HJ 310 Strang

(1) If $B = M^{-1}AM$ & $C = N^{-1}BN$

So that $C = N^{-1}(M^{-1}AM)N$

$$= (M \cdot N)^{-1} A (M \cdot N)$$

So for $T = M \cdot N$ we have $C = T^{-1}AT$. If B is similar to A and C is similar to B ~~then A is similar to A~~

then C is similar to A

(2) If $C = F^{-1}AF$ and also $C = G^{-1}BG$ then

$$F^{-1}AF = G^{-1}BG$$

so $B = GF^{-1}AFG^{-1} = (FG^{-1})^{-1}A(FG^{-1})$

Taking $M = FG^{-1}$ we have $B = M^{-1}AM$

So if C is similar to A and C is similar to B then A is similar to B.

(3) We are looking for a M such that

$$A = M^{-1}BM \quad \text{or} \quad MA = BM$$

let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $MA = BM$ is given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} = \begin{bmatrix} c & d \\ c & d \end{bmatrix}$$

so pick $d=0$ + $a=c$. So let $a=1$ + $b=2$

$$M = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$$

For the next pair of A + B we have

$$MA = BM$$

$$\Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a+b & a+b \\ c+d & c+d \end{bmatrix} = \begin{bmatrix} a-c & b-d \\ -a+c & -b+d \end{bmatrix}$$

$$\Rightarrow \begin{aligned} a+b &= a-c \\ a+b &= b-d \\ c+d &= -a+c \\ c+d &= -b+d \end{aligned} \quad \Rightarrow \quad \begin{aligned} b &= -c \\ a &= -d \\ d &= -a \\ c &= -b \end{aligned}$$

Thus we have the relation that $b=-c$ + $a=-d$.

Pick $a=1$ + $b=2$ to get $M = \begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix}$

For the next pair of A + B we have $MA=BM$ gives

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\Rightarrow \cancel{\begin{bmatrix} a+3b & 2c+4d \\ c+3d & 2c+4 \end{bmatrix}}$$

$$\Rightarrow \begin{bmatrix} a+3b & 2a+4b \\ c+3d & 2c+4d \end{bmatrix} = \begin{bmatrix} 4a+3c & 4b+3d \\ 2a+c & 2b+d \end{bmatrix}$$

$$\Rightarrow \begin{aligned} a+3b &= 4a+3c \\ 2a+\cancel{4b} &= \cancel{4b}+3d \\ c+3d &= 2a+c \\ 2c+4d &= 2b+d \end{aligned} \quad \Rightarrow \quad \begin{aligned} -3a+3b-3c &= 0 \\ 2a-3d &= 0 \\ 2a-3d &= 0 \\ \cancel{2b+2c} \quad 2b-2c-3d &= 0 \end{aligned}$$

Thus we have the system

$$\begin{bmatrix} -3 & 3 & -3 & 0 \\ 2 & 0 & 0 & -3 \\ 0 & 2 & -2 & -3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} 2 & 0 & 0 & -3 \\ -3 & 3 & -3 & 0 \\ 0 & 2 & -2 & -3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 0 & 0 & -3 \\ 0 & 3 & -3 & -9/2 \\ 0 & 2 & -2 & -3 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 0 & 0 & -3 \\ 0 & 1 & -1 & -3/2 \\ 0 & 0 & 0 & -6 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$(\frac{3}{2})(-3)$ $\quad (-\frac{3}{2})(2)+(-3) = -6$ $\quad \Rightarrow d=0,\ a=0,\ c=b$ so taking $b=c$

we have $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(4) ~~A has eigenvalues 0 + 1 + is a matrix~~

It A has eigenvalues 0 + 1 it has 2 linearly ~~dep~~ independent eigenvectors and therefore can be factorized into $A = S\Lambda S^{-1}$ which says that A + Λ are similar. Thus from problem 2 since every matrix with eigenvalues 0 + 1 are similar to $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ then they themselves are similar.

(5) $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ has $\lambda = 1$ only

$A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has $\lambda = -1$ + $+1$

$A_3 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ has $\lambda = 1$ + $\lambda = 0$

$A_4 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ has $\lambda = 1$ + $\lambda = 0$

$A_5 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ has $\lambda = 1$ + $\lambda = 0$

$A_6 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ has $\lambda = 1$ + $\lambda = 0$

Thus A_3, A_4, A_5 + A_6 are similar

(b) Two similar matrices will have the same eigenvalues. For the 2x2 matrices described in the book we ~~have~~ can have eigenvalues given by

$$M = \begin{bmatrix} j & l \\ k & m \end{bmatrix} \quad \text{w/ } j, l, k, m = 0, 1$$

$$\lambda^2 - \text{Tr}\,\lambda + D = 0$$

So trace$(M) = j + m$ only 4 choices

+ det$(M) = jm - kl$

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}$$

j l k m	Trace(M)	det(M)	λ's
0 0 0 0	0	0	0
0 0 0 1	1	0	$\frac{1\pm1}{2} = 1, 0$
0 0 1 0	0	0	0
0 0 1 1	1	0	1, 0
0 1 0 0	0	0	0
0 1 0 1	1	0	1, 0
0 1 1 0	0	−1	$\frac{0\pm\sqrt{4}}{2} = \pm 1$
0 1 1 1	1	−1	$\frac{1\pm\sqrt{1+4}}{2} = \frac{1\pm\sqrt{5}}{2}$
1 0 0 0	1	0	1, 0
1 0 0 1	2	1	$\frac{2\pm\sqrt{4-4}}{2} = 1$
1 0 1 0	1	0	1, 0
1 0 0 1	2	1	$\frac{2\pm\sqrt{4-4}}{2} = 1$
1 1 0 0	1	0	1, 0
1 1 0 1	2	1	1
1 1 1 0	1	−1	$\frac{1\pm\sqrt{5}}{2}$
1 1 1 1	2	0	$\frac{2\pm\sqrt{4}}{2} = \frac{2\pm2}{2} = 0, 2$

For each set of repeated eigenvalues we can have a matrix that is diagonalizable or not. These ~~diagonal~~ two matrices denote two different "families" since the diagonalizable matrices can be ~~decomposed~~ factored into a diagonal matrix, while the non-diagonal matrices can only be factored into a Jordan block structure like $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$. Thus we have the following sets of similar matrices w/ (λ_1, λ_2) given by

(λ_1, λ_2)	Property	Number of matrices in each family
$(0,0)$	diagonalizable	3 shared
$(0,0)$	non diagonalizable	3 shared
$(0,1)$ ~~(0,1)~~	diagonalizable	6
$(-1,1)$	diagonalizable	1
$\left(\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right)$	diagonalizable	2
$(1,1)$	diagonalizable	3 shared
$(1,1)$	non-diagonalizable	3 shared
$(0,2)$	diagonalizable	1

giving 8 families of similar matrices. One should check that in the cases when $(\lambda_1, \lambda_2) = (0,0)$ + $(\lambda_1, \lambda_2) = (1,1)$ that both types of familie (i.e. diagonal + non-diagonal) actually occur

(7) (a) If x ~~e~~ is in Nullspace of A, then

$Ax = 0$ So $M^{-1}x$ when multiplied ~~by~~ on the

left by $M^{-1}AM$ ~~is given~~ gives

$$M^{-1}AM(M^{-1}x) = M^{-1}Ax = M^{-1} \cdot 0 = 0$$

So $M^{-1}x$ is in the nullspace of $M^{-1}AM$.

(b) Since $\forall$ vector $x \in$ Nullspace of A there exists a vector $M^{-1}x$ in the Null space of $M^{-1}AM$ and to every vector x in the Nullspace of $M^{-1}AM$ there exists a vector Mx in the Nullspace of A (since $M^{-1}AMx$ must then ~~be~~ equal zero.) Thus the nullspace of A & $M^{-1}AM$ have the same number of elements & $\therefore$ the dimension of the nullspaces is the same

(8) No, the order of association of eigenvectors to eigenvalues could be different among the two matrices could be different. ~~If the order. also~~ If the ~~ordering~~ association is the same then I would think that $A = B$

With n independent eigenvectors again the answer is NO. to the question of $A = B$. The logic from the previous ~~to~~ discussion still holds. If A has eigenvalues $0, 0$ w/ ~~an~~ a single eigenvector $\propto (1,0)$ Then

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = M^{-1} A M \quad \text{or}$$

$$A = M \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} M^{-1}$$ with M a matrix ~~with~~ the 1st column of which ~~has~~ is the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & the 2nd column must be ~~orthogonal~~ linearly independent to the 1st column. This gives many possible A's. Consider

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & \cancel{2} \end{bmatrix} \quad \& \quad M_2 = \begin{bmatrix} 1 & a \\ 0 & b \end{bmatrix} \quad b \neq 0$$

then $$M_1^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \& \quad M_2^{-1} = \frac{1}{b}\begin{bmatrix} b & -a \\ 0 & 1 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \&$$

Then ~~$A_2 = M \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \frac{1}{b} \begin{bmatrix} b \\ \end{bmatrix}$~~

$$A_2 = M \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} M^{-1}$$

$$= \begin{bmatrix} 1 & a \\ 0 & b \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \frac{1}{b} \begin{bmatrix} b & -a \\ 0 & 1 \end{bmatrix}$$

$$= \frac{1}{b} \begin{bmatrix} 1 & a \\ 0 & b \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= \frac{1}{b} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \neq A_1 \quad \text{unless} \quad b = 1.$$

Thus in this case also there is the possibility of ~~more than~~ two different ~~eigenvecto~~ matrices with this propertie

(9) If $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ then $A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

& $A^3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

we guess that $A^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$

to check by induction $A^{k+1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & k+1 \\ 0 & 1 \end{bmatrix} \pm$$

If $k=0$ $A^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ + $k=-1$ gives

$A^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$. We can check that $A^{-1} A = I$

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ yes}$$

(10) If $J = \begin{bmatrix} c & 1 \\ 0 & c \end{bmatrix}$ then

$$J^2 = \begin{bmatrix} c & 1 \\ 0 & c \end{bmatrix}\begin{bmatrix} c & 1 \\ 0 & c \end{bmatrix} = \begin{bmatrix} c^2 & 2c \\ 0 & c^2 \end{bmatrix}$$

$$+ \ J^3 = \begin{bmatrix} c & 1 \\ 0 & c \end{bmatrix}\begin{bmatrix} c^2 & 2c \\ 0 & c^2 \end{bmatrix} = \begin{bmatrix} c^3 & 3c^2 \\ 0 & c^3 \end{bmatrix}$$

We guess that

$$J^k = \begin{bmatrix} c^k & kc^{k-1} \\ 0 & c^k \end{bmatrix}$$

Then $J^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$+ \ J^{-1} = \begin{bmatrix} c^{-1} & -c^{-2} \\ 0 & c^{-1} \end{bmatrix}$$

we can check $J^{-1}J = I$ to get

$$\begin{bmatrix} c^{-1} & -c^{-2} \\ 0 & c^{-1} \end{bmatrix}\begin{bmatrix} c & 1 \\ 0 & c \end{bmatrix} = \begin{bmatrix} 1 & c^{-1}-c^{-1} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{yes}$$

11 The text solved

$$\frac{du}{dt} = Ju = \begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

where the unknowns are (x, y, z)

Adding a 4th unknown w and the equation $\frac{dw}{dt} = 5w + x$ would result in the following system for $\underline{u} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$

$$\frac{du}{dt} = \begin{bmatrix} 5 & 1 & 0 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 5 & 0 \\ 1 & 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Following the same logic as before we can solve for $z(t)$ then $y(t)$, then $x(t)$ giving exactly the solutions found in the text namely

$$z = z(0)e^{+5t}$$
$$y = (y(0) + tz(0))e^{5t}$$
$$x = (x(0) + ty(0) + \tfrac{1}{2}t^2 z(0))e^{5t}$$

When ~~this~~ these functional forms are put back into the equation for $w(t)$ the following results

$$\frac{dw}{dt} = \left(x(0) + t y(0) + \tfrac{1}{2}t^2 z(0)\right)e^{\delta t} + \delta w(t)$$

or
$$\frac{dw}{dt} - \delta w(t) = \left(x(0) + t y(0) + \tfrac{1}{2}t^2 z(0)\right)e^{\delta t}$$

Which has a solution given by

$$w(t) = \left(w(0) + t x(0) + \frac{t^2}{2} y(0) + \frac{1}{6}t^3 z(0)\right)e^{\delta t}$$

Check

$$w'(t) = \left(x(0) + t y(0) + \frac{t^2}{2} z(0)\right)e^{\delta t} + \delta\,(\ \cdots\ )\,e^{\delta t}$$

$$= (\qquad\qquad\qquad)\,e^{\delta t} + \delta w(t)$$

(12) let $M = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix}$

Then $JM = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix}$

$$= \begin{bmatrix} m_{21} & m_{22} & m_{23} & m_{24} \\ 0 & 0 & 0 & 0 \\ m_{41} & m_{42} & m_{43} & m_{44} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$+ \quad Mk = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & m_{11} & m_{12} & 0 \\ 0 & m_{22} & m_{23} & 0 \\ 0 & m_{32} & m_{33} & 0 \\ 0 & m_{42} & m_{43} & 0 \end{bmatrix}$$

So if $JM = Mk$

$$\Rightarrow \begin{bmatrix} m_{21} & m_{22} & m_{23} & m_{24} \\ 0 & 0 & 0 & 0 \\ m_{41} & m_{42} & m_{43} & m_{44} \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & m_{11} & m_{12} & 0 \\ 0 & m_{22} & m_{23} & 0 \\ 0 & m_{32} & m_{33} & 0 \\ 0 & m_{42} & m_{43} & 0 \end{bmatrix}$$

So

$m_{21} = 0$

$m_{22} = m_{11}$

$m_{23} = m_{12}$

$m_{24} = 0$

~~my~~ $m_{22} = 0$

$m_{23} = 0$

$m_{41} = 0$

$m_{42} = m_{32}$

$m_{43} = m_{33}$

$m_{44} = 0$

$m_{42} = 0$

$m_{43} = 0$

Thus ~~that~~ we can deduce that

$m_{11} = 0$

$m_{12} = 0$

$m_{33} = 0$

$m_{32} = 0$

giving for M

$$M = \begin{bmatrix} 0 & 0 & m_{13} & m_{14} \\ m_{21} & 0 & 0 & 0 \\ m_{31} & 0 & 0 & m_{34} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since this matrix has a column of all zeros it is <u>not</u> invertible

$\therefore$ $M^{-1} J M = k$ is impossible

(13) When A is a Jordan block J_i it looks like

$$J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & 1 & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix} \quad \text{we have} \quad J_i^T = \begin{bmatrix} \lambda_i & & & & \\ 1 & \lambda_i & & & \\ & 1 & \lambda_i & & \\ & & 1 & \ddots & \\ & & & 1 & \lambda_i \end{bmatrix}$$

Dropping the "i" subscript for this part of the problem we are looking for a M ~~then~~ ~~pick~~ to make $\cancel{M_i^{-1} J_i M_i = J_i^T}$ ~~equivalently~~

$M^{-1} J M = J^T$ equivalently

$$\cancel{M_i =} \quad M_i = \begin{bmatrix} 1 & 1 & 0 & & & \\ 0 & 1 & 1 & & & \\ 0 & 0 & 1 & 1 & & \\ & & & \ddots & & \\ & & & & 1 & 1 \\ & & & & 0 & 1 \end{bmatrix}$$

~~Then~~ $$M_i^{-1} = \begin{bmatrix} 1 & -1 & & & \\ 0 & 1 & -1 & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & 1 & -1 \\ 0 & \cdots & \cdots & & 1 \end{bmatrix}$$

To try to find ~~the~~ a matrix M that makes J & J^T similar, lets consider a ~~[illegible]~~ very simple case to begin with

$$J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \qquad J^T = \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix}$$ then

If $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$JM = MJ^T$ is equivalent to

$$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} a\lambda + c & b\lambda + d \\ c\lambda & d\lambda \end{bmatrix} = \begin{bmatrix} a\lambda + b & b\lambda \\ c\lambda + d & d\lambda \end{bmatrix}$$

$\Rightarrow$ $c = b$ ~~a, b, c~~ everything else arbitrary.
$d = 0$

pick $a = 1$, $b = 1$, $c = 1$ & $d = 0$. Then $M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

consider $JM = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \lambda+1 & \lambda \\ \lambda & 0 \end{bmatrix}$

& $MJ^T = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix} = \begin{bmatrix} \lambda+1 & \lambda \\ \lambda & 0 \end{bmatrix}$ the same

For a 3×3 case we would have

$$J = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} \quad \text{so} \quad J^T = \begin{bmatrix} \lambda & 0 & 0 \\ 1 & \lambda & 0 \\ 0 & 1 & \lambda \end{bmatrix}$$

Then with $M = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$ we have

$$JM = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda m_{11} + m_{21} & \lambda m_{12} + m_{22} & \lambda m_{13} + m_{23} \\ \lambda m_{21} + m_{31} & \lambda m_{22} + m_{32} & \lambda m_{23} + m_{33} \\ \lambda m_{31} & \lambda m_{32} & \lambda m_{33} \end{bmatrix}$$

$$+ \; M J^T = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} \lambda & 0 & 0 \\ 1 & \lambda & 0 \\ 0 & 1 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} \lambda m_{11} + m_{12} & \lambda m_{12} + m_{13} & \lambda m_{13} \\ \lambda m_{21} + m_{22} & \lambda m_{22} + m_{23} & \lambda m_{23} \\ \lambda m_{31} + m_{32} & \lambda m_{32} + m_{33} & \lambda m_{33} \end{bmatrix}$$

So $m_{31} = m_{22}$

$m_{32} = 0$

$m_{22} = m_{13}$

$m_{32} = m_{23} \Rightarrow m_{23} = 0$

$m_{33} = 0$

$m_{23} = 0$

$m_{33} = 0$

with the other variables arbitrary. Let all non specified variables be 1

Thus

$$M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Thus I hypothesize that for general n

$$M = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & & 1 \cdot & 0 \\ 1 & \cdot & & \\ \vdots & 1 & 0 & \\ 1 & & & \end{bmatrix}$$

$i \rightarrow$

$M_{ij} = 1 \qquad i+j \leq n+1$

~~$2+n+1 = n+1$~~

$M_{ij} = 0 \qquad i+j > n+1$

lets check our requirement that $JM = MJ^T$ in this more general case the (i,j)th entry of ~~E~~ $JM = MJ^T$ gives in this case

$$\sum_{k=1}^{n} J_{ik} M_{kj} = \sum_{k=1}^{n} M_{ik} (J^T)_{kj} = \sum_{k=1}^{n} M_{ik} J_{jk} \qquad 1 \le i,j \le n$$

Since ~~J_{ik}~~ this becomes due to the ~~b~~ banded structure of J the following

$$J_{ii} M_{ij} + J_{i,i+1} M_{i+1,j} = M_{i,j} J_{j,j} + M_{i,j+1} J_{j,j+1}$$

since $J_{i,i} = J_{j,j}$ this simplifies to

$$J_{i,i+1} M_{i+1,j} = M_{i,j+1} J_{j,j+1}$$

Since $J_{i,i+1} = 1$ this becomes

$$M_{i+1,j} = M_{i,j+1}$$

Which will be true given the functional form on M ~~as~~ described above $M_{ij} = M(i+j)$. Thus the matrix M makes J & J^T similar

Now if J is a Jordan matrix it is composed of Jordan Blocks J_i as

$$J = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_s \end{bmatrix}$$ using M constructed of blocks as above we have

that $$M_0 = \begin{bmatrix} M_1 & & \\ & \ddots & \\ & & M_s \end{bmatrix}$$

So $JM_0 = M_0 J^T$ becomes (w/ M_0 a block matrix of M's)

$$\begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_s \end{bmatrix}\begin{bmatrix} M_1 & & \\ & \ddots & \\ & & M_s \end{bmatrix} = \begin{bmatrix} M_1 & & \\ & \ddots & \\ & & M_s \end{bmatrix}\begin{bmatrix} J_1^T & & \\ & \ddots & \\ & & J_s^T \end{bmatrix}$$

$$= \begin{bmatrix} J_1M_1 & & & \\ & J_2M_2 & & \\ & & \ddots & \\ & & & J_sM_s \end{bmatrix} = \begin{bmatrix} M_1J_1^T & & \\ & \ddots & \\ & & M_sJ_s^T \end{bmatrix}$$

Since we have equality of the subblocks involving J_i + M_i we have ~~...~~ found a matrix M (a block matrix) that diagonalizes J (w/ s ~~#~~ Jordan Blocks)

~~Let~~
Third let A be any matrix. Then by the Jordan Form for A we can decompose A into its Jordan blocks as

$M^{-1}AM = J$ or $A = MJM^{-1}$ so $A^T = (M^{-1})^T J^T M^T$

Further find the M_0 that makes J^T & J similar ~~the~~ from ~~part~~ the second part, i.e. $J^T = M_0^{-1} J M_0$ then

$$A^T = (M^{-1})^T M_0^{-1} J M_0 M^T$$

$$= (M^T)^{-1} M_0^{-1} M^{-1} A M M_0 M^T$$

$$= (M \cdot M_0 \cdot M^T)^{-1} A (M \cdot M_0 \cdot M^T)$$

So A^T & A are similar by the relationship above

(14) We can generate an ∞ # of matrices similar to a given matrix A by constructing an invertible M ~~such that~~ and computing

$$B = M^{-1} A M.$$

(15) $\det(A - \lambda I) = \cancel{\det(A - \lambda M^{-1} M)}$

$= \cancel{\det(M}$

$\cancel{= \det(M^{-1}(M^{-1}AM - \lambda I)M}$

$= \det(A - \lambda M \cdot M^{-1})$

$= \det(M(M^{-1}AM - \lambda I)M^{-1})$

$= \det(M) \cdot \det(M^{-1}AM - \lambda I) \cdot \det(M^{-1})$

$= \det(M) \det(M^{-1}AM - \lambda I) \det(M)^{-1}$

$= \det(M^{-1}AM - \lambda I)$

Thus A & $M^{-1}AM$ have the <u>same</u> characteristic polynomial & $\therefore$ the same eigenvalues

(16) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ + $\begin{bmatrix} d & c \\ b & a \end{bmatrix}$ are similar since they

have the same eigenvalues.

$\begin{bmatrix} b & a \\ d & c \end{bmatrix}$ + $\begin{bmatrix} c & d \\ a & b \end{bmatrix}$ are similar for the same reason

to show that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ + $\begin{bmatrix} b & a \\ d & c \end{bmatrix}$ are Not similar

let $a = 1$, $b = c = d = 0$ Then the two matrices are given by

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ + $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ which have different eigenvalues + ∴

are ~~same~~ not similar

(17) (a) True if it was ~~one~~ ~~would~~ ~~then~~ both A + B

~~$B = M^{-1}AM$~~ ~~but~~ we would have to have the

same eigenvalues but the invertible matrix cannot

have 0 as an eigenvalue while the singular

matrix must have 0 as an eigenvalue

(b) False. From example 1 ~~on~~ the projection matrix

$A = \frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$ ~~has eigendecomposition~~ $\cancel{A = S \ldots S^{-1}AS = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}}$ is symmetric

~~both are similar to it. The matrix of~~

but with $M = \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$ ~~the~~ a matrix it is similar to

$M^{-1}AM = \begin{bmatrix}1 & .5\\ 0 & 0\end{bmatrix}$ is not symmetric

(c) If A is similar to $-A$ ~~but~~ ~~have~~ then A & $-A$

must have the same eigenvalues thus

the eigenvalues must all be zero. ~~But we A'll could have~~

~~a~~ consider $A = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$ & $\cancel{M = \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}}$

& $-A = \begin{bmatrix}0 & -1\\ 0 & 0\end{bmatrix}$

then

$AM = M(-A)$

$$\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}a & b\\ c & d\end{bmatrix} = \begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}0 & -1\\ 0 & 0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -a \\ 0 & -c \end{bmatrix}$$

$\Rightarrow$ $c = 0$ & $d = -a$ b is arbitrary

Thus A & $-A$ are similar w/ a M given

by $M = \begin{bmatrix} a & b \\ 0 & -a \end{bmatrix}$

and $A \neq 0$.

(d) True, the eigenvalues of A & $A+I$ will be different & $\therefore$ they cannot be similar

(18) If B is invertible then AB has the same eigenvalues as BA. The eigenvalues of AB are given by

$$\det(AB - \lambda I) = 0$$

$$\Rightarrow \det(B^{-1}BAB - \lambda I) = 0$$

$$\Rightarrow \det\left(B^{-1}(BA - \lambda B \cdot B^{-1})B\right) = 0$$

$\Rightarrow \det(B^{-1})\det(BA-\lambda I)\det(B)=0$

$\Rightarrow \det(BA-\lambda I)=0$ $\therefore$ Since AB & BA have the same characteristic equation & $\therefore$ the same eigenvalues

(19) $$\begin{bmatrix} I & -A \\ 0 & I \end{bmatrix}\begin{bmatrix} AB & 0 \\ B & 0 \end{bmatrix}\begin{bmatrix} I & A \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & -A \\ 0 & I \end{bmatrix}\begin{bmatrix} AB & ABA \\ B & BA \end{bmatrix}$$

$(m+n)\times$

$$= \begin{bmatrix} 0 & 0 \\ B & BA \end{bmatrix}$$

(a) A is $m\times n$ & B is $n\times m$, so AB is $m\times m$ & BA is $n\times n$. The two I's in the 1st matrix are of size $m\times$

pg 312 Strang

(19) (a) If A is $m \times n$ + B is $n \times m$ then

AB is $m \times m$ + BA is $n \times n$, consistent sizes are given below

$$\begin{bmatrix} I_{m\times m} & -A_{m\times n} \\ 0_{n\times m} & I_{n\times n} \end{bmatrix} \begin{bmatrix} AB_{m\times m} & 0_{m\times n} \\ B_{n\times m} & 0_{n\times n} \end{bmatrix} \begin{bmatrix} I_{m\times m} & A_{m\times n} \\ 0_{n\times m} & I_{n\times n} \end{bmatrix}$$

(b) Since

$$\begin{bmatrix} I & -A \\ 0 & I \end{bmatrix} \begin{bmatrix} I & A \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

This block equation is $M^{-1} F M = G$

Based on the eigenvalue counting argument given in the text,

AB has m eigenvalues plus n zeros

BA has n eigenvalues plus m zeros

~~Then~~ when $m \geq n$ AB has the same eigenvalues as BA

but with $m-n$ zeros eigenvalues

(20)

(a) If A is similar to B then

$$\cancel{A = MBM^{-1}} \quad A = M^{-1}BM$$

Then $A^2 = M^{-1}BM \cdot M^{-1}BM = M^{-1}B^2M$

So A^2 is similar to B^2

(b) ~~If~~ ~~let~~ $\cancel{A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}$ $\cancel{A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}$

$\cancel{\& \; B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}$

~~let~~ ~~A be diago~~

Let A be ~~diagonal~~ diagonalizable with eigenvalues -1 and let

B be diagonalizable with eigenvalues $+1$

Then A^2 is ~~diagonalizable~~ has eigenvalues $+1$ & B^2 has

eigenvalues $+1$. They can be made similar while A & B

cannot (since they don't have the same eigenvalues)

For example, let $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ & $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

then since A & B are diagonal

the eigenvalues are the values found on the diagonal. Also $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ & $B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ So A^2 & B^2 are

similar while A & B are not.

(c) $\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$ & $\begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix}$ are both diagonalizable with the same

eigenvalues, therefore they are similar

(d) $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ ~~&~~ is diagonalizable but $\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ is not

the simplest matrix it can be reduced to is a Jordan form

matrix.

(e) considering the ~~...~~ requested transformation

$|A - \lambda I| = 0$ ~~is~~ has a sign introduced ~~exchanged~~ by the 1st row exchange i.e

$-|A' - \lambda I'| = 0$

The column exchange introduces another negative sign and ~~...~~ a

permutation of I'. This permutation of I' results in the identity

matrix back again so we have $(-1)^2 |A'' - \lambda I| = 0$

Which ~~...~~ shows that A & A'' have the same eigenvalues.

Pg 318 Strang

(1) If $A = U\Sigma V^T$ then $A^TA = V\Sigma^2 V^T$.

For this problem A^TA is given by

$$A^TA = \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix} = \begin{bmatrix} 5 & 20 \\ 20 & 80 \end{bmatrix}$$

Which has eigenvalues given by

$$\begin{vmatrix} 5-\lambda & 20 \\ 20 & 80-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (5-\lambda)(80-\lambda) - 400 = 0$$

$$\Rightarrow \cancel{400} - 5\lambda - 80\lambda + \lambda^2 - \cancel{400} = 0$$

$$\Rightarrow \lambda^2 - 85\lambda = 0 \Rightarrow \lambda = 0 \ \& \ \lambda = 85.$$

A^TA has eigenvectors given by (for $\lambda = 0$) ~~thus~~ $\cancel{\sigma_1^2 = 85}$

$$v_1 \propto \begin{bmatrix} -4 \\ 1 \end{bmatrix} \Rightarrow v_1 = \frac{1}{\sqrt{16+1}}\begin{bmatrix} -4 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{17}}\begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

for $\lambda = 85$

$$A^TA - 85I = \begin{bmatrix} -80 & 20 \\ 20 & -5 \end{bmatrix} \Rightarrow v_2 \propto \begin{bmatrix} 1 \\ 4 \end{bmatrix} \Rightarrow v_2 = \frac{1}{\sqrt{17}}\begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

~~Then~~ Since the book takes σ_1^2 to be the nonzero eigenvalue of A^TA

we have $\sigma_1^2 = 85$ & $v_1 = \frac{1}{\sqrt{17}}\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ &

$$v_2 = \frac{1}{\sqrt{17}}\begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

(2) (a) computing AA^T gives

$$\begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix} = \begin{bmatrix} 17 & 34 \\ 34 & 68 \end{bmatrix}$$

Which has an eigenvalue equal to 0 & a second eigenvalue given by

$$0 + \sigma_1^2 = 17 + 68 = 85$$ as expected from problem 1.

The eigenvectors of AA^T are then given by

$$AA^T - 85I = \begin{bmatrix} -68 & 34 \\ 34 & -17 \end{bmatrix} \quad \text{so} \quad u_1 \propto \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\Rightarrow \quad u_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

For u_2 we have

$$u_2 \propto \begin{bmatrix} -2 \\ 1 \end{bmatrix} \Rightarrow \quad u_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

(b) $$Av_1 = \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}\frac{1}{\sqrt{17}}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \frac{1}{\sqrt{17}}\begin{bmatrix} 17 \\ 34 \end{bmatrix} = \sqrt{17}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

w/ $\sigma_1^2 = 85 \Rightarrow \sigma_1 = \sqrt{85} = \sqrt{17}\cdot\sqrt{5}$ so Av_1 above

becomes

$$Av_1 = \sqrt{17}\cdot\sqrt{5}\cdot\frac{1}{\sqrt{5}}\begin{bmatrix}1\\2\end{bmatrix} = \sqrt{85}\cdot v_1 \quad \text{as requested}$$

All entries of the SVD are given by

$$\begin{bmatrix}1 & 4\\ 2 & 8\end{bmatrix} = \begin{bmatrix}1/\sqrt{5} & -2/\sqrt{5}\\ 2/\sqrt{5} & 1/\sqrt{5}\end{bmatrix}\begin{bmatrix}\sqrt{85} & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}1/\sqrt{17} & -4/\sqrt{17}\\ 4/\sqrt{17} & 1/\sqrt{17}\end{bmatrix}^T$$

Lets check this by computing the product of the matrices on the right hand side. We obtain remembering that $\sqrt{85} = \sqrt{5}\cdot\sqrt{17}$

$$\cancel{\begin{bmatrix}\sqrt{17} & 0\\ 2\sqrt{17} & 0\end{bmatrix}\begin{bmatrix}\sqrt{5} & -4\sqrt{17}\\ 0 & 0\end{bmatrix}} =$$

$$\cancel{\begin{bmatrix}1/\sqrt{5} & 2/\sqrt{5}\\ 2/\sqrt{5} & 1/\sqrt{5}\end{bmatrix}\begin{bmatrix}\sqrt{5} & -4\sqrt{5}\\ 0 & 0\end{bmatrix} = \begin{bmatrix}1 & -4\\ 2 & -8\end{bmatrix}}$$

$$\begin{bmatrix}1/\sqrt{5} & -2/\sqrt{5}\\ 2/\sqrt{5} & 1/\sqrt{5}\end{bmatrix}\begin{bmatrix}\sqrt{85} & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}1/\sqrt{17} & 4/\sqrt{17}\\ -4/\sqrt{17} & 1/\sqrt{17}\end{bmatrix}$$

$$= \begin{bmatrix}1/\sqrt{5} & -2/\sqrt{5}\\ 2/\sqrt{5} & 1/\sqrt{5}\end{bmatrix}\begin{bmatrix}\sqrt{5} & 4\sqrt{5}\\ 0 & 0\end{bmatrix} = \begin{bmatrix}1 & 4\\ 2 & 8\end{bmatrix} \quad \text{yes}$$

(3) Since this A has rank $= 1$ we have from the text that the matrices U + V contain orthonormal basis for all four subspaces $\{A = U\Sigma V^T\}$

- The 1st $r=1$ columns of V span the row space of A
  this is $v_1 = \frac{1}{\sqrt{17}}\begin{bmatrix} 1 \\ 4 \end{bmatrix}$

- The last $n-r = 2-1 = 1$ columns of ~~U~~ V span the nullspace of A
  this is $v_2 = \frac{1}{\sqrt{17}}\begin{bmatrix} -4 \\ 1 \end{bmatrix}$

- The 1st $r=1$ columns of U span the column space of A
  this is $u_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

- The last $m-r = 2-1 = 1$ columns of U span the null space of A^T. This ~~is~~ ~~v~~ $u_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

(4) (a) A^TA is given by

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

which has ~~eign~~ ~~ve~~ ~~ea~~ ~~eigen~~ eigenvalues given by

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (2-\lambda)(1-\lambda) - 1 = 0$$

$$2 - 3\lambda + \lambda^2 - 1 = 0$$

$$\Rightarrow \lambda^2 - 3\lambda + 1 = 0$$

So $\lambda = \frac{3 \pm \sqrt{9-4(1)(1)}}{2} = \frac{3 \pm \sqrt{5}}{2}$, the eigenvectors of A^TA

are given by the null space of

$$A^TA - \lambda I = \begin{bmatrix} 2-\left(\frac{3\pm\sqrt{5}}{2}\right) & 1 \\ 1 & 1-\left(\frac{3\pm\sqrt{5}}{2}\right) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1\mp\sqrt{5}}{2} & 1 \\ 1 & \frac{-2\mp\sqrt{5}}{2} \end{bmatrix}$$

~~Assume~~ One could do row reduction on this matrix to show that the 2nd row is a multiple of the 1st but if I assume I have done everything correctly, I know this is true and obtain for the ~~singular values~~ eigen vectors of $A^TA - \lambda I$ the following

$$\text{w/ } \sigma_1^2 \equiv \frac{3-\sqrt{5}}{2} \quad + \quad \sigma_2^2 \equiv \frac{3+\sqrt{5}}{2}$$

$$V_1 \propto \begin{bmatrix} \frac{2}{1+\sqrt{5}} \\ -1 \end{bmatrix} \quad + \quad V_2 \propto \begin{bmatrix} \frac{2}{1-\sqrt{5}} \\ -1 \end{bmatrix}$$

$$\text{or } V_1 \propto \begin{bmatrix} 2 \\ -(1+\sqrt{5}) \end{bmatrix} \quad + \quad V_2 \propto \begin{bmatrix} 2 \\ -(1-\sqrt{5}) \end{bmatrix}$$

So that $v_1 = \frac{1}{\sqrt{4+1+2\sqrt{5}+5}}\begin{bmatrix} 2 \\ -(1+\sqrt{5}) \end{bmatrix}$.

$$= \frac{1}{\sqrt{10+2\sqrt{5}}}\begin{bmatrix} 2 \\ -(1+\sqrt{5}) \end{bmatrix}$$

& $v_2 = \frac{1}{\sqrt{4+1-2\sqrt{5}+5}}\begin{bmatrix} 2 \\ -(1-\sqrt{5}) \end{bmatrix} = \frac{1}{\sqrt{10-2\sqrt{5}}}\begin{bmatrix} 2 \\ -(1-\sqrt{5}) \end{bmatrix}$

For the matrix AA^T we have the same matrix!! ($A^T = A$ in this case)

Thus $u_1 = v_1$ & $u_2 = v_2$. The singular value decomposition of A is then given by $A = U\Sigma V^T$ or

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{10+2\sqrt{5}}} & \frac{2}{\sqrt{10-2\sqrt{5}}} \\ \frac{-(1+\sqrt{5})}{\sqrt{10+2\sqrt{5}}} & \frac{-(1-\sqrt{5})}{\sqrt{10-2\sqrt{5}}} \end{bmatrix} \begin{bmatrix} \sqrt{\frac{3-\sqrt{5}}{2}} & 0 \\ 0 & \sqrt{\frac{3+\sqrt{5}}{2}} \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{10+2\sqrt{5}}} & \frac{2}{\sqrt{10-2\sqrt{5}}} \\ \frac{-(1+\sqrt{5})}{\sqrt{10+2\sqrt{5}}} & \frac{-(1-\sqrt{5})}{\sqrt{10-2\sqrt{5}}} \end{bmatrix}^T$$

which could be checked by direct calculation of the products of the matrices on the right hand side.

(5) Consider

$$Av_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ -(1+\sqrt{5}) \end{bmatrix}\frac{1}{\sqrt{10+2\sqrt{5}}}$$

$$= \begin{bmatrix} 2-1-\sqrt{5} \\ 2 \end{bmatrix}\frac{1}{\sqrt{}} = \begin{bmatrix} 1-\sqrt{5} \\ 2 \end{bmatrix}\frac{1}{\sqrt{}}$$

$$= +\frac{(1-\sqrt{5})}{2}\begin{bmatrix} 2 \\ -(1+\sqrt{5}) \end{bmatrix}\frac{1}{\sqrt{10+2\sqrt{5}}}$$

check $\quad \dfrac{-(1-\sqrt{5})(1+\sqrt{5})}{2} = \dfrac{-(1-5)}{2} = 2 \quad$ yes ✓

Thus the above becomes

$$Av_1 = \frac{(1-\sqrt{5})}{2\sqrt{2}\sqrt{5+\sqrt{5}}}\begin{bmatrix} 2 \\ -(1+\sqrt{5}) \end{bmatrix}$$

$$= \cancel{\frac{(1-\sqrt{5})}{2\sqrt{2}\sqrt{\sqrt{5}}\sqrt{1+\sqrt{5}}}} \qquad \cancel{\sqrt{\sqrt{5}(1+\sqrt{5})}}$$

$$= \frac{(1-\sqrt{5})}{2\sqrt{2}\,5^{1/4}\sqrt{1+\sqrt{5}}}\begin{bmatrix} 2 \\ -(\quad) \end{bmatrix}$$

$\leq$

~~Since~~ $\quad \dfrac{1-\sqrt{5}}{2\sqrt{2}\,5^{1/4}\sqrt{(1+\sqrt{5})(1-\sqrt{5})}}\sqrt{1-\sqrt{5}}$

... Finish ... check algebra to this point.

7 For $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ we have

$$A^T A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$ ✓

~~Then since this is the centered difference matrix~~ which has eigenvalues given by

$$\begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 2-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} - 1\begin{vmatrix} 1 & 0 \\ 1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)\left((2-\lambda)(1-\lambda) - 1\right) - (1-\lambda) = 0$$

$$\Rightarrow (1-\lambda)\left[\cancel{2} - 3\lambda + \lambda^2 - \cancel{2}\right] = 0$$

$$\Rightarrow \cancel{(1-\lambda)(2 - 3\lambda + \lambda^2) = 0}$$

~~which gives~~ $\cancel{\lambda = 1}$ ~~&~~ $\cancel{\lambda = \frac{3 \pm \sqrt{9-4}}{2} = \frac{3 \pm \sqrt{5}}{2}}$

$$\Rightarrow (1-\lambda)\lambda(\lambda - 3) = 0$$

So $\lambda = 1$, $\lambda = 0$, & $\lambda = 3$

Since ~~the~~ the SVD orders ~~these~~ ~~terms~~ with positive values 1st let $\sigma_1^2 = 3$, $\sigma_2^2 = 1$, Then the corresponding eigvectors are given by For $\sigma_1^2 = 3$ we have

$$\begin{bmatrix} -2 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 1 \\ 0 & 1 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

~~Pivot vars x_1, x_2,~~

~~Free vars x_3.~~

$\therefore$ ~~let $x_3 = 1$ to get~~

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

pivot vars x_1 + x_2, free vars x_3

let $x_3 = 1$ then

$x_1 = 1$ + $x_2 = 2$ $\therefore$ $v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

For $\sigma_1^2 = 1$ we have

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

pivot vars are x_1 + x_2 + free vars are x_3

$\therefore\ v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. For the zero eigenvalue we have

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \therefore \quad v_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Normalizing we have

$$v_1 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad v_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad v_3 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Computing AA^T we have

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Which has eigenvalues given by $\sigma_1^2 = 3$ & $\sigma_2^2 = 1$

The eigenvectors for $\sigma_1^2 = 3$ are given by the nullspace

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{so } u_1 \propto \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow u_1 = \frac{(\pm 1)}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The eigenvectors to $\sigma_2^2 = 1$ are given by the nullspace of

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{so} \quad u_2 \propto \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{or} \quad u_2 = \frac{(\pm 1)}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

~~Thus the SVD of A is given by~~ $\cancel{A = V \Sigma U^T}$ $\cancel{3\times3 \cdot 3\times3 = 3\times2}$

consider the SVD of A given by

$$A = U \Sigma V^T \qquad \text{equivalently}$$

$$AV = U\Sigma$$

$$\Rightarrow A = U \Sigma V^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}^T$$

$2\times2 \cdot 2\times3 \cdot 3\cdot3$

Which we will check by computing the product of the matrices on the right hand side. We have

$$= \begin{bmatrix} \sqrt{3}/\sqrt{2} & 1/\sqrt{2} & 0 \\ \sqrt{3}/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$$

To ~~B~~ determine the signs of u_1 + u_2 consider

$Av_i = \sigma_i u_i$ For $i=1$ we have

$$Av_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} = \begin{bmatrix} 3/\sqrt{6} \\ 3/\sqrt{6} \end{bmatrix} = \sqrt{3} \begin{bmatrix} \sqrt{3}/\sqrt{6} \\ \sqrt{3}/\sqrt{6} \end{bmatrix} = \sqrt{3} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

So $u_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

For $i=2$ we have

$$Av_2 = \sigma_2 u_2$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = 1 \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

So $u_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$.

Then the SVD of A is given by

$A = U \Sigma V^T$ $\quad$ U is 2×2

Σ is 2×3

+ V is 3×3

This gives

$$A = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}^T$$

which we can check by computing the product of these matrices

$$= \begin{bmatrix} \frac{\sqrt{3}}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{\sqrt{3}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}+\frac{1}{2} & 1 & \frac{1}{2}-\frac{1}{2} \\ \frac{1}{2}-\frac{1}{2} & 1 & \frac{1}{2}+\frac{1}{2} \end{bmatrix} \doteq \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \checkmark$$

(8) We desire $Av_1 = u_1$, $Av_2 = v_2$, $\cdots$, $Av_n = u_n$

so the singular values of A are all $\underline{1}$. Thus $\Sigma = \text{diag}(1, 1, \cdots, 1)$ $= I$

~~is~~ the $n \times n$ identity matrix. With this we have

$$A = U\Sigma V^T = UIV^T = UV^T$$

(9) $Av = 12u$ for $v = \frac{1}{2}(1,1,1,1)$ + $u = \frac{1}{3}(2,2,1)$.

Its single value must be 12 so

$$A = U\Sigma V^T = \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}(12)\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$= \frac{12}{6}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= 2\begin{bmatrix} 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \\ 2 & 2 & 2 & 2 \end{bmatrix}$$

(10) If A has orthogonal columns with length $\sigma_1, \ldots, \sigma_n$

Then $Ae_i = \sigma_i w_i$

So the matrix V is the identity $V = I$

$\Sigma = \text{diag}(\sigma_1, \sigma_2, \cdots, \sigma_n)$

So ~~...~~ $U = [w_1, w_2 \cdots, w_n]$

(11) The SVD expresses A as

$$A = U \Sigma V^T$$

The product $U\Sigma$ ~~scales~~ ~~each~~ multiplies each column of U by the corresponding magnitude σ in Σ. ~~The~~ ~~product~~ considering a block matrix factorization of ~~of~~ $U\Sigma$ into columns + V^T into rows (which are the columns of V) we have

$$A = \begin{bmatrix} \sigma_1 u_1 & \sigma_2 u_2 & \cdots & \sigma_n u_n \end{bmatrix} \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_n^T \end{bmatrix}$$

which when we compute this block matrix multiply gives

$$= \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \cdots + \sigma_n u_n v_n^T$$

(12) Since A is symmetric it has the following decomposition

$$A = U \Lambda U^T = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} u_1^T \\ u_2^T \end{bmatrix}$$

Its SVD would then be given by

$$U = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \quad \Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \quad V = \begin{bmatrix} u_1 & -u_2 \end{bmatrix}$$

So that $U\Sigma V^T = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} u_1^T \\ -u_2^T \end{bmatrix} = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} u_1^T \\ u_2^T \end{bmatrix}$

(13) If $A = QR$ then Q is an ~~orthogonal~~ orthonormal basis for the columns of A

~~then let $R = \Sigma V^T$~~

also ~~of~~ the SVD of A requires it to have a decomposition given by $A = U\Sigma V^T$ w/ U an orthonormal basis for the columns of A

~~so $QR = U\Sigma V^T$~~ I would then guess that the ~~matrix~~ matrix Q simply

~~$\Rightarrow R = Q^T U \Sigma V^T$~~ changes the basis ~~of~~ given by the matrix U

in the SVD. ~~so given it~~ in other words given the SVD of $\underline{\underline{R}}$

$R = \hat{U}\hat{\Sigma}\hat{V}^T$ A is then given by

$$A = Q\hat{U}\hat{\Sigma}\hat{V}^T = \cancel{(Q\hat{U})\hat{\Sigma}(v} \; (Q\hat{U})\hat{\Sigma}\hat{V}^T$$

which is the SVD of A!! only the matrix ~~Q~~ "U" changes between the SVD of A and that of R.

(14) To make A singular Add the matrix $-\sigma_2 \cdot I$ to A

or $-\sigma_2 U V^T$ since for invertible matrices $V = U$.

so $A' = \cancel{A - \sigma_2 I} \; A - \sigma_2 I = A - \sigma_2 U U^T$

$$= A - \sigma_2 U V^T$$

$$= U\begin{bmatrix} \sigma_1 & \\ & \sigma_2 \end{bmatrix}V^T - U\begin{bmatrix} -\sigma_2 & \\ & -\sigma_2 \end{bmatrix}V^T$$

$$= U\begin{bmatrix} \sigma_1 - \sigma_2 & 0 \\ 0 & 0 \end{bmatrix}V^T$$

Note σ_2 was chosen since it is the smallest singular value.

(15) (a) If $A \Rightarrow 4A$ the singular values get multiplied by 4 i.e. $\Sigma \Rightarrow 4\Sigma$

(b) If $A = U\Sigma V^T$ from the SVD

$$A^T = V\Sigma^T U^T = V\Sigma U^T$$

If $A = U\Sigma V^T$ from the SVD then

$$A^{-1} = (V^T)^{-1}\Sigma^{-1}U^{-1} = V\Sigma^{-1}U^T$$

(16) If A is square & invertible I believe this is correct. In other cases $I \neq UV^T$ which would be required for this to be true. Another way to see this is that the square of the singular values of $A+I$ should be the eigenvalues of

$$(A+I)^T(A+I) = A^TA + I^TI + A^TI + I^TA$$

$$= A^TA + I + A^T + A$$

$$\neq A^TA + I$$

Which would give ~~singular values~~ eigenvalues ~~of given by~~ $\cancel{\Sigma + I}$ equivalent to $\Sigma + I$.

In short, it is this inequality between $(A+I)^T(A+I)$ & $A^TA + I$ that is the reason