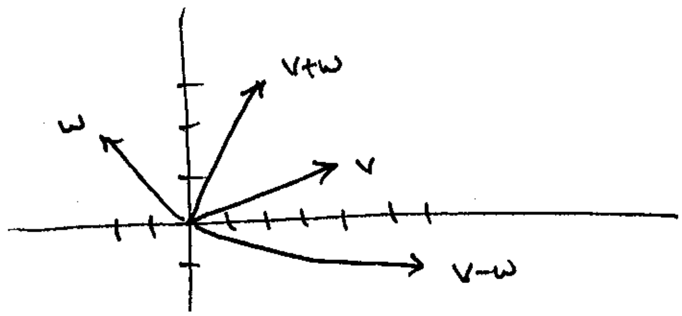
(1)



$v+w = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ $\qquad$ $v-w = \begin{bmatrix} 6 \\ -1 \end{bmatrix}$

(2) if $v+w = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ & $v-w = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

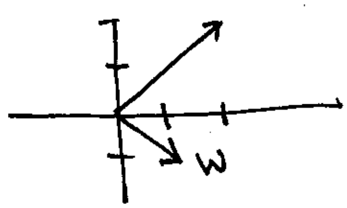
Then $v+w+(v-w) = \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$ By addition of the two vectors

$$2v = \begin{bmatrix} 4 \\ 4 \end{bmatrix} \Rightarrow v = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

in a similar way subtraction yeilds

$$(v+w)-(v-w) = 2w = \begin{bmatrix} 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

So $w = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, then in the xy plane



(3) $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ + $w = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$3v + w = \begin{bmatrix} 3(2)+1 \\ 3(1)+2 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$$

$$v - 3w = \begin{bmatrix} 2-3(1) \\ 1-3(2) \end{bmatrix} = \begin{bmatrix} -1 \\ -5 \end{bmatrix}$$

and $cv + dw = \begin{bmatrix} 2c + d(1) \\ c + 2d \end{bmatrix} = \begin{bmatrix} 2c+d \\ c+2d \end{bmatrix}$

(4) By adding components we have

$$u + v = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$$

$$u + v + w = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

+ $2u + 2v + w = \begin{bmatrix} 2-6+2 \\ 4+2-3 \\ 6-4-1 \end{bmatrix} = \begin{bmatrix} -2 \\ +3 \\ 1 \end{bmatrix}$

(5) $cv + dw = \begin{bmatrix} c \\ -2c+d \\ c-d \end{bmatrix}$

Since the 1st component of $cv+dw$ is 4 we get that

$c=4.$

Then with $c=4$ our linear combination becomes

$$4v + dw = \begin{bmatrix} 4 \\ -8+d \\ 4-d \end{bmatrix}$$

For this vector to equal $(4,2,-6)^T$ we must have

$d=10$

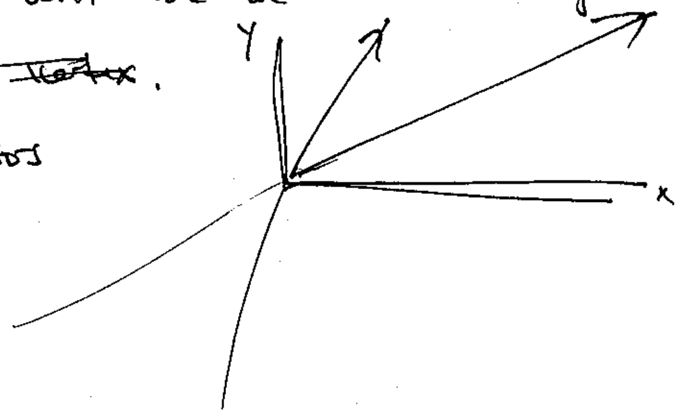
(6) $c\begin{bmatrix} 3 \\ 1 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3c \\ c+d \end{bmatrix}$

For $c=0,1,2$ + $d=0,1,2$ we have the vectors

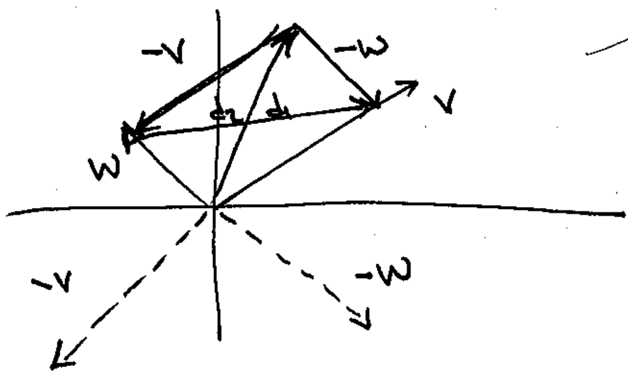
$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 6 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

(7) (a) The sum can be performed in reverse "backwards" along W and forwards along V.

(b) Considering both positive + negative vectors all linear combinations of V + W will be a "~~two triangles~~ "~~wedges~~" ~~connected at their vertex~~.
plane spanned by the two vectors
V + W



(8)



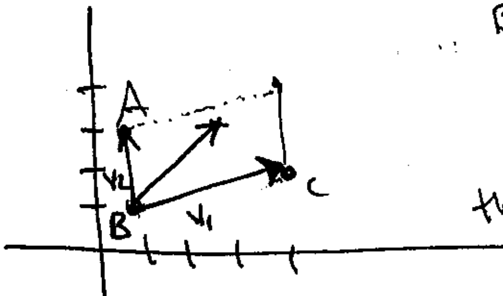
~~$B + V_1 + V_2$~~

$d_1 = V + W$

$d_2 = -V - W$

$d_1 + d_2 = 0$

(9)



let $A = (1,3)$ + $C = (4,2)$
$B = (1,1)$

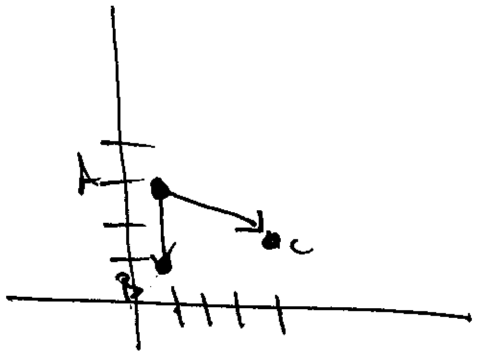
$V_1 = (4,2) - (1,1) = (3,1)$ = vector from B to C

$V_2 = (1,3) - (1,1) = (0,2)$ = vector from B to A

then the diagonal is given by
$V_1 + V_2 = (3,3)$

this points location is given by adding back in r_b giving $d = r_b + (3,3) = (1,1) + (3,3) = (4,4)$

Idea is to pick each of A, B, + C in turn making it the corner of ~~###~~ our parallelogram. For example, Picking B we get



The vector from pt A to pt B is given by

picking A as the corner point we have

$V_1 = r_B - r_A = (1,1) - (1,3) = (0,-2)$

+ the vector from pt A to pt C is given by

$V_2 = r_C - r_A = (4,2) - (1,3) = (3,-1)$

+ diagonal in this case

Thus our 4th point is given by $V_1 + V_2 = (3,-3)$

to which we must add r_A giving $(3,-3) + (1,3) = (4,0)$

Finally picking C as a corner point we have that, the vector from pt C to pt A is given by

$V_1 = r_A - r_C = (1,3) - (4,2) = (-3,1)$

+ the vector from pt C to pt B is given by

$V_2 = r_B - r_C = (1,1) - (4,2) = (-3,-1)$

So our 4th point for this parallelogram is given by

$V_1 + V_2 = (-3,1) + (-3,-1) = (-6,0)$

to which we must add r_C giving

$(4,2) + (-6,0) = (-2,2)$

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(10) $v = (4,2)$
$w = (-1,2)$

So $(\text{length of } v)^2 = 4^2 + 2^2 = 16 + 4 = 20$

+ $(\text{length of } w)^2 = 1 + 4 = 5$

+ $(\text{length of } v+w)^2 = (\text{length of } (3,4))^2 = 9 + 16 = 25$

so Pythagoras theorem holds true in this case

(11) $(\text{length of } v)^2 = 20$ as before

+ $(\text{length of } w)^2 = 5$ as before

$(\text{length of } v-w)^2 = (\text{length of } (5,0))^2 = 5^2 + 0^2 = 25$

To find a case when this fails pick

$v = (1,1)$ + $w = \cancel{(3,0)}\ (1,0)$

Then $(\text{length of } v)^2 = 2$
$(\text{length of } w)^2 = 1$

~~So~~ + $(\text{length } v-w)^2 = (\text{length } (0,1))^2 = 1 \neq \cancel{3}\ 2+1=3$

(12) As in Problem 11 let $v = (1,1)$ + $w = (1,0)$

Then $(\text{length } v)^2 = 2$, $(\text{length } w)^2 = 1$

+ $(\text{length}(v+w))^2 = (\text{length}(2,1))^2 = 4+1 = 5.$

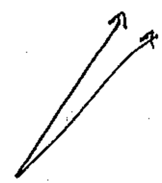
(13) $(\text{length } v)^2 = \sqrt{16+4} = \sqrt{20}$ (From

$(\text{length } w)^2 = \sqrt{5}$

$(\text{length}(v+w))^2 = \sqrt{25} = 5$

So is $\sqrt{20} + \sqrt{5} \overset{?}{\geq} 5$

$\Leftrightarrow$ $\cdot\cdot + \cdot\cdot \overset{?}{\geq} 5$ yes



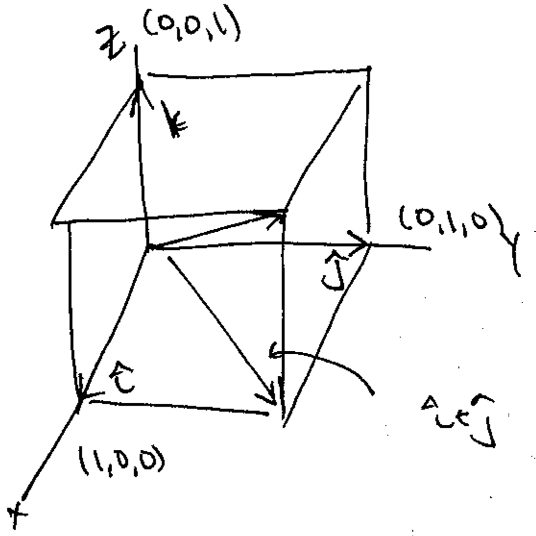
if v + w are <u>parallel</u>

Then $v + w = 2v$

+ the Δ inequality becomes

$$\underbrace{\|v\| + \|v\|}_{2\|v\|} \geq \|2v\| = 2\|v\|$$

(14)



$\hat{i}+\hat{j}+\hat{k}$ is the diagonal of the entire cube

$\hat{i}+\hat{j}$ is the diagonal of the bottom face

(15)

Take linear combinations of $\hat{i}, \hat{j}, \hat{k}$

i.e. consider

$\vec{r} = n\hat{i} + m\hat{j} + l\hat{k}$ ~~for $n,m,l = 0,1$~~

w/ $n=0,1$ $m=0,1$ + $l=0,1$

	n	m	l	result $\vec{r}$
corner →	0	0	0	(0,0,0)
	0	0	1	(0,0,1)
corner →	0	1	0	(0,1,0)
	0	1	1	(0,1,1)
corner →	1	0	0	(1,0,0)
	1	0	1	(1,0,1)
	1	1	0	(1,1,0)
	1	1	1	(1,1,1)

The other points are the other corners

The center-point has coordinates given by

$$\vec{c} = \frac{1}{8}\sum_{n,m,l=\{0,1\}} n\hat{i}+m\hat{j}+l\hat{k} =$$

$$= \frac{1}{8}(4,4,4) = \left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$$

The faces have coordinates given by all combination like

or

n	m	l	result $\vec{f}$
0	0	0	

$\left(\frac{1}{2},\frac{1}{2},0\right)$ $\left(\frac{1}{2},1,\frac{1}{2}\right)$ $\left(0,\frac{1}{2},\frac{1}{2}\right)$

$\left(\frac{1}{2},\frac{1}{2},1\right)$ $\left(\frac{1}{2},0,\frac{1}{2}\right)$ $\left(1,\frac{1}{2},\frac{1}{2}\right)$

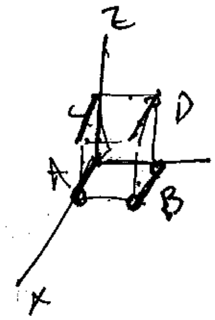
(16) From a typical corner of (0,1,0,0) we can generalize problem 15 to be that the corners of a cube in 4 dimensions are given by

~~note:~~

$$n(1,0,0,0) + m(0,1,0,0) + l(0,0,1,0) + p(0,0,0,1)$$

for all possible indices $n, m, l, + p \in \{0,1\}$

Thus since $n - p$ can take on 2 possible states the total # of such vectors (# of the vertices in 4-dimension) is given by



$2^4 = 16$ vertices.

A ~~the~~ faces in ~~3~~ 3 dimensions is ~~given~~ ~~by~~ defined by all ~~[illegible]~~ but one element of an 3 ~~tuple~~ tuple defined. For example $(n,m,0)$ + $(n,m,1)$ for $n,m = \{0,1\}$ are the top + bottom faces respectively

Thus each component has 2 ~~positi~~ possible values while

The other can take on any #. Thus the total # of faces is

$$2(\#\text{ components in vector}) = 2(4) = 8$$

The # of edges are determined by specifying two consecutive elements of our 4 vector

thus

$(1,0,0)$ + $(0,0,0)$ join one edge

Thus we can pick the component ~~element~~ where the edge will occur in 4 ways + then we have $2^3 = 8$ addit. so 32 edges

A $(1,0,0)$ $(0,0,0)$

C $(1,0,1)$ $(0,0,1)$

B $(1,1,0)$ $(0,1,0)$

D $(1,1,1)$ $(0,1,1)$

Check $d=3$

$$3\cdot 2^2 = 3\cdot 4 = 12 \checkmark$$

Thus $e=32$ $v=16$.

$f=8$ + 3 dim sides

Does this satisfy Eulers identity?

J

(17)

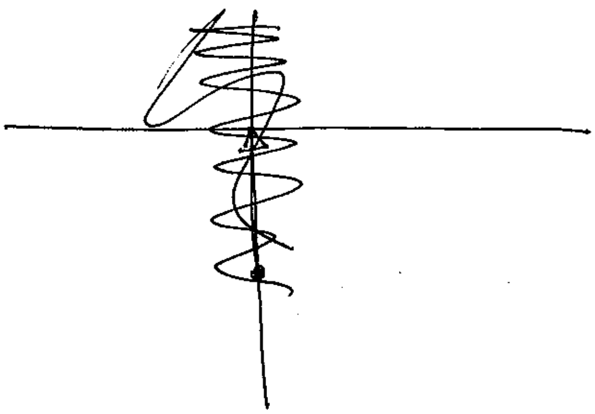
(a) Given every vector can be paired with a negative vector the sum must be zero

(b) $$\sum_{i\neq 4} v_i = \sum_{i\ \text{all}} v_i - v_4 = 0 - v_4 \leftarrow -$$

w/ v_4 the 4:00 vector

(c) $-\frac{v_1}{2}$

(18)



~~The sum shifted is still zero~~

$$\sum_{i\in\{1,2,\dots,12\}} v_i = 0$$

Add 12 copies of $\underbrace{(0,-1)}_{=-\hat{j}}$ to each vector

$$\sum_{i\in\{1,2,\dots\}} (v_i - \hat{j}) = -12\hat{j}$$

but now I set the vector 6:00 to zero &
the vector 12:00 is doubled

$$\sum_{i\in\{1,2,\dots,12\}} (v_i - \hat{j}) = \sum_{\substack{i\in\{1,2\dots 11\} \\ i\notin 6}} (v_i - \hat{j}) + (v_6 - \hat{j}) + (v_{12} - \hat{j}) = -12\hat{j}$$

$$\Rightarrow \sum_{i\in\{1,2,\dots,11\}} (v_i - \hat{j}) - \hat{j} + 2v_{12} - \hat{j} - v_{12} = -v_6 - 12\hat{j}$$

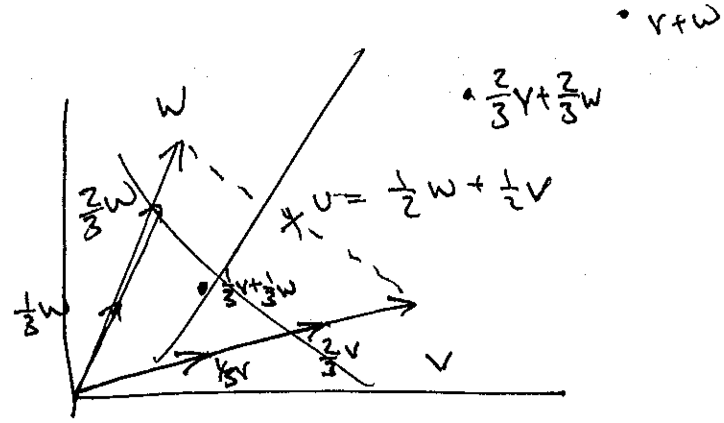
$$= \sum_{i\in\{1,2,3,\dots,11\}} (v_i - \hat{j}) + (0 - \hat{j}) + (2v_{12} - \hat{j}) = v_{12} - v_6 - 12\hat{j}$$

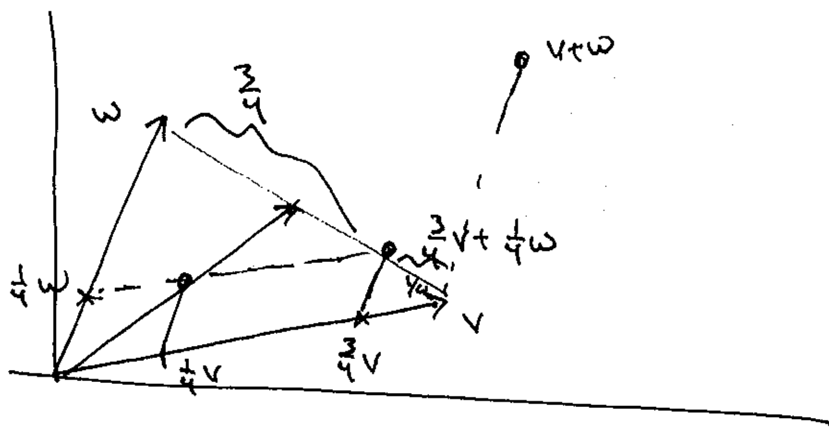
$$= (0,1) - (0,-1) - 12(0,1)$$

$$= -11(0,1) + (0,1)$$

$$= -10(0,1)$$

(19)

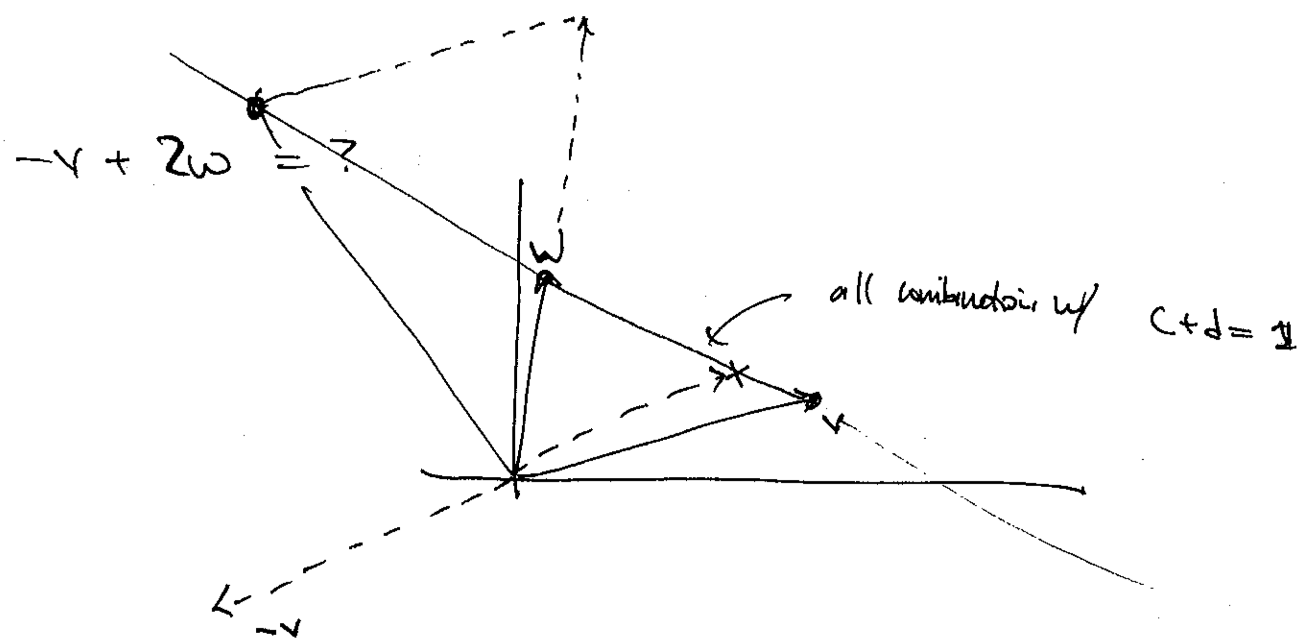




Since $u = \frac{1}{2}v + \frac{1}{2}w$

$\frac{1}{4}v + \frac{1}{4}w = \frac{1}{2}u$

(20)

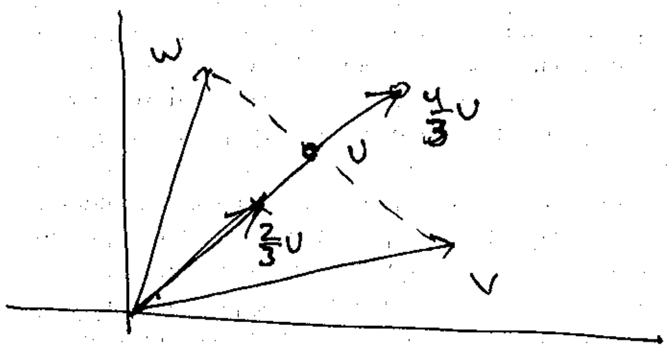


$\frac{1}{4}w + \frac{3}{4}v$

B

(21) $\frac{1}{3}v + \frac{1}{3}w = \frac{1}{3}\cdot 2\left(\frac{1}{2}v + \frac{1}{2}w\right) = \frac{2}{3}u$

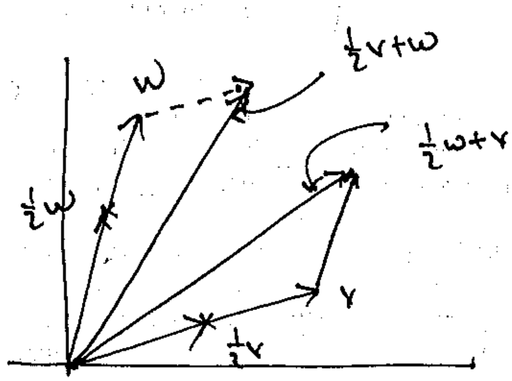
$\frac{2}{3}v + \frac{2}{3}w = \frac{2}{3}\cdot 2\left(\frac{1}{2}v + \frac{1}{2}w\right) = \frac{4}{3}u$



$cv + cw = 2c(u)$ or parallel (through the vector u)

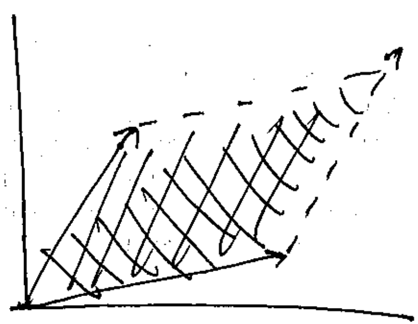
$c > 0$ means this is the positive only rays

(22) $\frac{1}{2}v + w$



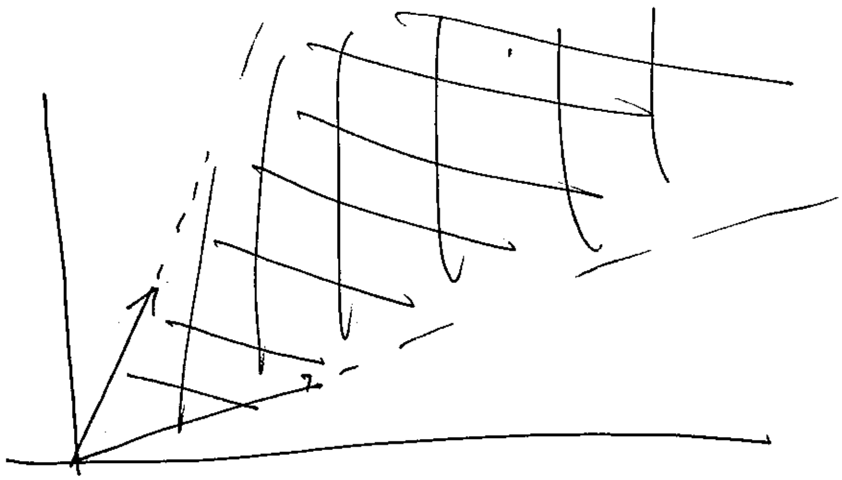
(a) Restricted by $0 \le c \le 1$ & $0 \le d \le 1$

we have the ~~wedge~~ trapezoid shown



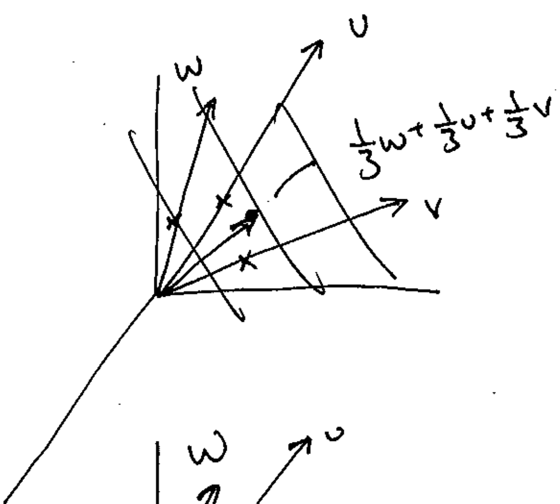
(b) Restricted by $0 \le c$ & $0 \le d$

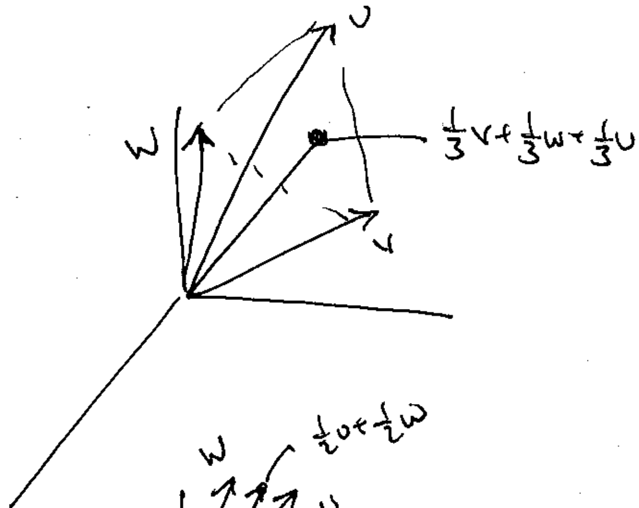
The cone looks like



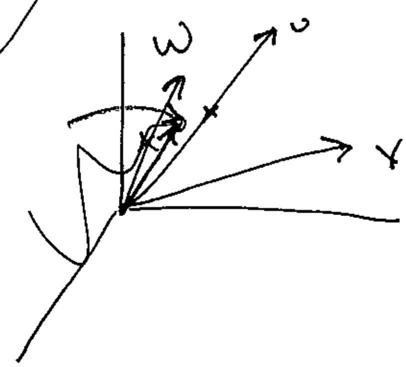
(23)

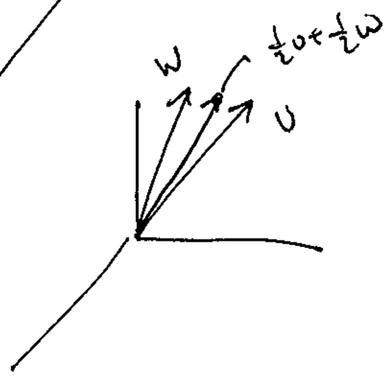
(a)



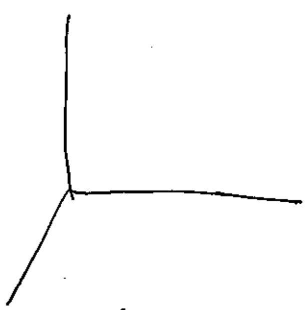


$\frac{1}{2}u+\frac{1}{2}w$





(b)



~~$(cu+dv+ew)\circ(v-u)=0$~~

~~$(cu+dv+ew)\circ(w-u)=0$~~

~~$(cu+dv+ew)\circ($~~

Guess $c+d+e=1$ See next pg

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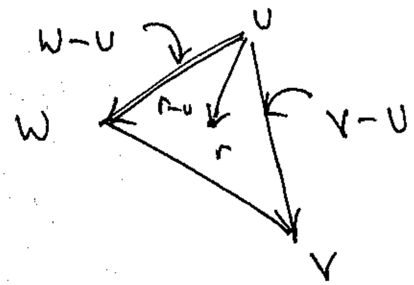
(23) (b) For $cu + dv + ew$ to be on the plane connecting $u, v,$ + w we must have...

To be on the plane connecting $u, v,$ + w we must have



$(r - u) \circ \vec{N} = 0$

w/ $\hat{n}$ a vector normal to the plane.

A vector normal to the plane is given by

$\vec{N} = (v - u) \times (w - u) = v \times w - v \times u - u \times w.$

$=$ (since $u \times u = 0$)

Now $(\vec{r} - \vec{u}) \circ \vec{N} = 0$

when $\vec{r} = cu + dv + ew$ is given by

$$((1-c)u + dv + ew) \circ (v \times w - v \times u - u \times w) = 0$$

Which is equivalent to

$$(1-c)\, u \circ v \times w + (1-c)\, u \circ v \times u - (1-c)\, u \circ u \times w$$

$$+ d v \circ v \times w - d v \circ v \times w - d v \circ u \times w$$

$$+ e w \circ v \times w - e w \circ v \times u - e w \circ u \times w = 0$$

But the following triple products are zero

This is due to the fact that $v \times w$ is a vector $\perp$ to v & w $\therefore$ $v \circ (v \times w) = 0$ & $w \circ (v \times w) = 0$

Thus we get

$$(1-c)\, u \circ v \times w - d v \circ u \times w - e w \circ v \times u = 0$$

But $v \circ u \times w = w \circ v \times u = - u \circ v \times w$ & the above

becomes $u \circ v \times w (1 - c - d - e) = 0$

So either $u \circ v \times w = 0$ or

$1-c-d-e=0$ but $u \cdot v \times w \neq 0$ or else the vectors are degenerate in 3 space & $\therefore$

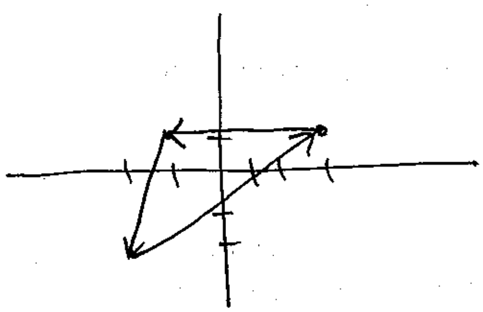
$$1-c-d-e=0 \quad \text{or}$$

$$1=c+d+e$$

To be inside the plane we must also have $c \geq 0$, $d \geq 0$ & $e \geq 0$

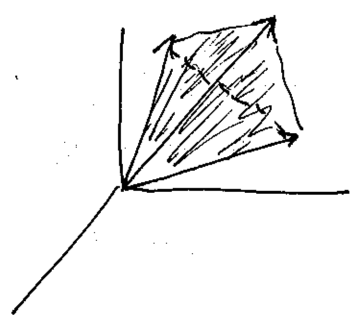
(24)

$$v-u+(w-v)+(u-w)=0$$



(25) $cu+dv+ew$ w/ $c \geq 0$, $d \geq 0$, $e \geq 0$

& $c+d+e \leq 1$



The vector

$$\tfrac{1}{2}u+\tfrac{1}{2}v+\tfrac{1}{2}w$$

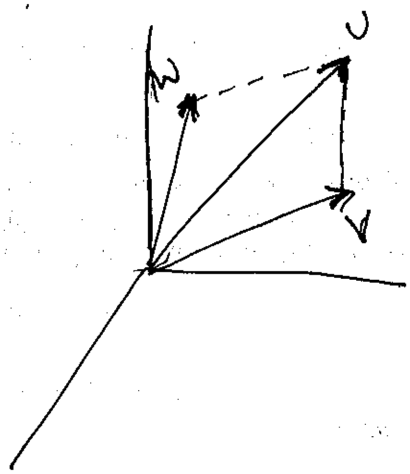
$= \tfrac{1}{2}(u+v+w)$ is in the same direction & of greater magnitude than $\frac{1}{3}(u+v+w)$ which is at the centroid of the Δ formed by the 3 points $u, v, + w$.

$\therefore$ it is outside the pyramid

~~in addition~~ because $\frac{1}{2}+\frac{1}{2}+\frac{1}{2} = \frac{3}{2} > 1$

(26) Not if they are not degenerate

(27)



~~All vectors~~ ~~$(u + dv + ew - w)$~~

~~the constraint that $1 = c + d + e$~~

Since the plane spanned by $u + v$

+ the plane spanned by $v + w$ intersect

in the line $\underline{\underline{v}}$ all vectors cv will

be in both planes

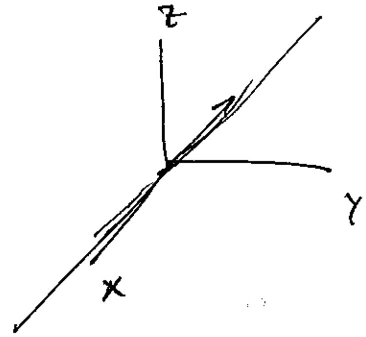
(28)

(a) pick 3 vectors colinear

like $U = (1,1,1)$

$V = (2,2,2)$

$W = (3,3,3)$

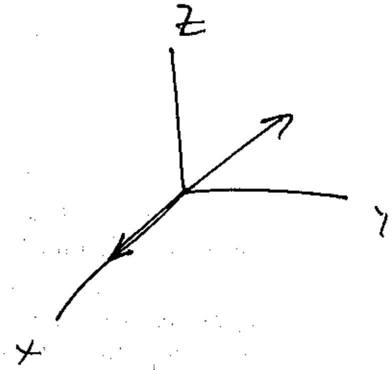


(b) Pick 2 vectors colinear w/ each other & the 3rd Not colinear with the 1st 2

$U = (1,1,1)$

$V = (2,2,2)$

$W = \cancel{(-1,-1,-1)}\ (1,0,0)$



(29) let c & d such that

$$c\begin{bmatrix}1\\2\end{bmatrix} + d\begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}14\\8\end{bmatrix}$$

$$\Rightarrow \quad \begin{aligned} c + 3d &= 14 \\ 2c + d &= 8 \end{aligned}$$

From High school math $\quad d = 8 - 2c$

So $\quad c + 3(8 - 2c) = 14$

$\Rightarrow \quad 24 + c - 6c = 14$

$\Rightarrow \quad 10 - 5c = 0 \quad \Rightarrow \quad c = 2$

Then $\quad d = 8 - 10 = -2$.

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(1)

$u \circ v = -.6(3) + .8(4) = \cancel{.36}\ \cancel{5.6}\ \cancel{-.24} - \cancel{24 + 3.2 = .8}$

$-1.8 + 3.2 = 1.4$

$u \circ w = -.6(4) + .8(3) = 0$

$v \circ w = 3(4) + 4(3) = 24$

$w \circ v = 24$

(2)

$\|u\| = \sqrt{.6^2 + .8^2} = \sqrt{.36 + .64} = 1$

$\|v\| = \sqrt{3^2 + 4^2} = 5$

$\|w\| = \sqrt{16 + 9} = 5$

$|u \circ v| = \cancel{5.6}\ \cancel{.8}\ 1.4 \overset{?}{\leq} \|u\| \cdot \|v\| = 1 \cdot 5$ yes

$|v \circ w| \overset{?}{\leq} \|v\| \cdot \|w\|$

$24 \leq 5^2 = 25$ yes

(3)

$\hat{v} = \dfrac{v}{\|v\|} = \dfrac{(3,4)}{5} = \left(\frac{3}{5}, \frac{4}{5}\right)$

$\hat{w} = \dfrac{w}{\|w\|} = \dfrac{(4,3)}{5} = \left(\frac{4}{5}, \frac{3}{5}\right)$

$\theta = \operatorname{acos}(\hat{v} \circ \hat{w}) = \operatorname{acos}\left(\frac{12}{25} + \frac{12}{25}\right) = \operatorname{acos}\left(\frac{24}{25}\right) = 16.26^\circ$

~~Which from~~

4 $u_1 = \frac{v}{\|v\|} = \frac{(3,1)}{\sqrt{9+1}} = \left(\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right)$

$u_2 = \frac{w}{\|w\|} = \frac{(2,1,2)}{\sqrt{4+1+4}} = \frac{(2,1,2)}{3} = \left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$

For v_1 to be $\perp$ to u_1 we must have

$v_1 \circ u_1 = 0$

$\Rightarrow$ $v_1 \circ (3,1)$ pick $\vec{v}_1 = (-1,3)$

Then a unit vector is given by $\hat{v}_1 = \frac{(-1,3)}{\sqrt{10}}$

For v_2 to be $\perp$ to w

$\Rightarrow$ $v_2 \circ (2,1,2) = 0$

Pick $v_2 = (-1,0,1)$ then a unit vector

is given by $\hat{v}_2 = \left(\frac{-1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$

(5) (a) $\cos\theta = v\circ v = 1$

$\Rightarrow \theta = 0$ ~~or~~

(b) $\cos\theta = w\circ(-w) = -1$

$\Rightarrow \theta = \pi$

(c) $\cos\theta = (v+w)\circ(v-w)$

$= v\circ v - w\circ w = 1-1 = 0$

$\Rightarrow \theta = \frac{\pi}{2}$

(6) (a) $v + w \Rightarrow \cos\theta = \frac{v}{\|v\|}\circ\frac{w}{\|w\|} = \frac{(1,\sqrt{3})}{2}\circ(1,0)$

$= \frac{1}{2}$

$\Rightarrow \theta = \frac{\pi}{3}$

(b) $\hat{v} = \frac{v}{\|v\|} = \frac{(2,2,-1)}{\sqrt{4+4+1}} = \frac{1}{3}(2,2,-1)$

$\hat{w} = \frac{w}{\|w\|} = \frac{(2,-1,2)}{\sqrt{9}} = \frac{1}{3}(2,-1,2)$

$\hat{v}\circ\hat{w} = \frac{1}{9}(4-2-2) = \cancel{\frac{4}{9}}\ 0$

$\therefore \theta = \cancel{\cos^{-1}(\frac{4}{9})}\ \cos^{-1}(0) \Rightarrow \theta = 90 = \frac{\pi}{2}$

(c) $\hat{v} = \frac{v}{\|v\|} = \frac{1}{2}(1, \sqrt{3})$

$\hat{w} = \frac{w}{\|w\|} = \frac{1}{2}(-1, \sqrt{3})$

$\hat{v} \circ \hat{w} = \frac{1}{4}(-1+3) = \frac{1}{2}$

$\theta = \cos^{-1}(1/2) = \frac{\pi}{3}$

(d) $\hat{v} = \frac{(3,1)}{\sqrt{10}}$ $\quad \hat{w} = \frac{(-1,-2)}{\sqrt{5}}$

$\hat{v} \circ \hat{w} = \frac{1}{\sqrt{50}}(-3-2) = -\frac{5}{\sqrt{50}} = -\frac{\sqrt{5^2}}{\sqrt{50}} = -\frac{\sqrt{5}}{\sqrt{10}}$

$= -\frac{1}{\sqrt{2}}$

$\Rightarrow \theta = \cancel{\frac{\pi}{4}} \; \frac{3\pi}{4}$

(7) (a) To be $\perp$ to $v = (2,-1)$ we must have

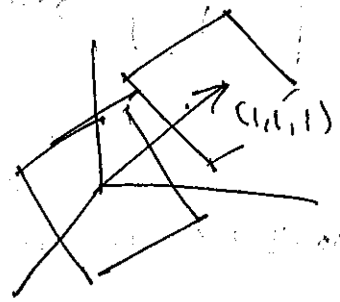
$(w_1, w_2) \circ (2,-1) = 0$

or ~~so~~ $2w_1 - w_2 = 0 \Rightarrow w_2 = 2w_1$

So $w = (w_1, 2w_1) = w_1(1,2)$

$\therefore$ all vectors $\propto (1,2)$ will be $\perp$ to $(2,-1)$

(b) They must be in a plane diagonal to the vector $(1,1,1)$



(8) (a) u is $\perp$ to v & w then v + w are $\parallel$.

False let $u = (1,0,0)$

$v = (0,1,0)$

& $w = (0,0,1)$

Then $u \perp v$ & $u \perp w$ but v & w are Not $\parallel$

(b) $u \perp$ to v & w Then u is $\perp$ to $v + 2w$.

Yes $u \perp v$ & $w \Rightarrow$ $u \circ v = 0$

& $u \circ w = 0$

$\therefore$ $u \circ (v + 2w) = u \circ v + 2(u \circ w) = 0 + 0 = 0$

(c) ~~combo~~ There is always a combination of $v + cu$ That is $\perp$ to u. ~~False~~ True to find such an element require to be true $\exists$ a c

$\ni$ $u \circ (v + cu) = 0 \Rightarrow u \circ v + c(u \circ u) = 0$

$\Rightarrow c = -\left(\frac{u \circ v}{u \circ u}\right)$

(9) If the product $\dfrac{V_2 W_2}{V_1 W_1} = -1$

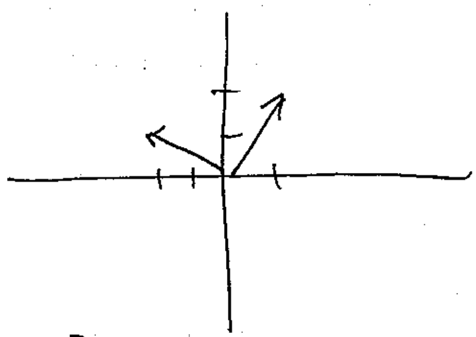
Then $V_2 W_2 = -V_1 W_1$

$\Rightarrow V_2 W_2 + V_1 W_1 = 0$

$\Rightarrow (V_1, V_2) \circ (W_1, W_2) = 0$

$\Rightarrow$ The vectors are $\perp$.

(10)



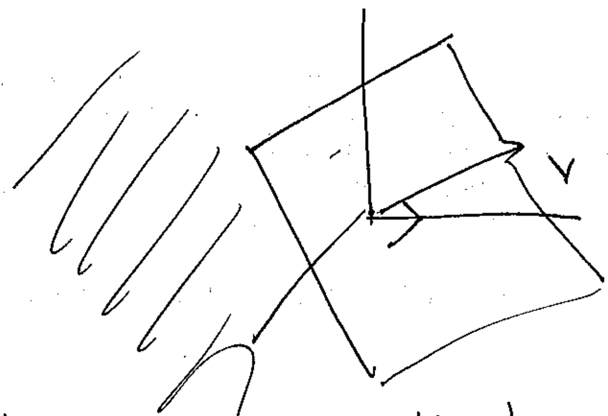
$S_1 = \frac{2}{1} = 2$

$S_2 = -\frac{1}{2}$

$S_1 \cdot S_2 = -1$

perpendicular

(11) $v \circ w < 0 \Rightarrow$ angle between v & w is between $\frac{\pi}{2}$ & π.



All vectors behind the plane shown will have a negative dot product w/ v.

(12) For $w - cv$ to be $\perp$ to v we must have

$$(w - cv) \circ v = 0$$

$$\Rightarrow w \circ v - c(v \circ v) = 0 \quad \Rightarrow \quad c = \frac{w \circ v}{v \circ v}$$

with the vector values given $c = \dfrac{(1,5)\circ(1,1)}{(1,1)\circ(1,1)} = \dfrac{6}{2} = 3$

So that $w - cv = (1,5) - 3(1,1) = (-2,2)$

(13) v & w must be $\perp$ to $(1,1,1)$ & each other

let $v = (-1,1,0)$ then $v \circ (1,1,1) = 0$

So w must be $\perp$ to v & to $(1,1,1)$

~~Let~~ ~~WE~~ w must be $\perp$ to $(-1,1,0)$ + $(1,1,1)$

let $w=(a,b,c)$ Then

$(-1,1,0)\circ w=0$

$\Rightarrow -a+b=0$

+ $(1,1,1)\circ w=0$

$\Rightarrow a+b+c=0$

let $c=1$ then $\quad -a+b=0$
$\quad a+b=-1$

$\Rightarrow a=b$ + $2b=-1 \Rightarrow b=\frac{-1}{2}$

So $a=\frac{-1}{2}$

$\therefore w=\left(\frac{-1}{2},\frac{-1}{2},1\right)$ will work.

(14) Find 3 vectors $\perp$ to $(1,1,1,1)$ + each other
u,v,w

let $u=(-1,1,0,0)$
+ $v=(0,0,1,-1)$

Then $u\circ(1,1,1,1)=0 \quad v\circ(1,1,1,1)=0$ + $u\circ v=0$

let $w = (a, b, c, d)$. For w to be $\perp$ to all 3 vectors thus far we must have

$$a+b+c+d = 0$$

$$+ \quad -a+b = 0 \quad \Rightarrow \quad a=b$$

$$c-d=0 \quad \Rightarrow \quad c=d$$

pick $d=3$ Then $c=3$

$2b+b=0 \Rightarrow b = \cancel{3}\ -3$ + $\cancel{a=3}$ $a=-3$

So $\cancel{w=(3,3,3,3)}$ $w = (-3,-3,3,3)$ is a vector that satisfies all the requirements.

(15) $\frac{1}{2}(x+y) = 5 \qquad \sqrt{xy} \le \frac{1}{2}(x+y)$

Schwarz:

$$|v \circ w| \le \|v\| \cdot \|w\|$$

$$\cos\theta = \frac{(\sqrt{2}, \sqrt{8})}{\sqrt{2+8}} \circ \frac{(\sqrt{8}, \sqrt{2})}{\sqrt{2+8}} = \frac{2(\sqrt{16})}{10} = \frac{8}{10} = \frac{2}{5}$$

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(15) $\frac{1}{2}(x+y) = 5$

$$\cos\theta = \frac{v}{\|v\|} \cdot \frac{w}{\|w\|} = \frac{(\sqrt{2},\sqrt{8})}{\sqrt{10}} \cdot \frac{(\sqrt{8},\sqrt{2})}{\sqrt{10}} = \frac{2\sqrt{16}}{10} = \frac{4}{5}$$

$$\theta = \cos^{-1}\left(\frac{4}{5}\right)$$

(16) $\|v\|_2 = \sqrt{1^2(9)} = 3$

$$u = \frac{v}{\|v\|} = \left(\frac{1}{3}, \frac{1}{3}, \dots, \frac{1}{3}\right)$$

let $w = (-1, 1, 0, \dots, 0)$

(17) $\cos\alpha = \frac{(1,0,-1)}{\sqrt{2}} \cdot (1,0,0) = \frac{1}{\sqrt{2}} \Rightarrow \alpha = \frac{\pi}{4}$

$\cos\beta = \frac{(1,0,-1)}{\sqrt{2}} \cdot (0,1,0) = 0 \Rightarrow \beta = \frac{\pi}{2}$

$\cos\theta = \frac{(1,0,-1)}{\sqrt{2}} \cdot (0,0,1) = \frac{-1}{\sqrt{2}} \Rightarrow \theta = \cancel{\frac{\pi}{4}}\ \frac{3\pi}{4}$

$\cos^2\alpha + \cos^2\beta + \cos^2\theta = \frac{1}{2} + \frac{1}{2} = 1$ ✓

(18) (1) $v \circ w = w \circ v$ multiply components + show

(2) ~~multi~~

$u \circ (v+w) = u \circ v + u \circ w$ multiply compont

(3) $(cv) \cdot w = c(v \circ w)$ multiply compont

$u = v + w$

$\|v+w\|^2 = (v+w) \circ (v+w) = v \circ v + v \circ w + w \circ v + w \circ w$

$= \quad \ldots$

(19) $\|v+w\|^2 = \|v\|^2 + 2\, v \circ w + \|w\|^2$

So $\|v+w\|^2 \le \|v\|^2 + 2|v \circ w| + \|w\|^2 \le \|v\|^2 + 2\|v\| \cdot \|w\| + \|w\|^2$

$= (\|v\| + \|w\|)^2$

or $\|v+w\| \le \|v\| + \|w\|$

(20) $\|v\|^2 + \|w\|^2 = \|v+w\|^2$

is equivalent to

$\|v\|^2 + \|w\|^2 = \|v\|^2 + 2 v \circ w + \|w\|^2$

or $\quad 0 = 2 v \circ w$

or $\quad 0 = v \circ w$

or $\quad 0 = v_1 w_1 + v_2 w_2 + v_3 w_3$

(21) $\cancel{\cos\alpha = \frac{v_1}{\|v\|}}$

$$\cos\beta = \frac{w_1}{\|w\|} \quad + \quad \sin\beta = \frac{w_2}{\|w\|}$$

$$\theta = \beta - \alpha$$

$$\cos\theta = \cos(\beta - \alpha) = \cos(\beta)\cos(\alpha) + \sin(\beta)\sin(\alpha)$$

$$= \frac{w_1}{\|w\|} \cdot \frac{v_1}{\|v\|} + \frac{w_2}{\|w\|} \cdot \frac{v_2}{\|v\|}$$

$$= \frac{v \circ w}{\|w\| \cdot \|v\|}$$ As requested

(22) $$\|v - w\|^2 = \|v\|^2 - 2\|v\| \cdot \|w\| \cos\theta + \|w\|^2$$

$$\underset{\shortparallel}{(v-w)\circ(v-w)} = \|v-w\|^2$$

$$\|v\|^2 - 2 v \circ w + \|w\|^2 = \|v\|^2 - 2\|v\| \cdot \|w\| \cos\theta + \|w\|^2$$

The ~~chance~~ gives

$$\cancel{v \circ w} \quad \cos\theta = \frac{v \circ w}{\|v\| \cdot \|w\|}$$

(23) (a) ~~Multiply~~ o

The left hand side is

$$\cancel{v_1^2 w_1^2} + 2v_1w_1v_2w_2 + \cancel{v_2^2 w_2^2} \le$$

The RHS is

$$\cancel{v_1^2 w_1^2} + v_1^2 w_2^2 + v_2^2 w_1^2 + \cancel{v_2^2 w_2^2}$$

Subtracting gives

$$v_1^2 w_2^2 - 2v_1w_1v_2w_2 + v_2^2 w_1^2 \overset{?}{\geq} 0$$

$$(v_1w_2 - v_2w_1)^2 \geq 0 \qquad \text{yes}$$

(24) $\cancel{|u \circ v| \le} 1$

Going from $|u_1|\cdot|v_2| \le \frac{1}{2}(u_1^2+v_2^2)$ is the step that uses example 6.

$(u_1, u_2) = (.6, .8)$ & $(v_1, v_2) = (.8, .6)$

$\cancel{u \circ v = .6(.8)+}$

1st check $\|u\| \overset{?}{=} 1$ $\quad \|v\| = 1$

$\|u\| = \sqrt{.6^2+.8^2}$ $\quad \|v\| = \sqrt{.8^2+.6^2} =$

Then $u \circ v = .6(.8) + .8(.6) = 2(.48) = 1 - 2(.02) = 1 - .04$

$= .96 = \cos\theta$

$|u \circ v| = .96 \le 1$ yes

(25) $\cos\theta$ is the x ~~coord~~ coordinate of a point on the unit circle & ~~thus~~ satisfies

$\cancel{\cos^2\theta + \sin^2\theta = 1}$ this value is always bounded

~~since~~ $\cancel{|\sin^2\theta| \to} 0$ by 1

(27) $||v|| = 5$ + $||w|| = 3$

$v \circ w$ will be largest or smallest when v + w are parallel or antiparallel respectivi. This would give

$v \circ w = 3 \cdot 5\,(+1) = 15$

+ $v \circ w = 3 \cdot 5\,(-1) = -15$

The largest + smallest of $||v-w||$ will be the same location as the largest + smallest of $||v-w||^2$

$||v-w||^2 = ||v||^2 - 2v \circ w + ||w||^2$

~~so~~

$||v-w||^2 = 25 + 9 - 2v \circ w$

if v + w are $\parallel$ we get the smallest

$||v-w||^2 = 25 + 9 - 2(5)(3) = 25 + 9 - 30$

$= 4$

~~The~~ If the are anti parallel we get the ~~sp~~ largest

$||v-w||^2 = 34 + 2(5)(3) = 34 + 30 = 64.$

(26) let $v = (x, y, z)$
$w = (z, x, y)$

Then $v \circ w = xz + xy + yz$

but if $z = -x - y$. The dot becomes

$$v \circ w = x(-x-y) + xy + y(-x-y) = -x^2 - xy - y^2$$

But $\|w\| = \|v\|$

So
$$\frac{v \cdot w}{\|w\| \cdot \|v\|} = \frac{v \circ w}{\|w\|^2} = \frac{-x^2 - xy - y^2}{x^2 + y^2 + z^2}$$

$$= \frac{-x^2 - xy - y^2}{x^2 + y^2 + (-x-y)^2} = \frac{-x^2 - xy - y^2}{x^2 + y^2 + x^2 + 2xy + y^2}$$

$$= -\left(\frac{x^2 + xy + y^2}{2(x^2 + y^2 + xy)}\right) = \frac{-1}{2}$$

The angle between v & w must be $\frac{3\pi}{4}$.

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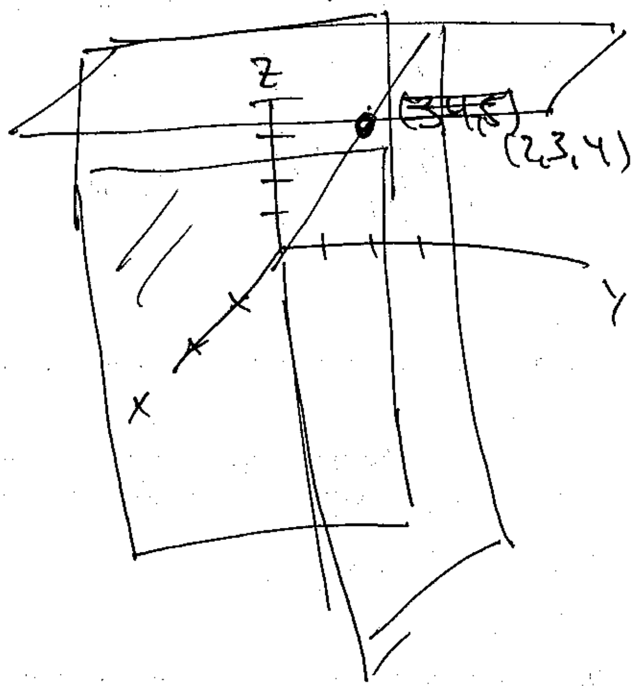
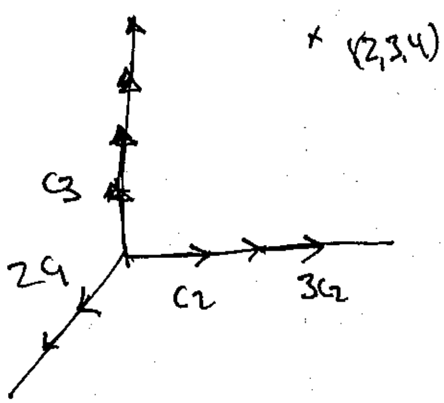
(1) The row picture involves drawing ~~each~~ the hyper-plane represented by each row & looking for the point of intersection if any. For this example we have



(2) The 3 columns in question are

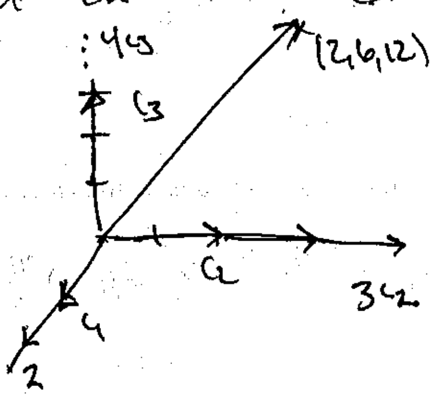
$$c_1 \; \begin{array}{ccc} c_1 & c_2 & c_3 \\ \begin{bmatrix}1\\0\\0\end{bmatrix} & \begin{bmatrix}0\\1\\0\end{bmatrix} & \begin{bmatrix}0\\0\\1\end{bmatrix} \end{array}$$

which produces



(3) The row picture is the same since the ~~planes~~ hyperplanes represented by each row have not changed. ~~The solution is~~. If the row picture has not changed the solution is still the same.

The column picture has change in that now our columns are pointing in the same directions as before but are at a ~~~~ different magnitude.



$$c_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$c_2 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

$$c_3 = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

+ of course the target vector b is different such that the solution for (x, y, z) stays the same

(4) With the new equations we have

$$x = 2$$
$$x + y = 5$$
$$z = 4$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 4 \end{bmatrix}$$

So the row picture ~~[illegible]~~ would be changed since the 2nd row now represents a different plane.
The column picture is also changed since now the columns are different than before

$$c_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad c_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad + \quad c_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$ Actually only c_1 has changed.

Correspondingly the coefficient matrix did change (only row 2)
The solution however did not change.

(5) If $x + y + 3z = 6$
& $x - y + z = 4$

Adding these two eqs gives $2x + 4z = 10$

or $x + 2z = 5$

let $x = 1$ & $z = 2$ then $y = x + z - 4 = 3 - 4 = -1$

so point $(1, -1, 2)$ is a point on this line

another point is given by $z = -2$, $x = 9$,

$y = 9 - 2 - 4 = 3$

or $(9, 3, -2)$

(6) * Satisfy the 3rd

3 solutions must satisfy

$$x + y + z = 2$$
$$x + 2y + z = 3$$

~~let~~ Subtract 1st from the 2nd

$y = 1$ which in each equation gives

~~x + z = 2~~ $x + z = 1$

$x + z$ $x + z = 1$

Thus pick 3 values of x calculate 3 values of z from $z = 1 - x$ + we have our 3 points

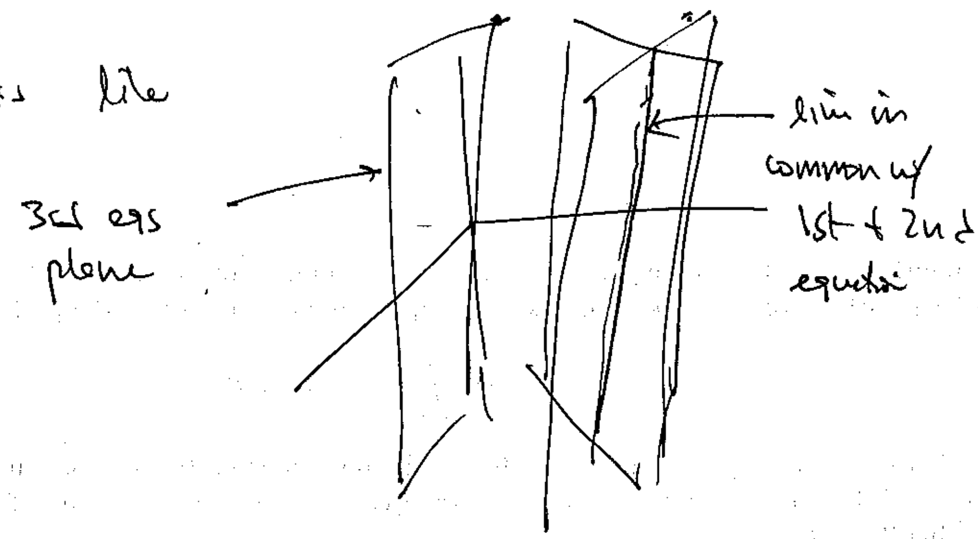
(7) ~~Something~~ ~~Other~~ Then 3 eqs would be

$$x + y + z = 2$$
$$x + 2y + z = 3$$
$$2x + 3y + 2z = 9$$

Then eq 1 + eq 2 − eq 3 would require

$0 = 4$ which clearly cannot happen

The picture looks like



(B) The 3rd column is the same as the 1st. The same 3 solutions found in problem 6 will also work here. In the column picture it is easier to see that given one solution ~~[illegible]~~ (x, y, z) another solution is easy to calculate w/

(z, y, x)

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(9)

(a) $\begin{bmatrix} 1 & 2 & 4 \\ -2 & 3 & 1 \\ -4 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} =$

$$= \begin{bmatrix} 1(2)+2(2)+4(3) \\ -2(2)+3(2)+1(3) \\ -4(2)+1(2)+2(3) \end{bmatrix} = \begin{bmatrix} 2+4+12 \\ -4+6+3 \\ -8+2+6 \end{bmatrix} = \begin{bmatrix} 18 \\ 5 \\ 0 \end{bmatrix}$$

(b) $$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1+2+1 \\ 1+2+2 \\ 1+4 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 5 \end{bmatrix}$$

(10)

(a) $Ax = \begin{bmatrix} 2 \\ -4 \\ -8 \end{bmatrix} + \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 12 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 18 \\ 5 \\ 0 \end{bmatrix}$ same as before

(b) $Ax = 1\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}$

$= \begin{bmatrix} 3 \\ 4 \\ 5 \\ 5 \end{bmatrix}$ same as before

We require 3 rows × 3 multiplications per row = 9 total multiplications.

(11) Skipped

(12) skipped

(13) (a) with n components
to produce a vector with m components

(b) in n - dimensional space
m - dimensional space

(14) (a) $Ax + By + Cz = f$ is a linear eq in 3 unknown (x, y, z)

(b) $Ax_0 + By_0 + Cz_0 = f$
$+ \; Ax_1 + By_1 + Cz_1 = f$

Multiply top by c + bottom by d + Add

$$A(cx_0 + dx_1) + B(cy_0 + dy_1) + C(cz_0 + dz_1) = (c+d)f$$

Then $c + d = 1$ (or convex combinations)

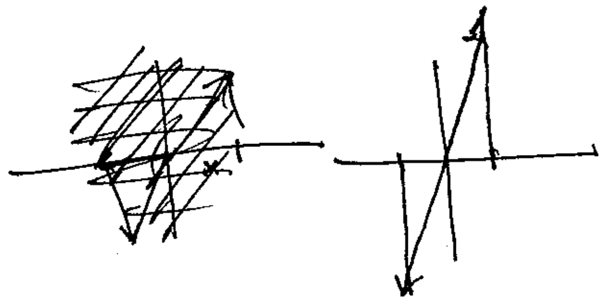
(c) When the Right hand side is zero (or homogeneous)

15 (a) $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(b) $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

16 (a) $R = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

(b) $R^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$



17 $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

$P^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = P^T$

18 $E = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

check $E\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ -3+5 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ ✓

In the 3x3 case

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Check $E\begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 7 \end{bmatrix}$ ✓

(19) $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

$$E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Check $E^{-1}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ -x+z \end{bmatrix}$ ✓

$$E\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 8 \end{bmatrix}$$

$E^{-1}E\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$ Same

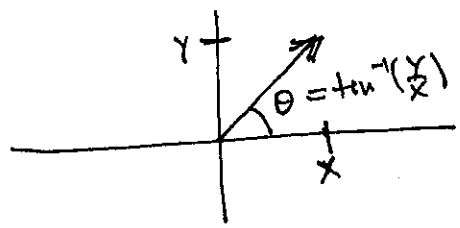
(20) $P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$P_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$P_1 \begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$

$P_2 \begin{bmatrix} 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

(21) $R =$



$z = \begin{pmatrix} x \\ y \end{pmatrix} \cong \begin{pmatrix} r \\ \theta \end{pmatrix}$

$z' = \begin{pmatrix} x' \\ y' \end{pmatrix} \cong \begin{pmatrix} r \\ \theta + 45° \end{pmatrix}$

so $x' = r\cos(\theta + 45)$

$y' = r\sin(\theta + 45)$

$\therefore \quad x' = r\left(\cos\theta\cos\left(\frac{\pi}{4}\right) - \sin\theta\sin\left(\frac{\pi}{4}\right)\right)$

$y' = r\left(\sin\theta\cos\left(\frac{\pi}{4}\right) + \cos\theta\sin\left(\frac{\pi}{4}\right)\right)$

$\Rightarrow \quad x' = r\left(\frac{1}{\sqrt{2}}\right)\cos\theta - r\left(\frac{1}{\sqrt{2}}\right)\sin\theta$

$y' = r\left(\frac{1}{\sqrt{2}}\right)\sin\theta + r\left(\frac{1}{\sqrt{2}}\right)\cos\theta$

Since $X = r\cos\theta$ + $y = r\sin\theta$ we have

$$x' = \frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}y$$
$$y' = \frac{1}{\sqrt{2}}y + \frac{1}{\sqrt{2}}x$$

$$\therefore \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

This mapping can also be determined by the two transformations given. Assume $R = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ Then

$R\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix}$ giving $a = \frac{\sqrt{2}}{2}$ + $c = \frac{\sqrt{2}}{2}$

Also $R\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}$ gives $b = -\frac{\sqrt{2}}{2}$ + $d = \frac{\sqrt{2}}{2}$

(22) $A = \begin{bmatrix} 1 & 4 & 5 \end{bmatrix}$

+ $x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Solution to $Ax = 0$ lie on a ~~hyper~~ plane through the origin.

The columns of A are only in 1 dimensional space

(23) Dot product of V w/ row 1 & then row 2?
is given by the code on the left

The code on the right finds column 1 times V(1) plus column 2 times V(2)

If A has 4 rows & 3 columns the left' code is

```
Do 10 I=1,4          Do 10 J=1,3
Do 10 J=1,3          Do 10 I=1,4
```

(24) The code on the left performs

$$B(1) = A(1,1)\,V(1)$$
$$B(1) = B(1) + A(1,2)\,V(2)$$
$$B(2) = A(2,1)\,V(1)$$
$$B(2) = B(2) + A(2,2)\,V(2)$$

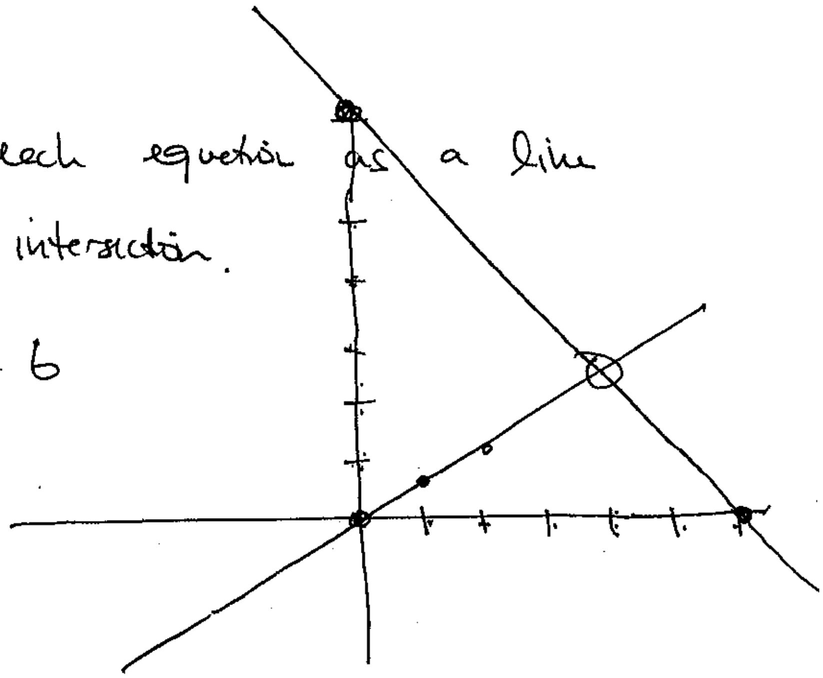
The code on the right performs

$$B(1) = A(1,1)\,V(1)$$
$$B(2) = A(2,1)\,V(1)$$
$$B(1) = B(1) + A(1,2)\,V(2)$$
$$B(2) = B(2) + A(2,2)\,V(2)$$

(25)

```
A = [1,2,3;4,5,6;7,8,9];
x = [1,1,1]';
A*x
```

(26) The row picture represents each equation as a line (constraint) + looks for the intersection.

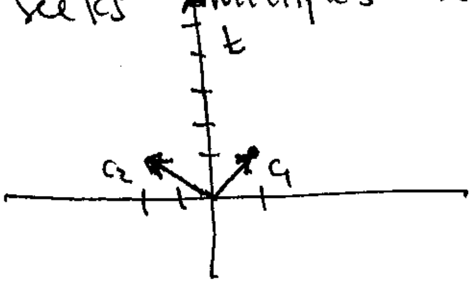


$$x - 2y = 0 \quad + \quad x + y = 6$$

x	y
0	0
0	0
1	1/2
2	1

x	y
0	6
6	0

The column picture draws each column in the x-y plane + seeks multipliers x + y to satisfy the constraints



$$c_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad c_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$t = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

(27) For 2 linear equations in 3 unknowns ~~x, y, z~~ x, y, z,
the row picture will show $\underline{\underline{2}}$ planes.
Those lie in a 3 dimensional space
The column picture is in a two dimensional space

(28) ~~4~~ Four linear equations in 2 unknowns, the row picture shows
4 lines. The column picture is in 4 dimensional
space. These equations have no solution unless the
R.H.S. is a combination of the columns.

(29) $$v_2 = Av_1 = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}\begin{bmatrix} .8 \\ .2 \end{bmatrix} = \begin{bmatrix} .64 + .06 \\ .16 + .14 \end{bmatrix} = \begin{bmatrix} .7 \\ .3 \end{bmatrix}$$

$$v_3 = Av_2 = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}\begin{bmatrix} .7 \\ .3 \end{bmatrix} = \begin{bmatrix} .56 + .09 \\ .14 + .21 \end{bmatrix}$$

$$= \begin{bmatrix} .65 \\ .35 \end{bmatrix}$$

All vectors ~~add~~ add to one

30. See matlab plots

31. ~~Sp~~ (See Next pg)

32. $v_0 = (a; b; 1-a-b)$

See Matlab code

33.

~~1~~ 2 ~~3~~ ~~4~~ 5 ~~6~~ 7 ~~8~~ 9

~~1 2 3 4 5 6 7 8 9~~

We know that

$M_3 \begin{pmatrix}1\\1\\1\end{pmatrix} = 15 \begin{pmatrix}1\\1\\1\end{pmatrix}$

$M_4 \begin{pmatrix}1\\1\\1\\1\end{pmatrix} = c \begin{pmatrix}1\\1\\1\\1\end{pmatrix}$ w/ c = sum of any row in a magic square

summing all elements in M_4

we get $\sum_{i=1}^{16} i = \frac{1}{2} n(n+1)$

$= \frac{1}{2}(16)(17) = 8(17)$

$= 80 + 56 = 136$

$\div$ this by 4 (the # of rows ~~&~~ & in M_4) gives what the row/column/diagonal sum should be

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10 ½

(31) From Numerical expt it looks like

$$x_1 \to .595$$

$$x_2 \to .404$$

$$A \begin{pmatrix} .595 \\ .404 \end{pmatrix} \cong \begin{pmatrix} .597 \\ .401 \end{pmatrix} \quad \text{thus}$$

$$As = s$$

$$\frac{136}{4} = \frac{120+16}{4} = 30+4 = 34$$

$\therefore \quad c = 34$

Now to compute M_3 w/ 1st row 8,3,4 we have

$$M_3 = \begin{bmatrix} 8 & 3 & 4 \\ 9 & & 5 \\ & & 6 \end{bmatrix}$$

1, 2, ~~3~~, ~~4~~, 5, 6, 7, ~~8~~, 9

1, 5, 9

2, 6, 7

Q: Is it true that the magic #'s can be written as permutations (column + rows) of $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$?

$$M_3 = \begin{bmatrix} 8 & 3 & 4 \\ 1 & 5 & 9 \\ 6 & 7 & 2 \end{bmatrix}$$ is <u>one</u> <u>ordering</u>!!

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(1) We ~~sh~~ should subtract 5 times the 1st equation

After this step we have

$-15+9=-6$

$$2x+3y = 11$$
$$-6y = 6$$

$-5+11=6$

$$\begin{bmatrix} 2 & 3 \\ 0 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$$

The two pivots are 2 & -6.

(2) The last eq gives $y=-1$

Then the 1st eq gives $2x-3=1$ or $2x=4$

or $x=2$.

Lets check $\begin{bmatrix} 2 \\ 10 \end{bmatrix}(2)+\begin{bmatrix} 3 \\ 9 \end{bmatrix}(-1) = \begin{bmatrix} 4-3 \\ 20-9 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$ yes ✓

If the RHS changes to $\begin{bmatrix} 4 \\ 44 \end{bmatrix}$

Then -5 times the 1st component added to the 2nd component

gives $44-20 = 24$

giving $$\begin{bmatrix}2 & 3\\ 0 & -6\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}4\\ 24\end{bmatrix}$$

~~The~~ Backsubstitution gives $y=-4$

The ~~[illegible]~~ - ~~[illegible]~~

$$x=\frac{4-3(-4)}{2}=\frac{4+12}{2}=8$$

~~[illegible]~~ also since this RHS is $4\times$ ~~the~~ the old RHS. The solution should be 4 times the old solution.

~~$v=\begin{pmatrix}1\\2\end{pmatrix}$~~ $v=\begin{pmatrix}1\\2\end{pmatrix}$ so $4v=\begin{pmatrix}4\\8\end{pmatrix}$ yes

(3) we should subtract $\frac{-1}{2}$ times the 1st eq from the 2nd eq giving

$$+4\left(-\tfrac{1}{2}\right)+5=3$$

or $$3y=0+\tfrac{1}{2}(6)=3 \rightarrow y=1$$

$$\begin{bmatrix}2 & -4\\ 0 & 3\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}6\\ 3\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}2 & -4\\ 0 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}6\\ 1\end{bmatrix} \Rightarrow \cancel{\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x\\ 1\end{bmatrix}}$$

$$\Rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

$\therefore \quad x = 5$ & $y = 1$.

If the right hand side goes to $(-6, 0)$ this is (-1) times the original RHS of $(6, 0)$ thus the solution is (-1) times the old solution or $x = -5$ & $y = -1$.

Check $\begin{bmatrix} 2 & -4 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} -5 \\ -1 \end{bmatrix} = \begin{bmatrix} -10+4 \\ 5-5 \end{bmatrix} = \begin{bmatrix} -6 \\ 0 \end{bmatrix}$ ✓

(4) We should multiply eq 1 by $\frac{c}{a}$ & subtract from equation #2 (assuming $a \neq 0$ otherwise pivoting is required)

elimination would then produce

$$\begin{bmatrix} a & b \\ 0 & d - b\left(\frac{c}{a}\right) \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ g - 7\left(\frac{c}{a}\right) \end{bmatrix}$$

or

$$\begin{bmatrix} a & b \\ 0 & \frac{1}{a}(ad-bc) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ g - 7\left(\frac{c}{a}\right) \end{bmatrix}$$

Thus the 2nd pivot is $\frac{1}{a}(ad-bc)$

(5) Given
$$3x + 2y = 7$$
$$+ \quad 6x + 4y = ?$$

To get no solution we need to pick ~~a~~ a value that will cause the 2nd eq to contradict the 1st eq. Since the coefficients of the 2nd eq are 2 times those in the 1st. Selecting a RHS. that is not $2 \cdot 7 = 14$ will give No solution i.e. pick say 3

$$3x + 2y = 7$$
$$6x + 4y = 3$$

To get ∞ # of solutions pick 14. (then this 2nd eq is a multiple of the 1st eq) The eqs then are

$$3x + 2y = 7$$
$$6x + 4y = 14.$$

(6) Pick b such that the determinant is singular or pick b such that the 2nd + 1st eq are multiples of each other

The 1st prescription gives pick b ∋

$2(8) - 4b = 0 \Rightarrow$ ~~8-2b=~~ $b = 4$

The 2nd prescription Notes that since ~~$a_{21} = 2a_{12}$~~ ~~we should~~ pick

~~$2a_{11}$~~ $a_{21} = 2a_{11}$ we should pick

$a_{22} = 2a_{12}$

or $8 = 2b \Rightarrow b = 4$. as before.

The only way for this system to be solvable ~~if~~ when $b=4$ is for the element g to <u>also</u> be 2 times b_1 or

$g = 2b_1 = 2(13) = 26$. This would give ∞ many solutions since in effect we have 2 unknowns + only one constraint.

(7) For elimination to break down permanently the system must be singular so the determinant must vanish $6a - 12 = 0$ or $a = 2$

Another way to see this is to require that the 2nd equation is a multiple of the 1st. Since the component

$a_{12} = \frac{1}{2} a_{22}$ but $b_1 \neq \frac{1}{2} b_2$ enforcing that

$a_{11} = \frac{1}{2} a_{21} = \frac{1}{2}(4) = 2$

will ~~yield~~ yield a system with <u>no</u> solutions.

Temporary break down occurs if the 1st pivot is zero but there exists a pivot below that is non zero. Selecting $a=0$ would have that condition true. The system then is after exchanging rows

$$4x + 6y = 6$$
$$3y = -3$$

So back-substitution gives $y=-1$ & then

$$x = \frac{6-6(-1)}{4} = \frac{12}{4} = 3$$

(8) Ellimination will break down if the rows are multiples of each other but the column entries are <u>not</u>. Selecting $k=3$ will make this true in which case there will be <u>No</u> solutions (the ~~same~~ equations contradict each other). There is also the possibility that $k=-3$ ~~which~~ which would see that the 2nd eq <u>is</u> a multiple of the 1st (with a multiplication factor of -1) This would have ∞ solutions. The final case is if $k=0$. Then ellimination would break down but only in the 1st step. Exchanging (Pivoting) gives

$$3x \qquad = -6$$
$$3y = 6$$

giving $x = -2$ + $y = 2$ or one solution.

(9) If $b_2 = 2b_1$ this system will have ∞ many solutions if $b_2 \neq 2b_1$ there will be No solution. This can be seen by performing elimination on the system

~~$3x - 2y$~~

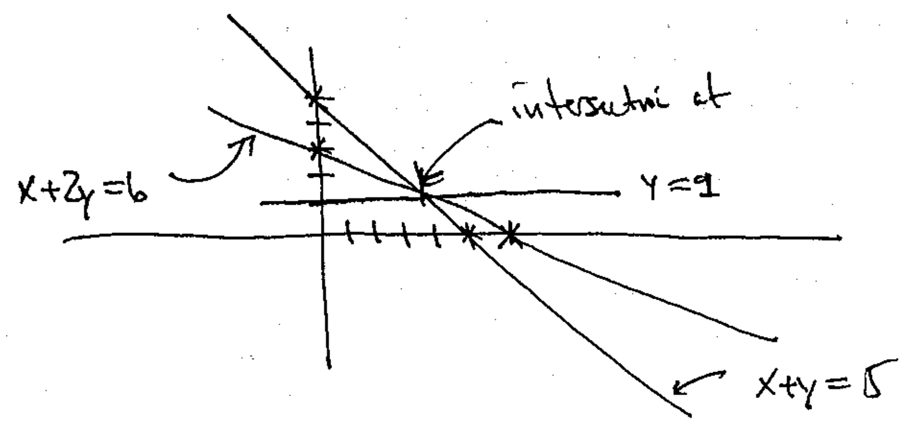
$$\begin{bmatrix} 3 & -2 \\ 0 & -4-(-2)(2) \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - 2b_1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - 2b_1 \end{bmatrix}$$

~~so~~ so if $b_2 = 2b_1$ ∞ many solutions

$b_2 \neq 2b_1$ No solution

(10)



$$x + y = 5$$
$$x + 2y = 6$$

Ellimination gives $y = 1$ & then $x = 4$. To find the c such that it will go through this additional for solution put $(x,y) = (4,1)$ into $c = 5x - 4y = 5(4) - 4(1) = 20 - 4 = 16$

(11)
$$2x + 3y + z = 1$$
$$4x + 7y + 5z = 7$$
$$-2y + 2z = 6$$

$$\Rightarrow \begin{bmatrix} 2 & 3 & 1 \\ 4 & 7 & 5 \\ 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} \\ \\ \\ \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ 6 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 16 \end{bmatrix}$$

Performing Back substitution gives

$z = 2$ then

$y = 5 - 3(2) = -1$

$x = \dfrac{1-1(2)-3(-1)}{2} = \dfrac{1-2+3}{2} = 1$

Zero must occur in the a_{31} position or the a_{21} position

(12)

$$\begin{aligned} 2x - 3y \phantom{{}+z} &= 3 \\ 4x - 5y + z &= 7 \\ 2x - y - 3z &= 5 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 2 & -3 & 0 & 3 \\ 4 & -5 & 1 & 7 \\ 2 & -1 & -3 & 5 \end{array}\right] \to \left[\begin{array}{ccc|c} 2 & -3 & 0 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & -3 & 2 \end{array}\right] \to \left[\begin{array}{ccc|c} 2 & -3 & 0 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -5 & 0 \end{array}\right]$$

The pivots are $2, 1, + -5$.

Backsubstitution gives $z = 0$, $y = 1$, $x = \dfrac{3-(-3)(1)}{2} = 3$

The row operations are

subtract 2 times row one from row 2

subtract 1 times row one from row 3

subtract 2 times row ~~one~~ two from row 3

13 $\left[\begin{array}{ccc|c} 2 & 5 & 1 & 0 \\ 4 & d & 1 & 2 \\ 0 & 1 & -1 & 3 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 2 & 5 & 1 & 0 \\ 0 & d-10 & 1-2 & 2 \\ 0 & 1 & -1 & 3 \end{array}\right]$

$\Rightarrow$ ~~If~~ $d=10$ we would have to perform a row swap. Assuming $d=10$, we have

$\left[\begin{array}{ccc|c} 2 & 5 & 1 & 0 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & -1 & 2 \end{array}\right]$ giving $z=-2$

$y = 3+1(-2) = 1$

& $x = \dfrac{-5(1)-1(-2)}{2} = \dfrac{-5+2}{2} = -3/2$

Exchanging rows 2 & 3 gives

$\left[\begin{array}{ccc|c} 2 & 5 & 1 & 0 \\ 0 & 1 & -1 & 3 \\ 0 & d-10 & -1 & 2 \end{array}\right] \to \left[\begin{array}{ccc|c} 2 & 5 & 1 & 0 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & d-11 & -3d+32 \end{array}\right]$

$\cancel{-1}(d-10)(\cancel{-1})-1 = d-11$

$3(-1)(d-10)+2 = -3d+32$

Then ~~No solution~~ No solution will exist if $d-11=0$ ~~for d~~

& $-3d+32 \neq 0$. The 1st eq gives $d=11$. checking the 2nd condition $-3(11)+32 = -33+32 = -1 \neq 0$ ✓

(14)

The matrix representation is

$$\left[\begin{array}{ccc|c} 1 & b & 0 & 0 \\ 1 & -2 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & b & 0 & 0 \\ 0 & -2-b & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$\Leftrightarrow$ If $b=-2$ we must perform a row exchange.

A missing pivot will result if the system is singular. (i.e. not uniquely invertible) if $-2-b=-1$ then the 2nd + 3rd rows are multiples of each other (by -1)

$-b=1 \Rightarrow b=-1$ + there are ~~more than~~ an ∞ # of solutions

Exchanging the 2nd + 3rd row we obtain

$$\left[\begin{array}{ccc|c} 1 & b & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -2-b & -1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & b & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1+b & 0 \end{array}\right]$$

~~1+b+b~~ + we see that if $1+b \neq 0$ we have a ~~new~~ full set of pivots + thus the only solution is $\underline{0}$.

If $b=-1$ we have

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

so $x + z = 0$

+ $y + z = 0$

an arbitrary solution is given by

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-z\\-z\\z\end{pmatrix}=z\begin{pmatrix}-1\\-1\\1\end{pmatrix} \quad \text{so} \quad \begin{pmatrix}-1\\-1\\1\end{pmatrix} \text{ is one solution}$$

(15) (a)

$$\begin{aligned} y + z &= 1 \\ 2x + y + z &= 2 \\ 3x + y + z &= 5 \end{aligned} \quad \Rightarrow \quad \begin{aligned} 2x + y + z &= 2 \\ y + z &= 1 \\ 3x + y & \end{aligned}$$

To require 2 row exchanges mean that when you eliminate variables you get ~~zero~~ zero pivots twice. If we start w/ a leading coefficient of 0 we will force ~~...~~ a row exchange

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & x \end{bmatrix} =$$

$$\begin{aligned} z &= 3 \\ y &= 2 \\ x &= 1 \end{aligned} \qquad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{matrix} 0 & 1 & 1 \\ 1 & 1 & ? \\ 1 & 1 & ? \end{matrix} \qquad \begin{matrix} 1 & 1 & ? \\ 0 & 0 & ? \\ x & x & x \end{matrix} \Rightarrow \begin{matrix} 1 & 1 & ? \\ 0 & 1 & 1 \\ 1 & 1 & ? \end{matrix}$$

$$\cancel{\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix} \Rightarrow \begin{matrix} a & b & c \\ 0 & e-b(\frac{d}{a}) & f-c(\frac{d}{a}) \\ 0 & h-b(\frac{g}{a}) & j-c(\frac{g}{a}) \end{matrix}}$$

($\frac{d}{a}$) We must have the 1st component 0 which forces the 1st row exchange so we begin with

$$\begin{bmatrix} 0 & a & b \\ c & d & e \\ f & g & h \end{bmatrix} \Rightarrow \begin{bmatrix} c & d & e \\ 0 & a & b \\ f & g & h \end{bmatrix}$$

Assuming $c \neq 0$

$$\Rightarrow \begin{bmatrix} c & d & e \\ 0 & a & b \\ 0 & g-d(\frac{f}{c}) & h-e(\frac{f}{c}) \end{bmatrix}$$

$\Rightarrow$ if $a=0$ have to ~~[illegible]~~ interchange again. It seems like a trivial system but any system the 1st row of which has 2 zeros

like:
$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix}$$
Requires 2 row exchanges

But by exchanging row 1 + row 3 ~~we can get that~~ ~~get a~~ system first we might be able to

Dad - Bose headphones [illegible]

~~c d e~~

$$\begin{bmatrix} 0 & a & b \\ c & d & e \\ f & g & h \end{bmatrix} \Rightarrow \begin{bmatrix} f & g & h \\ & & \\ & & \end{bmatrix}$$

Make a system that does not require any other row exchange.

So begin with

$$\begin{bmatrix} 0 & 0 & a \\ b & c & d \\ e & f & g \end{bmatrix}$$ to minimize row exchanges ~~...~~ (maybe) swap rows 1 + 3

$$= \begin{bmatrix} e & f & g \\ b & c & d \\ 0 & 0 & a \end{bmatrix} \quad \text{if } e \neq 0$$

It looks like you can always do just 1 row exchange if you are smart & the system is solvable

(b) $0x + 1y + 1z = 2$

~~$4y + 2z =$~~

$2y + 2z = 5$

$1x + 3x + 2z =$

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(16) If rows 1 + 2 are the same the 1st step of elimination will produce a 0=0 + you have to do a row exchange immediately which gives no 3rd pivot.
If columns 1 + 2 are the same the 2nd pivot is missing

(17) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 5 & 10 & 15 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(18) $\left[\begin{array}{ccc|c} 1 & 4 & -2 & 1 \\ 1 & 7 & -6 & 6 \\ 0 & 3 & q & t \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 4 & -2 & 1 \\ 0 & 3 & -4 & 5 \\ 0 & 3 & q & t \end{array}\right]$

$= \left[\begin{array}{ccc|c} 1 & 4 & -2 & 1 \\ 0 & 3 & -4 & 5 \\ 0 & 0 & q+4 & t-5 \end{array}\right]$

If $q+4=0$ then we have No 3rd pivot + the system is singular $(q=-4)$. If $q=-4$ then if $t-5=0$ or $t=5$ we will have ∞ many solutions

Assuming these two #'s we have

$$\begin{aligned} x + 4y - 2z &= 1 \\ 3y - 4z &= 5 \end{aligned} \Rightarrow \quad 3y - 4(1) = 5 \to 3y = 9 \to y = 3$$

Then $x = 1 - 4(3) + 2(1) = 1 - 12 + 2 = \cancel{18} - 9$

(19) System of linear eqs cannot have exactly 2 solutions. why.

(a) If (x,y,z) + $(\bar{x},\bar{y},\bar{z})$ are two solutions

then $A((x,y,z) - (\bar{x},\bar{y},\bar{z})) = b - b = 0$

Thus $(x,y,z) + c((x,y,z) - (\bar{x},\bar{y},\bar{z}))$ is another solution

i.e. any solution on the line between the two pts

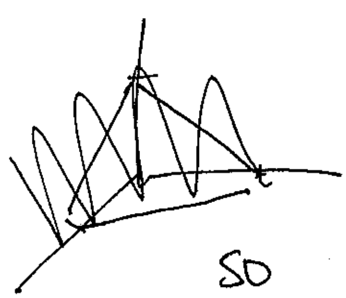
(x,y,z) + $(\bar{x},\bar{y},\bar{z})$ + thus there is an ∞ # of solutions.

(b) The line connecting them

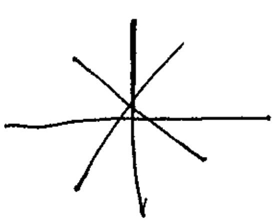
(20) ~~Consider the plane~~ $\cancel{x+y+z} = 1$

~~+ the plane~~ $\cancel{x} = 0$

~~+ the plane~~ $\cancel{y} = 0$



So



$x - y = 0$

$x + y = 0$

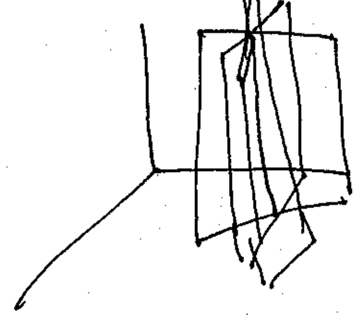
+ $x = 0$

in (x,y,z) Then

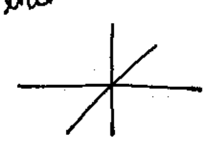
All 3 planes pass through $(0,0,z)$

for any z.

Any 3 planes that intersect in a single line i.e. pick 3 planes that all pass through one line



Viewed from the top we have something like



$$x+y+z=0$$
$$x-y-z=0$$

pick a 3rd eq that effectively contradicts one of the other two or a linear combination say

$$x+y+z=2.$$

(21)

$$\left[\begin{array}{cccc|c} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 5 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cccc|c} 2 & 1 & 0 & 0 & 0 \\ 0 & 3/2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 5 \end{array}\right]$$

$$-\frac{2}{3}(1)+2 = -\frac{2}{3}+\frac{6}{3} = \frac{4}{3}$$

$$\Rightarrow \left[\begin{array}{cccc|c} 2 & 1 & 0 & 0 & 0 \\ 0 & 3/2 & 1 & 0 & 0 \\ 0 & 0 & 4/3 & 1 & 0 \\ 0 & 0 & 1 & 2 & 5 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 2 & 1 & 0 & 0 & 0 \\ 0 & 3/2 & 1 & 0 & 0 \\ 0 & 0 & 4/3 & 1 & 0 \\ 0 & 0 & 0 & 5/4 & 5 \end{array}\right]$$

circled are pivots

$$-\frac{3}{4}(1)+2 = -\frac{3}{4}+\frac{8}{4} = \frac{5}{4}$$

Backsolving we obtain

$t = 4$

$\frac{4}{3}z + 4 = 0 \Rightarrow z = -3$

$\frac{3}{2}y + -3 = 0 \Rightarrow y = 2$

$2x + 2 = 0 \Rightarrow x = -1$

so $\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -3 \\ 4 \end{pmatrix}$

(22)

$$\left[\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & 5 \end{array}\right]$$

$-\frac{1}{2} + 2 = \frac{3}{2}$

$$\Rightarrow \left[\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 0 \\ 0 & 3/2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & 5 \end{array}\right]$$

$-\frac{2}{3} + 2 = \frac{-2}{3} + \frac{6}{3} = \frac{4}{3}$

$$\Rightarrow \left[\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 0 \\ 0 & 3/2 & -1 & 0 & 0 \\ 0 & 0 & 4/3 & -1 & 0 \\ 0 & 0 & -1 & 2 & 5 \end{array}\right]$$

$-\frac{3}{4} + 2 =$

$-\frac{3}{4} + \frac{8}{4} = \frac{5}{4}$

$$\Rightarrow \left[\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 0 \\ 0 & 3/2 & -1 & 0 & 0 \\ 0 & 0 & 4/3 & -1 & 0 \\ 0 & 0 & 0 & 5/4 & 5 \end{array}\right]$$

Back solving we have $t=4$ same as before.

... Now we see that Because the only difference between the two systems of equations is in the back ~~substitution~~ substitution or the upper triangle ~~portion~~ portion of the matrix. In general if two systems differ in the elements of the upper triangle elements one can use the <u>same</u> forward substitution but just different back substitution. Thus we have

$$\left[\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 0 \\ 0 & 3/2 & -1 & 0 & 0 \\ 0 & 0 & 4/3 & -1 & 0 \\ 0 & 0 & 0 & 5/4 & 5 \end{array}\right]$$

given $t=4$

$\frac{4}{3}z - 4 = 0 \Rightarrow z = 3$

$\frac{3}{2} y = \frac{+3}{3/2} = 2$

$\Rightarrow x = \frac{2}{2} = 1$

So this solution is $\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$

(23) ~~The 5th pivot would be the 5th~~

The ith row after $\overset{\text{Fo}}{\text{Z}}$ elimination would be
(+ i+1st)

$$\begin{matrix} \cdots 0 & p_i & 1 & 0 & \cdots & \\ \cdots 0 & 1 & 2 & 1 & 0 & \cdots \end{matrix}$$

So to elliminate the 1 in the i+1st row we multiply the ith row by $-\frac{1}{p_i}$ & Add to the i+1st row. This gives in the $(i+1)\times(i+1)$ th position the value of

$-\frac{1}{p_i}+2$ For the pivot of the i+1st row.

If we assume to be proven by the induction hypothesis that the ith pivot is given by $p_i = \frac{i+1}{i}$ then the i+1 st pivot is given by

$$-\frac{1}{p_i}+2 = -\frac{i}{i+1}+\frac{2(i+1)}{i+1} = \frac{i+2}{i+1}$$

Which ~~pro~~ proves this.

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(24) Any ~~[illegible]~~ row can be replaced with any linear combination of the others.

Add eq 1 to eq 2
+ " " eq 3

to obtain

$$x+y+z=0$$
$$x+2y+2z=0$$
$$x+y+4z=0$$

Add 2·eq 1 to eq 2
+ 3·eq 1 to eq 3 to obtain

$$x+y+z=0$$
$$2x+3y+3z=0$$
$$3x+3y+6z=0$$

Additional sets can be generated like this

(25) if $a=0$ there is No 1st pivot

if $a\neq 0$

$$\begin{bmatrix} a & 2 & 3 \\ 0 & a-2 & 1 \\ 0 & a-2 & a-3 \end{bmatrix}$$

If $a=2$ then there is No 2nd pivot

if $a \neq 2$

$$\begin{bmatrix} a & 2 & 3 \\ 0 & a-2 & 1 \\ 0 & 0 & a-4 \end{bmatrix}$$ if $a=4$ there will be No 3rd pivot.

– We see that if $a=0$ we would have an entire row of zeros

if $a=2$ we have two identical ~~rows~~ column

if $a=4$ we have two identical rows

(26) $$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 1 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 & 8 \\ 0 & 1 & 0 & 1 & s \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & -1 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 & 8 \\ 0 & 1 & 0 & 1 & s \end{array}\right]$$ ✓

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & -1 & 1 & 0 & -2 \\ 0 & 1 & 0 & 1 & s \\ 0 & 0 & 1 & 1 & 8 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & -1 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 & s-2 \\ 0 & 0 & 1 & 1 & 8 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & -1 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 & s-2 \\ 0 & 0 & 0 & 0 & s-6 \end{array}\right]$$

We must have $s=6$ for this eq to be solvable

+ from the above we have an ∞ ~~#~~ of solution

Check

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & -1 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 & 8 \\ 0 & 0 & 1 & 1 & s-2 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & -1 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 & 8 \\ 0 & 0 & 0 & 0 & \underbrace{s-6}_{=0} \end{array}\right]$$

We must have $s=6$ for this eq to be solvable

Then we get

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 2 \\ 0 & -1 & 0 & -1 & -10 \\ 0 & 0 & 1 & 1 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -6 \\ 0 & -1 & 0 & -1 & -10 \\ 0 & 0 & 1 & 1 & 8 \end{array}\right]$$

Then

$$\begin{aligned} a-d &= -6 \\ +b+d &= +10 \\ c+d &= 8 \end{aligned}$$

So

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} -6+d \\ 10-d \\ 8-d \\ d \end{pmatrix} = \begin{pmatrix} -6 \\ 10 \\ 8 \\ 0 \end{pmatrix} + d\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

Thus two matrices (pick $d=0$) & we get

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} -6 \\ 10 \\ 8 \\ 0 \end{pmatrix}$$

<u>Check:</u>

$a+b=4$ ✓

$c+d=8$ ✓

$a+c=2$ ✓

$b+d=10$ ✗

pick $d=1$ $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} -6+1 \\ 9 \\ 7 \\ 1 \end{pmatrix} = \begin{pmatrix} -5 \\ 9 \\ 7 \\ 1 \end{pmatrix}$

$a+b=4$ ✓

$c+d=8$

$a+c=2$ ✓

$b+d=10$ ✓

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(1) (a) $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & +7 & 1 \end{bmatrix}$

(c) $P_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(2) $E_{32}\begin{pmatrix} 1 \\ -5 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ -35 \end{pmatrix} = E_{32}E_{21}b$

$E_{21}E_{32}\cancel{b}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = E_{21}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ 0 \end{pmatrix}$

When E_{32} row 3 feels No effect from row 2.

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(3) E_{21}, E_{31}, E_{32}

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 0 \end{bmatrix}$$

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_{21}A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ -2 & 2 & 0 \end{bmatrix}$$

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \qquad E_{31}E_{21}A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 0 \end{bmatrix}$$

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \qquad E_{32}E_{31}E_{21}A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -2 \end{bmatrix} = U$$

then $M = E_{32}E_{31}E_{21} = \cancel{\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -10 & -2 & 1 \end{bmatrix} \checkmark$$

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(4) $$A' = \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 4 & 6 & 1 & 0 \\ -2 & 2 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & -4 \\ 0 & 4 & 0 & 2 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & -4 \\ 0 & 0 & -2 & 10 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & -4 \\ 0 & 0 & 1 & -5 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & -5 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & -5 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & -5 \end{array}\right]$$

(5) $$E \cdots E \begin{bmatrix} & & \\ & & \\ & & 7 \end{bmatrix} = \begin{bmatrix} \ddots & & \\ & & \\ & & 2 \end{bmatrix}$$

$$7 + \sum l_{ij} a_{ij} = 2$$

Then if I change the element a_{33} to 11, we change the pivot to $\underline{\underline{6}}$. ~~If you change a_{33} to~~ Because the pivot after ellimination is the sum of the a_{33} entry + other terms.

To make the pivot 0 Add minus 2 to both sides

Which is equivalent to having a a_{33} element of $7-2=5$

(6)
$$\begin{bmatrix} 1 & 5 & 7 \\ 1 & 5 & 7 \\ 1 & 5 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + y \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix} + z \begin{bmatrix} 7 \\ 7 \\ 7 \end{bmatrix}.$$

$$= (x + 5y + 7z) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Only one pivot is produced from elimination. The others are all zero.

(7)
$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -7 & 0 & 1 \end{bmatrix}$$

Add 7 times row 1 to row 3.

$$R_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 7 & 0 & 1 \end{bmatrix}$$

(8)
$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -7 & 0 & 1 \end{bmatrix} \quad R_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 7 & 0 & 1 \end{bmatrix}$$

Same as before

(9) (a) $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$M = P_{23} E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$

(b) $P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

$M = E_{31} P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$

Because subtracting row 1 from row 2 + then exchanging rows 2 + 3 gets ~~this~~ is the same as exchang rows 2 + 3 + subtracting row 1 from row 3.

(10) $a_{11} = a_{22} = a_{33} = 1$

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 4 \\ 1 & 1 & 1 \end{bmatrix}}_{\text{will produce two negative pivots}} \Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 1 \\ 0 & -1 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & -3 \end{bmatrix}$$

(11) Skipped

(12) ~~EB~~ The 3rd column of EB is the dot product of every row of E with the 3rd column of B. Since this is zero ~~all~~ the 3rd column of EB will be zero also

If the 3rd row of B is all zero then any non zero elts in the 1st or 2nd row of B can produce non zero elts

If $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ 0 & 0 & 0 \end{bmatrix}$ & $E = \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix}$

Then EB has elements in its 3rd row of

$$(EB)_{3j} = \sum_{k=1}^{3} e_{3k} b_{kj} = e_{31} b_{1j} + e_{32} b_{2j} + 0 \neq 0$$

for many choices of e + b.

(13) $E_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ then $E_{21}A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$

so $\cancel{E_{32} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \end{bmatrix}}$

$\frac{2}{3} + 2 = \frac{2}{3} + \frac{6}{3} =$

$3/8(-1) + 2 = \frac{-3}{8} + \frac{16}{8}$
$= \frac{13}{8}$

$E_{32} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2/3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ then $E_{32}E_{21}A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 8/3 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$

$E_{43} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3/8 & 1 \end{bmatrix}$ then $E_{43}E_{32}E_{21}A$

$= \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 8/3 & -1 \\ 0 & 0 & 0 & 13/8 \end{bmatrix}$

(14) $a_{ij} = 2i - 3j$

$A = \begin{bmatrix} -1 & -4 & -7 \\ 1 & -2 & -5 \\ 3 & 0 & -3 \end{bmatrix}$ $E_{12} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

giving

Then $E_{12}A = \begin{bmatrix} -1 & -4 & -7 \\ 0 & -6 & -12 \\ 3 & 0 & -3 \end{bmatrix}$

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

So that $E_{31}E_{21}A = \begin{bmatrix} -1 & -4 & -7 \\ 0 & -6 & -12 \\ 0 & -12 & -24 \end{bmatrix}$ This step destroys the zero in A_{32}

Then $E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

Producing $E_{32}E_{31}E_{21}A = \begin{bmatrix} -1 & -4 & -7 \\ 0 & -6 & -12 \\ 0 & 0 & 0 \end{bmatrix}$

15 (a) $X = 2Y$

$X + Y = 33$ so $\begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 0 \\ 33 \end{bmatrix}$

b) $5 = 2m + c$

& $7 = 3m + c$

giving: $\begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} m \\ c \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$ ...

(16) $$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 14 \end{bmatrix}$$

Skipped solving for a, b, + c

(17) $$EF = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}$$

$$FE = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ ac+b & c & 1 \end{bmatrix}$$

E^2 + F^2 skipped

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(18) $PQ = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

$QP = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

~~P exchanges rows 1 & 2~~ $(\cancel{1\ 2})(3)$

~~Q exchanges rows~~

$P^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$(PQ)^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

(19) (a) E times a column from B.

(b) let $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Then if $E = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

EB is $= \begin{bmatrix} 6 & 6 & 6 \\ 15 & 15 & 15 \\ 24 & 24 & 24 \end{bmatrix}$ each row is different.

(20) ~~EF~~ No since EF would 1st add ~~row~~ row 2 to row 1.
Then E would add row 1 to row 2, the net result
would be in row 1 = row 1 + row 2
in row 2 = row 1 + 2 row 2

But in the opposite order FE would Add row 1 to row 2
then add row 2 to row 1 & the net result would be
in row 1 have 2 row 1 + 1 row 2
in row 2 have row 1 + row 2

In matrices $E = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ & $F = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

So $EF = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ & $FE = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

(21) $(E_{21})_{21} = -1$
$(E_{21})_{22} = 1$

$$E = \begin{bmatrix} 1 & 0 & \cdots & & \\ -1 & 1 & 0 & \cdots & \\ 0 & 0 & 1 & 0 & \cdots \end{bmatrix}$$

(a) $\sum_{j=1}^{n} a_{3j} x_j$

$$EA = \begin{bmatrix} a_{11} & a_{12} & \cdots & \\ a_{21}-a_{11} & a_{22}-a_{12} & a_{23}-a_{13} & \cdots \end{bmatrix}$$

(b) $(EA)_{21} = \cancel{a_{22}-a_{12}}$
$a_{21} - a_{11}$

(c) $(Ex)_{21} = x_2 - x_1$

(d) $EAx = E(Ax) = \cancel{\;} (Ax)_1 = \sum_{j=1}^{n} a_{1j} x_j$

(22) $E = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

~~AE~~ = subtracting 2 times the 2nd column of A from the 1st column of A.

If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$AE = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ -5 & 4 \end{bmatrix}$$

(23) $$A' = \left[\begin{array}{cc|c} 2 & 3 & 1 \\ 4 & 1 & 17 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & 3 & 1 \\ 0 & -5 & 15 \end{array}\right]$$

$\Rightarrow$ ~~x_2~~ = The triangular system is

$$U = \begin{bmatrix} 2 & 3 \\ 0 & -5 \end{bmatrix}$$

Back substitution gives $x_2 = -3$

$$x_1 = \frac{1-3(-3)}{2} = 5$$

(24) $$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 3 & 4 & 2 \\ 3 & 5 & 7 & 6 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -2 & 0 \\ 0 & -1 & -2 & 6-3 \end{array}\right]$$

Since the last two eqs contradict each other. The 1st says that $-y-2z=0$ + the ~~3rd~~ 2nd says the $-y-2z=3\neq 0$ there can be no solution.

Selecting b to be 3 instead would give

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Then ~~x + 2y + 3z = 1~~

$$x - z = 1$$
$$+y + 2z = 0$$

So $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1+z \\ -2z \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + z\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ is a family of solutions

25 (a) $[A \,\vdots\, b \mid b^*]$

(b) The Augmented system is

$$\left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right] \cdots$$

(26) (a) From the reduced form shows

If $d=0$ then is No 3rd pivot, & the system is

If $c \neq 0$ then there is No solution,

If $c=0$ then are an ∞ many solutions

The #'s a & b have no effect on the solution

(27) $E_{31} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $E_{41} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

Then $E_{31} E_{41} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

& $E_{41} E_{31} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$ & they are equal

~~In general they are equal if (i,j)~~

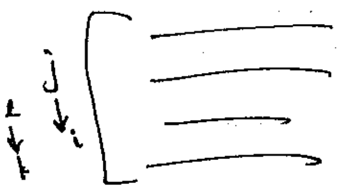
The matrix $E_{ij} E_{kl}$ is the identity with ones in both the locations (i,j) & (k,l)

$E_{32} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

So $E_{32} E_{31} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ & $E_{31} E_{32} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ which are the same.

The matrix E_{ij} can be described in words as producing the original matrix but with the ith row replaced by the ith + the jth $(i>j)$

Similarly E_{kl} can be desc as adding the lth row to the kth $(k>l)$



So ~~$E_{kl}E_{ij} = E_{kl}E_{ij}$~~ $E_{ij}E_{kl}$ is the matrix with the lth row added to the kth + then the jth row added to the ith.

These will commute as long as $j \neq k$, since in this case we add l to j + then j to ith when in the opposite order we add j to ith + then j to lth. Check

$(i,j) = (3,2)$ $(k,l) = (2,1)$

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad E_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Then $$E_{32}E_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Which are Not equal.

$$E_{21}E_{32} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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① $A \in \mathbb{R}^{3\times 5}$ $B \in \mathbb{R}^{5\times 3}$ $C \in \mathbb{R}^{5\times 1}$ $D \in \mathbb{R}^{3\times 1}$

(a) Allowed + of dimension 5×5

(b) Not allowed B + C cannot be added

(c) Allowed of dimension 3×1

(d) $A\cdot C$ is of dimension 3×1

$B\cdot D$ is of dimension 5×1

But these cannot be added so this is not allowed.

(e) ABABD

AB is 3×3

ABA is 3×5

ABAB is 3×3

ABABD is 3×1 + allowed

② (a) 3rd column of AB is the matrix A times the 3rd column of B (All rows of A + 3rd column of B only)

(b) 1st row of A only with all columns of B.

(c) row 3 of A + column 4 of B

(d) row 1 of A times ~~row~~ column 1 of (DE). The latter requires ~~the 1st row of D with all columns of E~~
all of D with the 1st column of E

(3) skipped (example ~~proof~~ of ~~are~~ distributive law)

(4) skipped (example of associative law)

(5) $$A^2 = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2b \\ 0 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2b+b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3b \\ 0 & 1 \end{bmatrix}$$

$\vdots$

predict $A^5 = \begin{bmatrix} 1 & 5b \\ 0 & 1 \end{bmatrix}$

& $A^n = \begin{bmatrix} 1 & nb \\ 0 & 1 \end{bmatrix}$

Then $$A \cdot A^n = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & nb \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & (n+1)b \\ 0 & 1 \end{bmatrix}$$

which effectively proves this by induction. This makes sense because

~~If~~ $\cancel{A = \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix}}$

$\cancel{A^2 =} [$

A is effectively the operator that adds ~~row~~ b times row 2 to row 1.

n-applications of this rule is the same as adding n times row 2 to row 1

If $A = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}$ ... ~~then~~ Since this matrix adds

~~Then~~ 2 of row 1 to row 2, with ellimination of row 2 multiple applications of this operator will result in

$$A^n = \begin{bmatrix} 2^n & 2^n \\ 0 & 0 \end{bmatrix}$$

Check $A^2 = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 0 & 0 \end{bmatrix}$

$$A^3 = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 4 & 4 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 8 & 8 \\ 0 & 0 \end{bmatrix}$$

$$\vdots$$

$$A^n = \begin{bmatrix} 2^n & 2^n \\ 0 & 0 \end{bmatrix}$$

$$A^{n+1} = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2^n & 2^n \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2^{n+1} & 2^{n+1} \\ 0 & 0 \end{bmatrix}$$

effectively proving this result by induction

(6) The 60 skipped example that

$$(A+B)^2 \neq A^2 + 2AB + B^2$$

The correct result is that

$$(A+B)^2 = A^2 + AB + BA + B^2$$

(7) (a) True. The column of AB are the A times the columns of B

(b) False let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ which has rows 1 + 3 the same

Then $AB = \begin{bmatrix} 4 & 4 & 4 \\ 10 & 10 & 10 \\ 16 & 16 & 16 \end{bmatrix}$ which does not have rows 1 + 3 the same.

(c) ~~BA~~ Yes. The rows of AB are the rows of A times each column of B, If the rows of A are the same then in AB they must be the same also since they are multiplying by the same elements.

(d) $(AB)^2 = AB \cdot AB \neq A^2B^2$

Any two matrices that ~~does~~ do not commute will have this property

consider the two in problem 6

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$$

Then $AB = \begin{bmatrix} 7 & 0 \\ 0 & 0 \end{bmatrix}$

& $BA = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \neq AB.$

(8) D is a diagonal matrix & multiplication on the left by a diagonal matrix is equivalent to multiplication of the rows of the matrix we are multiplying by

Thus DA is 3 times 1st row of A or $\begin{bmatrix} 3a & 3b \\ 5c & 5d \end{bmatrix}$
5 times 2nd row of A

EA is the 2nd row of A with zeros for the 2nd row or $\begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix}$

The columns of AD are the columns of A multiplied by the elements of D ie

$$AD = \begin{bmatrix} 3a & 5b \\ 3c & 5d \end{bmatrix}$$

& $AE = \begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix}$

is the zero column followed by the 1st row of A.

(9) (a)

$$AF = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a & a+b \\ c & c+d \end{bmatrix}$$

Then

$$E(AF) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a & a+b \\ c & c+d \end{bmatrix}$$

$$= \begin{bmatrix} a & a+b \\ a+c & a+b+c+d \end{bmatrix}$$

(b) Which is the same as $(EA)F$. matrix multiplication is associative

(10)

$$FA = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ c & d \end{bmatrix}$$

$$E(FA) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a+c & b+d \\ c & d \end{bmatrix}$$

$$= \begin{bmatrix} a+c & b+d \\ a+2c & b+2d \end{bmatrix} \neq F(EA)$$

Showing that commutation is <u>Not</u> obeyed in matrix multiplication.

(11) (a) Pick $B = 4 \cdot I = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

(b) Not possible unless $B = 0$

(c) BA has rows 1 + 3 ...

$$B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

(d) $B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

(12) $AB = BA$

$$\Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{array}{l} b = 0 \\ + \ c = 0 \end{array}$$

If $AC = CA$

$$\Rightarrow \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & d \\ 0 & 0 \end{bmatrix} \Rightarrow \quad a=d$$

So $A = a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(13) $(A-B)^2 = (A-B)(A-B) = A^2 - AB - BA + B^2$.

$A^2 - B^2$ No

$(B-A)^2$ Yes

$A^2 - 2AB + B^2$ No

$A(A-B) - B(A-B)$ Yes

$A^2 - AB - BA + B^2$ Yes

(14) (a) Yes/True

(b) $(m\times n)(n\times p)$ $(n\times p)\times(m\times n)$

$p=m$

False A would be $m\times n$

+ B would be $n\times m$

(c) Yes/True

(d) $AB=B \Rightarrow (A-I)B=0$

$A=I$ or $B=0$

True

(15) (a) A is $m \times n$

computing Ax

$m \cdot n = mn$ multiplication

(b) $A \cdot B$

$m \cdot n \cdot p$ multiplications

(c) A^2

m^3

(16) Pu $(AB)C = A(BC)$

let C have only one column $\underline{c}$

AB has columns $Ab_1, Ab_2, \ldots, Ab_n$

+ Bc has 1 column $c_1 b_1 + c_2 b_2 + c_3 b_3 + \cdots + c_n b_n$

So $(AB)c = Ab_1 c_1 + Ab_2 c_2 + \cdots \; Ab_n c_n$

while $A(Bc) = A(c_1 b_1 + c_2 b_2 + \cdots + c_n b_n)$

linearity makes the two equal $(AB)c = A(Bc)$

The same is true $\forall$ columns of the matrix C

(17)

(a) column 2 of AB is each row of A times only the 2nd column of B

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(b) row 2 of AB = row 2 of A times (dot) every column of B.

$$= \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$$

(c) row 2 of A^2 = row 2 of A times every column of A

$$= (6-6, -3+2) = (0, -1)$$

(d) row 2 of AAA = row 2 of AA times every column of A row 2 of AA is computed above giving

$$(-3, 2)$$

(18) (a)
$$\begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$$

(c)
$$\begin{bmatrix} 1 & 1/2 & 1/3 \\ 2 & 1 & 2/3 \\ 3 & 3/2 & 1 \end{bmatrix}$$

(b)
$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

(19)
(a) Diagonal matrix
$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

(b) lower triangular matrix
$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 6 & 8 & 10 \end{bmatrix}$$

(c) Symmetric matrix
$$\begin{bmatrix} 1 & 4 & -1 \\ 4 & 2 & 7 \\ -1 & 7 & 3 \end{bmatrix}$$

(d) Every row is the same
$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

The zero matrix belongs to all 4 classes

(20)
(a) a_{11}

(b) $\dfrac{a_{31}}{a_{11}}$

(c) $a_{32}' = a_{32} - \left(\dfrac{a_{31}}{a_{11}}\right) a_{13}$

(d) $a_{22}' = a_{22} - \left(\dfrac{a_{21}}{a_{11}}\right) a_{12}$

(21) $A^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$A^3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$A^4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$Av = \begin{pmatrix} y \\ z \\ t \\ 0 \end{pmatrix} \qquad A^2 v = \begin{pmatrix} z \\ t \\ 0 \\ 0 \end{pmatrix} \qquad A^3 v = \begin{pmatrix} z \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad A^4 v = \underline{0}$

(22) $A = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$A^2 = (\frac{1}{2})^2 \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = A$

$\therefore \quad A^n = A$

$AB = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$

$(AB)^2 = \frac{1}{4}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

So $(AB)^n = 0 \qquad n \geq 2$

(23)

(a) $A^2 = -I$

$$A^2 = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$$

So $A = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}$ is an example

(b) $BC = -CB$ (w/ $BC \neq 0$)

$$\begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = -\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} c_{11} - 2c_{21} & c_{12} - 2c_{22} \\ c_{11} - c_{21} & c_{12} - c_{22} \end{bmatrix} = -\begin{bmatrix} c_{11} + c_{12} & -2c_{11} - c_{12} \\ c_{21} + c_{22} & -2c_{21} - c_{22} \end{bmatrix}$$

$\Rightarrow$ G_1 ~~subtracting~~

$$\begin{bmatrix} c_{11} - 2c_{21} & c_{12} - 2c_{22} \\ c_{11} - c_{21} & c_{12} - c_{22} \end{bmatrix} = \begin{bmatrix} -c_{11} - c_{12} & 2c_{11} + c_{12} \\ -c_{21} - c_{22} & 2c_{21} + c_{22} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2c_{11} - c_{21} & -2c_{22} - 2c_{11} \\ c_{11} + c_{22} & c_{12} - 2c_{21} - 2c_{22} - \end{bmatrix} = 0$$

$\Rightarrow \quad c_{11} = -c_{22}$

$\& \quad c_{21} = 2c_{11}$

$\& \quad \cancel{2} \quad c_{12} = 2c_{21} + 2c_{22}$

$= 4c_{11} - 2c_{11} = 2c_{11}$

let $c_{11} = 1$ Then $c_{22} = -1$

$c_{21} = 2$ & $c_{12} = 2$ giving

$$C = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

Check:

$$BC = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ -1 & 3 \end{bmatrix}$$

$$\& \; CB = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 1 & -3 \end{bmatrix} = -\begin{bmatrix} -3 & 4 \\ -1 & 3 \end{bmatrix} \checkmark$$

(24)

(a) Use the matrix from problem 22

let $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ $\qquad 1^2 = (-1)^2 = 1$

(b) $A^2 \neq 0$ but $A^3 = 0$ $\qquad 1^3 = 1$ what are the other two cube roots of unity?

$e^{i\frac{2\pi}{3}}$ Are they complex?

let

let $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ Then $A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq 0.$

So $A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(25) $A_1 = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$

multiplying the 1st row by 2 + add to the 2nd row.

repeating this we would get

$\begin{bmatrix} 4 & 3 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 8 & 7 \\ 0 & 1 \end{bmatrix} \quad \dots$

$$A_1^2 = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 0 & 1 \end{bmatrix} \checkmark$$

$$A_1^3 = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 7 \\ 0 & 1 \end{bmatrix}$$

By the induction lets guess

$$A_1^n = \begin{bmatrix} 2^n & 2^n-1 \\ 0 & 1 \end{bmatrix}$$

$$A_1^{n+1} = \begin{bmatrix} 2^n & 2^n-1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2^{n+1} & 2^n+2^n-1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2^{n+1} & 2^{n+1}-1 \\ 0 & 1 \end{bmatrix} \checkmark$$

$$A_2^2 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \quad A_2^3 = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} \quad A_2^4 = \begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix}$$

$$A_2^5 = \begin{bmatrix} 16 & 16 \\ 16 & 16 \end{bmatrix}$$

Guess $A_2^n = \begin{bmatrix} 2^{n-1} & 2^{n-1} \\ 2^{n-1} & 2^{n-1} \end{bmatrix}$

Then

Check

$$A_2^{n+1} \stackrel{?}{=} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2^{n-1} & 2^{n-1} \\ 2^{n-1} & 2^{n-1} \end{bmatrix}$$

$$= \begin{bmatrix} 2^n & 2^n \\ 2^n & 2^n \end{bmatrix} \checkmark$$

$$A_3^2 = \begin{bmatrix} a^2 & ab \\ 0 & 0 \end{bmatrix} \qquad A_3^3 = \begin{bmatrix} a^3 & a^2b \\ 0 & 0 \end{bmatrix}$$

$$A_3^4 = \begin{bmatrix} a^4 & a^3b \\ 0 & 0 \end{bmatrix}$$

So guess $A_3^n = \begin{bmatrix} a^n & a^{n-1}b \\ 0 & 0 \end{bmatrix}$

$$A_3^{n+1} = \begin{bmatrix} a^{n+1} & a^n b \\ 0 & 0 \end{bmatrix} \checkmark$$

(26)

$$AB = \left[\begin{array}{c|c} 1 & 0 \\ 2 & 4 \\ 2 & 1 \end{array}\right]\left[\begin{array}{ccc} 3 & 3 & 0 \\ \hline 1 & 2 & 1 \end{array}\right] = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}\begin{bmatrix} 3 & 3 & 0 \end{bmatrix}$$

"1" × 2 · 2 × "3"

$$+ \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 3 & 0 \\ 6 & 6 & 0 \\ 6 & 6 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 1 \\ 4 & 8 & 4 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 1 \\ 10 & 14 & 4 \\ 7 & 8 & 1 \end{bmatrix}$$

(27) Row times column is dot product:

(Row 3 of A) ∘ (column 1 of B)

(Row 3 of A) ∘ (column 2 of B)

Column times row is full matrix:

Break A into "blocks" of size 1×3 (block columns)

& B into "blocks" of size 3×1 (block rows)

$$AB = \left[\begin{array}{c|c|c} x & x & x \\ 0 & x & x \\ 0 & 0 & x \end{array}\right]\left[\begin{array}{ccc} x & x & x \\ \hline 0 & x & x \\ \hline 0 & 0 & x \end{array}\right]$$

$$= \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} x & x & x \end{bmatrix} + \begin{bmatrix} x \\ x \\ 0 \end{bmatrix}\begin{bmatrix} 0 & x & x \end{bmatrix} + \begin{bmatrix} x \\ x \\ x \end{bmatrix}\begin{bmatrix} 0 & 0 & x \end{bmatrix}$$

column 1 of A | row 1 of B | column 2 of A | row 2 of B | column 3 of A | row 3 of B

Requested products are

$$\begin{bmatrix} x \\ x \\ 0 \end{bmatrix}\begin{bmatrix} 0 & x & x \end{bmatrix} = \begin{bmatrix} 0 & x & x \\ 0 & x & x \\ 0 & 0 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} x \\ x \\ x \end{bmatrix}\begin{bmatrix} 0 & 0 & x \end{bmatrix} = \begin{bmatrix} 0 & 0 & x \\ 0 & 0 & x \\ 0 & 0 & x \end{bmatrix}$$

(28) Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ + $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \end{bmatrix}$

(1) $$AB = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} B = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{23} & b_{33} & b_{34} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}\begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}\begin{bmatrix} b_{12} \\ b_{22} \\ b_{23} \end{bmatrix}$$

$$+ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & \end{bmatrix}$$

(1) $$A\begin{bmatrix} | & | & | & | & | \end{bmatrix} = \begin{bmatrix} A| & A| & A| & A| \end{bmatrix}$$

(2) $$\begin{bmatrix} — \\ — \end{bmatrix} B = \begin{bmatrix} —B \\ —B \end{bmatrix}$$

(3) $\begin{bmatrix} \text{—} \\ \text{—} \end{bmatrix} \begin{bmatrix} | & | & | & | \end{bmatrix} = \begin{bmatrix} \text{—}| & \text{—}| & \text{—}| & \text{—}| \\ \text{—}| & \text{—}| & \text{—}| & \text{—}| \end{bmatrix}$

(4) $\begin{bmatrix} | & | & | \end{bmatrix} \begin{bmatrix} \text{—} \\ \text{—} \\ \text{—} \end{bmatrix} = \begin{bmatrix} | \text{—} \end{bmatrix} + \begin{bmatrix} | \text{—} \end{bmatrix} + \begin{bmatrix} | \text{—} \end{bmatrix}$

(29) $A = [a_1 | a_2 | \cdots | a_n]$ be the n columns of A

let $x = \begin{bmatrix} x_1 \\ \hline x_2 \\ \hline \vdots \\ \hline x_n \end{bmatrix}$ be partitions of x

Then $Ax = [a_1 | a_2 | \cdots | a_n] \begin{bmatrix} x_1 \\ \hline x_2 \\ \hline \vdots \\ \hline x_n \end{bmatrix} = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$

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(30)

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -2 & 0 & 1 \\ 8 & 5 & 3 \end{bmatrix}$$

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$$

$$E_{21}A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 8 & 5 & 3 \end{bmatrix}$$

$$E_{31}E_{21}A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix} = EA$$

Now $E_{31}E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$

(31)

$$EA = \begin{bmatrix} 1 & 0 \\ -c/a & I \end{bmatrix}\begin{bmatrix} a & b^T \\ c & D \end{bmatrix} = \begin{bmatrix} a & b^T \\ 0 & -\frac{cb^T}{a} + D \end{bmatrix}$$

In problem 30 $\quad c = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$; $a = 2$; $b^T = [1 \ 0]$

+ $\quad D = \begin{bmatrix} 0 & 1 \\ 5 & 3 \end{bmatrix}$

Then $D - \frac{cb^T}{a} = \begin{bmatrix} 0 & 1 \\ 5 & 3 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -2 \\ 8 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 1 \\ 5 & 3 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -2 & 0 \\ 8 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 5 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & \cancel{0} \\ 1 & 3 \end{bmatrix}$$

which agrees with problem 30 ✓

(32) $(A + iB)(x + iy) = Ax + iBx + iAy - By$

$$\begin{bmatrix} A & -B \\ B & A \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} Ax - By \\ Bx + Ay \end{bmatrix}$$

(33) $(a+ib)(c+id) = ac + iad + ibc - bd$

$$= ac - bd + i(ad + bc)$$

~~~~ rather than compute all 4 terms ac, bd, ad, bc

~~~~ we can get

$(ad + bc)$ as $(a+b)(c+d) - ac - bd$

$$\cancel{ac} + ad + bc + \cancel{bd} - \cancel{ac} - \cancel{bd}$$
$$= ad + bc$$

Just compute the 3 products $(a+b)(c+d)$, ac, bd only

then ~~add~~ + subtract to get $ad+bc$.

(a) A ~~dot~~ product requires ~~[illegible]~~ $n-1$ or $O(n)$ additions + $O(n)$ multiplications
$n \times n$ matrix multiplication requires n^2 dot products

+ $\therefore$ $O(n^3)$ additions + $O(n^3)$ multiplications

(b) AC + BD each require $\cancel{O(n^3)}$ $O(n^3)$ additions + multiplications.
the sum is $O(n^2)$ so the β term

$AC - BD$ is $\cancel{O(n^3)}$ $2n^3$

Sim. S requires $O(2n^3)$ additions thus in total

we require $O(4n^3)$ additions

(c) <u>3M method</u>:

$A+B$ requires n^2 additions
$C+D$ " n^2 additions

$(A+B)(C+D)$ requires n^3 additions

$AC + BD$ require $2n^3$ additions

$(A+B)(C+D) - AC - BD$ requires $3n^2$ additions

Giving in total $3n^3 + 5n^2 = O(3n^3)$ calculations

when $n=2$ we get

$$4n^3\Big|_{n=2} = 4 \cdot 8 = \cancel{24}\ 32$$

or $$3n^3\Big|_{n=2} = 3 \cdot 8 = 24$$

(34) $Ax = [Ax_1 \,\vdots\, Ax_2 \,\vdots\, Ax_3] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

(35) $$A\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$A\begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$A\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

Then $b = \begin{pmatrix}3\\5\\8\end{pmatrix} = 3\begin{pmatrix}1\\0\\0\end{pmatrix} + 5\begin{pmatrix}0\\1\\0\end{pmatrix} + 8\begin{pmatrix}0\\0\\1\end{pmatrix}$

So $$= 3A\begin{pmatrix}1\\1\\1\end{pmatrix} + 5A\begin{pmatrix}0\\1\\1\end{pmatrix} + 8A\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$= A\begin{pmatrix}3\\3\\3\end{pmatrix} + A\begin{pmatrix}0\\5\\5\end{pmatrix} + A\begin{pmatrix}0\\0\\8\end{pmatrix}$$

$$= A\begin{pmatrix}3\\8\\16\end{pmatrix}$$

So $x = \begin{pmatrix}3\\8\\16\end{pmatrix}$

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

The concatenation of the 3 columns

x_1, x_2, x_3

r_1

$r_1 + r_2$

$\underline{r_1} + r_2 + r_3$

So $$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

Check $$A^{-1}A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & +1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$

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(1) $A^{-1} = \frac{1}{-12}\begin{bmatrix} 0 & -3 \\ -4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1/4 \\ 1/3 & 0 \end{bmatrix}$

Skipped

(2) $P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ $P^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = P^T$ ✓

check $P \cdot P^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ✓

$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ $P^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \stackrel{?}{=} P^T$ ✓

Check $\cancel{P \cdot P^{-1}} = \Big[$

(3) $= \begin{bmatrix} 10 & 20 \\ 20 & 50 \end{bmatrix} \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 10 & 20 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\Rightarrow \quad c = -1/5 \qquad a = \dfrac{1 - 20(-1/5)}{10} = \dfrac{1+4}{10} = \dfrac{1}{2}$

rest skipped.

(4) try to solve for the column $\begin{bmatrix} a \\ c \end{bmatrix}$ which gives

$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ The last eq is a contradiction & thus there is No solution

(5) $A^2 = I$

let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓

let, $A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

let $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ then $A^2 = I$

let $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ then $A^2 = I$

Any matrix + $A = B \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} B^{-1}$

Then $A^2 = \underbrace{B \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \underbrace{B^{-1} B}_{I} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}_{I} B^{-1}}_{I} = I.$

(6) (a) $AB = AC$ multiply by A^{-1}

$B = C$.

(b) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $B \neq C$ + Note A is not invertible since $\det(A) = 0$

$AB = AC$

let $B = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ + $C = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$

Then $AB = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$ + $AC = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$ + $B \neq C$

(7) (a) ~~$3x+Px$~~ Think that A partitioned into 3 rows

+ that x partitioned into 1 column then

$$Ax = \begin{bmatrix} r_1^T \\ r_2^T \\ r_3^T \end{bmatrix} x = \begin{bmatrix} r_1^T x \\ r_2^T x \\ r_3^T x \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

If $r_1^T x = 1$ + $r_2^T x = 0$

Then $r_3^T x = (r_1^T + r_2^T) x = r_1^T x + r_2^T x = 1 + 0 = 1 \neq 0$.

Thus $r_3^T x \neq 0$ + $Ax = (1, 0, 0)$ cannot have a solution

(b) All R.H.S. when $b_3 = b_1 + b_2$ which is not true in part (a)

(c) Row 3 is elliminated to produce a row of zeros

No 3rd pivot.

(8) A invertible

+ $B = PA = \begin{bmatrix} P & 1 & 0 & \cdots & & \\ 1 & 0 & \cdots & & & \\ 0 & 0 & 1 & 0 & \cdots & \\ & & & \ddots & & \\ & & & & 1 & \end{bmatrix}$

So $B^{-1} = A^{-1}P^{-1}$ thus B <u>is</u> invertible +

$P^{-1} = P^T = P$

$A^{-1}P$ is exchange of the 1st two colums of A^{-1}.

exchanging rows 1st 2 in A exchanges columns $1 + 2$ in A^{-1}.

9 $A^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1/5 \\ 0 & 0 & 1/4 & 0 \\ 0 & 1/3 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \end{bmatrix}$

check $AA^{-1} = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 3 & 0 \\ 0 & 4 & 0 & 0 \\ 5 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1/5 \\ 0 & 0 & 1/4 & 0 \\ 0 & 1/3 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ ✓

$B^{-1} = \begin{bmatrix} [\ \ ]^{-1} & \underline{\underline{0}} \\ \underline{\underline{0}} & [\ \ ]^{-1} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{bmatrix} 6 & -5 \\ -7 & 6 \end{bmatrix} \end{bmatrix}$

12 +

Check

$$B \cdot B^{-1} = \begin{bmatrix} 3 & 2 & 0 & 0 \\ 4 & 3 & 0 & 0 \\ 0 & 0 & 6 & 5 \\ 0 & 0 & 7 & 6 \end{bmatrix} \begin{bmatrix} 3 & -2 & 0 & 0 \\ -4 & 3 & 0 & 0 \\ 0 & 0 & 6 & -5 \\ 0 & 0 & -7 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \checkmark$$

(10) (a) let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

(b) ~~A~~ let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

(11) If $C = AB$ is invertible

Then $C^{-1} = B^{-1}A^{-1}$

& $A^{-1} = BC^{-1}$

(12) & $M = ABC$

$M^{-1} = \cancel{A}\, C^{-1}B^{-1}A^{-1}$

So $B^{-1} = CM^{-1}A$

(13) Subtract row 1 from row 2

So ~~$\cancel{B}$~~ $M = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ $\quad M^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$

$+ \; B^{-1} = A^{-1} M^{-1} = A^{-1} \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$

(14) ~~It~~ Assume it had an inverse then $A^{-1}A = I$

but what eacs column of A is all zeros say the ith

colum ~~A~~ has $A^{-1} a_i = A^{-1} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \neq e_i$

thus A cannot have an inverse

(15) $$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad-bc & -a\cancel{b}+ab \\ cd-cd & -bc+da \end{bmatrix}$$

$$= \begin{bmatrix} ad-bc & 0 \\ 0 & -bc+da \end{bmatrix}$$

$$= (ad-bc) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{(ad-bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$+\begin{bmatrix} d & -b \\ -c & d \end{bmatrix}^{-1} = \frac{1}{(ad-bc)}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

16 (a) $$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & -1 & 1 \end{bmatrix} \checkmark$$

My Guess $$E = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & -1 & 1 \end{bmatrix}$$

(b) My Guess

$\cancel{E =}$ $$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

As a product / sequence of operation

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$ Same ✓

(17) (a) let $\cancel{x_3 = -(x_1 + x_2)}$ $x = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

Then block multiply Ax in terms of the columns of A

$$Ax = a_1x_1 + a_2x_2 + a_3x_3 = a_1x_1 + a_2x_2 + (a_1 + a_2)x_3$$

$$= a_1 + a_2 - (a_1 + a_2) = 0$$

& $\underline{x} \neq 0$

(b) ~~That Not.~~

let $\cancel{A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 5 \end{bmatrix}}$ let $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 3 & 5 \end{bmatrix}$

Then $E_{31}A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 5 \end{bmatrix}$ $E_{32}E_{31}A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}$

~~& the columns are not~~

(b) looks like yes

let $A = [a_1 | a_2 | a_3]$

Then $EA = [Ea_1 | Ea_2 | Ea_3]$

Q: does $Ea_1 + Ea_2 = Ea_3$, A: yes

Since $a_3 = a_1 + a_2$ & $\therefore$ $Ea_3 = E(a_1 + a_2)$

$$= Ea_1 + Ea_2$$

Why is there No 3rd pivot? Because since after elimination

the element in $a_{33} = a_{31} + a_{32}$ ~~but~~ ~~since~~ by

the argument that elimination preserves the column sum

identity. But since the purpose of elimination is

to make $a_{31} = a_{32} = 0$ $a_{33} = 0 + 0 = 0$

& $\therefore$ there is No 3rd pivot.

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(18) If B is the inverse of A^2

Then $BA^2 = I$ & $A^2B = I$

$\Rightarrow (BA)A = I$ $\Rightarrow A(AB) = I$

Showing BA is the inverse of A ~~showing~~ showing AB is the inverse ~~of~~ A.

thus $AB = BA$

(19)

$$\begin{bmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} a & b & b & b \\ b & a & b & b \\ b & b & a & b \\ b & b & b & a \end{bmatrix}$$

pick ~~a = 1/4~~ & ~~b = -1~~

4 row 1 + row 2 + row 3 + row 4

−row 1 + 4 row 2 + row 3 + row 4

−row 1 + row 2 + 4 row 3 + row 4

−row 1 + row 2 + row 3 + 4 row 4

Check

$$\cancel{\begin{bmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{bmatrix}^{1/4}}$$

pick ~~a~~ $\cancel{|a| = \frac{1}{4}}$ $\cancel{|b| = -\frac{1}{4}}$ I can't seem to "see" the correct values by ~~can~~ attempting to "descramble" the rows above

But since we ~~or~~ "told" the form of the inverse we can compute its action on A + ~~~~ compute constraints on a + b

$$\begin{bmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{bmatrix} \begin{bmatrix} a & b & b & b \\ b & a & b & b \\ b & b & a & b \\ b & b & b & a \end{bmatrix}$$

$$= \begin{bmatrix} 4a-3b & 4b-2b-a & 4b-2b-a & 2b-a \\ 2b-a & 4a-3b & '' & '' \\ '' & & & \\ & & & \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus $4a-3b = 1$

\+ $2b - a = 0 \Rightarrow a = 2b$

so $\cancel{4a-3}\ 4(2b) - 3b = 1 \Rightarrow 5b = 1 \Rightarrow b = 1/5$

\+ $a = 2/5$

$$\therefore \begin{bmatrix} & & & \\ & & & \end{bmatrix}^{-1} = \begin{bmatrix} 2/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 2/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 2/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 2/5 \end{bmatrix}$$

(20) 16 2x2 matrices

~~0~~ ~~0000~~ ~~0001~~ ~~0010~~ ~~0011~~ ~~0~~

To be invertable the determinent must be Non Zero

$$a_{11}a_{22} - a_{12}a_{21} \neq 0$$

Enumerating all of the possibilities

a_{11}	a_{12}	a_{21}	a_{22}	~~Invertable~~ $a_{11}a_{22} - a_{12}a_{21} \neq 0$
0	0	0	0	No
0	0	0	1	No
0	0	1	0	No
0	0	1	1	No
0	1	0	0	No
0	1	0	1	No
0	1	1	0	Yes
0	1	1	1	~~No~~ Yes
1	0	0	0	No
1	0	0	1	Yes
1	0	1	0	No
1	0	1	1	Yes
1	1	0	0	No
1	1	0	1	Yes
1	1	1	0	Yes
1	1	1	1	No

These invertible matrices become:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

(21) $$[A\ I] = \begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{bmatrix}$$

So $\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}^{-1} = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix}$

$$[A\ I] = \begin{bmatrix} 1 & 3 & 1 & 0 \\ 3 & 8 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & -1 & -3 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & -3 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 10 & 3 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$

so $$\begin{bmatrix} 1 & 3 \\ 3 & 8 \end{bmatrix}^{-1} = \begin{bmatrix} 10 & 3 \\ -3 & -1 \end{bmatrix}$$

22
$$\begin{bmatrix} 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1/2 & 0 & 1/2 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 3/2 & 1 & -1/2 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 2/3 & -1/3 & 2/3 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \cancel{\begin{bmatrix} 1 & 0 & -1/3 & 2/3 & -1/3 & 0 \\ 0 & 1 & 2/3 & -1/3 & 2/3 & 0 \\ 0 & 0 & 5/3 & 1/6 & -1/3 & 1 \end{bmatrix}}$$

$-\frac{1}{2}\left(\frac{2}{3}\right) = -\frac{1}{3}$

$-\frac{1}{2}\left(-\frac{1}{3}\right) = \frac{1}{6}$

$+\frac{1}{2} \quad + 1/2 = \frac{4}{6} = \frac{2}{3}$

$-\frac{1}{2}\left(\frac{2}{3}\right) = -\frac{1}{3}$

$\frac{2}{3}\left(-\frac{1}{2}\right) + 2 = -\frac{1}{3} + \frac{6}{3} = \frac{5}{3}$

$-\frac{1}{3}\left(-\frac{1}{2}\right) + 0 = \frac{1}{6}$

$\frac{2}{3}\left(-\frac{1}{2}\right) = -\frac{1}{3}$

$$\Rightarrow \cancel{\begin{bmatrix} 1 & 0 & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 1 & \frac{1}{10} & -\frac{1}{5} & \frac{3}{5} \end{bmatrix}}$$

$$\Rightarrow \begin{bmatrix} 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & -\frac{2}{3} & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & -\frac{2}{3} & 1 \end{bmatrix}$$

$\frac{6}{3} - \frac{2}{3}$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 1 & \frac{1}{4} & -\frac{1}{2} & \frac{3}{4} \end{bmatrix}$$

$\frac{2}{3}\left(-\frac{1}{2}\right) =$

$-\frac{1}{3}\left(-\frac{1}{2}\right) + \frac{1}{2} = \frac{1}{6} + \frac{1}{2} = \frac{1}{6} + \frac{3}{6} = \frac{2}{3}$

$-\frac{1}{2}\left(\frac{2}{3}\right) + 0 =$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & \frac{3}{4} & -\frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{4} & -\frac{1}{2} & \frac{3}{4} \end{bmatrix}$$

$-\frac{2}{3}\left(\frac{1}{4}\right) - \frac{1}{3} = -\frac{1}{3}\left(\frac{3}{2}\right) = -\frac{1}{2}$

$-\frac{2}{3}\left(-\frac{1}{2}\right) + \frac{2}{3} = \frac{1}{3}(1+2) = 1$

$-\frac{2}{3}\left(+\frac{3}{4}\right) + 0 = \cancel{\frac{4}{3}} -\frac{1}{2}$

$\frac{1}{3}\left(\frac{1}{4}\right) + \frac{2}{3} = \frac{1}{3}\left(\frac{9}{4}\right) = \frac{3}{4}$

$\frac{1}{3}\left(-\frac{1}{2}\right) - \frac{1}{3} = \frac{1}{3}\left(-\frac{3}{2}\right) = -\frac{1}{2}$

$\frac{1}{3}\left(\frac{3}{4}\right) + 0 = \frac{1}{4}$

Lets check

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 3/4 & -1/2 & 1/4 \\ -1/2 & 1 & -1/2 \\ 1/4 & -1/2 & 3/4 \end{bmatrix} = \begin{bmatrix} 3/2-\frac{1}{2} & -1+1 & \frac{1}{2}-\frac{1}{2} \\ \frac{3}{4}-1+\frac{1}{4} & -\frac{1}{2}+2-\frac{1}{2} & \frac{1}{4}-1+\frac{3}{4} \\ -\frac{1}{2}+\frac{1}{2} & 1-1 & -\frac{1}{2}+\frac{3}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ yes } \checkmark$$

(23) Form the augmented matrix + Perform Gauss Jordan elliminatio

$$\left[\begin{array}{ccc|ccc} 1 & a & b & 1 & 0 & 0 \\ 0 & 1 & c & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$\Rightarrow$ From the form of the matrix A
Its inverse is given by

$$A^{-1} = \begin{bmatrix} 1 & - & -b-c \; \cdots \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}$$

Gauss Jordan elliminatio gives:

$$\begin{bmatrix} 1 & 0 & b-ac & 1 & -a & 0 \\ 0 & 1 & c & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & -a & -b+ac \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$-b-ac$

Check:

$$\therefore \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -a & -b+ac \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \cancel{-b}\cancel{-ac}+\cancel{ac}\cancel{+b} \\ 0 & 1 & -c+c \\ 0 & 0 & 1 \end{bmatrix} = I \quad \checkmark$$

(24) For the 1st A the agumented matrix becomes

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 1/2 & 1/2 & 1/2 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$\Rightarrow$ ... Skipped.

(25)

$$E_{21}A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \quad \text{w/ } E_{21} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

$$E_{12}E_{21}A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{w/ } E_{12} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$+\ D^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$ we have

$$D^{-1}E_{12}E_{21}A = I$$

Giving $A^{-1} = D^{-1}E_{12}E_{21}$

$$A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -1 & 1/2 \end{bmatrix}$$

Check:

$$\begin{bmatrix} 3 & -1 \\ -1 & 1/2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

(26)

$$\left[\begin{array}{cccccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cccccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cccccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & -3 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\therefore \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

(27) $$\begin{bmatrix} 0 & 2 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 & 1/2 \\ 0 & 2 & 1 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 & 1/2 \\ 0 & 1 & 1/2 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -1/2 & 1/2 \\ 0 & 1 & 1/2 & 0 \end{bmatrix}$$

So $A^{-1} = \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 0 \end{bmatrix}$

Then $\begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(28) (a) ~~4 zeros~~

with 13 zeros true

with 4 zeros False

$$\begin{matrix} 0 & 1 & 1 & 1 \\ x & x & x & x \\ x & x & x & x \\ x & x & x & x \end{matrix}$$

~~4~~, 5, 6, 7, 8

$$\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$$

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(b) False consider

$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $\det(A) = 0$

(c) True

(d) True as $(A^2)^{-1} = (A^{-1})^2$

(29)

$$\begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix}$$

if $c=2$ the matrix is Not invertible since it ~~has~~ ~~2~~ would have 2 rows the same

$$\left[\begin{array}{ccc|ccc} 2 & c & c & 1 & 0 & 0 \\ c & c & c & 0 & 1 & 0 \\ 8 & 7 & c & 0 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} 1 & c/2 & c/2 & 1/2 & 0 & 0 \\ c & c & c & 0 & 1 & 0 \\ 8 & 7 & c & 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & c/2 & c/2 & 1/2 & 0 & 0 \\ 0 & c-\frac{c^2}{2} & c-\frac{c^2}{2} & -\frac{c}{2} & 1 & 0 \\ 0 & 7-4c & c-4c & -4 & 0 & 1 \end{bmatrix}$$

If $c-\frac{c^2}{2} = 0$ there is No 2nd pivot

$\Rightarrow$ $c=0$ or $c=2$

In the case $c=0$ we have a row of all zeros & A is $\therefore$ Not invertible. Assuming $c \neq 0$ & $c \neq 2$ we have

$$\left[\begin{array}{ccc|ccc} 1 & c/2 & c/2 & 1/2 & 0 & 0 \\ 0 & 1 & 1 & \frac{-c/2}{c-c^2/2} & \frac{1}{c-c^2/2} & 0 \\ 0 & 7-4c & -3c & -4 & 0 & 1 \end{array}\right]$$

$$= \frac{-c/2}{c(1-\frac{c}{2})} = \frac{-1}{2(1-c/2)} = \frac{-1}{(2-c)}$$

$$\begin{array}{ccc} 1/2 & 0 & 0 \\ \frac{-1}{2-c} & \frac{2}{c(2-c)} & 0 \\ -4 & 0 & 1 \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} - \frac{c}{2}\left(-\frac{1}{2-c}\right) & \left(-\frac{c}{2}\right)\left(\frac{2}{c(2-c)}\right) & 0 \\ 0 & 1 & 1 & \frac{-1}{2-c} & \frac{2}{c(2-c)} & 0 \\ 0 & 0 & \underbrace{-7+4c-3c}_{-7+c} & +\frac{(7+4c)}{2-c} - 4 & \frac{(-7+4c)(2)}{c(2-c)} & 1 \end{array}\right]$$

Thus $c \neq 7$ or else we have two 0's columns the same & $\therefore$ no inverse

Simplifying each expression in turn we obtain

$$\frac{1}{2}\left[\frac{2-c+c}{2-c}\right] = \frac{1}{2-c}$$

$$-\frac{c}{2}\left(\frac{2}{c(2-c)}\right) = \frac{-1}{2-c}$$

$$\frac{7+4c}{2-c} - \frac{4(2-c)}{2-c} = \frac{7+4c-8+4c}{2-c} = \frac{-1+8c}{2-c}$$

Thus the above is

$$\begin{bmatrix} 1 & 0 & 0 & \frac{1}{2-c} & \frac{-1}{2-c} & 0 \\ 0 & 1 & 1 & \frac{-1}{2-c} & \frac{2}{c(2-c)} & 0 \\ 0 & 0 & -7+c & \frac{-1+8c}{2-c} & \frac{2}{c}\frac{(-7+4c)}{(2-c)} & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2-c} & \frac{-1}{2-c} & 0 \\ 0 & 1 & 1 & \frac{-1}{2-c} & \frac{2}{c(2-c)} & 0 \\ 0 & 0 & 1 & \frac{-1+8c}{(-7+c)(2-c)} & \frac{2}{c}\frac{(-7+4c)}{(2-c)(-7+c)} & \frac{1}{(-7+c)} \end{bmatrix}$$

continuing we have ...

Stopped in computing full inverse

30. If $a=b$ we have equal columns/rows
If $a=0$ we have no 1st pivot

To see if there are any others consider the ~~augo~~ augmented matrix & perform ~~the~~ Gauss Jordan ellimination on it.

$$\left[\begin{array}{ccc|ccc} a & b & b & 1 & 0 & 0 \\ a & a & b & 0 & 1 & 0 \\ a & a & a & 0 & 0 & 1 \end{array}\right]$$

Assume $a \neq 0$
or else No 1st pivot exists

$$= \left[\begin{array}{cccccc} \cancel{1} & \cancel{b/a} & \cancel{b/a} & \cancel{1/a} & \cancel{0} & \cancel{0} \\ a & a & b & 0 & 1 & 0 \\ a & a & a & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cccccc} 1 & b/a & b/a & 1/a & 0 & 0 \\ 0 & a-b & 0 & -1 & 1 & 0 \\ 0 & a-b & a-b & -1 & 0 & 1 \end{array}\right]$$

~~Assume $a-b \neq 0$ or else No 2nd pivot.~~

$$\Rightarrow \left[\begin{array}{cccccc} a & b & b & 1 & 0 & 0 \\ 0 & a-b & 0 & -1 & 1 & 0 \\ 0 & a-b & a-b & -1 & 0 & 1 \end{array}\right]$$

Assume $a-b \neq 0$
or else No 2nd pivot exists

$$\Rightarrow \left[\begin{array}{cccccc} a & 0 & b & 1-\frac{b}{a-b} & \frac{-b}{a-b} & 0 \\ 0 & a-b & 0 & -1 & 1 & 0 \\ 0 & 0 & a-b & 0 & -1 & 1 \end{array}\right]$$

$\frac{-b}{a-b}$

$$\Rightarrow \left[\begin{array}{cccccc} a & 0 & 0 & 1-\frac{b}{a-b} & 0 & \frac{-b}{a-b} \\ 0 & a-b & 0 & -1 & 1 & 0 \\ 0 & 0 & a-b & 0 & -1 & 1 \end{array}\right]$$

$\frac{-b}{a-b}(-1) - \frac{b}{a-b} = 0$

$$\Rightarrow \left[\begin{array}{cccccc} 1 & 0 & 0 & \frac{1}{a}\left(1-\frac{b}{a-b}\right) & 0 & \frac{-b}{a(a-b)} \\ 0 & 1 & 0 & \frac{-1}{a-b} & \frac{1}{a-b} & 0 \\ 0 & 0 & 1 & 0 & \frac{-1}{a-b} & \frac{1}{a-b} \end{array}\right]$$

(31) $A^{-1} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ By inspection

check $\begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ ✓

The 5×5 inverse would be

$$A^{-1} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(32) $x = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 1 \end{pmatrix}$ & $x = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 1 \end{pmatrix}$

33 See matlab code...

(34)

$$\begin{bmatrix} I & 0 \\ C & I \end{bmatrix}^{-1} = \begin{bmatrix} I & 0 \\ -C & I \end{bmatrix}$$

then checking
$$\begin{bmatrix} I & 0 \\ C & I \end{bmatrix}\begin{bmatrix} I & 0 \\ -C & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \checkmark$$

$$\begin{bmatrix} A & 0 \\ C & D \end{bmatrix}^{-1} = ?$$

$$\begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix}\begin{bmatrix} A & 0 \\ C & D \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$$

$$\begin{bmatrix} A^{-1} & 0 \\ 0 & D^{-1} \end{bmatrix}\begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix}\begin{bmatrix} A & 0 \\ C & D \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

Thus
$$\begin{bmatrix} A & 0 \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & 0 \\ 0 & D^{-1} \end{bmatrix}\begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix}$$

$$= \begin{bmatrix} A^{-1} & 0 \\ -D^{-1}CA^{-1} & D^{-1} \end{bmatrix}$$

$$\begin{bmatrix}0 & I\\ I & 0\end{bmatrix} \Rightarrow \begin{bmatrix}I & 0\\ 0 & I\end{bmatrix}$$

So $$\begin{bmatrix}0 & I\\ I & 0\end{bmatrix}\begin{bmatrix}0 & I\\ I & 0\end{bmatrix} = \begin{bmatrix}I & 0\\ 0 & I\end{bmatrix}$$

$$\therefore \left(\begin{bmatrix}0 & I\\ I & 0\end{bmatrix}\begin{bmatrix}0 & I\\ I & 0\end{bmatrix}\right)^{-1} = \begin{bmatrix}0 & I\\ I & 0\end{bmatrix}^{-1}\begin{bmatrix}0 & I\\ I & 0\end{bmatrix}^{-1} = \begin{bmatrix}I & -0\\ 0 & I\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}0 & I\\ I & 0\end{bmatrix}^{-1} = \begin{bmatrix}I & -0\\ 0 & I\end{bmatrix}\begin{bmatrix}0 & I\\ I & 0\end{bmatrix} = \begin{bmatrix}-0 & I\\ I & 0\end{bmatrix}$$

Check $$\begin{bmatrix}0 & I\\ I & 0\end{bmatrix}\begin{bmatrix}-0 & I\\ I & 0\end{bmatrix} = \begin{bmatrix}I & 0\\ 0 & I\end{bmatrix} \checkmark$$

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(1) $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 5 \end{bmatrix}$ let $E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$

$$E_{31}A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{bmatrix} = U$$

Then $\cancel{A}\ E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$ & $L = E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$

so $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{bmatrix}$

& for $A = LDU$ we have

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}$$

(2) For the given A

$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ so $E_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Now $E_{21}A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 4 & 0 \end{bmatrix}$

Then $E_{32}E_{21}A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -6 \end{bmatrix}$ w/ $E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ & $E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$

So $U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -6 \end{bmatrix}$

$$A = E_{32}^{-1} E_{21}^{-1} U = LU$$

w/ $L = E_{32}^{-1} E_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix}$$

Then $A = LU$ is

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 5 \\ 0 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -6 \end{bmatrix}$$

+ $A = LDU$ is

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -6 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3/2 \\ 0 & 0 & 1 \end{bmatrix}$$

(3) $E_{21}A = \begin{bmatrix}1 & 0 & 1\\ 0 & 2 & 0\\ 3 & 4 & 5\end{bmatrix}$ when $E_{21} = \begin{bmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$

& $E_{31}E_{21}A = \begin{bmatrix}1 & 0 & 1\\ 0 & 2 & 0\\ 0 & 4 & 2\end{bmatrix}$ when $E_{31} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -3 & 0 & 1\end{bmatrix}$

Finally $E_{32}E_{31}E_{21}A = \underbrace{\begin{bmatrix}1 & 0 & 1\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}}_{\equiv U}$ when $E_{32} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & -2 & 1\end{bmatrix}$

Now the inverse of each elimination matrix is given by

$$A = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}U = \begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 3 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 2 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 1\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}$$

$$= \begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 3 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 2 & 1\end{bmatrix} \; U$$

$$= \begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 3 & 2 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 1\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}$$

Defining $A = LDU$ we have

$$= \begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 3 & 2 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

(4) (a) our elimination matrices are

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

then $$E_{21}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b & c & 1 \end{bmatrix}$$

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -b & 0 & 1 \end{bmatrix} \quad \text{w/} \quad E_{31}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b & 0 & 1 \end{bmatrix}$$

then $$E_{31}E_{21}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{bmatrix}$$

Now define

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -c & 1 \end{bmatrix} \quad \text{w/} \quad E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{bmatrix}$$

then $$E_{32}E_{31}E_{21}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This product is

$$E = E_{32}E_{31}E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -c & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -b & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -b & -c & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ -b+ac & -c & 1 \end{bmatrix}$$

(b) $L = E_{21}^{-1} E_{31}^{-1} E_{32}^{-1}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}$$

5. $$\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 1 & 0 \\ l & 1 \end{bmatrix}\begin{bmatrix} d & e \\ 0 & f \end{bmatrix}$$

$$= \begin{bmatrix} d & e \\ dl & le+f \end{bmatrix}$$

Thus $d=0$ + $e=1$ but this would require $2 = dl$

but $2=0$, which is not possible.

The 2nd ~~matrix~~

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l & 1 & 0 \\ m & n & 1 \end{bmatrix}\begin{bmatrix} d & e & g \\ 0 & f & h \\ 0 & 0 & i \end{bmatrix}$$

Multiplying out the two matrices on the right we have

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} d & e & g \\ ld & el+f & lg+h \\ md & em+fn & mg+nh+i \end{bmatrix}$$

Then equating the 1st row we have that

$d=1$; $e=1$; $g=0$; $l=1$. This gives

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1+f & h \\ m & m+fn & nh+i \end{bmatrix}$$

giving $f=0$; $h=2$; $m=1$; ~~$2=1+f$~~ But with this specification we see that the (3,2) equation is then

$2 = 1+0 = 1$ Which ~~can~~ is in direct contradiction. Showing that a decomposition like this cannot exist.

This can also be seen by performing row reduction on the need matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$ we will obtain ~~need~~ as an intermediate state ~~one~~ with a zero as a pivot.

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(6)

$$\begin{bmatrix} 1 & c & 0 \\ 2 & 4 & 1 \\ 3 & 5 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & c & 0 \\ 0 & 4-2c & 1 \\ 0 & 5-3c & 1 \end{bmatrix}$$

If $4-2c=0$ or $c=2$ then there is no pivot in the 2nd row.

Assuming $c \neq 2$ we then get

$$\begin{bmatrix} 1 & c & 0 \\ 0 & 4-2c & 1 \\ 0 & 0 & 1-\frac{(5-3c)}{4-2c} \end{bmatrix}$$

The element $a(3,3)$ is $\dfrac{1(4-2c)-5+3c}{4-2c}$

$$= \frac{4-2c-5+3c}{4-2c} = \frac{-1+c}{4-2c}$$

If this element $=0$ we have No 3rd pivot + ellimination fails. The pivot equals zero if $c=1$

(7) $A = \begin{bmatrix} 2 & 4 & 8 \\ 0 & 3 & 9 \\ 0 & 0 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 4 & 8 \\ 0 & 3 & 9 \\ 0 & 0 & 7 \end{bmatrix}$

So $L = I$ $\qquad = I \underbrace{\begin{bmatrix} 2 & & \\ & 3 & \\ & & 7 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}}_{U'}$

+ $U = A$

(8) $~~A~~$ $\underbrace{E_{21}A = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}}_{=U}$ w/ $E_{21} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

Then

$$A = E_{21}^{-1}U = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

Note $U = L^T$ "cholesky" factorization.

$B = \cdots$

$$E_{21}B = \begin{bmatrix} 1 & 4 & 0 \\ 0 & -4 & 4 \\ 0 & 4 & 0 \end{bmatrix} \quad \text{w/} \quad E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_{32}E_{21}B = \underbrace{\begin{bmatrix} 1 & 4 & 0 \\ 0 & -4 & 4 \\ 0 & 0 & 4 \end{bmatrix}}_{U} \quad \text{w/} \quad E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Then $B = E_{21}^{-1}E_{32}^{-1}U = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 & 0 \\ 0 & -4 & 4 \\ 0 & 0 & 4 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 & 0 \\ 0 & -4 & 4 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that $L = U^T$ in this last case also

(9) If $E_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ then

$$E_{21}A = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

of course $a \neq 0$ for there to be a pivot in the 1st column

If $E_{31} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$$E_{31}E_{21}A = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & b-a & c-a & c-a \\ a & b & c & d \end{bmatrix}$$

If $E_{41} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$

$$E_{41}E_{31}E_{21}A = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & b-a & c-a & c-a \\ 0 & b-a & c-a & d-a \end{bmatrix}$$

Now $b \neq a$ for there to be a pivot in the $(2,2)$ position.

Assuming $b-a \neq 0$ we can get

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad + \quad E_{42} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

We have that

$$E_{42}E_{32}E_{41}E_{31}E_{21}A = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & c-b & d-b \end{bmatrix}$$

Now $c-b \neq 0$ for them to have a nonzero pivot in the (3,3) position.

Let $E_{43} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$ then

$$E_{43}E_{42}E_{32}E_{41}E_{31}E_{21}A = \underbrace{\begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix}}_{U} \equiv U$$

Finally $d \neq c$ for the 4th pivot to be non zero.

The 4 conditions are $a \neq 0,\ b \neq a,\ c \neq b,\ d \neq c$.

Then

$$A = \underbrace{E_{21}^{-1}\ \cancel{E_{31}^{-1}E_{41}^{-1}E_{32}^{-1}E_{42}^{-1}U}\ E_{31}^{-1}E_{41}^{-1}E_{32}^{-1}E_{42}^{-1}E_{43}^{-1}}_{\equiv L}\ U$$

So $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$

$\cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ -1 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ -1 & -1 & -1 & 1 \end{bmatrix}$$

So

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ -1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ -1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b-a & 0 & 0 \\ 0 & 0 & c-b & 0 \\ 0 & 0 & 0 & d-c \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(10) let $E_* = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$

+ obviously $a \neq 0$ or else there is No pivot in column 1.

Then

$$E_* A = \begin{bmatrix} a & r & r & r \\ 0 & b-r & s-r & s-r \\ 0 & b-r & c-r & t-r \\ 0 & b-r & c-r & d-r \end{bmatrix}$$

Now $b \neq r$ or else there is No pivot in the 2nd row

let

$$E_{**} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

to get

$$E_{**} E_* A = \begin{bmatrix} a & r & r & r \\ 0 & b-r & s-r & s-r \\ 0 & 0 & c-s & t-s \\ 0 & 0 & c-s & d-s \end{bmatrix}$$

Then $c \neq s$ or ~~lx~~ else there is No 3rd pivot

$E_{***} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$ then we have

$$E_{***}E_{**}E_{*}A = \underbrace{\begin{bmatrix} a & r & r & r \\ 0 & b-r & s-r & s-r \\ 0 & 0 & c-s & t-s \\ 0 & 0 & 0 & d-t \end{bmatrix}}_{U}$$

& $d \neq t$ or else there is No 4th pivot.

$$A = E_{*}^{-1}E_{**}^{-1}E_{***}^{-1}U$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}U$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}U$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}U$$

(11) Skipped

(12) Skipped

(13) (a) $L = \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix}$ ~~Ans E~~ $\cancel{= \begin{bmatrix} 1 & 0 & 0 \\ -\ell_{21} & 1 & 0 \\ -\ell_{31} & -\ell_{32} & 1 \end{bmatrix}}$

Multiply By $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -\ell_{21} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ then by $E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\ell_{31} & 0 & 1 \end{bmatrix}$

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\ell_{32} & 1 \end{bmatrix}$$

Then $E_{32}E_{31}E_{21}L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

$$\therefore L^{-1} = E_{32}E_{31}E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\ell_{32} & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\ell_{31} & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -\ell_{21} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\ell_{31} & -\ell_{32} & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -\ell_{21} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\ell_{21} & 1 & 0 \\ -\ell_{31}+\ell_{32}\ell_{21} & -\ell_{32} & 1 \end{bmatrix}$$

(b) ~~It is already in upper triangular so no operations are performed~~

Apply $E_{32}E_{31}E_{21}$ to I gives L^{-1}

~~(c) Applying these steps to LU will produce U~~

(c) Applying these steps to LU give U

(14) (a) since $A = LDU$ + $A = L_1D_1U_1$ we have $LDU = L_1D_1U_1$

$\Rightarrow LD = L_1D_1U_1U^{-1}$

$\Rightarrow L_1^{-1}LD = D_1U_1U^{-1}$

Now L is lower Δ and so is L_1^{-1}. So $L \cdot L_1^{-1}$ is lower triangular
U_1 is upper triangular + U^{-1} is also so U_1U^{-1} is upper triangular.
A diagonal matrix multiplied on the left multiplys the <u>columns</u> of the matrix multiplied by. A diagonal matrix multiplied on the left multiplies the rows of the matrix thus $L_1^{-1}LD$ is lower triangular + $D_1U_1U^{-1}$ is upper triangular

(b) The diagonal elements of $L_1^{-1}LD$ are given by?
Now both matrices in $L_1^{-1}L$ have ones on their main diagonal so the product will have 1's on its main diagonal.

$$\begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ x & x & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ y & 1 & 0 \\ y & y & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ & 1 & \\ & & 1 \end{bmatrix}$$

~~But~~ Multiplying by D ~~gives~~ produces diagonal elts on the diagonal

$$L_1^{-1}LD = \begin{bmatrix} d_1 & 0 & 0 \\ x & d_2 & 0 \\ x & x & d_3 \end{bmatrix}$$

Now ~~By~~ By the same argument U_1 + U_1 have ones on their diagonal so U_1U^{-1} has 1's on its diagonal + $D_1U_1U^{-1}$ has the diagonal elements + D_1 on U_1U^{-1}. Thus the diagonal element of both factorizations must be the same. $D = D_1$

~~Thus~~ ~~$AU^{-1} = L_1^{-1}AU_1^{-1}$~~

$\Leftrightarrow$ ~~$L_1^{-1}A =$~~

Since $L_1^{-1}LD$ is lower triangular + $D_1U_1U^{-1}$ is upper triangle with equal Diagonal elements, looking at the off diagonal elements of $L_1^{-1}LD$ + $D_1U_1U^{-1}$ we see that $(L_1^{-1}LD)_{ij} = 0 \quad i<j \ + \ i>j$

$$L_1^{-1}LD = \begin{bmatrix} d_1 & 0 & 0 \\ x & d_2 & 0 \\ x & x & \ddots d_3 \end{bmatrix} = \begin{bmatrix} d_1 & x & x \\ 0 & d_2 & x \\ 0 & 0 & d_3 \end{bmatrix} = D_1U_1U^{-1}$$

thus $L_1^{-1}LD$ is diagonal only

say $L_1^{-1}LD = D \Rightarrow L_1^{-1}L = I \Rightarrow L_1 = L$ ✓

(15) If $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ with $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ with $E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

Then $A = E_{21}^{-1}E_{32}^{-1}U = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}}_{\equiv L}\underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{\equiv L^T}$$

If $A = \begin{bmatrix} a & a & 0 \\ a & a+b & b \\ 0 & b & b+c \end{bmatrix}$

then $E_{21}A = \begin{bmatrix} a & a & 0 \\ 0 & b & b \\ 0 & b & b+c \end{bmatrix}$ when $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\underbrace{E_{32}E_{21}A = \begin{bmatrix} a & a & 0 \\ 0 & b & b \\ 0 & 0 & c \end{bmatrix}}_{\equiv U}$$ where $E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

Then $A = E_{21}^{-1}E_{32}^{-1}U = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ +1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}}\begin{bmatrix} a & a & 0 \\ 0 & b & b \\ 0 & 0 & c \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} a & a & 0 \\ 0 & b & b \\ 0 & 0 & c \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

(16) L + U could be stored in two arrays one for the diagonal elements + one for the super diagonal.

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 3 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 3 & 4 \end{bmatrix}$$

when $E_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 3 & 4 \end{bmatrix} \quad \text{when } E_{32} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \underbrace{\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\equiv U} \quad \text{when } E_{43} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$A = E_{21}^{-1} E_{32}^{-1} E_{43}^{-1} U$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} U$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} U$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(17) When $A = \begin{bmatrix} x & x & x & x \\ x & x & x & 0 \\ 0 & x & x & x \\ 0 & 0 & x & x \end{bmatrix}$

The LU Factorization of A will ~~produce~~ require

E_{21}, E_{32}, E_{43}

Then the zero at position $A(2,4)$ will not stay zero in forming U since the factor E_{21} will kill it. The zeros at $(3,1)$, $(4,1)$, $(4,2)$ will stay zero in the U factor. The notation in the elimination matrices says that the L factor

$L = E_{43}^{-1} E_{32}^{-1} E_{21}^{-1}$ will be non zero at $(2,1), (3,2), (4,3)$ for the diagonal elements

$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ 0 & x & 1 & 0 \\ 0 & 0 & x & 1 \end{bmatrix}$ Thus All 3 original zeros are still zero in the L factor

When $B = \begin{bmatrix} x & x & x & 0 \\ x & x & 0 & x \\ x & 0 & x & x \\ 0 & x & x & x \end{bmatrix}$

we will require $E_{21}, E_{31}, E_{32}, E_{42}, E_{43}$

$\begin{bmatrix} x & x & x & 0 \\ 0 & x & x & x \\ x & 0 & x & x \\ 0 & x & x & x \end{bmatrix} \begin{bmatrix} x & x & x & 0 \\ 0 & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \end{bmatrix} \begin{bmatrix} x & x & x & 0 \\ 0 & x & x & x \\ 0 & 0 & x & x \\ 0 & x & x & x \end{bmatrix} \begin{bmatrix} x & x & x & 0 \\ 0 & x & x & x \\ 0 & 0 & x & x \\ 0 & 0 & x & x \end{bmatrix}$

$$\begin{bmatrix} x & x & x & 0 \\ 0 & x & x & x \\ 0 & 0 & x & x \\ 0 & 0 & 0 & x \end{bmatrix}$$

Thus the only zero that remains unaltered in the U factor is the one at A(1,4). Assembling L from the elementary ellimination matrices we have

$L = E_{21}^{-1} E_{31}^{-1} E_{32}^{-1} E_{42}^{-1} E_{43}^{-1}$ has ones on the diagonals + nonzero elements at each of (2,1), (3,1), (3,2), (4,2), (4,3)

giving for $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ x & x & 1 & 0 \\ 0 & x & x & 1 \end{bmatrix}$

+ the zeros at (4,1), (1,4), +(2,3) remain zero

Thus in summary if we store L in the lower ~~triangular~~ section of M ~~and~~ U in the upper triangular section of M, we see (ignoring the ones) + U in the upper triangular section of M. we see that ~~matrix~~ for the matrix A, L ~~results in~~ requires no fill in

But U requires a single element fill in.

For the matrix B L requires one element of fill in (3,1) + U requires one element of fill in (2,3)

(18)

$$\begin{bmatrix} x & x & x \\ x & x & x \\ x & x & x \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ x & & 1 \end{bmatrix} \begin{bmatrix} x & x & x \\ 0 & & \\ 0 & & \end{bmatrix}$$

The matrices E_{21} + E_{31} result in the two known entries in L in the 1st column.

(19) If $A = \begin{bmatrix} 5 & 3 & 1 \\ 3 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$$E_{23}A = \begin{bmatrix} 5 & 3 & 1 \\ 2 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{when} \quad E_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_{13}E_{23}A = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{when} \quad E_{13} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_{12}E_{13}E_{23}A = \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}}_{L} \quad \text{when} \quad E_{12} = \begin{bmatrix} +1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So $A = E_{23}^{-1} E_{13}^{-1} E_{12}^{-1} L$

$$= \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad ''$$

$$= \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}}_{L}$$

(20) $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

$E_{21}A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ w/ $E_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$E_{34}E_{21}A = \underbrace{\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}}_{\text{After one elliminaton step}}$ w/ $E_{34} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(A nice parallel version of a direct LU factorization would result from doing elliminations in blocks)

~~Elliminating the elements at (1,2) + (4,3) gives~~

$\cancel{E_{12}E_{34}E_{21}A} = \begin{bmatrix} \cancel{1} & \cancel{0} & \cancel{0} & \cancel{0} \\ \cancel{0} & \cancel{-1} & \cancel{1} & \\ & & & \end{bmatrix}$

Now $E_{32}E_{34}E_{21}A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ which has x_3 directly solveable,

when $E_{32} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Thus

$$E_{32}E_{34}E_{21}Ax = E_{32}E_{34}E_{21}b = E_{32}E_{34}\begin{bmatrix} 5 \\ -2 \\ 8 \\ 2 \end{bmatrix}$$

$$= E_{32}\begin{bmatrix} 5 \\ -2 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 2 \\ 2 \end{bmatrix}$$

So $$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 2 \\ 2 \end{bmatrix}$$

$\Rightarrow$ $x_3 = 1$ which is then passed up + down for elimination

$$\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix} - 1\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

$\Rightarrow$ $x_2 = 3$ + $x_1 = 2$

$x_3 + x_4 = 2$ $\Rightarrow$ $x_4 = 1$

Giving $x_1 = 2,\ x_2 = 3,\ x_3 = 1,\ x_4 = 1$

(21) This sub matrix would have pivots 2 + 7. The addition of another ~~row~~ row at the bottom of the matrix will just affect the pivots after the ones computed on all previous rows. The addition of an extra column does not affect the pivots ~~at~~ at all + is just ~~extra~~ ~~extra~~ carried along ~~...~~ in the calculation.

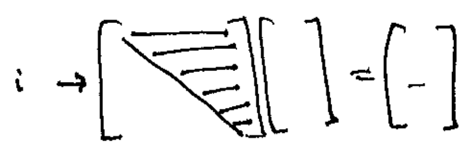
(22) The 1st 3 pivots of M are 2, 7, 6. This is because Elimination on M must 1st be elimination on A. ~~...~~ The row (0 0 0 9) + column (0 0 0) so the matrix becomes

$$\begin{matrix} & & & 0 \\ & A & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 9 \end{matrix}$$

(24) See Next pg

(27) c(k) is the Right hand side, t is the expression

$$t_k = \sum_{j=k+1}^{n} U(k,j)\,x$$



so $(t - c(k))/U(k,k)$ is the expression for x_k

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(24) $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & c & 1 \\ 0 & 1 & 1 \end{bmatrix}$

$\Rightarrow E_{21}A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & c-6 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ so $c \neq 6$ or else LU is Not possible with out pivots

$E_{32}E_{21}A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & c-6 & 1 \\ 0 & 0 & 1-\frac{1}{c-6} \end{bmatrix}$

$= \begin{bmatrix} 1 & 2 & 0 \\ 0 & c-6 & 1 \\ 0 & 0 & \frac{c-6-1}{c-6} \end{bmatrix}$

$\frac{c-7}{c-6}$ so $c \neq 7$ or we don't have our 3rd pivot

$c=6$ will require a $PA=LU$ decomposition

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① $A = \begin{bmatrix} 1 & 0 \\ 8 & 2 \end{bmatrix}$ $\quad A^T = \begin{bmatrix} 1 & 8 \\ 0 & 2 \end{bmatrix}$ $\quad \cancel{A^{-1} = \begin{bmatrix} 2 & 0 \\ -8 & 1 \end{bmatrix} (2-8)}$

$$= \cancel{-\frac{1}{6}\begin{bmatrix} 2 & 0 \\ -8 & 1 \end{bmatrix}}$$

Check $\cancel{A \cdot A^{-1}} = \cancel{-\frac{1}{6}\begin{bmatrix} 1 & 0 \\ 8 & 2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ -8 & 1 \end{bmatrix}}$

$$= \cancel{-\frac{1}{6}\,2}$$

$A^{-1} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ -8 & 1 \end{bmatrix}$ $\quad$ Check $AA^{-1} = \begin{bmatrix} 1 & 0 \\ 8 & 2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ -8 & 1 \end{bmatrix}\frac{1}{2}$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\frac{1}{2}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$(A^{-1})^T = \frac{1}{2}\begin{bmatrix} 2 & -8 \\ 0 & 1 \end{bmatrix}$

$(A^T)^{-1} = \frac{1}{2}\begin{bmatrix} 2 & -8 \\ 0 & 1 \end{bmatrix}$

II $\quad A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad A^T = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

$A^{-1} = \frac{1}{-1}\begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix}$ $\quad \cancel{(A^{-1})^T = \begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix}} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$

&

$(A^{-1})^T = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$ $\quad (A^T)^{-1} = \frac{1}{-1}\begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$

(2) Skipped

If $AB = BA$

$$(AB)^T = B^T A^T$$
$$\| $$
$$(BA)^T$$
$$\|$$
$$A^T B^T$$

(3) (a) $((AB)^{-1})^T = (B^{-1})^T (A^{-1})^T$

(b) U upper triangular U^{-1} is upper triangular so $(U^{-1})^T$ is lower triangular

(4) $A^2 = 0$

let $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ then $A^2 = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

But $A^T A$ has element (i,j) given by

$$(A^T A)_{ij} = \sum_{k=1}^{n} (A^T)_{ik} A_{kj} = \sum_{k=1}^{n} A_{ki} A_{kj} =$$ dot product of the ith + jth row of A ~~cs~~

In general if $(A^T A)_{ii} = 0$

$$\Rightarrow \sum_{k=1}^{n} A_{ki}^2 = 0 \Rightarrow A_{ki} = 0 \quad \forall\, k + i \Rightarrow A \equiv 0$$

(5) (a) $x^T A y = [4\ 5\ 6]\begin{bmatrix}0\\1\\0\end{bmatrix} = 5$

(b) $x^T A = [4\ 5\ 6]$

(c) $x^T = [0\ 1]$

$Ay = \begin{bmatrix}2\\5\end{bmatrix}$

(6) $M = \begin{bmatrix}A & B\\C & D\end{bmatrix}$

$M^T = \begin{bmatrix}A^T & C^T\\B^T & D^T\end{bmatrix}$

If $C = B = C^T = B^T$ + $A = A^T$ + $D = D^T$ M is a symmetric matrix

(7) (a) False unless A is symmetric since

$\begin{bmatrix}0 & A\\A & 0\end{bmatrix}^T = \begin{bmatrix}0 & A^T\\A^T & 0\end{bmatrix} \overset{?}{=} \begin{bmatrix}0 & A\\A & 0\end{bmatrix}$ iff $A^T = A$

(b) ~~False~~ $(AB)^T = B^T A^T = BA \neq AB$ Ness
False

(c) True if A^{-1} is symmetric then A is symmetric
this statement is the contrapositive of the above statement

(d) $(ABC)^T = C^T B^T A^T = CBA$ + this statement is true but
the transpose of $(ABC)^T$ is $((ABC)^T)^T = ABC$ so
the statement is false

(8) Since in each of the n rows of P we can select one of the rows to put a 1 in.

For example for the 1st row we have n choices of columns to place a 1 in, For the 2nd row we then have $n-1$ choices for where we can place a 1. For the 3rd row we have $n-2$, ...

Thus in total we have $n(n-1)(n-2)\cdots 1 = n!$ different permutation matrices

(9) Give examples where $P_1P_2 \neq P_2P_1$

let $P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

P_1 exchanges rows $1+2$ $\qquad$ P_2 exchanges rows $2+3$

Then $P_1P_2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ + $P_2P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ Showing $P_1P_2 \neq P_2P_1$

For an example where $P_3P_4 = P_4P_3$

$P_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\quad P_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

P_3 exchanges rows $1+2$ $\qquad$ P_4 exchanges rows $3+4$

Then $P_3P_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

& $P_4P_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ Thus $P_3P_4 = P_{43}$ since

in effect the actions of the P's are independent from each other.

(10) 12 even permutations   1st list all the permutations:

~~(1,2,3,4)~~ → $4! = 24$

~~(2,1,4,3)~~

(~~4,3,2,~~1)   ~~(1,2,3,4)~~

~~(2~~

~~(1,2,3,4) → (2,1,4,3) →~~ (3,4,1,2)

(1,2,3,4)	(1,2,3,4)✓	(3,1,2,4)✓	The even permutations are the ones with an even # of exchanges ~~to~~ ~~obtain~~ from (1,2,3,4) to obtain the given permutation
(2,1,4,3)	(1,2,4,3)	(3,1,4,2)	
(4,3,2,1)	(1,3,2,4)	(3,2,1,4)✓	
(3,4,1,2)	(1,3,4,2)✓	(3,2,4,1)	
(1,4,2,3)	(2,1,3,4)	(4,1,2,3)✓	(3,4,1,2)✓
(1,4,3,2)✓	(2,1,4,3)✓	(4,1,3,2)	(3,4,2,1)
	(2,3,1,4)✓	(4,2,1,3)✓	
	(2,3,4,1)	(4,2,3,1)	
	(2,4,1,3)	(4,3,1,2)✓	
	(2,4,3,1)✓	(4,3,2,1)	

Giving in summary

(1,2,3,4)
(1,3,4,2)
(2,1,4,3)
(2,3,1,4)
(2,4,3,1)
(3,1,2,4)
(3,2,1,4)
(4,1,2,3)
(4,2,1,3)
(4,3,1,2)
(1,4,3,2)
(3,4,1,2)

(11) Given $A = \begin{bmatrix} 0 & 0 & 6 \\ 1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix}$

If $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ Then PA is upper triangular

w/ $PA = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$

$P_1 A = \begin{bmatrix} 0 & 0 & 6 \\ 0 & 4 & 5 \\ 1 & 2 & 3 \end{bmatrix}$ If $P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

~~Where~~ ~~π~~

Then $P_1 A P_2 = \begin{bmatrix} 6 & 0 & 0 \\ 5 & 4 & 0 \\ 3 & 2 & 1 \end{bmatrix}$ is lower triangular with $P_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

Multiplying A on the right by a permutation matrix exchanges columns of A.

(12) (a) $\cancel{x^T y} = (Px)^T (Py) = x^T P^T P y = x^T y$

since $P^T = P^{-1}$.

we are only exchanging the order in the summation

(b) $x = (1,2,3)$

$y = (1,1,2)$

$Px \circ y \neq x \circ Py$

let $P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ Then $Px = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ & $Py = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

so $Px \circ y = 3 + 2 + 2 = 7$

... didn't work

& $x \circ Py = 2 + 2 + 3 = 7$

let $P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ Then $Px = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ & $Py = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

Then $Px \circ y = 3 + 1 + 4 = 8$

$x \circ Py = 1 + 2 + 3 = 6$ so $Px \circ y \neq x \circ Py$

(13) (a) ~~P is simply a reshuffling of the rows of A.~~
~~Repeated application of P must simply reorder the~~
~~Assume it did not. Then $P^k \neq I \; \forall k$. But~~
~~$P^k = I \Rightarrow P^{k-1} = P^{-1} = P^T$~~
~~Effectively we have permuted round back to the identity~~

Since there are only a finite # of permutation matrices taking $P^k \; \forall k$ must eventually cycle, Not all elements can be unique or else there would be as many of them.

Then two powers of P must be identical $P^s = P^t$

Multiply by $(P^t)^{-1} = (P^{-1})^t$ to get

$I = (P^s)(P^{-1})^t = P^{s-t}$ & we must have $s-t \le n!$

(b) let $P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$.

Thus the 1st two rows exchange rows & the last 3 rows exchange rows the L.C.M. of 2 & 3 is 6 which should be the "order" of this permutation matrix

$\le$ [illegible]

lets check $P^2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & \\ 0 & 0 & & & 1 \end{bmatrix}$

let $P = \xcancel{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$

Then $P^2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$

$P^4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$ $\qquad \cancel{P^6 \neq [}$ $\qquad P^5 = I$.

let $P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$ $\qquad P^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$

$P^4 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$ $\qquad P^6 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = I$ ✓!!

(14) (a) P sends row 1 to row 4

P^T sends row 1 to row 4 since these two elements are symmetric in the matrix

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ & & & \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

When $P = P^T$ row exchanges come in pairs w/ No overlap

(b) P moves all 4 rows

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad \text{Then} \quad P = P^T$$

(15) (a) let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ or $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = I \cdot U$

So $L = I$ + $U \neq I$

16

(a) $(A^2 - B^2)^T = A^2 - B^2$ symmetric

(b) $((A+B)(A-B))^T = \cancel{A+B}\ (A-B)^T(A+B)^T$

$= (A-B)(A+B)$ Not ness symmetric

(c) $(ABA)^T = A^TB^TA^T = ABA$ symm.

(d) $(ABAB)^T = B^TA^TB^TA^T = BABA$ Not ness symmetric

17

(a) A is $5\times 5 \to 25$ total elements

w/ 5 on the diagonal + $\frac{25-5}{2}$ on each side of the off diagonal $= 10$.

plus the 5 elements on the diagonal = 15 total degrees of freedom.

(b) In L we have 10 elements free to choose (the lower diagonal elements)

In U we have the diagonal elements only since the factorization of A is

$A = LDL^T$

(c) The diagonal in this case must be zero but one triangular region of the non diagonal ~~terms~~ terms are free so total of 10 #'s can be choosen

18

(a) $(R^TAR)^T = R^TA^TR = R^TAR$ so this matrix is symmetric

R^TAR is of size $n\times m \cdot m\times m \cdot m\times n$ so a matrix of size $n\times n$

(b) A diagonal element for R^TR is given by

$$(R^TR)_{ii} = \sum_{k=1}^{m} (R^T)_{ik} R_{ki} = \sum_{k=1}^{m} R_{ki}R_{ki} = \sum_{k=1}^{m} R_{ki}^2 > 0$$

If the ~~column~~ ith column of R is

(19) If $A = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}$

with $E_{21} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$ we have $E_{21}A = \begin{bmatrix} 1 & 3 \\ 0 & 7 \end{bmatrix}$

Then $A = E_{21}^{-1} \begin{bmatrix} 1 & 3 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 7 \end{bmatrix}$

$$= \underbrace{\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}}_{\equiv L}\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix}}_{D}\underbrace{\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}}_{L^T}$$

If $A = \begin{bmatrix} 1 & b \\ b & c \end{bmatrix}$

multiplying on the left by $E_{21} = \begin{bmatrix} 1 & 0 \\ -b & 1 \end{bmatrix}$ we have

$$E_{21}A = \begin{bmatrix} 1 & b \\ 0 & c-b^2 \end{bmatrix}$$

Assuming $c-b^2 \neq 0$ so that we have a 2nd pivot we see that

$$A = E_{21}^{-1}\begin{bmatrix} 1 & b \\ 0 & c-b^2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}}_{L}\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & c-b^2 \end{bmatrix}}_{D}\underbrace{\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}}_{L^T}$$

If $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ Then multiplying by $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

we get that

$$E_{21}A = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$ then multiply by $$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2/3 & 1 \end{bmatrix}$$

we get $$E_{32}E_{21}A = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & 0 & \underbrace{2-2/3}_{4/3} \end{bmatrix}$$

so that we have

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2/3 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & 0 & 4/3 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & -2/3 & 1 \end{bmatrix}}_{L}\underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3/2 & 0 \\ 0 & 0 & 4/3 \end{bmatrix}}_{D}\underbrace{\begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 1 & -2/3 \\ 0 & 0 & 1 \end{bmatrix}}_{L^T}$$

(20) If $$A = \begin{bmatrix} 2 & 4 & 8 \\ 4 & 3 & 9 \\ 8 & 9 & 0 \end{bmatrix}$$

let $$E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$$

$-16+9 = -7$

-16

Then $$EA = \begin{bmatrix} 2 & 4 & 8 \\ 0 & -5 & -7 \\ 0 & -7 & -32 \end{bmatrix}$$

The symmetric 2x2 matrix that remains is

$$\begin{bmatrix} -5 & -7 \\ -7 & -32 \end{bmatrix}$$

If $A = \begin{bmatrix} 1 & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$

Then if $E = \begin{bmatrix} 1 & 0 & 0 \\ -b & 1 & 0 \\ -c & 0 & 1 \end{bmatrix}$

Then $EA = \begin{bmatrix} 1 & b & c \\ 0 & d-b^2 & e-cb \\ 0 & e-cb & f-c^2 \end{bmatrix}$ Now if $d-b^2 \neq 0$ we can continue

The submatrix is $\begin{bmatrix} d-b^2 & e-cb \\ e-cb & f-c^2 \end{bmatrix}$

(21) If $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & \underline{4} \\ 2 & 3 & 4 \end{bmatrix}$

let $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

Then $PA = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 3 & 4 \\ 0 & 1 & 1 \end{bmatrix}$

let $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Then $E_{21}PA = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

Then defining $U = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

we have $PA = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

Checking we multiply to obtain

$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 3 & 4 \\ 0 & 0 & 1 \end{bmatrix}$ which is correct.

If $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Then $E_{21} A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ so we should permute 1st

let $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ Then $PA = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 2 & 4 & 1 \end{bmatrix}$

Then let $E = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$ so

$$EPA = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \equiv U$$

so that $PA = E^{-1}U = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

(22) let $A = \cancel{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}} \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

Then let $P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = A^T$

The $PA = I$

so P is given + $L = U = I$.

(23) when $A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 3 & 8 \\ 2 & 1 & 1 \end{bmatrix}$

we have it $P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ that

$$PA = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & 8 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & +3 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}}_{U}$$

To factor it into $A = L_1 P_1 U_1$

We ~~see that~~ will elliminate the elt

(2,2)

By
$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E}\underbrace{\begin{bmatrix} 0 & 1 & 2 \\ 0 & 3 & 8 \\ 2 & 1 & 1 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 2 \\ 2 & 1 & 1 \end{bmatrix}}$$
$$= P_1 \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

when $P_1 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

thus $EA = P_1 U$

or $A = E^{-1} P_1 U$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

(24) For

$$A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$$ we have $A(1,1) < tol$ so

We exchange rows 1+2 to get

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$$ so $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Then LU factorization proceeds normally. Since this is already in "U" form $L = I$

For $A = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 3 & 4 \\ 0 & 5 & 6 \end{bmatrix}$

$A(1,1)$ is less than tol ~~so~~ $A(2,1) > tol$

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So A is $A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 0 & 1 \\ 0 & 5 & 6 \end{bmatrix}$

Now $A(2,2) < tol$ so $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

& A becomes

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 1 \end{bmatrix}$$

which is in U form so the code stops w/ $L = I$

(25) If $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 2 & 5 & 4 \end{bmatrix}$

since $A(1,1) <$ tol we have $P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

& $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 2 & 5 & 4 \end{bmatrix}$

Then $L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ +2 & 0 & 1 \end{bmatrix}$ & $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 3 & 4 \end{bmatrix}$

Next $A(2,2) >$ tol so normal LU proceeds &

$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}$ w/ $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix}$

which is in U form & the code stops

~~This test~~ lets check

$$LU = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 2 & 5 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \quad \text{yes} \checkmark$$

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(27) If $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 2 & 5 & 8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 3 & 6 \end{bmatrix}$

By the action of $E = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

But given as stated $P_1 U_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{bmatrix}$

we thus have that

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 2 & 5 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 2 & 5 & 8 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}}_{L_1} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{P_1} \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{bmatrix}}_{U_1}$$

(28) Since $L = U = I$

$L \cdot U = I$

So $PA = I$

which since A itself is a permutation matrix to make $PA = I$

P must be $A^{-1} = A^T$ since A is a permutation matrix

is to multiply by the inverse or A^T

(29) $I \neq P_1 P_2 P_3$ with P_i a permutation matrix (not the identity permutation) since the inverse of any permutation matrix is (another permutation) equal to P^T an odd # of permutations can not return us to the identity only an even # ~~of~~ can.

(30) (a) $E_{21} A = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 4 \\ 0 & 4 & 9 \end{bmatrix}$ with $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

so $E_{21}^T = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

+ $E_{21} A E_{21}^T = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 4 \\ 0 & 4 & 9 \end{bmatrix} \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 4 & 9 \end{bmatrix}$

continuing let $E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ we have

$$E_{32} E_{21} A E_{21}^T \cancel{E} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

Then $E_{32} E_{21} A E_{21}^T E_{32}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\equiv D}$

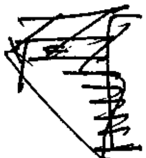
inverting as suggested in the book gives

$$A = E_{21}^{-1} E_{32}^{-1} D (E_{32}^{-1})^T (E_{21}^{-1})^T$$

bihy $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} D (E_{32}^{-1})^T (E_{21}^{-1})^T$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} D \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = LDL^T$$

(31) ~~$(Ax)^T y = x^T A^T y$~~



$$y = Ax = \begin{bmatrix} y_{BC} \\ y_{CS} \\ y_{BS} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_B \\ x_C \\ x_S \end{bmatrix}$$

(a) $A^T y = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} y_{BC} \\ y_{CS} \\ y_{BS} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_B \\ x_C \\ x_S \end{bmatrix}$

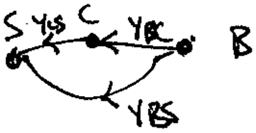
$$= \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_B \\ x_C \\ x_S \end{bmatrix}$$

$$\begin{bmatrix} y_{BC} + y_{BS} \\ -y_{BC} + y_{CS} \\ -y_{CS} - y_{BS} \end{bmatrix}$$

$$(Ax)^T y = \begin{bmatrix} x_B - x_C \\ x_C - x_S \\ x_B - x_S \end{bmatrix}^T \begin{bmatrix} x_B - x_C \\ x_C - x_S \\ x_B - x_S \end{bmatrix} = (x_B - x_C)^2 + (x_C - x_S)^2 + (x_B - x_S)^2$$

(31) x = voltages
y = current

(a) $$A^T y = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} y_{BC} \\ y_{CS} \\ y_{BS} \end{bmatrix} = \begin{bmatrix} y_{BC} + y_{BS} \\ -y_{BC} + y_{CS} \\ -y_{CS} - y_{BS} \end{bmatrix}$$



(b) $$(Ax)^T y = x^T A^T y$$

$$Ax = \begin{bmatrix} x_B - x_C \\ x_C - x_S \\ x_B - x_S \end{bmatrix}$$

So $$(Ax)^T y = y_{BC}(x_B - x_C) + y_{CS}(x_C - x_S) + y_{BS}(x_B - x_S)$$

Now $A^T y$ is given above so

$$x^T A^T y = x_B(y_{BC} + y_{BS}) + x_C(-y_{BC} + y_{CS}) + x_S(-y_{CS} - y_{BS})$$

$$= y_{BC}(x_B - x_C) + y_{CS}(x_C - x_S) + y_{BS}(x_B - x_S)$$

So they are the same

(32) $\begin{bmatrix} y_S \\ y_R \\ y_L \end{bmatrix} = \begin{bmatrix} 1 & 50 \\ 40 & 1000 \\ 2 & 50 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

~~cost~~ total cost $= 700 y_S + 3 y_R + 3000 y_L$

$$A^T = \begin{bmatrix} 1 & 40 & 2 \\ 50 & 1000 & 50 \end{bmatrix}$$

$$A^T y = \begin{bmatrix} y_S + 40 y_r + 2 y_L \\ 50 y_S + 1000 y_r + 50 y_L \end{bmatrix}$$

A ~~single~~ car ~~truck~~ represents ~~repairs~~

$$A^T \begin{bmatrix} 700 \\ 3 \\ 3000 \end{bmatrix} = \begin{bmatrix} 700 + 120 + 6000 \\ 35000 + 3000 + 150{,}000 \end{bmatrix} = \begin{bmatrix} 6820 \\ 188{,}000 \end{bmatrix}$$

150,000
35,000
3,000
188

(33) $A = \begin{bmatrix} 1 & 50 \\ 40 & 1000 \\ 2 & 50 \end{bmatrix}$ ~~$A x \cdot y$ is the cost to produce of the x_1 trucks & x_2 cars times the~~

~~$x \cdot A^T y$~~

Ax is the amount of raw materials needed to produce x_1 trucks & x_2 cars.

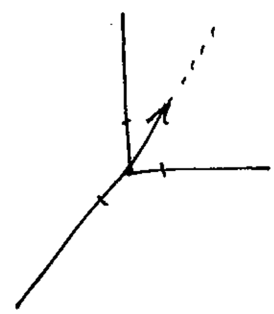
$Ax \circ y$ is the total cost to produce x_1 trucks & x_2 cars

$x \circ (A^T y)$ is the same ~~thing~~ thing because $A^T y$ is the cost to produce 1 truck & 1 car. $x \cdot A^T y$ is the product of the two

(34) $P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$P^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ $\quad P^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$Pa = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = a$ ✓



~~Then~~ The angle between $(2,3,-5)$ is $Pv = (-5,2,3)$

$25+9+4$

$29+9 = 38$

$\cancel{\cos\theta = \frac{-10+6-15}{\sqrt{4+9+25}\sqrt{25+4+9}}} =$

Since $P^3 a = a$ 3 applications rotate a us back to where we started

& each application of P rotates us $\frac{360}{3} = 120$ degrees

(35) $A = \begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix}$

$EH = A$ w/ E elementary row operation, + H a symmetric matrix

$H = E^{-1}A$ let $E^{-1} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

Then $E^{-1}A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$

so $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ Check: $\begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix}$ ✓

(36) $A = EH$ $A = LDU$

$= \cancel{(LC)(C^{T}DU)}$ $\underbrace{(LC)}_{\text{lower triangular}}\underbrace{(U^TDU)}_{\text{symmetric}}$

$C = (U^{-1})^T$ so since U is upper unit triangle U^{-1} is also

+ $(U^{-1})^T$ is lower unit triangle

$\therefore$ $LC = L \cdot (U^{-1})^T$ is unit lower triangular