatrix}$$

And more specifically $B^{10}\begin{bmatrix}3\\1\end{bmatrix} = B^{10}(x_2) = \lambda_2^{10} x_2 = (.9)^{10} x_2 = (.9)^{10}\begin{bmatrix}-3\\1\end{bmatrix}$

$$B^{10}\begin{bmatrix}+3\\-1\end{bmatrix} = B^{10}(-x_1) = -B^{10}x_1 = -(\lambda_1)^{10} x_1 = -(.3)^{10}\begin{bmatrix}-3\\1\end{bmatrix} = (.3)^{10}\begin{bmatrix}3\\-1\end{bmatrix}$$

$$\& \; B^{10}\begin{bmatrix}6\\0\end{bmatrix} = B^{10}(-x_1+x_2) = -B^{10}x_1 + B^{10}x_2 = -(\lambda_1^{10})x_1 + \lambda_2^{10}x_2$$

$$= -(.3)^{10}\begin{bmatrix}-3\\1\end{bmatrix} + (.9)^{10}\begin{bmatrix}3\\1\end{bmatrix} = (.3)^{10}\begin{bmatrix}-3\\1\end{bmatrix} + (.9)^{10}\begin{bmatrix}3\\1\end{bmatrix}$$

(22) A has eigenvalues given by the roots to

$$\begin{vmatrix}2-\lambda & 1\\ 1 & 2-\lambda\end{vmatrix} = 0 \Rightarrow (2-\lambda)^2 - 1 = 0 \Rightarrow 2-\lambda = \pm 1$$

$\Rightarrow \lambda = 1, 3$ with eigenvector for $\lambda_1 = 1$ given by the nullspace of $\begin{bmatrix}2-1 & 1\\ 1 & 2-1\end{bmatrix} = \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$ or the span of $\begin{bmatrix}1\\-1\end{bmatrix}$

The eigenvector for $\lambda_2 = 3$ is given by the nullspace of $\begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}$ or the span of $\begin{bmatrix}1\\1\end{bmatrix}$

Thus $S = \begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}$ & so $S^{-1} = \frac{1}{2}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$

So $A^k = S\Lambda^k S^{-1} = \begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 3^k\end{bmatrix}\begin{bmatrix}1/2 & -1/2\\ 1/2 & 1/2\end{bmatrix}$.

$$= \begin{bmatrix}1 & 3^k\\ -1 & 3^k\end{bmatrix}\begin{bmatrix}1/2 & -1/2\\ 1/2 & 1/2\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1+3^k & -1+3^k\\ -1+3^k & 1+3^k\end{bmatrix}$$

(23) The eigenvalues of B are given by $3 + 2$ since B is upper triangular & $\therefore$ the eigenvalues can be read at the diagonals. The eigenvector for $\lambda = 2$ is given by the nullspace of

$$\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix} \quad \text{or} \quad \begin{bmatrix}1\\ -1\end{bmatrix}.$$

The eigenvector for $\lambda = 3$ is given by the nullspace of

$$\begin{bmatrix}0 & 1\\ 0 & -1\end{bmatrix} \quad \text{or} \quad \begin{bmatrix}1\\ 0\end{bmatrix}$$

Thus $S = \begin{bmatrix}1 & 1\\ -1 & 0\end{bmatrix}$ so $S^{-1} = \begin{bmatrix}0 & -1\\ 1 & 1\end{bmatrix}\left(\frac{1}{1}\right)$ & $\Lambda = \begin{bmatrix}2 & 0\\ 0 & 3\end{bmatrix}$

Thus $B^k = S\Lambda^k S^{-1} = \begin{bmatrix}1 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}2^k & 0\\ 0 & 3^k\end{bmatrix}\begin{bmatrix}0 & -1\\ 1 & 1\end{bmatrix}$

$$= \begin{bmatrix}2^k & 3^k\\ -2^k & 0\end{bmatrix}\begin{bmatrix}0 & -1\\ 1 & 1\end{bmatrix} = \begin{bmatrix}3^k & 3^k - 2^k\\ 0 & 2^k\end{bmatrix}$$

(24) If $A = S\Lambda S^{-1}$ then $|A| = |S\Lambda S^{-1}| = |S||\Lambda|\cdot|S^{-1}| = |S|\cdot|\Lambda|\cdot|S|^{-1} =$ ~~$|A|$~~
$=$ ~~$|A|$~~ $= |\Lambda|$

But since Λ is diagonal its determinant is the product of the diagonal elements & $\therefore$ $|A| = \prod_{i=1}^{n} \lambda_i$. This quick proof works only when A is diagonalizable.

(25) $AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} q & r \\ s & t \end{bmatrix} = \begin{bmatrix} aq+bs & ar+bt \\ cq+sd & cr+dt \end{bmatrix}$

So $Tr(AB) = aq + bs + cr + dt$

& $BA = \begin{bmatrix} q & r \\ s & t \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} qa+rc & qb+rd \\ sa+tc & \cancel{qb+rd}\ sb+td \end{bmatrix}$

So $Tr(BA) = aq + rc + \cancel{\ } sb + td$ which are the same

Choose A as S & B as ΛS^{-1} then $S(\Lambda S^{-1})$ has the same trace as $(\Lambda S^{-1})S = \Lambda$ which has its trace given by $\sum_{i=1}^{m} \lambda_i$

This again assumes that A is diagonalizable. For general $m \times m$ matrices we have that the terms in the product matrix AB is given by $\sum_{k=1}^{m} a_{ik} b_{kj}$ & the terms in the product BA are given by $\sum_{k=1}^{M} b_{ik} a_{kj}$ so the trace of $AB = \sum_{i=1}^{m}\left(\sum_{k=1}^{M} a_{ik} b_{ki}\right)$

and the trace of BA is given by $\sum_{i=1}^{m}\left(\sum_{k=1}^{m} b_{ik}a_{ki}\right)$ which is the same as the trace of AB

(26) $AB - BA = I$ is impossible since the trace of the left hand side is given by $Tr(AB) - Tr(BA) = 0$ while the right hand side has a Trace equal to m if I is the $m \times m$ identity matrix.

let $A = E = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ + $B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ so that

$$AB = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \quad \& \quad BA = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

so that $AB - BA = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ which has trace two

(27) If $A = S\Lambda S^{-1}$ + $B = \begin{bmatrix} A & 0 \\ 0 & 2A \end{bmatrix}$ then

$$B = \begin{bmatrix} S\Lambda S^{-1} & 0 \\ 0 & S(2\Lambda)S^{-1} \end{bmatrix} = \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix}\begin{bmatrix} \Lambda & 0 \\ 0 & 2\Lambda \end{bmatrix}\begin{bmatrix} S^{-1} & 0 \\ 0 & S^{-1} \end{bmatrix}$$

Check: The right hand side is given by $\begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix}\begin{bmatrix} \Lambda S^{-1} & 0 \\ 0 & 2\Lambda S^{-1} \end{bmatrix}$

which equals

$$\begin{bmatrix} S\Lambda S^{-1} & 0 \\ 0 & S(2\Lambda)S^{-1} \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & 2A \end{bmatrix}$$

Thus the eigenvalue matrix for the block matrix $\begin{bmatrix} A & 0 \\ 0 & 2A \end{bmatrix}$ is given by $\begin{bmatrix} \Lambda & 0 \\ 0 & 2\Lambda \end{bmatrix}$ & the eigenvector matrices are given by

$$S = \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} \quad \& \quad S^{-1} = \begin{bmatrix} S^{-1} & 0 \\ 0 & S^{-1} \end{bmatrix}$$

(28) let $\mathcal{S} = \{ A \mid A \text{ is } 4\times4 \ \& \ \Lambda_A = S^{-1}AS \text{ for a fixed given } S \}$.

Then if A_1 & $A_2 \in \mathcal{S}$ we have that

$$\cancel{A_1 + A_2} \quad A_1 + A_2 = S\Lambda_1 S^{-1} + S\Lambda_2 S^{-1} = S(\Lambda_1 + \Lambda_2)S^{-1}$$

so $A_1 + A_2 \in \mathcal{S}$

& if $A_1 \in \mathcal{S}$ then $cA_1 = S(c\Lambda_A)S^{-1}$ so $cA_1 \in \mathcal{S}$

Thus $\mathcal{S}$ is a subspace. If $S = I$ then ~~A~~ the only possible A's in $\mathcal{S}$ are the diagonal ones its dimension is then 4.

(29) Suppose $A^2 = A$ then the column space of A must contain eigenvectors with $\lambda = 1$

~~The Nullspace or kernel of A contains the eigen vectors with $\lambda = 0$.~~

~~Thus since the dim Column space + dim (Nullity) = n~~

In fact all columns of A are eigenvectors w/ eigenvalue equal to 1.

~~If A is nxn then A~~ All vectors in the column space are eigenvectors with $\lambda = 1$. The vectors with $\lambda = 0$ lie in the Nullspace & from the first fundamental theorem of linear algebra The dim Column space + dim Nullity = n

Thus A will be diagonalizable since we are gurrented to have enough (n) eigen vectors.

(30) ~~Because~~ When A has a non empty nullspace we do indeed get $n-r$ linearly independent eigen vectors. But if x is not in the nullspace ~~then it is in the column space but~~ then is no gurrentee that $Ax = \lambda x$ for any constant λ. Thus the r vectors in the column space of A may have no basis of the column space such that $Ax = \lambda x$

In addition the null space & the column space can overlap if for instance one of the nullspace vectors is a column of the orginal A.

(31) The eigenvectors of A are given by for $\lambda = 1$ by the null space of

$$\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} \quad \text{or the span of} \quad \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The eigenvector of A for $\lambda = 9$ are given by the nullspace of

$$\begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix} \quad \text{or the span of} \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Thus $S = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ so $S^{-1} = \frac{1}{1+1}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

& $\therefore$ $R = S\sqrt{\Lambda}\,S^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$

So $R = \begin{bmatrix} 1 & 3 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}+\frac{3}{2} & -\frac{1}{2}+\frac{3}{2} \\ -\frac{1}{2}+\frac{3}{2} & \frac{1}{2}+\frac{3}{2} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

Note that RR is given by $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix}$

which should be A but is not... I don't know why.

Since if $R = S\sqrt{\Lambda}S^{-1}$ then $R \cdot R = S\sqrt{\Lambda}S^{-1}S\sqrt{\Lambda}S^{-1}$

$$= S\Lambda S^{-1}$$

The square root of Λ would require square roots of 9 & -1 the latter would be imaginary & then $R = S\sqrt{\Lambda}S^{-1}$ could not be real since S & S^{-1} are & $\sqrt{\Lambda}$ would not be $\therefore$ the product $S\sqrt{\Lambda}S^{-1}$ could not be real.

(32) $x^T x = x^T I x = x^T(AB - BA)x = x^T ABx - x^T BAx$

$$= (Ax)^T(Bx) + (Bx)^T(Ax) = 2(Ax)^T(Bx) \leq 2\|Ax\| \cdot \|Bx\|$$

where I have used $A^T = A$ & $B^T = -B$ to simplify

$$x^T ABx = (Ax)^T(Bx) \quad \& \quad x^T BAx = -(Bx)^T(Ax)$$

Thus $\|x\|^2 \leq 2\|Ax\| \cdot \|Bx\|$

$$\Rightarrow \frac{1}{2} \leq \frac{\|Ax\|}{\|x\|} \cdot \frac{\|Bx\|}{\|x\|}$$

(33) If A & B have the same ~~eigen~~ independent eigenvectors & the same λ's then $A = S\Lambda S^{-1}$ & $B = S\Lambda S^{-1}$ so $A = B$.

(34) If S is such that $A = S\Lambda_1 S^{-1}$ & $B = S\Lambda_2 S^{-1}$ then

$$AB = S\Lambda_1 S^{-1} \cdot S\Lambda_2 S^{-1} = S(\Lambda_1\Lambda_2)S^{-1} = S(\Lambda_2\Lambda_1)S^{-1}$$ since

diagonal matrices are commutative & $\therefore$

$$AB = S\Lambda_2 S^{-1} \cdot S\Lambda_1 S^{-1} = BA$$

(35) If A is diagonalizable then $A = S\Lambda S^{-1}$

Then the product matrix $(A - \lambda_1 I)(A - \lambda_2 I)\cdots(A - \lambda_n I)$

$$= (S\Lambda S^{-1} - \lambda_1 S \cdot S^{-1})(S\Lambda S^{-1} - \lambda_2 SS^{-1})\cdots(S\Lambda S^{-1} - \lambda_n SS^{-1})$$

$$= S(\Lambda - \lambda_1 I)S^{-1} \cdot S(\Lambda - \lambda_2 I)S^{-1} \cdot S(\cdots \; S(\Lambda - \lambda_n I)S^{-1}$$

$$= S(\Lambda - \lambda_1 I)(\Lambda - \lambda_2 I)\cdots(\Lambda - \lambda_n I)S^{-1}$$

But if we consider the product ~~A~~ $(\Lambda - \lambda_1 I)(\Lambda - \lambda_2 I)\cdots(\Lambda - \lambda_n I)$

is the product of diagonal matrices & $\therefore$ is given by

$$\begin{bmatrix} 0 & & & \\ & \lambda_2 - \lambda_1 & & \\ & & \lambda_3 - \lambda_1 & \\ & & & \ddots \\ & & & & \lambda_n - \lambda_1 \end{bmatrix} \begin{bmatrix} \lambda_1 - \lambda_2 & & & \\ & 0 & & \\ & & \lambda_3 - \lambda_2 & \\ & & & \ddots \\ & & & & \lambda_n - \lambda_2 \end{bmatrix} \cdots \begin{bmatrix} \lambda_1 - \lambda_n & & & \\ & \lambda_2 - \lambda_n & & \\ & & \ddots & \\ & & & 0 \end{bmatrix}$$

which gives
$$\begin{bmatrix} 0\,(\lambda_1-\lambda_2)\cdots(\lambda_1-\lambda_n) & & & \\ & (\lambda_2-\lambda_1)\,0\,(\lambda_2-\lambda_3)\cdots(\lambda_2-\lambda_n) & & 0 \\ & 0 & \ddots & \\ & & & (\lambda_n-\lambda_1)\cdot(\lambda_n-\lambda_2)\cdots(\lambda_n-\lambda_{n-1})\,0 \end{bmatrix}$$

which gives the zero matrix since each diagonal element has a zero in the product.

$\therefore \quad (A-\lambda_1 I)(A-\lambda_2 I)(A-\lambda_3 I)\cdots(A-\lambda_n I)$

(36) If $A = \begin{bmatrix} -3 & 4 \\ -2 & 3 \end{bmatrix}$ then $|A-\lambda I| = \begin{vmatrix} -3-\lambda & 4 \\ -2 & 3-\lambda \end{vmatrix}$

$= (-3-\lambda)(3-\lambda)+8 = \cancel{-9} \quad -(3+\lambda)(3-\lambda)+8 = -9+\lambda^2+8$

$= \lambda^2-1$.

Consider: A^2-I which gives

$$\begin{bmatrix} -3 & 4 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} -3 & 4 \\ -2 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 9-8 & -12+12 \\ 6-6 & -8+9 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \underline{\underline{0}}$$

Thus $A^2 = I$ + it looks like $A^{-1} = A$. To check this directly we have

$$A^{-1} = \frac{1}{-9+8}\begin{bmatrix} 3 & -4 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ -2 & 3 \end{bmatrix} = A \quad \text{as claimed}$$

(37) (a) Always. ~~The eigenvector with eigenvalue $\lambda \neq 0$~~ An ~~vector~~ vector in the nullspace of A is automatically an eigenvector with eigenvalue 0.

(b) ~~If the eigenvectors w/ $\lambda \neq 0$ span the column space then Av can be written as a linear combination of eigenvectors~~ with λ's $\neq 0$

$\cancel{Av = \sum_i c_i x_i}$

So $\cancel{A^2 v = \sum_i \lambda_i c_i x_i}$

The eigenvectors with $\lambda \neq 0$ will ~~always~~ span the column space if there are r independent eigenvectors. ~~then the algebraic multiplicity,~~

(2) Solving $\frac{dz}{dt} = z$ w/ $z(0) = -2$

the equation gives $z(t) = z_0 e^{t} \Rightarrow z(t) = -2e^{t}$.

then $\frac{dy}{dt} = 4y + 3z = 4y + -6e^{t}$

to solve $\frac{dy}{dt} = 4y - 6e^{t}$

we have a homogeneous equation $\frac{dy}{dt} = 4y$ and a

particular solution $\frac{dy}{dt} - 4y = -6e^{t}$ to solve for.

The homogeneous solution is given by $y(t) = c_2 e^{4t}$ + a

particular solution can be found by ~~[illegible]~~ substituting $y(t) = A e^{t}$ into

the particular solution i.e. $Ae^{t} - 4Ae^{t} = -6e^{t}$

$$\Rightarrow -3A = -6 \Rightarrow A = 2$$

Thus the total solution for y is $y(t) = c_2 e^{4t} + 2e^{t}$

then to satisfy the initial condition of $y(0) = 5$ we have that

c_2 must be given by $c_2 + 2 = 5 \Rightarrow c_2 = 3$

Thus
$$z(t) = -2e^{t}$$
$$y(t) = 3e^{4t} + 2e^{t}$$

(3) let $v = y'$

Then $y'' = 5v + 4y$

so $\frac{d}{dt}\begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} y' \\ 5y' + 4y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix}$

So our coefficient matrix A is given by $\begin{bmatrix} 0 & 1 \\ 4 & 5 \end{bmatrix}$. The eigenvalue are given by

$\lambda_1 + \lambda_2 = 5$ & $\lambda_1 \cdot \lambda_2 = -4$

$\lambda_1 = 5 - \lambda_2$ so $(5 - \lambda_2)\lambda_2 = -4$

$\Rightarrow -\lambda_2^2 + 5\lambda_2 + 4 = 0$

$\lambda_2^2 - 5\lambda_2 - 4 = 0$

$\Rightarrow \lambda_2 = \dfrac{5 \pm \sqrt{25 - 4(-4)}}{2} = \dfrac{5 \pm \sqrt{25 + 16}}{2} = \dfrac{5 \pm \sqrt{41}}{2}$

we can verify this by substituting $e^{\lambda t}$ into $y'' = 5y' + 4y$

obtaining $\lambda^2 = 5\lambda + 4$ which is equivalent to

$\lambda^2 - 5\lambda - 4 = 0$ the same as earlier

(4) $$\frac{dr}{dt} = 6r - 2w$$

$$\frac{dw}{dt} = 2r + w$$

Then in matrix form we have

$$\frac{d}{dt}\begin{bmatrix} r \\ w \end{bmatrix} = \begin{bmatrix} 6 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} r \\ w \end{bmatrix}$$

The coefficient matrix above has ~~the following~~ eigenvalues & eigenvectors given by $\lambda_1 \cdot \lambda_2 = 10$ & $\lambda_1 + \lambda_2 = 7$

Thus $\lambda = 2$ & 5 & the eigenvectors for $\lambda = 2$ are given by the null space of

$$\begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix} \to \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$$

which has a vector given by $\underline{X} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

The eigenvectors for $\lambda = 5$ is given by the nullspace of

$$\begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

& is given by $\underline{X} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Then the total solution $v(t)$ is given by a linear combination of

$$\begin{bmatrix}1\\2\end{bmatrix}e^{2t} + \begin{bmatrix}2\\1\end{bmatrix}e^{5t} \quad \text{i.e.}$$

$$v(t) = c_1\begin{bmatrix}1\\2\end{bmatrix}e^{2t} + c_2\begin{bmatrix}2\\1\end{bmatrix}e^{5t}$$

with $v(0) = \begin{bmatrix}30\\30\end{bmatrix} = c_1\begin{bmatrix}1\\2\end{bmatrix} + c_2\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}1 & 2\\2 & 1\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}$

$$\Rightarrow \begin{bmatrix}c_1\\c_2\end{bmatrix} = \frac{1}{1-4}\begin{bmatrix}1 & -2\\-2 & 1\end{bmatrix}\begin{bmatrix}30\\30\end{bmatrix} = \frac{-1}{3}\begin{bmatrix}1 & -2\\-2 & 1\end{bmatrix}\begin{bmatrix}30\\30\end{bmatrix}$$

$$= \begin{bmatrix}-1 & 2\\2 & -1\end{bmatrix}\begin{bmatrix}10\\10\end{bmatrix} = \begin{bmatrix}-10+20\\20-10\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}10\\10\end{bmatrix}$$

Thus $v(t) = 10\begin{bmatrix}1\\2\end{bmatrix}e^{2t} + 10\begin{bmatrix}2\\1\end{bmatrix}e^{5t}$

So the population of rabbits + wolves is given by

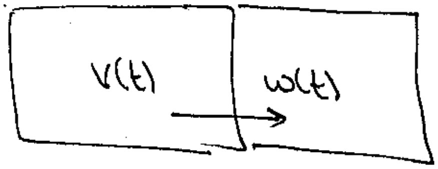
$$r(t) = 10e^{2t} + 20e^{5t}$$

$$+ \; w(t) = 20e^{2t} + 10e^{5t}$$

After a long time the ratio of rabbits to wolves is given by

$$\frac{r(t)}{w(t)} = \frac{10e^{2t} + 20e^{5t}}{20e^{2t} + 10e^{5t}} \to 2$$

(5)



$$\frac{dw}{dt} = v - w \quad \& \quad \frac{dv}{dt} = w - v$$

Consider $\quad y = v + w$

$$\frac{dy}{dt} = \frac{dv}{dt} + \frac{dw}{dt} = w - v + v - w = 0$$

So $y(t) = v(t) + w(t)$ is a constant $\therefore y(t) = y(0) = v(0) + w(0) = 30 + 10 = 40$

Defining $\underline{U} = \begin{bmatrix} v \\ w \end{bmatrix}$ then $\frac{dU}{dt} = \begin{bmatrix} w - v \\ v - w \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix}$

The coefficient matrix is given by $A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$

Its eigenvalues are given by $\begin{vmatrix} -1-\lambda & 1 \\ 1 & -1-\lambda \end{vmatrix} = 0$

$$\Rightarrow (1+\lambda)^2 - 1 = 0$$

$$\lambda^2 + 2\lambda = 0 \Rightarrow \lambda = 0 \ \& \ \lambda = -2$$

Its eigenvectors are given by for $\lambda = -2$ the nullity of the following matrix

$$\begin{bmatrix} -1+2 & 1 \\ 1 & -1+2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$ which has a nullvector

given by $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. The eigenvector for $\lambda = 0$ gives us

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$ which has a null space given by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Thus the total solution is given by

$$u(t) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Given the initial conditions of $v(0) = 30$ + $w(0) = 10$ to find that $c_1 + c_2$

or given by $$\begin{bmatrix} 30 \\ 10 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

pick $c_2 = 20$ + $c_1 = 10$ so that $u(t)$ is given by

$$u(t) = 10 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t} + 20 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

We can check that $v(t) + w(t) = 40$ by adding the two functions giving $$10e^{-2t} + 20 + -10e^{-2t} + 20 = 40$$

as expected

When $t=1$ we have

$$u(1) = \begin{bmatrix} v(1) \\ w(1) \end{bmatrix} = \begin{bmatrix} 10e^{-2}+20 \\ -10e^{-2}+20 \end{bmatrix}$$

(6) Now that our coefficient matrix is (-1) times A means that the eigenvectors of $Ax=\lambda x$ become $-Ax=-\lambda x$.

So the ~~eigenvalues~~ eigenvectors of ~~are the~~ $-A$ are the same as that of A, and the eigen~~vectors~~values are the negative of the eigenvalues of A.

Thus the two eigenvalues of $-A$ are given by $\lambda=0$ & $\lambda=2$.

and the eigenvectors $\begin{bmatrix}1\\1\end{bmatrix}$ & $\begin{bmatrix}1\\-1\end{bmatrix}$ so again the solution is given by

$$\begin{bmatrix} v(t) \\ w(t) \end{bmatrix} = 10\begin{bmatrix}1\\-1\end{bmatrix}e^{+2t} + 20\begin{bmatrix}1\\1\end{bmatrix}$$

then $v(t) = 10e^{2t}+20 \longrightarrow +\infty$ as $t \to +\infty$

(7) let the vector $u(t)=\begin{bmatrix}y\\y'\end{bmatrix}$ then $\frac{du}{dt} = \begin{bmatrix}y'\\y''\end{bmatrix} = \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}y\\y'\end{bmatrix}$

which has as its solution

$$\begin{bmatrix} y(t) \\ y'(t) \end{bmatrix} = e^{At}\begin{bmatrix} y(0) \\ y'(0) \end{bmatrix}$$

We can evaluate e^{At} ~~by~~ using the definition in terms of Taylor series

$$e^{At} = I + At + \frac{A^2t^2}{2} + \frac{A^3t^3}{6} + \cdots$$

Now $A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

So $e^{At} = I + At = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$

So $\begin{bmatrix} y(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} y(0) \\ y'(0) \end{bmatrix} = \begin{bmatrix} y(0) + y'(0)t \\ y'(0) \end{bmatrix}$

The 1st component ~~with~~ gives $y(t) = y(0) + y'(0)t$

(8) given $y = e^{\lambda t}$ then into our differential equation gives

$$\lambda^2 = 6\lambda - 9$$

$\Rightarrow \lambda^2 - 6\lambda + 9 = 0$

$\Rightarrow (\lambda - 3)^2 = 0$ so $\lambda = 3$ is a double root.

The matrix representation for $y'' = 6y' - 9y$ is given by

$$\frac{d}{dt}\begin{bmatrix} y(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -9 & 6 \end{bmatrix}\begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}$$

which has eigenvalues given by $(\lambda-3)^2=0$ as earlier.

To look for eigenvectors we consider

$$\begin{bmatrix} -3 & 1 \\ -9 & 3 \end{bmatrix}$$ which has $$\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$ as the <u>only</u> eigenvector

To show that te^{3t} is a second solution, evaluate the differential equation

$$y = te^{3t}$$
$$y' = e^{3t} + 3te^{3t}$$
$$y'' = 3e^{3t} + 3e^{3t} + 9te^{3t} = 6e^{3t} + 9te^{3t}$$

Then $6y' - 9y = 6e^{3t} + 18te^{3t} - 9te^{3t}$

$= 6e^{3t} + 9te^{3t}$ which is y'' showing how $y(t)$

satisfies the D.E.

9 (a) $$\frac{d}{dt}(u_1^2+u_2^2+u_3^2) = 2u_1u_1' + 2u_2u_2' + 2u_3u_3'$$

$$= 2u_1(cu_2-bu_3) + 2u_2(au_3-cu_1) + 2u_3(bu_1-au_2)$$

$$= 2c\cancel{u_1u_2} - 2b\cancel{u_1}u_3 + 2a\cancel{u_2}u_3 - 2c\cancel{u_1u_2}$$
$$+ 2b\cancel{u_1}u_3 - 2a\cancel{u_2}u_3$$

$$= 0$$

Since $u_1^2+u_2^2+u_3^2 = \|u\|^2$, $\|u\|^2$ must be constant

(b) $\|e^{At}u(0)\| = \|u(0)\|$ so e^{At} is an ~~an~~ orthogonal matrix.

So ~~Q~~ when A is skew symmetric $Q = e^{At}$ is an orthogonal matrix.

(10) When $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ we have 2 eigenvectors

(a) $\begin{bmatrix} 1 \\ i \end{bmatrix}$ with eigenvalue $\lambda = i$

+ $\begin{bmatrix} 1 \\ -i \end{bmatrix}$ with eigenvalue $\lambda = -i$

to superimpose $\begin{bmatrix} 1 \\ i \end{bmatrix}$ + $\begin{bmatrix} 1 \\ -i \end{bmatrix}$ into $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ we have

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ i \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -i \end{bmatrix} \quad \text{so } c_1 = \tfrac{1}{2} + c_2 = \tfrac{1}{2}$$

(b) Thus the solution to $\frac{du}{dt} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$

is given by $u(t) = c_1 e^{it} \begin{bmatrix} 1 \\ i \end{bmatrix} + c_2 e^{-it} \begin{bmatrix} 1 \\ -i \end{bmatrix}$

w/ $c_1 = c_2 = \frac{1}{2}$

$$u(t) = \frac{1}{2} e^{it} \begin{bmatrix} 1 \\ i \end{bmatrix} + \frac{1}{2} e^{-it} \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

with $e^{it} = \cos t + i\sin t$

$+ \; e^{-it} = \cos t - i\sin t$

$$v(t) = \frac{1}{2}(\cos t + i\sin t)\begin{bmatrix} 1 \\ i \end{bmatrix} + \frac{1}{2}(\cos t - i\sin t)\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2\cos t \\ -\sin t - \sin t \end{bmatrix} = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$$

(11) (a) $\frac{d^2y}{dt^2} = -y$. Is solved by $y(t) = A\cos t + B\sin(t)$

To have $y(0) = 1$ + $y'(0) = 0$ we must have

$y(t) = \cos t$

(b) If we write the matrix form of $y'' = -y$.

$$\underline{v} = \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix} \quad \text{then} \quad \frac{dv}{dt} = \begin{bmatrix} y'(t) \\ y''(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}$$

From part (a) $y(t) = \cos t$ + $y'(t) = -\sin t$

Then $\underline{v} = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}$ + $\frac{dv}{dt} = \begin{bmatrix} -\sin t \\ -\cos t \end{bmatrix}$

+ which equals $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} = \begin{bmatrix} -\sin t \\ -\cos t \end{bmatrix}$

+ $v(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ showing this vector solution $\underline{v}$ solves the differential equal + initial conditions also

(12) If A is invertible then a particular solution to

$$\frac{du}{dt} = Au - b \quad \text{then if } u \text{ is constant} \quad \frac{du}{dt} = 0$$

$$0 = Au - b \Rightarrow u = A^{-1}b$$

(a) $\frac{du}{dt} = 2u - 8$

The particular solution is given by $2u = 8 \Rightarrow u = 4$.

& the homogenous solution is given by $\frac{du}{dt} = 2u \Rightarrow u = ce^{2t}$

So the complete solution is given by

$u(t) = 4 + ce^{2t}$

(b) $\frac{du}{dt} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} u - \begin{bmatrix} 8 \\ 6 \end{bmatrix}$

Then a particular solution is given by (Assuming u is a constant)

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} u = \begin{bmatrix} 8 \\ 6 \end{bmatrix} \Rightarrow \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

a particular solution is given by the solution to

$$\frac{du}{dt} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} u . \quad \& \quad \cancel{u(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

The coefficient matrix A is given by $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

Which has ~~eigenvectors~~ eigenvalues 2 + 3. With eigenvectors

$\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ then the total solution is then

$$c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{3t}$$

So that the total solution (particular plus homogenous) is given by

$$\underline{u} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{3t} + \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

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(13) Assume c is not an eigenvalue of A.

Let $u = e^{ct} v$ where v is a vector (constant)

Then $\frac{du}{dt} = c e^{ct} v$

and $Au = A e^{ct} v = e^{ct} A v$. So that the

equation $\frac{du}{dt} = Au - e^{ct} b$ ~~given~~ by becomes

$$c e^{ct} v = e^{ct} A v - e^{ct} b$$

$$\Rightarrow \quad cv = Av - b$$

$$\text{or} \quad (A - cI) v = b$$

$$\Rightarrow \quad v = (A - cI)^{-1} b.$$

Since c is not an eigenvalue of A, $A - cI$ is invertible

Showing that $u = e^{ct} v = e^{ct} (A - cI)^{-1} b$ is a particular

solution to the differential equation $\frac{du}{dt} = Au - e^{ct} b$

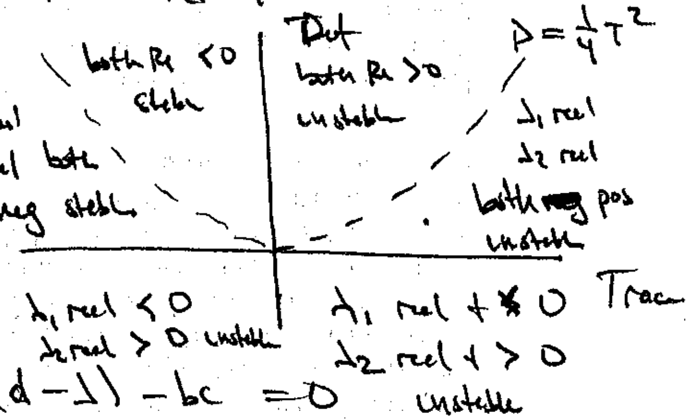
If c is an eigenvalue of A, then $A - cI$ is not invertible

& $\exists$ a nonzero v such that $Av = cv$. ~~showing~~

So that when $e^{ct} v$ is substituted into our differential equation

gives $cv = Av - b$ or $0 = -b$ a contradiction.

(14) For a differential equation to be stable, we require that $U \to 0$ as $t \to \infty$. This will happen when all the eigenvalues of A ~~are~~ ~~negative~~ have negative real parts. For two by two systems this breaks down to the condition that $T = a + d < 0$ & $D = ad - bc > 0$.



(ex) Since the eigenvalues are given by

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0 \Rightarrow (a-\lambda)(d-\lambda) - bc = 0$$

$$\Rightarrow \lambda^2 - (a+d)\lambda + ad - bc = 0$$

$$\Rightarrow \lambda^2 - T\lambda + D = 0$$

which has roots given by

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2} \qquad \text{let } \lambda_1 = \frac{T - \sqrt{T^2-4D}}{2}$$

$$\& \quad \lambda_2 = \frac{T + \sqrt{T^2-4D}}{2}$$

So $T^2 - 4D$ separates real from complex eigenvalues

plotting $T^2 - 4D = 0$ on the ~~trace~~ ~~v.s.~~ Determinant v.s. trace axis gives the plot above.

(a) For $\lambda_1 < 0$ & $\lambda_2 > 0$

let $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ or $A' = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$

(b) For $\lambda_1 > 0$ and $\lambda_2 > 0$ let A given be

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(c) For complex λ's w/ real part $a > 0$, let

$$a + d = \lambda_1 + \lambda_2$$

$$ad - bc = \lambda_1 \cdot \lambda_2$$

let $\lambda_1 = 1 + i$ & $\lambda_2 = 1 - i$ Then $\lambda_1 + \lambda_2 = 2$

& $\lambda_1 \cdot \lambda_2 = 2$

Then let $A =$

$$A = S\Lambda S^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1+i+1-i & 1+i-1+i \\ 1+i-1+i & 1+i+1-i \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 2i \\ 2i & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$$ which is not real.

Q: How do I impose the fact that A must be real?

let $A = \begin{bmatrix} 2 & 2 \\ -1 & 0 \end{bmatrix}$ then $|A| = 2$ + $Tr(A) = 2$

Check the eigenvalues are given by $1 \pm i$.

$$\begin{vmatrix} 2-\lambda & 2 \\ -1 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 2\lambda + 2 = 0$$

$$\lambda = \frac{2 \pm \sqrt{4-4(2)}}{2} = \frac{2 \pm 2\sqrt{1-2}}{2}$$

$$= 1 \pm i \quad \text{yes}$$

This A works

(15) Consider $e^{At} = I + At + \frac{1}{2}A^2t^2 + \frac{1}{6}A^3t^3 + \frac{1}{24}A^4t^4 + \frac{1}{5!}A^5t^5 + \cdots$

Then $\frac{d}{dt}$ to both sides gives

$$\frac{d}{dt}e^{At} = A + A^2 + \frac{1}{2}A^3t^2 + \frac{1}{6}A^4t^3 + \frac{1}{4!}A^5t^4 + \cdots$$

$$= A(I + A + \tfrac{1}{2}A^2t^2 + \tfrac{1}{6}A^3t^3 + \tfrac{1}{24}A^4t^4 + \cdots)$$

$$= Ae^{At}$$

(16) For $B = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$ then $B^2 = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

So $e^{Bt} = I + Bt + \frac{B^2t^2}{2} + \frac{B^3t^3}{6} + \cdots$

so $e^{Bt} = I + Bt = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} t$

$$= \begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix}$$

Then $\frac{d}{dt} e^{Bt} = \frac{d}{dt} \begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$

(17) The solution at time $t+T$ can also be written as

$e^{A(t+T)} u(0)$ so $e^{At} \cdot e^{AT} u(0) = e^{A(t+T)} u(0)$

so $e^{At} \cdot e^{AT} = e^{A(t+T)}$

(18) For $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ we have $\lambda_1 + \lambda_2 = 1$ &
$\lambda_1 \cdot \lambda_2 = 0$

Thus $\lambda_1 = 0$ & $\lambda_2 = 1$. The eigenvector for $\lambda_1 = 0$

is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and for $\lambda_2 = 1$ is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Thus $S = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$ so $S^{-1} = \frac{1}{1} \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$ & $\Lambda = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Thus $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$

Then since $e^{At} = I + A + \frac{A^2t^2}{2} + \frac{A^3t^3}{3!} + \cdots$

$$= I + S\Lambda S^{-1}t + S\Lambda^2 S^{-1}\frac{t^2}{2} + S\Lambda^3 S^{-1}\frac{t^3}{6} + \cdots$$

$$= S\left[I + \Lambda t + \frac{\Lambda^2 t^2}{2} + \frac{\Lambda^3 t^3}{3!} + \frac{\Lambda^4 t^4}{4!} + \cdots\right] S^{-1}$$

$$= S\begin{bmatrix} 1 & 0 \\ 0 & 1 + t + \frac{t^2}{2} + \frac{t^3}{3!} + \cdots \end{bmatrix} S^{-1}$$

$$= S\begin{bmatrix} 1 & 0 \\ 0 & e^t \end{bmatrix} S^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & e^t \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \cancel{\begin{bmatrix} 1 & e^t \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}} = \begin{bmatrix} 1 & e^t \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^t & -1+e^t \\ 0 & 1 \end{bmatrix}$$

Note also that $e^{At} = S e^{\Lambda t} S^{-1}$ which may be a quicker way of deriving the above

(19) if $A^2 = A$ then

$$e^{At} = I + At + A^2\frac{t^2}{2} + A^3\frac{t^3}{6} + \cdots$$

$$= I + At + A\frac{t^2}{2} + A\frac{t^3}{6} + \cdots$$

all terms have only single power of A in them

$$= I + \cancel{At} \; A\left(t + \frac{t^2}{2} + \frac{t^3}{6} + \cdots\right)$$

$$= I + A(e^t - 1)$$

For $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ we see that $A^2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

So the above formula gives

$$e^{At} = \cancel{I+} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}(e^t - 1)$$

$$= \begin{bmatrix} \cancel{\#} + e^t & e^t - 1 \\ 0 & 1 \end{bmatrix}$$ the same as before

(20) For $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ $\quad e^A = \begin{bmatrix} e & e-1 \\ 0 & 1 \end{bmatrix}$ from problem 18

For $B = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$ $\quad e^B = I + B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ since $B^2 = 0$

For $A+B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad e^{A+B} = I + (e^1 - 1)(A+B)$ since $(A+B)^2 = (A+B)$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (e^1-1)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} e^1 & 0 \\ 0 & 1 \end{bmatrix}$$

Now consider

$$e^A \cdot e^B = \begin{bmatrix} e & e-1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e & -e+e-1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e & -1 \\ 0 & 1 \end{bmatrix}$$

$$\neq e^{A+B} = \begin{bmatrix} e & 0 \\ 0 & 1 \end{bmatrix}$$

$$\& \ e^B e^A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} e & e-1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e & e-1-1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e & e-2 \\ 0 & 1 \end{bmatrix}$$

$$\neq e^A e^B$$

(21) $A = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$

we have eigenvalues given by $\lambda = 1$, $\lambda = 3$ w/ eigenvectors

(for $\lambda = 1$)

$\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$ of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and for $\lambda = 3$ $\begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix}$ of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Then $S = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ so $S^{-1} = \frac{1}{2}\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$ w/ $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

Thus $e^{At} = S e^{\Lambda t} S^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} e^{t} & 0 \\ 0 & e^{3t} \end{bmatrix}\frac{1}{2}\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} e^{t} & e^{3t} \\ 0 & 2e^{3t} \end{bmatrix}\frac{1}{2}\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2e^{t} & -e^{t} + e^{3t} \\ 0 & 2e^{3t} \end{bmatrix} = \begin{bmatrix} e^{t} & -\frac{1}{2}e^{t} + \frac{1}{2}e^{3t} \\ 0 & e^{3t} \end{bmatrix}$$

when $t = 0$ $e^{A \cdot 0} = e^{0} = I$

& the right hand side of the above gives the same (the identity matrix)

(22) If $A = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$ then $A^2 = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} = A$$

So from problem 19

$$e^{At} = I + (e^t - 1)A$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (e^t - 1)\begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} e^t & 3(e^t - 1) \\ 0 & 1 \end{bmatrix}$$

(23) (a) ~~The inverse of~~ Since $(e^{At})^{-1} = (e^{-At})$ e^{At} is never singular. Lets check e^{-At} is the inverse of e^{At}.

$$e^{At} \cdot e^{-At} = \left(I + At + \frac{A^2t^2}{2} + \frac{A^3t^3}{3!} + \cdots\right)\left(I - At + \frac{A^2t^2}{2} - \frac{A^3t^3}{6} + \cdots\right)$$

$$= I - \cancel{At} + \frac{A^2\cancel{t^2}}{2} - \frac{A^3t^3}{\cancel{6}} + \frac{A^4t^4}{24} - \cdots$$

$$\cancel{+ At + \frac{A^3t^3}{2}}$$

$$+ \cancel{At} - A^2\cancel{t^2} + \frac{A^3\cancel{t^3}}{2} - \frac{A^4t^4}{6} + \cdots$$

$$\frac{A^2\cancel{t^2}}{2} - \frac{A^3\cancel{t^3}}{2} + \frac{A^4t^4}{4} \cdots$$

$$+ \frac{A^3\cancel{t^3}}{6}$$

$$= \cancel{\;\;} \; I$$

Which we can show analytically like

$$\left(\sum_{k_1 \geq 0} \frac{A^{k_1} t^{k_1}}{k_1!}\right)\left(\sum_{k_2 \geq 0} \frac{A^{k_2}(-1)^{k_2} t^{k_2}}{k_2!}\right)$$

$$= \sum_{\substack{\cancel{k \geq 0} \\ k_1, k_2 \geq 0 \\ \cancel{k = k_1 + k_2}}} \frac{A^{k_1+k_2} \, t^{k_1+k_2}}{k_1! \, k_2!} (-1)^{k_2} \qquad = \sum_{k \geq 0} \sum_{k_2 \geq 0}^{k} \frac{A^k t^k (-1)^{k_2}}{(k-k_2)! \, k_2!}$$

let $k = k_1 + k_2$

$k_1 = k - k_2$

$$= \sum_{k \geq 0} A^k t^k \left(\sum_{k_2 = 0}^{k} \frac{(-1)^{k_2}}{(k-k_2)! \, k_2!} \right)$$

$$= I + \sum_{k \geq 1} A^k t^k \Big(\underbrace{\sum_{k_2=0}^{k} \frac{(-1)^{k_2}}{(k-k_2)! \, k_2!}}_{Z_k} \Big)$$

Show $Z_k \equiv 0 \quad \forall \quad k \geq 1$

$$Z_1 = \sum_{k_2=0}^{1} \frac{(-1)^{k_2}}{(1-k_2)! \, k_2!} = 1 - 1 = 0$$

$$Z_2 = \sum_{k_2=0}^{2} \frac{(-1)^{k_2}}{(2-k_2)! \, k_2!} = \frac{1}{2!} - \frac{1}{1} + \frac{1}{2} = 0$$

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(1) $A = \begin{bmatrix} 1 & 2 & 4 \\ 4 & 3 & 0 \\ 8 & 6 & 5 \end{bmatrix} = M + N$ w/ $M^T = M$, $N^T = -N$

For a square matrix $M = \frac{A + A^T}{2} = \frac{1}{2}\begin{bmatrix} 1 & 2 & 4 \\ 4 & 3 & 0 \\ 8 & 6 & 5 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 & 4 & 8 \\ 2 & 3 & 6 \\ 4 & 0 & 5 \end{bmatrix}$

$$= \frac{1}{2}\begin{bmatrix} 2 & 6 & 12 \\ 6 & 6 & 6 \\ 12 & 6 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 6 \\ 3 & 3 & 3 \\ 6 & 3 & 5 \end{bmatrix}$$

Then N must be given by

$N = A - M = A - \frac{1}{2}(A + A^T) = \frac{1}{2}(A - A^T)$ which

in this case gives

$$\frac{1}{2}\begin{bmatrix} 1 & 2 & 4 \\ 4 & 3 & 0 \\ 8 & 6 & 5 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 & 4 & 8 \\ 2 & 3 & 6 \\ 4 & 0 & 5 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 0 & -2 & -4 \\ 2 & 0 & -6 \\ 4 & 6 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -3 \\ 2 & 3 & 0 \end{bmatrix}$$

Thus $A = M + N$ is given by

$$\begin{bmatrix} 1 & 2 & 4 \\ 4 & 3 & 0 \\ 8 & 6 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 6 \\ 3 & 3 & 3 \\ 6 & 3 & 5 \end{bmatrix} + \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -3 \\ 2 & 3 & 0 \end{bmatrix}$$

(2) If C is symmetric then A^TCA is also symmetric since

$$(A^TCA)^T = A^TC^TA = A^TCA$$

When A is 6×3, A^T is 3×6 and C must be 6×6 and A^TCA is 3×3

(3) The dot product of Ax with y equals

$(Ax)^Ty = x^TA^Ty = x^TAy$ which is the dot product of x with Ay.. If A is not symmetric then

$$(Ax)^Ty = x^TA^Ty.$$

(4) Note A is symmetric so has real eigenvalues & orthogonal eigenvectors. The eigenvalues of A are given by

$$\begin{vmatrix} -2-\lambda & 6 \\ 6 & 7-\lambda \end{vmatrix} = 0$$

$\rightarrow$ $(-2-\lambda)(7-\lambda) - 36 = 0$

$\rightarrow$ $\lambda^2 - 7\lambda + 2\lambda - 14 - 36 = 0$

$\lambda^2 - 5\lambda - 50 = 0$

$\rightarrow$ $(\lambda+5)(\lambda-10) = 0$ $\rightarrow$ $\lambda = -5$ & $\lambda = 10$

The eigenvector for $\lambda = -5$ is given by the nullspace of

$\begin{bmatrix} +3 & 6 \\ 6 & 12 \end{bmatrix}$ which is $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

The eigenvector ~~for~~ $\lambda = 10$ is given by the nullspace of

$\begin{bmatrix} -12 & 6 \\ 6 & -3 \end{bmatrix}$ which is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ which is orthogonal to

the previously computed eigenvector as it must be. To obtain a orthogonal matrix we need to normalize each vector giving

$$Q = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \quad \text{so} \quad Q^{-1} = Q^T = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

Thus $A = Q \Lambda Q^T = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -5 & 0 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \cdot \frac{1}{\sqrt{5}}$

which we can check by multiplying as

$$= \frac{1}{5} \begin{bmatrix} -10 & 10 \\ 5 & 20 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} -20+10 & 10+20 \\ 10+20 & -5+40 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} -10 & 30 \\ 30 & 35 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 6 \\ 6 & 7 \end{bmatrix}$$

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(5) For $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{bmatrix}$ since $A = A^T$ the eigenvalues must be real and the eigenvectors will be orthogonal. To find the eigenvalues solve

$$\begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & -1-\lambda & -2 \\ 2 & -2 & -\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)\left[-\lambda(-1-\lambda)-4\right] + 2\left[2(1+\lambda)\right] = 0$$

$$\Rightarrow \lambda(1-\lambda)(1+\lambda) - 4(1-\lambda) + 4(1+\lambda) = 0$$

$$= \lambda(1-\lambda^2) - \cancel{4} + 4\lambda + \cancel{4} + 4\lambda = 0$$

$$= -\lambda^3 + \lambda + 8\lambda = 0$$

$$= \lambda(\lambda^2 - 9) = 0 \qquad \text{so} \quad \lambda = 0 \ \& \ \lambda = \pm 3.$$

For $\lambda_1 = -3$ the eigenvector is given by the nullspace of

$$\begin{bmatrix} 4 & 0 & 2 \\ 0 & 2 & -2 \\ 2 & -2 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & -1 \\ 1 & -1 & 3/2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \Leftrightarrow$$ which has a null space given by

$\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$. For $\lambda_2 = 0$ we have an eigenvector given by the span of the following.

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & -2 & -4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

which has a null space given by $\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$. For $\lambda_3 = 3$ we have a eigenvector given by the nullspace of the following matrix

$$\begin{bmatrix} -2 & 0 & 2 \\ 0 & -4 & -2 \\ 2 & -2 & -3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1/2 \\ 1 & -1 & -3/2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1/2 \\ 0 & -1 & -1/2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$$ which has a nullspace given by

$\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$. Thus the vector of all eigenvectors is given by

$$\hat{Q} = \begin{bmatrix} -1 & 2 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & 2 \end{bmatrix}$$ to make $\hat{Q}$ an orthogonal matrix we

must normalize each vector by its length so $Q = \begin{bmatrix} -1 & 2 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & 2 \end{bmatrix} \frac{1}{\sqrt{4+4+1}} = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & 2 \end{bmatrix}$

so $Q^{-1} = Q^T = \begin{bmatrix} -1 & 2 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & 2 \end{bmatrix} \frac{1}{3}$ & $\Lambda = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

so that $A = Q \Lambda Q^T$ with the definitions given above

~~which we can check by comput~~

(6) For $A = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}$ it is symmetric and therefore its eigenvalues will be real and its eigenvectors are orthogonal. To find the eigenvalues consider $\begin{vmatrix} 9-\lambda & 12 \\ 12 & 16-\lambda \end{vmatrix} = 0$

$(9-\lambda)(16-\lambda) - 144 = 0$

$\lambda^2 - 25\lambda + 9(16) - 144 = 0$

$$\begin{array}{r} 90 \\ 54 \\ \hline 144 \end{array}$$

$\Rightarrow \lambda^2 - 25\lambda = 0 \Rightarrow \lambda = 0$ & $\lambda = 25$.

For $\lambda = 0$ the eigenvector is given by

$\begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 4 \\ 3 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix}$ giving

$v = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$

The eigenvalue given by 25. Hence an eigenvector given by

$9-25=-16$

$$\begin{bmatrix} -16 & 12 \\ 12 & -9 \end{bmatrix} \Rightarrow \begin{bmatrix} -4 & 3 \\ 4 & -3 \end{bmatrix} \Rightarrow \begin{bmatrix} -4 & 3 \\ 0 & 0 \end{bmatrix}$$ given an

eigenvector given by $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$. Thus the matrix of eigenvectors given by

$$Q = \frac{1}{\sqrt{9+16}}\begin{bmatrix} -4 & 3 \\ 3 & 4 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -4 & 3 \\ 3 & 4 \end{bmatrix}.$$

Then $Q^{-1} = Q^T = \frac{1}{5}\begin{bmatrix} -4 & 3 \\ 3 & 4 \end{bmatrix}$

Thus ~~any~~ all orthogonal matrices are given by multiples of the Q above

7 (a) let $A = \begin{bmatrix} 1 & x \\ x & 1 \end{bmatrix}$ then A is symmetric & to

have negative eigenvalues we must have $\lambda^2 - T\lambda + D = 0$

have negative roots $(T=2)$ $D = 1-x^2$ so

$$\lambda^2 - 2\lambda + (1-x^2) = 0$$

then $$\lambda = \frac{2 \pm \sqrt{4-4(1-x^2)}}{2} = 1 \pm \sqrt{1-(1-x^2)} = 1 \pm \sqrt{x^2} = 1 \pm |x|$$
$$= 1 \pm x$$

so to have a negative eigenvalue we must have $|x| > 1$

Thus pick $x=2$ & our matrix is given by

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

(b) For symmetric matrices the number of positive eigenvalues equals the number of positive pivots & the number of negative eigenvalues equals the number of negative pivots. Because we have one positive ~~pivot~~ eigenvalue & one negative eigenvalue we must have one positive pivot & one negative pivot.

(c) From the formula for the eigenvalues we have ~~one~~

$\lambda = 1 \pm x$ so one is positive & one is negative.

Also the trace$(A) = 2$ so A can't have two negative eigenvalues or else the trace would be negative.

(8) If $A^3 = 0$ ~~then~~ the eigenvalues of A must be zero.

~~[illegible]~~ so

~~we know that~~ let $A = \begin{bmatrix} \cancel{0} & \cancel{1} \\ \cancel{0} & \cancel{0} \end{bmatrix}$ ~~then~~

$$\cancel{A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}$$

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(B) If $A^3 = 0$, then $\lambda = 0$ must be an eigenvalue of A. Since A operating on the columns of A^2 results in the zero matrix each column of A^2 is an eigenvector of A w/ eigenvalue zero. It is easy to find a 2×2 matrix that has

$A^2 = 0$. For example $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is one such

matrix. ~~The A that are A ≠ 0 that such that $A^3 = 0$~~ I don't see why all the eigenvalues of A must be zero. because

$$|A^3| = 0$$

But $|A^3| = |A|^3$ which says that $(\prod_i \lambda_i)^3$ so all that is really required is to have one eigenvalue from A zero, and then the product will be zero.

If A is symmetric then it has an eigenvalue decomposition with real eigenvalues Λ so $A = Q\Lambda Q^T$ and orthogonal eigenvectors. In this case

$$A^3 = Q\Lambda^3 Q^T = 0 \Rightarrow \Lambda^3 = 0 \Rightarrow \Lambda = 0$$

so A is identically zero

(9) The characteristic eq of a 3x3 matrix A is ~~given~~ ~~by~~ a 3rd order polynomial. As such ~~since~~ it can have at most two complex roots (that are complex conjugates) and still be a real polynomial. Thus A must have one real ~~eigenvect~~ eigenvalue. Another way to see this is to consider the trace(A) which must be real $= \lambda + \bar{\lambda} + \lambda_3$ since $\lambda + \bar{\lambda}$ is real λ_3 (the 3rd eigenvalue) must also be real.

(10) It is not stated the x must be real. For example consider the matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ then $\lambda^2 + 1 = 0$ & $\lambda = \pm i$

with eigenvectors given by (for $\lambda = -i$)

$$\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \to \begin{bmatrix} i & 1 \\ 0 & 0 \end{bmatrix} \quad \text{so} \quad x_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

and for $\lambda_2 = +i$ x_2 will be the complex conjugate of x_1 or

$\begin{bmatrix} -i \\ 1 \end{bmatrix}$ Then the expression $\frac{x^T A x}{x^T x}$ will be complex

(since the x's are)

(11) For $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ the spectral theorem requires calculating

$Q\Lambda Q^T$. First the eigenvalues

$$\begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = 0 \Rightarrow (3-\lambda)^2 - 1 = 0$$

$$(3-\lambda-1)(3-\lambda+1) = 0$$

$$\Rightarrow \lambda = 2 \text{ or } \lambda = 4$$

The eigenvectors are given by for $\lambda = 2$

$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ so $x_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ & for $\lambda = 4$

$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ so $x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Then $Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ & $Q^{-1} = Q^T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

So $A = Q\Lambda Q^T = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ 4 & 4 \end{bmatrix}$

$$\Rightarrow \quad A = \frac{1}{2}\begin{bmatrix}1\\-1\end{bmatrix}[2 \;\; -2] + \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix}[4 \;\; 4]$$

$$= 2\left(\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}\right)\left(\frac{1}{\sqrt{2}}[1 \;\; -1]\right) + 4\left(\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}\right)\left(\frac{1}{\sqrt{2}}[1 \;\; 1]\right)$$

For B we perform the same manipulations

$$\begin{vmatrix}9-\lambda & 12\\12 & 16-\lambda\end{vmatrix} = 0 \quad \Rightarrow \quad (9-\lambda)(16-\lambda) - 144 = 0$$

$$\lambda^2 - 25\lambda = 0$$

$$\Rightarrow \lambda = 0 \;\&\; \lambda = 25.$$

Now in the spectral theorem for A. The decomposition

$\lambda_1 x_1 x_1^T + \lambda_2 x_2 x_2^T + \dots + \lambda_n x_n x_n^T$ means that all

eigenvalues w/ $\lambda = 0$ don't contribute to the decomposition above.

Thus we will only look for the $\lambda = 25$ eigenvector

$$\begin{bmatrix}9-25 & 12\\12 & 16-25\end{bmatrix} = \begin{bmatrix}-16 & 12\\12 & -9\end{bmatrix} \Rightarrow \begin{bmatrix}-4 & 3\\4 & -3\end{bmatrix} \Rightarrow \begin{bmatrix}-4 & 3\\0 & 0\end{bmatrix}$$

$$\Rightarrow \quad x_2 = \frac{1}{5}\begin{bmatrix}3\\4\end{bmatrix}$$

Thus $B = 25\left(\frac{1}{5}\begin{bmatrix}3\\4\end{bmatrix}\right)\left(\frac{1}{5}[3 \;\; 4]\right)$

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(12) ~~For~~ For $A = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$ Because $A^T = -A$ A must have

~~0~~ ~~of~~ imaginary eigenvalues. ~~This~~ Given by

$\begin{vmatrix} -\lambda & b \\ -b & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + b^2 = 0 \Rightarrow \lambda = \pm ib$

~~~~ Consider the following 3x3 skew-symmetric matrix

$B = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix}$ which has eigenvalues given by

$$\begin{vmatrix} -\lambda & 1 & 2 \\ -1 & -\lambda & 3 \\ -2 & -3 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow -\lambda \begin{vmatrix} -\lambda & 3 \\ -3 & -\lambda \end{vmatrix} + 1 \begin{vmatrix} 1 & 2 \\ -3 & -\lambda \end{vmatrix} - 2 \begin{vmatrix} 1 & 2 \\ -\lambda & 3 \end{vmatrix} = 0$$ ✓

$$\Rightarrow -\lambda(\lambda^2 + 9) + 1(-\lambda + 6) - 2(3 + 2\lambda) = 0$$ ✓

$$\Rightarrow -\lambda^3 - 9\lambda - \lambda + \cancel{6} - \cancel{6} - 4\lambda = 0$$ ✓

$$\Rightarrow -\lambda^3 - 14\lambda = 0$$ ✓

$$\Rightarrow \lambda(\lambda^2 + 14) = 0 \quad \Rightarrow \lambda = 0 \ \& \ \lambda = \pm i\sqrt{14}$$
~~~~

(13) The matrix M is skew symmetric and also orthogonal since

$$M^TM = \frac{1}{3}\begin{bmatrix} 0 & -1 & -1 & -1 \\ 1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 0 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} = I$$

Thus its eigenvalues are pure imaginary and have $|\lambda| = +1$. The 4 eigenvalues are given by solving

$$\begin{vmatrix} -\lambda & 1/\sqrt{3} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & -\lambda & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\lambda & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow -\lambda\begin{vmatrix} -\lambda & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & -\lambda & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\lambda \end{vmatrix} + \frac{1}{\sqrt{3}}\begin{vmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & -\lambda & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\lambda \end{vmatrix}$$

$$- \frac{1}{\sqrt{3}}\begin{vmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\lambda & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\lambda \end{vmatrix} + \frac{1}{\sqrt{3}}\begin{vmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\lambda & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & -\lambda & -\frac{1}{\sqrt{3}} \end{vmatrix} = 0$$

~~But This~~ ⇒ continuing we could find the result, but this is very difficult. It is easier to recognize that

$\sum_i \lambda_i = 0$ By the eigenvalue trace theorem, but also $|\lambda_i| = 1$

since M is orthogonal, + λ_i are pure imaginary ~~complex~~. The only pure imaginary numbers w/ $|\lambda| = 1$ are given by $\pm i$. Thus ~~we must to~~ have 4 of them we must have ~~+i+i~~ $+i, +i, -i, -i$ as the eigenvalues of M.

(14) Given $A = \begin{bmatrix} 2i & 1 \\ 1 & 0 \end{bmatrix}$ its eigenvalues must satisfy

$$\det \begin{bmatrix} 2i-\lambda & 1 \\ 1 & -\lambda \end{bmatrix} = -2i\lambda + \lambda^2 - 1 = 0$$

$$\Rightarrow (\lambda^2 - 2i\lambda - 1) = 0$$

$$\Rightarrow$$

$$(\lambda - i)(\lambda - i) = 0$$

So $\lambda = i$ is a double root. Looking for its eigenvectors we ~~have~~ consider

$$\begin{bmatrix} 2i-i & 1 \\ 1 & -i \end{bmatrix} = \begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix} \Rightarrow \begin{bmatrix} i & 1 \\ 0 & 0 \end{bmatrix}$$

with only one eigenvector $x_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$ so A is Not diagonalizable.

(18) $Bx = \lambda x$ is $\begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} = \lambda \begin{bmatrix} y \\ z \end{bmatrix}$

$\Rightarrow$ $Az = \lambda y$

& $A^T y = \lambda z$

(a) multiplying the 1st equation above by A^T gives

$A^T A z = \lambda A^T y = \lambda^2 z$ so λ^2 is an eigenvalue of $A^T A$.

(b) If $A = I$ then ~~the~~ ~~or~~ λ^2 is an eigenvalue of I which are only 1's. Thus $\lambda = \pm 1$, are the eigenvalues of B. Since B is of size 4×4 we need 4 eigenvalues & they are $1, 1, -1, -1$. The eigenvectors of B ~~are~~ can be obtained from the ~~eigenvectors~~ ~~of~~ system above. Thus z must be an eigenvector of I & $\therefore$ is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. In the same way $A^T y = \lambda z$ $\Rightarrow$ ~~$Iy = \pm 1 z = \pm 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$~~ gives 4 systems for y (providing the 4 eigenvectors of B) they are

(Since $A^T = I$ we can drop this ~~obtaining~~)

$$y = -1\begin{bmatrix}1\\0\end{bmatrix} \quad \text{for} \quad \lambda = -1 \ \& \ z = \begin{bmatrix}1\\0\end{bmatrix}$$

$$y = -1\begin{bmatrix}0\\1\end{bmatrix} \quad \text{for} \quad \lambda = -1 \ \& \ z = \begin{bmatrix}0\\1\end{bmatrix}$$

$$y = 1\begin{bmatrix}1\\0\end{bmatrix} \quad \text{for} \quad \lambda = 1 \ \& \ z = \begin{bmatrix}1\\0\end{bmatrix}$$

$$y = 1\begin{bmatrix}0\\1\end{bmatrix} \quad \text{for} \quad \lambda = 1 \ \& \ z = \begin{bmatrix}0\\1\end{bmatrix}$$

Thus the eigenvectors/eigenvalue system is given by

$$Q = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad \text{with}$$

$$\Lambda = \operatorname{diag}(-1,-1,1,1)$$

(16) If $A = \begin{bmatrix}1\\1\end{bmatrix}$ then from $A^T A z = \lambda^2 z$ we have

$$\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} z = \lambda^2 z$$

$$\Rightarrow 2z = \lambda^2 z$$

$$\Rightarrow \lambda^2 = 2 \quad \Rightarrow \quad \lambda = \pm\sqrt{2} \quad \text{if } z \neq 0$$

w/ z any vector. but z is 1×1 so $z = 1$. From the size definition of B.
Also $z=0$ w/ any λ will work.
To evaluate y consider $A^T y = -\sqrt{2}$ (for $\lambda = -\sqrt{2}$)

or $\begin{bmatrix} 1 & 1 \end{bmatrix} y = -\sqrt{2}$

so $y = -\frac{\sqrt{2}}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

and consider $\begin{bmatrix} 1 & 1 \end{bmatrix} y = +\sqrt{2}$, for $(\lambda = +\sqrt{2})$

so $y = +\frac{\sqrt{2}}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Finally consider if $z=0$ (λ unknown ~~$\lambda \neq 0$~~) to obtain

$\begin{bmatrix} 1 & 1 \end{bmatrix} y = 0$

$\Rightarrow$ $y = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. ~~To solve for λ consider~~ ~~$A^T y = \lambda z$~~
~~or $\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$~~

Then the eigensystem for B is given by

$$Q = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -1 \\ 1 & 1 & 0 \end{bmatrix}$$ w/ $Q^{-1} = Q^T$ as required

and $\Lambda = \mathrm{diag}(-\sqrt{2}, +\sqrt{2}, 0)$

Where I have taken $\lambda_3 = 0$ since $B\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = 0 \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

(17) Every 2×2 symmetric can be written as

$$A = \lambda_1 x_1 x_1^T + \lambda_2 x_2 x_2^T$$

$$= \lambda_1 P_1 + \lambda_2 P_2$$

w/ P_1 + P_2 projection matrices. when $\|x_1\| = 1$ + $\|x_2\| = 2$

(a) Now $P_1 + P_2 = x_1 x_1^T + x_2 x_2^T = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} x_1^T \\ x_2^T \end{bmatrix}$

$$= Q \cdot Q^T = I$$

since Q is an orthogonal matrix

(b) $P_1 P_2 = x_1 x_1^T (x_2 x_2^T) = x_1 (x_1^T x_2) x_2^T = 0$

since $x_1^T x_2 = 0$ as x_1 + x_2 are orthogonal.

(18) Suppose $Ax = \lambda x$ and $Ay = 0y$ w/ $\lambda \neq 0$

y is in Nullspace + x is in the column space

$$x^T A = \lambda x^T$$

$$x^T A y = \lambda x^T y$$

Since $Ay = 0$ then $\lambda x^T y = 0$ since $\lambda \neq 0$ then $x^T y = 0$

~~Also~~ Also y is in the null ~~space~~ space + x ~~is~~ in the column space

but since $A = A^T$ x in the column space means x is in the

~~null space~~ row space but the row space + the nullspace are orthogonal

so $x^T y = 0$

~~If the second eigenvalue is a nonzero number β, apply the argument to $A - \beta I$. For that matrix its eigenvalue is now zero + its eigenvector is the same eigenvector that has β as its eigenvalue. Thus $x^T y = 0$ from the previous part~~

If the second eigenvector is not zero say β, then we have

$$Ay = \beta y \quad + \quad Ax = \lambda x$$

so we ~~have~~ $\cancel{(A - \beta I) y = 0} \cdot y$ consider the matrix $B = A - \beta I$

$$\text{so} \quad Bx = (A - \beta I)x = Ax - \beta x = \lambda x - \beta x = (\lambda - \beta)x$$

$$+ \quad By = (A - \beta I)y = Ay - \beta y = \beta y - \beta y = 0$$

So x is an eigenvector of B (w/ eigenvalue $\lambda - \beta$)

+ y is an eigenvector of B (w/ eigenvalue 0)

so x + y are orthogonal by the previous arguments.

(19) For $B = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{bmatrix}$ which is Not symmetric

it has eigenvalues given by

$$\begin{vmatrix} -1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & \lambda-\lambda \end{vmatrix} = 0$$

$$= (-1-\lambda)\begin{vmatrix} 1-\lambda & 0 \\ 0 & \lambda-\lambda \end{vmatrix} + 1\begin{vmatrix} 0 & 1-\lambda \\ 0 & 0 \end{vmatrix}$$

$$= -(1+\lambda)(1-\lambda)(\lambda-\lambda) = 0$$

So $\lambda = -1, \lambda, +1$

Which has eigenvectors given by (For $\lambda = -1$)

$$\cancel{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda+1 \end{bmatrix}}$$

~~which has two eigenvectors given by~~

$\cancel{x_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} + \cancel{x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}$

~~Thus~~

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & \lambda+1 \end{bmatrix}$$

Which has an eigenvector given by $x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

For $\lambda = d$ the eigenvector is determined by

$$\begin{bmatrix} -1-d & 0 & 1 \\ 0 & 1-d & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which has an eigenvector given by

$$v = \begin{bmatrix} 1 \\ 0 \\ 1+d \end{bmatrix}$$

For $\lambda = +1$ the eigenvector is given by the matrix

$$\begin{bmatrix} -2 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & d-1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad (d \neq 1)$$

$$\Longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{so} \quad v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

In fact this vector is valid ~~on~~ as an eigenvector for an eigenvector regardless of the value of d.

Thus the eigenvector matrix is given by

$$S = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1+d & 0 \end{bmatrix}$$

The eigenvectors are not perpendicular

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(20) For $A = \begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}$

$\lambda_1 + \lambda_2 = 0$ & $\lambda_1 \cdot \lambda_2 = -9 - 16 = -25$

So $\lambda_1 = -5$ & $\lambda_2 = +5$. Then we have one eigenvalue that is negative & one that is positive. The pivots of this matrix are given by -3 & $\frac{25}{3}$ as can be seen from

$$\begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix} \to \begin{bmatrix} -3 & 4 \\ 0 & 3 + \frac{4}{3}(4) \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ 0 & \frac{25}{3} \end{bmatrix}$$

$\boxed{\cancel{+\frac{4}{3}(4) + 3 = \frac{16}{3} + \frac{9}{3} = \frac{25}{3}}}$ Thus we have one negative pivot

and one positive pivot.

(21) (a) False, let $A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$ then A has eigenvalues given by $\lambda = \pm 1$. The eigenvectors are given by

$\lambda_1 = -1$ & $\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$ so $v_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$\lambda_2 = 1$ & $\begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}$ so $v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Thus A has real eigenvalues & eigenvectors but is not symmetric

(b) If A & B are symmetric then

$$(AB)^T = B^T A^T = BA \neq AB$$

so No. consider $A = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$ & $B = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$

then $AB = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & -1 \\ \cancel{4} & 1 \end{bmatrix}$ which is not symmetric
-4

(c) If A is symmetric then $A^T = A$ ~~so~~ & considering

$$\cancel{A \cdot A^{-1} = I}$$

so ~~taking the transpose of both sides gives~~

$$\cancel{(A^{-1})^T A^T = I} \qquad \cancel{(A^{-1})^T} =$$

$$\cancel{(A^T)^{-1} A = I}$$

$$(A^{-1})^T = (A^T)^{-1} = A^{-1}$$

so A^{-1} is symmetric since the inverse & the transpose commutes

Another way to see this ~~~~ is to consider the decomposition

$$A = Q\Lambda Q^T$$ since A is symmetric. Then

$$A^{-1} = (Q\Lambda Q^T)^{-1} = (Q^{-1})^T \Lambda^{-1} Q^{-1} = Q\Lambda^{-1}Q^T$$

which is symmetric.

(d) False. Let. $A = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$ Then the eigenvalues are

$$(1-\lambda)^2 + -4 = 0$$

$$(1-\lambda-2)(1-\lambda+2) = 0$$

$\Rightarrow \lambda = -1$ $\quad \lambda = 3$ with eigenvectors given by

So $\lambda = 3$ has an eigenvector given by

$$\begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix} \quad + \quad x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

For $\lambda = -1$ the eigenvector is given by

$$\begin{bmatrix} 2 & -2 \\ -2 & +2 \end{bmatrix} \quad + \quad x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

So the eigenvector matrix S is given by

$$S = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$ which is not symmetric

(22) A normal matrix is defined by $A^T A = AA^T$. If A is skew symmetric $A^T = -A$ so that $A^T A = -A^2$ & $AA^T = A(-A) = -A^2$ & they are equivalent.

- Every orthogonal matrix is normal since

$$AA^T = AA^{-1} = I = A^T A = A^{-1}A = I$$

since $A^T = A^{-1}$ for an orthogonal matrix.

For $\begin{bmatrix} a & 1 \\ -1 & d \end{bmatrix}$ to be normal we must have

$$A^T A = \begin{bmatrix} a & -1 \\ 1 & d \end{bmatrix}\begin{bmatrix} a & 1 \\ -1 & d \end{bmatrix} = AA^T = \begin{bmatrix} a & 1 \\ -1 & d \end{bmatrix}\begin{bmatrix} a & -1 \\ 1 & d \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a^2+1 & a-d \\ a-d & 1+d^2 \end{bmatrix} = \begin{bmatrix} a^2+1 & -a+d \\ -a+d & 1+d^2 \end{bmatrix}$$

So we require $a-d = -a+d$

or $2a = 2d \Rightarrow a = d$

(23) If $AA^T = A^TA$ then $A + A^T$ shre the same eigenvectors ~~then~~ and $A + A^T$ always share the same eigenvalues.

They must have the same S and Λ so A and A^T are the same. This is Not true since they maybe in different ~~$A = S\Lambda S^{-1}$~~ orders. Thus If $\lambda = \lambda_1$ for A has an eigenvector x_1 ~~$A^T = S\Lambda$~~ Then A^T may have $\lambda = \lambda_1$ the eigenvector of A given by $x_{I(1)} \neq x_1$ thus we may have to select a different vector to get the eigenvector of A^T that corresponds to the eigenvector for $\lambda = \lambda_1$.

(24) A is invertible, orthogonal since

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

it is a permutation since it is a reordering of the identity

not a projection since $A^2 \neq A$, ~~this~~ it is to be diagonal we must have our ~~algebraic~~ geometric multiplicity equal to the algebraic multiplicity. The eigenvalues are given by

$$\begin{vmatrix} -\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = 0 \rightarrow -\lambda \begin{vmatrix} 1-\lambda & 0 \\ 0 & -\lambda \end{vmatrix} + 1 \begin{vmatrix} 0 & 1 \\ 1-\lambda & 0 \end{vmatrix}$$

$$\Rightarrow \; -\lambda(-\lambda)(1-\lambda) + (1-\lambda)(-1) = 0$$

$$\Rightarrow \; \lambda^2(1-\lambda) - 1 + \lambda = 0$$

$$\Rightarrow \; \lambda^2 - \lambda^3 - 1 + \lambda = 0$$

$$\Rightarrow \; \lambda^3 - \lambda^2 - \lambda + 1 = 0$$ which factors into $(\lambda-1)(\lambda^2-1) = 0$

$$\Rightarrow (\lambda-1)^2(\lambda+1) = 0$$

So $\lambda = 1$ ~~is an eigenvalue~~

~~$\lambda^2 - 1$~~

~~$\lambda - 1 \,\big)\, \lambda^3 - \lambda^2 - \lambda + 1$~~

~~$\lambda^3 - \lambda^2$~~

~~$-\lambda + 1$~~

Thus $\lambda = 1$ has an ~~two~~ algebraic multiplicity of 2

& $\lambda = -1$ has an algebraic multiplicity of 1.

The eigenvectors for $\lambda = 1$ we obtain for ~~eig~~

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix} \quad \text{~~twos~~ ~~only~~ two} \Longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Which has two ~~elements~~ elements of the nullspace given by

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

This is the only eigenvalue to worry about so A is diagonalizable. 11

To be Markov every element must be nonnegative and each column must add to one. Since this is true A is also markov.

For B, it is not invertible since every ~~multiple~~ row is the same multiple of every other row. It is not orthogonal since

$$B^T B = \frac{1}{9}\begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix} = B, \text{ so } B \text{ is a projection}$$

it is not a permutation. To be diagonalizable we must have ~~an~~ ~~eig~~ a ~~eig~~ geometric multiplicity equal to the algebraic multiplicity

We 1st must find the eigenvalues of B

$$\begin{vmatrix} \frac{1}{3}-\lambda & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3}-\lambda & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3}-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (\tfrac{1}{3}-\lambda)\left[(\tfrac{1}{3}-\lambda)^2-\tfrac{1}{9}\right]-\tfrac{1}{3}\left[\tfrac{1}{3}(\tfrac{1}{3}-\lambda)-\tfrac{1}{9}\right]+\tfrac{1}{3}\left[\tfrac{1}{9}-\tfrac{1}{3}(\tfrac{1}{3}-\lambda)\right]=0$$

$$\Rightarrow (\tfrac{1}{3}-\lambda)^3-\tfrac{1}{9}(\tfrac{1}{3}-\lambda)-\tfrac{1}{9}(\tfrac{1}{3}-\lambda)+\tfrac{1}{27}+\tfrac{1}{27}-\tfrac{1}{9}(\tfrac{1}{3}-\lambda)$$

$$= (\tfrac{1}{3}-\lambda)^3-\tfrac{1}{3}(\tfrac{1}{3}-\lambda)+\tfrac{2}{27}=0$$

$$\frac{\cancel{1}}{\cancel{27}} - 3\left(\frac{1}{3}\right)^2\lambda + 3\left(\frac{1}{3}\right)\lambda^2 - \lambda^3 - \frac{\cancel{1}}{\cancel{9}} + \frac{\lambda}{3} + \frac{2\cancel{}}{\cancel{27}} = 0$$

$$-\frac{\lambda}{3}$$

$\Rightarrow \lambda^3 - \lambda^2 = 0$

$\Rightarrow \lambda(\lambda^2 - 1) = 0 \quad \Rightarrow \quad \lambda = 0, \lambda = \pm 1.$

Since each eigenvector is unique this matrix is diagonalizable.

It is ~~[illegible]~~ also Markov.

Thus for A, LU, QR, $S\Lambda S^{-1}$, $Q\Lambda Q^T$ are all possible

For B, QR, $S\Lambda S^{-1}$, + $Q\Lambda Q^T$ are all possible

We ~~[illegible]~~ also note that the fact that $A + B$ are ~~is~~ symmetric gives us the information that B will have a factorization given by $A = S\Lambda S^{-1} = Q\Lambda Q^T$

If $A^T = A$ then   pg 293 Strang   1

(28) ~~For~~ $A = Q\Lambda Q^T$ is a possible decomposition so that ~~need to hold A symmetric~~ so $b = 1$. To hold $A = S\Lambda S^{-1}$ we must have ~~...~~ enough eigenvectors (i.e. geometric multiplicity equals the algebraic multiplicity for every eigenvalue).

The eigenvectors must satisfy $\lambda_1 + \lambda_2 = 2$

& $\lambda_1 \cdot \lambda_2 = -b$

$\Rightarrow \lambda_2 = 2 - \lambda_1$ so

$$\lambda_1(2-\lambda_1) = -b \Rightarrow -\lambda_1^2 + 2\lambda_1 = -b$$

$$\Rightarrow \lambda_1^2 - 2\lambda_1 - b = 0$$

$$\therefore \lambda_1 = \frac{2 \pm \sqrt{4+4(1)(b)}}{2} = 1 \pm \sqrt{1+b}$$

So if ~~...~~ $b \neq -1$ we will have two distinct eigenvalues & $\therefore$ enough eigenvectors for diagonalization so $A = S\Lambda S^{-1}$!

If $b = -1$ however our matrix becomes $\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$

which has eigenvectors for $\lambda = 1$ given by the null space

of $\begin{bmatrix} 2-1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} +1 & -1 \\ 1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ which only

has one eigenvector given by $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ so A is not diagonalizable.

(26) One eigenvector is given by $\lambda = 1$, another is given by

$$2 + 10^{-15} = \lambda_1 + \lambda_2 = 1 + \lambda_2$$

$$\Rightarrow \lambda_2 = 1 + 10^{-15}$$

this eigenvector is then given by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

The angle between the eigenvectors is given by

$$\cos\theta = \frac{\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}{\left\| \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\| \cdot \left\| \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\|} = \frac{1}{1 \cdot \sqrt{2}} = \frac{1}{\sqrt{2}}$$

So $\theta = \frac{\pi}{4}$.

(1) ~~A~~ For 2x2 symmetric systems the eigenvalues are positive iff

$a > 0$ + $\frac{ac-b^2}{a} > 0$

For A_1 ~~these~~ these expressions are given by $a = 5 > 0$ ✓

$$\frac{ac-b^2}{a} = \frac{5(7)-36}{5} = \frac{-1}{5} > 0 \quad \text{No.}$$

The eigenvalues are not both positive.

For A_2 these expressions are given by $a = -1 > 0$ No.

thus the eigenvalues are Not positive.

For A_3 $a = 1 > 0$ yes

$$\frac{1(100)-10^2}{1} = 0 > 0 \quad \text{false.}$$

Therefore the eigenvalues are not positive

For A_4 $a = 1 > 0$ yes

+ $\frac{ac-b^2}{a} = \frac{1(101)-10^2}{1} = 1 > 0$ yes so the eigenvalues are both positive

If $ac - b^2 > 0$ then $ac > b^2 > 0$

so a + c must have the same sign (either both be positive or negative) thus if c is positive then so is a and vice versa.

(2) For the 1st A we must have

$a = 1 > 0$ yes &

$$\frac{ac-b^2}{a} > 0 \Rightarrow \frac{9-b^2}{1} = 9-b^2 > 0 \Rightarrow b^2 < 9$$

$$\Rightarrow \quad |b| < 3$$

Factoring A into LU we have

$$\cancel{A = \begin{bmatrix} 1 & 0 \\ -b & 1 \end{bmatrix}\begin{bmatrix} 1 & b \\ 0 & \end{bmatrix}}$$

$$A = \begin{bmatrix} 1 & b \\ b & 9 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & b \\ 0 & 9-b^2 \end{bmatrix}$$

so $$A = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}\begin{bmatrix} 1 & b \\ 0 & 9-b^2 \end{bmatrix}$$

to factor into LDL^T we have

$$A = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 9-b^2 \end{bmatrix}\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$$

For the second matrix $A = \begin{bmatrix} 2 & 4 \\ 4 & c \end{bmatrix}$ to be

positive definite we have ~~that~~ the requirement that

$$2 > 0 \quad \checkmark$$

$$\frac{2 \cdot c - 16}{2} > 0 \Rightarrow \quad c > 8$$

To factor A into LU we have

$$A = \begin{bmatrix} 2 & 4 \\ 4 & c \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 4 \\ 0 & c-8 \end{bmatrix}$$

So $\quad A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 0 & c-8 \end{bmatrix}$

to factor into LDL^T we have

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & c-8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

3 What is the quadratic form $ax^2 + 2bxy + cy^2$

For $\quad A = \begin{bmatrix} 1 & 2 \\ 2 & 7 \end{bmatrix}$

$$x^T A x = [x \; y] \begin{bmatrix} 1 & 2 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = [x \; y] \begin{bmatrix} x + 2y \\ 2x + 7y \end{bmatrix}$$

$$= x^2 + 2xy + 2xy + 7y^2 = x^2 + 4xy + 7y^2$$

From this expression we recognize that the quadratic form ~~for~~ the second A ~~is~~ given by $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ will have a quadratic form of $x^2 + 2(2)xy + 4y^2$.

To complete the square we write $A = Q \Lambda Q^T$ to obtain $f = \lambda_1 X^2 + \lambda_2 Y^2$. ~~The~~ We also can do this manually. For example in the 1st problem $f = x^2 + 4xy + 7y^2$

$$= (x+2y)^2 - 4y^2 + 7y^2$$

$$= (x+2y)^2 + 3y^2.$$

For the 2nd problem $f = x^2 + 4xy + 4y^2$

$$= (x+2y)^2.$$

(4) ~~To write~~ Following the hint we write f has a sum of squares as follows

$$f = (x+2y)^2 - 4y^2 + 3y^2$$

$$= (x+2y)^2 - y^2.$$

Selecting $x = -2y$ gives $y = \frac{-x}{2}$ and then on this line we have $f(x, -\frac{x}{2}) = 0 - \frac{x^2}{4} = -\frac{x^2}{4} < 0$

so $f(1, -\frac{1}{2}) = \frac{-1}{4} < 0$

(5) For $f(x,y) = 2xy$ the matrix A with this quadratic form is given by $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Its eigenvalues must have $\lambda_1 + \lambda_2 = 0$ & $0 - 1 = \lambda_1 \cdot \lambda_2$

so $\lambda_1 = -1$ & $\lambda_2 = +1$.

(6) One quick test $A^T A$ will be positive definite if A has linearly independent columns. Thus for

$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ $A^T A$ will be positive definite

$4 + 9 = 1$

$$A^T A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 13 \end{bmatrix}$$

check it positive definite using the upper left determinant rule

$1 > 0$ yes

& $13 - 4 = 9 > 0$ yes $A^T A$ is S.P.D.

For $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix}$ again the columns are linearly independent 6

So A^TA will be S.P.D. Also consider

$$A^TA = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 1+2+2 \\ 1+2+2 & 1+4+1 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 5 & 6 \end{bmatrix}$$

So check $6 > 0$

& $6^2 - 5^2 = 36 - 25 = 11 > 0$ ✓

So A^TA is S.P.D

For $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$ the columns of A are <u>Not</u>

linearly independent so A^TA might not be Positive definite

$$A^TA = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 3 \\ 3 & 5 & 4 \\ 3 & 4 & 5 \end{bmatrix}$$

lets check for positive definites by looking at the signs of the determinants of the principal submatrices we have

$2 > 0$ ✓

$10 - 9 > 0$ ✓

$2(25-16) - 3(15-12) + 3(12-15)$

$= 2(9) - 3(3) + 3(-3) = 0$ which is not greater than zero so ~~expect~~ ~~that~~ this matrices is not S.P.D.

(7) $x^T A^T A x = (Ax)^T (Ax) = \|Ax\|^2 = 0$

iff $Ax = 0 \Rightarrow x = 0$ since A has linearly independent columns

(8) $f(x,y) = 3(x^2 + 4xy + 4y^2) + 4y^2$

$= 3x^2 + 12xy + 12y^2 + 4y^2$

$= 3x^2 + 12xy + 16y^2$

so $f(x,y) = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 3 & 6 \\ 6 & 16 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

The pivots of A are given by

$$\begin{bmatrix} 3 & 6 \\ 6 & 16 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 6 \\ 0 & 4 \end{bmatrix}$$ so the pivots are $3 + 4$.

~~$-2(4\cdot 6) + 16$~~

(9) For $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 4(x_1 - x_2 + 2x_3)^2$

we expand the functional form $4(x_1 - x_2 + 2x_3)^2$ to get

$$\cancel{4(x_1^2} \; 4(x_1 - x_2 + 2x_3)(x_1 - x_2 + 2x_3)$$

$$= 4(x_1^2 - \underline{x_1x_2} + \underline{\underline{2x_1x_3}} - \underline{x_2x_1} + x_2^2 - \underline{\underline{2x_2x_3}}$$

$$+ \underline{\underline{2x_3x_1}} - \underline{\underline{2x_3x_2}} + 4x_3^2)$$

$$= 4x_1^2 - 8x_1x_2 + 16x_1x_3 - 16x_2x_3 + 16x_3^2 + 4x_2^2$$

So our matrix A can be given by

$$A = \begin{bmatrix} 4 & -4 & 8 \\ -4 & 4 & -8 \\ 8 & -8 & 16 \end{bmatrix}$$

~~A has only one nonzero pivot = 4, and nonzero~~

A can be written in canonical form as

$$A = Q\Lambda Q^T = \begin{bmatrix} 1/\sqrt{6} & & \\ -1/\sqrt{6} & q_2 & q_3 \\ 2/\sqrt{6} & & \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \\ & q_2^T & \\ & q_3^T & \end{bmatrix} \quad 1+1+4 = 6$$

w/ q_2 + q_3 vectors chosen to ~~complete the~~ provide a orthogonal basis for $\mathbb{R}^3$

Then the eigenvalues are 4, 0, 0 + $\det(A) = 4 \cdot 0 \cdot 0 = 0$.

~~The~~ If I add the 1st row to the 2nd + 2 times the 1st row to the 3rd I get

$$A \to \begin{bmatrix} 4 & -4 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ so the only pivot is 4

+ A has rank = 1

(10) (a) For $f = 2(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3)$

is given by $\underline{x}^T A x$ with $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

This A has principal submatrices with determinants given by

$2 > 0$

$4 + 1 = 5 > 0$

$2(4 - 1) + 1(-2) = \cancel{10}\,6 - 2 = \cancel{8}\,4 > 0$

Thus A is S.P.D. For (b) we have

$f = x^T A x$ w/ A given by

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

To check if this A is S.P.D consider the principal submatrix determinants

& . $D_1 = 2 > 0$

$D_2 = 4 + 1 = 5 > 0$

$D_3 = 2(4 - 1) + 1(-2 - 1) - 1(1 + 2) = 6 - 3 - 3 = 0 \not> 0$

Since D_3 is not positive this matrix A is nonnegative.

(11) $D_1 = 2$

$D_2 = 10 - 4 = 6$

$D_3 = 2(40 - 9) - 2(16) = 2(31) - 36 = 62 - 36 = 26 > 0$

The pivots of A can be found from

$$A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 3 \\ 0 & 3 & 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 2 & 0 \\ 0 & 1 & 3 \\ 0 & 3 & 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 2 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & -1 \end{bmatrix}$$

There is something wrong here ... I'm not sure what's going on

(12) For $A = \begin{bmatrix} c & 1 & 1 \\ 1 & c & 1 \\ 1 & 1 & c \end{bmatrix}$ to have all principal submatrices have

a positive determinant we require $D_1 = c > 0$

+ $D_2 = c^2 - 1 > 0$

+ $D_3 = c \begin{vmatrix} c & 1 \\ 1 & c \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 1 & c \end{vmatrix} + 1 \begin{vmatrix} 1 & 1 \\ c & 1 \end{vmatrix}$

$$= c(c^2-1) - 1(c-1) + (1-c)$$

$$= c^3 - c - c + 1 + 1 - c$$

$$= c^3 - 3c + 2$$

$$\begin{array}{r} c^2 + c - 2 \\ c-1 \,\overline{)\, c^3 + 0c^2 - 3c + 2} \\ \underline{c^3 - c^2 \qquad\qquad} \\ c^2 - 3c + 2 \\ \underline{c^2 - c \qquad} \\ -2c + 2 \end{array}$$

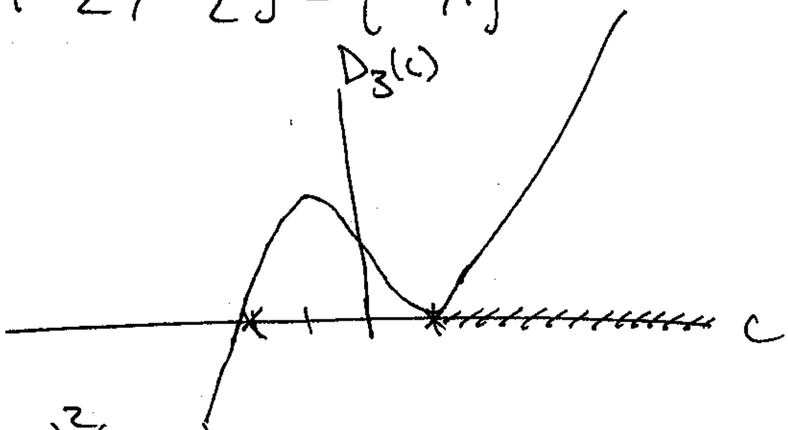
so $D_3 = (c-1)(c^2+c-2) > 0$

Looking for the roots to c^2+c-2 we see that

$$c = \frac{-1 \pm \sqrt{1^2 - 4(-2)}}{2} = \frac{-1 \pm \sqrt{1+8}}{2} = \frac{-1 \pm 3}{2}$$

So $c = \left\{ \frac{-1-3}{2}, \frac{-1+3}{2} \right\} = \{-2, 1\}$

Thus



$D_3 = (c-1)^2(c+2)$

Thus to have $D_1 > 0$ we must have $c > 0$

to have $D_2 = c^2 - 1 > 0$ we must have $|c| > 1$

to have $D_3 = (c-1)^2(c+2) > 0$ we must have (viewing the above graph that $c > 1$.

For the B ~~matrix~~ matrix we have

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & d & 4 \\ 3 & 4 & 5 \end{bmatrix}$$ to have the principle submatrices to be positive we must have

$D_1 = 1 > 0$ yes

$D_2 = d - 4 > 0 \Rightarrow d > 4$ yes

$$D_3 = 1\begin{vmatrix} d & 4 \\ 4 & 5 \end{vmatrix} - 2\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} + 3\begin{vmatrix} 2 & 3 \\ d & 4 \end{vmatrix}$$

$$D_3 = 5d - 16 - 2(10-12) + 3(8-3d)$$
$$= 5d - 16 - 20 + 24 + \overset{24}{\cancel{23}} - 9d$$
$$= -4d + \overset{12}{\cancel{23}} > 0$$

So $-4d > -12$

$\Rightarrow$ $d < 3$

since

Both d cannot be greater than 4 and less than 3

(13) let $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$

~~So if~~ $\cancel{A = \begin{bmatrix} 2 & 2 \\ 2 & 10 \end{bmatrix}}$ $\quad \cancel{\lambda_1 + \lambda_2 = 12}$

$\cancel{\lambda_1 \cdot \lambda_2 = 20 - 4} = 16$

& we require $a > 0$ & $c > 0$

& $a + c > 2b$ & $\lambda_1 \cdot \lambda_2 < 0$ since one eigenvalue is positive & one is negative. This is equivalent to

$ac - b^2 < 0$

Thus we are looking for a matrix that satisfies the following

$$a > 0 \quad \& \quad c > 0$$
$$a + c > 2b \quad \& \quad ac - b^2 < 0$$

~~Solve for $a > 2b =$~~ c

$a+c > 2b$ + $ac < b^2$

pick $b = \frac{1}{2}$, then take $a = 10$ + $c = \frac{1}{100}$

Then $a + c = 10 + \frac{1}{100} = \frac{10 \cdot 10^2 + 1}{100} = \frac{1001}{100} > 1$ yes

+ $a \cdot c = \frac{10}{100} = \frac{1}{10} < b^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$ yes.

Thus our matrix is given by

$$A = \begin{bmatrix} 10 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{100} \end{bmatrix}$$

(14) The eigenvalues of A^{-1} are the ~~reciprocal~~ reciprocal of the eigenvalues of A. Since ~~A~~ the eigenvalues of A are all positive then the eigenvalues of A^{-1} are all positive.

For $A^{-1} = \frac{1}{ac - b^2} \begin{bmatrix} c & -b \\ -b & a \end{bmatrix}$

the tests are $\frac{c}{ac - b^2} > 0$

+ $\frac{1}{ac-b^2}(ac + b^2) = \frac{ac + b^2}{ac - b^2}$

The tests for A being positive definite require that $a > 0$ + $ac - b^2 > 0$

~~...~~ these two conditions imply that $c > 0$ (since $ac > b^2 > 0$ if $a > 0 \Rightarrow c > 0$ + if $c > 0 \Rightarrow a > 0$)

Thus $\frac{c}{ac-b^2}$ is the ratio of two positive ~~things~~ numbers + $\therefore$ is positive. Also $\frac{a+b^2}{ac-b^2}$ is positive since a + c are both positive. Thus A^{-1} passes the positive definite tests

(15) Consider $x^T(A+B)x = x^TAx + x^TBx > 0$, so since both $x^TAx > 0$ + $x^TBx > 0$, $\forall\, x \neq 0$ $x^T(A+B)x > 0$ $\forall\, x \neq 0$. Thus $A+B$ is S.P.D.

(16) If y = all zeros then the block matrix multiplication will select only the A matrix + thus A must be positive definite

How does one show that $C - B^TA^{-1}B$ must be positive definite?

(17) For $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}\begin{bmatrix} 4 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$= 4x_1^2 + 0 \cdot x_2^2 + 5x_3^2 + 2x_1x_2 + 2x_1x_3 + 4x_2x_3$

Thus pick $x_3 = 0$ + we have

$4x_1^2 + 2x_1x_2$

So let $x_1 = 1$ + then $x_2 = -5$ + the functional becomes

$4 + 2(-5) = -6 < 0$.

Also if $(x_1, x_2, x_3) = (0, 1, 0) \neq 0$ then $x^T A x = 0$

(18) If a_{11} is smaller than all the eigenvalues of A. If it ~~is~~ was then $A - a_{11}I$ would have eigenvalues of $\lambda - a_{11}$ which would all be positive. ~~for all eigenvalues of A.~~

~~But it has a zero on the main diagonal.~~

Thus the matrix $A - a_{11}I$ would be positive definite

But this is a contradiction since it has a zero on the main diagonal. A zero on the main diagonal at (i,i) will produce a zero in the inner product when evaluated on a vector with a one in the ith component.

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(21) ~~For the A given obviously s=4 is an eigenvalue.~~

Since I can't see any "easy" way of computing the eigenvalues for general s, I'll just do the long calculation

$$\begin{vmatrix} s-\lambda & -4 & -4 \\ -4 & s-\lambda & -4 \\ -4 & -4 & s-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (s-\lambda)\begin{vmatrix} s-\lambda & -4 \\ -4 & s-\lambda \end{vmatrix} + 4\begin{vmatrix} -4 & -4 \\ -4 & s-\lambda \end{vmatrix} - 4\begin{vmatrix} -4 & -4 \\ s-\lambda & -4 \end{vmatrix} = 0$$

$$\Rightarrow (s-\lambda)\left[(s-\lambda)^2 - 16\right] + 4\left[-4(s-\lambda) - 16\right] - 4(16 + 4(s-\lambda)) = 0$$

$$\Rightarrow (s-\lambda)^3 - 16\cancel{(s-\lambda)} - 16\cancel{(s-\lambda)} - 64 - 64 - 16(s-\lambda) = 0$$

$$\Rightarrow \cancel{(s-\lambda)^3 - 16(s-\lambda) - 128 = 0} \qquad 3 \cdot 16 = 30 + 18$$

$$\Rightarrow (s-\lambda)^3 - 48(s-\lambda) - 128 = 0$$

$$\Rightarrow s^3 - 3s^2\lambda + 3s\lambda^2 - \lambda^3 - 48s + 48\lambda - 128 = 0$$

$$\Rightarrow -\lambda^3 + 3s\lambda^2 + (-3s^2 + 48)\lambda + s^3 - 128 = 0$$

~~I know that $\lambda =$~~ one would then have to impose conditions on s so that all the roots of the above are positive. Since this seems difficult, I will instead look at the leading determinant principal ~~diagonal~~ sub matrices.

For example $D_1 = s > 0$ ~~For for sto assure that~~ +

$D_2 = s^2 - 16 > 0 \Rightarrow |s| > 4$

and finally $D_3 = s \begin{vmatrix} s & -4 \\ -4 & s \end{vmatrix} + 4 \begin{vmatrix} -4 & -4 \\ -4 & s \end{vmatrix} - 4 \begin{vmatrix} -4 & -4 \\ s & -4 \end{vmatrix}$

$$= s(s^2-16) + 4(-4s - 16) - 4(16 + 4s)$$

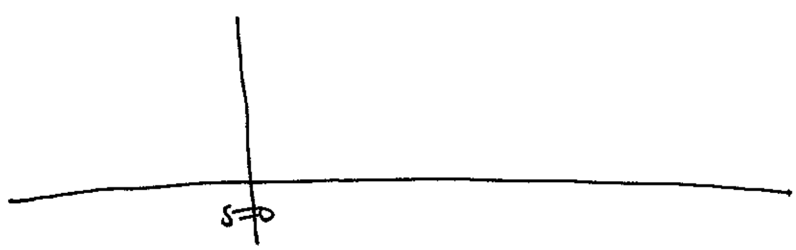
$$= s^3 - 16s - 8s - 64 - 64 - 16s$$

$$= s^3 - 40s - 128 > 0$$

consider $s^3 - 40s - 128$ at $s = 4$ + we get

$64 - 40 \cdot 4 - 128 < 0$

plotting the function $s^3 - 40s - 128$ gives the following graph



From the graph we see that for $s > s_0 =$ — $D_1, D_2, + D_3$ will all be positive + ∴ the matrix A will ~~not~~ be positive definite + have only positive eigenvalues

For B again consider the ~~principal~~ determinants of the principal submatrices.

We have $D_1 = t > 0$

$$D_2 = t^2 - 9 > 0 \Rightarrow |t| > 3$$

$$D_3 = t\begin{vmatrix} t & 4 \\ 4 & t \end{vmatrix} - 3\begin{vmatrix} 3 & 0 \\ 4 & t \end{vmatrix}$$

$$= t(t^2-16) - 3(3t)$$

$$= t^3 - 16t - 9t$$

$$= t^3 - 25t = t(t^2-25) > 0$$

$\Rightarrow |t| > 5$. Thus if $t > 5$ B will be positive definite & $\therefore$ has only positive eigenvalues

(22) From $A = Q\Lambda Q^T$ compute $Q\Lambda^{1/2}Q^T$ for each matrix.

For $A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$ we have eigenvalues given by

$\lambda = 1$ & $\lambda = 9$ with eigenvectors given by $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ & $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Then $Q = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ w/ $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$

So $Q\Lambda^{1/2}Q^T = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\frac{1}{\sqrt{2}}$

$$= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 3 & 3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \equiv R$$

consider $R^2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} = A$

For $A = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix}$ we have eigenvalues given by

$\lambda = 4$ + $\lambda = 16$ with eigenvectors given by $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ + $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Then $Q\Lambda^{1/2}Q^T = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$$= \frac{1}{2}\begin{bmatrix} 2 & 4 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 6 & 2 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \equiv R$$

So $R^2 = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix}$ which is A

(23) $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \iff \left(\frac{1}{a}\right)^2 x^2 + \left(\frac{1}{b}\right)^2 y^2 = 1$

So $\lambda_1 = \left(\frac{1}{a}\right)^2$ + $\lambda_2 = \left(\frac{1}{b}\right)^2$

$\Rightarrow$ $a = \frac{1}{\sqrt{\lambda_1}}$ + $b = \frac{1}{\sqrt{\lambda_2}}$

The ellipse $9x^2+16y^2=1$ has axes w/ half-lengths given by $a=\frac{1}{3}$ & $b=\frac{1}{4}$.

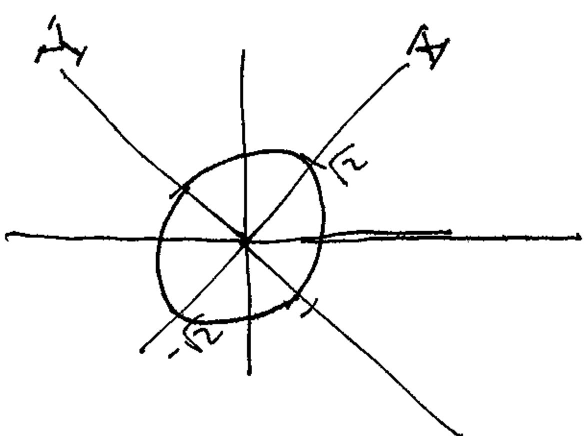
(24) $x^2+xy+y^2=1$ is generated from

$$[x \;\; y]\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 1$$

This $A=\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$ has eigenvalues given by $\lambda=\frac{1}{2}$ & $\lambda=\frac{3}{2}$

w/ eigenvalues $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ & $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ respectively. So this ellipse's half-lengths are given by $a=\frac{1}{\sqrt{\lambda_1}}$ & $b=\frac{1}{\sqrt{\lambda_2}}$ giving

$a=\sqrt{2}$ & $b=\sqrt{\frac{2}{3}}$. Thus this ellipse looks like



(25) From $C=\begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}$ then

$$A=CC^T=\begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 3 \\ 3 & 5 \end{bmatrix}$$

Given $A = \begin{bmatrix} 4 & 8 \\ 8 & 25 \end{bmatrix}$ we 1st factor $A = LDL^T$

$$\begin{array}{r} 25 \\ -16 \\ \hline 9 \end{array}$$

$$\begin{bmatrix} 4 & 8 \\ 8 & 25 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & 8 \\ 0 & 9 \end{bmatrix} \quad \text{so}$$

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 4 & 8 \\ 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \equiv LDL^T$$

Then $C = LD^{1/2} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}$

(26) From $C = L\sqrt{D}$ since L has ones on its main diagonal C has the square roots of the pivots on the diagonal

For $A = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 9 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$

So $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

So we have $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix}$$

For the second matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 7 \end{bmatrix}$ we have

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 7 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 5 \end{bmatrix}$$

So $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 5 \end{bmatrix}$

So $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{5} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{5} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & \sqrt{5} \end{bmatrix}$$

(27) $A = LDL^T$

$x^TAx = x^TLDL^Tx$ or in the case of a 2x2 system we have

$$\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 1 & 0 \\ b/a & 1 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & \frac{ac-b^2}{a} \end{bmatrix}\begin{bmatrix} 1 & b/a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow ax^2 + 2bxy + cy^2 = \begin{bmatrix} x+\frac{b}{a}y & y \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & \frac{ac-b^2}{a} \end{bmatrix}\begin{bmatrix} x+\frac{b}{a}y \\ y \end{bmatrix}$$

$$= a\left(x+\frac{b}{a}y\right)^2 + \left(\frac{ac-b^2}{a}\right)y^2$$

Thus we ~~are~~ are completing the square. With $a=2, b=4,$ & $c=10$ we have

$$2x^2 + 8xy + 10y^2 = 2(x^2+4xy+4y^2) - 8y^2 + 10y^2$$

$$= 2(x+2y)^2 + 2y^2$$

While the R.H.S. gives

$$= 2\left(x+\frac{4}{2}y\right)^2 + \left(\frac{20-16}{2}\right)y^2$$

$$= 2(x+2y)^2 + 2y^2$$

(28) ~~[illegible]~~ Define $Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

we have that $|Q| = 1$ & $Q^T = Q^{-1}$ so

(a) $|A| = 2 \cdot 5 = 10$

(b) Eigenvalues of A are 2 & 5

(c) The eigenvectors of A are given by

$$\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \; \& \; \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

(d) λ_1 & $\lambda_2 > 0$ so A is positive definite

(29) $f_1 = \frac{1}{4}x^4 + x^2y + y^2$ & $f_2 = x^3 + xy - x$

Then $A_1 = \begin{bmatrix} \frac{\partial^2 f_1}{\partial x^2} & \frac{\partial^2 f_1}{\partial x \partial y} \\ \frac{\partial^2 f_1}{\partial x \partial y} & \frac{\partial^2 f_1}{\partial y^2} \end{bmatrix}$

Now $\frac{\partial f_1}{\partial x} = x^3 + 2xy \Rightarrow \frac{\partial^2 f_1}{\partial x^2} = 3x^2 + 2y$

& $\frac{\partial f_1}{\partial y} = x^2 + 2y \Rightarrow \frac{\partial^2 f_1}{\partial y^2} = 2$

& $\frac{\partial f_1}{\partial x \partial y} = 2x$

Thus

$$A_1 = \begin{bmatrix} 3x^2+2y & 2x \\ 2x & 2 \end{bmatrix}$$

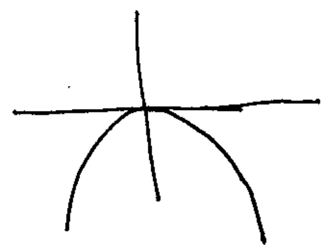
The minimum point of f_1 is the location where

$\frac{\partial f_1}{\partial x} = 0 \Rightarrow x^3+2xy=0 \Rightarrow x(x^2+2y)=0 \Rightarrow x=0$ or $y=-\frac{1}{2}x^2$

& $\frac{\partial f_1}{\partial y} = 0 \Rightarrow x^2+2y=0 \Rightarrow y=-\frac{1}{2}x^2$ which ~~when put into the 1st equation gives~~ is the same equation found

$\Rightarrow$ ✓

above. Thus the minimum lies along the curve $y=-\frac{1}{2}x^2$



For $f_2 = x^3+xy-x$ we have

$$\frac{\partial f_2}{\partial x} = 3x^2+y-1 \Rightarrow \frac{\partial^2 f_2}{\partial x^2} = 6x$$

$$\frac{\partial f_2}{\partial y} = x \Rightarrow \frac{\partial^2 f_2}{\partial y^2} = 0 \quad \text{so} \quad \frac{\partial^2 f_2}{\partial x \partial y} = 1$$

So $$A_2 = \begin{bmatrix} \frac{\partial^2 f_2}{\partial x^2} & \frac{\partial^2 f_2}{\partial x \partial y} \\ \frac{\partial^2 f_2}{\partial x \partial y} & \frac{\partial^2 f_2}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 6x & 1 \\ 1 & 0 \end{bmatrix}$$

looking for the extrema we have to solve

$$\frac{\partial f_2}{\partial x} = 3x^2+y-1 = 0$$

& $\frac{\partial f_2}{\partial y} = x = 0$ so $x=0 \Rightarrow y-1=0 \Rightarrow y=1$

So the point is $(x,y) = (0,1)$ and the A_2 matrix is then

$$A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{at} \quad (0,1)$$

(3D) The function $z = \cancel{4x^2 + 12xy + 6y}$ $ax^2 + 2bxy + cy^2$ can be written as

$$z = [x \; y]\begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = [x\,y]A\begin{bmatrix} x \\ y \end{bmatrix}$$

Then A will be positive definite iff $a > 0$ & $\frac{ac-b^2}{a} > 0$

Then A will have only positive eigenvalues & z will be "bowl" opening upwards. If $a < 0$ + $\frac{ac-b^2}{a} < 0$ both eigenvalues of A will be negative, and z will be a "bowl" opening downwards. If a + $\frac{ac-b^2}{a}$ are of different signs, then z will be a saddle at $(0,0)$. Thus the product of a + $\frac{ac-b^2}{a}$ must be negative so the test is

$$a\left(\frac{ac-b^2}{a}\right) = ac - b^2 < 0$$

equivalently we have

$$\left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 < 0$$ then $(0,0)$ will be a saddle point.

(31) For $z = 4x^2 + 12xy + cy^2$

For z to have a ~~....~~ ~~...~~ saddle from problem 30 we require that

$$\cancel{4c - 6 < 0} \Rightarrow \cancel{c < \frac{3}{2}} \qquad 8x + 12y$$

$$\left.\frac{\partial^2 z}{\partial x^2}\right|_{(0,0)} = 8 \qquad \left.\frac{\partial^2 z}{\partial y^2}\right|_{(0,0)} = 2c \qquad \frac{\partial^2 z}{\partial x \partial y} = 12$$

$$8(2c) - 12^2 < 0$$

$$= 4^2 c - 12 \cdot 12 < 0$$

$$= c - 3 \cdot 3 < 0 \Rightarrow c < 9$$ will have a saddle point

If $c > 9$ z will be a bowl. When $c = 9$ z is given by

$$z = 4x^2 + 12xy + 9y^2 = (2x + 3y)^2$$